

Generalized Minkowski weights and Chow rings of T -varieties

Ana Botero

Abstract. We give a combinatorial characterization of Fulton’s operational Chow cohomology groups of a complete, \mathbb{Q} -factorial, rational T -variety of complexity one in terms of so called generalized Minkowski weights in the contraction-free case. We also describe the intersection product with Cartier invariant divisors in terms of the combinatorial data. In particular this provides a new way of computing top intersection numbers of invariant Cartier divisors combinatorially.

Introduction

Any algebraic variety X has Chow “homology” groups $A_*(X) = \bigoplus_k A_k(X)$ and Chow “cohomology” groups $A^*(X) = \bigoplus_k A^k(X)$. The latter are the operational groups defined in [8, Chapter 17]. These cohomology groups have a natural graded ring structure, written with a cup product “ \cup ”, and $A_*(X)$ is a module over $A^*(X)$, written with a cap product “ \cap ”. If X is complete, composing with the degree homomorphism from $A_0(X)$ to \mathbb{Z} one has a a Kronecker duality homomorphism

$$A^k(X) \longrightarrow \text{Hom}(A_k(X), \mathbb{Z}), \quad z \longmapsto (a \mapsto \deg(z \cap a)).$$

In general, this map is not an isomorphism.

However, we will see that for a large class of T -varieties of complexity one, the Kronecker duality homomorphism is indeed an isomorphism, at least after tensoring with \mathbb{Q} . This will follow from Poincaré duality and a combinatorial property related to the T -variety called shellability (see Proposition 3.5).

We now briefly recall the definition and combinatorial characterization of T -varieties. We refer to [2–4, 16] for further details on general T -varieties, and to [12, 14, 15] for the case of complexity one.

A T -variety is a normal algebraic variety X defined over an algebraically closed field of characteristic zero with an effective action of an algebraic torus T . The complexity of a T -variety is defined as $\dim X - \dim T$. Hence T -varieties of complexity zero correspond exactly to toric varieties. As for toric varieties, one can encode T -varieties of arbitrary complexity in terms of some combinatorial data. For this, one uses the language of p -divisors and divisorial fans (see Section 1 for details). In the case of complexity one, one even has a simpler description in terms of so called fancy divisors. This case is

in some ways close to the case of toric varieties. For example, one has a characterization of invariant Cartier divisors in terms of so called divisorial Cartier support functions (see Section 2.1). These are collections of piecewise-affine functions defined on the slices of the divisorial fan sharing the same recession function. Such a characterization is not known in higher complexity. This is one of the reasons why we decided to restrict to the complexity-one case in the present article.

In order to state our results, let X be a complete, rational T -variety of complexity one with divisorial fan \mathcal{S} and recession fan Σ . We can think of \mathcal{S} as a collection of complete rational polyhedral complexes $(\mathcal{S}_p)_{p \in \mathbb{P}^1}$, all having recession fan Σ , and $\mathcal{S}_p \neq \Sigma$ only for finitely many $p \in \mathbb{P}^1$. The \mathcal{S}_p are called the slices of \mathcal{S} . Let P be the set of points such that $\mathcal{S}_p \neq \Sigma$. We assume that $|P| \geq 2$.

It has been shown in [15] that the Chow groups $A_k(X)$ are generated by invariant subvarieties corresponding to elements in the complexes \mathcal{S}_p for $p \in P$, and in Σ . Roughly, polyhedra in the \mathcal{S}_p correspond to vertical cycles, cones in Σ correspond to horizontal cycles, and one has to keep track of which subvarieties are contracted along the map $r: \tilde{X} \rightarrow X$ (see Section 1 for the definition of the map r). Relations are given by the divisors of invariant rational functions on subvarieties of dimension $k + 1$. This gives an explicit presentation of the Chow groups $A_k(X)$ by an exact sequence (see Theorem 3.6).

It now follows from the Kroenecker duality map, that one can associate to a given Chow cohomology class a certain function on the set of polyhedra of the \mathcal{S}_p 's and the set of cones in Σ . We call these functions *generalized Minkowski weights* (see Definition 4.1). The name is inspired by [10], where the authors study the operational Chow cohomology rings of complete toric varieties and show that they can be identified with so called *Minkowski weights*, which are certain functions on the set of cones of the fan defining the toric variety. This association is bijective whenever the Kroenecker duality map is an isomorphism.

We say that X is contraction-free if $\tilde{X} \simeq X$. In this case, and under some other mild assumptions, the Chow cohomology groups are generated in codimension one, and a Stanley–Reisner type presentation of the Chow groups is given in [14, Theorem 4]. We use this description together with Poincaré duality to show that in this case, the Kroenecker duality map is an isomorphism after tensoring with \mathbb{Q} .

The above is summarized in the following theorem, which is our first main result.

Theorem 4.4. *Let X be a complete rational T -variety of complexity one. Then there is a tropicalization map*

$$\text{trop}: A^k(X) \longrightarrow M_k(X)$$

between the Chow cohomology groups $A^k(X)$ and the group of k -codimensional generalized Minkowski weights $M_k(X)$. If X is projective, \mathbb{Q} -factorial and contraction-free, then the induced tropicalization map

$$\text{trop}: A^k(X)_{\mathbb{Q}} \longrightarrow M_k(X)_{\mathbb{Q}},$$

where $M_k(X)_{\mathbb{Q}}$ denotes \mathbb{Q} -valued generalized Minkowski weights, is a bijection.

Let us comment on the above assumptions. \mathbb{Q} -factoriality is needed for Poincaré duality to hold and, as the authors in [14] remark, their contraction-free assumption is necessary for their result, as they are using that $A^k(X)$ is generated by divisors, a fact which is no longer true in the non-contraction-free case. This is why we need to impose all of these conditions in the above theorem. We however expect that the Kronecker duality map is still an isomorphism also in the not necessarily contraction-free case.

For the rest of the introduction, assume that X is projective, \mathbb{Q} -factorial and contraction-free. The ring structure on $A^*(X)_{\mathbb{Q}}$ makes the group of generalized Minkowski weights $M_*(X)_{\mathbb{Q}} = \bigoplus_k M_k(X)_{\mathbb{Q}}$ into a commutative ring. The second main result of this article is a combinatorial description of the intersection pairing between Chow cohomology classes and T -invariant Cartier divisors.

Before we state our result, as was mentioned above, a T -Cartier divisor on X corresponds to a divisorial Cartier support function h on \mathcal{S} (Definition 2.5). We denote by $\text{CaSF}(\mathcal{S})$ the set of divisorial Cartier support functions on \mathcal{S} . In Section 5 we define in the contraction-free case an intersection pairing

$$\text{CaSF}(\mathcal{S}) \times M_k(X) \longrightarrow M_{k+1}(X), \quad (h, c) \longmapsto h \cdot c. \tag{0.1}$$

We can think of this pairing as an analogue of the so called *corner locus* of a tropical cycle in tropical geometry. Our second main result states that this pairing is compatible with the isomorphism trop of above and hence recovers the algebraic intersection product.

Theorem 5.6. *Assume that X is projective, \mathbb{Q} -factorial and contraction-free. Let $h \in \text{CaSF}(\mathcal{S})$ and let D_h its associated Cartier divisor. Set $[D_h] \in A^1(X)_{\mathbb{Q}}$ for its corresponding cohomology class. Then for any $z \in A^k(X)_{\mathbb{Q}}$ we have*

$$\text{trop}([D_h] \cup z) = h \cdot \text{trop}(z)$$

as generalized Minkowski weights in $M_{k+1}(X)_{\mathbb{Q}}$.

We expect that the pairing (0.1) generalizes to the non-contraction-free case such that Theorem 5.6 continues to hold. This will be part of a future project.

Now, to any invariant Cartier divisor D_h we associate a measure μ_h on a real vector space $\text{Tot}(\mathcal{S})$ by means of the pairing (0.1) (see Definition 6.1). It is supported on the vertices of the polyhedral complexes \mathcal{S}_p and the rays of Σ . Moreover, μ_h is positive if D_h is nef.

For any integer ℓ and any $p \in P$ we denote by $\mathcal{S}_p(\ell)$ and by $\Sigma(\ell)$ the set of ℓ -dimensional polyhedra in \mathcal{S}_p and the set of ℓ -dimensional cones in Σ , respectively.

As an application of Theorem 5.6 we obtain a way to compute the top intersection number of D_h as an integral with respect to the measure μ_h .

Corollary 6.2. *Assume that X is projective, \mathbb{Q} -factorial and contraction-free and let $\dim(X) = n + 1$. Let D_h be an invariant Cartier divisor associated to a Cartier divisorial support function h with recession function $\text{rec}(h) = \bar{h}$, and let μ_h be the associated*

measure. Then

$$D_h^{n+1} = \int_{\text{Tot}(\mathcal{S})} -h\mu_h = \sum_{p \in P} \sum_{F \in \mathcal{S}_p(0)} -h(v_F)\mu_h(F) \sum_{\tau \in \Sigma(1)} -\bar{h}(v_\tau)\mu_h(\tau),$$

where for $F \in \mathcal{S}_p(0)$, $v_F = F$, and for $\tau \in \Sigma(1)$, v_τ denotes the primitive vector spanning the ray τ .

This article is a first step towards a convex-geometrical description of the intersection theory of nef b-divisors (b stands for birational) on complexity-one T -varieties in the spirit of [7]. Indeed, we expect that nef b-divisors on such a T -variety can be characterized in terms of some “concave” functions on the divisorial fan. We then expect to associate a Monge–Ampère type measure to such a function, as a weak limit of measures of the form μ_h from above. We plan to pursue these ideas in the future.

Finally, in [5, 6] the authors study equivariant operational K-theory and Grothendieck transformations from bivariant operational K-theory to Chow. These results are later used in [18], where the author describes the operational K-theory of complete toric varieties in terms of so called Grothendieck weights (a K-theoretic analog of the Minkowski weights) and gives a product formula in terms of these weights. It would be interesting to try to characterize the operational K-theory of complete T -varieties of complexity one in terms of some generalized Grothendieck weights, a K-theoretic analog of the generalized Minkowski weights introduced in the present article.

Outline of the paper

In Section 1 we recall some basic facts of T -varieties and the combinatorial framework to describe them. In particular, we recall the definitions of p -divisors and of divisorial fans. In Section 2 we restrict to the complexity one case and we describe the invariant subvarieties induced by the polyhedra in the slices of the divisorial fan. It turns out that these invariant subvarieties generate the pseudoeffective cone of the T -variety and we give a list of its generators (Theorem 2.3). For this we mainly follow [15]. Then we recall the classification of T -invariant Cartier divisors in terms of Cartier divisorial support functions, following the work in [4, 16].

In Section 3 we recall the definition of the Chow homology and cohomology groups of an algebraic variety. In Section 3.1 we show that the Kroenecker duality map is an isomorphism for projective, rational, \mathbb{Q} -factorial, contraction-free T -varieties after tensoring with \mathbb{Q} (see Proposition 3.5). Then, following [15], we give in the general, not necessarily contraction-free case, an explicit presentation of $A_k(X)$ in terms of an exact sequence. In Section 4 we prove our first main result. We give the definition of the groups of generalized Minkowski weights and we show that they can be identified with the Chow cohomology groups in the projective, \mathbb{Q} -factorial, contraction-free case. This yields Theorem 4.4.

In Section 5, we prove our second main result. We restrict to the \mathbb{Q} -factorial, contraction free case and define an intersection pairing between Cartier divisorial support functions and generalized Minkowski weights. We show that this pairing is compatible with the tropicalization isomorphism from Theorem 4.4. This gives Theorem 5.6.

In Section 6 we associate a measure μ_h to any Cartier divisorial support function h using the intersection pairing. As an application of Theorem 5.6 we obtain a way of computing top intersection numbers of invariant Cartier divisors in terms of this measure. We get Corollary 6.2. We end the article with an example (see Example 6.3).

Conventions. We denote by k an algebraically closed field of characteristic 0. A variety means an integral, separated scheme of finite type over k . Varieties are assumed to be reduced and irreducible. A subvariety is a closed subscheme which is a variety.

1. Preliminaries on T -varieties

We use standard notation from toric geometry. Let N be a lattice of dimension n , $M = N^\vee$ its dual lattice and $T = \text{Spec}(N) \simeq (k^*)^n$ the n -dimensional torus. For any ring R we put $N_R = N \otimes_{\mathbb{Z}} R$ and $M_R = M \otimes_{\mathbb{Z}} R$.

Definition 1.1. A T -variety is a normal variety X together with an effective algebraic torus action $T \times X \rightarrow X$. Its *complexity* is defined as $\dim X - n$.

Thus T -varieties of complexity zero correspond to toric varieties.

We recall the general framework for describing T -varieties of any complexity. For details we refer to [2–4].

We use notations and basic facts regarding convex cones and polyhedra from [17]. In particular, for any polyhedron $\Lambda \in N_{\mathbb{Q}}$, its *recession cone* $\text{rec}(\Lambda)$ is the set of direction vectors of all rays contained in Λ

$$\text{rec}(\Lambda) := \{v \in N_{\mathbb{Q}} : v + \Lambda \subseteq \Lambda\}.$$

Fix a polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$. Consider the semigroup (under Minkowski addition)

$$\text{Pol}_{\mathbb{Q}}^+(N, \sigma) = \{\Lambda \subseteq N_{\mathbb{Q}} : \Lambda \text{ is a polyhedron with } \text{rec}(\Lambda) = \sigma\}.$$

We also allow $\emptyset \in \text{Pol}_{\mathbb{Q}}^+(N, \sigma)$.

Let Y be a normal variety. We denote by $\text{CaDiv}(Y)$ the group of Cartier divisors on Y and by $\text{CaDiv}_{\mathbb{Q}}(Y)$ the group of \mathbb{Q} -Cartier divisors on Y . A *polyhedral divisor on (Y, N)* is a formal sum

$$\mathcal{D} = \sum_Z \Lambda_Z \cdot Z,$$

for $\Lambda_Z \in \text{Pol}_{\mathbb{Q}}^+(N, \sigma)$ and $Z \in \text{CaDiv}_{\mathbb{Q}}(Y)$, such that only finitely many Λ_Z differ from σ . For a polyhedral divisor \mathcal{D} consider the evaluation

$$\mathcal{D}(u) = \sum_{Z \text{ s.t. } \Lambda_Z \neq \emptyset} \min\langle \Lambda_Z, u \rangle Z \in \text{CaDiv}_{\mathbb{Q}}(Y)$$

for $u \in \sigma^\vee \cap M$. This is a finite sum.

The polyhedral divisor \mathcal{D} on (Y, N) is called a *p-divisor* if $\mathcal{D}(u)$ is semiample for all $u \in \sigma^\vee \cap M$, as well as big for $u \in \text{relint}(\sigma^\vee) \cap M$.

To a *p-divisor* \mathcal{D} one associates a sheaf of rings

$$\mathcal{O}_Y(\mathcal{D}) = \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(\mathcal{D}(u)).$$

Then $X = \text{Spec } \Gamma(Y, \mathcal{O}_Y(\mathcal{D}))$ is an affine *T*-variety of complexity $\dim Y$. Also the relative spectrum $\tilde{X} = \text{Spec}_Y \mathcal{O}_Y(\mathcal{D})$ is a *T*-variety of complexity $\dim Y$ and there is an equivariant map $r: \tilde{X} \rightarrow X$. The variety X is said to be *contraction-free* if $\tilde{X} = X$.

Altman and Hausen show that any affine *T*-variety arises from a *p-divisor* in this way [2, Theorem 3.4].

Given two *p-divisors* \mathcal{D} and \mathcal{D}' on (Y, N) we write $\mathcal{D}' \subseteq \mathcal{D}$ if each coefficient of \mathcal{D}' is contained in the corresponding coefficient of \mathcal{D} . In this case, we have a map

$$X(\mathcal{D}') \rightarrow X(\mathcal{D}),$$

and \mathcal{D}' is said to be a *face* of \mathcal{D} if this map is an open embedding.

The intersection $\mathcal{D} \cap \mathcal{D}'$ of two polyhedral divisors is the polyhedral divisor

$$\sum_Z (\Lambda_Z \cap \Lambda'_Z) \cdot Z.$$

Definition 1.2. A *divisorial fan* \mathcal{S} is a finite set of *p-divisors* on (Y, N) such that the intersection of any two *p-divisors* of \mathcal{S} is a face of both and \mathcal{S} is closed under taking intersections.

We denote by $X(\mathcal{S})$ the scheme obtained by gluing the affine *T*-varieties $X(\mathcal{D})$ and $X(\mathcal{D}')$ along the open subvarieties $X(\mathcal{D} \cap \mathcal{D}')$ for each $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$. As in the toric case, $X(\mathcal{S})$ turns out to be a *T*-variety of complexity $\dim Y$.

Analogously, define the variety $\tilde{X}(\mathcal{S})$ by gluing the affine *T*-varieties $\tilde{X}(\mathcal{D})$ and $\tilde{X}(\mathcal{D}')$ along the open subvarieties $\tilde{X}(\mathcal{D} \cap \mathcal{D}')$ for each $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$.

We have an induced proper birational morphism

$$r: \tilde{X}(\mathcal{S}) \longrightarrow X(\mathcal{S})$$

and $X(\mathcal{S})$ is said to be *contraction-free* if $r = \text{id}$. The set

$$\Sigma(\mathcal{S}) = \{ \text{rec}(\mathcal{D}) \mid \mathcal{D} \in \mathcal{S} \}$$

is the *recession fan* of \mathcal{S} , where $\text{rec}(\mathcal{D})$ stands for the common recession cone of the coefficients of \mathcal{D} .

Altmann, Hausen and Süß show that any *T*-variety arises from a divisorial fan in this way [3, Theorem 5.6].

The *degree* of a *p-divisor* \mathcal{D} , denoted $\text{deg}(\mathcal{D})$, is the Minkowski sum $\sum_Z \Lambda_Z$ of its coefficients.

We now specialize to the case where Y is a smooth projective curve.

Definition 1.3. A *marked fancy divisor* on Y is a formal sum

$$\Xi = \sum_{p \in Y} \Xi_p \otimes [p],$$

together with a complete fan Σ in $N_{\mathbb{Q}}$ and a subset $K \subseteq \Sigma$ such that

- (1) for all $p \in Y$, the coefficient Ξ_p is a complete polyhedral subdivision of $N_{\mathbb{Q}}$ with $\text{rec}(\Xi_p) = \{\text{rec}(\Lambda_p) \mid \Lambda_p \in \Xi_p\} = \Sigma$.
- (2) for a cone $\sigma \in K$ of full dimension, the element $\mathcal{D}^\sigma = \sum_p \mathcal{D}_p^\sigma \otimes [p]$ is a p -divisor, where \mathcal{D}_p^σ denotes the unique polyhedron in Ξ_p whose recession cone is equal to σ .
- (3) for a full-dimensional cone $\sigma \in K$ and a face $\tau \prec \sigma$ we have that $\tau \in K$ if and only if $\text{deg}(\mathcal{D}^\sigma) \cap \tau \neq \emptyset$.
- (4) if τ is a face of σ then $\tau \in K$ implies that $\sigma \in K$.

The elements of $K \subseteq \Sigma$ are called *marked cones*.

It is shown in [12, Proposition 1.6] that any complete T -variety of complexity one corresponds to a marked fancy divisor. Given a divisorial fan \mathcal{S} one associates a marked fancy divisor $\Xi(\mathcal{S})$ by taking the polyhedral subdivisions \mathcal{S}_p given by \mathcal{S} and letting K consist of all cones $\sigma \in \Sigma$ such that there exists $\mathcal{D} \in \mathcal{S}$ with $\text{rec}(\mathcal{D}) = \sigma$ and such that no coefficient of \mathcal{D} equals \emptyset . The variety \tilde{X} is given as a marked fancy divisor by the same subdivisions as for X but with $K = \emptyset$. Thus the cones in K capture the information of which orbits are identified via the contraction map r .

Intuitively, one can think of a T -variety of complexity one in the following way. For a point $p \in Y$, the fiber over p corresponds to a union of toric varieties corresponding to the polyhedral subdivision \mathcal{S}_p . The polyhedral subdivision at a general fiber is $\Sigma = \text{rec}(\mathcal{S})$ and hence the generic fiber is the toric variety corresponding to Σ .

2. Subvarieties of T -varieties of complexity one

Let X be a $(n + 1)$ -dimensional complete, rational T -variety of complexity one with divisorial fan \mathcal{S} over $Y = \mathbb{P}^1$, recession fan Σ in $N_{\mathbb{Q}}$ and subset $K \subseteq \Sigma$. Assume that K is a proper subset of Σ , i.e., $K \neq \Sigma$.

For $p \in \mathbb{P}^1$ we write \mathcal{S}_p for the corresponding complete polyhedral subdivision of $N_{\mathbb{Q}}$. Let P be the set of points of \mathbb{P}^1 such that $\mathcal{S}_p \neq \Sigma$. P is a finite set. We assume that P has size at least two.

Notation 2.1. We use standard notation in toric and tropical geometry. For a cone $\sigma \in \Sigma$ we set $N_\sigma = N \cap \text{span}(\sigma)$ and $N(\sigma) = N/N_\sigma$. For any integer $k \geq 0$ we denote by $\Sigma(k)$ the set of cones of Σ of dimension k . Also for a polyhedral subdivision \mathcal{S}_p we denote by $\mathcal{S}_p(k)$ the set of k -dimensional polyhedra in \mathcal{S}_p . If $\tau \in \Sigma(k)$ is a cone. Then for every cone $\sigma \in \Sigma(k + 1)$ with $\tau \prec \sigma$ we denote by $v_{\tau,\sigma}$ the *lattice normal vector of σ relative to τ* .

This is the image in $N(\tau)$ of the unique generator of N_σ/N_τ that points in the direction of σ . If $k = 0$ we also write v_σ for $v_{\{0\},\sigma}$.

For $F \in \mathcal{S}_p$ denote by $N_{F,\mathbb{R}}$ the linear space spanned by all $x - y$ for $x, y \in \sigma$. Further, denote by $N_F = N_{F,\mathbb{R}} \cap N$ the induced sublattice and by $N(F)$ the quotient N/N_F . If $F \in \mathcal{S}_p(k)$ then for every $G \in \mathcal{S}_p(k + 1)$ denote by $v_{F,G}$ the *lattice normal vector of G relative to F* . This is the image in $N(F)$ of the unique generator of N_G/N_F that points in the direction of G .

Be aware that the notation v_F for $F \in \mathcal{S}_p$ is given below in Notation 2.2.

On \tilde{X} we have various subvarieties arising from the combinatorial structure. We briefly recall the description of these here from [15, Section 3] (see also [11]).

- For $p \in \mathbb{P}^1$ and a polyhedron $F \in \mathcal{S}_p$, there is a T -orbit $\text{orb}(p, F) \subseteq \tilde{X}$ of dimension $\text{codim}(F)$. The closure of this orbit in \tilde{X} is denoted $Z_{p,F}$. It is an invariant subvariety of an irreducible component of the fiber. The irreducible component is a toric variety corresponding to the fan $\Sigma(p, F) \subseteq N(F)_\mathbb{Q}$ consisting of all polyhedral in \mathcal{S}_p containing F . The corresponding character lattice is $M(F) = N(F)^\vee$.
- Let η be the generic point of \mathbb{P}^1 . For any cone $\sigma \in \Sigma$ we have a T -orbit $\text{orb}(\eta, \sigma)$ of dimension $\text{codim}(\sigma) + 1$. The closure of this orbit in \tilde{X} is denoted by B_σ . Then B_σ dominates \mathbb{P}^1 with general fiber given by the subvariety of X_Σ corresponding to σ . B_σ is itself a T -variety of complexity one (see proof of [15, Theorem 4.1]). Hence, B_σ corresponds to a divisorial fan $\mathcal{S}(\sigma)$. Over each $p \in \mathbb{P}^1$, the polyhedral complexes $\mathcal{S}(\sigma)_p \subseteq N(\sigma)_\mathbb{Q}$ consist of all polyhedra in \mathcal{S}_p with recession cone containing σ . The recession fan of $\mathcal{S}(\sigma)$ is the star $\Sigma(\sigma)$. For example, for $\tau \in \Sigma(1)$, B_τ is a horizontal divisor in the sense of [14].

On X , some of the subvarieties are contracted, namely those B_σ where σ is in K . In this case, $r(B_\sigma)$ is contracted to a subvariety of one dimension less. We denote by W_σ the corresponding orbit closure $r(B_\sigma)$ in X . Then $\dim(W_\sigma) = \text{codim}(\sigma)$.

Denote also by the same symbols $Z_{p,F}$ and B_σ the subvarieties $r(Z_{p,F})$ and $r(B_\sigma)$ for $\sigma \notin K$, respectively.

Let us introduce a bit more notation.

Notation 2.2. Let $F \in \mathcal{S}_p$ with $\text{rec}(F) = \sigma$. Then from the description of the invariant subvariety B_σ given above, it follows that the image of F in $\mathcal{S}(\sigma)_p$ is zero-dimensional. We denote this image by v_F . It is an element in $N(\sigma)_\mathbb{Q}$. Let $\mu(F)$ be the smallest positive integer in $N(\sigma)$ such that $\mu(F)v_F$ is a lattice point.

If $\sigma \in K$ we consider the index

$$s_\sigma = [\text{Stab}(r(Z_{p,F})): \text{Stab}(Z_{p,F})]$$

(note that since the map r is equivariant, any $t \in T$ in the stabilizer of $Z_{p,F}$ will be in the stabilizer of its image).

For any integer $k \geq 0$ let $\overline{\text{Eff}}_k(X) \subseteq N_k(X)$ be the pseudoeffective cone of X inside the group of k -cycles modulo numerical equivalence. The following is [15, Proposition 3.5].

Theorem 2.3. *For $k \geq 0$, the pseudoeffective cone $\overline{\text{Eff}}_k(X)$ of X is rational polyhedral, generated by classes of invariant subvarieties. Moreover, a list of generators is provided by the following classes.*

- B_σ with $\sigma \in \Sigma(n - k + 1)$, $\sigma \notin K$,
- $Z_{p,F}$ with $p \in \mathbb{P}^1$, $F \in \mathcal{S}_p(n - k)$, $\text{rec}(F) \notin K$,
- W_σ with $\sigma \in \Sigma(n - k)$, $\sigma \in K$.

Remark 2.4. Note that in the theorem above, subvarieties of the form $Z_{p,F}$ with $\text{rec}(F) \in K$ do not appear. Indeed, for a cone $\sigma \in K$, one can show that $r_*(Z_{p,F})$ is numerically equivalent to $(s_{\text{rec}(F)}/\mu(F)) \cdot W_{\text{rec}(F)}$ for any $p \in \mathbb{P}^1$ and any F with $\text{rec}(F) = \sigma$ (see [15, Lemma 3.4]). Hence all such classes $Z_{p,F}$ are proportional and we need only to remember the representative $W_{\text{rec}(F)}$.

2.1. T -Cartier divisors

We now briefly describe T -invariant Cartier divisors on X in terms of divisorial support functions on \mathcal{S} . For details we refer to [4, Section 7] and [16].

First, recall that given a piecewise affine function f on a polyhedral subdivision \mathcal{S}_p of $N_\mathbb{Q}$ with recession fan Σ , its *recession function* $\text{rec}(f)$ is the piecewise linear function on Σ defined by

$$\text{rec}(f): |\Sigma| \longrightarrow \mathbb{R}, \quad u \longmapsto \lim_{\lambda \rightarrow \infty} \frac{f(v_0 + \lambda u) - f(v_0)}{\lambda}.$$

Definition 2.5. A *divisorial support function* on \mathcal{S} is a collection $h = (h_p)_{p \in \mathbb{P}^1}$ of continuous, piecewise affine functions $h_p: |\mathcal{S}_p| \rightarrow \mathbb{Q}$ such that

- (1) h_p has integral slope and integral translation on every polyhedron in the polyhedral complex \mathcal{S}_p of $N_\mathbb{Q}$. In other words, for any $F \in \mathcal{S}_p$, we have

$$h_p|_F(u) = \langle m_F, u \rangle + \ell_F$$

with $(m_F, \ell_F) \in M \times \mathbb{Z}$.

- (2) All h_p have the same recession function $\text{rec}(h)$.
- (3) The set of points $p \in \mathbb{P}^1$ such that $\text{rec}(h) \neq h_p$ is finite.

The set of all divisorial support functions on \mathcal{S} is denoted by $\text{SF}(\mathcal{S})$.

To any divisor D on \mathbb{P}^1 we can associate a divisorial support function $\text{SF}(D)$ by $\text{SF}(D)_p \equiv \text{coeff}_p D$. Also, for any $u \in M$ we define its corresponding support function $\text{SF}(u)_p \equiv u$.

Definition 2.6. A divisorial support function $h \in \text{SF}(\mathcal{S})$ is called *principal* if $h = \text{SF}(u) + \text{SF}(D)$ for some $u \in M$ and some principal divisor D on \mathbb{P}^1 . It is called *Cartier* if its restriction to any affine T -invariant open subset is principal. We denote by $\text{CaSF}(\mathcal{S})$ the set of divisorial Cartier support functions. It is a free abelian group.

From the description of invariant subvarieties given above we see that there are two classes of T -invariant prime Weil divisors on X . The *vertical* invariant prime divisors arising as the closure of a family of n -dimensional T -orbits. These are parametrized by pairs (p, v) of points $p \in \mathbb{P}^1$ and vertices v of the polyhedral subdivision \mathcal{S}_p . And then there are the *horizontal* invariant prime divisors arising as the closure of a family of $(n - 1)$ -dimensional T -orbits. These are parametrized by rays τ of Σ .

To a given divisorial support function $h = (h_p)_{p \in \mathbb{P}^1}$, one associates a T -invariant Weil divisor

$$D_h = - \sum_{\tau \in \Sigma(1)} h(v_\tau) B_\tau - \sum_{\substack{(p,v) \\ v \in \mathcal{S}_p(0)}} \mu(v) h_p(v) Z_{p,v}, \tag{2.1}$$

where one omits the prime divisors contracted by r . The following is Proposition 3.10 of [16].

Proposition 2.7. *The map*

$$h \mapsto D_h$$

induces an isomorphism of free abelian groups $\text{CaSF}(\mathcal{S}) \simeq T\text{-Ca}(X)$.

Example 2.8. If $h = \text{SF}(u)$ for $u \in M$ then

$$D_h = \sum_{\tau \in \Sigma(1)} \langle u, v_\tau \rangle B_\tau - \sum_{\substack{(p,v) \\ v \in \mathcal{S}_p(0)}} \mu(v) \langle u, v \rangle Z_{p,v}.$$

Let $p \neq \infty$ be any point in \mathbb{P}^1 . If $h = \text{SF}([p] - [\infty])$ then

$$D_h = \sum_{v \in \mathcal{S}_\infty(0)} -\mu(v) Z_{\infty,v} + \sum_{v \in \mathcal{S}_p(0)} \mu(v) Z_{p,v}.$$

Consider the Picard group $\text{Pic}(X)$ consisting of all Cartier divisors on X modulo linear equivalence. If h is principal then D_h is 0 in $\text{Pic}(X)$. Furthermore, if we consider the equivalence relation on $\text{CaSF}(\mathcal{S})$ given by $h \sim h'$ if and only if $h - h'$ is principal then it follows as a particular case of [16, Corollary 3.15] that the map $h \mapsto D_h$ induces an isomorphism

$$(\text{CaSF}(\mathcal{S}) / \sim) \simeq \text{Pic}(X).$$

Example 2.9. It follows from Example 2.8 that all special fibers (with corresponding multiplicities) are linearly equivalent.

Remark 2.10. It follows from Theorem 2.3 above that the Picard group $\text{Pic}(X)$ is generated by classes $Z_{p,v}$, for $v \in \mathcal{S}_p(0)$, and B_τ , for $\tau \in \Sigma(1)$, $\tau \notin K$ since the divisor class corresponding to a contracted ray can be written as a combination of the other classes (see also [16, Corollary 3.17]).

We end this section with the following important definition.

Definition 2.11 (Restriction to subvarieties). Let $h \in \text{CaSF}(\mathcal{S})$.

For $\tau \in \Sigma$ consider the embedding $\iota: B_\tau \hookrightarrow \tilde{X}$ of the corresponding subvariety. Recall that B_τ is itself a rational complete T -variety of complexity one with divisorial fan $\mathcal{S}(\tau)$. Then $\iota^* D_h$ is an invariant Cartier divisor on B_τ corresponding to a divisorial Cartier support function on $\text{CaSF}(\mathcal{S}(\tau))$. We denote this function by $h(\tau)$.

Similarly, for a point $p \in \mathbb{P}^1$ and $F \in S_p$ such that $Z_{p,F}$ is an invariant subvariety of a toric subvariety $j: X_{\Sigma(p,F)} \hookrightarrow \tilde{X}$, we consider the pullback of $j^* D_h$. This gives a toric Cartier divisor and hence a virtual support function on $\Sigma(p, F)$, which we denote by $h_{p,F}$.

3. Chow groups

For any normal, irreducible algebraic variety Y over k , the Chow group $A_q(Y)$ of Y is defined to be $Z_q(Y)/R_q(Y)$, where $Z_q(Y)$ is the free abelian group generated by all q -dimensional closed subvarieties of X , and $R_q(X)$ is the subgroup generated by divisors $[\text{div}(f)]$ of non-zero rational functions f on $(q + 1)$ -dimensional subvarieties W of Y . When an algebraic group Γ acts on Y one can form the Γ -invariant Chow group $A_q^\Gamma(Y) = Z_q^\Gamma(Y)/R_q^\Gamma(Y)$, with $Z_q^\Gamma(Y)$ the free abelian group generated by Γ -invariant closed subvarieties of Y , and $R_q^\Gamma(Y)$ the subgroup generated by all divisors of eigenfunctions on Γ -invariant $(q + 1)$ -dimensional subvarieties. Recall that a function f in $R(W)^*$ is said to be an eigenfunction if $g \cdot f = \chi(g)f$ for all $g \in \Gamma$, for some character $\chi = \chi_f$ on Γ . The following useful result is [9, Theorem 1].

Theorem 3.1. *If a connected solvable algebraic group Γ acts on Y , then the canonical homomorphism $A_k^\Gamma(Y) \rightarrow A_k(Y)$ is an isomorphism.*

We obtain the following corollary.

Corollary 3.2. *Let X be a complete rational T -variety of complexity one. Then the Chow groups $A_k(X)$ are generated by the classes described in Theorem 2.3.*

In order to have a complete combinatorial description of the Chow groups $A_k(X)$ of a complete rational T -variety of complexity one needs to find the relations between these generators. This will be done in Theorem 3.6 in the following section.

Before, we recall the general definition of Fulton’s operational Chow cohomology groups of any normal, irreducible algebraic variety Y as above.

Let $A_*(Y) = \bigoplus_q A_q(Y)$. The Chow groups $A_q(Y)$ are called Chow “homology” groups, since they are covariant with respect to proper maps. In order to have a natural intersection product, one considers the Chow “cohomology” groups $A^q(Y)$ (see [8, Chapter 17]). They satisfy the following expected functorial properties.

- (1) One has “cup products” $A^p(Y) \otimes A^q(Y) \rightarrow A^{p+q}(Y)$, $a \otimes b \mapsto a \cup b$, making $A^*(Y) = \bigoplus_q A^q(Y)$ into a graded associative ring.
- (2) There are contravariant graded ring maps $f^*: A^*(Y) \rightarrow A^*(Y')$ for arbitrary morphisms $f: Y' \rightarrow Y$.

- (3) One has “cap products” $A^q(Y) \otimes A_m(Y) \rightarrow A_{m-q}(Y)$, $c \otimes z \mapsto c \cap z$, making $A_*(Y)$ into an $A^*(Y)$ -module and satisfying the usual projection formula.
- (4) When Y is non-singular of dimension n , then the “Poincaré duality map” $A^q(Y) \rightarrow A_{n-q}(Y)$, $c \mapsto c \cap [Y]$ is an isomorphism, and the ring structure on $A^*(Y)$ is that determined by the intersection products of cycles of Y .
- (5) Vector bundles on Y have Chern classes in $A^*(Y)$.

Remark 3.3. The Poincaré duality map is still an isomorphism if one assumes some mild singularities (e.g., \mathbb{Q} -factorial), and after tensoring with \mathbb{Q} (see [13, Section 3]).

An element in $A^q(Y)$ determines, by the cap product, a homomorphism from $A_q(Y)$ to $A_0(Y)$; and if Y is complete, one can compose with the degree map from $A_0(Y)$ to \mathbb{Z} and one has a natural “Kroenecker duality” homomorphism

$$A^q(Y) \longrightarrow \text{Hom}(A_q(Y), \mathbb{Z}), \quad z \longmapsto [a \mapsto \text{deg}(z \cap a)]. \tag{3.1}$$

As was mentioned in the introduction, this map is in general not an isomorphism. We will see in the next section that for projective, \mathbb{Q} -factorial, rational, contraction-free T -varieties of complexity one, this map is indeed an isomorphism after tensoring with \mathbb{Q} (Proposition 3.5).

3.1. The Chow group of a complexity-one T -variety

Let X be a $(n + 1)$ -dimensional complete, rational T -variety of complexity one with divisorial fan \mathcal{S} over $Y = \mathbb{P}^1$, recession fan Σ in $N_{\mathbb{Q}}$ and subset $K \subseteq \Sigma$. Assume that K is a proper subset of Σ , i.e., $K \neq \Sigma$. We say that a cone $\sigma \in \Sigma$ is *contracted* if $\sigma \in K$.

As before, for $p \in \mathbb{P}^1$ we write \mathcal{S}_p for the corresponding complete polyhedral subdivision of $N_{\mathbb{Q}}$. Let P be the set of points of \mathbb{P}^1 such that $\mathcal{S}_p \neq \Sigma$. P is a finite set and we assume that P has size at least two.

For an integer $k \geq 0$ we define the following sets.

$$\begin{aligned} V_k &= \{F \in \mathcal{S}_p(n - k), p \in P \text{ such that } \text{rec}(F) \notin K\}, \\ R_k &= \{\sigma \in \Sigma(n - k + 1), \sigma \notin K\}, \\ T_k &= \{\sigma \in \Sigma(n - k), \sigma \in K\}. \end{aligned}$$

Remark 3.4. We always have $T_n = \emptyset$. If $X = \tilde{X}$ is contraction free, then for all k , we have $T_K = \emptyset$.

We begin by showing that when X is a projective, \mathbb{Q} -factorial, rational, contraction-free T -variety of complexity one, then the Kroenecker duality map is in fact an isomorphism after tensoring with \mathbb{Q} .

Proposition 3.5. *Let X be a projective, \mathbb{Q} -factorial, rational, contraction-free T -variety of complexity one. After tensoring with \mathbb{Q} , the Kroenecker duality map*

$$A^q(X)_{\mathbb{Q}} \longrightarrow \text{Hom}(A_q(X)_{\mathbb{Q}}, \mathbb{Q}), \quad z \longmapsto [a \mapsto \text{deg}(z \cap a)] \tag{3.2}$$

is an isomorphism.

Proof. By Poincaré duality $A^q(X)_{\mathbb{Q}} \simeq A_{n-q}(X)_{\mathbb{Q}}$ (see Remark 3.3) it suffices to show that the induced pairing

$$A_{n-q}(X)_{\mathbb{Q}} \times A_q(X)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

is perfect. By [14, Theorem 4], we have the following Stanley–Reisner type description of the homology groups:

$$A_*(X)_{\mathbb{Q}} = \frac{\mathbb{Q}[x_{\omega} \mid \omega \in R_n \sqcup V_n]}{\mathcal{J}},$$

where \mathcal{J} is generated by the linear forms with integer coefficients $\sum n_{\omega}x_{\omega}$ such that the corresponding invariant divisor $\sum_{\omega \in V_n} n_{\omega}Z_{\omega} + \sum_{\omega \in R_n} n_{\omega}B_{\omega}$ is principal, together with the square-free monomials $\prod_{\omega \in R_n \sqcup V_n} x_{\omega}$ such that the intersection $\bigcap_{\omega \in V_n} Z_{\omega} \cap \bigcap_{\omega \in R_n} B_{\omega}$ is empty. In particular, square-free monomials not contained in \mathcal{J} are of the form

$$x_F = \prod_{\tau \in \text{rec}(F)(1)} x_{\tau} \cdot \prod_{v \in F(0)} x_v$$

for some $F \in \mathcal{S}_p$ (see [14, Lemma 4.3]).

As a key fact to prove this result the authors define a technical combinatorial property called “shellability” [14, Definition 3.8]. The authors show in [14, Proposition 3.11] that the divisorial fans of projective, \mathbb{Q} -factorial, contraction-free rational T -varieties of complexity one are shellable, which allows them to prove the above representation.

Now, in order to show the proposition, it thus suffices to show that the pairing

$$\left(\frac{\mathbb{Q}[x_{\omega} \mid \omega \in R_n \sqcup V_n]}{\mathcal{J}} \right)_{n-q} \times \left(\frac{\mathbb{Q}[x_{\omega} \mid \omega \in R_n \sqcup V_n]}{\mathcal{J}} \right)_q \longrightarrow \mathbb{Q}, \tag{3.3}$$

where $(\cdot)_{\ell}$ denotes the ℓ 'th graded piece, is perfect. Let $b \in \left(\frac{\mathbb{Q}[x_{\omega} \mid \omega \in R_n \sqcup V_n]}{\mathcal{J}} \right)_{n-q}$ be non-zero. Without loss of generality, we can assume that $b = x_F$ for $F \in \mathcal{S}_p$. Let $G \in \mathcal{S}_p(n)$ be a maximal polyhedron containing F and set

$$x_{F'} = \prod_{\rho \in \text{rec}(G)(1) \setminus \text{rec}(F)(1)} x_{\rho} \cdot \prod_{v \in G(0) \setminus F(0)} x_v.$$

Since \mathcal{S}_p is simplicial we have

$$x_{F'} \in \left(\frac{\mathbb{Q}[x_{\omega} \mid \omega \in R_n \sqcup V_n]}{\mathcal{J}} \right)_q \quad \text{and} \quad x_F \cdot x_{F'} = x_G \neq 0.$$

Hence the pairing in (3.3) is perfect, which is what we wanted to show. ■

We now consider a more general complete, rational (not necessarily contraction-free) T -variety of complexity one. In this case, the Chow homology groups are not longer generated by divisors (see [14] for an example). Nevertheless, in [15, Theorem 4.1], the author gives a nice representation in terms of generators and relations.

Theorem 3.6. For a rational, complete T -variety X of complexity one, there is for any $0 \leq k \leq \dim(X)$ an exact sequence

$$\bigoplus_{F \in V_{k+1}} M(F) \oplus \bigoplus_{\tau \in R_{k+1}} (M(\tau) \oplus \mathbb{Z}^P / \mathbb{Z}) \oplus \bigoplus_{\tau \in T_{k+1}} M(\tau) \longrightarrow \mathbb{Z}^{V_k} \oplus \mathbb{Z}^{R_k} \oplus \mathbb{Z}^{T_k} \longrightarrow A_k(X) \longrightarrow 0, \tag{3.4}$$

where the maps are given in the following way.

Let $F \in V_{k+1}$. Then we can choose a point $p \in p$ such that $Z_{p,F}$ corresponds to a $(k + 1)$ -dimensional invariant subvariety of a toric subvariety with character lattice $M(F)$. Then $m \in M(F)$ gets sent to

$$\sum_{\substack{G \in S_p(n-k) \\ F < G, \text{rec}(G) \notin K}} \langle m, v_{F,G} \rangle Z_{p,G} + \sum_{\substack{G \in S_p(n-k) \\ F < G, \text{rec}(G) \in K}} \langle m, v_{F,G} \rangle \frac{s_{\text{rec}(G)}}{\mu(G)} W_{\text{rec}(G)},$$

where $v_{F,G}$ is defined in Notation 2.1.

Let $\tau \in R_{k+1}$. Then B_τ corresponds to a $(k + 1)$ -dimensional T -variety of complexity one. Then on the one hand side, a character $m \in M(\tau)$ is mapped to

$$\sum_{\substack{F \in V_k \\ \text{rec}(F) = \tau}} \mu(v_F) \langle m, v_F \rangle Z_{p,F} + \sum_{\substack{\sigma \in R_k \\ \tau < \sigma}} \langle m, v_{\tau,\sigma} \rangle B_\sigma$$

where $v_{\tau,\sigma}$ is defined in Notation 2.1 and v_F in Notation 2.2. On the other hand, a generator of $\mathbb{Z}^P / \mathbb{Z}$ corresponds to a divisor $[p] - [\infty]$. This is mapped to

$$\sum_{\substack{F \in S_p(n-k) \\ \text{rec}(F) = \tau}} \mu(v_F) Z_{p,F} - \sum_{\substack{F \in S_\infty(n-k) \\ \text{rec}(F) = \tau}} \mu(v_F) Z_{\infty,F}. \tag{3.5}$$

Finally, if $\tau \in T_{k+1}$, then every cone $\sigma \in \Sigma$ containing τ is also contracted. In particular, such a $(n - k)$ -dimensional cone σ lies in T_k . Then $m \in M(\tau)$ gets mapped to

$$\sum_{\substack{\sigma \in \Sigma(n-k) \\ \tau < \sigma}} \langle m, v_{\tau,\sigma} \rangle W_\sigma.$$

Proof. This is [15, Theorem 4.1]. ■

Remark 3.7. We make the following three remarks.

- (i) Note that the surjectivity of the map follows from Theorem 2.3. The exactness of the sequence is then a consequence of the fact that the maps are exactly given by choosing an invariant subvariety Z of dimension $k + 1$, choosing a rational function on Z and taking its divisor. We refer for details to the proof in *loc. cit.*
- (ii) The maps describe the relations between the invariant cycles in $A_k(X)$. For example, note that equation (3.5) is saying that for $\tau \in R_{k+1}$, any two special fibers of B_τ are rationally equivalent.
- (iii) Since the set P of points in \mathbb{P}^1 such that the polyhedral complex S_p is different from Σ is finite, Theorem 3.6 gives a finite presentation of the Chow groups.

4. Generalized Minkowski weights

Let X be a $(n + 1)$ -dimensional complete, rational T -variety of complexity one with divisorial fan \mathcal{S} over $Y = \mathbb{P}^1$, recession fan Σ in $N_{\mathbb{Q}}$ and subset $K \subseteq \Sigma$. Assume that K is a proper subset of Σ , i.e., $K \neq \Sigma$.

As before, for $p \in \mathbb{P}^1$ we write \mathcal{S}_p for the corresponding complete polyhedral subdivision of $N_{\mathbb{Q}}$. Let P be the set of points of \mathbb{P}^1 such that $\mathcal{S}_p \neq \Sigma$. P is a finite set and we assume that P has size at least two.

For an integer $k \geq 0$ let R_k, V_k, T_k be the sets defined in Section 3.1. Recall that we have a Kronecker duality map

$$A^k(X) \longrightarrow \text{Hom}(A_k(X), \mathbb{Z}),$$

which by Proposition 3.5 is an isomorphism after tensoring with \mathbb{Q} if X is projective, \mathbb{Q} -factorial and contraction-free.

Then it follows from Theorem 3.6 that to an element in $A^k(X)$ we can associate an integer-valued function on $R_k \cup V_k \cup T_k$ satisfying some properties, and that this association is bijective whenever the Kronecker duality map is an isomorphism. We will describe such functions explicitly.

Definition 4.1. A weight on X of codimension k is a map

$$c: V_k \cup R_k \cup T_k \longrightarrow \mathbb{Z}.$$

It is called a *generalized Minkowski weight of codimension k* if it satisfies the following three conditions.

- (1) For $F \in V_{k+1}$ choose a point p such that $Z_{p,F}$ corresponds to an invariant $(k + 1)$ -dimensional subvariety of a toric variety with lattice of characters $M(F)$. Then for all $m \in M(F)$

$$\sum_{\substack{G \in \mathcal{S}_p(n-k) \\ F \prec G, \text{rec}(G) \notin K}} \langle m, v_{F,G} \rangle c(G) + \sum_{\substack{G \in \mathcal{S}_p(n-k) \\ F \prec G, \text{rec}(G) \in K}} \langle m, v_{F,G} \rangle \frac{S_{\text{rec}(G)}}{\mu(G)} c(\text{rec}(G)) = 0.$$

- (2) Let $\tau \in R_{k+1}$. Then for all $m \in M(\tau)$

$$\sum_{\substack{F \in V_k \\ \text{rec}(F) = \tau}} \mu(v_F) \langle m, v_F \rangle c(F) + \sum_{\substack{\sigma \in R_k \\ \tau \prec \sigma}} \langle m, v_{\tau,\sigma} \rangle c(\sigma) = 0. \tag{4.1}$$

Assume $\infty \in P$. Then for any point $p \in P$

$$\sum_{\substack{F \in \mathcal{S}_p(n-k) \\ \text{rec}(F) = \tau}} \mu(v_F) c(F) - \sum_{\substack{F \in \mathcal{S}_{\infty}(n-k) \\ \text{rec}(F) = \tau}} \mu(v_F) c(F) = 0. \tag{4.2}$$

- (3) For all $\tau \in T_{k+1}$ and for all $m \in M(\tau)$

$$\sum_{\substack{\sigma \in \Sigma(n-k) \\ \tau \prec \sigma}} \langle m, v_{\tau,\sigma} \rangle c(\sigma) = 0.$$

We denote by $W_k(X)$ the set of weights on X of codimension k . It is an abelian group under the addition of functions. We denote by $M_k(X)$ the subgroup of generalized Minkowski weight of codimension k and we refer to the linear conditions (1), (2) and (3) as *balancing conditions*.

Remark 4.2. The term *balanced* comes from tropical geometry, where it plays a fundamental role. Indeed, tropical cycles are always *balanced* and this condition is necessary for example in order to have a well-defined tropical intersection theory.

Definition 4.3. We define a tropicalization map

$$\text{trop}: A^k(X) \longrightarrow W_k(X)$$

by

$$z \longmapsto \text{trop}(z),$$

where

$$\text{trop}(z): V_k \cup R_k \cup T_k \longrightarrow \mathbb{Z}$$

is given by

$$\gamma \longmapsto z \cap [V_\gamma].$$

Here, V_γ denotes the invariant subvariety corresponding to γ .

The following is our first main result.

Theorem 4.4. *Let X be a complete rational T -variety of complexity one. Then the tropicalization map*

$$\text{trop}: A^k(X) \longrightarrow M_k(X)$$

is well-defined. If X is projective, \mathbb{Q} -factorial and contraction-free, then the induced tropicalization map

$$\text{trop}: A^k(X)_{\mathbb{Q}} \longrightarrow M_k(X)_{\mathbb{Q}}, \tag{4.3}$$

where $M_k(X)_{\mathbb{Q}}$ denotes \mathbb{Q} -valued generalized Minkowski weights, is a bijection.

Proof. By the above discussion, this follows directly from Proposition 3.5 and Theorem 3.6. ■

We will now study the restriction of weights to invariant subvarieties.

Definition 4.5. Let $c \in W_k(X)$.

- For $\tau \in \Sigma$ let $\iota: B_\tau \hookrightarrow \tilde{X}$ the inclusion of the corresponding subvariety. Recall that $\dim(B_\tau) = \text{codim}(\tau) + 1$. We define a weight c^τ on B_τ in the following way. Consider the sets

$$\begin{aligned} V_k(\tau) &= \{F \in S_p(n-k), p \in P \text{ such that } \tau \prec \text{rec}(F), \text{rec}(F) \notin K\}, \\ R_k(\tau) &= \{\sigma \in \Sigma(n-k+1), \tau \prec \sigma, \sigma \notin K\}, \\ T_k(\tau) &= \{\sigma \in \Sigma(n-k), \tau \prec \sigma, \sigma \in K\}. \end{aligned}$$

Then

$$c^\tau: V_k(\tau) \cup R_k(\tau) \cup T_k(\tau) \longrightarrow \mathbb{Z}$$

is given by

$$\gamma \longmapsto c(\gamma).$$

The weight c^τ can be thought of as the restriction of c to the invariant subvariety B_τ .

- On the other hand, let $p \in \mathbb{P}^1$ and $G \in S_p$ such that $Z_{p,G}$ is an invariant subvariety of a toric subvariety $X_{\Sigma(p,G)}$. Then if $\text{rec}(G) \notin K$, we define $c^{p,G}$ as a function

$$c^{p,G}: \Sigma(p, G) = \{F \in S_p(n - k), G \prec F\} \longrightarrow \mathbb{Z}$$

defined by

$$F \longmapsto c^{p,G}(F),$$

where $c^{p,G}(F) = c(F)$ if $\text{rec}(F) \notin K$. If $\text{rec}(F) \in K$, then by the proof of Theorem 3.6, one can write $Z_{p,F}$ in $A_k(X)$ as an integral linear combination

$$Z_{p,F} = \sum_{\gamma} a_{\gamma} V_{\gamma},$$

the sum being over all γ in $V_k \cup R_k \cup T_k$. Then we set

$$c^{p,G}(F) = \sum_{\gamma} a_{\gamma} c(\gamma).$$

Then $c^{p,G}$ can be thought of as the restriction of c to the invariant subvariety $F_{p,G}$.

Lemma 4.6. *Let $c \in M_k(X)$ be a generalized Minkowski weight.*

- (1) *For any $\tau \in \Sigma$ the restriction c^τ is a generalized Minkowski weight in $M_k(B_\tau)$.*
- (2) *For any point $p \in \mathbb{P}^1$ and $G \in S_p$ such that $Z_{p,G}$ is an invariant subvariety of a toric subvariety $X_{\Sigma(p,G)}$, and such that $\text{rec}(G) \notin K$, the restriction $c^{p,G}$ is a Minkowski weight on the fan $\Sigma(p, G)$ in the sense of [10].*

Proof. Let $c = \text{trop}(z)$ for some $z \in A^k(X)$.

For (1) we claim that

$$\text{trop}(t^*z) = c^\tau.$$

Indeed, it follows from the description of B_τ as a T -variety that the sequence

$$\mathbb{Z}^{V_k(\tau)} \oplus \mathbb{Z}^{R_k(\tau)} \oplus \mathbb{Z}^{T_k(\tau)} \longrightarrow A_k(B_\tau) \longrightarrow 0$$

is exact. Hence, by construction, the domain of $\text{trop}(t^*z) \in M_k(B_\tau)$ is indeed $V_k(\tau) \cup R_k(\tau) \cup T_k(\tau)$.

Let $\gamma \in V_k(\tau) \cup R_k(\tau) \cup T_k(\tau)$. We have to check that

$$\text{trop}(t^*Z)(\gamma) = c^\tau(\gamma).$$

But this follows from the definition of c^τ and the projection formula.

For (2) we proceed similarly. Let $j : X_{\Sigma(p,G)} \hookrightarrow \tilde{X}$ be the inclusion of the toric subvariety. Then $\text{trop}(j^*z)$ is a Minkowski weight on the fan $\Sigma(p, G)$ in the sense of [10]. We claim that

$$\text{trop}(j^*Z) = c^{p,G}.$$

Indeed, we have to check that for $F \in S_p(n - k)$ with $G \prec F$, we have that

$$\text{trop}(j^*Z)(F) = c^{p,G}(F).$$

Again, this follows from the definition of $c^{p,G}$ and the projection formula. ■

5. Intersecting with Cartier divisors

Let X be a $(n + 1)$ -dimensional complete, rational T -variety of complexity one with divisorial fan \mathcal{S} over $Y = \mathbb{P}^1$ and recession fan Σ in $N_{\mathbb{Q}}$. In this section we assume that X is projective, \mathbb{Q} -factorial and contraction free.

As before, for $p \in \mathbb{P}^1$ we write S_p for the corresponding complete polyhedral subdivision of $N_{\mathbb{Q}}$. Let P be the set of points of \mathbb{P}^1 such that $S_p \neq \Sigma$. P is a finite set and we assume that P has size at least two.

The goal of this section is to define a pairing

$$\text{CaSF}(\mathcal{S}) \times M_k(X) \longrightarrow M_{k+1}(X), \quad (h, c) \longmapsto h \cdot c$$

and to show that it is compatible with the isomorphism (4.3) from Theorem 4.4.

We can think of this pairing as a rational analogue of the so called *corner locus* of a tropical cycle in the sense of [1, Section 6].

In order to define this pairing we use the combinatorial data of \mathcal{S} and Σ . Note that since we are assuming that X is contraction free, we have that $T_k(X) = \emptyset$ for any integer $k \geq 0$.

Definition 5.1. Let $h = (h_p)_{p \in \mathbb{P}^1} \in \text{CaSF}(\mathcal{S})$ be a Cartier divisorial support function on \mathcal{S} with recession function $\text{rec}(h) = \bar{h}$. And let $c \in M_k(X)$ a generalized Minkowski weight of codimension k . First we define a pairing

$$\text{CaSF}(\mathcal{S}) \times M_k(X) \longrightarrow W_{k+1}(X), \quad (h, c) \longmapsto h \cdot c, \tag{5.1}$$

where $h \cdot c : V_{k+1} \cup R_{k+1} \rightarrow \mathbb{Z}$ is defined in the following way.

- Let $F \in V_{k+1}$. Choose a point $p \in P$ such that $Z_{p,F}$ corresponds to an invariant subvariety of a toric variety. Then we set

$$(h \cdot c)(F) = \sum_{\substack{G \in S_p(n-k) \\ F \prec G}} -h_{p,F}(v_{F,G})c(G),$$

where $h_{p,F}$ is the function defined in Definition 2.11.

- Let $\tau \in R_{k+1}$. For any $\sigma \in R_k$ containing τ and for any $F \in V_k$ with $\text{rec}(F) = \tau$ we consider lifts of $v_{\tau,\sigma}$ and v_F in $N_{\mathbb{R}}$. We denote these lifts by the same symbols, in order to avoid the notation becoming even heavier. Then we set

$$\begin{aligned} (h \cdot c)(\tau) &= \sum_{\substack{F \in V_k \\ \text{rec}(F) = \tau}} -\mu(v_F)h(v_F)c(F) + \sum_{\substack{\sigma \in R_k \\ \tau < \sigma}} -\bar{h}_\sigma(v_{\tau,\sigma})c(\sigma) \\ &+ \bar{h}_\tau \left(\sum_{\substack{F \in V_k \\ \text{rec}(F) = \tau}} \mu(v_F)v_F c(F) + \sum_{\substack{\sigma \in R_k \\ \tau < \sigma}} v_{\tau,\sigma} c(\sigma) \right), \end{aligned}$$

where for $\gamma \in \Sigma$, \bar{h}_γ denotes a linear function on $N_{\mathbb{R}}$ which agrees with \bar{h} on γ .

Note that since c is a generalized Minkowski weight, the pairing (5.1) is well defined, i.e., it does not depend on the choice of lifts.

We will now proceed to show that $h \cdot c \in W_{k+1}(X)$ satisfies the balancing conditions (1), (2) and (3) from Definition 4.1, and is thus a generalized Minkowski weight. This will follow from the following three important lemmas.

Lemma 5.2. *Let $c \in M_k(X)$ and $h \in \text{CaSF}(S)$. Then the following holds true.*

- (1) *For all $\tau \in \Sigma$ we have*

$$h(\tau) \cdot c^\tau = (h \cdot c)^\tau$$

as weights in $W_{k+1}(B_\tau)$, where $h(\tau)$ denotes the function in Definition 2.11.

- (2) *Let $p \in P$ and $G \in S_p$ such that $Z_{p,G}$ is an invariant subvariety of a toric subvariety $X_{\Sigma(p,G)}$. Then*

$$h_{p,G} \cdot c^{p,G} = (h \cdot c)^{p,G}$$

as Minkowski weights in the fan $\Sigma(p, G)$. Here, the intersection pairing $h_{p,G} \cdot c^{p,G}$ is the classical one for fans, as defined in [1, Definition 3.4].

Proof. Part 2 follows from toric geometry and the compatibility of the intersection pairing with the pullback (see e.g. [8, Chapter 17]).

For part 1 we first assume that $\bar{h}|_\tau = 0$. We have to show that for all $\omega \in R_{k+1}(\tau) \cup V_{k+1}(\tau)$ we have that

$$(h(\tau) \cdot c^\tau)(\omega) = (h \cdot c)^\tau(\omega).$$

- Let $F \in V_{k+1}(\tau)$ and choose a point $p \in P$ such that $Z_{p,F}$ corresponds to an invariant subvariety of a toric variety. Then

$$\begin{aligned} (h \cdot c)^\tau(F) &= (h \cdot c)(F) = \sum_{\substack{G \in S_p(n-k) \\ F < G}} -h_{p,F}(v_{F,G})c(G) \\ &= \sum_{\substack{G \in S_p(n-k) \\ F < G}} -h(\tau)_{p,F}(v_{F,G})c^\tau(G) \\ &= (h(\tau) \cdot c^\tau)(F). \end{aligned}$$

- Let $\gamma \in R_{k+1}(\tau)$. Then

$$\begin{aligned}
 (h \cdot c)^\tau(\gamma) &= (h \cdot c)(\gamma) \\
 &= \sum_{\substack{F \in V_k \\ \text{rec}(F)=\gamma}} -\mu(v_F)h(v_F)c(F) + \sum_{\substack{\sigma \in R_k \\ \gamma < \sigma}} -\bar{h}_\sigma(v_{\gamma,\sigma})c(\sigma) \\
 &\quad + \bar{h}_\gamma \left(\sum_{\substack{F \in V_k \\ \text{rec}(F)=\gamma}} \mu(v_F)v_F c(F) + \sum_{\substack{\sigma \in R_k \\ \gamma < \sigma}} v_{\gamma,\sigma}c(\sigma) \right) \\
 &= \sum_{\substack{F \in V_k(\tau) \\ \text{rec}(F)=\gamma}} -\mu(v_F)h(\tau)(v_F)c^\tau(F) + \sum_{\substack{\sigma \in R_k(\tau) \\ \gamma < \sigma}} -\overline{h(\tau)}_\sigma(v_{\gamma,\sigma})c^\tau(\sigma) \\
 &\quad + \overline{h(\tau)}_\gamma \left(\sum_{\substack{F \in V_k(\tau) \\ \text{rec}(F)=\gamma}} \mu(v_F)v_F c^\tau(F) + \sum_{\substack{\sigma \in R_k(\tau) \\ \gamma < \sigma}} v_{\gamma,\sigma}c^\tau(\sigma) \right) \\
 &= (h(\tau) \cdot c^\tau)(\gamma).
 \end{aligned}$$

If $\bar{h}|_\tau = m_\tau \neq 0$ then we consider the divisorial support function $h - m_\tau$. Then by linearity of m_τ we have that for all $\omega \in R_{k+1}(\tau) \cup V_{k+1}(\tau)$

$$(h \cdot c)^\tau(\omega) = ((h - m_\tau) \cdot c)^\tau(\omega) = ((h - m_\tau)(\tau) \cdot c^\tau)(\omega) = (h(\tau) \cdot c^\tau)(\omega).$$

This finishes the proof of the lemma. ■

Lemma 5.3. *Let $h, h' \in \text{CaSF}(\mathcal{S})$. Then for any $c \in M_k(X)$ the following holds true.*

- (1) $(h + h') \cdot c = h \cdot c + h' \cdot c,$
- (2) $h \cdot (h' \cdot c) = h' \cdot (h \cdot c).$

Proof. Part 1 follows easily from the definition of the pairing. We now prove part 2. As before, we have to show that

$$(h \cdot (h' \cdot c))(\omega) = (h' \cdot (h \cdot c))(\omega)$$

for any $\omega \in V_{k+2} \cup R_{k+2}$.

- Let $F \in V_{k+2}$ and choose a point $p \in P$ such that $Z_{p,F}$ corresponds to an invariant subvariety of a toric variety. We compute

$$\begin{aligned}
 (h \cdot (h' \cdot c))(F) &= \sum_{\substack{G \in S_p(n-k-1) \\ F < G}} -h_{p,F}(v_{F,G})(h' \cdot c)(G) \\
 &= \sum_{\substack{G \in S_p(n-k-1) \\ F < G}} -h_{p,F}(v_{F,G}) \left(\sum_{\substack{H \in S_p(n-k) \\ G < H}} -h'_{p,G}(v_{G,H})c(H) \right).
 \end{aligned}$$

Now, for $H \in S_p(n-k)$, $G \in S_p(n-k-1)$ with $F < G < H$ we let $G' \in S_p(n-k-1)$ the unique polyhedron such that $F < G' < H$. Let $\alpha_{F,H}$ and $\beta_{F,H}$ be elements such that

$$v_{G,H} = \alpha_{F,H} v_{F,G} + \beta_{F,H} v_{F,G'}, \tag{5.2}$$

$$v_{G',H} = \alpha_{F,H} v_{F,G'} + \beta_{F,H} v_{F,G}. \tag{5.3}$$

Using (5.2) we obtain

$$\begin{aligned} (h \cdot (h' \cdot c))(F) &= \sum_{\substack{G \in S_p(n-k-1) \\ F < G}} -h_{p,F}(v_{F,G}) \left(\sum_{\substack{H \in S_p(n-k) \\ G < H}} -h'_{p,G}(v_{G,H}) c(H) \right) \\ &= \sum_{\substack{G \in S_p(n-k-1) \\ F < G}} \sum_{\substack{H \in S_p(n-k) \\ G < H}} c(H) h_{p,F}(v_{F,G}) h'_{p,G}(v_{G,H}) \\ &= \sum_{\substack{G \in S_p(n-k-1) \\ F < G}} \sum_{\substack{H \in S_p(n-k) \\ G < H}} c(H) \alpha_{F,H} h_{p,F}(v_{F,G}) h'_{p,F}(v_{F,G}) \\ &\quad + \sum_{\substack{G \in S_p(n-k-1) \\ F < G}} \sum_{\substack{H \in S_p(n-k) \\ G < H}} c(H) \beta_{F,H} h_{p,F}(v_{F,G}) h'_{p,F}(v_{F,G'}) \\ &= \sum_{\substack{H \in S_p(n-k) \\ F < H}} c(H) \alpha_{F,H} \left(\sum_{\substack{G \in S_p(n-k-1) \\ F < G < H}} h_{p,F}(v_{F,G}) h'_{p,F}(v_{F,G}) \right) \\ &\quad + c(H) \beta_{F,H} \left(\sum_{\substack{G \in S_p(n-k-1) \\ F < G < H}} h_{p,F}(v_{F,G}) h'_{p,F}(v_{F,G'}) \right). \end{aligned}$$

Next, we interchange the roles of G and G' in the inner sum of the second summand of the last expression and revert the argument. Using (5.3) we get

$$\begin{aligned} (h \cdot (h' \cdot c))(F) &= \sum_{\substack{H \in S_p(n-k) \\ F < H}} c(H) \alpha_{F,H} \left(\sum_{\substack{G \in S_p(n-k-1) \\ F < G < H}} h_{p,F}(v_{F,G}) h'_{p,F}(v_{F,G}) \right) \\ &\quad + c(H) \beta_{F,H} \left(\sum_{\substack{G \in S_p(n-k-1) \\ F < G < H}} h_{p,F}(v_{F,G'}) h'_{p,F}(v_{F,G}) \right) \\ &= \sum_{\substack{G \in S_p(n-k-1) \\ F < G}} \sum_{\substack{H \in S_p(n-k) \\ G < H}} c(H) h'_{p,F}(v_{F,G}) h_{p,G}(\alpha_{F,H} v_{F,G} + \beta_{F,H} v_{F,G'}) \\ &= \sum_{\substack{G \in S_p(n-k-1) \\ F < G}} \sum_{\substack{H \in S_p(n-k) \\ G < H}} c(H) h'_{p,F}(v_{F,G}) h_{p,G}(v_{G,H}) \\ &= (h' \cdot (h \cdot c))(F), \end{aligned}$$

which is what we wanted to show.

- Let $\tau \in R_{k+2}$. By the restriction Lemma 5.2 we may assume $n = 2, k = 1$. Then

$$(h \cdot (h' \cdot c))(\{0\}) = \sum_{F \in V_2} -\mu(v_F)h(v_F)(h' \cdot c)(F) + \sum_{\sigma \in R_2} -\bar{h}_\sigma(v_\sigma)(h' \cdot c)(\sigma).$$

Let

$$A := \sum_{F \in V_2} -\mu(v_F)h(v_F)(h' \cdot c)(F) \quad \text{and} \quad B := \sum_{\sigma \in R_2} -\bar{h}_\sigma(v_\sigma)(h' \cdot c)(\sigma).$$

By definition of the pairing, we have

$$A = \sum_{F \in V_2} -\mu(v_F)h(v_F) \left(\sum_{\substack{G \in S_p(1) \\ F \prec G}} -h'_{p,F}(v_{F,G})c(G) \right).$$

We can write the above sum as a sum over all one-dimensional faces G in $S_p(1)$ by distinguishing whether $\text{rec}(G) = \{0\}$ or $\text{rec}(G) \in \Sigma(1)$. Note however that in the former case, the terms in the sum cancel each other out since $v_{v_1,G} = -v_{v_2,G}$, where v_1, v_2 are the vertices corresponding to a bounded edge G . Hence, we can write the sum in A as a sum over all unbounded edges G in $S_p(1)$. For such a G we write v_G for its unique vertex. Then we get

$$A = \sum_{\substack{G \in S_p(1) \\ \{0\} \neq \text{rec}(G)}} \mu(v_G)h(\text{rec}(G))(v_G)h'_{p,v_G}(v_{v_G,G})c(G). \tag{5.4}$$

Now, for B , by Lemma 5.2 and the definition of the pairing, we have

$$\begin{aligned} B &= \sum_{\sigma \in R_2} -\bar{h}_\sigma(v_\sigma)(h' \cdot c)(\sigma) \\ &= \sum_{\sigma \in R_2} -\bar{h}_\sigma(v_\sigma)(h'(\sigma) \cdot c^\sigma)(\{0\}) \\ &= \sum_{\sigma \in R_2} \bar{h}_\sigma(v_\sigma) \left(\sum_{\substack{G \in S_p(1) \\ \sigma = \text{rec}(G)}} -\mu(v_G)h'(\sigma)(v_G)c^\sigma(G) + \sum_{\substack{\gamma \in \Sigma(2) \\ \sigma \prec \gamma}} -\bar{h}'_\sigma(\gamma)(v_{\sigma,\gamma})c^\sigma(\gamma) \right) \\ &= \sum_{\sigma \in R_2} \bar{h}_\sigma(v_\sigma) \left(\sum_{\substack{G \in S_p(1) \\ \sigma = \text{rec}(G)}} -\mu(v_G)h'(\sigma)(v_G)c(G) + \sum_{\substack{\gamma \in \Sigma(2) \\ \sigma \prec \gamma}} -\bar{h}'_\gamma(v_{\sigma,\gamma})c(\gamma) \right) \end{aligned}$$

Now, note that for $\sigma \in R_2$ and for any $G \in S_p(1)$ such that $\text{rec}(G) = \sigma$, we have $\bar{h}_\sigma(v_\sigma) = h_{p,v_G}(v_{v_G,G}) - h(\sigma)(v_G)$. Also, for $G, G' \in S_p(1)$ such that $\text{rec}(G) + \text{rec}(G') \in \Sigma(2)$ we have

$$\bar{h}'_{\text{rec}(G)+\text{rec}(G')}(v_{\text{rec}(G')}) = h'_{p,v_G}(v_{v_G,G'}) - h'(\text{rec}(G'))(v_{G'})$$

Then we can write the above sum as a sum over pairs of one-dimensional faces in the following way.

$$\begin{aligned}
 B &= \sum_{\substack{G, G' \in \mathcal{S}_p(1) \\ \text{rec}(G) + \text{rec}(G') \in \Sigma(2)}} (h_{p, v_G}(v_{v_G, G}) - h(\text{rec}(G))(v_G)) \left[\mu(v_G) h'_{p, v_G}(v_{v_G, G}) c(G) \right. \\
 &\quad \left. + (h'_{p, v_{G'}}(v_{v_{G'}, G'}) - h'(\text{rec}(G'))(v_{G'})) c(\text{rec}(G) + \text{rec}(G')) \right] \\
 &= \sum_{\substack{G \in \mathcal{S}_p(1) \\ \{0\} \neq \text{rec}(G)}} -\mu(v_G) c(G) h(\text{rec}(G))(v_G) h'_{p, v_G}(v_{v_G, G}) + Z(h, h'), \tag{5.5}
 \end{aligned}$$

where Z is symmetric in h, h' , i.e., $Z(h, h') = Z(h', h)$.

From (5.4) and (5.5) we get

$$(h \cdot (h' \cdot c))(\{0\}) = A + B = Z(h, h') = Z(h', h).$$

We can now redo the above argument for $(h' \cdot (h \cdot c))(\{0\})$ and we obtain

$$(h' \cdot (h \cdot c))(\{0\}) = Z(h', h) = Z(h, h') = (h \cdot (h' \cdot c))(\{0\}),$$

which is what we wanted to show. ■

Lemma 5.4. *Let $c \in W_k(X)$ be a weight of codimension k . Then c is a generalized Minkowski weight if and only if for all principal $h \in \text{CaSF}(\mathcal{S})$ we have that $h \cdot c = 0$.*

Proof. Assume that c is a generalized Minkowski weight and let $h \in \text{CaSF}(\mathcal{S})$ be principal. We have to show that $h \cdot c = 0$. By the first part of Lemma 5.3 it suffices to show this for the case $h = \text{SF}(u)$ for $u \in M$, and for the case $h = \text{SF}([p] - [\infty])$, $p \in P$.

The case $h = \text{SF}(u)$ follows easily by linearity and the definition of the pairing.

Assume now that $h = \text{SF}([p] - [\infty])$.

Let $\tau \in R_{k+1}$. Then we have

$$h \cdot c(\tau) = \sum_{\substack{F \in \mathcal{S}_p(n-k) \\ \text{rec}(F) = \tau}} -\mu(v_F) c(F) + \sum_{\substack{F \in \mathcal{S}_\infty(n-k) \\ \text{rec}(F) = \tau}} \mu(v_F) c(F) = 0 \tag{5.6}$$

by condition (2) of a generalized Minkowski weight.

On the other hand, for $F \in V_{k+1}$, let $\tilde{p} \in P$ be a point such that $Z_{\tilde{p}, F}$ corresponds to an invariant subvariety of a toric variety. We may assume $p = \tilde{p}$. Then $h \cdot c(F) = 0$ by the definition of the pairing since $h_{p, F} = 0$ in this case.

Conversely, assume that $h \cdot c = 0$ for all principal h . We have to show that c satisfies the balancing conditions of Definition 4.1.

- Let $F \in V_{k+1}$ and choose a point $p \in P$ such that $Z_{p, F}$ corresponds to an invariant subvariety of a toric variety with lattice of characters $M(F) = N(F)^\vee$. Choose a representative \tilde{m} , of m in M . Let $h = \text{SF}(\tilde{m}) + \text{SF}([p] - [\infty])$. We compute

$$0 = -(h \cdot c)^{p, F}(\{0\}) = h_{p, F} \cdot c^{p, F}(\{0\})$$

$$\begin{aligned}
 &= \sum_{\alpha \in \Sigma(p,F)(1)} h_{p,F}(v_\alpha) c^{p,F}(\alpha) \\
 &= \sum_{\substack{G \in \mathcal{S}_p(n-k) \\ F < G}} \langle m, v_{F,G} \rangle c(G).
 \end{aligned}$$

Hence, c satisfies condition (1) of a generalized Minkowski weight.

- Let $\tau \in R_{k+1}$ and let m be any element in $M(\tau)$. Choose any representative $\tilde{m} \in M$ of m . Consider the principal divisor $h = \text{SF}(\tilde{m})$. Note that we have $h|_\tau = 0$. We compute

$$\begin{aligned}
 0 = -h \cdot c(\tau) &= \sum_{\substack{F \in V_k \\ \text{rec}(F)=\tau}} \mu(v_F) h(v_F) c(F) + \sum_{\substack{\sigma \in R_k \\ \tau < \sigma}} \bar{h}_\sigma(v_{\tau,\sigma}) c(\sigma) \\
 &= \sum_{\substack{F \in V_k \\ \text{rec}(F)=\tau}} \mu(v_F) \langle m, v_F \rangle c(F) + \sum_{\substack{\sigma \in R_k \\ \tau < \sigma}} \langle m, v_{\tau,\sigma} \rangle c(\sigma).
 \end{aligned}$$

Moreover, for $h = \text{SF}([p] - [\infty])$, then as in (5.6) we have

$$0 = -h \cdot c(\tau) = \sum_{\substack{F \in \mathcal{S}_p(n-k) \\ \text{rec}(F)=\tau}} \mu(v_F) c(F) - \sum_{\substack{F \in \mathcal{S}_\infty(n-k) \\ \text{rec}(F)=\tau}} \mu(v_F) c(F).$$

This shows that c also satisfies the balancing condition (2). Hence c is a generalized Minkowski weight. ■

Proposition 5.5. *Let $c \in M_k(X)$ and $h \in \text{SF}(S)$. Then $h \cdot c \in M_{k+1}(X)$. It follows that the pairing (5.1) induces a pairing*

$$\text{CaSF}(S) \times M_k(X) \longrightarrow M_{k+1}(X), \quad (h, c) \longmapsto h \cdot c.$$

Proof. Let $h' \in \text{SF}(S)$ be principal. By Lemma 5.4 it suffices to show that $h' \cdot (h \cdot c) = 0$. By Lemma 5.3 we have

$$h' \cdot (h \cdot c) = h \cdot (h' \cdot c) = 0,$$

concluding the proof. ■

The next result states that the pairing in Proposition 5.5 is compatible with the tropicalization isomorphism (4.3) from Theorem 4.4.

Theorem 5.6 (Compatibility with tropicalization). *As before we assume that X is a projective, \mathbb{Q} -factorial, rational, contraction free T -variety of complexity one. Let $h \in \text{CaSF}(S)$ and D_h its associated Cartier divisor. We set $[D_h] \in A^1(X)_\mathbb{Q}$ for its corresponding cohomology class. Furthermore, for $z \in A^k(X)_\mathbb{Q}$ we set $c = \text{trop}(z) \in M_k(X)_\mathbb{Q}$, where trop is the isomorphism (4.3) from Theorem 4.4. Then*

$$\text{trop}([D_h] \cup z) = h \cdot c$$

as generalized Minkowski weights in $M_{k+1}(X)_\mathbb{Q}$.

Proof. Let $\omega \in R_{k+1} \cup V_{k+1}$. We have to show that

$$\text{trop}([D_h] \cup z)(\omega) = h \cdot c(\omega).$$

- Let $F \in V_{k+1}$. Choose a point $p \in P$ such that $Z_{p,F}$ corresponds to an invariant subvariety of a toric variety. Recall that $h_{p,F}$ denotes the (toric) virtual support function on the fan $\Sigma(p, F)$ corresponding to the pullback of D_h to $X_{\Sigma(p,F)}$. Then

$$[D_h] \cap [Z_{p,F}] = \sum_{\substack{G \in S_p(n-k) \\ F \prec G}} h_{p,F}(v_{F,G}) [Z_{p,G}].$$

Hence

$$\begin{aligned} \text{trop}([D_h] \cup z)(F) &= \text{deg}([D_h] \cup z \cap [Z_{p,F}]) \\ &= \sum_{\substack{G \in S_p(n-k) \\ F \prec G}} h_{p,F}(v_{F,G}) \text{deg}(z \cap [Z_{p,G}]). \end{aligned}$$

Note that by definition we have $\text{deg}(z \cap [Z_{p,G}]) = c^{p,F}(G)$. Then, using Lemma 5.2, we obtain

$$\begin{aligned} \text{trop}([D_h] \cup z)(F) &= \sum_{\substack{G \in S_p(n-k) \\ F \prec G}} h_{p,F}(v_{F,G}) \text{deg}(Z \cap [Z_{p,G}]) \\ &= \sum_{\substack{G \in S_p(n-k) \\ F \prec G}} h_{p,F}(v_{F,G}) c^{p,F}(G) \\ &= h_{p,F} \cdot c^{p,F}(F) \\ &= (h \cdot c)^{p,F}(F) \\ &= (h \cdot c)(F). \end{aligned}$$

- Let $\tau \in R_{k+1}$. By definition, we have

$$\text{trop}([D_h] \cup z)(\tau) = \text{deg}([D_h] \cup z \cap [B_\tau]).$$

The restriction $h(\tau) \in \text{CaSF}(\mathcal{S}(\tau))$ is the Cartier support function corresponding to $[D_h] \cap [B_\tau]$. Then

$$[D_h] \cap [B_\tau] = [D_{h(\tau)}] = - \sum_{\sigma \in R_k(\tau)} \overline{h(\tau)}_\sigma(v_{\tau,\sigma}) [B_\sigma] - \sum_{\substack{F \in V_k(\tau) \\ \text{rec } F = \tau}} \mu(v_F) h(\tau)(v_F) [Z_{p,F}].$$

Hence the weight of $\text{trop}([D_h] \cup z)$ at τ is

$$\begin{aligned} \text{deg}([D_h] \cup z \cap [B_\tau]) &= - \sum_{\sigma \in R_k(\tau)} \overline{h(\tau)}_\sigma(v_{\tau,\sigma}) \text{deg}(z \cap [B_\sigma]) \\ &\quad - \sum_{\substack{F \in V_k(\tau) \\ \text{rec } F = \tau}} \mu(v_F) h(\tau)(v_F) \text{deg}(z \cap [Z_{p,F}]). \end{aligned}$$

On the other hand we have

$$\deg(z \cap [B_\sigma]) = c^\tau(\sigma) \quad \text{and} \quad \deg(z \cap [Z_{p,F}]) = c^\tau(F).$$

Hence, using Lemma 5.2 we get

$$\begin{aligned} \text{trop}([D_h] \cup c)(\tau) &= - \sum_{\sigma \in R_k(\tau)} \overline{h(\tau)}_\sigma(v_{\tau,\sigma}) \deg(z \cap [B_\sigma]) \\ &\quad - \sum_{\substack{F \in V_k(\tau) \\ \text{rec } F = \tau}} \mu(v_F)h(\tau)(v_F) \deg(z \cap [Z_{p,F}]) \\ &= - \sum_{\sigma \in R_k(\tau)} \overline{h(\tau)}_\sigma(v_{\tau,\sigma})c^\tau(\sigma) - \sum_{\substack{F \in V_k(\tau) \\ \text{rec } F = \tau}} \mu(v_F)h(\tau)(v_F)c^\tau(F) \\ &= (h(\tau) \cdot c^\tau)(\tau) \\ &= (h \cdot c)^\tau(\tau) \\ &= (h \cdot c)(\tau). \end{aligned}$$

■

6. The measure associated to an invariant Cartier divisor and top intersection numbers

Let X be a $(n + 1)$ -dimensional complete, rational T -variety of complexity one with divisorial fan \mathcal{S} over $Y = \mathbb{P}^1$ and recession fan Σ in $N_{\mathbb{Q}}$. As in the last section, we assume that X is projective, \mathbb{Q} -factorial and contraction free.

As before, for $p \in \mathbb{P}^1$ we write \mathcal{S}_p for the corresponding complete polyhedral subdivision of $N_{\mathbb{Q}}$ and P for the finite set of points of \mathbb{P}^1 such that $\mathcal{S}_p \neq \Sigma$. P , which is assumed to have size at least two.

The goal of this section is to define a (discrete) Measure μ_h associated to a Cartier divisorial Support function $h \in \text{CaSF}(\mathcal{S})$ based on the combinatorial intersection pairing (5.1) and to compute the top intersection number D_h^{n+1} of the corresponding invariant divisor D_h in terms of this measure.

let $[X] \in A_{n+1}(X)$ denote the top Chow homology class. By Poincaré duality and Theorem 4.4, we may consider the associated Minkowski weight $c_{[X]} \in M_0(X)$. This is given by

$$c_{[X]}: V_0 \longrightarrow \mathbb{Z}, \quad F \longmapsto 1.$$

Definition 6.1. Let $h \in \text{CaSF}(\mathcal{S})$. The *Total space* $\text{Tot}(\mathcal{S})$ of the divisorial fan \mathcal{S} is the real vector space

$$\text{Tot}(\mathcal{S}) := \bigoplus_{p \in P} N_{\mathbb{R}} \bigoplus N_{\mathbb{R}}.$$

Then to h we associate a discrete measure μ_h on $\text{Tot}(\mathcal{S})$ given in the following way. Consider the codimension n generalized Minkowski weight defined inductively by

$$h^n \cdot c_{[X]} = h \cdot (h^{n-1} \cdot c_{[X]}).$$

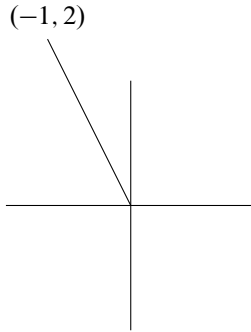


Figure 1. Fan for \mathbb{H}_2 .

Then μ_h is the measure supported on the set

$$V_n \cup R_n,$$

which is in bijection with the set $\bigsqcup_{p \in P} S_p(0) \sqcup \bigsqcup_{\tau \in \Sigma(1)} v_\tau$.

The weights on V_n are given by

$$F \mapsto \mu(v_F)(h^n \cdot c_{[X]}(F));$$

and the weights on R_n are given by

$$\tau \mapsto h^n \cdot c_{[X]}(\tau).$$

Note that the measure μ_h is positive if D_h is nef.

We are now able to compute top intersection numbers of invariant Cartier divisors on X in terms of their associated measures.

Corollary 6.2. *Let D_h be a T -invariant Cartier divisor on X associated to $h \in \text{CaSF}(\mathcal{S})$ and let $\text{rec}(h) = \bar{h}$ be its recession function.*

(1) *The top intersection number D_h^{n+1} can be computed inductively by*

$$h^{n+1} \cdot c_{[X]}(\{0\}) = h^n \cdot (h \cdot c_{[X]})(\{0\}).$$

(2) *Moreover, it can be computed as an integral with respect to the measure μ_h*

$$D_h^{n+1} = \int_{\text{Tot}(\mathcal{S})} -h \mu_h = \sum_{p \in P} \sum_{F \in S_p(0)} -h(v_F) \mu_h(F) + \sum_{\tau \in \Sigma(1)} -\bar{h}(v_\tau) \mu_h(\tau).$$

Proof. The first part follows from Theorem 5.6 and the second from the definition of the measure μ_h and the intersection pairing. ■

Example 6.3. Consider the Hirzebruch surface $\mathbb{H}_2 = \text{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. It is a toric variety whose corresponding fan is given in Figure 1.

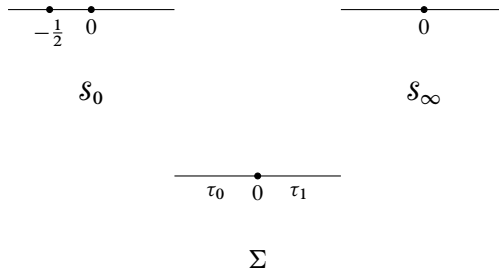


Figure 2. Divisorial fan for \mathbb{H}_2 as a \mathbb{C}^* -surface.

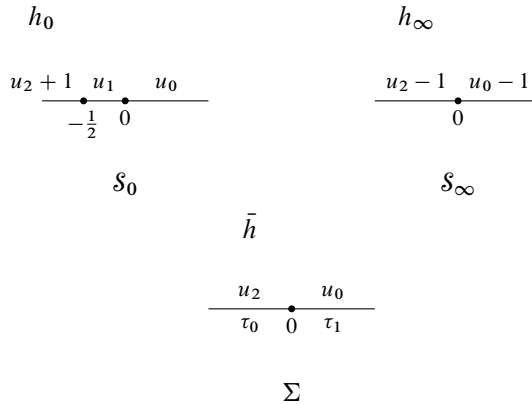


Figure 3. Divisorial support function h .

We can perform a toric downgrade and consider \mathbb{H}_2 as a T -surface, where T is a one-dimensional torus, as is done in [16, Example 2.9]. Here one considers the embedding $T \hookrightarrow T_{\mathbb{H}_2}$ corresponding to the exact sequence

$$0 \longrightarrow N \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} N \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} \mathbb{Z} \longrightarrow 0.$$

This yields the divisorial fan $\mathcal{S} = (\mathcal{S}_p)_p$ with recession fan Σ depicted in the following Figure 2.

We consider the T -invariant Cartier divisor D_h on \mathbb{H}_2 from [16, Example 3.24]. This is a very ample divisor whose corresponding Cartier divisorial support function $h = (h_p)_p$ on \mathcal{S} with recession function $\text{rec}(h) = \bar{h}$ is given in the following Figure 3.

Here, the $u_i \in M$ are given by $u_0 \equiv 0$, $u_1: x \mapsto x$, $u_2: x \mapsto 3x$.

We denote by $v_{0,-\frac{1}{2}}$, $v_{0,0}$ the vertices in \mathcal{S}_0 and by $v_{\infty,0}$ the vertex in \mathcal{S}_∞ . From (2.1) we obtain that the Weil divisor D_h corresponding to h can be written as

$$D_h = 3B_{\tau_0} + Z_{0,-\frac{1}{2}} + Z_{\infty,0}.$$

Here, B_{τ_0} denotes the horizontal divisor corresponding to $\tau_0 \in \Sigma(1)$ and $Z_{0,-\frac{1}{2}}$, respectively $Z_{\infty,0}$ the vertical divisors corresponding to the vertices $v_{0,-\frac{1}{2}}$ and $v_{\infty,0}$ in \mathcal{S}_0 and \mathcal{S}_∞ , respectively.

It follows from [16, Proposition 3.31] that the self-intersection number of D_h can be computed as

$$D_h^2 = 2 \cdot \text{vol}(h^*),$$

where h^* is the dual function of h , and vol is some well-defined volume function (we refer to [16, Section 3] for details). An easy calculation using the description of h^* given in Figure 3 of *loc. cit.* and the definition of vol gives

$$D_h^2 = 2 \cdot (1 + 1 + 3) = 10.$$

We will now compute D_h^2 using Proposition 6.2. First we compute $c_1 := h \cdot c_{[\mathbb{H}_2]}$. Using the definition of the intersection pairing in Section 5, we get that c_1 is the generalized Minkowski weight on $V_1 \cup R_1$ with values on V_1 given by

$$c_1(v_{0,-\frac{1}{2}}) = 1, \quad c_1(v_{0,0}) = 1, \quad c_1(v_{\infty,0}) = 3$$

and with values on R_1 given by

$$c_1(\tau_1) = 3, \quad c_1(\tau_0) = 2.$$

It is easy to check that c_1 satisfies the balancing conditions from Definition 4.1. Indeed, we only have to consider condition (2) (since conditions (1) and (3) are empty).

Let $\tau = \{0\} \in \Sigma(0)$. For the first part of condition (2) (equation (4.1)) we compute

$$\begin{aligned} & \sum_{F \in \mathcal{S}_p(0)} \mu(v_F)v_F c_1(F) + \sum_{\tau \in \Sigma(1)} v_\tau c_1(\tau) \\ &= -c_1(v_{0,-\frac{1}{2}}) + c_1(\tau_1) - c_1(\tau_0) \\ &= -1 + 3 - 2 = 0. \end{aligned}$$

Now, to check the second part of condition (2) (equation (4.2)) we compute

$$\begin{aligned} \sum_{F \in \mathcal{S}_0(0)} \mu(v_F)c_1(F) &= 2 \cdot c_1(v_{0,-\frac{1}{2}}) + 1 \cdot c_1(v_{0,0}) = 3 = 1 \cdot c_1(v_{\infty,0}) \\ &= \sum_{F \in \mathcal{S}_\infty(0)} \mu(v_F)c_1(F). \end{aligned}$$

Hence c_1 satisfies the balancing conditions from Definition 4.1.

By Proposition 6.2 we get

$$\begin{aligned} D_h^2 &= \int_{\text{Tot}(\mathcal{S})} h\mu_h = \sum_{F \in \mathcal{S}_p(0)} -h(v_F)\mu_h(F) + \sum_{\tau \in \Sigma(1)} -h_\tau(v_\tau)\mu_h(\tau) \\ &= c_1(v_{0,-\frac{1}{2}}) + c_1(v_{\infty,0}) + 3 \cdot c_1(\tau_0) \\ &= 1 + 3 + 3 \cdot 2 = 10, \end{aligned}$$

which, as we have seen, is compatible with the results of [16].

Acknowledgments. We thank an anonymous referee for their careful reading of this manuscript and their many insightful comments and suggestions.

Funding. This work was partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 491392403 – TRR 358.

References

- [1] L. Allermann and J. Rau, [First steps in tropical intersection theory](#). *Math. Z.* **264** (2010), no. 3, 633–670 Zbl [1193.14074](#) MR [2591823](#)
- [2] K. Altmann and J. Hausen, [Polyhedral divisors and algebraic torus actions](#). *Math. Ann.* **334** (2006), no. 3, 557–607 Zbl [1193.14060](#) MR [2207875](#)
- [3] K. Altmann, J. Hausen, and H. Süß, [Gluing affine torus actions via divisorial fans](#). *Transform. Groups* **13** (2008), no. 2, 215–242 Zbl [1159.14025](#) MR [2426131](#)
- [4] K. Altmann, N. O. Ilten, L. Petersen, H. Süß, and R. Vollmert, [The geometry of \$T\$ -varieties](#). In *Contributions to algebraic geometry*, pp. 17–69, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012 Zbl [1316.14001](#) MR [2975658](#)
- [5] D. Anderson, R. Gonzales, and S. Payne, [Equivariant Grothendieck-Riemann-Roch and localization in operational \$K\$ -theory](#). *Algebra Number Theory* **15** (2021), no. 2, 341–385 Zbl [1474.19001](#) MR [4243651](#)
- [6] D. Anderson and S. Payne, [Operational \$K\$ -theory](#). *Doc. Math.* **20** (2015), 357–399 Zbl [1344.14010](#) MR [3398716](#)
- [7] A. M. Botero and J. I. B. Gil, [Toroidal \$b\$ -divisors and Monge-Ampère measures](#). *Math. Z.* **300** (2022), no. 1, 579–637 Zbl [1511.14011](#) MR [4359537](#)
- [8] W. Fulton, [Intersection theory](#). 2nd edn., *Ergeb. Math. Grenzgeb. (3) 2*, Springer, Berlin, 1998 Zbl [0885.14002](#) MR [1644323](#)
- [9] W. Fulton, R. MacPherson, F. Sottile, and B. Sturmfels, [Intersection theory on spherical varieties](#). *J. Algebraic Geom.* **4** (1995), no. 1, 181–193 Zbl [0819.14019](#) MR [1299008](#)
- [10] W. Fulton and B. Sturmfels, [Intersection theory on toric varieties](#). *Topology* **36** (1997), no. 2, 335–353 Zbl [0885.14025](#) MR [1415592](#)
- [11] J. Hausen and H. Süß, [The Cox ring of an algebraic variety with torus action](#). *Adv. Math.* **225** (2010), no. 2, 977–1012 Zbl [1248.14008](#) MR [2671185](#)
- [12] N. O. Ilten and H. Süß, [Polarized complexity-1 \$T\$ -varieties](#). *Michigan Math. J.* **60** (2011), no. 3, 561–578 Zbl [1230.14075](#) MR [2861089](#)
- [13] S. L. Kleiman, [Intersection theory and enumerative geometry: a decade in review](#). In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, pp. 321–370, Proc. Sympos. Pure Math. 46, American Mathematical Society, Providence, RI, 1987 Zbl [0664.14031](#) MR [0927987](#)
- [14] A. Laface, A. Liendo, and J. Moraga, [On the topology of rational \$\mathbb{T}\$ -varieties of complexity one](#). *Mosc. Math. J.* **20** (2020), no. 2, 405–422 Zbl [1460.14017](#) MR [4088800](#)
- [15] B. I. U. Nødland, [Chow groups and pseudoeffective cones of complexity-one \$T\$ -varieties](#). *Manuscripta Math.* **169** (2022), no. 1-2, 141–165 Zbl [1505.14018](#) MR [4462660](#)
- [16] L. Petersen and H. Süß, [Torus invariant divisors](#). *Israel J. Math.* **182** (2011), 481–504 Zbl [1213.14084](#) MR [2783981](#)
- [17] R. T. Rockafellar, [Convex analysis](#). Princeton Math. Ser. 28, Princeton University Press, Princeton, NJ, 1970 Zbl [0193.18401](#) MR [0274683](#)
- [18] A. Shah, [Minkowski weights and the Grothendieck group of a toric variety](#). 2020, arXiv:[2002.02043v3](#)

Communicated by Gavril Farkas

Received 14 November 2023.

Ana Botero

Department of Mathematics, Bielefeld University, Universitätsstraße 25, 133615 Bielefeld,
Germany; abotero@math.uni-bielefeld.de