Propagation of polarization sets for systems of MHD type

Rayhana Darwich

Abstract. Polarization sets were introduced by Dencker (1982) as a refinement of wavefront sets to the vector-valued case. He also clarified the propagation of polarization sets when the characteristic variety of the pseudodifferential system under study consists of two hypersurfaces intersecting tangentially (1992), or transversally (1995). In this paper, we consider the case of more than two intersecting characteristic hypersurfaces that are intersecting transversally (and we give a note on the tangential case).

Mainly, we consider two types of systems which we name "systems of generalized transverse type" and "systems of MHD type", and we show that we can get a result for the propagation of polarization set similar to Dencker's result for systems of transversal type. Furthermore, we give an application to the MHD equations.

1. Introduction

In [10], Hörmander defined the wavefront set of a distribution u, denoted by $\mathrm{WF}(u)$, which is a refinement of the singular support of a distribution. The wavefront sets does not only show the location of singularity, but also the direction in which the singularity occurs. Concerning the propagation of the wavefront sets, many results were given. For example, in [10], Hörmander gave the result for the propagation of the wavefront set for pseudodifferential operator P of real principal type, where he stated that the wavefront set of u; the solution of $Pu = f$, is invariant under the bicharacteristic flow given that f is smooth. Moreover, in [5], Dencker studied the propagation of singularities for pseudodifferential operator P on a smooth manifold X, having characteristics of variable multiplicity. He considered the characteristic set to be union of hypersurfaces S_j, $j = 1, \ldots, r_0$ tangent at $\bigcap_{j=1}^{r_0} S_j$. Under some assumptions he proved that the wavefront set of the solution of the considered pseudodifferential operator is invariant under the union of the Hamilton flows on S_j, $j = 1, \ldots, r_0$, given that Pu is smooth on X.

In [11], Hörmander defined locally the wavefront set of distributional sections u in $\mathcal{D}'(X; E)$, where $E \to X$ is a vector bundle over the smooth manifold X. He defined the wavefront set of u locally as $\bigcup \mathrm{WF}(u_j)$ where $(u_1, \ldots, u_N)$ are the components of u with respect to a local trivialization of E. However, this definition does not specify

in which components u is singular, that is why Dencker defined in [2] the polarization set for vector-valued distribution u which we will denote by Pol(u). The polarization set also shows the location and the direction of the singularity as the wavefront set, but it additionally shows the most singular components of a distribution. Hence, the polarization set of a distribution is a refinement of the wavefront set, and the projection of Pol(u) \ 0 on the cotangent bundle T^*X gives the wavefront set of u. Similarly, the H^r-polarization set, Pol$^r(u)$, is defined as a refinement of the H^r-wavefront set, where the H^r-wavefront set of a distribution u; denoted by WF$^r(u)$, shows the location and direction where the distribution is not in the Sobolev space $H^r(X)$.

In [2], Dencker defined systems of pseudodifferential operators of real principal type; note that the definition of systems of pseudodifferential operators of real principal type differs from the case of scalar pseudodifferential operators of real principal type, and he defined Hamilton orbits for systems of real principal type which are certain line bundles, and then he proved that the polarization set of a solution u of systems of real principal type P will be union of Hamilton orbits, given that Pu is smooth. In [8], Gérard pointed out that the above result also holds for H^r-polarization sets.

Moreover, in [6], Dencker considered pseudodifferential system having its characteristic set is union of two non-radial hypersurfaces intersecting tangentially at an involutive manifold of exactly order $k_0 \geq 1$. He also assumed that the principal symbol vanishes of first order on the two-dimensional kernel at the intersection, and he assumed a Levi type of condition. Then, he defined systems satisfying these conditions to be systems of uniaxial type. Outside the intersection of the hypersurfaces the system will be of real principal type, hence the propagation result of the polarization set is already known there. In this article, Dencker has also proved a propagation result of the polarization set at the intersection. In [7], Dencker considered pseudodifferential system having its characteristic set is union of two non-radial hypersurfaces intersecting transversally at an involutive manifold of codimension 2. He also assumed that the principal symbol vanishes of first order on the two-dimensional kernel at the intersection. Systems satisfying these conditions are systems of transversal type. In this article, Dencker has proved a propagation result of the polarization set at the intersection. Outside the intersection the system is of real principal type.

We worked on extending Dencker's result stated above to pseudodifferential systems having their characteristic sets is union of several non-radial hypersurfaces intersecting transversally at an involutive manifold of codimension 2; not necessary just two hypersurfaces as in the case of systems of transversal type and systems of uniaxial type. Note that even if we assumed that the hypersurfaces are intersecting tangentially of exactly order $k_0 \geq 1$ instead of intersecting transversally, we get a similar result, and for the proof we use the same weight and metric introduced by Dencker in [5] for the symbol classes $S(\vartheta, g)$ of the Weyl calculus. We have considered two cases for that: the first case is the case where we have r_0 non-radial hypersurfaces intersecting transversally at a manifold Σ_2, and every two hypersurfaces does not intersect outside Σ_2, and we assumed that the r_0th-differential of the determinant of the principal symbol is different than zero at the intersection, and

the ith-differential of the determinant of the principal symbol vanishes at the intersection for $i < r_0$. Moreover, we assumed that the dimension of the kernel of the principal symbol to be r_0 at the intersection, and we assumed a condition similar to the Levi type condition considered in [6]. We called systems satisfying the above conditions systems of generalized transverse type, and we proved that we have a similar propagation result of the polarization set as that for systems of transversal type. The second case, is the case where we also have r_0 non-radial hypersurfaces intersecting transversally at a manifold Σ_2, every two hypersurfaces does not intersect outside Σ_2, and a condition similar to the Levi type condition considered in [6], but here we assumed that the $(r_0 + 1)$th-differential of the determinant of the principal symbol is different than zero at the intersection, and the ith-differential of the determinant of the principal symbol vanishes at the intersection for $i < r_0 + 1$. Moreover, we assumed that the dimension of the kernel of the principal symbol to be $r_0 + 1$ at the intersection. We also assumed some additional conditions that we did not assume in the case of systems of generalized transverse type. We defined systems satisfying these conditions to be systems of MHD type. We named them systems of MHD type because we have first noticed such systems when we considered the linearized ideal MHD equations. Thus we will have a section in which we study the propagation of polarization sets for the linearized ideal MHD equations.

In our work, we will assume that we have $P \in \Psi_{\mathrm{phg}}^m(X)$ an $N \times N$ system of classical pseudodifferential operators on a smooth manifold X of order m. Let $p = \sigma(P)$ be the principal symbol, $\det p$ the determinant of p, and $\Sigma = (\det p)^{-1}(0)$ the characteristics of P. We consider Σ to be union of several hypersurfaces non-radial in the covariable ξ that are intersecting transversally at an involutive manifold Σ_2. Now, we state our main theorem in this paper regarding the propagation of polarization sets for systems of generalized transverse type, and systems of MHD type, but its proof will be postponed to Sections 3, and 4 to prove it for systems of generalized transverse type, and systems of MHD type respectively. Let

$$r_u^*(\nu) = \sup\{r \in \mathbb{R} : u \in H^r \text{ at } \nu\}, \quad \nu \in T^*X \setminus 0$$

be the regularity function. The H^r-polarization set, where H^r is the usual Sobolev space is defined as follows.

Definition 1.1. For $u \in \mathscr{D}'(X, \mathbb{C}^N)$, the H^r-polarization set is given by

$$\mathrm{Pol}^r(u) = \bigcap \mathcal{N}_B \subseteq (T^*X \setminus 0) \times \mathbb{C}^N,$$

where $\mathcal{N}_B = \ker \sigma(B)$, and the intersection is taken over those $1 \times N$ systems $B \in \Psi_{\mathrm{phg}}^0$ such that $Bu \in H^r$.

Theorem 1.2. *Let $P \in \Psi_{\mathrm{phg}}^m$ be an $N \times N$ system of generalized transverse type (or of MHD type) at $\nu_0 \in \Sigma_2$, and let $A \in \Psi_{\mathrm{phg}}^0$ be an $N \times N$ system such that the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at ν_0, and $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ is a hypersurface near ν_0, where $\pi_1 : T^*X \times \mathbb{C}^N \to T^*X$ is the projection along the fibers. Assume that*

$u \in \mathcal{D}'(X, \mathbb{C}^N)$ *satisfies* $\min(r^*_{Pu} + m - 1, r^*_{Au}) > r$ *at* v_0. *Then,* $\mathrm{Pol}^r(u)$ *is a union of* $\mathcal{C}^\infty$ *line bundles in* $\mathcal{N}_A \cap \mathcal{N}_P$ *over bicharacteristics of* $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ *near* v_0.

The plan of this paper is as follows: in Section 2, we mention previous results on the propagation of polarization sets. More precisely, we will state Dencker's propagation result for systems of real principal type which was proven in [2], and Dencker's propagation result for systems of uniaxial type that was proven in [6]. Also, we state Dencker's result for the propagation of polarization sets for systems of transversal type; see [7]. Note that in [6,7], Dencker proved several results for the propagation of polarization sets under different conditions. Here we just mention the result which is similar to the result in our main theorem. In Sections 3 and 4, we define systems of generalized transverse type, and systems of MHD type, and we prove Theorem 1.2 for both types of systems. In Section 5, we give an application for the results in [9], and for the propagation of polarization sets for systems of MHD type, so we divide it into two subsections. First, we give the set of equations describing the ideal MHD, and we linearize it. In Section 5.1, we write the linearized ideal MHD equations in the form of a wave equation, and we give the characteristic variety of this wave equation which was calculated in [16] under some assumptions. Then, we calculate the transport equation under these assumptions as an application to Hansen's and Röhrig's results in [9]. In Section 5.2, we return to the linearized ideal MHD equations, and we calculate the eigenvalues and their multiplicities which are not constant. Then, we study the propagation of polarization sets, where we observe different cases, some in which our system is of real principal type, some in which our system is of uniaxial type, and one where our system is of MHD type.

2. Previous results

In this section, we state some previous results regarding the propagation of polarization sets. More precisely, we state the results for systems of real principal type, systems of uniaxial type, and systems of transversal type proven in [2, 6, 7], respectively. First, we will state the definition of the wavefront set, the H^r-wavefront set, and the polarization set. For the H^r-polarization set, see Definition 1.1.

As mentioned in the introduction, the definition of the wavefront set was first given by Hörmander in [10].

Definition 2.1. If $u \in \mathcal{D}'(X)$, we have

$$\mathrm{WF}(u) = \bigcap \mathrm{Char}\, A,$$

where the intersection is taken over all properly supported pseudodifferential operators $A \in \Psi^0(X)$ such that $Au \in \mathcal{C}^\infty(X)$, and $\mathrm{Char}\, A$ denotes the characteristic set of A given by

$$\mathrm{Char}\, A = \{(x, \xi) \in T^*(X) \setminus 0;\ a(x, \xi) = 0\}.$$

Here a is the principal symbol of A.

Similarly, the H^r-wavefront set was defined by Gérard in [8] using the Sobolev space.

Definition 2.2. If $u \in \mathcal{D}'(X)$ we have

$$\mathrm{WF}^r(u) = \bigcap \mathrm{Char}\, A,$$

where the intersection is taken over all properly supported pseudodifferential operators $A \in \Psi^0(X)$ such that $Au \in H^r_{\mathrm{loc}}(X)$.

In [2], Dencker generalized the notion of wavefront set to polarization set when we have vector-valued distribution.

Definition 2.3. For $u \in \mathcal{D}'(X, \mathbb{C}^N)$, the polarization set of u is given by

$$\mathrm{Pol}(u) = \bigcap \mathcal{N}_B \subseteq (T^*X \setminus 0) \times \mathbb{C}^N,$$

where $\mathcal{N}_B = \ker \sigma(B)$, and the intersection is taken over those $1 \times N$ systems $B \in \Psi^0_{\mathrm{phg}}$ such that $Bu \in \mathcal{C}^\infty$.

2.1. Systems of real principal type

First, we want to state Dencker's result regarding the propagation of polarization sets for systems of real principal type. For the definition of real principal type, we will differentiate between two cases, the scalar case, and the case of system of pseudodifferential operators.

Definition 2.4. We say that $P \in \Psi^m(X)$ is of real principal type if the principal symbol $\sigma(P) = p$ is real and the Hamilton field $H_p = \sum \partial_{\xi_j} p \partial_{x_j} - \partial_{x_j} p \partial_{\xi_j}$ is non-vanishing, and does not have the radial direction $\sum \xi_j \partial_{\xi_j}$ when $p = 0$.

Definition 2.5 (Case of system of pseudodifferential operators). An $N \times N$ system P of pseudodifferential operators on X with principal symbol $p(x, \xi)$ is of real principal type at $(y, \eta) \in T^*X \setminus 0$ if there exists an $N \times N$ symbol $\tilde{p}(x, \xi)$ such that

$$\tilde{p}(x, \xi) p(x, \xi) = q(x, \xi) \cdot \mathrm{Id}_N \tag{2.1}$$

in a neighborhood of (y, η) where $q(x, \xi)$ is a scalar symbol of real principal type and Id_N is the identity in $\mathbb{C}^N$.

Assume $P(x, D)$ to be an $N \times N$ system of classical pseudodifferential operators on an n-dimensional smooth manifold X of order m. The symbol of P is an asymptotic sum of homogeneous terms: $p(x, \xi) + p_{m-1}(x, \xi) + p_{m-2}(x, \xi) + \cdots$ where p is the principal symbol of P and p_j is homogeneous of degree j. Assume P to be of real principal type, thus there exists an $N \times N$ symbol $\tilde{p}$, and a scalar symbol q of real principal type such that (2.1) is satisfied. Let

$$\Sigma = \{(x, \xi) : \det p(x, \xi) = 0\}$$

be the characteristic set of P. To state the result of the propagation of polarization set given by Dencker in [2], we have to introduce first the connection he defined, and give the definition of the Hamilton orbit. Let

$$D_P w = H_q w + \frac{1}{2}\{\tilde{p}, p\}w + i\,\tilde{p}\,p^s_{m-1}w,$$

where w is $\mathcal{C}^\infty$ function on $T^*X \setminus 0$ with values in $\mathbb{C}^N$, $\{,\}$ is the Poisson bracket, that is $\{\tilde{p}, p\} = H_{\tilde{p}}p$, and p^s_{m-1} is the subprincipal symbol of P defined by

$$p^s_{m-1} = p_{m-1} - (2i)^{-1}\sum \partial_{x_j}\partial_{\xi_j}p.$$

D_P is a connection on $\mathcal{N}_P$ over Σ, that is, if $w \in \ker p$ at one point of a bicharacteristic of Σ, then $D_P w \in \ker p$ along the bicharacteristic if and only if $w \in \ker p$ there. Hence, each parallel section (that is w such that $D_P w = 0$) is uniquely determined by one point. D_P depends on the choice of $\tilde{p}$ and q, however Dencker showed that different choices of $\tilde{p}$ and q only change the solution of $D_P w = 0$ in $\mathcal{N}_P$ by a scalar factor. This motivated him to define the following.

Definition 2.6 (Hamilton Orbit). A Hamilton orbit of a system P of real principal type is a line bundle $L \subseteq \mathcal{N}_P|_\gamma$, where γ is an integral curve of the Hamilton field of Σ, and L is spanned by $\mathcal{C}^\infty$ section w satisfying $D_P w = 0$.

Theorem 2.7 (Dencker's propagation result). *Let P be an $N \times N$ system of pseudodifferential operators on a manifold X, and let $u \in \mathcal{D}'(X, \mathbb{C}^N)$. Assume that P is of real principal type at $(y, \eta) \in \Sigma$, and that $(y, \eta) \notin \mathrm{WF}(Pu)$. Then, over a neighborhood of (y, η) in Σ, $\mathrm{Pol}(u)$ is a union of Hamilton orbits of P.*

In [8], Gérard stated that we have similar propagation result for the H^r-polarization sets for systems of real principal type.

Theorem 2.8. *Let P be an $N \times N$ system of pseudodifferential operators on a manifold X of order m, and let $u \in \mathcal{D}'(X, \mathbb{C}^N)$. Assume that P is of real principal type at $(y, \eta) \in \Sigma$, and that $(y, \eta) \notin \mathrm{WF}^r(Pu)$. Then over a neighborhood of $(y, \eta) \in \Sigma$, $\mathrm{Pol}^{r+m-1}(u)$ is a union of Hamilton orbits of P.*

2.2. Systems of uniaxial type

Let $P \in \Psi^m_{\mathrm{phg}}(X)$ be an $N \times N$ system of classical pseudodifferential operators on a smooth manifold X. Let p be the principal symbol of P. Let $\Sigma = (\det p)^{-1}(0)$ be the characteristics of P, and let

$$\Sigma_2 = \big\{(x, \xi) \in \Sigma : d(\det p) = 0 \text{ at } (x, \xi)\big\},$$

and $\Sigma_1 = \Sigma \setminus \Sigma_2$. Assume that we have

$\Sigma = S_1 \cup S_2$, where S_1 and S_2 are non-radial hypersurfaces tangent at

$\Sigma_2 = S_1 \cap S_2$ of exactly order $k_0 \geq 1$, $\hfill (2.2)$

microlocally near $\nu_0 \in \Sigma_2$. This means that the Hamilton field of S_j does not have the radial direction $\langle \xi, \partial_\xi \rangle$, and it means also that the k_0th jets of S_1 and S_2 coincide on Σ_2, but no $(k_0 + 1)$th jet does. Note that we have P is of real principal type at Σ_1, since $d(\det p) \neq 0$ there; see Definition 2.5. (2.2) gives us that Σ_2 has to be a manifold of codimension ≥ 2. We assume that

$$\Sigma_2 \text{ is an involutive manifold of codimension } d_0 \geq 2. \tag{2.3}$$

Moreover, we assume that

$$\text{the (complex) dimension of the fiber of } \mathcal{N}_p \text{ is equal to 2 at } \Sigma_2 \tag{2.4}$$

and

$$d^2(\det p) \neq 0 \text{ at } \Sigma_2, \tag{2.5}$$

that is, p vanishes of first order on the kernel. We want to consider the limits of $\mathcal{N}_P|_{\Sigma_1}$ when we approach Σ_2, so let

$$\mathcal{N}_P^j = \mathcal{N}_P|_{S_j \setminus \Sigma_2}, \tag{2.6}$$

$T_{\Sigma_2}\Sigma := T_{\Sigma_2}S_1 = T_{\Sigma_2}S_2$ (note here Σ is not a manifold), and $\partial \Sigma_1 := T_{\Sigma_2}\Sigma/T\Sigma_2$. Here $\partial \Sigma_1$ is the normal bundle of Σ_2 in S_1 which is equal to the normal bundle of Σ_2 in S_2. Let $i_0 : \Sigma_2 \to \partial \Sigma_1$ denotes the zero section of $\partial \Sigma_1$. By the tubular neighborhood theorem we know that there exists a diffeomorphism Φ from some neighborhood $\mathcal{U} \subset S_j$ of Σ_2 to a neighborhood $\mathcal{U}_0 \subset \partial \Sigma_1$ of the zero section of $\partial \Sigma_1$, and Φ identifies Σ_2 itself with the zero section. Actually, the tubular neighborhood theorem says that if $N \subset M$ is a submanifold and M and N both have empty boundary, and

$$\mathcal{V}N := (TM|_N)/TN \to N,$$

is the normal bundle of the submanifold $N \subset M$, then there is a diffeomorphism Φ from some neighborhood $\mathcal{U} \subset M$ of N to a neighborhood $\mathcal{U}_0 \subset \mathcal{V}N$ of the zero-section in the total space of its normal bundle, and Φ identifies N itself with the zero-section.

Before giving the definition of systems of uniaxial type, we need to give the definition of the limit polarizations.

Definition 2.9. For $j = 1, 2$, we define the limit polarizations

$$\partial \mathcal{N}_P^j = \left\{ (\nu, \rho, z) \in \partial \Sigma_1 \times \mathbb{C}^N : \rho \neq 0 \text{ and } z = \lim_{k \to \infty} z_k \right\},$$

where $z_k \in \ker p(\nu_k)$ and $\nu_k \in S_j \setminus \Sigma_2$ satisfy $(\nu - \nu_k)/|\nu - \nu_k| \to \rho/|\rho|$ when $k \to \infty$.

$\partial \mathcal{N}_P^j$ is conical in ξ and ρ, and homogeneous in the fiber, but it may have (complex) dimension > 1 at (ν, ρ). We assume that the fiber of

$$\partial \mathcal{N}_P^1 \cap \partial \mathcal{N}_P^2 = \{0\} \quad \text{over } \partial \Sigma_1 \setminus (\Sigma_2 \times 0). \tag{2.7}$$

This condition means that no element in $\mathcal{N}_P|_{\Sigma_2}$ can be the limit of polarization vectors on both characteristic surfaces, along the same direction. Dencker showed that (2.7) implies

that $\partial \mathcal{N}_P^j$ is a complex line bundle over $\partial \Sigma_1 \setminus (\Sigma_2 \times 0)$, if we assume (2.2)–(2.5). Now, we give the definition of systems of uniaxial type.

Definition 2.10. The system P is of uniaxial type at $\nu_0 \in \Sigma_2$, if (2.2)–(2.5) and (2.7) hold microlocally near ν_0.

If $P \in \Psi_{\mathrm{phg}}^m$ is of uniaxial type and $Pu \in H^r$ near $\nu \in \Sigma_1$, then we already know the result as Gérard stated in [8] that $\mathrm{Pol}^{r+m-1}(u)$ is a union of Hamilton orbits in $\mathcal{N}_P$ near ν because P is of real principal type at Σ_1. Now, we want to give Dencker's result for the propagation of the polarization set when we approach Σ_2. Here S_1 and S_2 are tangent at Σ_2, so their Hamilton fields are parallel on Σ_2. Since Σ_2 is involutive, the Hamilton fields are tangent to Σ_2. Therefore, Σ and Σ_2 are foliated by the bicharacteristics of Σ. Also, Dencker proved that $\partial \Sigma_1 \setminus (\Sigma_2 \times 0)$ is foliated by limit bicharacteristics, which are liftings of bicharacteristics in Σ_2, and that $\partial \mathcal{N}_P^1 \cup \partial \mathcal{N}_P^2$ is foliated by limit Hamilton orbits, which are liftings of limits of Hamilton orbits, and are unique line bundles over limit bicharacteristics. The following theorem is Theorem 8.11 in [6].

Theorem 2.11. *Let $P \in \Psi_{\mathrm{phg}}^m$ be an $N \times N$ system of uniaxial type at $\nu_0 \in \Sigma_2$, and let $A \in \Psi_{\mathrm{phg}}^0$ be a $1 \times N$ system such that the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at ν_0. Assume that $u \in \mathcal{D}'(X, \mathbb{C}^N)$ satisfies $\min(r_{Pu}^* + m - 1, r_{Au}^*) > r$ at ν_0. Then, $\mathrm{Pol}^r(u)$ is a union of $\mathcal{C}^\infty$ line bundles in $\mathcal{N}_A \cap \mathcal{N}_P$ over bicharacteristics of Σ in $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ near ν_0, where $\pi_1 : T^*X \setminus 0 \times \mathbb{C}^N \to T^*X$ is the projection along the fibers.*

Moreover, in [6], Dencker showed under what assumptions we get $\mathrm{Pol}^r(u)$ is union of limits of Hamilton orbits in $\mathcal{N}_A \cap \mathcal{N}_P$ near $\nu_0 \in \Sigma_2$.

2.3. Systems of transversal type

Finally, we want to state Dencker's result regarding the propagation of polarization sets for systems of transversal type; see [7]. Let $P \in \Psi_{\mathrm{phg}}^m$ be an $N \times N$ system of classical pseudodifferential operators on a smooth manifold X, $p = \sigma(P)$ be the principal symbol, and $\Sigma = (\det p)^{-1}(0)$ be the characteristics of P. Let

$$\Sigma_2 = \{(x, \xi) \in \Sigma : d(\det p) = 0 \text{ at } (x, \xi)\},$$

and $\Sigma_1 = \Sigma \setminus \Sigma_2$. For systems of transversal type we have Σ is a union of two non-radial hypersurfaces intersecting transversally at Σ_2. More precisely, the systems of transversal type is defined as the following.

Definition 2.12. The system P is of transversal type at $\nu_0 \in \Sigma_2$ if

Σ_2 is a non-radial involutive manifold of codimension 2,

$\det p = e \cdot q$, where $e \neq 0$ and q is real valued with Hessian having rank 2

and positivity 1,

$\dim \ker p = 2$ on Σ_2,

microlocally near ν_0.

Similar to the case of systems of uniaxial type, if $P \in \Psi^m_{\text{phg}}$ is of transversal type and $Pu \in H^r$ near $\nu \in \Sigma_1$, then P is of real principal type at ν. Let $\mathcal{N}^j_P$ be as in (2.6). In [7], Dencker modified slightly the definition of limit polarizations.

Definition 2.13. For $j = 1, 2$, the limit polarizations is defined by

$$\partial \mathcal{N}^j_P = \{(\nu, z) \in \Sigma_2 \times \mathbb{C}^N : z = \lim_{k \to \infty} z_k\},$$

where $z_k \in \ker p(\nu_k)$ and $S_j \setminus \Sigma_2 \ni \nu_k \to \nu$.

$\partial \mathcal{N}^j_P$ is conical in ξ and linear in the fibers. Dencker showed that $\partial \mathcal{N}^j_P$ is a $\mathcal{C}^\infty$ line bundle over Σ_2, $j = 1, 2$, and that

$$\partial \mathcal{N}^1_P \cap \partial \mathcal{N}^2_P = \{0\} \quad \text{over } \Sigma_2.$$

Here, S_1 and S_2 are transversal at Σ_2, so their Hamilton fields are non-parallel on Σ_2. Since Σ_2 is involutive of codimension 2, the Hamilton fields of S_j are tangent to Σ_2 and generate the two-dimensional foliation of Σ_2. Moreover, $\partial \mathcal{N}^j_P$ is foliated by limit Hamilton orbits which are limits of Hamilton orbits in $\mathcal{N}^j_P$, and are unique line bundles over bicharacteristics in S_j at Σ_2 for $j = 1, 2$. The following theorem is [7, Theorem 7.1].

Theorem 2.14. *Let $P \in \Psi^m_{\text{phg}}$ be an $N \times N$ system of transversal type at $\nu_0 \in \Sigma_2$, and let $A \in \Psi^0_{\text{phg}}$ be a $1 \times N$ system such that the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at ν_0, and $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ is a hypersurface near ν_0. Assume that $u \in \mathcal{D}'(X, \mathbb{C}^N)$ such that $Pu \in H^{r-m+1}$ and $Au \in H^r$ at ν_0. Then $\text{Pol}^r(u)$ is a union of (limit) Hamilton orbits in $\mathcal{N}_A \cap \mathcal{N}_P$. Here $\pi_1 : T^*X \times \mathbb{C}^N \to T^*X$ is the projection along the fibers.*

Note that in this case $M_A = S_j$ for some j, and $\mathcal{N}_A \cap \mathcal{N}_P$ is a union of (limit) Hamilton orbits.

3. Propagation of polarization sets for systems of generalized transverse type

In this section, we generalize Dencker's result stated in Section 2.3 by considering the system to have its characteristic set is union of r_0 hypersurfaces intersecting transversally at an involutive manifold of codimension 2, with $r_0 \geq 2$. Let $P \in \Psi^m_{\text{phg}}(X)$ be an $N \times N$ system of classical pseudodifferential operators on a smooth manifold X. Let $p = \sigma(P)$ be the principal symbol and $\Sigma = (\det p)^{-1}(0)$ the characteristic set. Assume microlocally near $(x_0, \xi_0) \in \Sigma$ that

$$\Sigma = \bigcup_{j=1}^{r_0} S_j, \quad r_0 \geq 2, \text{ where } S_j \text{ are non-radial hypersurfaces}$$

$$\text{intersecting transversally at } \Sigma_2 = \bigcap_{j=1}^{r_0} S_j, \tag{3.1}$$

$$\Sigma_2 \text{ is an involutive manifold of codimension 2,} \tag{3.2}$$

and assume that every two hypersurfaces does not intersect outside Σ_2, that is

$$S_i \cap S_j = \Sigma_2, \quad \text{for } i \neq j. \tag{3.3}$$

Moreover, assume that

$$\text{the dimension of the fiber of } \mathcal{N}_P \text{ is equal to } r_0 \text{ at } \Sigma_2 \tag{3.4}$$

and

$$d^i(\det p) = 0 \text{ for } i < r_0 \quad \text{and} \quad d^{r_0}(\det p) \neq 0 \text{ at } \Sigma_2. \tag{3.5}$$

We want to consider the limits of $\mathcal{N}_P|_{\Sigma_1}$ when we approach Σ_2, so let

$$T_{\Sigma_2}\Sigma := \bigcup_{j=1}^{r_0} T_{\Sigma_2} S_j$$

(note that Σ is not a manifold), and $\partial\Sigma_1 := T_{\Sigma_2}\Sigma / T\Sigma_2$. Before giving the definition of systems of generalized transverse type, we need to give the definition of the limit polarizations.

Definition 3.1. For $j = 1, \ldots, r_0$, we define the limit polarizations

$$\partial\mathcal{N}_P^j = \left\{ (\nu, \rho, z) \in \partial\Sigma_1 \times \mathbb{C}^N : \rho \neq 0 \text{ and } z = \lim_{k \to \infty} z_k \right\},$$

where $z_k \in \ker p(\nu_k)$ and $\nu_k \in S_j \setminus \Sigma_2$ satisfy $(\nu - \nu_k)/|\nu - \nu_k| \to \rho/|\rho|$ when $k \to \infty$.

$\partial\mathcal{N}_P^j$ is conical in ξ and ρ, and homogeneous in the fiber. We will assume that the fiber of

$$\partial\mathcal{N}_P^1 \cap \cdots \cap \partial\mathcal{N}_P^{r_0} = \{0\} \quad \text{over } \partial\Sigma_1 \setminus (\Sigma_2 \times 0). \tag{3.6}$$

This condition means that no element in $\mathcal{N}_P|_{\Sigma_2}$ can be the limit of polarization vectors on all characteristic surfaces along the same direction.

Definition 3.2. The system P is of generalized transverse type at $\nu_0 \in \Sigma_2$, if (3.1)–(3.6) hold microlocally near ν_0.

Proposition 3.3. *Let $P \in \Psi_{phg}^1$ be an $N \times N$ system of generalized transverse type at $\nu_0 \in \Sigma_2$. Then by choosing suitable symplectic coordinates, we may assume that $X = \mathbb{R} \times \mathbb{R}^{n-1}$, $\nu_0 = (0; (0, \ldots, 1))$, and*

$$S_j = \left\{ (t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau + \beta_j(t, x, \xi) = 0 \right\}, \quad j = 1, \ldots, r_0, \tag{3.7}$$

microlocally near ν_0. Here β_j are real and homogeneous of degree 1 in ξ; with $\beta_1 \equiv 0$, satisfies in a conical neighborhood of ν_0

$$c|\xi'| \leq |\beta_j| \leq C|\xi'|, \quad j = 2, \ldots, r_0, \ 0 < c < C, \tag{3.8}$$

where $(\tau, \xi', \xi'') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$, which gives $\Sigma_2 = \{\tau = 0, \xi' = 0\}$. By conjugating P

with an elliptic, scalar Fourier integral operators, and multiplying with elliptic $N \times N$ systems of order 0, we may assume that

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \mod \mathscr{C}^{\infty},$$

microlocally near v_0, where $E \in \Psi_{\text{phg}}^1$ is an elliptic $(N - r_0) \times (N - r_0)$ system and

$$F \cong \text{Id}_{r_0} D_t + K(t, x, D_x) \mod \mathscr{C}^{\infty}. \tag{3.9}$$

Here,

$$K(t, x, D_x) \in \mathscr{C}^{\infty}(\mathbb{R}, \Psi_{\text{phg}}^1)$$

is an $r_0 \times r_0$ system, such that $k = \sigma(K)$ has $0, \beta_2, \ldots, \beta_{r_0}$ as eigenvalues.

Proof. We will prove it in a way similar to how Dencker proved the normal form for systems of uniaxial type. Since the result is local, we may assume $X = \mathbb{R}^n$. Because Σ_2 is involutive, we may choose symplectic, homogeneous coordinates $(x, \xi) \in T^*\mathbb{R}^n$ near $v_0 \in \Sigma_2$, so that $v_0 = (0; (0, \ldots, 1))$ and

$$\Sigma_2 = \{(x, \xi) \in T^*\mathbb{R}^n : \xi' = 0\},$$

where $\xi = (\xi', \xi'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$. We may also assume

$$S_1 = \{(x, \xi) \in T^*\mathbb{R}^n : \xi_1 = 0\},$$

near v_0. Now, we rename $x_1 = t$, $x_2 = x'$, and $(x_3, \ldots, x_n) = x''$. Since S_j is intersecting transversally with S_1 at Σ_2, we obtain

$$S_j = \{(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau + \beta_j(t, x, \xi) = 0\},$$

with β_j real and homogeneous of degree 1 in ξ, $\beta_1 \equiv 0$, and

$$c|\xi'| \leq |\beta_j - \beta_k| \leq C|\xi'|, \quad j \neq k,$$

in a conical neighborhood of v_0. By taking $k = 1$, we obtain (3.8) in a conical neighborhood of v_0. Using that $\dim \mathscr{N}_P = r_0$ at Σ_2, we can find an $N \times N$ elliptic matrix b homogeneous of degree 0 in the ξ variables which maps $\text{Im } p$ to $\{z \in \mathbb{C}^N; z_j = 0, j \leq r_0\}$ over Σ_2 near v_0, and we can choose an $N \times N$ matrix a homogeneous of degree 0 in the ξ variables such that a^{-1} maps $\ker p$ onto $\{z \in \mathbb{C}^N; z_j = 0, j > r_0\}$ over Σ_2 near v_0. Then we have

$$bpa = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & e \end{pmatrix} \tag{3.10}$$

such that e is an $(N - r_0) \times (N - r_0)$ matrix which is elliptic at v_0, and s_{11}, s_{12}, s_{21}, vanish on Σ_2 near v_0.

Now, we choose $N \times N$ systems of pseudodifferential operators A and B with principal symbols a, and b respectively. Then

$$BPA = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & E \end{pmatrix}$$

where its principal symbol is given by (3.10). As E is a system of order 1 which is elliptic at Σ_2, choose J to be its microlocal parametrix of order -1. Multiply BPA from the left with

$$B_1 = \begin{pmatrix} \mathrm{Id}_{r_0} & -S_{12}J \\ 0 & \mathrm{Id}_{N-r_0} \end{pmatrix}.$$

Multiply also $B_1 BPA$ from the right by

$$A_1 = \begin{pmatrix} \mathrm{Id}_{r_0} & 0 \\ -J S_{21} & \mathrm{Id}_{N-r_0} \end{pmatrix}.$$

Hence, we get

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \quad \mathrm{mod}\ \mathcal{C}^{\infty},$$

microlocally near ν_0, where $E \in \Psi^1_{\mathrm{phg}}$ is an elliptic $(N - r_0) \times (N - r_0)$ system. If f is the principal symbol for F, then conditions (3.5) and (3.7) imply

$$\det f = c\tau \prod_{i=2}^{r_0} (\tau + \beta_i), \quad 0 \neq c \in S^{-1},$$

thus $\partial_\tau^{r_0}(\det f) = \det(\partial_\tau f) \neq 0$ at Σ_2. By [6, Theorem A.3], and homogeneity, we may find homogeneous system $C_0 \in S^0$ such that

$$f = C_0\big(\tau\, \mathrm{Id}_{r_0} + k(t, x, \xi)\big),$$

where $\det C_0 \neq 0$ at Σ_2. By multiplication with an elliptic system, we may assume $C_0 \equiv \mathrm{Id}_{r_0}$. Thus, $\det f = \tau \prod_{i=2}^{r_0} (\tau + \beta_j)$, which implies that $k(t, x, \xi)$ has the eigenvalues 0, $\beta_2, \ldots, \beta_{r_0}$. If $f_0 \in S^0$ is the term homogeneous of degree 0 in the expansion of F, then [6, Theorem A.4], and homogeneity give

$$f_0 = B_{-1} f + B_0,$$

where $B_0 \in \mathcal{C}^{\infty}(\mathbb{R}, S^0)$ is independent of τ, and $B_{-1} \in S^{-1}$. By multiplying f with an operator with symbol $\mathrm{Id}_{r_0} - B_{-1}$, we may assume $B_{-1} \equiv 0$. By induction over lower order terms we obtain (3.9). ∎

We want to introduce symbol classes adapted to the functions β_j defined in (3.8) for $j = 2, \ldots, r_0$. Let

$$\vartheta(\xi) = \langle \xi' \rangle, \tag{3.11}$$

where $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$, thus $\vartheta \approx 1 + |\beta_j|$. Consider the metric

$$g(dx, d\xi) = |dx|^2 + |d\xi'|^2 \langle \xi' \rangle^{-2} + |d\xi''|^2 \langle \xi \rangle^{-2}, \tag{3.12}$$

and $h^2 = \sup g/g^\sigma = \langle \xi' \rangle^{-2}$. We get that g is σ temperate, and $\beta_j \in S(\vartheta, g)$. Check [13, Chapter XVIII] to know more about the symbol classes $S(\vartheta, g)$ of the Weyl calculus. Moreover, using Taylor's formula we can write

$$\beta_j = a_j \xi', \tag{3.13}$$

with $a_j \in S^0$ is homogeneous of degree 0 in ξ.

Proposition 3.4. *Let*

$$P = \mathrm{Id}_{r_0} D_t + K(t, x, D_x) \tag{3.14}$$

be an $r_0 \times r_0$ system with $K \in \mathcal{C}^\infty(\mathbb{R}, \Psi_{\mathrm{phg}}^1)$, such that the eigenvalues of $k = \sigma(K)$ are $0, \beta_2, \dots, \beta_{r_0}$, with β_j as in (3.8), that is $\beta_j \neq 0$. Then P is of generalized transverse type if and only if $k \in \mathcal{C}^\infty(\mathbb{R}, S(\vartheta, g))$.

Proof. First, let $k = (k_{ij})_{1 \leq i, j \leq r_0}$. Let

$$\alpha = (\alpha_1, \dots, \alpha_{r_0}) \in \mathcal{C}^\infty(\mathbb{R}, S^1),$$

with $\alpha_i = (k_{i1}, \dots, k_{ir_0})$ for $i = 1, \dots, r_0$ homogeneous of degree 1 in ξ. By homogeneity,

$$k \in \mathcal{C}^\infty\big(\mathbb{R}, S(\vartheta, g)\big) \Leftrightarrow \alpha = O(\beta_j),$$

for every $j = 2, \dots, r_0$.

Assume that $\alpha = O(\beta_j)$ and $(v, \rho) \in \partial \Sigma_1$, $\rho \neq 0$. Choose $\Sigma \setminus \Sigma_2 \ni v_l \to v$ such that $(v - v_l)|v - v_l|^{-1} \to \rho/|\rho|$, $l \to \infty$. Let us define

$$\gamma_s^{ij}(v, \rho) := \lim_{v_l \to v} \frac{k_{ij}}{\beta_s}(v_l) \quad \text{for } s \in \{2, \dots, r_0\} \text{ and } 1 \leq i, j \leq r_0. \tag{3.15}$$

Since $\alpha = O(\beta_j)$ does not depend on τ, so the above definition is independent of the choice of v_l. We get

$$\partial \mathcal{N}_P^1(v, \rho) = \ker \big((\gamma_s^{ij}(v, \rho))_{1 \leq i, j \leq r_0}\big), \quad \forall s \in \{2, \dots, r_0\},$$

where $(\gamma_s^{ij}(v, \rho))_{1 \leq i, j \leq r_0}$ denote the matrix with entries $\gamma_s^{ij}(v, \rho)$ for $1 \leq i, j \leq r_0$.

$$\partial \mathcal{N}_P^s(v, \rho) = \ker \big(- \mathrm{Id}_{r_0} + (\gamma_s^{ij}(v, \rho))_{1 \leq i, j \leq r_0}\big) \quad \text{for } s \in \{2, \dots, r_0\}.$$

It is easy to see that the condition (3.6) is satisfied.

On the other hand, assume that $\alpha \neq O(\beta_j)$ at $v \in \Sigma_2$. Then, there exists a sequence $v_l = (t_l, x_l; 0, \xi_l) \to v$, such that

$$|\alpha(v_l)| > l|\beta_j(v_l)|, \quad \forall l \in \mathbb{N}.$$

It is no restriction to assume that $\{(\nu - \nu_l)|\nu - \nu_l|^{-1}\}$ has a limit $0 \neq \rho \in \partial\Sigma_1|_\nu$ as $l \to \infty$, and that

$$\varepsilon^{i,j} = \lim_{l \to \infty} k_{ij}(\nu_l)/|\alpha(\nu_l)| \text{ exists.}$$

Since $\beta_j(\nu_l)/|\alpha(\nu_l)| \to 0$, we get that

$$\partial\mathcal{N}_P^s(\nu, \rho) \supseteq \ker\left((\varepsilon^{i,j})_{1 \leq i, j \leq r_0}\right) \quad \text{for } s = 1, \dots, r_0.$$

Now, we want to show that $\ker((\varepsilon^{i,j})_{1 \leq i,j \leq r_0}) \neq \{0\}$. Since, we have $\det k = 0$, we get $\det((\varepsilon^{i,j})_{1 \leq i,j \leq r_0}) = 0$. Hence, we get that the rank of $(\varepsilon^{i,j})_{1 \leq i,j \leq r_0}$ is less than or equal to $r_0 - 1$, which gives in turn that $\dim \ker((\varepsilon^{i,j})_{1 \leq i,j \leq r_0})$ is greater than or equal to 1. ∎

We will introduce two spaces that we are going to use. These spaces were also given by Dencker in [7] and he showed the relation between these two spaces by a lemma that we will also state. Let $H^{r,s}$ be the space of $u \in \mathcal{S}'$ (here u depends on t and x) satisfying

$$\|u\|_{r,s}^2 = (2\pi)^{-n} \int |\hat{u}(\tau, \xi)|^2 \langle(\tau, \xi)\rangle^{2r} \langle(\tau, \xi')\rangle^{2s} d\tau d\xi < \infty. \tag{3.16}$$

We say that $u \in H^{r,s}$ at $\nu \in T^*\mathbb{R}^n \setminus 0$, that is, $\nu \notin \mathrm{WF}^{r,s}(u)$, if $u = u_1 + u_2$, where $u_1 \in H^{r,s}$ and $\nu \notin \mathrm{WF}(u_2)$.

Similarly, when we have u depends only on x then the norm becomes

$$\|u\|_{r,s}^2 = (2\pi)^{-n} \int |\hat{u}(\xi)|^2 \langle\xi\rangle^{2r} \langle\xi'\rangle^{2s} d\xi < \infty.$$

We have $S^0 \subset S(1, g)$. Note that the spaces $H^{r,s}$ is a particular case of the spaces $B_{p,k}$ introduced by Hörmander; check [12], where $p = 2$ and $k(\tau, \xi) = \langle\tau, \xi\rangle^r \langle\tau, \xi'\rangle^s$.

Proposition 3.5. *Assume that P is an $r_0 \times r_0$ system of pseudodifferential operators of order 1 on $\mathbb{R}^n$, on the form (3.14), with $K \in \mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}(S(\vartheta, g)))$ near $\nu_0 \in \Sigma_2$. Let $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^{r_0})$ and assume $Pu \in H^{r,s}$ at ν_0. Then, for every $\delta > 0$, and $N \in \mathbb{N}$, we can find c_δ and $C_{\delta,N} > 0$ and $v_\delta \in H^{r,s+1}$ at ν_0, such that $u_\delta = u - v_\delta$ satisfies*

$$|\hat{u}_\delta(\tau, \xi)| \leq C_{\delta,N} \langle(\tau, \xi)\rangle^{-N}, \quad \text{when } |\tau| > c_\delta\left(\langle\xi\rangle^\delta + \langle\xi'\rangle\right). \tag{3.17}$$

Proof. Follow the proof of [7, Proposition A.1]. ∎

Let $H_*^{r,s}$ be the Banach space of $u \in \mathcal{S}'$, satisfying

$$\left(\|u\|_{r,s}^*\right)^2 = (2\pi)^{-n} \int |\hat{u}(\tau, \xi)|^2 \langle\xi\rangle^{2r} \langle\xi'\rangle^{2s} d\tau d\xi < \infty.$$

If we have u depends only on x then the norm becomes

$$\left(\|u\|_{r,s}^*\right)^2 = (2\pi)^{-n} \int |\hat{u}(\xi)|^2 \langle\xi\rangle^{2r} \langle\xi'\rangle^{2s} d\xi < \infty.$$

Hence, $\|u\|_{r,s}^* = \|u\|_{r,s}$ when u depends only on x.

If $u \in H_{r,s}^*$ then we get $u|_{t=\rho} \in H^{r,s}$ for almost all ρ, by Fubini's theorem. If $u \in \mathcal{S}'$ satisfies (3.17), then

$$\|u\|_{r-\delta s_-,s}^* \le C_{r,s}\big(\|u\|_{r,s} + 1\big) \le C_{r,s}'\big(\|u\|_{r+\delta s_+,s}^* + 1\big), \quad \forall r, s \in \mathbb{R},$$

where $s_\pm = \max(\pm s, 0)$. Hence, we lose only $O(\delta)$ derivatives when taking restriction to $\{t = r\}$, for almost all r.

Definition 3.6. Let $u \in \mathcal{S}'(\mathbb{R}^n)$, and assume $\xi \ne 0$ in WF(u). We say that $u \in H_*^{r,s}$ at (t_0, x_0, ξ_0), that is, $(t_0, x_0, \xi_0) \notin \mathrm{WF}_*^{r,s}(u)$, if there exists $\phi(t, x, \xi) \in \mathcal{C}^\infty(\mathbb{R}, S_{1,0}^0)$ such that $\phi(t, x, D_x)u \in H_*^{r,s}$ and $\underline{\lim}_{\lambda\to\infty}|\phi(t_0, x_0, \lambda\xi_0)| \ne 0$.

We have

$$(t_0, x_0, \xi_0) \notin \mathrm{WF}_*^{r,s}(u) \Rightarrow (x_0, \xi_0) \notin \mathrm{WF}^{r,s}(u_\rho), \tag{3.18}$$

for almost all ρ close to t_0, where $u_\rho = u|_{t=\rho}$. If $\xi \ne 0$ in WF(u), then from [3, Lemma 2.3], we get that

$$\pi_0(\mathrm{WF}^{r,0})(u) = \mathrm{WF}_*^{r,0}(u),$$

where $\pi_0(t, x; \tau, \xi) = (t, x, \xi)$. The following lemma gives the result for the more general wavefront sets.

Lemma 3.7. *Assume that $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies* (3.17). *Then Au satisfies* (3.17), *for any $A \in \mathcal{C}^\infty(\mathbb{R}, \Psi_{\delta,0}^m)$ $\forall m$ and $\forall \delta > 0$. We also obtain*

$$\mathrm{WF}_*^{r-\delta s_-,s}(u) \subseteq \pi_0\big(\mathrm{WF}^{r,s}(u)\big) \subseteq \mathrm{WF}_*^{r+\delta s_+,s}(u),$$

where $s_\pm = \max(\pm s, 0)$ and $\pi_0(t, x; \tau, \xi) = (t, x, \xi)$. Since $u \in \mathcal{C}^\infty$ in $\pi_0^{-1}(\Sigma_2) \setminus \Sigma_2$ by (3.17), *we find*

$$\iota_0\big(\mathrm{WF}_*^{r-\delta s_-,s}(u)\big) \subseteq \mathrm{WF}^{r,s}(u) \subseteq \iota_0\big(\mathrm{WF}_*^{r+\delta s_+,s}(u)\big) \quad \text{on } \Sigma_2, \tag{3.19}$$

where $\iota_0(t, x, \xi) = (t, x; 0, \xi)$.

Proof. See [7, Lemma A.3]. ∎

Note that $H^r = H^{r,0}$ is the usual Sobolev space. Changing the notation, let $x_1 = t$, $x' = x_2$, then $x = (x_1, x', x'') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$. Introduce the symbol classes

$$S^{r,s} = S(\langle\xi\rangle^r h^{-s}, g)$$

where $h^{-2} = 1 + |\xi_1|^2 + |\xi'|^2$ and $\langle\xi\rangle$ are weights for the metric g defined by

$$g_{x,\xi}(dx, d\xi) = |dx|^2 + |d\xi|^2 h^2.$$

Let $\Psi^{r,s} = \mathrm{Op}\, S^{r,s}$ be the corresponding pseudodifferential operators, which maps $H^{r,s}$ into L^2. Returning to the old notation where using t instead of x_1, and assume that $P \in \mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}(S(\vartheta, g)))$ be as in (3.14), we get $P \in \Psi^{0,1}$.

In order to prove the main theorem, we need to study the regularity of a Cauchy problem that we will state. Consider the following $N \times N$ system

$$Q = q \operatorname{Id}_N + \sum_{i=0}^{r_0-1} Q_i.$$

Here q is a scalar operator with symbol

$$q(t, x; \tau, \xi) = \tau \prod_{j=2}^{r_0} (\tau + \beta_j), \tag{3.20}$$

where $\beta_j \in S(\vartheta, g)$ is homogeneous and satisfies (3.8). We will assume

$$Q_i = \sum_{k=0}^{i} A_{i-k}^i D_t^k,$$

with $A_{k'}^i \in \operatorname{Op} S(\vartheta^{k'}, g)$. We are going to study the following Cauchy Problem:

$$\begin{aligned} Qu &= f, \\ D_t^k u|_{t=0} &= u_k, \quad \text{for } k = 0, \dots, r_0 - 1. \end{aligned} \tag{3.21}$$

We are going to assume that the $\xi \neq 0$ in WF(u). Hence, the restrictions are well defined.

Proposition 3.8. *Assume that $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N)$ satisfies (3.21), and $\xi \neq 0$ in WF(u). If $u_k \in H^{r,s-k}$ at (x_0, ξ_0) for $k = 0, \dots, r_0 - 1$, $f \in H_*^{r,s-r_0+1}$ at (t, x_0, ξ_0) for $0 \leq t \leq t_0$, and $\xi_0' = 0$, then $u \in H_*^{r,s}$ at (t_0, x_0, ξ_0).*

Proof. By conjugating with an elliptic, scalar operator with symbol in $S(\langle \xi \rangle^r \vartheta^s, g)$, we may assume that $r = s = 0$. We will reduce to a first order symmetric system. Let $v_k = \lambda^{k-1} D_t^{k-1} u$ for $k = 1, \dots, r_0$, where $\lambda \in \operatorname{Op} S(\vartheta^{-1}, g)$ has symbol ϑ^{-1}. Hence, $v_k = \lambda D_t v_{k-1}$ for $k = 2, \dots, r_0$. Then $V = {}^t(v_1, \dots, v_{r_0}) \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^{r_0 N})$, $\xi \neq 0$ in WF(V), and V satisfies

$$PV = F, \quad V|_{t=0} = V_0. \tag{3.22}$$

Here $P = \operatorname{Id}_{r_0 N} D_t + K$, $F = (0, \dots, 0, \lambda^{r_0-1} f)$, $V_0 = (u_0, \lambda u_1, \dots, \lambda^{r_0-1} u_{r_0-1})$, and $K \in \operatorname{Op} S(\vartheta, g)$ has principal symbol $k = (k_{ij})_{1 \leq i,j \leq r_0}$ such that

$$\begin{aligned} k_{ii+1} &= -\vartheta \operatorname{Id}_N \quad \text{for } 1 \leq i \leq r_0 - 1, \\ k_{r_0 j} &= \sum \beta_{i_1} \cdots \beta_{i_{r_0-j+1}} \vartheta^{j-r_0} \operatorname{Id}_N \quad \text{for } 2 \leq j \leq r_0, \\ k_{ij} &= 0 \quad \text{elsewhere,} \end{aligned} \tag{3.23}$$

where the sum in (3.23) is such that $2 \leq i_1 \leq j$ and $i_{k-1} + 1 \leq i_k \leq j + k - 1$ for $2 \leq k \leq r_0 - j + 1$.

We find $V_0 \in H^{0,0} = L^2$ at (x_0, ξ_0) and $F \in H_*^{0,0}$ at (t, x_0, ξ_0) for $0 \leq t \leq t_0$. Thus the result follows from the next proposition, which we will state after the following lemma. ∎

Lemma 3.9. *When the above assumptions are satisfied we get*

$$\int_0^\varepsilon \|V\|(t)dt \le C\varepsilon^{1/2}\left(\int_0^\varepsilon \|D_t V + KV\|(t)dt + \|V\|(0)\right)$$

for $V \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^{r_0 N})$, if $\varepsilon > 0$ is sufficiently small. Here $\|.\|(t)$ denotes the L^2 norm in the x variables, depending on t.

Proof. We are going to prove it in a similar way as the proof of Lemma 5.2 in [4]. As k is diagonalizable in $S(1, g)$, we get that $k = \sum_{j=1}^{r_0} \beta_j \pi_j$, with $\beta_1 = 0$, where $\pi_j(t, x, \xi) \in S(1, g)$ is the projection on the eigenvectors corresponding to the eigenvalues β_j along the others when $\xi' \ne 0$. k is symmetrizable with symmetrizer $M = \sum \pi_j^* \pi_j$, that is $M > 0$ and Mk is symmetric. Note that k is symmetrizable means there exists symmetric $N \times N$ system $M(t, x, \xi) \in S(1, g)$ such that $0 < c \le M$ and $Mk - (Mk)^* \in S(1, g)$. If we put $\|V\|(t)$ to be the L^2 norm in the x-variables, depending on t, and we put

$$\|V\|_M^2(t) = \langle MV, V\rangle = \int \langle MV(t, x), \overline{V(t, x)}\rangle dx$$

then

$$c \le \frac{\|V\|_M^2(t)}{\|V\|^2(t)} \le C. \tag{3.24}$$

If $D_t V + KV = F$, we obtain

$$\partial_t \|V\|_M^2(t) = \langle (\partial_t M - i(MK - K^* M))V, V\rangle(t) + \langle MF, V\rangle(t) + \langle MV, F\rangle(t)$$
$$\le C\left(\|V\|_M^2(t) + \|F\|_M^2(t)\right).$$

By Grönwall's inequality we get, for bounded t,

$$\|V\|_M^2(t) \le C\left(\|V\|_M^2(0) + \int_0^t \|F\|_M^2(s)ds\right),$$

so (3.24) and integration gives the result. ∎

Proposition 3.10. *Let $P = D_t \operatorname{Id}_N + K$ where $K \in \operatorname{Op} S(\vartheta, g)$ has symbol which is diagonalizable in $S(1, g)$ with eigenvalues $0, \beta_2, \ldots, \beta_{r_0} \mod S(1, g)$, and $\beta_j \in \mathcal{C}^\infty$ is homogeneous, satisfying (3.8) for $j = 2, \ldots, r_0$. Assume that $V \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N)$, $\xi \ne 0$ in $\mathrm{WF}(V)$ and V satisfies (3.22). If $V_0 \in L^2$ at (x_0, ξ_0), $F \in H_*^{0,0}$ at (t, x_0, ξ_0) for $0 \le t \le t_0$, and $\xi_0' = 0$, then $V \in H_*^{0,0}$ at (t_0, x_0, ξ_0).*

The condition on k means that there exists a basis of eigenvectors $\{v_j\} \in S(1, g)$, with eigenvalues $0, \beta_2, \ldots, \beta_{r_0} \mod S(1, g)$.

The above proposition is similar to [6, Proposition 7.2]. To prove [6, Proposition 7.2], Dencker used the parametrix constructed in [5] for $P = D_t \operatorname{Id}_N + K$, where $K \in \operatorname{Op} S(\vartheta, g)$ has principal symbol k satisfying the conditions in [6, Proposition 7.2], with ϑ and g are given as in [6, (3.8) and (3.9)], respectively, and he used the microlocal uniqueness; see [5].

In our case, case of generalized transverse type, we are using different weight and metric, but we can still construct a parametrix for the $N_0 \times N_0$ matrix, where $K \in \mathrm{Op}\, S(\vartheta, g)$ has principal symbol k satisfying the conditions in Proposition 3.10, and we can prove microlocal uniqueness as in [5]. The steps are similar to that in [5], except some details are changed. Thus, we will not write about the construction of the parametrix and the microlocal uniqueness and we invite the readers to check [5].

Proof of Proposition 3.10. The argument is all the same as in the proof of [6, Proposition 7.2], except that to get $U \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}, \mathbb{C}^N)) \cap L^2$, we need to prove [4, (5.6)] for our case, which we proved in Lemma 3.9. For the parametrix and the microlocal uniqueness part we have mentioned above that it can be proven in our case in a similar way. ∎

Note that here S_j are transversal at Σ_2, so their Hamilton fields are non-parallel on Σ_2. That is why we considered in the main theorem, Theorem 1.2, that M_A is a hypersurface near ν_0, so one can consider $M_A = S_j$ for some j. Now, we will prove Theorem 1.2 for the case of systems of generalized transverse type.

Proof. By multiplication and conjugation with elliptic, scalar pseudodifferential operators, we may assume that $m = 1$ and $r = 0$, and using the normal form we can assume that $N = r_0$, and P is of the form in Proposition 3.4. By using [6, Theorem A.4] for all the terms in the expansion of A, we obtain that $A \in \mathcal{C}^\infty(\mathbb{R}, \Psi^0_{\mathrm{phg}})$. As the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is 1 at $\nu_0 \in \Sigma_2$, and the dimension of the fiber of $\mathcal{N}_P$ is r_0 at Σ_2, we get $\mathrm{rank}\,\sigma(A) = r_0 - 1$ at Σ_2. Hence, we can conjugate by suitable elliptic systems in $\mathcal{C}^\infty(\mathbb{R}, \Psi^0_{\mathrm{phg}})$ to get that $Au \cong (u_1, \ldots, u_{r_0-1}, 0) \in H^\varepsilon$ in a conical neighborhood U of ν_0, for some $\varepsilon > 0$. Then, we find $\pi_1(\mathrm{Pol}^0(u)) = \mathrm{WF}^0(u_{r_0})$ in U. By shrinking U and decreasing ε, we may assume $Pu \in H^\varepsilon$ in U. Remember that we have $P \in \Psi^{0,1}$, hence we get $Qu = {}^t P^{\mathrm{co}} Pu \in H^{\varepsilon, -r_0+1}$ there, where ${}^t P^{\mathrm{co}}$ is the adjugate matrix of P. Let $Q = (q_{ij})_{i,j=1}^{r_0}$, and $P = (P_{ij})_{i,j=1}^{r_0}$. Since $q_{r_0 i}$ are in $\mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}\, S(\vartheta^{r_0-1}, g))$ for $i = 1, \ldots, r_0 - 1$, we find that $q_{r_0 r_0} u_{r_0} \in H^{\varepsilon, -r_0+1}$ in U. Similarly, we find that $P_{r_0 r_0} u_{r_0} \in H^{\varepsilon, -1}$, which in turns gives $D_t^{k-1} P_{r_0 r_0} u_{r_0} \in H^{\varepsilon, -k}$ for $k = 1, \ldots, r_0 - 1$. We want to prove that $u_{r_0} \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ for $t < t_0$, implies $u_{r_0} \in H^0$ at $(t_0, x_0; 0, \xi_0) = \nu_0$.

Thus assume that $u_{r_0} \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ when $t < t_0$. We may assume that $\delta \leq 1$, then Lemma 3.7 gives that $u_{r_0}, P_{r_0 r_0} u_{r_0}, D_t^{k-1} P_{r_0 r_0} u_{r_0}$ for $k = 2, \ldots, r_0 - 1$ and $q_{r_0 r_0} u_{r_0}$ satisfies (3.17). Then $\xi \neq 0$ in $\mathrm{WF}(u_{r_0})$, and assuming that $(r_0 - 1)\delta \leq \varepsilon$ in (3.17), we find that $P_{r_0 r_0} u_{r_0} \in H^{0,-1}_*$, $D_t^{k-1} P_{r_0 r_0} u_{r_0} \in H^{0,-k}_*$ for $k = 1, \ldots, r_0 - 1$ and $q_{r_0 r_0} u_{r_0} \in H^{0,-r_0+1}_*$ in $\pi_0(U \cap \Sigma_2)$, and $u_{r_0} \in H^0_*$ at $(t, x_0, \xi_0) \in \pi_0(U \cap \Sigma_2)$ for $t < t_0$. Since $P_{r_0 r_0} \cong D_t \mod \mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}\, S(\vartheta, g))$, we get $P_{r_0 r_0} u_{r_0} \cong D_t u_{r_0} \in H^{0,-1}_*$ which implies, using $D_t^{k-1} P_{r_0 r_0} u_{r_0} \in H^{0,-k}_*$, that $D_t^k u_{r_0} \in H^{0,-k}_*$ for $k = 1, \ldots, r_0 - 1$. This gives

$$(D_t^k u_{r_0})|_{t=r} \in H^{0,-k}_* \text{ at } (x_0, \xi_0), \quad \text{for } k = 0, \ldots, r_0 - 1,$$

for almost all $r < t_0$, close to t_0. Proposition 3.8 (with $N = 1$ and $Q = q_{r_0 r_0}$) gives $u_{r_0} \in H^0_*$ at (t_0, x_0, ξ_0), and Lemma 3.7 gives $u_{r_0} \in H^0$ at $(t_0, x_0; 0, \xi_0)$. ∎

4. Propagation of polarization sets for systems of MHD type

In this section, we define systems of MHD type which are also systems having their characteristic sets are union of r_0 hypersurfaces intersecting transversally at an involutive manifold of codimension 2 and $r_0 \geq 2$. However, they satisfy some assumptions different than that in case of systems of generalized transverse type. We named them systems of MHD type because we first noticed these types of systems when we considered studying the propagation of polarization sets of the linearized ideal MHD equations. Let $P \in \Psi^m_{\text{phg}}(X)$ be an $N \times N$ system of classical pseudodifferential operators of order m on a smooth manifold X. Let $p = \sigma(P)$ be the principal symbol of P, $\det p$ the determinant of p and $\Sigma = (\det p)^{-1}(0)$ the characteristic set of P. Assume microlocally near $\nu_0 = (x_0, \xi_0) \in \Sigma$, that (3.1)–(3.3); which we will restate below, are satisfied. That is, we have microlocally near $\nu_0 = (x_0, \xi_0) \in \Sigma$

$$\Sigma = \bigcup_{j=1}^{r_0} S_j, \quad r_0 \geq 2, \text{ where } S_j \text{ are non-radial hypersurfaces}$$

$$\text{intersecting transversally at } \Sigma_2 = \bigcap_{j=1}^{r_0} S_j, \tag{4.1}$$

$$\Sigma_2 \text{ is an involutive manifold of codimension 2,} \tag{4.2}$$

and every two hypersurfaces does not intersect outside Σ_2, that is

$$S_i \cap S_j = \Sigma_2, \quad \text{for } i \neq j. \tag{4.3}$$

Also, assume that

$$d^j(\det p) = 0 \text{ for } j \leq r_0 \text{ and } d^{r_0+1}(\det p) \neq 0 \text{ at } \Sigma_2, \tag{4.4}$$

$$\text{the dimension of the fiber of } \mathcal{N}_P \text{ is equal to } r_0 + 1 \text{ at } \Sigma_2, \tag{4.5}$$

$$d(\det p) = 0 \text{ and } d^2(\det p) \neq 0 \text{ at } S_{i_0} \setminus \Sigma_2, \text{ for only one } i_0 \in \{1, \ldots, r_0\}, \tag{4.6}$$

$$d(\det p) \neq 0 \text{ at } S_j \setminus \Sigma_2 \text{ for each } j \in \{1, \ldots, r_0\}, \text{ such that } j \neq i_0. \tag{4.7}$$

Moreover, assume that ${}^t P^{\text{co}}$ the adjugate matrix of P can be written as

$$ {}^t P^{\text{co}} = R L_1 + L_2, \tag{4.8}$$

with R being a scalar pseudodifferential operator of order m, with $\sigma(R)$ vanishing on S_{i_0} to the first order. L_1, and L_2 are $N \times N$ system of pseudodifferential operators of order $m(N-2)$, and $m(N-3)$ respectively. Assume also that

$$\sigma(L_1) p = f \, \text{Id}_N, \tag{4.9}$$

with $\Sigma = \{f = 0\}$, and $df \neq 0$ at $S_i \setminus \Sigma_2$ for $i = \{1, \ldots, r_0\}$. We are using same notation as the previous section.

In fact, the condition that every two hypersurfaces does not intersect outside Σ_2 gives us that Σ_2 is of codimension 2 in case of systems of generalized transverse type and in case of systems of MHD type. However, this is not true in general in the case where the hypersurfaces intersect tangentially as the case of systems of uniaxial type in [6].

Definition 4.1. The system P is of MHD type at $v_0 \in \Sigma_2$, if (4.1)–(4.9), and (3.6) hold microlocally near v_0.

Note that for systems of generalized transverse type, in proving the main theorem we dealt with $Q = {}^t P^{co} P = \det p \, \mathrm{Id}_{r_0}$ where P was $r_0 \times r_0$ matrix having eigenvalues 0, $\beta_2, \ldots, \beta_{r_0}$. Thus, we choose q in (3.20) to be $\det p$ which is equal to $\tau \prod_{j=2}^{r_0} (\tau + \beta_j)$ in that case. However, when we have a system satisfying (4.1)–(4.7), one can see in the following proposition that P can be written as $(r_0 + 1) \times (r_0 + 1)$ matrix having eigenvalues $0, \beta_2, \ldots, \beta_{r_0}$ where we have outside the intersection, Σ_2, 0 is double eigenvalue, and the rest are simple. Thus, if we want to follow the same procedure done for systems of generalized transverse type to prove the main theorem and take $Q = {}^t P^{co} P = \det p \, \mathrm{Id}_{r_0+1}$, we have to choose $q = \det p = \tau^2 \prod_{j=2}^{r_0} (\tau + \beta_j)$ in (3.20), and by that we get in Proposition 3.10 a matrix which is not diagonalizable. Hence, we cannot get the result about the regularity of the solution of the Cauchy problem in the same way we did for systems of generalized transverse type. That is why we considered additionally the conditions (4.8) and (4.9) for systems of MHD type, so we can in the proof of the main theorem deal with $Q = LP = f \, \mathrm{Id}_{r_0+1}$, where $L = L_1 + L_2$, instead of $Q = {}^t P^{co} P$, and here f will be equal to $\tau \prod_{j=2}^{r_0} (\tau + \beta_j)$.

We want to write systems of MHD in a normal form.

Proposition 4.2. Let $P \in \Psi^1_{phg}$ be an $N \times N$ system of MHD type at $v_0 \in \Sigma_2$. Then by choosing suitable symplectic coordinates, we may assume that $X = \mathbb{R} \times \mathbb{R}^{n-1}$, $v_0 = (0; (0, \ldots, 1))$, and

$$S_j = \{(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau + \beta_j(t, x, \xi) = 0\}, \quad j = 1, \ldots, r_0, \quad (4.10)$$

microlocally near v_0. Here β_j are real and homogeneous of degree 1 in ξ; with $\beta_1 \equiv 0$, satisfies in a conical neighborhood of v_0

$$c|\xi'| \leq |\beta_j| \leq C|\xi'|, \quad j = 2, \ldots, r_0, \ 0 < c < C, \quad (4.11)$$

where $(\tau, \xi', \xi'') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$, which gives $\Sigma_2 = \{\tau = 0, \xi' = 0\}$. By conjugating P with elliptic, scalar Fourier integral operators, and multiplying with elliptic $N \times N$ systems of order 0, we may assume that

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \quad \mathrm{mod} \ \mathcal{C}^\infty,$$

microlocally near v_0, where $E \in \Psi^1_{phg}$ is an elliptic $(N - r_0 - 1) \times (N - r_0 - 1)$ system,

and

$$F \cong \mathrm{Id}_{r_0+1} \, D_t + K(t, x, D_x) \quad \mathrm{mod} \ \mathcal{C}^\infty. \tag{4.12}$$

Here $K(t, x, D_x) \in \mathcal{C}^\infty(\mathbb{R}, \Psi^1_{\mathrm{phg}})$ is an $(r_0 + 1) \times (r_0 + 1)$ system, and the eigenvalues of $k(t, x, \xi)$; the principal symbol of $K(t, x, D_x)$, are 0 (double), $\beta_2, \ldots, \beta_{r_0}$.

Proof. In a similar way as in the proof of Proposition 3.3, we get (4.10), (4.11), and

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \quad \mathrm{mod} \ \mathcal{C}^\infty,$$

microlocally near ν_0, where $E \in \Psi^1_{\mathrm{phg}}$ is an elliptic $(N - r_0 - 1) \times (N - r_0 - 1)$ system. If f is the principal symbol for F, then conditions (4.6), (4.7), and (4.10) imply

$$\det f = c \tau^2 \prod_{i=2}^{r_0} (\tau + \beta_i), \quad 0 \neq c \in S^{-1},$$

thus $\partial_\tau^{r_0+1} (\det f) = \det(\partial_\tau f) \neq 0$ at Σ_2. Same as before we get (4.12). ∎

We use the same weight and metric introduced in the previous section; see (3.11) and (3.12). Thus, we have $\beta_j \in S(\vartheta, g)$.

Proposition 4.3. *Let*

$$P = \mathrm{Id}_{r_0+1} \, D_t + K(t, x, D_x) \tag{4.13}$$

be an $(r_0 + 1) \times (r_0 + 1)$ system with $K \in \mathcal{C}^\infty(\mathbb{R}, \Psi^1_{\mathrm{phg}})$, such that the eigenvalues of $k = \sigma(K)$ are 0 (double), $\beta_2, \ldots, \beta_{r_0}$, with β_j as in (4.11), that is $\beta_j \neq 0$. Then P satisfies (3.6) if and only if $k \in \mathcal{C}^\infty(\mathbb{R}, S(\vartheta, g))$.

Proof. Same proof as in Proposition 3.4, just we replace r_0 by $r_0 + 1$ when needed. ∎

Proposition 4.4. *Assume that P is an $(r_0 + 1) \times (r_0 + 1)$ system of pseudodifferential operators of order 1 on $\mathbb{R}^n$, on the form (4.13), with K is in $\mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}(S(\vartheta, g)))$ near $\nu_0 \in \Sigma_2$, and $k(t, x, \xi)$ has the eigenvalues 0 (double), $\beta_1, \ldots, \beta_{r_0}$. Let $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^{r_0+1})$ and assume $Pu \in H^{r,s}$ at ν_0. Then, for every $\delta > 0$, and $N \in \mathbb{N}$, we can find c_δ and $C_{\delta, N} > 0$ and $v_\delta \in H^{r, s+1}$ at ν_0, such that $u_\delta = u - v_\delta$ satisfies*

$$\left| \hat{u}_\delta(\tau, \xi) \right| \leq C_{\delta, N} \langle (\tau, \xi) \rangle^{-N}, \quad \text{when } |\tau| > c_\delta \left(\langle \xi \rangle^\delta + \langle \xi' \rangle \right). \tag{4.14}$$

Proof. Same proof as for Proposition 3.5, with replacing r_0 by $r_0 + 1$ when needed. ∎

Now, we are ready to prove Theorem 1.2 for systems of MHD type.

Proof of Theorem 1.2. Note that as $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ is a hypersurface near ν_0, we have $M_A = S_j$ for some j. By multiplication and conjugation with elliptic, scalar pseudodifferential operators we may assume that $m = 1$ and $r = 0$, and using the normal form we can assume that $N = r_0 + 1$, and P is of the form in Proposition 4.3. By using

[6, Theorem A.4] for all the terms in the expansion of A, we obtain that $A \in \mathcal{C}^{\infty}(\mathbb{R}, \Psi^0_{\text{phg}})$. As the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is 1 at $\nu_0 \in \Sigma_2$, and the dimension of the fiber of $\mathcal{N}_P$ is $r_0 + 1$ at Σ_2, we get rank $\sigma(A) = r_0$ at Σ_2. Hence, we can conjugate by suitable elliptic systems in $\mathcal{C}^{\infty}(\mathbb{R}, \Psi^0_{\text{phg}})$ to get that $Au \cong {}^t(u_1, \ldots, u_{r_0}, 0) \in H^{\varepsilon}$ in a conical neighborhood U of ν_0, for some $\varepsilon > 0$. Then, we find $\pi_1(\text{Pol}^0(u)) = \text{WF}^0(u_{r_0+1})$ in U. By shrinking U and decreasing ε, we may assume $Pu \in H^{\varepsilon}$ in U. Remember that we have $P \in \Psi^{0,1}$. Let $L = L_1 + L_2$, where L_1 and L_2 are given in (4.8), and let $Q = LP$. We have $Qu = LPu \in H^{\varepsilon, -r_0+1}$ there. Let $Q = (q_{ij})_{i,j=1}^{r_0+1}$, and $P = (P_{ij})_{i,j=1}^{r_0+1}$. By (4.8) and (4.9) we have q_{ij} are in $\mathcal{C}^{\infty}(\mathbb{R}, \text{Op}\, S(\vartheta^{r_0-1}, g))$ for $i \neq j$, we find that $q_{r_0+1 r_0+1} u_{r_0+1} \in H^{\varepsilon, -r_0+1}$ in U. Similarly, we find that $P_{r_0+1 r_0+1} u_{r_0+1} \in H^{\varepsilon, -1}$, which in turns gives $D_t^{k-1} P_{r_0+1 r_0+1} u_{r_0+1} \in H^{\varepsilon, -k}$ for $k = 1, \ldots, r_0 - 1$. We want to prove that $u_{r_0+1} \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ for $t < t_0$, implies $u_{r_0+1} \in H^0$ at $(t_0, x_0; 0, \xi_0) = \nu_0$.

Thus assume that $u_{r_0+1} \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ when $t < t_0$. We may assume that $\delta \leq 1$, then Lemma 3.7 gives that $u_{r_0+1}, P_{r_0+1 r_0+1} u_{r_0+1}, D_t^{k-1} P_{r_0+1 r_0+1} u_{r_0+1}$ for $k = 2, \ldots, r_0 - 1$ and $q_{r_0+1 r_0+1} u_{r_0+1}$ satisfies (4.14). Then $\xi \neq 0$ in $\text{WF}(u_{r_0+1})$, and assuming that $(r_0 - 1)\delta \leq \varepsilon$ in (4.14), we find that

$$P_{r_0+1 r_0+1} u_{r_0+1} \in H^{0,-1}_*, \qquad D_t^{k-1} P_{r_0+1 r_0+1} u_{r_0+1} \in H^{0,-k}_* \text{ for } k = 1, \ldots, r_0 - 1$$

and $q_{r_0+1 r_0+1} u_{r_0+1} \in H^{0,-r_0+1}_*$ in $\pi_0(U \cap \Sigma_2)$, and $u_{r_0+1} \in H^0_*$ at $(t, x_0, \xi_0) \in \pi_0(U \cap \Sigma_2)$ for $t < t_0$. Since $P_{r_0+1 r_0+1} \cong D_t \mod \mathcal{C}^{\infty}(\mathbb{R}, \text{Op}\, S(\vartheta, g))$, we get $P_{r_0+1 r_0+1} u_{r_0+1} \cong D_t u_{r_0+1} \in H^{0,-1}_*$ which implies using $D_t^{k-1} P_{r_0+1 r_0+1} u_{r_0+1} \in H^{0,-k}_*$ that $D_t^k u_{r_0+1} \in H^{0,-k}_*$ for $k = 1, \ldots, r_0 - 1$. This gives

$$(D_t^k u_{r_0+1})|_{t=r} \in H^{0,-k}_* \text{ at } (x_0, \xi_0), \quad \text{for } k = 0, \ldots, r_0 - 1$$

for almost all $r < t_0$, close to t_0. Proposition 3.8 (with $N = 1$ and $Q = q_{r_0+1 r_0+1}$) gives $u_{r_0+1} \in H^0_*$ at (t_0, x_0, ξ_0), and Lemma 3.7 gives $u_{r_0+1} \in H^0$ at $(t_0, x_0; 0, \xi_0)$. ∎

5. Application

Magnetohydrodynamics, or MHD couples Maxwell's equations with hydrodynamics to describe the behavior of electrically conducting fluids under the influence of electromagnetic fields. In this section, we want to consider the simplest form of MHD, which is the Ideal MHD to study the propagation of polarization set of the solution of the linearized ideal MHD equations. Ideal MHD, assumes that the fluid has so little resistivity that it can be treated as a perfect conductor. See [16] to know more about MHD equations.

We will show, under some assumptions, that the linearized ideal MHD equations is of real principal type. As we mentioned before, systems of real principal type were defined by Dencker; see [2], who studied the propagation of their solutions, and showed that the propagation of polarization sets is governed by a certain connection on sections of the kernel subbundle, ker p_2 where p_2 is the principal symbol of the system. In [9], Hansen

and Röhrig merged the theory of real principal type systems with the calculus of Fourier integral operators and constructed a Fourier integral solution for system of real principal type, and derived a transport equation for the principal symbol of this solution (note that disregarding half densities this transport equation is the connection introduced by Dencker).

We will divide this section into two subsections: first, we introduce the ideal MHD equations and its linearization. In Section 5.1, we write the linearized ideal MHD equations in the form of a wave equation $P\beta = 0$ where β is the displacement vector and P is a second order 3×3 system; see [16, Lecture 20], and we show that under some assumptions, the characteristic variety of P is disjoint union of the Shear Alfvén wave, the slow magnetosonic wave and the fast magnetosonic wave; see [16, Lecture 24]. Moreover, we show that, under the considered assumptions, P is of real principal type and we calculate the transport equation on Σ; the characteristic set of P, as an application to the result in [9]. In Section 5.2, we return to the linearized ideal MHD equations, and we study the propagation of polarization sets. It turns out that we can consider different cases, some in which we have our system is of real principal type, some in which our system is of uniaxial type, and we have a case where our system is of MHD type.

The set of equations describing the ideal MHD are

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \operatorname{div} u = 0, \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p + H \times \operatorname{curl} H = 0, \\ \partial_t H - \nabla \times (u \times H) = 0, \\ \partial_t p + u \cdot \nabla p + \gamma p \operatorname{div} u = 0, \end{cases} \tag{5.1}$$

where $\rho,\ p \in \mathbb{R}$ denotes the density and the pressure respectively. $u \in \mathbb{R}^3$ is the fluid velocity, $H \in \mathbb{R}^3$ is the magnetic field, and γ is the adiabatic index, see [16, Lecture 20]. Assuming a stationary equilibrium the linearized equations of (5.1) about (ρ, H, p) is:

$$\partial_t \dot{\rho} = -\rho \operatorname{div} \dot{u} - \dot{u} \cdot \nabla \rho, \tag{5.2a}$$

$$\rho \partial_t \dot{u} = -\nabla \dot{p} - H \times \operatorname{curl} \dot{H} - \dot{H} \times \operatorname{curl} H, \tag{5.2b}$$

$$\partial_t \dot{H} = \nabla \times (\dot{u} \times H), \tag{5.2c}$$

$$\partial_t \dot{p} = -\gamma p \operatorname{div} \dot{u} - \dot{u} \cdot \nabla p, \tag{5.2d}$$

where (ρ, H, p) are the values in the equilibrium state (that is the solutions of the Ideal MHD equations when $\partial / \partial t = 0$, and as we assumed stationary equilibrium we have $u = 0$). Note that we used that

$$\nabla p + H \times \operatorname{curl} H = 0, \tag{5.3}$$

which we get from the stationary equilibrium assumption, and $\dot{u}$ can be written as

$$\dot{u} = \frac{\partial \beta}{\partial t}, \tag{5.4}$$

where β is the displacement.

5.1. The ideal MHD wave equation and the transport equation

We can write the linearized ideal MHD equations (5.2) in the form of a wave equation; see [16, Lecture 20]. Substituting (5.4) in (5.2a), (5.2c), and (5.2d) respectively, and integrating with respect to t we get

$$\dot{\rho} = -\rho \operatorname{div} \beta - \beta \cdot \nabla \rho, \tag{5.5a}$$

$$\dot{H} = \nabla \times (\beta \times H), \tag{5.5b}$$

$$\dot{p} = -\gamma p \operatorname{div} \beta - \beta \cdot \nabla p. \tag{5.5c}$$

Replace now (5.5b), (5.5c), and (5.4) in (5.2b) to get

$$\rho \frac{\partial^2 \beta}{\partial t^2} = \gamma \nabla(p \operatorname{div} \beta) + \nabla(\beta \cdot \nabla p)$$
$$+ \left(\nabla \times \left(\nabla \times (\beta \times H)\right)\right) \times H + (\nabla \times H) \times \left(\nabla \times (\beta \times H)\right). \tag{5.6}$$

Equation (5.6) is the ideal MHD wave equation. Consider from now on P where

$$P\beta = -\rho \frac{\partial^2 \beta}{\partial t^2} + \gamma \nabla(p \operatorname{div} \beta) + \nabla(\beta \cdot \nabla p) + \left(\nabla \times \left(\nabla \times (\beta \times H)\right)\right) \times H$$
$$+ (\nabla \times H) \times \left(\nabla \times (\beta \times H)\right) = 0.$$

Now, we want to give the characteristic variety of P under some assumptions; for this part we refer to [16, Lectures 23 and 24].

Lemma 5.1. *Assume $c^2 = \gamma p/\rho > 0$, $0 < |H|^2 \neq \rho c^2$, $\xi \cdot H \neq 0$, and $\xi \times H \neq 0$. The characteristic variety of P is disjoint union of the Shear Alfvén wave, the slow magnetosonic wave, and the fast magnetosonic wave characteristic varieties $\{q_1 = 0\}$, $\{q_2 = 0\}$, and $\{q_3 = 0\}$ respectively, where*

$$\begin{cases} q_1 = \rho \tau^2 - (H \cdot \xi)^2, \\ q_2 = \rho\left(\tau^2 - c_s^2(x, \xi)\right), \\ q_3 = \rho\left(\tau^2 - c_f^2(x, \xi)\right), \end{cases}$$

with

$$c_f^2(x, \xi) := \frac{1}{2}\left((c^2 + h^2)\xi^2 + \sqrt{(c^2 - h^2)^2 \xi^4 + 4b^2 c^2 \xi^2}\right),$$

$$c_s^2(x, \xi) := \frac{1}{2}\left((c^2 + h^2)\xi^2 - \sqrt{(c^2 - h^2)^2 \xi^4 + 4b^2 c^2 \xi^2}\right),$$

where $c^2 = \gamma p/\rho > 0$, and $h^2 = |H|^2/\rho$ are considered to be the sound speed and the Alfvén speed respectively, and $b^2 = |\xi \times H|^2/\rho$.

Proof. We have

$$p_2 \beta = \rho \tau^2 \beta - \gamma p \xi(\xi \cdot \beta) - \left(\xi \times \left(\xi \times (\beta \times H)\right)\right) \times H, \tag{5.7}$$

with p_2 is the principal symbol of P. Considering $v = H/\sqrt{\rho}$, equation (5.7) can be written as

$$\rho\big(\tau^2 - (\xi \cdot v)^2\big)\beta = \rho\big((c^2 + h^2)(\xi \cdot \beta) - (v \cdot \beta)(\xi \cdot v)\big)\xi - \rho v(\xi \cdot \beta)(\xi \cdot v). \quad (5.8)$$

Without loss of generality, let $v = |H|/\sqrt{\rho}\,\hat{e}_z$, $\xi = \xi_\perp \hat{e}_x + \xi_\parallel \hat{e}_z$ with $\xi^2 = \xi_\perp^2 + \xi_\parallel^2$, and $\beta = \beta_x \hat{e}_x + \beta_y \hat{e}_y + \beta_z \hat{e}_z$ with $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ are unit vectors that points in the direction of the x-axis, y-axis, and z-axis respectively. Substituting this in equation (5.8), we find

$$x\text{-component: } \rho\tau^2\beta_x = \rho c^2\xi_\parallel\xi_\perp\beta_z + \rho(h^2\xi^2 + c^2\xi_\perp^2)\beta_x, \quad (5.9a)$$

$$y\text{-component: } \rho\tau^2\beta_y = \rho h^2\xi_\parallel^2\beta_y, \quad (5.9b)$$

$$z\text{-component: } \rho\tau^2\beta_z = \rho(c^2\xi_\parallel\xi_\perp)\beta_x + \rho c^2\xi_\parallel^2\beta_z. \quad (5.9c)$$

Notice that the y-component decouples from the x- and z-components. This immediately gives the shear Alfvén wave:

$$\rho\tau^2 = \rho h^2\xi_\parallel^2.$$

The characteristic equation for the coupled x- and z-component is

$$\rho^2\tau^4 - \rho^2\tau^2\xi^2(c^2 + h^2) + \rho^2 c^2 h^2\xi_\parallel^2\xi^2 = 0. \quad (5.10)$$

Hence, we get

$$\rho\tau^2 = \frac{\rho}{2}\big((c^2 + h^2)\xi^2 \pm \sqrt{(c^2 - h^2)\xi^4 + 4c^2 h^2\xi_\perp^2\xi^2}\big).$$

Still we want to prove that $\{q_1 = 0\}$, $\{q_2 = 0\}$, and $\{q_3 = 0\}$ are disjoint. Dividing (5.10) by ρ^2, it can be written as

$$(\tau^2 - h^2\xi_\parallel^2)(\tau^2 - c^2\xi^2) - \tau^2 h^2\xi_\perp^2.$$

Consider $R(X) = (X^2 - h^2\xi_\parallel^2)(X^2 - c^2\xi^2) - X^2 h^2\xi_\perp^2$, $\{R \le 0\} = [c_s^2, c_f^2]$ and $R(X) \le 0$ for $X \in [\min(h^2\xi_\parallel^2, c^2\xi^2), \max(h^2\xi_\parallel^2, c^2\xi^2)]$. Thus,

$$c_f^2 \ge \max(h^2\xi_\parallel^2, c^2\xi^2) \ge h^2\xi_\parallel^2,$$

$$c_s^2 \le \min(h^2\xi_\parallel^2, c^2\xi^2) \le h^2\xi_\parallel^2.$$

As $h^2\xi_\parallel^2 \ne 0$, we have $R(h^2\xi_\parallel^2) = -h^2\xi_\parallel^2 c^2\xi^2 \le 0$. Hence, $c_s^2 < h^2\xi_\parallel^2 < c_f^2$. ∎

Suppose that the conditions of Lemma 5.1 are satisfied. Now, we are interested in calculating the transport equation as in [9]. The full symbol of P is $p_2 + p_1 + p_0$, with p_2 is the principal symbol of P homogeneous of degree 2, and p_1 and p_0 are homogeneous terms of degree 1 and 0 respectively. One can check that the principal symbol of P is

$$p_2 = \big(\rho\tau^2 - (H \cdot \xi)^2\big)\,\mathrm{Id}_3 - (\gamma p + |H|^2)\xi \otimes \xi + (H \cdot \xi)(\xi \otimes H + H \otimes \xi), \quad (5.11)$$

and

$$p_1 = i\gamma(\nabla p) \otimes \xi + i\xi \otimes (\nabla p) + i\big(\nabla(H \cdot \xi) \cdot H + (H \cdot \xi)\operatorname{div} H\big)\operatorname{Id}_3$$
$$+ \frac{i}{2}\big(\xi \otimes \nabla|H|^2\big) + i\big(\nabla|H|^2 \otimes \xi\big) - 2i(H \cdot \nabla)(H \otimes \xi)$$
$$- i\big(\nabla(H \cdot \xi)\big) \otimes H - i(H \cdot \xi)(\nabla \otimes H) - i(\nabla \cdot H)(\xi \otimes H).$$

Using (5.3), we get

$$p_1 = i\gamma(\nabla p) \otimes \xi + i\xi \otimes (\nabla p) - i(\nabla p) \otimes \xi$$
$$+ i\big(\nabla(H \cdot \xi) \cdot H + (H \cdot \xi)\operatorname{div} H\big)\operatorname{Id}_3 + \frac{i}{2}\big(\xi \otimes \nabla|H|^2\big)$$
$$+ \frac{i}{2}\big(\nabla|H|^2 \otimes \xi\big) - i(H \cdot \nabla)(H \otimes \xi)$$
$$- i\big(\nabla(H \cdot \xi)\big) \otimes H - i(H \cdot \xi)(\nabla \otimes H) - i(\nabla \cdot H)(\xi \otimes H). \tag{5.12}$$

One can check (5.11), (5.12), and the calculations given below by using "Mathematica" for example.

Let Γ_1, Γ_2, and Γ_3 be disjoint conic neighborhoods of $\{q_1 = 0\}$, $\{q_2 = 0\}$, and $\{q_3 = 0\}$ respectively. Set $q = q_1$ in Γ_1, $q = q_2$ in Γ_2, and $q = q_3$ in Γ_3.

Proposition 5.2. *P is of real principal type with respect to the Hamilton field H_q of $\Sigma = \{q = 0\}$.*

Proof. We have q_1, q_2, and q_3 are scalar real principal type. Let

$$w^2 := |H|^2 \xi \otimes \xi + |\xi|^2 H \otimes H - (H \cdot \xi)(H \otimes \xi + \xi \otimes H).$$

In Γ_1 we take

$$\tilde{p}_2 = \operatorname{Id}_3 + \frac{q_1}{q_2 q_3}\big(\gamma p + |H|^2\big)\xi \otimes \xi - \frac{q_1}{q_2 q_3}\big((H \cdot \xi)(\xi \otimes H + H \otimes \xi)\big) + \frac{(H \cdot \xi)^2 w^2}{q_2 q_3}$$

to get $\tilde{p}_2\, p_2 = q_1 \operatorname{Id}_3$. In Γ_2 we take

$$\tilde{p}_2 = \frac{q_2}{q_1}\operatorname{Id}_3 + \frac{1}{q_3}\big(\gamma p + |H|^2\big)\xi \otimes \xi - \frac{1}{q_3}\big((H \cdot \xi)(\xi \otimes H + H \otimes \xi)\big) + \frac{(H \cdot \xi)^2 w^2}{q_1 q_3}$$

to get $\tilde{p}_2\, p_2 = q_2 \operatorname{Id}_3$. In Γ_3 we take

$$\tilde{p}_2 = \frac{q_3}{q_1}\operatorname{Id}_3 + \frac{1}{q_2}\big(\gamma p + |H|^2\big)\xi \otimes \xi - \frac{1}{q_2}\big((H \cdot \xi)(\xi \otimes H + H \otimes \xi)\big) + \frac{(H \cdot \xi)^2 w^2}{q_1 q_2}$$

to get $\tilde{p}_2\, p_2 = q_3 \operatorname{Id}_3$. ∎

Remarks. Now, we write p_2 and $\tilde{p}_2$ in terms of π_1, π_2, and π_3, the orthogonal projectors with $\pi_1 + \pi_2 + \pi_3 = 1$, and we give the values of some terms on the characteristic set Σ.

- The principal symbol of P calculated before can be written as

$$p_2 = q_1\pi_1 + q_2\pi_2 + q_3\pi_3,$$

where

$$\pi_1 = \mathrm{Id}_3 + \frac{w^2}{(H \cdot \xi)^2 - |H|^2|\xi|^2},$$

$$\pi_2 = \frac{1}{\rho c_s^2(x,\xi) - \rho c_f^2(x,\xi)}\left((\gamma p + |H|^2)\xi \otimes \xi - (H \cdot \xi)(\xi \otimes H + H \otimes \xi)\right.$$
$$\left. + \frac{(H \cdot \xi)^2 w^2}{\rho c_s^2(x,\xi) - (H \cdot \xi)^2}\right),$$

and

$$\pi_3 = \frac{1}{\rho c_f^2(x,\xi) - \rho c_s^2(x,\xi)}\left((\gamma p + |H|^2)\xi \otimes \xi - (H \cdot \xi)(\xi \otimes H + H \otimes \xi)\right.$$
$$\left. + \frac{(H \cdot \xi)^2 w^2}{\rho c_f^2(x,\xi) - (H \cdot \xi)^2}\right),$$

with π_1, π_2 and π_3 are orthogonal projectors and $\pi_1 + \pi_2 + \pi_3 = \mathrm{Id}_3$.

- In Γ_1,

$$\tilde{p}_2 = \pi_1 + \frac{q_1}{q_2}\pi_2 + \frac{q_1}{q_3}\pi_3,$$

and set $\pi = \pi_1$.
In Γ_2,

$$\tilde{p}_2 = \pi_2 + \frac{q_2}{q_1}\pi_1 + \frac{q_2}{q_3}\pi_3,$$

and set $\pi = \pi_2$.
In Γ_3,

$$\tilde{p}_2 = \pi_3 + \frac{q_3}{q_1}\pi_1 + \frac{q_3}{q_2}\pi_2,$$

and set $\pi = \pi_3$.
On Σ, $\tilde{p}_2 = \pi$, $p_2\pi = 0 = \pi p_2$, and $p_2 a = 0$ if and only if $a = \pi a$.

In what follows, let $X = \mathbb{R} \times \mathbb{R}^3$, $\Lambda \subset T^*X \setminus 0$ be a closed Lagrangian submanifold of the characteristic set of P, and let $\Omega_\Lambda^{1/2}$ denote the half-density bundle of Λ. $S^{\mu+1}(\Lambda, (\Omega_\Lambda^{1/2})^3)$ is the space of symbols of the space of Lagrangian distributions $I^\mu(X, \Lambda; (\Omega_X^{1/2})^3)$; see [14, Section 25.1] to read more about Lagrangian distributions.

From [9], we know that there is a first order differential operator $\mathcal{T}_{P,H_q}$ on Λ, uniquely determined by P and H_q which maps a a 3-vector of half densities with $p_2 a = 0$ to 3-vector of half densities where

$$\mathcal{T}_{P,H_q}a = \mathcal{L}_{H_q}a + \frac{1}{2}\{\tilde{p}_2, p_2\}a + i\tilde{p}_2 p^s a.$$

Here

$$p^s = p_1 - \frac{1}{2i} \sum_j \frac{\partial^2 p_2}{\partial x_j \, \partial \xi_j}, \tag{5.13}$$

is the subprincipal symbol of P, $\mathcal{L}_{H_q}$ is the Lie derivative with respect to H_q, and $\{.\}$ is the Poisson bracket.

Lemma 5.3. $\tilde{p}_2 \, p^s \pi = 0$ *on* Σ.

Proof. Differentiating p_2 we get

$$\sum_{j=1}^3 \frac{\partial^2 p_2}{\partial x_j \, \partial \xi_j} = -2\big(\nabla(H \cdot \xi) \cdot H + (H \cdot \xi) \operatorname{div} H\big) \operatorname{Id}_3 - \big(\gamma \nabla p + \nabla(|H|^2)\big) \otimes \xi$$
$$- \xi \otimes \big(\gamma \nabla p + \nabla |H|^2\big) + (\operatorname{div} H + H \cdot \nabla)(\xi \otimes H + H \otimes \xi)$$
$$+ \big(\nabla(H \cdot \xi)\big) \otimes H + (H \cdot \xi)\big(\nabla \otimes H + (\nabla \otimes H)^\mathsf{T}\big) + H \otimes \big(\nabla(H \cdot \xi)\big).$$

Therefore, the subprincipal symbol is given as follows:

$$2i \, p^s = \gamma(\xi \otimes \nabla p - \nabla p \otimes \xi) + \operatorname{div} H(\xi \otimes H - H \otimes \xi)$$
$$+ (H \cdot \xi)\big(-(\nabla \otimes H)^\mathsf{T} + (\nabla \otimes H)\big) + (H \cdot \nabla)(H \otimes \xi - \xi \otimes H)$$
$$+ \big(\big(\nabla(H \cdot \xi)\big) \otimes H - H \otimes \nabla(H \cdot \xi)\big) + 2(\nabla p \otimes \xi - \xi \otimes \nabla p).$$

We have

$$2i \, \tilde{p}_2 \, p^s \pi = 2i \, \pi p^s \pi \quad \text{on } \Sigma.$$

Using that $2i p^s$ is a 3×3 skew-symmetric matrix with zero entries on the diagonal, and π is a symmetric matrix we get that $2i \pi p^s \pi$ is a 3×3 skew-symmetric matrix. Therefore to prove that it vanishes, it suffices to show that its rank is < 2. Since π is projection we have rank $\pi = \operatorname{trace} \pi$. Calculating the trace of π we get that rank $\pi = 1$ and hence we proved the lemma. ∎

Lemma 5.4. *Let* $\Lambda \subset \Sigma$ *be a conic Lagrangian submanifold. Let* $a \in S^{\mu+1}(\Lambda, (\Omega_\Lambda^{1/2})^3)$. *Then,* $\{\tilde{p}_2, p_2\}\pi a = -2(H\pi)a$ *on* Σ.

Proof. We will prove the result for $\pi = \pi_1$ and the same argument applies for π_2 and π_3. We have in a conic neighborhood of $\{q_1 = 0\}$, $p_2 = q_1 \pi_1 + q_2 \pi_2 + q_3 \pi_3$, and $\tilde{p}_2 = \pi_1 + \frac{q_1}{q_2} \pi_2 + \frac{q_1}{q_3} \pi_3$. Using that $\pi^2 = \pi$, and $\{q_1, q_1\} = 0$ we get

$$\{\tilde{p}_2, p_2\}\pi_1 a = \{\pi_1, q_1\}\pi_1 a + q_2\{\pi_1, \pi_2\}\pi_1 a + q_3\{\pi_1, \pi_3\}\pi_1 a + \pi_2\{q_1, \pi_2\}\pi_1 a$$
$$+ \frac{q_3}{q_2}\pi_2\{q_1, \pi_3\}\pi_1 a + \frac{q_2}{q_3}\pi_3\{q_1, \pi_2\}\pi_1 a + \pi_3\{q_1, \pi_3\}\pi_1 a.$$

Using that $H\pi_j = H\pi_j^2 = (H\pi_j)\pi_j + \pi_j(H\pi_j)$ for $j = 2, 3$, we get $\pi_j\{q_1, \pi_j\}\pi_1 a = \{q_1, \pi_j\}\pi_1 a$ for $j = 2, 3$, and using $0 = H(\pi_2\pi_3) = \pi_2(H\pi_3) + (H\pi_2)\pi_3$ and $0 = H(\pi_3\pi_2) = \pi_3(H\pi_2) + (H\pi_3)\pi_2$ we get $\pi_2\{q_1, \pi_3\}\pi_1 a = \pi_3\{q_1, \pi_2\}\pi_1 a = 0$.

Using $\pi_2 = \mathrm{Id}_3 - \pi_1 - \pi_3$, $\{\pi_1, \mathrm{Id}_3\} = \{q_1, \mathrm{Id}_3\} = 0$ and $\{\pi_1, \pi_1\} = 0$ we get

$$\{\tilde{p}_2, p_2\}\pi_1 a = -2(H\pi_1)\pi_1 a + (q_3 - q_2)\{\pi_1, \pi_3\}\pi_1 a.$$

Now, we want to prove that $\{\pi_1, \pi_3\}\pi_1 a = 0$. We have

$$\partial_{\xi_i}\pi_1 = \partial_{\xi_i}\pi_1^2 = \pi_1 \partial_{\xi_i}\pi_1 + \partial_{\xi_i}\pi_1 \pi_1,$$

and similarly $\partial_{x_i}\pi_1 = \partial_{x_i}\pi_1^2 = \pi_1 \partial_{x_i}\pi_1 + \partial_{x_i}\pi_1 \pi_1$. Also, $0 = \partial_{\xi_i}(\pi_1 \pi_3) = \pi_1 \partial_{\xi_i}\pi_3 + \partial_{\xi_i}\pi_1 \pi_3$. Hence, $\pi_1 \partial_{\xi_i}\pi_3 = -\partial_{\xi_i}\pi_1 \pi_3$. Similarly, we have $\pi_1 \partial_{x_i}\pi_3 = -\partial_{x_i}\pi_1 \pi_3$. Combining these together we get

$$\{\pi_1, \pi_3\}\pi_1 a = \pi_1\{\pi_1, \pi_3\}\pi_1 a.$$

We have

$$\pi_1\{\pi_1, \pi_3\}\pi_1 a = \sum_{i=1}^{3} \pi_1(\partial_{\xi_i}\pi_1 \partial_{x_i}\pi_3 - \partial_{x_i}\pi_1 \partial_{\xi_i}\pi_3)\pi_1 a.$$

Moreover,

$$\begin{aligned}
\pi_1 \partial_{\xi_i}\pi_1 \partial_{x_i}\pi_3 \pi_1 a &= \pi_1 \partial_{\xi_i}\pi_1(\pi_3 \partial_{x_i}\pi_3 + \partial_{x_i}\pi_3 \pi_3)\pi_1 a = \pi_1(\partial_{\xi_i}\pi_1 \pi_3)\partial_{x_i}\pi_3 \pi_1 a \\
&= -\pi_1 \partial_{\xi_i}\pi_3(\partial_{x_i}\pi_3 \pi_1)a \pi_1(\partial_{\xi_i}\pi_3 \pi_3)\partial_{x_i}\pi_1 a \\
&= \pi_1(\partial_{\xi_i}\pi_3 - \pi_3 \partial_{\xi_i}\pi_3)\partial_{x_i}\pi_1 a = \pi_1 \partial_{\xi_i}\pi_3 \partial_{x_i}\pi_1 a.
\end{aligned}$$

Using that $\pi_1 a = a$ on Σ, we get $\pi_1 \partial_{\xi_i}\pi_1 \partial_{x_i}\pi_3 \pi_1 a = \pi_1 \partial_{\xi_i}\pi_3 \partial_{x_i}\pi_1 \pi_1 a$. Therefore,

$$\pi_1\{\pi_1, \pi_3\}\pi_1 a = \sum_{i=1}^{3} \pi_1(\partial_{\xi_i}\pi_3 \partial_{x_i}\pi_1 - \partial_{x_i}\pi_1 \partial_{\xi_i}\pi_3)\pi_1 a.$$

We have $\partial_{\xi_i}\pi_3 \partial_{x_i}\pi_1 - \partial_{x_i}\pi_1 \partial_{\xi_i}\pi_3$ is a 3×3 skew-symmetric matrix with the entries in the diagonal equal to zero. So same as before we get $\pi_1\{\pi_1, \pi_3\}\pi_1 a = 0$ as the rank of π_1 equal to 1. Hence, the lemma is proved. ∎

Proposition 5.5. *Let $\Lambda \subset \Sigma$ be a conic Lagrangian submanifold. Let $a \in S^{\mu+1}(\Lambda, (\Omega_\Lambda^{1/2})^3)$ with $p_2 a = 0$. Then*

$$\mathcal{T}_{P,H} a = \mathcal{L}_H a - (H\pi)a \quad \text{on } \Lambda.$$

5.2. Propagation of polarization sets for the linearized ideal MHD equations

Note that (5.2) is hyperbolic symmetric with symmetrizer

$$S = \begin{pmatrix}
\gamma p & 0 & 0 & 0 & 0 & 0 & 0 & -\rho \\
0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-\rho & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\rho^2}{\gamma p}
\end{pmatrix}. \tag{5.14}$$

The principal symbol of (5.2) is

$$\begin{cases} \tau\dot{\rho} + \rho(\xi \cdot \dot{u}) = 0, \\ \tau\dot{u} + \rho^{-1}\dot{p}\xi + \rho^{-1}H \times (\xi \times \dot{H}) = 0, \\ \tau\dot{H} + (\xi \cdot \dot{u})H - (H \cdot \xi)\dot{u} = 0, \\ \tau\dot{p} + \gamma p(\xi \cdot \dot{u}) = 0. \end{cases} \tag{5.15}$$

We use here the notation $\xi = (\xi_1, \xi_2, \xi_3)$ for the spatial frequencies and $\xi = |\xi|\hat{\xi}$, $\dot{u}_\| = \hat{\xi} \cdot \dot{u}$, $\dot{u}_\perp = \dot{u} - \dot{u}_\|\hat{\xi} = -\hat{\xi} \times (\hat{\xi} \times \dot{u})$.

We write (5.15) in the general form $\tau\dot{U} + A(U, \xi)\dot{U} = 0$ with parameters $U = (\rho, H, p)$, and $\dot{U} = (\dot{\rho}, \dot{u}, \dot{H}, \dot{p})$.

We have the following result.

Lemma 5.6. *Assume that $c^2 = \gamma p/\rho > 0$. The eigenvalues of $A(U, \xi)$ are*

$$\begin{cases} \lambda_0 = \lambda_4 = 0, \\ \lambda_{\pm 1} = \pm c_s(\hat{\xi})|\xi|, \\ \lambda_{\pm 2} = \pm(\xi \cdot H)/\sqrt{\rho}, \\ \lambda_{\pm 3} = \pm c_f(\hat{\xi})|\xi|, \end{cases}$$

with $\hat{\xi} = \xi/|\xi|$ and

$$c_f^2(\hat{\xi}) := \frac{1}{2}\big((c^2 + h^2) + \sqrt{(c^2 - h^2)^2 + 4b^2c^2}\big), \tag{5.16}$$

$$c_s^2(\hat{\xi}) := \frac{1}{2}\big((c^2 + h^2) - \sqrt{(c^2 - h^2)^2 + 4b^2c^2}\big), \tag{5.17}$$

where $h^2 = |H|^2/\rho$, $b^2 = |\hat{\xi} \times H|^2/\rho$.

Moreover, if we assume that $0 < |H|^2 \neq \rho c^2$, then we have

(i)   *When $\xi \cdot H \neq 0$ and $\xi \times H \neq 0$: $\lambda_0 = \lambda_4$ is double eigenvalue of $A(U, \xi)$, and the eigenvalues $\lambda_{\pm 1}$, $\lambda_{\pm 2}$ and $\lambda_{\pm 3}$ are simple eigenvalues of $A(U, \xi)$.*

(ii) *When $\xi \cdot H = 0$, $\xi \neq 0$: $\lambda_{\pm 3}$ are simple eigenvalues, while $\lambda_0 = \lambda_{\pm 1} = \lambda_{\pm 2} = \lambda_4$ is a multiple eigenvalue.*

(iii) *When $\xi \times H = 0$, $\xi \neq 0$: when $|H|^2 < \rho c^2$ (resp. $|H|^2 > \rho c^2$), $\lambda_{\pm 3}$ (resp. $\lambda_{\pm 1}$) are simple; $\lambda_{+2} \neq \lambda_{-2}$ are double, equal to $\lambda_{\pm 1}$ (resp. $\lambda_{\pm 3}$) depending on $\xi \cdot H$, λ_0 is double equal to λ_4.*

The proof of this lemma is very similar to the explanation given in [15, Appendix A] except here we have the additional eigenvalue $\lambda_4 = 0$. Also, here we will not state all the eigenspaces as in [15].

Proof. Let $\dot{U} = (\dot{\rho}, \dot{u}, \dot{H}, \dot{p})$. The eigenvalue equation $A(U, \xi)\dot{U} = \lambda \dot{U}$ reads

$$\begin{cases} \lambda \dot{\rho} = \rho \dot{u}_{\|}, \\ \rho \lambda \dot{u}_{\|} = \dot{p} + H_{\perp} \cdot \dot{H}_{\perp}, \\ \lambda \rho \dot{u}_{\perp} = -H_{\|} \dot{H}_{\perp}, \\ \lambda \dot{H}_{\perp} = \dot{u}_{\|} H_{\perp} - H_{\|} \dot{u}_{\perp}, \\ \lambda \dot{H}_{\|} = 0, \\ \lambda \dot{p} = \gamma p \dot{u}_{\|}. \end{cases}$$

On $\{\dot{\rho} = 0, \dot{u} = 0, \dot{H}_{\perp} = 0, \dot{p} = 0\} = \mathbb{E}_0(\xi)$, A is equal to $\lambda = 0$. From now on we work on $\mathbb{E}_0^{\perp} = \{\dot{H}_{\|} = 0\}$ which is invariant by A.

Consider $v = H/\sqrt{\rho}$, $\dot{v} = \dot{H}/\sqrt{\rho}$, $\dot{\alpha} = \dot{p}/\rho$, $\alpha = p/\rho$, and $\dot{\sigma} = \dot{\rho}/\rho$. The characteristic system reads:

$$\begin{cases} \lambda \dot{\sigma} = \dot{u}_{\|}, \\ \lambda \dot{u}_{\|} = \dot{\alpha} + v_{\perp} \cdot \dot{v}_{\perp}, \\ \lambda \dot{u}_{\perp} = -v_{\|} \dot{v}_{\perp}, \\ \lambda \dot{v}_{\perp} = \dot{u}_{\|} v_{\perp} - v_{\|} \dot{u}_{\perp}, \\ \lambda \dot{\alpha} = \gamma \alpha \dot{u}_{\|}. \end{cases}$$

Take a basis of $\xi^{\perp}$ such that $v_{\perp} = (b, 0)$ and let $a = v_{\|}$. In such a basis, the matrix of the system is

$$\lambda - \tilde{A} := \begin{pmatrix} \lambda & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & -b & 0 & -1 \\ 0 & 0 & \lambda & 0 & a & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & a & 0 \\ 0 & -b & a & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & a & 0 & \lambda & 0 \\ 0 & -\gamma\alpha & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}. \tag{5.18}$$

The characteristic roots satisfy

$$\lambda(\lambda^2 - a^2)\big((\lambda^2 - a^2)(\lambda^2 - c^2) - \lambda^2 b^2\big) = 0.$$

Thus either

$$\lambda = 0 \text{ or}$$
$$\lambda^2 = a^2 \text{ or}$$
$$\lambda^2 = c_f^2(\hat{\xi}) = \frac{1}{2}\left(c^2 + h^2 + \sqrt{(c^2 - h^2)^2 + 4b^2 c^2}\right) \text{ or}$$
$$\lambda^2 = c_s^2(\hat{\xi}) = \frac{1}{2}\left(c^2 + h^2 - \sqrt{(c^2 - h^2)^2 + 4b^2 c^2}\right),$$

with $h^2 = a^2 + b^2 = |H|^2/\rho$.

As in Lemma 5.1, if we consider $R(X) = (X - a^2)(X - c^2) - b^2 X$, $\{R \leq 0\} = [c_s^2(\hat{\xi}), c_f^2(\hat{\xi})]$, and $R(X) \leq 0$ for $X \in [\min(a^2, c^2), \max(a^2, c^2)]$. Thus,

$$c_f^2(\hat{\xi}) \geq \max(a^2, c^2) \geq a^2,$$

$$c_s^2(\hat{\xi}) \leq \min(a^2, c^2) \leq a^2.$$

At the case $v_\perp \neq 0$ that is $w = \hat{\xi} \times v \neq 0$: we have the basis such that (5.18) holds is smooth in ξ. In this basis, $w = (0, b)$, $b = |v_\perp| > 0$. Since $R(c^2) = -b^2 c^2 < 0$ there holds $c_s^2(\hat{\xi}) < c^2 < c_f^2(\hat{\xi})$. Suppose that $a \neq 0$. Then $R(a^2) = -a^2 c^2 < 0$ and $c_s^2(\hat{\xi}) < a^2 < c_f^2(\hat{\xi})$. Moreover, $c_s^2(\hat{\xi}) c_f^2(\hat{\xi}) = a^2 c^2$ and $c_s^2(\hat{\xi}) > 0$. However, when $a = 0$, we get $c_s^2(\hat{\xi}) = 0$, but $c_f^2(\hat{\xi}) > c^2 > 0$.

When $a \neq 0$, and $b = 0$, the eigenvalues of $\tilde{A}$ are $\pm c$ (simple), 0 (simple), and $\pm h$ (double). Assume that $c^2 \neq h^2$. Note that when $b = 0$, then $|a| = h$ and

$$\text{when } c^2 > h^2 : c_f(\hat{\xi}) = c, \quad c_s(\hat{\xi}) = h,$$

$$\text{when } c^2 < h^2 : c_f(\hat{\xi}) = h, \quad c_s(\hat{\xi}) = c. \qquad \blacksquare$$

Let Q be a pseudodifferential operator of order 1, such that $Q\dot{U} = 0$ be the system of the linearized ideal MHD equations, and $q = \sigma(Q)$ be its principal symbol. We have $\det q = \tau^2(\tau^2 - c_s^2(\hat{\xi})|\xi|^2)(\tau^2 - c_f^2(\hat{\xi})|\xi|^2)(\tau^2 - (\xi \cdot H)^2/\rho)$.

Proposition 5.7. *When we have Σ is disjoint union of the hypersurfaces*

$$S_1 = \{q_1 = \tau = 0\}, \quad S_2 = \{q_2 = \tau - c_s(\hat{\xi})|\xi| = 0\}, \quad S_3 = \{q_3 = \tau + c_s(\hat{\xi})|\xi| = 0\},$$

$$S_4 = \{q_4 = \tau - (\xi \cdot H)/\sqrt{\rho} = 0\}, \quad S_5 = \{q_5 = \tau + (\xi \cdot H)/\sqrt{\rho} = 0\},$$

$$S_6 = \{q_6 = \tau - c_f(\hat{\xi})|\xi| = 0\}, \quad S_7 = \{q_7 = \tau + c_f(\hat{\xi})|\xi| = 0\};$$

that is when we are outside the intersection of any of these hypersurfaces, then Q is of real principal type. Note that we have this case when $\xi \cdot H \neq 0$, and $\xi \times H \neq 0$.

Proof. Let $\Gamma_1, \ldots, \Gamma_7$ be the disjoint conic neighborhoods of $S_1, \ldots, S_7$, respectively.

Let ${}^t q^{\mathrm{co}}$ be the adjugate matrix (transpose of the cofactor matrix) of q. We can check by using "Mathematica" for example that ${}^t q^{\mathrm{co}}$ can be written as

$$ {}^t q^{\mathrm{co}} = \tau M, \tag{5.19}$$

with M being an 8×8 matrix.

In Γ_1, we choose

$$\tilde{q} = \left(1/\prod_{i=2}^{7} q_j\right) M,$$

so we get

$$\tilde{q} q = \tau \, \mathrm{Id}_8 \, .$$

In Γ_j, for $j = 2, \ldots, 7$ we choose

$$\tilde{q} = \left(1 / \left(q_1^2 \prod_{\substack{i=2 \\ i \neq j}}^{7} q_i\right)\right) {}^t q^{\text{co}},$$

so we get

$$\tilde{q} q = q_j \operatorname{Id}_8 .$$

As q_j for $j = 1, \ldots, 7$ are of real principal type we get the result. ∎

Remember that from Lemma 5.6, we know that $c_f^2(\widehat{\xi}) \neq c_s^2(\widehat{\xi})$ and that $c_f^2(\widehat{\xi}) \neq 0$.

Proposition 5.8. *If $\tau \neq 0$, and $\tau^2 \neq c_f^2(\widehat{\xi})|\xi|^2$, or if $\tau \neq 0$ and $\tau^2 \neq c_s^2(\widehat{\xi})|\xi|^2$, then our system is of uniaxial type at Σ_2.*

Proof.

First case. If $\tau \neq 0$, and $\tau^2 \neq c_f^2(\widehat{\xi})|\xi|^2$, we have Σ is union of two hypersurfaces

$$S_1 = \{\tau - c_s(\widehat{\xi})|\xi| = 0\} \sqcup \{\tau + c_s(\widehat{\xi})|\xi| = 0\},$$
$$S_2 = \{\tau - (\xi \cdot H)/\sqrt{\rho} = 0\} \sqcup \{\tau + (\xi \cdot H)/\sqrt{\rho} = 0\}$$

intersecting at

$$\Sigma_2 = \{\tau = |\xi||H|/\sqrt{\rho}, \ \xi \times H = 0, \ |H|^2 < \rho c^2, \ \xi \neq 0\}$$
$$\sqcup \{\tau = -|\xi||H|/\sqrt{\rho}, \ \xi \times H = 0, \ |H|^2 < \rho c^2, \ \xi \neq 0\}.$$

Second case. If $\tau \neq 0$, and $\tau^2 \neq c_s^2(\widehat{\xi})|\xi|^2$, we have Σ is union of two hypersurfaces

$$S_1 = \{\tau - c_f(\widehat{\xi})|\xi| = 0\} \sqcup \{\tau + c_f(\widehat{\xi})|\xi| = 0\},$$
$$S_2 = \{\tau - (\xi \cdot H)/\sqrt{\rho} = 0\} \sqcup \{\tau + (\xi \cdot H)/\sqrt{\rho} = 0\}$$

intersecting at

$$\Sigma_2 = \{\tau = |\xi||H|/\sqrt{\rho}, \ \xi \times H = 0, \ |H|^2 > \rho c^2, \ \xi \neq 0\}$$
$$\sqcup \{\tau = -|\xi||H|/\sqrt{\rho}, \ \xi \times H = 0, \ |H|^2 > \rho c^2, \ \xi \neq 0\}.$$

In the first and in the second case we have: S_1 and S_2 are tangent of order 1 at Σ_2, the codimension of Σ_2 is three, the (complex) dimension of $\mathcal{N}_Q$ is equal to 2 at Σ_2, $d^2(\det q) \neq 0$ at Σ_2, and $d^i(\det q) = 0$ at Σ_2 for $i < 2$. Hence, the conditions (2.2)–(2.5) are satisfied. It remains only to prove (2.7). In [6], Dencker mentioned that by [6, proposition 3.2], we only have to verify

$$\partial_\rho q : \ker q \mapsto \operatorname{Im} q \quad \text{at } \Sigma_2$$

when $\rho \in T_{\Sigma_2}\Sigma$, since the order of tangency of S_1 and S_2 is 1. $T_{\Sigma_2}\Sigma$ is characterized as those $\rho \in T_{\Sigma_2}X$ such that $\partial_\rho^2(\det q) = 0$. Thus $T_{\Sigma_2}\Sigma$ is spanned by

$$D_1 = \xi_2\partial_{\xi_1} - \xi_1\partial_{\xi_2}, \quad D_2 = \xi_3\partial_{\xi_1} - \xi_1\partial_{\xi_3}, \quad D_3 = \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2},$$

$$D_4 = H_2\partial_{\xi_1} - H_1\partial_{\xi_2}, \quad D_5 = H_1\partial_{\xi_3} - H_3\partial_{\xi_1}, \quad D_6 = H_2\partial_{\xi_3} - H_3\partial_{\xi_2},$$

$$D_7 = \partial_t, \quad D_8 = \xi_1\tau\partial_\tau + |\xi|^2\partial_{\xi_1}, \quad D_9 = \xi_2\tau\partial_\tau + |\xi|^2\partial_{\xi_2}, \quad D_{10} = \xi_3\tau\partial_\tau + |\xi|^2\partial_{\xi_3},$$

$$D_{11} = \tau H_1\partial_\tau + (\xi \cdot H)\partial_{\xi_1}, \quad D_{12} = \tau H_2\partial_\tau + (\xi \cdot H)\partial_{\xi_2}, \quad D_{13} = \tau H_3\partial_\tau + (\xi \cdot H)\partial_{\xi_3}.$$

We can check that if ${}^t(v_1, \ldots, v_8) \in \ker q$ at Σ_2, then we find

$$D_j q \, {}^t(v_1, \ldots, v_8) = {}^t(0, \ldots, 0), \quad j = 1, \ldots, 13, \tag{5.20}$$

so, (2.7) is satisfied. For D_7 we clearly have $D_7 q \, {}^t v = 0$. We will show how one can get (5.20) for the other D_j, in particular we show the proof for D_1, and one can prove it for the other D_j in a similar way. Let $v \in \ker q$ at Σ_2 so we have

$$
\begin{cases}
\pm \frac{|H||\xi|}{\sqrt{\rho}} v_1 + \rho\xi_1 v_2 + \rho\xi_2 v_3 + \rho\xi_3 v_4 = 0, \\[4pt]
\pm \frac{|H||\xi|}{\sqrt{\rho}} v_2 - \rho^{-1}(H_3\xi_3 + H_2\xi_2)v_5 + \rho^{-1}\xi_1 H_2 v_6 + \rho^{-1}\xi_1 H_3 v_7 + \rho^{-1}\xi_1 v_8 = 0, \\[4pt]
\pm \frac{|H||\xi|}{\sqrt{\rho}} v_3 + \rho^{-1}\xi_2 H_1 v_5 - \rho^{-1}(H_3\xi_3 + H_1\xi_1)v_6 + \rho^{-1}\xi_2 H_3 v_7 + \rho^{-1}\xi_2 v_8 = 0, \\[4pt]
\pm \frac{|H||\xi|}{\sqrt{\rho}} v_4 + \rho^{-1}\xi_3 H_1 v_5 + \rho^{-1}\xi_3 H_2 v_6 - \rho^{-1}(H_2\xi_2 + H_1\xi_1)v_7 + \rho^{-1}\xi_3 v_8 = 0, \\[4pt]
-(H_2\xi_2 + H_3\xi_3)v_2 + \xi_2 H_1 v_3 + \xi_3 H_1 v_4 \pm \frac{|H||\xi|}{\sqrt{\rho}} v_5 = 0, \\[4pt]
\xi_1 H_2 v_2 - (H_1\xi_1 + H_3\xi_3)v_3 + \xi_3 H_2 v_4 \pm \frac{|H||\xi|}{\sqrt{\rho}} v_6 = 0, \\[4pt]
\xi_1 H_3 v_2 + \xi_2 H_3 v_3 - (H_1\xi_1 + H_2\xi_2)v_4 \pm \frac{|H||\xi|}{\sqrt{\rho}} v_7 = 0, \\[4pt]
\gamma p\xi_1 v_2 + \gamma p\xi_2 v_3 + \gamma p\xi_3 v_4 \pm \frac{|H||\xi|}{\sqrt{\rho}} v_8 = 0.
\end{cases}
\tag{5.21}
$$

We have

$$D_1 q v = D_1 \begin{pmatrix} \tau v_1 + \rho\xi_1 v_2 + \rho\xi_2 v_3 + \rho\xi_3 v_4 \\ \tau v_2 - \rho^{-1}(H_3\xi_3 + H_2\xi_2)v_5 + \rho^{-1}\xi_1 H_2 v_6 + \rho^{-1}\xi_1 H_3 v_7 + \rho^{-1}\xi_1 v_8 \\ \tau v_3 + \rho^{-1}\xi_2 H_1 v_5 - \rho^{-1}(H_3\xi_3 + H_1\xi_1)v_6 + \rho^{-1}\xi_2 H_3 v_7 + \rho^{-1}\xi_2 v_8 \\ \tau v_4 + \rho^{-1}\xi_3 H_1 v_5 + \rho^{-1}\xi_3 H_2 v_6 - \rho^{-1}(H_2\xi_2 + H_1\xi_1)v_7 + \rho^{-1}\xi_3 v_8 \\ -(H_2\xi_2 + H_3\xi_3)v_2 + \xi_2 H_1 v_3 + \xi_3 H_1 v_4 + \tau v_5 \\ \xi_1 H_2 v_2 - (H_1\xi_1 + H_3\xi_3)v_3 + \xi_3 H_2 v_4 + \tau v_6 \\ \xi_1 H_3 v_2 + \xi_2 H_3 v_3 - (H_1\xi_1 + H_2\xi_2)v_4 + \tau v_7 \\ \gamma p\xi_1 v_2 + \gamma p\xi_2 v_3 + \gamma p\xi_3 v_4 + \tau v_8 \end{pmatrix}$$

$$
= D_1 \begin{pmatrix} \tau \nu_1 \mp \frac{|H||\xi|}{\sqrt{\rho}} \nu_1 \\ \tau \nu_2 \mp \frac{|H||\xi|}{\sqrt{\rho}} \nu_2 \\ \tau \nu_3 \mp \frac{|H||\xi|}{\sqrt{\rho}} \nu_3 \\ \tau \nu_4 \mp \frac{|H||\xi|}{\sqrt{\rho}} \nu_4 \\ \tau \nu_5 \mp \frac{|H||\xi|}{\sqrt{\rho}} \nu_5 \\ \tau \nu_6 \mp \frac{|H||\xi|}{\sqrt{\rho}} \nu_6 \\ \tau \nu_7 \mp \frac{|H||\xi|}{\sqrt{\rho}} \nu_7 \\ \tau \nu_8 \mp \frac{|H||\xi|}{\sqrt{\rho}} \nu_8 \end{pmatrix} = {}^t(0, \ldots, 0),
$$

for $\nu \in \ker q$ at Σ_2. ∎

Proposition 5.9. *If $\tau^2 \neq c_f^2(\hat{\xi})|\xi|^2$, and $\xi \times H \neq 0$, then our system is of MHD type at Σ_2.*

Proof. When $\tau^2 \neq c_f^2(\hat{\xi})|\xi|^2$, then Σ is union of the five hypersurfaces $S_1 = \{\tau = 0\}$, $S_2 = \{\tau - c_s(\hat{\xi})|\xi| = 0\}$, $S_3 = \{\tau + c_s(\hat{\xi})|\xi| = 0\}$, $S_4 = \{\tau - (\xi \cdot H)/\sqrt{\rho} = 0\}$, and $S_5 = \{\tau + (\xi \cdot H)/\sqrt{\rho} = 0\}$, intersecting at $\Sigma_2 = \bigcap_{j=1}^5 S_j = \{\tau = 0, \xi \cdot H = 0, \xi \neq 0\}$. We want to prove that our system is of MHD type at Σ_2. We have S_j intersect transversally at Σ_2, the codimension of Σ_2 is equal to two, (4.3) is satisfied, $d^6(\det q) \neq 0$, and $d^i(\det q) = 0$ for $i < 6$ at Σ_2, and the dimension of the fiber of $\mathcal{N}_Q$ is equal to 6 at Σ_2. (4.6) is satisfied for $i_0 = 1$. Hence, still we want to check (3.6). Again, we will prove this by proving the following

$$
\partial_\rho q : \ker q \mapsto \operatorname{Im} q \quad \text{at } \Sigma_2,
$$

when $\rho \in T_{\Sigma_2}\Sigma$. $T_{\Sigma_2}\Sigma$ is spanned by $D_1 = \xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2} + \xi_3 \partial_{\xi_3}$, $D_2 = \tau \partial_\tau$, $D_3 = \partial_t$, $D_4 = \tau \partial_{\xi_1}$, $D_5 = \tau \partial_{\xi_2}$, $D_6 = \tau \partial_{\xi_3}$, $D_7 = \tau \partial_{x_1}$, $D_8 = \tau \partial_{x_2}$, $D_9 = \tau \partial_{x_3}$, $D_{10} = (\xi \cdot H) \partial_{x_1}$, $D_{11} = (\xi \cdot H) \partial_{x_2}$, $D_{12} = (\xi \cdot H) \partial_{x_3}$, $D_{13} = (\xi \cdot H) \partial_{\xi_1}$, and $D_{14} = (\xi \cdot H) \partial_{\xi_3}$ (note that we have not mentioned $(\xi \cdot H) \partial_{\xi_2}$ as it can be written in terms of D_{13}, D_{14}, and D_5). We have for $(\nu_1, \ldots, \nu_8) \in \ker q$ at Σ_2

$$
D_i q \, {}^t(\nu_1, \ldots, \nu_8) = {}^t(0, \ldots, 0) \text{ at } \Sigma_2 \quad \text{for } i = 1, \ldots, 14.
$$

From (5.19) we know that (4.8), and (4.9) are satisfied with $R = D_t$, $\sigma(L_1) = M$, and $f = \tau(\tau^2 - c_s^2(\hat{\xi})|\xi|^2)(\tau^2 - c_f^2(\hat{\xi})|\xi|^2)(\tau^2 - (\xi \cdot H)^2/\rho)$. ∎

Note. As an application for systems of generalized transverse type, one can consider the linearized isentropic MHD equations, which is 7×7 matrix; check [15, Appendix A] where the first order term of the linearized isentropic MHD equations and its eigenvalues are given, and then we can easily check the type of the system as we did in this section for linearized ideal MHD equations.

References

[1] R. Darwich, *Propagation of polarization sets for systems of generalized transverse type and for systems of MHD type*. Ph.D. thesis, Georg-August-Universität Göttingen, 2023

[2] N. Dencker, On the propagation of polarization sets for systems of real principal type. *J. Functional Analysis* **46** (1982), no. 3, 351–372 Zbl 0487.58028 MR 0661876

[3] N. Dencker, On the propagation of singularities for pseudodifferential operators of principal type. *Ark. Mat.* **20** (1982), no. 1, 23–60 Zbl 0503.58031 MR 0660124

[4] N. Dencker, On the propagation of polarization in conical refraction. *Duke Math. J.* **57** (1988), no. 1, 85–134 Zbl 0669.35116 MR 0952227

[5] N. Dencker, The propagation of singularities for pseudo-differential operators with self-tangential characteristics. *Ark. Mat.* **27** (1989), no. 1, 65–88 Zbl 0699.35272 MR 1004722

[6] N. Dencker, The propagation of polarization in double refraction. *J. Funct. Anal.* **104** (1992), no. 2, 414–468 Zbl 0767.58041 MR 1153993

[7] N. Dencker, The propagation of polarization for systems of transversal type. *Ark. Mat.* **33** (1995), no. 2, 249–279 Zbl 0856.58042 MR 1373024

[8] C. Gérard, *Propagation de la polarisation pour des problèmes aux limites*. Ph.D. thesis, Faculté des sciences d'Orsay, Université de Paris-Sud, 1986

[9] S. Hansen and D. Röhrig, Lagrangian solutions to systems of real principal type. *Math. Nachr.* **277** (2004), 47–55 Zbl 1162.58304 MR 2103431

[10] L. Hörmander, Linear differential operators. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pp. 121–133, Gauthier-Villars Éditeur, Paris, 1971 Zbl 0223.35083 MR 0513000

[11] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and fourier analysis*. Classics Math., Springer, Berlin, 2003 Zbl 1028.35001 MR 1996773

[12] L. Hörmander, *The analysis of linear partial differential operators. II. Differential operators with constant coefficients*. Classics Math., Springer, Berlin, 2005 Zbl 1062.35004 MR 2108588

[13] L. Hörmander, *The analysis of linear partial differential operators. III. Pseudo-differential operators*. Classics Math., Springer, Berlin, 2007 Zbl 1115.35005 MR 2304165

[14] L. Hörmander, *The analysis of linear partial differential operators. IV. Fourier integral operators*. Classics Math., Springer, Berlin, 2009 Zbl 1178.35003 MR 2512677

[15] G. Métivier and K. Zumbrun, Hyperbolic boundary value problems for symmetric systems with variable multiplicities. *J. Differential Equations* **211** (2005), no. 1, 61–134 Zbl 1073.35155 MR 2121110

[16] D. D. Schnack, *Lectures in magnetohydrodynamics*. Lecture Notes in Phys. 780, Springer, Berlin, 2009 Zbl 1178.76001 MR 2724902

Rayhana Darwich

Mathematical Institute, Georg-August-Universität Göttingen, Bunsenstrasse 3-5, 37073 Göttingen, Germany; Lebanese Association for Scientific Research (LASeR), Azmi Beik Street, 1300 Tripoli, Lebanon; rayhana.darwich@mathematik.uni-goettingen.de, rayhana.darwich@laser-lb.org