Doc. Math. 29 (2024), 815–830
DOI 10.4171/DM/962

Toric principal bundles, Tits buildings and reduction of structure group

Shaoyu Huang and Kiumars Kaveh

Abstract. A toric principal G-bundle is a principal G-bundle over a toric variety together with a torus action commuting with the G-action. Recently, extending the Klyachko classification of toric vector bundles, Kaveh–Manon give a classification of toric principal G-bundles using *piecewise linear maps* to the (extended) Tits building of G. In this paper, we use the Kaveh–Manon classification to give a description of the (equivariant) automorphism group of a toric principal bundle as well as a simple criterion for (equivariant) reduction of structure group, recovering results of Dasgupta et al. Finally, motivated by the equivariant splitting problem for toric principal bundles, we introduce the notion of *Helly's number* of a building and pose the problem of giving sharp upper bounds for Helly's numbers of Tits buildings of semisimple algebraic groups G.

1. Introduction

Throughout $\mathbf{k}$ denotes an algebraically closed field. Let G be a linear algebraic group over $\mathbf{k}$. Also let T denote a torus over $\mathbf{k}$ and X_Σ the T-toric variety corresponding to a fan Σ. A *toric principal G-bundle $\mathcal{P}$* over X_Σ is a principal G-bundle over X_Σ together with a T-action on $\mathcal{P}$, lifting its action on X_Σ, such that the T-action and the G-action commute. The (isomorphism classes of) toric principal $\mathrm{GL}(r)$-bundles are in one-to-one correspondence with the (isomorphism classes of) rank r toric vector bundles.

Throughout the paper we fix a point x_0 in the open torus orbit in X_Σ. It gives an identification of the open orbit with the torus T. By a *framed* toric principal bundle $(\mathcal{P}, p_0)$ we mean a toric principal bundle $\mathcal{P}$ together with the choice of a point p_0 in the fiber $\mathcal{P}_{x_0}$. This choice gives an identification of the fiber $\mathcal{P}_{x_0}$ with the group G.

When G is a reductive algebraic group, in [9], the authors give a classification of framed toric principal G-bundles on X_Σ in terms of *piecewise linear maps* from $|\Sigma|$, the support of Σ, to $\widetilde{\mathfrak{B}}(G)$, *the cone over the Tits building of G* (see [9, Theorems 2.2 and 2.4]). This classification is in the spirit of the Klyachko classification of toric vector bundles [10]. In [3] toric principal bundles are classified with certain data of cocycles and homomorphisms. The classification in [3] is in the spirit of the Kaneyama classification of toric vector bundles [7].

Mathematics Subject Classification 2020: 14M25 (primary); 32L05 (secondary).
Keywords: toric variety, principal bundle, reduction of structure group, Tits building, Helly's theorem.

In [5], the authors use the Kaneyama type classification in [3], to obtain interesting results on (equivariant) automorphism group, (equivariant) reduction of structure group and stability of toric principal bundles. In the present paper, we use the classification in [9] to give short proofs for some of the results in [5]. The following is a more specific description of the content of the paper:

- We give a short proof of a result of Dasgupta et al. [5, Proposition 5.1] describing the equivariant automorphism group of a (non-framed) toric principal bundle, as an intersection of certain parabolic subgroups of G (Theorem 4.2).
- We give a simple criterion for the equivariant reduction of structure group of a toric principal bundle (see Definition 5.1 and Theorem 5.4). More precisely, we show that a toric principal G-bundle has an equivariant reduction of structure group to a closed subgroup K, if and only if, for some choice of a frame p_0, the image of the corresponding piecewise linear map $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ lies in $\widetilde{\mathfrak{B}}(K)$. As corollaries we recover the results in [5] regarding reduction of structure group and splitting of toric principal bundles (see Corollaries 5.5 and 5.6).
- We introduce the notion of *Helly's number* of a building and pose the problem of finding sharp upper bounds for it (see Definition 6.1). For the Tits building of a group G, this Helly's number is directly related to the problem of splitting of toric principal G-bundles over projective spaces (Corollary 6.5).

2. Preliminaries on Tits buildings

2.1. Tits building of a linear algebraic group

In this section we review some basic facts that we need about the Tits buildings associated to linear algebraic groups.

A *building* is a pair $(\Delta, \mathcal{A})$ consisting of a simplicial complex Δ and a family $\mathcal{A}$ of subcomplexes A (*apartments*) satisfying certain conditions. Readers can find the general definition of building in the appendix (Definition A.1).

To a linear algebraic group G over a field $\mathbf{k}$ there corresponds a building called *Tits building* of G. We denote it by $\Delta(G)$. The set of simplices in $\Delta(G)$ is the set of parabolic subgroups of G ordered by reverse inclusion. The apartments in $\Delta(G)$ correspond to maximal tori in G. For a maximal torus $H \subset G$, the corresponding apartment consists of parabolic subgroups containing H. Clearly, Borel subgroups correspond to the maximal simplices, i.e., chambers, in $\Delta(G)$. Since every parabolic subgroup contains the solvable radical $R(G)$ of G, $\Delta(G)$ and $\Delta(G/R(G))$ are isomorphic as simplicial complexes.

Example 2.1 (Tits building of $\mathrm{GL}(r)$). Consider $G = \mathrm{GL}(r)$. Any parabolic subgroup P in $\mathrm{GL}(r)$ is the stabilizer of a flag $F_\bullet = (\{0\} = F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_k = \mathbb{C}^r)$. This gives a one-to-one correspondence between the simplices in the Tits building of $\mathrm{GL}(r)$ and flags in $\mathbb{C}^r$. In particular, Borel subgroups are stabilizers of complete flags and correspond to chambers in $\Delta(\mathrm{GL}(r))$. A *frame* L in $\mathbb{C}^r$ is a direct sum decomposition of $\mathbb{C}^r = \bigoplus_{i=1}^r L_i$ into one-dimensional subspaces L_i. In other words, a *frame* is an equivalence class of vector spaces bases up to scaling basis elements by non-zero scalars. We say that a flag $F_\bullet$ is *adapted to a frame* L if each subspace F_i is spanned by some of the L_j. The apartments in the Tits building of $\mathrm{GL}(r)$ correspond to frames in $\mathbb{C}^r$. The apartment corresponding to a frame L consists of all the flags adapted to it.

Example 2.2 (Tits building of $\mathrm{Sp}(2r)$). Consider $G = \mathrm{Sp}(2r) \subset \mathrm{GL}(2r)$. We denote by $\langle\cdot,\cdot\rangle$ the standard skew symmetric bilinear form $\sum_i x_i \wedge y_i$ on $\mathbb{C}^{2r}$. We call a flag $F_\bullet = (\{0\} = F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_k = \mathbb{C}^{2r})$, an *isotropic flag* if for each $0 \le j \le k$ we have

$$F_j^\perp = F_{k-j}.$$

Any parabolic subgroup of $\mathrm{Sp}(2r)$ is the stabilizer of an isotropic flag.

We say that a basis $B = \{e_1, \dots, e_r, f_1, \dots, f_r\}$ for $\mathbb{C}^{2r}$ is a *normal basis* if the following holds:

$$\begin{aligned}
\langle e_i, e_j\rangle &= 0, \quad \forall i, j,\\
\langle f_i, f_j\rangle &= 0, \quad \forall i, j,\\
\langle e_i, f_i\rangle &= 1, \quad \forall i,\\
\langle e_i, f_j\rangle &= 0, \quad \forall i, j, i \ne j.
\end{aligned}$$

One knows that normal bases exist. If $B = \{e_1, \dots, e_r, f_1, \dots, f_r\}$ is a normal basis then $\{t_1 e_1, \dots, t_r e_r, t_1^{-1} f_1, \dots, t_r^{-1} f_r\}$, for any non-zero $t_1, \dots, t_r$, is also a normal basis. We call a normal basis, up to multiplication by non-zero scalars t_i, a *normal frame*. The normal frames are in one-to-one correspondence with maximal tori of G and hence with apartments in $\Delta(G)$. The apartment corresponding to a normal frame L consists of all the isotropic flags that are adapted to L.

When G is semisimple, the simplicial complex $\Delta(G)$ has a natural geometric realization. Namely, there is a topological space $\mathfrak{B}(G)$ together with a triangulation in which simplices in the triangulation (which are subsets of $\mathfrak{B}(G)$ homeomorphic to standard simplices) are in one-to-one correspondence with the simplices in $\Delta(G)$ and intersect according to how simplices in $\Delta(G)$ intersect. It is constructed as follows. For each maximal torus $H \subset G$ let $\Lambda^\vee(H)$ be its cocharacter lattice and let $\Lambda^\vee_{\mathbb{R}}(H) = \Lambda^\vee(H) \otimes_{\mathbb{Z}} \mathbb{R}$. The apartment corresponding to H is the triangulation of the unit sphere in $\Lambda^\vee_{\mathbb{R}}(H)$ obtained by intersecting it with the Weyl chambers and their faces. Two simplices, in different apartments, are glued together if the corresponding faces represent the same parabolic subgroup in G.

Definition 2.3 (Geometric realization of the Tits building). The topological space $\mathfrak{B}(G)$ is obtained by gluing the unit spheres in the $\Lambda^\vee_{\mathbb{R}}(H)$, for all maximal tori H, along their common simplices.

While in our notation, we distinguish between the building as an abstract simplicial complex, i.e. $\Delta(G)$, and as a topological space, i.e., $\mathfrak{B}(G)$, by abuse of terminology we refer to both $\Delta(G)$ and $\mathfrak{B}(G)$ as the Tits building of G.

Definition 2.4 (Cone over the Tits building of a semisimple group). Let G be semisimple. Similar to the construction of $\mathfrak{B}(G)$, we construct the topological space $\widetilde{\mathfrak{B}}(G)$ by gluing the vector spaces $\Lambda^\vee_{\mathbb{R}}(H)$, along their common faces of Weyl chambers. We think of $\widetilde{\mathfrak{B}}(G)$ as the cone over $\mathfrak{B}(G)$ and call it the *cone over the Tits building of* G.

Now let G be a linear algebraic group and let $G_{\mathrm{ss}} = G/R(G)$ be the semisimple quotient of G. The previous construction in the semisimple case works in this case as well and we can define $\widetilde{\mathfrak{B}}(G)$ (respectively $\mathfrak{B}(G)$) to be the topological space obtained by gluing the vector spaces $\Lambda^\vee_{\mathbb{R}}(H)$ (respectively unit spheres in the $\Lambda^\vee_{\mathbb{R}}(H)$), for all maximal tori $H \subset G$, along their common faces of Weyl chambers (respectively intersections of common faces with the unit spheres). When G is reductive, the topological space $\widetilde{\mathfrak{B}}(G)$ is the Cartesian product of $\widetilde{\mathfrak{B}}(G_{\mathrm{ss}})$ with the real vector space $\Lambda^\vee(Z) \otimes_{\mathbb{Z}} \mathbb{R}$, where $Z = Z(G)^\circ$ is the connected component of the identity in the center of G.

Definition 2.5 (Extended Tits building of a linear algebraic group). For a linear algebraic group G, we refer to $\widetilde{\mathfrak{B}}(G)$ (above) as the *extended Tits building* of G. Also, for a maximal torus H, we refer to $\Lambda^\vee_{\mathbb{R}}(H)$ as the *cone* over the apartment of H and denote it by $\tilde{A}_H$. When G is semisimple, the extended Tits building $\widetilde{\mathfrak{B}}(G)$ is the cone over the Tits building of G.

We denote by $\widetilde{\mathfrak{B}}_{\mathbb{Z}}(G)$ the subset of $\widetilde{\mathfrak{B}}(G)$ obtained by gluing the lattices $\Lambda^\vee(H)$, for all maximal tori H, and call it the set of *lattice points in the extended Tits building of* G.

Remark 2.6. Our choice of terminology *the extended Tits building* is motivated by a similar term, namely *the extended Bruhat–Tits building*, from the theory of Bruhat–Tits buildings for algebraic groups over valued fields (see [13] as well as [12, Remark 1.23]).

We will see in Section 2.2 that the set $\widetilde{\mathfrak{B}}_{\mathbb{Z}}(G)$ of lattice points in $\widetilde{\mathfrak{B}}(G)$ can be identified with the set of one-parameter subgroups of G modulo certain equivalence relation (Definition 2.7 and Proposition 2.9).

2.2. One-parameter subgroups and Tits building

In this section, following [9, Section 1.3], we present a natural way to realize the extended Tits building of G in terms of one-parameter subgroups of G. More precisely, we see that the set of lattice points $\widetilde{\mathfrak{B}}_{\mathbb{Z}}(G)$ in $\widetilde{\mathfrak{B}}(G)$ can naturally be identified with certain equivalence classes of one-parameter subgroups in G (Proposition 2.9). For details and proofs we refer the reader to [9, Section 1.3]. This construction of the Tits building of a linear algebraic group from one-parameter subgroups also appears, in slightly different form, in [11, Section 2.2].

Definition 2.7. Let λ_1, λ_2 be algebraic one-parameter subgroups of G. We say that λ_1 is *equivalent* to λ_2 and write $\lambda_1 \sim \lambda_2$ if $\lim_{s\to 0} \lambda_1(s)\lambda_2(s)^{-1}$ exists in G.

It is easy to see this is indeed an equivalence relation.

Definition 2.8 (Parabolic subgroup associated to a one-parameter subgroup). For a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$, let

$$P_\lambda = \left\{ g \in G \mid \lim_{s \to 0} \lambda(s) g \lambda(s)^{-1} \text{ exists in } G \right\}.$$

One shows that P_λ is a parabolic subgroup in G.

Alternatively, P_λ can be described in terms of the equivalence relation $\sim$ (see [9, Proposition 1.8]):

$$P_\lambda = \{ g \in G \mid g \lambda g^{-1} \sim \lambda \}. \tag{2.1}$$

It is straightforward to check that if $\lambda_1 \sim \lambda_2$ then $P_{\lambda_1} = P_{\lambda_2}$. Thus to each equivalence class of one-parameter subgroups there corresponds a parabolic subgroup. One also shows that, for a maximal torus $H \subset G$, no two one-parameter subgroups in $\Lambda^\vee(H)$ are equivalent. Moreover, if a one-parameter subgroup $\lambda \in \Lambda^\vee(H)$ lies in the relative interior of a face of a Weyl chamber, the parabolic subgroup P_λ is exactly the parabolic subgroup corresponding to this face. Putting these facts together one obtains the following [9, Corollary 1.11].

Proposition 2.9. *The set $\widetilde{\mathfrak{B}}_{\mathbb{Z}}(G)$ can naturally be identified with the set of equivalence classes of one-parameter subgroups of G.*

Remark 2.10. The above realization of the (extended) Tits building of G in terms of equivalence classes of one-parameter subgroups (Proposition 2.9) is analogous to the description of the Tits building of a symmetric space as the set of equivalence classes of geodesics (see [6, Section 3]).

Example 2.11. Consider $G = \mathrm{Sp}(2r)$. A one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ is given by a diagonal matrix

$$\mathrm{diag}(t^{v_1}, \ldots, t^{v_r}, t^{-v_r}, \ldots, t^{-v_1}), \quad v_i \in \mathbb{Z}$$

under some ordered normal basis $\{e_1, \ldots, e_r, f_r, \ldots, f_1\}$. After reordering and switching, we may assume $v_1 \geq \cdots \geq v_r \geq 0 \geq -v_r \cdots \geq -v_1$, which will still gives us a normal basis. For $i = 1, \ldots, r$, let $v_{r+i} = -v_{r+1-i}$. Consider indices $i_1, \ldots, i_k = 2r$ such that $v_1 = \cdots = v_{i_1} > v_{i_1+1} = \cdots = v_{i_2} > \cdots > v_{i_{k-1}+1} = \cdots = v_{i_k} = v_{2r}$. For $j = 1, \ldots, k$, we let $c_j = v_{i_j}$ and $F_j = V_{i_j}$ which is spanned by first i_j vectors in ordered normal basis. Then we get an isotropic flag $F_\bullet = (\{0\} = F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_k = \mathbb{C}^{2r})$ and a labeling $c_\bullet = (c_1 > \cdots > c_k)$ with $c_j = -c_{k+1-j}$. We call $(F_\bullet, c_\bullet)$, where $c_\bullet = (c_1 > \cdots > c_k)$ is a sequence with $c_j = -c_{k+1-j}$, a *labeled isotropic flag*. The extended Tits building $\widetilde{\mathfrak{B}}(G)$ can be realized as the collection of labeled isotropic flags.

A homomorphism of linear algebraic groups naturally induces a map between the corresponding extended Tits buildings. The above realization of the extended Tits building in terms of equivalence classes of one-parameter subgroups gives an easy way to construct this map.

Definition 2.12. Let $\alpha : G \to G'$ be a homomorphism of linear algebraic groups. If $\lambda : \mathbb{G}_m \to G$ is a one-parameter subgroup of G, then $\alpha \circ \lambda$ is a one-parameter subgroup of G'. The map $\lambda \mapsto \alpha \circ \lambda$ respects the equivalence classes and thus gives a well-defined map $\widehat{\alpha} : \widetilde{\mathfrak{B}}_{\mathbb{Z}}(G) \to \widetilde{\mathfrak{B}}_{\mathbb{Z}}(G')$. This extends to a map $\widehat{\alpha} : \widetilde{\mathfrak{B}}(G) \to \widetilde{\mathfrak{B}}(G')$. The map $\widehat{\alpha}$ sends an extended apartment for G to an extended apartment for G'. This is because the image of a torus in G is a torus in G' and every torus lies in a maximal torus.

Finally, we use the above to make the observation that the extended Tits building does not change under semidirect product with a unipotent group. In particular, the extended Tits building of a parabolic subgroup and its Levi subgroup coincide.

Proposition 2.13. *For a linear algebraic group G, suppose there exist subgroups $L, U \subset G$ such that $G = L \ltimes U$ (in particular, U is normalized by L). If U is unipotent then $\widetilde{\mathfrak{B}}(L)$ and $\widetilde{\mathfrak{B}}(G)$ can be identified via the map $\widehat{\iota}$ where $\iota : L \to G$ is the inclusion.*

Proof. Since L is a closed subgroup, it is straightforward to see that $\widehat{\iota} : \widetilde{\mathfrak{B}}(L) \to \widetilde{\mathfrak{B}}(G)$ is an embedding. It remains to show $\widehat{\iota}$ is surjective. Let $\gamma : \mathbb{G}_m \to G$ be a one-parameter subgroup in G. Since $G = L \ltimes U$, there exist a one-parameter subgroup $\gamma_L : \mathbb{G}_m \to L \simeq G/U$ and a morphism $\gamma_U : \mathbb{G}_m \to U$ such that $\gamma(s) = \gamma_L(s)\gamma_U(s)$, $\forall s \in \mathbb{G}_m$. Since the unipotent group U can be embedded in $\mathrm{GL}(r)$ as a subvariety of upper triangular matrices with 1's on the diagonal, $\lim_{s\to 0} \gamma_U(s)$ exists in U. Therefore,

$$\lim_{s\to 0} \gamma(s)\gamma_L^{-1}(s) = \lim_{s\to 0} \gamma_U(s) \in U \subset G.$$

This shows $\gamma \sim \gamma_L$ and hence $\widehat{\iota}$ is surjective. ∎

3. Preliminaries on toric principal bundles

In this section we review the classification of (framed) toric principal bundles in [9]. Let $T \cong \mathbb{G}_m^n$ denote an n-dimensional algebraic torus over an algebraically closed field $\mathbf{k}$. We let M and N denote its character and cocharacter lattices respectively. We also denote by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ the $\mathbb{R}$-vector spaces spanned by M and N. Let Σ be a (finite rational polyhedral) fan in $N_{\mathbb{R}}$ and let X_Σ be the corresponding toric variety. Also U_σ denotes the invariant affine open subset in X_Σ corresponding to a cone $\sigma \in \Sigma$. We denote the support of Σ, that is the union of all the cones in Σ, by $|\Sigma|$. For each i, $\Sigma(i)$ denotes the subset of i-dimensional cones in Σ. In particular, $\Sigma(1)$ is the set of rays in Σ. For each ray $\rho \in \Sigma(1)$ we let v_ρ be the primitive vector along ρ, i.e., v_ρ is the shortest non-zero integral vector on ρ.

Throughout the paper we fix a point x_0 in the open torus orbit in X_Σ. It gives an identification of the torus T with the open orbit via $t \mapsto t \cdot x_0$.

We start by recalling the notion of a principal bundle. Let G be an algebraic group, a *principal G-bundle over a variety X* is a fiber bundle $\mathcal{P}$ over X with an action of G such that G preserves each fiber and the action is free and transitive. Throughout, we take the action of G on $\mathcal{P}$ to be a *right* action.

Let G, G' be algebraic groups and $\mathcal{P}$ (respectively $\mathcal{P}'$) be a principal G-bundle (respectively G'-bundle) over X. A *morphism of principal bundles with respect to a homomorphism of algebraic groups* $\alpha : G \to G'$ is a bundle map $F : \mathcal{P} \to \mathcal{P}'$ such that

$$F(z \cdot g) = F(z) \cdot \alpha(g), \quad \forall z \in \mathcal{P}, \ \forall g \in G.$$

We refer to a morphism between toric principal G-bundles, with respect to the identity homomorphism $G \to G$, simply as a *morphism of principal G-bundles*. We note that any morphism of principal G-bundles is an isomorphism.

Definition 3.1 (Toric principal bundle). Let X_Σ be the toric variety associated to a fan Σ and G an algebraic group. A *toric principal G-bundle over X_Σ* is a principal G-bundle $\mathcal{P}$ together with a torus action lifting that of X_Σ, such that the T-action and the G-action on $\mathcal{P}$ commute. More precisely, $\forall t \in T$, $\forall x \in X_\Sigma$, $\forall z \in \mathcal{P}_x$ we have:

$$t : \mathcal{P}_x \to \mathcal{P}_{t \cdot x},$$
$$t \cdot (z \cdot g) = (t \cdot z) \cdot g.$$

Recall that we have fixed a point x_0 in the open torus orbit in X_Σ. We call a toric principal G-bundle $\mathcal{P}$ together with a choice of a point $p_0 \in \mathcal{P}_{x_0}$ a *framed toric principal G-bundle*.

Definition 3.2. A *morphism of toric principal bundles* is a morphism F of principal bundles (with respect to some homomorphism α as above) that is also T-equivariant. A *morphism of framed principal bundles* $(\mathcal{P}, p_0) \to (\mathcal{P}', p_0')$ is a morphism F that sends $p_0 \in \mathcal{P}_{x_0}$ to $p_0' \in \mathcal{P}'_{x_0}$.

The following is the main combinatorial gadget to classify (framed) toric principal bundles. It can be thought of as a generalization of a real-valued piecewise linear function $\varphi : |\Sigma| \to \mathbb{R}$.

Definition 3.3 (Piecewise linear map). Let G be a linear algebraic group with $\widetilde{\mathfrak{B}}(G)$, the extended Tits building of G. Let Σ be a fan in $N_\mathbb{R}$, we say that a map $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ is a *piecewise linear map* if:

(a) For each cone $\sigma \in \Sigma$, there exists a maximal torus H_σ (not necessarily unique) such that $\Phi(\sigma)$ lies in an extended apartment $\widetilde{A}_\sigma = \Lambda^\vee_\mathbb{R}(H_\sigma)$.

(b) For each cone $\sigma \in \Sigma$, the restriction $\Phi|_\sigma : \sigma \to \widetilde{A}_\sigma$ is an $\mathbb{R}$-linear map.

We say that a piecewise linear map Φ is *integral* if Φ sends lattice points to lattice points, i.e., for any $\sigma \in \Sigma$, $\Phi(\sigma \cap N) \subset \Lambda^\vee(H_\sigma)$.

Definition 3.4 (Equivariant triviality). We say that a toric principal bundle $\mathcal{P}$ on an affine toric variety U_σ is *equivariantly trivial* if there exists a toric principal G-bundle isomorphism between $\mathcal{P}$ and $U_\sigma \times G$, where T acts on $U_\sigma \times G$ via an algebraic group homomorphism $\phi_\sigma : T \to G$ by:

$$t \cdot (x, g) = \big(t \cdot x, \phi_\sigma(t)g\big), \quad \forall t \in T, \ \forall x \in U_\sigma, \ \forall g \in G.$$

Definition 3.5 (Local equivariant triviality). Let $\mathcal{P}$ be a toric principal G-bundle on a toric variety X_Σ. We say that $\mathcal{P}$ is *locally equivariantly trivial* if for any $\sigma \in \Sigma$, the restriction $\mathcal{P}\,|_{U_\sigma}$, to the affine open chart U_σ, is equivariantly trivial.

The following gives a classification of locally equivariantly trivial framed toric principal bundles in terms of piecewise linear maps [9, Theorem 2.4].

Theorem 3.6. *Let G be a linear algebraic group over* **k**.

(a) *There is a one-to-one correspondence between the isomorphism classes of locally equivariantly trivial framed toric principal G-bundles $\mathcal{P}$ over X_Σ and the integral piecewise linear maps $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$.*

(b) *Moreover, let $\alpha : G \to G'$ be a homomorphism of linear algebraic groups. Let $(\mathcal{P}, p_0)$ (respectively $(\mathcal{P}', p_0'))$ be a locally equivariantly trivial framed toric principal G-bundle (respectively G'-bundle) with corresponding piecewise linear map $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ (respectively $\Phi' : |\Sigma| \to \widetilde{\mathfrak{B}}(G')$). Then there is a (necessarily unique) morphism of framed toric principal bundles $F : \mathcal{P} \to \mathcal{P}'$ with respect to α if and only if $\Phi' = \widehat{\alpha} \circ \Phi$.*

The idea of the proof of Theorem 3.6 is as follows: Let $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ be a piecewise linear map. For each cone $\sigma \in \Sigma$, the integral linear map $\Phi_{|\sigma}$ gives an algebraic group homomorphism $T_\sigma \to H_\sigma$ where T_σ is the stabilizer of the orbit O_σ. Extend this to a homomorphism $\phi_\sigma : T \to H_\sigma$. On each affine chart U_σ, consider the trivial toric principal bundle $\mathcal{P}_\sigma = U_\sigma \times G$ where T acts on G via ϕ_σ. For two cones $\sigma, \sigma' \in \Sigma$ with $\tau = \sigma \cap \sigma'$, define the transition function $\psi_{\sigma,\sigma'} : U_\tau = U_\sigma \cap U_{\sigma'} \to G$ by defining it on the open orbit by $\psi_{\sigma,\sigma'}(t \cdot x_0) = \phi_{\sigma'}(t)\phi_\sigma(t)^{-1}$. One shows that this extends to a regular function $\psi_{\sigma,\sigma'} : U_\tau \to G$. The toric principal bundle $\mathcal{P}$, associated to Φ, is obtained by gluing the $\mathcal{P}_\sigma$ via the transition functions $\phi_{\sigma,\sigma'}$.

It is shown in [4, Theorem 4.1] that if G is reductive then any toric principal G-bundle is locally equivariantly trivial. Thus Theorem 3.6 immediately implies the following.

Corollary 3.7. *Let G be a reductive algebraic group over* **k**.

(a) *There is a one-to-one correspondence between the isomorphism classes of framed toric principal G-bundles $\mathcal{P}$ over X_Σ and the integral piecewise linear maps $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$.*

(b) *Moreover, let $\alpha : G \to G'$ be a homomorphism of reductive algebraic groups. Let $\mathcal{P}$ (respectively $\mathcal{P}'$) be a framed toric principal G-bundle (respectively G'-bundle) with corresponding piecewise linear map $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ (respectively $\Phi' : |\Sigma| \to \widetilde{\mathfrak{B}}(G'))$. Then there is a morphism of framed toric principal bundles $F : \mathcal{P} \to \mathcal{P}'$ with respect to α if and only if $\Phi' = \widehat{\alpha} \circ \Phi$.*

Remark 3.8 (Toric principal bundles over $\mathbb{C}$). Using analytic methods, it is also shown in [3] that when the base field $\mathbf{k} = \mathbb{C}$, the local equivariant triviality of toric principal bundles holds for any linear algebraic group. Hence Corollary 3.7 also holds for linear algebraic groups over $\mathbb{C}$.

The following is a simple corollary of Theorem 3.6 (b).

Lemma 3.9. *Let $(\mathcal{P}, p_0)$ be a locally equivariantly trivial framed toric principal G-bundle with corresponding integral piecewise linear map $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$. Then for any $g_0 \in G$, the corresponding integral piecewise linear map for the framed toric principal G-bundle $(\mathcal{P}, p_0 \cdot g_0)$ is $\widehat{\alpha}_{g_0} \circ \Phi$, where $\alpha_{g_0} : G \to G$ is the conjugation homomorphim $x \mapsto g_0^{-1} x g_0$.*

Proof. The right action by g_0 gives a morphism of framed toric principal bundles from $(\mathcal{P}, p_0)$ to $(\mathcal{P}, p_0 \cdot g_0)$ with respect to the conjugation homomorphism $\alpha_{g_0} : G \to G$. Theorem 3.6 (b) then implies that the piecewise linear map of $(\mathcal{P}, p_0 \cdot g_0)$ is $\widehat{\alpha}_{g_0} \circ \Phi$. ∎

4. Equivariant automorphism group

In this section we use the classification of framed toric principal bundles (Theorem 3.6) to give a short proof of a result of Dasgupta et al. [5, Proposition 5.1] describing the equivariant automorphism group of a toric principal bundle.

Definition 4.1. Let $\mathcal{P}$ be a toric principal G-bundle over a toric variety X_Σ. A *T-equivariant automorphism F* on $\mathcal{P}$ is a morphism of principal G-bundles $F : \mathcal{P} \to \mathcal{P}$ which is T-equivariant. In other words, in the sense of Definition 3.2, F is a morphism of toric principal bundles with respect to the identity homomorphism $\mathrm{id} : G \to G$. We let $\mathrm{Aut}_T(\mathcal{P})$ denote the group of T-equivariant automorphisms of $\mathcal{P}$.

Theorem 4.2. *Let $\mathcal{P}$ be a locally equivariant trivial toric principal G-bundle over a toric variety X_Σ. Pick a frame $p_0 \in \mathcal{P}_{x_0}$ and let $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ be the piecewise linear map associated to $(\mathcal{P}_{x_0}, p_0)$. We have:*

$$\mathrm{Aut}_T(\mathcal{P}) \cong \bigcap_{\rho \in \Sigma(1)} P_\rho,$$

where P_ρ is the parabolic subgroup in G corresponding to $\Phi(v_\rho) \in \widetilde{\mathfrak{B}}_{\mathbb{Z}}(G)$ (see (2.1)).

Proof. Let $F \in \mathrm{Aut}_T(\mathcal{P})$ and let $F(p_0) = p_0'$. Let Φ, Φ' be the piecewise linear maps corresponding to the framed bundles $(\mathcal{P}, p_0)$, $(\mathcal{P}, p_0')$ respectively. There exists a $g_0 \in G$ such that $p_0' = p_0 \cdot g_0$. By Lemma 3.9 we have $\Phi' = \widehat{\alpha}_{g_0} \circ \Phi$ where $\alpha_{g_0} : G \to G$ is the conjugation by g_0. It is straightforward to check that $F \mapsto g_0$ gives an injective homomorphism $\eta : \mathrm{Aut}_T(\mathcal{P}) \to G$. It is injective because, firstly F is determined by its values on the open orbit. Moreover, by T and G-equivariance, F is determined on the open orbit by its value at the single point p_0. We need to show that the image coincides with $\bigcap_{\rho\in\Sigma(1)} P_\rho$. Note that Theorem 3.6 (b) implies that $\Phi' = \Phi$ because the automorphism F is equivariant with respect to the identity $\mathrm{id} : G \to G$. It follows that g_0 is in the image of η if and only of $\widehat{\alpha}_{g_0} \circ \Phi = \Phi$. This means that, for any lattice point $x \in |\Sigma| \cap N$ we have $g_0^{-1}\Phi(x)g_0 \sim \Phi(x)$. In view of piecewise linearity of Φ this is equivalent to:

$$g_0^{-1}\Phi(v_\rho)g_0 \sim \Phi(v_\rho), \quad \forall \rho \in \Sigma(1),$$

where v_ρ is the shortest non-zero integral vector on ρ. In view of (2.1), this is the case if and only of $g_0 \in \bigcap_{\rho\in\Sigma(1)} P_\rho$. ■

5. Equivariant reduction of structure group

In this section we address the question of reduction of structure group for toric principal bundles.

Definition 5.1 (Equivariant reduction of structure group). Let K be a closed subgroup of a linear algebraic group G. We say that a toric principal G-bundle $\mathcal{P}$ over X_Σ has an *equivariant reduction of structure group to* K if there exists a toric principal K-bundle $\mathcal{P}'$ over X_Σ such that there is an isomorphism of toric principal G-bundles between $\mathcal{P}$ and $\mathcal{P}' \times^K G$, where $\mathcal{P}' \times^K G$ is the quotient of $\mathcal{P}' \times G$ by the right action of K given by:

$$(p, g) \cdot k = (pk, k^{-1}g), \quad \forall p \in \mathcal{P}, \ \forall k \in K, \ \forall g \in G.$$

The group G acts on $\mathcal{P}' \times^K G$ by right multiplication on the second component and with this action $\mathcal{P}' \times^K G$ is a principal G-bundle. If $\mathcal{P}$ admits an equivariant reduction of structure group to a maximal torus in G, then we say $\mathcal{P}$ *splits equivariantly*.

Remark 5.2. Let $\iota : K \hookrightarrow G$ be the inclusion map and $F : \mathcal{P}' \to \mathcal{P}' \times^K G$ be defined by $F(p') = (p', 1)$, where 1 is the identity element in G. It is not difficult to see that F is a morphism of principal bundles with respect to the homomorphism ι since

$$F(p' \cdot k) = (p' \cdot k, 1) = (p' \cdot kk^{-1}, k \cdot 1) = (p', k) = (p', 1) \cdot k = F(p') \cdot \iota(k).$$

Remark 5.3. A toric principal G-bundle $\mathcal{P}$ over X_Σ has an equivariant reduction of structure group to K just means $\mathcal{P}$ has equivariant trivializations whose transition functions all lie in K.

The inclusion map $\iota : K \hookrightarrow G$, gives an embedding $\widehat{\iota} : \widetilde{\mathfrak{B}}(K) \hookrightarrow \widetilde{\mathfrak{B}}(G)$ (see Definition 2.12). For any extended apartment $\widetilde{A}_H \subset \widetilde{\mathfrak{B}}(G)$, the preimage of $\widetilde{A}_H$ under $\widehat{\iota}$ lies in an extended apartment in $\widetilde{\mathfrak{B}}(K)$.

Theorem 5.4 (Criterion for equivariant reduction of structure group). *A locally equivariantly trivial toric principal G-bundle $\mathcal{P}$ over X_Σ has an equivariant reduction of structure group to K if and only if there exists a $p_0 \in \mathcal{P}_{x_0}$ such that the image of Φ lies in $\widetilde{\mathfrak{B}}(K)$.*

Here $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ *is the integral piecewise linear map corresponding to the framed bundle* $(\mathcal{P}, p_0)$.

Proof. Suppose $\mathcal{P}$ has an equivariant reduction of structure group to K. Then there exists a toric principal K-bundle $\mathcal{P}'$ over X_Σ such that $\mathcal{P} \simeq \mathcal{P}' \times^K G$ as toric G-principal bundles. Let $\Phi' : |\Sigma| \to \widetilde{\mathfrak{B}}(K)$ be the corresponding integral linear map of $(\mathcal{P}', p_0')$ for some $p_0' \in \mathcal{P}'_{x_0}$. Then $\widehat{\iota} \circ \Phi' : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ is an integral piecewise linear map as well. From Theorem 3.6 (b), we know $\widehat{\iota} \circ \Phi'$ is the integral piecewise linear map corresponding to $(\mathcal{P}' \times^K G, (p_0', 1))$, i.e., there exists a $(p_0', 1) \in \mathcal{P}_{x_0}$ such that the image of $\widehat{\iota} \circ \Phi'$ lies in $\widetilde{\mathfrak{B}}(K)$. Conversely, suppose there exists a $p_0 \in \mathcal{P}_{x_0}$ such that the image of Φ, the integral piecewise linear map corresponding to $(\mathcal{P}, p_0)$, lies in $\widetilde{\mathfrak{B}}(K)$, where $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$. Since the image of Φ lies in $\widetilde{\mathfrak{B}}(K)$, we have a piecewise linear map $\Phi' : |\Sigma| \to \widetilde{\mathfrak{B}}(K)$ such that $\Phi = \widehat{\iota} \circ \Phi'$. Let $\mathcal{P}'$ be the framed toric principal bundle corresponding to Φ'. As above, by Remark 5.2 and Theorem 3.6 (b), $(\mathcal{P}' \times^K G, (p_0', 1))$ is the framed toric principal G-bundle corresponding to $\widehat{\iota} \circ \Phi'$. Therefore, $\mathcal{P} \cong \mathcal{P}' \times^K G$ as toric G-principal bundles. ■

Corollary 5.5 (Criterion for equivariant splitting). *A locally equivariantly trivial toric principal* G*-bundle* $\mathcal{P}$ *over* X_Σ *splits equivariantly if and only if for some (and hence any)* $p_0 \in \mathcal{P}_{x_0}$ *the image of* Φ *lies in an extended apartment* $\widetilde{A}_H$ *for some maximal torus* $H \subset G$. *Here* $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ *is the integral piecewise linear map corresponding to the framed bundle* $(\mathcal{P}, p_0)$.

Proof. By definition, $\widetilde{\mathfrak{B}}(H)$ is the extended apartment $\widetilde{A}_H$. The claim follows from this and Theorem 5.4. ■

Theorem 5.4 readily implies the following result of Dasgupta et al. [5, Theorem 6.9].

Corollary 5.6. *Let* K *be a closed subgroup of a linear algebraic group* G. *Let* $\mathcal{P}'$ *be a locally equivariantly trivial toric principal* K*-bundle over* X_Σ. *If* $\mathcal{P} = \mathcal{P}' \times^K G$ *splits equivariantly (as a* G*-bundle), then* $\mathcal{P}'$ *splits equivariantly (as a* K*-bundle).*

Proof. As before let $\iota : K \hookrightarrow G$ denote the inclusion map. We consider $\widetilde{\mathfrak{B}}(K)$ as a subset of $\widetilde{\mathfrak{B}}(G)$ via the embedding $\widehat{\iota} : \widetilde{\mathfrak{B}}(K) \hookrightarrow \widetilde{\mathfrak{B}}(G)$. As explained above, for any frame $(p_0', 1) \in \mathcal{P}_{x_0}$, the image of the piecewise linear map Φ corresponding to $(\mathcal{P}' \times^K G, (p_0', 1))$ lies in $\widetilde{\mathfrak{B}}(K)$. Since this bundle splits equivariantly, Corollary 5.5 implies that this image moreover lies in $\widetilde{\mathfrak{B}}(H)$, for some maximal torus $H \subset G$. Now since the connected component of the identity in $H \cap K$ is a torus, it is contained in some maximal torus $H' \subset K$. This means that $\widetilde{\mathfrak{B}}(K) \cap \widetilde{\mathfrak{B}}(H) \subset \widetilde{\mathfrak{B}}(H')$ which, in light of Corollary 5.5, implies that $\mathcal{P}'$ also splits equivariantly. ■

In [10, Theorem 6.1.2] as well as [8, Corollary 3.5], it is shown that any toric vector bundle of rank r over $\mathbb{P}^n$ splits equivariantly, for $r < n$. In our language, any toric principal $\mathrm{GL}(r)$-bundle over $\mathbb{P}^n$ splits equivariantly, for $r < n$. As observed in [5, Theorem 6.1], this combined with Corollary 5.6 gives us the following.

Corollary 5.7. *Let K be a closed subgroup of* $\mathrm{GL}(r)$. *Any toric principal K-bundle on $\mathbb{P}^n$ splits equivariantly if $r < n$.*

Proof. Let $\mathcal{P}$ be a toric principal K-bundle on $\mathbb{P}^n$ where $r < n$. One knows that $\mathcal{P} \times^K \mathrm{GL}(r)$ splits equivariantly. Then by Corollary 5.6, $\mathcal{P}$ also splits equivariantly. ∎

Finally, from Theorem 5.4 we obtain a short proof of [5, Proposition 6.4] about reduction of the structure group of a toric principal P-bundle, where P is a parabolic subgroup, to its Levi subgroup. In fact, we give a slightly more general version of this result for any linear algebraic group that can be written as a semidirect product of a subgroup and a unipotent subgroup.

Corollary 5.8 (Equivariant reduction of structure group to a Levi). *Let P be a linear algebraic group that can be written as a semidirect product $P = L \ltimes U$ of subgroups L and U where U is unipotent. Let $\mathcal{P}$ be a locally equivariantly trivial toric principal P-bundle. Then $\mathcal{P}$ has an equivariant reduction of structure group to L. This in particular applies to the Levi decomposition $P = L \ltimes R_u(P)$ of a parabolic subgroup P.*

Proof. From Proposition 2.13, $\widetilde{\mathfrak{B}}(P) \simeq \widetilde{\mathfrak{B}}(L)$. By Theorem 5.4, $\mathcal{P}$ has an equivariant reduction of structure group to L. ∎

Example 5.9 (Toric principal bundles over $\mathbb{P}^1$). Let $\mathcal{P}$ be a toric principal G-bundle over $X_\Sigma = \mathbb{P}^1$. The fan Σ consists of two cones $\sigma_1 = \langle 1 \rangle$ and $\sigma_2 = \langle -1 \rangle$ in 1-dimensional space. For any $p_0 \in \mathcal{P}_{x_0}$, the corresponding integral piecewise linear map Φ gives us two simplices $\Phi(\sigma_1)$ and $\Phi(\sigma_2)$. Since any two simplices lie in an apartment, there exists a maximal torus $H \subset G$ such that $\Phi(|\Sigma|) \subset \widetilde{A}_H$ and hence $\mathcal{P}$ splits equivariantly.

Example 5.10 (Toric orthogonal principal bundle). Let $\mathcal{P}$ be a toric principal $\mathrm{SO}(r)$-bundle. From Corollary 5.7 it follows that any toric principal $\mathrm{SO}(r)$-bundle over $\mathbb{P}^n$ splits equivariantly when $r < n$.

6. Helly's number of a building

In this section we introduce Helly's number of the Tits building of a linear algebraic group. More generally, we define Helly's number for an (abstract) building.

The classical Helly's theorem in convex geometry asserts the following: let S be a finite collection of convex subsets in $\mathbb{R}^n$ such that any $n + 1$ of these convex subsets have non-empty intersection, then the intersection of all the convex sets in S is non-empty.

Motivated by this theorem, one defines Helly's number for any collection of sets. Let $\mathcal{F}$ be a collection of sets. *Helly's number* $h(\mathcal{F})$ of $\mathcal{F}$ is the minimal positive integer h such that if a finite subcollection $S \subset \mathcal{F}$ satisfies $\bigcap_{X \in S'} X \neq \emptyset$ for all $S' \subset S$ with $|S'| \leq h$, then $\bigcap_{X \in S} X \neq \emptyset$. Helly's theorem about convex sets tells us that for the collection $\mathcal{F}$ of compact convex subsets of $\mathbb{R}^n$, we have $h(\mathcal{F}) \leq n + 1$. In fact, it is not hard to see that $h(\mathcal{F}) = n + 1$ [2].

Motivated by [10, Section 6], we give an analogous definition for the collection of parabolic subgroups of a linear algebraic group G. The difference with the usual notion of Helly's number is that instead of asking that a collection of parabolic subgroups have a non-empty intersection, we ask that their intersection contains a maximal torus.

Definition 6.1 (Helly's number of a Tits building). Let G be a linear algebraic group. We define *Helly's number* $h(G)$ of G to be the minimal positive integer k such that the following holds: if S is a collection of parabolic subgroups of G such that the intersection of any k elements in S contains a maximal torus, then the intersection of all the elements in S contains a maximal torus.

Remark 6.2. It is not difficult to see that the above Helly's number is different from usual Helly's number for the collection of parabolic subgroups of G. That is, a finite intersection of parabolic subgroups may have non-empty intersection but does not contain a maximal torus.

More generally we define Helly's number of an abstract building.

Definition 6.3 (Helly's number of a building). Let Δ be a building. We define *Helly's number* $h(\Delta)$ of Δ to be the minimal positive integer k such that the following holds: if S is a collection of simplices of Δ such that any k simplices in S lie in an apartment, then all of the simplices in S lie in the same apartment.

In [10, Section 6], Klyachko shows that

$$h\big(\mathrm{GL}(r)\big) = r + 1.$$

Therefore, for $G \hookrightarrow \mathrm{GL}(r)$, we have $h(G) \leq r + 1$. A natural question is how to find a sharp upper bound for $h(G)$ for any semisimple algebraic group G. More generally, we pose the following problem.

Problem 6.4. For a building Δ, give a sharp upper bound for Helly's number $h(\Delta)$.

From Corollary 5.5, we have the following corollary.

Corollary 6.5. *Let G be a reductive algebraic group. Then any toric principal G-bundle on $\mathbb{P}^k$ splits equivariantly when $k \geq h(G)$.*

Proof. Let $(\mathcal{P}, p_0)$ be a framed toric principal G-bundle over $\mathbb{P}^k$. Let $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}(G)$ be the integral piecewise linear map corresponding to $(\mathcal{P}, p_0)$ where Σ is the fan of $\mathbb{P}^k$. In the fan Σ, there are $k + 1$ rays and any collection of k rays lies in some maximal cone σ. Since $\Phi(\sigma)$ lies in an extended apartment $\widetilde{A}_\sigma$, we see that the images of any collection of k rays lies in an extended apartment. Since $k \geq h(G)$, the image of any $h(G)$ rays also lies in an extended apartment. By the definition of $h(G)$, we then conclude that the images of all the $k + 1$ rays of Σ belong to the same apartment. Now Corollary 5.5, implies that $\mathcal{P}$ splits equivariantly. ∎

Example 6.6. Let $G = \mathrm{Sp}(2)$. Since $\mathrm{Sp}(2) \subset \mathrm{GL}(2)$, $h(G) \leq 2 + 1 = 3$. Consider three isotropic flags

$$F_1 = \big(\{0\} \subsetneqq \{e_1\} \subsetneqq \mathbb{C}^2\big),$$
$$F_2 = \big(\{0\} \subsetneqq \{f_1\} \subsetneqq \mathbb{C}^2\big),$$
$$F_3 = \big(\{0\} \subsetneqq \{e_1 + f_1\} \subsetneqq \mathbb{C}^2\big),$$

where $\{e_1, f_1\}$ is a normal basis of $\mathbb{C}^2$. Any 2 of these flags are adapted to a normal frame, but all of them are not adapted to any normal frame. This shows $h(G) > 2$. Therefore, $h(G) = 3$.

Appendix

For the sake of completeness in this appendix we give the defining axioms of an (abstract) building.

Definition A.1 (Building). A building is a pair $(\Delta, \mathcal{A})$ consisting of a simplicial complex Δ and a family $\mathcal{A}$ of subcomplexes A (*apartments*) satisfying the following conditions:

(i) each simplex of Δ or any apartment A is contained in a maximal simplex (*chamber*), and each chamber of Δ or A has the same finite dimension n;

(ii) each apartment A is connected, in the sense that for any two chambers C, D in A there is a sequence of chambers of A starting with C and ending with D, the intersection of any two successive members of which is an $(n-1)$-simplex;

(iii) any $(n-1)$-simplex of Δ (respectively, of any apartment A) is contained in more than 2 chambers of Δ (respectively, in exactly 2 chambers of A);

(iv) any two chambers C, D of Δ are contained in some apartment;

(v) if two simplices C, C' of Δ are contained in two apartments A, A', then there is an isomorphism from A onto A' fixing both C and C' pointwise.

Extending the construction of Tits building of a linear algebraic group as the collection of its parabolic subgroups, there is a group theoretic way to construct buildings using the notion of a *Tits system* or a (B, N) *pair*. A Tits system is a structure on groups of Lie type and roughly speaking says that such groups have structure similar to that of the general linear group over a field.

Definition A.2 (Tits system). A *Tits system* or (B, N) *pair* is a collection (G, B, N, S), where B and N are subgroups of a group G and S is a subset of $N/(B \cap N)$ satisfying the following conditions:

(i) $H = B \cap N$ generates G;

(ii) $H \lhd N$;

(iii) S generates $W = N/H$ and consists of elements of order 2;

(iv) $sBw \subset BwB \cup BswB$, $\forall s \in S, w \in W$;

(v) $sBs \not\subset B$, $\forall s \in S$.

A subgroup of G is called *parabolic* if it contains a conjugate of B. The collection of all parabolic subgroups in a Tits system can be given the structure of a building [1, Section 6.2].

Acknowledgments. We thank Roman Fedorov, Mainak Poddar, Jyoti Dasgupta, Bivas Khan, Christopher Manon, and Sam Payne for useful discussions and comments.

Funding. The second author is partially supported by NSF grant (DMS-2101843) and a Simons Collaboration Grant for Mathematicians.

References

[1] P. Abramenko and K. S. Brown, *Buildings. Theory and applications*. Grad. Texts in Math. 248, Springer, New York, 2008 Zbl 1214.20033 MR 2439729

[2] I. Bárány and G. Kalai, Helly-type problems. *Bull. Amer. Math. Soc. (N.S.)* **59** (2022), no. 4, 471–502 Zbl 1503.52009 MR 4478031

[3] I. Biswas, A. Dey, and M. Poddar, A classification of equivariant principal bundles over non-singular toric varieties. *Internat. J. Math.* **27** (2016), no. 14, article no. 1650115 Zbl 1360.32014 MR 3593677

[4] I. Biswas, A. Dey, and M. Poddar, Tannakian classification of equivariant principal bundles on toric varieties. *Transform. Groups* **25** (2020), no. 4, 1009–1035 Zbl 1457.14111 MR 4166679

[5] J. Dasgupta, B. Khan, I. Biswas, A. Dey, and M. Poddar, Classification, reduction, and stability of toric principal bundles. *Transform. Groups* (2023), DOI 10.1007/s00031-023-09812-5

[6] L. Ji, Buildings and their applications in geometry and topology. In *Differential geometry*, pp. 89–210, Adv. Lect. Math. (ALM) 22, Int. Press, Somerville, MA, 2012 Zbl 1275.20025 MR 3076052

[7] T. Kaneyama, On equivariant vector bundles on an almost homogeneous variety. *Nagoya Math. J.* **57** (1975), 65–86 Zbl 0283.14008 MR 0376680

[8] T. Kaneyama, Torus-equivariant vector bundles on projective spaces. *Nagoya Math. J.* **111** (1988), 25–40 Zbl 0820.14010 MR 0961215

[9] K. Kaveh and C. Manon, Toric principal bundles, piecewise linear maps and Tits buildings. *Math. Z.* **302** (2022), no. 3, 1367–1392 Zbl 1510.14036 MR 4492498

[10] A. A. Klyachko, Equivariant bundles over toric varieties. *Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989), no. 5, 1001–1039 Zbl 0706.14010 MR 1024452

[11] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*. 3rd edn., Ergeb. Math. Grenzgeb. (2) 34, Springer, Berlin, 1994 Zbl 0797.14004 MR 1304906

[12] B. Rémy, A. Thuillier, and A. Werner, Bruhat-Tits buildings and analytic geometry. In *Berkovich spaces and applications*, pp. 141–202, Lecture Notes in Math. 2119, Springer, Cham, 2015 Zbl 1326.51005 MR 3330766

[13] J. Tits, Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, pp. 29–69, Proc. Sympos. Pure Math. 33, American Mathematical Society, Providence, RI, 1979 Zbl 0415.20035 MR 0546588

Communicated by Eckhard Meinrenken

Received 17 July 2023; revised 6 January 2024.

Shaoyu Huang
Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA; shh123@pitt.edu

Kiumars Kaveh
Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA; kaveh@pitt.edu