

The cubic Dirac operator on compact quotients of the oscillator group

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Abstract. We study Kostant’s cubic Dirac operator $D^{1/3}$ on locally symmetric Lorentzian manifolds of the form $\Gamma \backslash \text{Osc}_1$, where Osc_1 is the four-dimensional oscillator group and $\Gamma \subset \text{Osc}_1$ is a cocompact lattice. These quotients are the only four-dimensional, compact Lorentzian G -homogeneous spaces for a solvable but non-abelian Lie group G . We determine the spectrum of $D^{1/3}$. We also give an explicit decomposition of the regular representation of Osc_1 on L^2 -sections of the spinor bundle into irreducible subrepresentations and we determine the eigenspaces of $D^{1/3}$.

1. Introduction

This paper is a contribution to the spectral theory of the cubic Dirac operator on compact locally symmetric Lorentzian manifolds. The cubic Dirac operator was introduced by Kostant [13] as a purely algebraic object on reductive spaces endowed with a naturally reductive metric. However, it can be also considered as a geometric Dirac operator $D^{1/3}$ belonging to a family of Dirac operators D^t . These operators are induced by metric connections with skew torsion [1, 10]. For $t = 1/2$, the corresponding connection is the Levi-Civita connection. The square of the cubic Dirac operator satisfies a simple formula. Actually it is equal to minus the Casimir operator up to terms of order zero [13]. In the special case, where the naturally reductive space is just a quotient of a Lie group G by a discrete subgroup Γ , the metric is induced by a bi-invariant metric on G . In this case, the metric connection ∇^t is characterised by its torsion T^t , which equals $T^t(X, Y) = (2t - 1)[X, Y]$ for all left-invariant vector fields X and Y on G .

The spectrum of geometric Dirac operators on pseudo-Riemannian manifolds has been calculated for some specific examples. For the Dirac operator $D = D^{1/2}$ associated with the Levi-Civita connection, the point spectrum of the pseudo-Riemannian torus $T^{p,q}$ has been computed, see [4] for $T^{1,2}$ and [15] for the general case. Kunstmann [14] studied the spectrum for pseudo-Riemannian spheres. In the case of even dimension of the manifold or even index of the metric, he computed the point spectrum and proved that the imaginary axis belongs to the continuous spectrum and that the residual spectrum is empty. Reincke [23] explicitly computed the full spectrum of D on $R^{p,q}$, the flat torus $T^{p,q}$ and products of the form $T^{1,1} \times F$, where F is an arbitrary compact, even-dimensional Riemannian spin manifold.

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Here we will consider the Dirac operator on four-dimensional compact homogeneous spaces G/Γ , where G is a solvable Lie group endowed with a bi-invariant Lorentzian metric and Γ is a (cocompact) lattice in G . More exactly, G will be the four-dimensional oscillator group.

Homogeneous spaces of the form G/Γ for solvable G not only serve as examples of compact homogeneous Lorentzian manifolds but also play a central role in their classification. Indeed, Baues and Globke [3] proved the following result. Let $M = G/H$ be a compact homogeneous pseudo-Riemannian manifold, and let G be connected and solvable. Then H is a lattice in G and the pseudo-Riemannian metric on M pulls back to a bi-invariant metric on G . In the Lorentzian case Lie groups G of this type admitting a bi-invariant metric were classified by Medina [18]. They are products of an abelian group by a so-called oscillator group, which is a certain semi-direct product of a Heisenberg group by the real line. Combining these results of Baues and Globke and Medina, one obtains a classification of all solvable Lie groups G for which there exist compact Lorentzian G -homogeneous spaces. This classification in the Lorentzian case can already be found in [25]. If we restrict ourselves to four-dimensional manifolds, we now see that the group G is isomorphic either to the abelian group \mathbb{R}^4 or to the four-dimensional oscillator group Osc_1 . Thus M is a flat Lorentzian torus or a quotient of Osc_1 by a lattice and the metric on M is induced by the bi-invariant metric on Osc_1 .

Let us explain the four-dimensional oscillator group Osc_1 in more detail. This group is a semi-direct product of the 3-dimensional Heisenberg group H by the real line \mathbb{R} , where \mathbb{R} acts trivially on the centre $Z(H)$ of H and by rotation on $H/Z(H)$. In particular, it is solvable. As mentioned above, it admits an essentially unique bi-invariant Lorentzian metric. This metric is a particular case of a plane wave metric. More details can be found in Section 3.3.

The Lie algebra \mathfrak{osc}_1 of Osc_1 is spanned by a basis X, Y, Z, T where Z spans the centre and the remaining basis elements satisfy the relations $[X, Y] = Z$, $[T, X] = Y$, $[T, Y] = -X$. This Lie algebra is strongly related to the one-dimensional quantum harmonic oscillator. Actually, the Lie algebra spanned by the differential operators $Q := x$, $P := -id/dx$, iid , and $iH = i(P^2 + Q^2)/2$ is isomorphic to \mathfrak{osc}_1 .

The oscillator group contains lattices. Each lattice L in Osc_1 gives rise to a compact locally symmetric Lorentzian manifold $L \backslash \text{Osc}_1$. Notice that for a lattice from now on we take the quotient of the left action and therefore write the subgroup on the left. The problem of classifying lattices in Osc_1 was first considered by Medina and Revoy [19]. Note, however, that the result in [19] is not correct due to a wrong description of the automorphism group of an oscillator group. Lattices of Osc_1 (as subgroups) were classified up to automorphisms of Osc_1 by Fischer [6] and up to inner automorphisms of Osc_1 by Fischer and Kath [7]. Here we will not consider arbitrary lattices but we will concentrate on basic lattices. See Section 4 for a justification of this assumption.

Let X be a quotient of Osc_1 by a basic lattice L . We fix a spin structure on X and consider the spinor bundle Σ . We use the canonical indefinite inner product on smooth sections of $\Sigma(X)$ to define a Krein space $L^2(\Sigma(X))$. Then the cubic Dirac operator is

defined on $L^2(\Sigma(X))$ and $iD^{1/3} : L^2(\Sigma(X)) \rightarrow L^2(\Sigma(X))$ is essentially selfadjoint (as an operator on a Krein space). Our aim is to determine the spectrum of this operator.

The scalar curvature of $X = L \backslash \text{Osc}_1$ is zero. This implies that the zero order terms in the formula for the square of $D^{1/3}$ vanish. So $(D^{1/3})^2 = -\Omega$, where Ω denotes the Casimir operator on $L^2(\Sigma(X))$. To determine the point spectrum of $D^{1/3}$, we will consider $L^2(\Sigma(X))$ as a representation of Osc_1 . We provide an explicit decomposition of $L^2(\Sigma(X))$ into a (discrete) direct sum of irreducible subrepresentations with finite multiplicities, see Theorem 5.2. The cubic Dirac operator preserves each summand and its square $-\Omega$ acts on it by scalar multiplication. So we first determine the eigenvalues of $-\Omega$. For this we use results from [7]. We consider a finite cover \tilde{X} of X such that the pullback of the spin structure of X to \tilde{X} becomes trivial. We proceed by decomposing $L^2(\Sigma(\tilde{X})) = L^2(\tilde{X}) \otimes \Delta$ according to [7] and then we determine the space of sections that are invariant under all deck transformations of the cover $\tilde{X} \rightarrow X$. This indeed allows us to determine the point spectrum of $-\Omega$, see Theorem 6.1. In order to determine the point spectrum of $D^{1/3}$, we finally show that all square roots of eigenvalues of $-\Omega$ are eigenvalues of $D^{1/3}$. Using the decomposition of $L^2(\Sigma(X))$, we can also prove that the entire spectrum of $D^{1/3}$ is all of \mathbb{C} . From general properties of Dirac operators, it follows, moreover, that the residual spectrum is empty.

In summary, we obtain: Let $L \subset \text{Osc}_1$ be a basic lattice and $X := L \backslash \text{Osc}_1$ be the quotient space. We endow Osc_1 with its bi-invariant metric and fix a spin structure on X . Let $D^t : L^2(\Sigma(X)) \rightarrow L^2(\Sigma(X))$ be the Dirac operator with respect to the metric connection with torsion $T^t = (2t - 1)[\cdot, \cdot]$ on \mathfrak{osc}_1 . In particular, $D^{1/3}$ is the cubic Dirac operator. Then:

- The spectrum of D^t is equal to \mathbb{C} , see Section 6.6.
- The residual spectrum of D^t is empty, see Section 6.6.
- The point spectrum of $D^{1/3}$ will be explicitly determined. It depends on the spin structure. It is discrete and contains only real and purely imaginary numbers, see Section 6.2.
- The eigenspaces of $D^{1/3}$ are determined explicitly, see Section 6.2.
- The point spectrum of D^t is computed in terms of the eigenvalues of $D^{1/3}$, see Section 6.6.

For arbitrary lattices, it is no longer true that the point spectrum is always discrete. In Section 6.5, we give examples of shifted lattices for which the point spectrum of $D^{1/3}$ on the quotient has accumulation points.

2. Basic notions

2.1. Invariant connections on Lie groups

In this short subsection, we want to recall some basic facts on connections on Lie groups. Let G be a simply-connected Lie group endowed with a bi-invariant semi-Riemannian

metric $\langle \cdot, \cdot \rangle_G$. Let \mathfrak{g} denote the Lie algebra of G . The metric $\langle \cdot, \cdot \rangle_G$ on G corresponds to an $\text{ad}(\mathfrak{g})$ -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . As a Lie group, G is endowed with a canonical left-invariant connection ∇^0 , for which all left-invariant vector fields are parallel. On $(G, \langle \cdot, \cdot \rangle_G)$, this connection is also known as Ambrose–Singer connection, i.e., the unique metric connection on G whose torsion and curvature are parallel.

We use ∇^0 to define a one-parameter family of connections ∇^t by

$$\nabla_X^t Y = \nabla_X^0 Y + t[X, Y] = t[X, Y]$$

for left-invariant vector fields X, Y . These connections are metric since $\langle \cdot, \cdot \rangle$ is $\text{ad}(\mathfrak{g})$ -invariant. The torsion of ∇^t equals $T^t(X, Y) = (2t - 1)[X, Y]$ for all left-invariant vector fields on G . The $\text{ad}(\mathfrak{g})$ -invariance of $\langle \cdot, \cdot \rangle$ implies that the tensor $\langle T^t(\cdot, \cdot), \cdot \rangle_G$ is skew-symmetric and that all connections ∇^t define the same divergence operator on vector fields. For $t = 1/2$, ∇^t is torsion-free and metric. Thus it is the Levi-Civita connection of $\langle \cdot, \cdot \rangle_G$.

Let us explain the relation between the connection ∇^t of G and the canonical connections of G understood as a reductive homogeneous space K/H for $K := G \times G$ and $H := \Delta G \subset G \times G$. Let \mathfrak{k} and \mathfrak{h} denote the Lie algebras of K and H , respectively. Then

$$\mathfrak{m}_t := \{((t - 1)v, tv) \mid v \in \mathfrak{g}\} \subset \mathfrak{k}, \quad t \in \mathbb{R}$$

is a one-parameter family of $\text{Ad}(H)$ -invariant complements of \mathfrak{h} in \mathfrak{k} . Each of the decompositions $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_t$ defines a canonical connection on $G = K/H$ in the sense of Wang. This canonical connection coincides with the left-invariant connection ∇^t of the Lie group G introduced above. For more information, see [2, Section 5.2].

2.2. Spin structures on quotients of Lie groups

Here we gather some facts on spin structures on quotients of Lie groups by discrete subgroups. Let G be a simply-connected Lie group endowed with a bi-invariant semi-Riemannian metric $\langle \cdot, \cdot \rangle_G$, an orientation and a time orientation. As above, let \mathfrak{g} denote the Lie algebra of G and $\langle \cdot, \cdot \rangle$ the induced scalar product on \mathfrak{g} .

Let $\Gamma \subset G$ be a discrete subgroup and consider the quotient $\Gamma \backslash G$. The metric, the orientation and the time orientation on G induce a metric, an orientation and a time orientation on the quotient. The tangent bundle of $\Gamma \backslash G$ is trivial. Let $\pi : G \rightarrow \Gamma \backslash G$ denote the projection, then

$$\Gamma \backslash G \times \mathfrak{g} \longrightarrow T(\Gamma \backslash G), \quad (\Gamma g, X) \longmapsto d\pi dL_g(X)$$

is an isomorphism of vector bundles. Consequently, the bundle of oriented and time-oriented orthonormal frames on $\Gamma \backslash G$ equals

$$P_{\text{SO}^+}(\Gamma \backslash G) = \Gamma \backslash G \times \text{SO}^+(\mathfrak{g}),$$

where $\text{SO}^+(\mathfrak{g})$ denotes the identity component of $\text{SO}(\mathfrak{g}) := \text{SO}(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. The spin struc-

tures on $\Gamma \backslash G$ are classified by $H^1(\Gamma \backslash G, \mathbb{Z}_2)$, see [4, Satz 2.6.] or [8]. Since

$$H^1(\Gamma \backslash G, \mathbb{Z}_2) = \text{Hom}(H_1(\Gamma \backslash G), \mathbb{Z}_2) = \text{Hom}(\pi_1(\Gamma \backslash G), \mathbb{Z}_2) = \text{Hom}(\Gamma, \mathbb{Z}_2),$$

we obtain a one-to-one correspondence between isomorphism classes of spin structures on $\Gamma \backslash G$ and homomorphisms $\varepsilon : \Gamma \rightarrow \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, see [8, Section 2.2] for the more general case of a covering map. The spin structure corresponding to ε equals

$$P_{\text{Spin}^+}^\varepsilon(\Gamma \backslash G) := G \times_{\Gamma, \varepsilon} \text{Spin}^+(\mathfrak{g}) := \Gamma \backslash (G \times \text{Spin}^+(\mathfrak{g})),$$

where the action of Γ on $G \times \text{Spin}^+(\mathfrak{g})$ is given by

$$\gamma \cdot (g, \alpha) = (\gamma g, (-1)^{\varepsilon(\gamma)} \alpha),$$

see [4, Folgerung 2.3] or [9, Proposition 1.4.2]. Let $\Delta_{\mathfrak{g}}$ denote the spinor representation of $\text{Spin}^+(\mathfrak{g})$. We obtain

$$\Sigma(\Gamma \backslash G) = G \times_{\Gamma, \varepsilon} \Delta_{\mathfrak{g}}$$

for the spinor bundle associated with $P_{\text{Spin}^+}^\varepsilon(\Gamma \backslash G)$. We identify vector fields on $\Gamma \backslash G$ with functions $X : G \rightarrow \mathfrak{g}$ that are invariant under left translation by Γ and we identify smooth sections of $\Sigma(\Gamma \backslash G)$ with smooth functions $\psi : G \rightarrow \Delta_{\mathfrak{g}}$ that are invariant under the action of Γ , that is

$$\psi(\gamma g) = (-1)^{\varepsilon(\gamma)} \psi(g) \tag{2.1}$$

for all $\gamma \in \Gamma$.

2.3. The cubic Dirac operator

Every connection ∇^t on G induces a connection on $\Sigma(\Gamma \backslash G)$, which we will also denote by ∇^t . Consider a smooth section $\psi : G \rightarrow \Delta_{\mathfrak{g}}$ of $\Sigma(\Gamma \backslash G)$, see (2.1). Let $X \in \mathfrak{g}$ be a left-invariant vector field on G . Then X can also be considered as a vector field on $\Gamma \backslash G$. Let “ \cdot ” denote the Clifford multiplication. Then $\nabla_X^t \psi = X(\psi) - (t/2)\gamma(X) \cdot \psi$, where

$$\gamma(X) = - \sum_{a < b} \langle X, [e_a, e_b] \rangle e^a e^b$$

[20, p. 164]. Here and in the following, $\{e_a \mid a = 1, \dots, n\}$ denotes a basis of \mathfrak{g} and $\{e^a \mid a = 1, \dots, n\}$ its dual basis with respect to $\langle \cdot, \cdot \rangle$. These elements of \mathfrak{g} can also be understood as vector fields on $\Gamma \backslash G$ or, equivalently, as constant maps $G \rightarrow \mathfrak{g}$, $g \mapsto e^a$ and $G \rightarrow \mathfrak{g}$, $g \mapsto e_a$.

The Dirac operator corresponding to ∇^t is equal to

$$D^t = \sum e^a \cdot \nabla_{e_a}^t.$$

If we apply this to a smooth section ψ of the spinor bundle, we obtain

$$D^t \psi = \sum e^a \cdot e_a(\psi) - \frac{3t}{2} \hat{\gamma} \cdot \psi, \tag{2.2}$$

for

$$\hat{\gamma} = - \sum_{a < b < c} \langle [e_a, e_b], e_c \rangle e^a e^b e^c$$

where $e^a e^b e^c$ is understood as an element of the Clifford algebra [1, equation (5)]. If we use multiplication of spinors by alternating differential forms instead of multiplication by elements of the Clifford algebra, we obtain

$$\gamma(X) \cdot \psi = T^0(X, \cdot, \cdot) \cdot \psi, \quad \hat{\gamma} \cdot \psi = T^0 \cdot \psi, \tag{2.3}$$

where T^0 is the torsion of ∇^0 .

For $t = 1/3$ we obtain the cubic Dirac operator $D^{1/3}$. The square of this operator is related to the Casimir operator $\Omega = \sum e_a e^a \in \mathcal{U}(\mathfrak{g})$ with respect to $\langle \cdot, \cdot \rangle$ by

$$(D^{1/3})^2 = -\Omega - \frac{1}{24} \text{tr ad}(\Omega) = -\Omega + \frac{1}{24} \sum_{a,b} \langle [e_a, e_b], [e^a, e^b] \rangle = -\Omega + \frac{1}{6} \text{Scal},$$

see [13]. Here Scal denotes the scalar curvature of $\langle \cdot, \cdot \rangle_G$ on $\Gamma \backslash G$.

2.4. Krein spaces

As we will see in the next subsection, the natural scalar product on the spinor bundle of a Lorentzian spin manifold is indefinite. So sections of the spinor bundle do not constitute a Hilbert space in a natural way. Therefore we will work in Krein spaces as it is done in [4]. For a general theory of such spaces see [5, 16].

Let K be a complex vector space and $\langle \cdot, \cdot \rangle$ a possibly indefinite inner product on K . We define symmetric operators and selfadjoint operators on K in the same way as in the definite case.

Definition 2.1. A Krein space $(K, \langle \cdot, \cdot \rangle)$ consists of a complex vector space K and an indefinite inner product $\langle \cdot, \cdot \rangle$ on K such that there exists a selfadjoint linear map $J : K \rightarrow K$ with the following properties:

- (1) $(\cdot, \cdot) := \langle \cdot, J \cdot \rangle$ is a positive definite inner product that makes K a Hilbert space,
- (2) $J^2 = \text{id}$.

A linear map J that satisfies this condition is called a *fundamental symmetry*.

On K , we consider the strong topology. It is defined to be the norm topology of the Hilbert space $(K, (\cdot, \cdot))$, where $(\cdot, \cdot) = \langle \cdot, J \cdot \rangle$ for any fundamental symmetry J . Although, in general, the linear map J is not uniquely determined by $(K, \langle \cdot, \cdot \rangle)$, the strong topology is well defined, i.e., independent of J .

If A is a closed linear operator with a dense domain, then $\text{spec}(A)$ denotes the spectrum of A and $\text{spec}_p(A)$, $\text{spec}_c(A)$, and $\text{spec}_r(A)$ denote the discrete, continuous, and residual spectra, respectively. If A is a closable operator, then we mean by the spectrum of A always the spectrum of its closure.

Fact 2.2 ([16]). *If A is a closed selfadjoint operator on a Krein space, then the complex conjugate of $\text{spec}(A)$ satisfies*

- (1) $\overline{\text{spec}_r(A)} \subset \text{spec}_p(A)$,
- (2) $\overline{\text{spec}_p(A)} \subset \text{spec}_p(A) \cup \text{spec}_r(A)$,
- (3) $\overline{\text{spec}_c(A)} \subset \text{spec}_c(A)$.

2.5. Dirac operators on Krein spaces

On Riemannian spin manifolds, there exists a scalar product on the spinor bundle such that the Dirac operator is essentially selfadjoint. On pseudo-Riemannian manifolds this is not the case. Therefore let us first recall some basic facts on inner products on the space of spinor fields on a pseudo-Riemannian manifold. Since we are interested in Dirac operators on Lorentzian manifolds, we restrict the explanations to the case where the metric of the manifold has Lorentzian signature. Moreover, we will concentrate on the case where the manifold is a quotient of a Lie group G by a discrete subgroup Γ although most of the results could be stated as well for general Lorentzian manifolds.

Then, on the spinor module $\Delta_{\mathfrak{g}}$, there exists a scalar product $\langle \cdot, \cdot \rangle_{\Delta}$ of split signature such that

$$\langle X \cdot u, v \rangle_{\Delta} = \langle u, X \cdot v \rangle_{\Delta} \tag{2.4}$$

for all $X \in \mathfrak{g}$, where “ \cdot ” denotes the Clifford multiplication. This scalar product is unique up to multiplication by a real number different from zero. It defines a scalar product on the bundle $\Sigma(\Gamma \backslash G)$, which we also denote by $\langle \cdot, \cdot \rangle_{\Delta}$. We choose a time-oriented left-invariant vector field ξ on G with $\langle \xi, \xi \rangle_G = -1$. This vector field defines a vector field on the quotient $\Gamma \backslash G$, which we also denote by ξ . We use ξ to define a map

$$J_{\xi} : \Sigma(\Gamma \backslash G) \longrightarrow \Sigma(\Gamma \backslash G), \quad u \longmapsto \xi \cdot u$$

and a positive definite scalar product $\langle \cdot, \cdot \rangle_{\Delta, \xi} := \langle \cdot, J_{\xi}(\cdot) \rangle_{\Delta}$. It satisfies

$$\langle \xi \cdot u, v \rangle_{\Delta, \xi} = \langle u, \xi \cdot v \rangle_{\Delta, \xi}, \quad \langle X \cdot u, v \rangle_{\Delta, \xi} = -\langle u, X \cdot v \rangle_{\Delta, \xi}, \quad X \in \xi^{\perp}.$$

The stabiliser of a timelike vector is a maximal compact subgroup of the Lorentz group. Therefore the vector field ξ defines a reduction of the frame bundle of $\Gamma \backslash G$ to a maximal compact subgroup of the Lorentz group. The scalar product $\langle \cdot, \cdot \rangle_{\Delta, \xi}$ is invariant under this subgroup.

The volume form of the metric $\langle \cdot, \cdot \rangle_G$ induces a measure μ on $\Gamma \backslash G$, which is invariant under G . We define inner products on the space of compactly supported smooth sections of $\Sigma(\Gamma \backslash G)$ by

$$\langle \varphi, \psi \rangle := \int_{\Gamma \backslash G} \langle \varphi, \psi \rangle_{\Delta} d\mu, \quad (\varphi, \psi)_{\xi} := \int_{\Gamma \backslash G} (\varphi, \psi)_{\Delta, \xi} d\mu = \langle \varphi, J_{\xi} \circ \psi \rangle.$$

The first one is indefinite, the second one is positive definite.

We can identify spinors with smooth functions with values in $\Delta_{\mathfrak{g}}$ satisfying (2.1). If we identify, in addition, $(\Delta_{\mathfrak{g}}, \langle \cdot, \cdot \rangle_{\Delta, \xi})$ with the standard unitary space by choosing an orthonormal basis, then the scalar product $\langle \cdot, \cdot \rangle_{\xi}$ on smooth sections of $\Sigma(\Gamma \backslash G)$ becomes the standard L^2 -product on functions (with several components).

We define $L^2_{\xi}(\Sigma(\Gamma \backslash G))$ as the completion of the space of compactly supported smooth sections in $\Sigma(\Gamma \backslash G)$ with respect to the norm induced by $\langle \cdot, \cdot \rangle_{\xi}$.

We want to compare the spaces $L^2_{\xi}(\Sigma(\Gamma \backslash G))$ for different choices of ξ . Let r_{ξ} be the left-invariant Riemannian metric on G defined by reversing the sign of $\langle \cdot, \cdot \rangle_G$ in direction of ξ . More exactly,

$$r_{\xi}(s\xi + X, t\xi + Y) = st + \langle X, Y \rangle_G$$

for X, Y in the orthogonal complement of ξ with respect to $\langle \cdot, \cdot \rangle_G$. Let ξ_1 and ξ_2 be time-oriented left-invariant vector fields with $\langle \xi_1, \xi_1 \rangle = \langle \xi_2, \xi_2 \rangle = -1$. Since r_{ξ_1} and r_{ξ_2} are left-invariant, they are quasi-isometric, i.e., there exists a constant $C > 0$ such that

$$\frac{1}{C} r_{\xi_1}(X, X) \leq r_{\xi_2}(X, X) \leq C r_{\xi_1}(X, X)$$

for all $X \in TG$. Of course, also the metrics induced by r_{ξ_1} and r_{ξ_2} on $\Gamma \backslash G$ are quasi-isometric. This implies that the spaces $L^2_{\xi_1}(\Sigma(\Gamma \backslash G))$ and $L^2_{\xi_2}(\Sigma(\Gamma \backslash G))$ are the same in the following sense. They coincide as vector spaces (whose elements are equivalence classes of Cauchy series in the space of compactly supported smooth sections of $\Sigma(\Gamma \backslash G)$), and the identity $I : L^2_{\xi_1}(\Sigma(\Gamma \backslash G)) \rightarrow L^2_{\xi_2}(\Sigma(\Gamma \backslash G))$ is a bounded isomorphism with bounded inverse [23, Theorem 3.8].

Let us fix ξ as above and put $K := L^2(\Sigma(\Gamma \backslash G)) := L^2_{\xi}(\Sigma(\Gamma \backslash G))$ as a vector space. The map J_{ξ} can be extended to K , and we can define an indefinite inner product $\langle \cdot, \cdot \rangle = \langle \cdot, J_{\xi} \cdot \rangle$. Then $(K, \langle \cdot, \cdot \rangle)$ is a Krein space [4, Satz 3.16]. Its definition is independent of ξ . For any time-oriented left-invariant vector field ξ' the map $J_{\xi'}$ is a fundamental symmetry in the sense of Definition 2.1.

Now we consider the Dirac operator D^t defined by (2.2) in the case where $\langle \cdot, \cdot \rangle_G$ is Lorentzian. We can think of D^t as an operator on K . Then $iD^t : K \rightarrow K$ is essentially self-adjoint. This can be seen as follows. We have noticed that ∇^t defines the same divergence operator as the Levi-Civita connection $\nabla^{1/2}$. Furthermore, the Riemannian metric r_{ξ} on $\Gamma \backslash G$ that is obtained by reversing the sign in direction of a time-oriented left-invariant vector field ξ is complete. Indeed, r_{ξ} is left-invariant on G , hence (G, r_{ξ}) is a homogeneous Riemannian manifold and therefore complete. Hence $(\Gamma \backslash G, r_{\xi})$ is also complete. Now the assertion follows from [4, Satz 3.19].

2.6. The right regular representation

Let $(G, \langle \cdot, \cdot \rangle_G)$ be as above and let Γ be a cocompact discrete subgroup of G . The right regular representation ρ of G on $L^2(\Gamma \backslash G)$ is the unitary representation given by

$$(\rho(g)(\varphi))(x) = \varphi(xg). \tag{2.5}$$

It is a classical result that $(\rho, L^2(\Gamma \backslash G))$ is a discrete direct sum of irreducible unitary representations of G with finite multiplicities, see e.g. [24].

Let F be an automorphism of G . For a representation (σ, V) of G we define a representation

$$F^*(\sigma, V) := (\sigma \circ F, V). \tag{2.6}$$

Then

$$L^2(\Gamma \backslash G) \xrightarrow{\sim} F^*(L^2(F(\Gamma) \backslash G)), \quad f \mapsto f \circ F^{-1} \tag{2.7}$$

is an equivalence of representations.

Recall that a smooth section of the spinor bundle $\Sigma(\Gamma \backslash G)$ is identified with a smooth Γ -invariant function $\varphi : G \rightarrow \Delta_{\mathfrak{g}}$. In this way we can also define an action of G on $L^2(\Sigma(\Gamma \backslash G))$ by (2.5).

3. The oscillator group and its Lie algebra

3.1. The oscillator group

The 4-dimensional oscillator group is a semi-direct product of the 3-dimensional Heisenberg group H by the real line. Usually, the Heisenberg group H is defined as the set $H = \mathbb{C} \times \mathbb{R}$ with multiplication given by

$$(\xi_1, z_1) \cdot (\xi_2, z_2) = (\xi_1 + \xi_2, z_1 + z_2 + \frac{1}{2}\omega(\xi_1, \xi_2)),$$

where $\omega(\xi_1, \xi_2) := \Im(\overline{\xi_1}\xi_2)$. Hence in explicit terms, the oscillator group is understood as the set $\text{Osc}_1 = H \times \mathbb{R}$ with multiplication defined by

$$(\xi_1, z_1, t_1) \cdot (\xi_2, z_2, t_2) = (\xi_1 + e^{it_1}\xi_2, z_1 + z_2 + \frac{1}{2}\omega(\xi_1, e^{it_1}\xi_2), t_1 + t_2).$$

If we identify $\text{Osc}_1 \cong \mathbb{R}^4$ as sets, then the Lebesgue measure is left- and right-invariant with respect to multiplication in Osc_1 .

Let us consider the automorphisms of this group. For $\eta \in \mathbb{C}$, let $C_\eta : \text{Osc}_1 \rightarrow \text{Osc}_1$ be the conjugation by $(\eta, 0, 0)$. Then

$$C_\eta : (\xi, z, t) \mapsto (\xi + \eta - e^{it}\eta, z + \frac{1}{2}\omega(\eta + \xi, \xi - e^{it}\eta), t).$$

Furthermore, we define an automorphism T_u of Osc_1 for $u \in \mathbb{R}$ by

$$T_u : (\xi, z, t) \mapsto (\xi, z + ut, t). \tag{3.1}$$

Finally, consider an \mathbb{R} -linear isomorphism S of \mathbb{C} such that $S(i\xi) = \epsilon i S(\xi)$ for an element $\epsilon \in \{1, -1\}$ and for all $\xi \in \mathbb{C}$. Then $\epsilon = \text{sgn}(\det S)$ and also

$$F_S : (\xi, z, t) \mapsto (S\xi, \det(S)z, \epsilon t) \tag{3.2}$$

is an automorphism of Osc_1 . Each automorphism F of Osc_1 is of the form

$$F = T_u \circ C_\eta \circ F_S$$

for suitable $u \in \mathbb{R}$, $\eta \in \mathbb{C}$ and $S \in \text{GL}(2, \mathbb{R})$ as considered above [6]. Besides C_η also F_S is an inner automorphism if $S \in \text{SO}(2, \mathbb{R})$.

In some of our computations we will use a slightly different multiplication rule for the oscillator group. It looks more complicated than the usual one but it will make the computations easier. We use the well known fact that the Heisenberg group H is isomorphic to the set $H(1)$ of elements $M(x, y, z)$ parametrized by $x, y, z \in \mathbb{R}$ with group multiplication

$$M(x, y, z)M(x', y', z') = M(x + x', y + y', z + z' + xy').$$

We define an action l of \mathbb{R} on $H(1)$ by

$$l(t)(M(x, y, z)) = M(x \cos t - y \sin t, x \sin t + y \cos t, z + \frac{xy}{2}(\cos(2t) - 1) + \frac{x^2 - y^2}{4} \sin(2t))$$

and consider the semi-direct product

$$\text{Osc}_1^M := H(1) \rtimes_l \mathbb{R}. \tag{3.3}$$

The image of an element $t \in \mathbb{R}$ under the identification of \mathbb{R} with the second factor of G in (3.3) is denoted by (t) . It is easy to check that

$$\phi : \text{Osc}_1 \longrightarrow \text{Osc}_1^M, \quad (x + iy, z, t) \longmapsto M(-y, x, z - \frac{1}{2}xy)(t) \tag{3.4}$$

is an isomorphism. Again we can identify $\text{Osc}_1^M \cong \mathbb{R}^4$. Then ϕ preserves the Lebesgue measure.

3.2. The oscillator algebra

The Lie algebra $\mathfrak{o}\mathfrak{s}c_1$ of the four-dimensional oscillator group is spanned by elements Z, X, Y, T , whose non-vanishing commutators are

$$[T, X] = Y, \quad [T, Y] = -X, \quad [X, Y] = Z.$$

The following result about the centre of the universal enveloping algebra of \mathfrak{g} is known, see [21]. We give a short self-contained proof.

Proposition 3.1. *The centre $Z(\mathcal{U}(\mathfrak{g}))$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} is generated by $\Omega_0 := X^2 + Y^2 + 2ZT$ and $Z \in \mathfrak{g}$.*

Proof. Obviously, Ω_0 belongs to $Z(\mathcal{U}(\mathfrak{g}))$. By the Poincaré–Birkhoff–Witt theorem, the symmetrisation map

$$\text{sym} : S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow Z(\mathcal{U}(\mathfrak{g})) \tag{3.5}$$

induces an isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \cong \text{gr}(Z(\mathcal{U}(\mathfrak{g})))$. Hence it suffices to show that $S(\mathfrak{g})^{\mathfrak{g}}$ is generated by the centre of \mathfrak{g} and the preimage of Ω_0 under sym . Obviously, it suffices to show this for any $S^k(\mathfrak{g})^{\mathfrak{g}}$.

Instead of \mathfrak{g} , we consider its complexification $\mathfrak{g}_{\mathbb{C}}$. The vectors $Z, T, N_+ := X + iY, N_- := X - iY$ constitute a basis of $\mathfrak{g}_{\mathbb{C}}$. Their non-vanishing Lie brackets are

$$[T, N_+] = -iN_+, \quad [T, N_-] = iN_-, \quad [N_+, N_-] = -2iZ.$$

In this new basis, we have $\Omega_0 = N_+N_- + 2ZT + iZ$. Hence $\Omega_S := N_+N_- + 2ZT \in S(\mathfrak{g})^{\mathfrak{g}}$ is a preimage of Ω_0 under sym . Let ω be in $S(\mathfrak{g})^{\mathfrak{g}}$ and assume that ω is homogeneous. Then $\omega = \sum_{k,l} N_+^k p_{k,l}(Z, T) N_-^l$, where $p_{k,l}$ is a homogeneous polynomial in Z and T . Since

$$\text{ad}(T)(N_+^k p(Z, T) N_-^l) = i(l - k) N_+^k p(Z, T) N_-^l,$$

we obtain $\omega = \sum_{k=0}^n N_+^k p_k(Z, T) N_-^k$ if ω is invariant. Moreover, in this case

$$\text{ad}(N_+)(N_+^k p(Z, T) N_-^k) = i N_+^{k+1} \frac{\partial}{\partial T} p(Z, T) N_-^k - 2ik N_+^k Z p(Z, T) N_-^{k-1}$$

yields $\frac{\partial}{\partial T} p_n(Z, T) = 0$, thus $p_n(Z, T) = a_n Z^{m_n}$ for some $m_n \in \mathbb{N}$. Now we consider $\omega' := \omega - a_n \Omega_S^n Z^{m_n}$. Then ω' is in $S(\mathfrak{g})^{\mathfrak{g}}$ and of the form $\omega' = \sum_{k=0}^{n-1} N_+^k p'_k(Z, T) N_-^k$. We proceed inductively and obtain

$$\omega = a_n \Omega_S^n Z^{m_n} + a_{n-1} \Omega_S^{n-1} Z^{m_{n-1}} + \dots + a_0 Z^{m_0}. \quad \blacksquare$$

Remark 3.2. The symmetrisation map (3.5) is not a homomorphism. For the sake of completeness let us determine Duflo’s factor for \mathfrak{g} although we will not use it in the present paper. See [20] for a general introduction to this subject. For $\xi = zZ + n_+N_+ + n_-N_- + tT$, we have

$$\text{ad}(\xi) = \begin{pmatrix} 0 & 2in_- & -2in_+ & 0 \\ 0 & -it & 0 & in_+ \\ 0 & 0 & it & -in_- \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis Z, N_+, N_-, T . This gives $J(\xi) = \det(j(\text{ad}(\xi))) = j(-it)j(it)$, where

$$j(z) = \frac{\sinh z/2}{z/2}.$$

Hence Duflo’s factor equals

$$J^{1/2}(\xi) = \frac{\sinh(it/2)}{it/2} = \frac{\sin(t/2)}{t/2}.$$

3.3. The bi-invariant metric and the cubic Dirac operator

On Osc_1 , there exists a 2-parameter family of bi-invariant metrics. The metrics are defined by the ad-invariant scalar products on $\mathfrak{o} \oplus \mathfrak{c}_1$ given by $\text{span}\{X, Y\} \perp \text{span}\{Z, T\}$, and

$$\langle X, X \rangle = \langle Y, Y \rangle = r, \quad \langle X, Y \rangle = \langle Z, Z \rangle = 0, \quad \langle T, T \rangle = s, \quad \langle Z, T \rangle = r$$

for $r > 0$ and $s \in \mathbb{R}$. It is well known that there is only one bi-invariant Lorentzian metric on Osc_1 up to isometric Lie group isomorphisms [19]. The above defined family of metrics arises as the orbit of such a metric under the action of the automorphism group of Osc_1 . The Casimir operator corresponding to the metric with parameters $r > 0$ and $s \in \mathbb{R}$ is equal to $\frac{1}{r}(\Omega_0 - sZ^2)$, where $\Omega_0 \in Z(\mathcal{U}(\mathfrak{g}))$ is as defined in Proposition 3.1.

Each isometric isomorphism $(G, \langle \cdot, \cdot \rangle_G) \rightarrow (G, \langle \cdot, \cdot \rangle'_G)$ defines an isomorphism of the associated spinor bundles. By equation (2.2), this isomorphism commutes with the cubic Dirac operator. In particular, the spectrum of D^t is independent of the choice of the metric in this 2-parameter family. We make it as simple as possible and set $r = 1$ and $s = 0$, i.e.,

$$\langle X, X \rangle = \langle Y, Y \rangle = \langle Z, T \rangle = 1, \quad \langle X, Y \rangle = \langle Z, Z \rangle = \langle T, T \rangle = 0.$$

The Casimir operator Ω of this metric equals

$$\Omega = \Omega_0 = X^2 + Y^2 + 2ZT.$$

The dual basis of $e_1 = Z, e_2 = X, e_3 = Y, e_4 = T$ is $e^1 = T, e^2 = X, e^3 = Y, e^4 = Z$. Hence the scalar curvature Scal^t of ∇^t induced on G vanishes since

$$\text{Scal}^t = (t - t^2) \sum_{a,b} \langle [e_a, e_b], [e^a, e^b] \rangle = 0.$$

In particular, this yields $(D^{1/3})^2 = -\Omega$ for the square of the cubic Dirac operator.

Now we examine the actual Dirac operator $D^{1/3}$ according to (2.2). The element $\hat{\gamma}$ corresponding to the torsion T^0 by (2.3) is equal to

$$\hat{\gamma} = - \sum_{a < b < c} \langle [e_a, e_b], e_c \rangle e^a e^b e^c = -\langle [X, Y], T \rangle XYZ = -XYZ.$$

Thus we obtain

$$D^{1/3}\psi = X \cdot X(\psi) + Y \cdot Y(\psi) + Z \cdot T(\psi) + T \cdot Z(\psi) + \frac{1}{2}XYZ \cdot \psi.$$

Let $\Delta = \mathbb{C}^4$ denote the spinor module of the metric Lie algebra \mathfrak{osc}_1 . We can choose a basis u_1, \dots, u_4 of Δ such that the Clifford multiplication by Z, X, Y and T is given with respect to this basis by

$$Z = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad T = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \quad X = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & iC \\ -iC & 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

In particular,

$$\hat{\gamma} = -XYZ = -i \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}. \tag{3.6}$$

According to Section 2.5, there is an indefinite scalar product $\langle \cdot, \cdot \rangle_\Delta$ on Δ satisfying (2.4) and this scalar product is uniquely defined up to a constant. We fix it by $\langle u_1, u_2 \rangle_\Delta = \langle u_3, u_4 \rangle_\Delta = 1$, and $\langle u_i, u_j \rangle_\Delta = 0$ for all other indices.

We choose the timelike left-invariant vector field $\xi = \frac{1}{\sqrt{2}}(Z - T)$ in order to define a definite $\mathfrak{spin}(3)$ -invariant scalar product: $(u, v)_\Delta := \langle u, \xi \cdot v \rangle_\Delta$. The vectors u_1, \dots, u_4 constitute an orthonormal basis with respect to $(\cdot, \cdot)_\Delta$.

3.4. Unitary representations of the oscillator group

The irreducible unitary representations of \mathfrak{osc}_1 can be determined by applying a generalised version of Kirillov’s orbit method. An explicit description of these representations can be found in [12, Section 4.3], where the oscillator Lie algebra is called diamond Lie algebra. Let us recall this description. Note that the case $c < 0$ in item (iii) does not appear in [12]. The infinite-dimensional representations will be given only on the Lie algebra level.

Every irreducible unitary representation of the oscillator group is equivalent to one of the following representations, see also [7]:

- (i) $\mathcal{C}_d := (\sigma_d, \mathbb{C}), \sigma_d(\xi, z, t) = e^{2\pi i d t}, d \in \mathbb{R}$,
- (ii) $\mathcal{S}_a^\tau := (\sigma := \sigma_a^\tau, L^2(S^1)), a > 0, \tau \in \mathbb{R}/\mathbb{Z} \cong [0, 1)$ where σ is given by

$$\begin{aligned} \sigma_*(Z)(\varphi) &= 0, \\ \sigma_*(X + iY)(\varphi) &= 2\pi i a e^{-it} \varphi, \\ \sigma_*(X - iY)(\varphi) &= 2\pi i a e^{it} \varphi, \\ \sigma_*(T)(\varphi) &= \varphi' + i\tau \varphi \end{aligned}$$

for $\varphi = \varphi(t) \in C^\infty(S^1) \subset L^2(S^1)$. The orthonormal system $\phi_n := e^{int}, n \in \mathbb{Z}$ satisfies

$$\begin{aligned} \sigma_*(X + iY)(\phi_n) &= 2\pi i a \phi_{n-1}, \\ \sigma_*(X - iY)(\phi_n) &= 2\pi i a \phi_{n+1}, \\ \sigma_*(T)(\phi_n) &= i(n + \tau)\phi_n. \end{aligned}$$

- (iii) For $c > 0, d \in \mathbb{R}$, we consider the Hilbert space

$$\mathcal{F}_c(\mathbb{C}) := \left\{ \varphi : \mathbb{C} \rightarrow \mathbb{C} \text{ holomorphic} \mid \int_{\mathbb{C}} |\varphi(\xi)|^2 e^{-\pi c |\xi|^2} c d\xi < \infty \right\}$$

with scalar product

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\mathbb{C}} \varphi_1(\xi) \overline{\varphi_2(\xi)} e^{-\pi c |\xi|^2} c d\xi \tag{3.7}$$

for $\varphi_1, \varphi_2 \in \mathcal{F}_c(\mathbb{C})$. Then the representation $\sigma := \sigma_{c,d}$ on $\mathcal{F}_c(\mathbb{C})$ is given by

$$\begin{aligned} \sigma_*(Z)(\varphi) &= 2\pi ic\varphi, \\ \sigma_*(X + iY)(\varphi) &= 2\pi c\xi\varphi, \\ \sigma_*(X - iY)(\varphi) &= -2\partial\varphi, \\ \sigma_*(T)(\varphi) &= 2\pi id\varphi - i\xi\partial\varphi. \end{aligned}$$

The functions $\psi_n := \frac{(\sqrt{\pi c\xi})^n}{\sqrt{n!}}$, $n \geq 0$, constitute a complete orthonormal system of $\mathcal{F}_c(\mathbb{C})$ and we have

$$\sigma_*(Z)(\psi_n) = 2\pi ic\psi_n, \quad \sigma_*(T)(\psi_n) = (2\pi d - n)i\psi_n$$

and, for $A_+ := \sigma_*(X + iY)$ and $A_- := \sigma_*(X - iY)$,

$$A_+(\psi_n) = 2\sqrt{\pi c(n + 1)}\psi_{n+1}, \quad n \geq 0, \tag{3.8}$$

$$A_-(\psi_0) = 0, \quad A_-(\psi_n) = -2\sqrt{\pi cn}\psi_{n-1}, \quad n \geq 1. \tag{3.9}$$

Furthermore, for $c < 0, d \in \mathbb{R}$, we consider

$$\mathcal{F}_c(\mathbb{C}) := \left\{ \varphi : \mathbb{C} \rightarrow \mathbb{C} \text{ anti-holomorphic} \mid \int_{\mathbb{C}} |\varphi(\xi)|^2 e^{\pi c|\xi|^2} |c|d\xi < \infty \right\}$$

with scalar product given by (3.7) with c replaced by $-c$, now for $\varphi_1, \varphi_2 \in \mathcal{F}_c(\mathbb{C})$. The representation $\sigma := \sigma_{c,d}$ on $\mathcal{F}_c(\mathbb{C})$ is given by

$$\begin{aligned} \sigma_*(Z)(\varphi) &= 2\pi ic\varphi, \\ \sigma_*(X + iY)(\varphi) &= -2\bar{\partial}\varphi, \\ \sigma_*(X - iY)(\varphi) &= -2\pi c\bar{\xi}\varphi, \\ \sigma_*(T)(\varphi) &= 2\pi id\varphi + i\bar{\xi}\bar{\partial}\varphi. \end{aligned}$$

Here, the functions $\psi_n := \frac{(\sqrt{\pi|c|\bar{\xi}})^n}{\sqrt{n!}}$, $n \geq 0$, constitute a complete orthonormal system and we have

$$\sigma_*(Z)(\psi_n) = 2\pi ic\psi_n, \quad \sigma_*(T)(\psi_n) = (2\pi d + n)i\psi_n.$$

Now, (3.8) and (3.9) hold for $A_+ := \sigma_*(X - iY)$ and $A_- := \sigma_*(X + iY)$. We will use the notation $\mathcal{F}_{*,d} := (\sigma_{c,d}, \mathcal{F}_c(\mathbb{C}))$ for all $c \neq 0$ and $d \in \mathbb{R}$.

Let F be an automorphism of Osc_1 . In (2.6), we defined the pullback of a representation of G by F . Table 1 shows (the equivalence class of) $F^*(\sigma, V)$ for the case that V is one of the irreducible unitary representations of \mathfrak{osc}_1 and F is one of the (outer) automorphisms T_u or F_S introduced in Section 3.1 by (3.1) and (3.2). Note that C_η as an inner automorphism does not change (σ, V) .

F	$F^*\mathcal{C}_d$	$F^*\mathcal{S}_a^\tau$	$F^*\tilde{\mathcal{F}}_{c,d}$
T_u	\mathcal{C}_d	\mathcal{S}_a^τ	$\tilde{\mathcal{F}}_{c,d+uc}$
F_S	$\mathcal{C}_{\epsilon d}$	$\mathcal{S}_{ \det S ^{1/2}a}^{\epsilon\tau}$	$\tilde{\mathcal{F}}_{\det(S)c,\epsilon d}$

Table 1. Action of outer automorphisms.

4. Straight and basic lattices

In Section 6 we will study the spectrum of the cubic Dirac operator on compact quotients of the oscillator group. More exactly, we consider quotients of Osc_1 by discrete uniform subgroups of Osc_1 . We will call such subgroups lattices. This is justified by the fact that the group Osc_1 is solvable and therefore a quotient by a discrete subgroup is of finite measure (for the measure inherited from Haar measure on Osc_1) if and only if it is compact. The lattices of the oscillator group are known. They were classified up to automorphisms of Osc_1 by Fischer [6]. Since here we are interested in the spectrum of the quotient and therefore in the right regular representation, we need a classification up to inner automorphisms, which can be found in [7].

To avoid too much technical effort, we will concentrate on straight lattices, where a lattice in Osc_1 is called *straight* if it is generated by a lattice in H and an element δ of the centre of Osc_1 . It can be shown that each lattice in Osc_1 contains a sublattice of finite index which is a straight, see [6, Section 8]. In other words, each lattice in Osc_1 is virtually straight.

Moreover, we will assume that the lattice is unshifted and normalised in the sense of [7]. Let us recall these notions. For a lattice $L \subset \text{Osc}_1$, the projection of $L \cap H$ to $H/Z(H) \cong \mathbb{R}^2$ is a lattice in \mathbb{R}^2 (see [17, Theorem 4] or [22, proof of Theorem 2.21]). Then $L \subset \text{Osc}_1$ is called *normalised* if this projection has covolume one with respect to the standard metric of \mathbb{R}^2 . This can also be expressed as follows: L is normalised if and only if the commutator subgroup $[L \cap H, L \cap H] \subset Z(H)$ is generated by $(0, 1, 0) \in \text{Osc}_1$. A normalised straight lattice is called *unshifted* if δ can be chosen in the \mathbb{R} -factor of $\text{Osc}_1 = H \rtimes \mathbb{R}$, i.e., $\delta = (0, 0, 2\pi\kappa)$. This leads us to the following definition.

Definition 4.1. A lattice of Osc_1 is called a basic lattice if it is normalised and generated by a lattice in the Heisenberg group and an element $(0, 0, 2\pi\kappa) \in \mathbb{R} \subset H \rtimes \mathbb{R}$.

The additional assumptions to be normalised and unshifted are justified by the fact that each straight lattice can be normalised and shifted by (outer) automorphisms of Osc_1 . More exactly, the following holds. Let \mathcal{M}^{str} denote the set of all isomorphism classes of straight lattices of Osc_1 with respect to inner automorphisms of Osc_1 and let $\mathcal{B} \subset \mathcal{M}^{\text{str}}$ be the set of isomorphism classes of basic lattices. For a basic lattice L , we define $\kappa = \kappa(L) \in \mathbb{N}_{>0}$ by $L = \langle L \cap H, \delta = (0, 0, 2\pi\kappa) \rangle$, see Definition 4.1. Furthermore, we define a number $r = r(L) \in \mathbb{N}_{>0}$ by $L \cap Z(H) = \mathbb{Z} \cdot (0, 1/r, 0)$. This number is indeed an integer and has the following meaning. Since L is normalised, $[L \cap H, L \cap H] \subset Z(H)$ is generated by $(0, 1, 0) \in \text{Osc}_1$ as above explained. On the other hand, $H \cap L$ is a lattice

in H . Thus it is isomorphic to a discrete Heisenberg group

$$H_1^r(\mathbb{Z}) = \langle \alpha, \beta, \gamma \mid \alpha\beta\alpha^{-1}\beta^{-1} = \gamma^r, \alpha\gamma = \gamma\alpha, \beta\gamma = \gamma\beta \rangle$$

for some unique $r \in \mathbb{N}_{>0}$, which coincides with r as chosen before.

Proposition 4.2. *The map*

$$\mathcal{B} \times \mathbb{R}_{>0} \times \mathbb{R}/\mathbb{Z} \longrightarrow \mathcal{M}^{\text{strt}}$$

$$(L, a, s) \longmapsto F_S(T_u(L)), \quad u = \frac{s}{2\pi\kappa r}, \quad S = a \cdot I_2$$

is a bijection. Here I_2 denotes the identity on \mathbb{R}^2 and T_u and F_S are the automorphisms of Osc_1 defined by (3.1) and (3.2), respectively.

Proof. The assertion follows from [7, Theorem 4.12]. Indeed, the property to be straight is invariant under automorphisms. Therefore we can restrict the bijections in [7, Theorem 4.12] to straight lattices. In the notation of [7], we thus obtain bijections from $(\mathcal{M}_0 \cap \mathcal{M}^{\text{strt}}) \times \mathbb{R}/\mathbb{Z} = \mathcal{B} \times \mathbb{R}/\mathbb{Z}$ to $\mathcal{M}_1 \cap \mathcal{M}^{\text{strt}}$ and from $(\mathcal{M}_1 \cap \mathcal{M}^{\text{strt}}) \times \mathbb{R}_{>0}$ to $\mathcal{M}^{\text{strt}}$. It remains to check that the composition of these bijections has the form asserted in the proposition. Let L be a basic lattice. Then we have $q = 1$ and $x_\delta = y_\delta = 0$ in item (1) in [7, Definition 4.8], which implies $v = w = 0$ and therefore $s_0 = 1$. The assertion follows. ■

The computation of the spectrum relies on the decomposition of the right regular representation into irreducible subrepresentations. Once this decomposition is known for basic lattices, the decomposition for arbitrary lattices can be derived using Proposition 4.2. Indeed, according to Table 1, the decomposition of $L^2(F_S(T_u(L)) \setminus \text{Osc}_1)$ can be computed from that of $L^2(L \setminus \text{Osc}_1)$. That is why we focus on basic lattices here.

Proposition 4.3 ([7, Remark 5.3]). *A basic lattice is isomorphic via an inner automorphism of Osc_1 to a lattice generated by*

$$l_1 := \left(\frac{1}{\sqrt{v}}, 0, 0\right), \quad l_2 := \left(-\frac{\mu}{\sqrt{v}} + i\sqrt{v}, 0, 0\right), \quad l_3 := \left(0, \frac{1}{r}, 0\right), \quad l_4 := (0, 0, 2\pi\kappa),$$

for some $\mu, v \in \mathbb{R}, v > 0$ and $r, \kappa \in \mathbb{N}_{>0}$.

We will denote this lattice by $L_r(\kappa, \mu, v)$. In [7], it is denoted by $L_r(2\pi\kappa, \mu, v, 0, 0)$, but here we do not need the last two parameters since we only consider straight lattices.

Remark 4.4. The lattices $L_r(\kappa, \mu, v)$ and $L_{r'}(\kappa', \mu', v')$ differ by an inner automorphism of Osc_1 if and only if $r = r', \kappa = \kappa'$ and (μ, v) and (μ', v') are on the same orbit of the $\text{SL}(2, \mathbb{Z})$ -action on the Poincaré half plane [7, Theorem 4.15].

Now let L be a basic lattice. By Proposition 4.3 we may assume that $L = L_r(\kappa, \mu, v)$ in the following. Let us rewrite the first two generators using the matrix

$$T_{\mu,v} := \begin{pmatrix} \sqrt{v} & \frac{\mu}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix}.$$

Identifying $\mathbb{C} \cong \mathbb{R}^2$ and using the standard basis e_1, e_2 of \mathbb{R}^2 , we obtain

$$l_1 = (T_{\mu,v}^{-1}e_1, 0, 0), \quad l_2 = (T_{\mu,v}^{-1}e_2, 0, 0).$$

As an abstract group, the lattice L is isomorphic to the direct product of the discrete Heisenberg group $H_1^r(\mathbb{Z}) = \langle l_1, l_2, l_3 \mid l_1 l_2 l_1^{-1} l_2^{-1} = l_3^r, l_1 l_3 = l_3 l_1, l_3 l_2 = l_2 l_3 \rangle$ and \mathbb{Z} .

We fix a spin structure on $X = L \backslash \text{Osc}_1$. As explained in Section 2, it is determined by a homomorphism $\varepsilon : L \rightarrow (\mathbb{Z}_2, +)$. We will use the notation

$$\varepsilon_i := \varepsilon(l_i), \quad i = 1, \dots, 4$$

and write $\varepsilon = (\varepsilon_1, \dots, \varepsilon_4)$. Note that a map $\varepsilon : L \rightarrow \mathbb{Z}_2$ is a homomorphism if and only if $r\varepsilon_3 = 0$. Let again $\Delta = \mathbb{C}^4$ denote the spinor module of the metric Lie algebra $\mathfrak{o}\mathfrak{s}\mathfrak{c}_1$.

Remark 4.5. If ε_1 and ε_2 are different, we may assume that $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$ by changing μ and ν if necessary. Indeed, let be given the lattice $L = L_r(\kappa, \mu, \nu)$ and a spin structure defined by $\varepsilon = (\varepsilon_1, \dots, \varepsilon_4)$. We define μ' and ν' by $\mu' + i\nu' = -(\mu + i\nu)^{-1}$ and consider $L' := L_r(\kappa, \mu', \nu')$. In particular, we have

$$l_1 = (T_{\mu,v}^{-1}e_1, 0, 0), \quad l_2 = (T_{\mu,v}^{-1}e_2, 0, 0), \quad l'_1 = (T_{\mu',\nu'}^{-1}e_1, 0, 0), \quad l'_2 = (T_{\mu',\nu'}^{-1}e_2, 0, 0).$$

Let $S \in \text{SO}(2)$ be the multiplication by $(\mu - i\nu)/|\mu - i\nu|$ on $\mathbb{R}^2 \cong \mathbb{C}$. Then

$$T_{\mu',\nu'}^{-1} = ST_{\mu,v}^{-1} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $F_S : \text{Osc}_1 \rightarrow \text{Osc}_1$ denote the inner automorphism defined by S according to (3.2). Then $F_S(l_2) = l'_2$, $F_S(l_1) = (l'_2)^{-1}$ and $F_S(l_j) = l'_j$ for $j = 3, 4$. In particular, F_S induces an isometry from $L \backslash \text{Osc}_1$ to $L' \backslash \text{Osc}_1$. If we pull back the spin structure on $L \backslash \text{Osc}_1$ defined by ε by the inverse of this isometry, we obtain the spin structure on $L' \backslash \text{Osc}_1$ defined by $\varepsilon' = (\varepsilon_2, \varepsilon_1, \varepsilon_3, \varepsilon_4)$. In particular, the spectra of $D^{1/3}$ on $L \backslash \text{Osc}_1$ with respect to ε and on $L' \backslash \text{Osc}_1$ with respect to ε' coincide.

5. The right regular representation for basic lattices

5.1. Strategy

Let L be a basic lattice of Osc_1 . We consider $X := L \backslash \text{Osc}_1$. The aim of this subsection is to decompose the representation $L^2(\Sigma(X))$ of Osc_1 into irreducible components. We want to apply the results of [7] for the decomposition of the right regular representation on L^2 -functions. In [7], the decomposition of $L^2(L \backslash \text{Osc}_1)$ is determined for arbitrary lattices, where first the computation is reduced to the case of unshifted and normalised lattices and then explicit formulas are given in this case. In particular, [7, Proposition 7.2] describes the decomposition for basic lattices. In order to apply these results, we consider a finite covering of X such that the lifted spin structure becomes trivial.

More exactly, we consider the covering $\tilde{X} = L' \backslash \text{Osc}_1$ of X , where L' is the subgroup of L generated by l_1^2, \dots, l_4^2 . Since L is a basic lattice, this subgroup is normal. We obtain that $X = I \backslash \tilde{X}$, where $I \subset \text{Iso}(\tilde{X})$ is the finite group generated by the actions of l_1, \dots, l_4 on \tilde{X} . The spin structure on X lifts to a spin structure on \tilde{X} , which is now the trivial one since $2\varepsilon_j = 0, j = 1, \dots, 4$. Therefore the associated spinor bundle of \tilde{X} equals $\tilde{X} \times \Delta$ and sections in this bundle can be identified with functions from Osc_1 to Δ that are invariant under left translation by elements of the lattice L' . To recover the sections in the spinor bundle of X from these sections we have to find those sections in $\tilde{X} \times \Delta$ that are invariant under the action of the group I of deck transformations, where this action is defined as follows. Let $\varphi = f \otimes u \in C^\infty(\tilde{X}) \otimes \Delta$ be a (local) section. Then $l.\varphi = l^* f \otimes (-1)^{\varepsilon(l)} u = (-1)^{\varepsilon(l)} l^* f \otimes u$, where l^* denotes the left translation by $l \in L$. Thus we can identify

$$L^2(\Sigma(X)) = L^2(\Sigma(\tilde{X}))^I = (L^2(\tilde{X}) \otimes \Delta)^I = L^2(\tilde{X})^I \otimes \Delta, \tag{5.1}$$

where an element $[l] \in I$ induced by $l \in L$ acts on $L^2(\tilde{X})$ by

$$[l].f = (-1)^{\varepsilon(l)} l^* f. \tag{5.2}$$

Consequently, we can obtain a decomposition of $L^2(\Sigma(X))$ into irreducible subspaces in the following way. Since l^* commutes with $\rho(g)$ for all $g \in G$, we may first decompose $L^2(\tilde{X})$ and then determine the invariants of I . More exactly, we proceed as follows. First we decompose $L^2(\tilde{X})$ according to [7]. Then, for each isotypic component, we determine the subspace of sections that are invariant under the action of l_1, \dots, l_4 by (5.2). Finally, tensoring by Δ gives the result.

As said above, the explicit formulas in [7] for the decomposition work under the assumption that the lattice is normalised. However, note, that our new lattice L' generated by

$$l_1^2 := (2T_{\mu,v}^{-1}e_1, 0, 0), \quad l_2^2 := (2T_{\mu,v}^{-1}e_2, 0, 0), \quad l_3^2 := (0, \frac{2}{r}, 0), \quad l_4^2 := (0, 0, 4\pi\kappa),$$

is not normalised. Indeed, the projection of $L' \cap H$ to $H/Z(H) \cong \mathbb{R}^2$ is generated by the projections of l_1^2 and l_2^2 . Thus it has covolume 4. Therefore we apply the automorphism F_S for $S = \frac{1}{2} \cdot \text{id}$ on Osc_1 . We have $F_S(\xi, z, t) = (\frac{1}{2}\xi, \frac{1}{4}z, t)$. This yields $F_S(l_1^2) = l_1, F_S(l_2^2) = l_2, F_S(l_3^2) = (0, \frac{1}{2r}, 0), F_S(l_4^2) = (0, 0, 4\pi\kappa) = l_4^2$. Hence $F_S(L')$ is the basic lattice $L_{r'}(\kappa', \mu, \nu)$ for $r' = 2r$ and $\kappa' = 2\kappa$. Thus the formulas in [7] apply to $F_S(L')$.

Proposition 5.1. *We have*

$$L^2(\Sigma(X)) \cong F_S^*(L^2(F_S(L') \backslash \text{Osc}_1)^{F_S(I)}) \otimes \Delta, \tag{5.3}$$

where F_S^* is understood as the pullback of a representation as defined by (2.6) and $F_S(I)$ is the finite group of isometries of $F_S(L') \backslash \text{Osc}_1$ generated by $[F_S(l_1)], \dots, [F_S(l_4)]$, where $[F_S(l_j)]$ acts on $L^2(F_S(L') \backslash \text{Osc}_1)$ by $[F_S(l_j)].f = (-1)^{\varepsilon_j} F(l_j)^* f$ for $j = 1, \dots, 4$.

Proof. We have seen above that $L^2(\Sigma(X)) = L^2(\tilde{X})^I \otimes \Delta = L^2(L' \setminus \text{Osc}_1)^I \otimes \Delta$. Now the assertion follows from (2.7) and the fact that the equivalence $f \mapsto f \circ F_S^{-1}$ maps $L^2(L' \setminus \text{Osc}_1)^I$ to $L^2(F_S(L') \setminus \text{Osc}_1)^{F_S(I)}$. ■

5.2. The decomposition of $L^2(\Sigma(X))$

As explained in Section 3.3 we have to describe the decomposition of $L^2(\Sigma(X))$ into irreducible subrepresentations up to equivalence. We will use the representations introduced in Section 3.4. To formulate the result we need the following notations.

Let

$$\|(k, l)\|_{\mu, \nu} := (\nu k^2 + \frac{1}{\nu}(-\mu k + l)^2)^{\frac{1}{2}} = \|T_{\mu, \nu}^{-1}(l, k)^T\|,$$

and

$$\begin{aligned} \alpha(\mu, \nu, a) &:= \#\{(k, l) \in \mathbb{Z}^2 \mid \|(k, l)\|_{\mu, \nu} = a\}, \\ \alpha_0(\mu, \nu, a) &:= \#\{(k, l) \in \mathbb{Z}^2 \mid \|(k, l)\|_{\mu, \nu} = a, k \text{ even}, l \text{ even}\} = \alpha(\mu, \nu, a/2), \\ \alpha_1(\mu, \nu, a) &:= \#\{(k, l) \in \mathbb{Z}^2 \mid \|(k, l)\|_{\mu, \nu} = a, k \text{ even}, l \text{ odd}\}, \\ \alpha_2(\mu, \nu, a) &:= \#\{(k, l) \in \mathbb{Z}^2 \mid \|(k, l)\|_{\mu, \nu} = a, k \text{ odd}, l \text{ odd}\}. \end{aligned} \tag{5.4}$$

Moreover,

$$\mathbb{A}(\mu, \nu) := \{a \in \mathbb{R}_{>0} \mid \alpha(\mu, \nu, a) \neq 0\}.$$

Theorem 5.2. *On $X = L_r(\kappa, \mu, \nu) \setminus \text{Osc}_1$ we consider the spin structure given by $\varepsilon = (\varepsilon_1, \dots, \varepsilon_4)$, where we assume $(\varepsilon_1, \varepsilon_2) = (0, 1)$ if $\varepsilon_1 \neq \varepsilon_2$. Then we have $L^2(\Sigma(X)) = 4\mathcal{H}_0 \oplus 4\mathcal{H}_1$ with*

$$\mathcal{H}_0 \cong \bigoplus_{n \in \mathbb{Z}} \mathbf{m}_{\mathcal{E}}(n) \mathcal{E}_{\frac{n}{4\pi\kappa}} \oplus \bigoplus_{a \in \mathbb{A}(\mu, \nu)} \bigoplus_{K=0}^{2\kappa-1} \mathbf{m}_{\mathcal{S}}(a, K) \mathcal{S}_{a/2}^{K/(2\kappa)}, \tag{5.5}$$

$$\mathcal{H}_1 \cong \bigoplus_{m \in \mathbb{Z}_{\neq 0}} \bigoplus_{n \in \mathbb{Z}} \mathbf{m}_{\mathcal{F}}(m, n) \mathcal{F}_{\frac{r|m}{2}, \frac{n}{4\pi\kappa}}, \tag{5.6}$$

where

$$\mathbf{m}_{\mathcal{E}}(n) = \begin{cases} 1, & \text{if } \varepsilon = (0, 0, 0, n) \in \mathbb{Z}_2^4, \\ 0, & \text{else,} \end{cases} \tag{5.7}$$

$$\mathbf{m}_{\mathcal{S}}(a, K) = \begin{cases} \alpha_0(\mu, \nu, a), & \text{if } \varepsilon = (0, 0, 0, K) \in \mathbb{Z}_2^4, \\ \alpha_1(\mu, \nu, a), & \text{if } \varepsilon = (0, 1, 0, K) \in \mathbb{Z}_2^4, \\ \alpha_2(\mu, \nu, a), & \text{if } \varepsilon = (1, 1, 0, K) \in \mathbb{Z}_2^4, \\ 0, & \text{else,} \end{cases} \tag{5.8}$$

$$\mathbf{m}_{\mathcal{F}}(m, n) = \begin{cases} \frac{r|m|}{2}, & \text{if } (\varepsilon_3, \varepsilon_4) = (m, n) \in \mathbb{Z}_2^2, \\ 0, & \text{else.} \end{cases} \tag{5.9}$$

Proof. We proceed according to the strategy outlined in Section 5.1. By Proposition 5.1, we have to determine the decomposition of $F_S^*(L^2(F_S(L') \setminus \text{Osc}_1)^{F_S(L)})$, where $F_S(\xi, z, t) = (\frac{1}{2}\xi, \frac{1}{4}z, t)$ and L' is the lattice generated by l_1^2, \dots, l_4^2 . It turns out that calculations on Osc_1^M are easier than on Osc_1 . Therefore we transform the lattice $F_S(L)$ by the isomorphism $\phi : \text{Osc}_1 \rightarrow \text{Osc}_1^M$ defined in (3.4). We obtain

$$\begin{aligned} \gamma_1 &:= \phi \circ F_S(l_1) = M\left(0, \frac{1}{2\sqrt{v}}, 0\right), \\ \gamma_2 &:= \phi \circ F_S(l_2) = M\left(-\frac{\sqrt{v}}{2}, -\frac{\mu}{2\sqrt{v}}, \frac{1}{8}\mu\right), \\ \gamma_3 &:= \phi \circ F_S(l_3) = M\left(0, 0, \frac{1}{4r}\right) = M\left(0, 0, \frac{1}{2r'}\right), \\ \gamma_4 &:= \phi \circ F_S(l_4) = (2\pi\kappa) = (\pi\kappa'), \end{aligned}$$

where $r' = 2r$ and $\kappa' = 2\kappa$. We denote the lattices $(\phi \circ F_S)(L)$ and $(\phi \circ F_S)(L')$ by Γ and Γ' , respectively.

The push-forward of a representation (σ, V) of Osc_1 is a representation $(\phi^{-1})^*(\sigma, V) = (\sigma \circ \phi^{-1}, V)$ of Osc_1^M . In the following, we will identify these representations with each other and omit $(\phi^{-1})^*$ in the notation. In particular, we identify the representation

$$L^2(F_S(L') \setminus \text{Osc}_1)^{F_S(L)} = L^2(F_S(L') \setminus \text{Osc}_1)^{F_S(L)}$$

with $L^2(\Gamma' \setminus \text{Osc}_1^M)^\Gamma$.

It is natural to use the push-forwards of the irreducible representations of Osc_1 as models for the irreducible representations of Osc_1^M . Then the irreducible unitary representations of Osc_1^M are $(\phi^{-1})^*\mathcal{C}_d$, $(\phi^{-1})^*\mathcal{S}_a^\tau$ and $(\phi^{-1})^*\mathcal{F}_{c,d}$. According to the above remark we simply write \mathcal{C}_d , \mathcal{S}_a^τ and $\mathcal{F}_{c,d}$ instead of $(\phi^{-1})^*\mathcal{C}_d$, $(\phi^{-1})^*\mathcal{S}_a^\tau$ and $(\phi^{-1})^*\mathcal{F}_{c,d}$. Our task now is to describe explicitly the irreducible subrepresentations of the right regular representation $L^2(\Gamma' \setminus \text{Osc}_1^M)$ of Osc_1^M . Here we can use the results of [7]. The representation $L^2(\Gamma' \setminus \text{Osc}_1^M)$ is equivalent to $\mathcal{H}'_0 \oplus \mathcal{H}'_1$, where the subrepresentations \mathcal{H}'_0 and \mathcal{H}'_1 are given as follows:

$$\begin{aligned} \mathcal{H}'_0 &\cong \bigoplus_{n \in \mathbb{Z}} \mathcal{C}_{\frac{n}{2\pi\kappa'}} \oplus \bigoplus_{a \in \mathbb{A}(\mu, v)} \bigoplus_{K=0}^{\kappa'-1} \alpha(\mu, v, a) \mathcal{S}_a^{K/\kappa'}, \\ \mathcal{H}'_1 &\cong \bigoplus_{m \in \mathbb{Z}_{\neq 0}} |m|r' \cdot \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{r'm, \frac{n}{2\pi\kappa'}}. \end{aligned}$$

The subspaces of \mathcal{H}'_0 and \mathcal{H}'_1 corresponding to the irreducible submodules of $L^2(\Gamma' \setminus \text{Osc}_1^M)$ in the above formulas are explicitly given as follows. For a function $f : \text{Osc}_1^M \rightarrow \mathbb{C}$, we will denote by $f(x, y, z, t)$ the image of $M(x, y, z)(t)$ under f . Then the representation \mathcal{H}'_0 is the direct sum of subspaces

$$\overline{\text{span}}\{\varphi_n(x, y, z, t) := \varphi_n(t) := e^{2\pi i dt}\} \cong \mathcal{C}_d$$

for $d = \frac{n}{2\pi\kappa'}$ and

$$\bigoplus_{\substack{k,l \in \mathbb{N} \\ \|(k,l)\|_{\mu,\nu} = a}} \overline{\text{span}}\{\phi_{l,n}^k(x, y, z, t) = \exp(2\pi i(x, y)T_{\mu,\nu}^{-1}(l, k)^\top)\varphi_n(t) \mid n \in K + \kappa'\mathbb{Z}\} \cong \alpha(\mu, \nu, a)\mathcal{S}_a^{K/\kappa'}$$

for $a \in \mathbb{A}(\mu, \nu)$ and $K = 0, \dots, \kappa' - 1$.

Next we want to describe the decomposition of \mathcal{H}'_1 . For $m \in \mathbb{Z}_{\neq 0}$ and $n \in \mathbb{Z}$, we consider the subspace

$$\mathcal{W}_{m,n} := \overline{\text{span}}\{\theta_{m,n,k} \mid k \in \mathbb{Z}_{r'|m} = \mathbb{Z}/r'|m\mathbb{Z}\}$$

for

$$\theta_{m,n,k}(x, y, z, t) := e^{2\pi i r' m z} e^{i n t / \kappa'} \sum_{j \in \frac{k}{r'm} + \mathbb{Z}} e^{-\pi r'|m|(x+j\sqrt{\nu})^2} e^{\pi i r' j m (j\mu + 2\sqrt{\nu}y)}.$$

We denote by A_+ the ladder operator $X + \text{sgn}(m)iY$. Then \mathcal{H}'_1 decomposes into the direct sum of subspaces

$$\mathcal{W}_{m,n} \oplus A_+(\mathcal{W}_{m,n}) \oplus A_+^2(\mathcal{W}_{m,n}) \oplus \dots \cong |m|r' \cdot \mathcal{F}_{r'm, \frac{n}{2\pi\kappa'}}$$

for $m \in \mathbb{Z}_{\neq 0}$ and $n \in \mathbb{Z}$.

In order to obtain $L^2(\Gamma' \setminus \text{Osc}_1^M)^\Gamma$, we determine the elements in the isotypic components of

$$L^2(\Gamma' \setminus \text{Osc}_1^M) \cong \mathcal{H}'_0 \oplus \mathcal{H}'_1$$

that are invariant under $\gamma_1, \dots, \gamma_4$. We compute

$$\begin{aligned} \gamma_1 \cdot M(x, y, z)(t) &= M(x, y + \frac{1}{2\sqrt{\nu}}, z)(t), \\ \gamma_2 \cdot M(x, y, z)(t) &= M(x - \frac{\sqrt{\nu}}{2}, y - \frac{\mu}{2\sqrt{\nu}}, z + \frac{1}{8}\mu - \frac{\sqrt{\nu}}{2}y)(t), \\ \gamma_3 \cdot M(x, y, z)(t) &= M(x, y, z + \frac{1}{2r'})(t), \\ \gamma_4 \cdot M(x, y, z)(t) &= M(x, y, z)(t + \pi\kappa'), \end{aligned}$$

where we used that κ' is even. Thus, the action defined by (5.2) is now given by

$$\gamma_j \cdot \varphi_n = (-1)^{\varepsilon_j} \varphi_n, \quad j = 1, 2, 3, \quad \gamma_4 \cdot \varphi_n = (-1)^{n+\varepsilon_4} \varphi_n \tag{5.10}$$

and

$$\begin{aligned} \gamma_1 \cdot \phi_{l,n}^k &= (-1)^{k+\varepsilon_1} \phi_{l,n}^k, & \gamma_2 \cdot \phi_{k,n}^l &= (-1)^{l+\varepsilon_2} \phi_{l,n}^k, \\ \gamma_3 \cdot \phi_{l,n}^k &= (-1)^{\varepsilon_3} \phi_{l,n}^k, & \gamma_4 \cdot \phi_{l,n}^k &= (-1)^{n+\varepsilon_4} \phi_{l,n}^k. \end{aligned} \tag{5.11}$$

Furthermore,

$$\begin{aligned}
 \gamma_1 \cdot \theta_{m,n,k} &= (-1)^{k+\varepsilon_1} \theta_{m,n,k}, & \gamma_2 \cdot \theta_{m,n,k} &= (-1)^{\varepsilon_2} \theta_{m,n,k+rm}, \\
 \gamma_3 \cdot \theta_{m,n,k} &= (-1)^{m+\varepsilon_3} \theta_{m,n,k}, & \gamma_4 \cdot \theta_{m,n,k} &= (-1)^{n+\varepsilon_4} \theta_{m,n,k}.
 \end{aligned}
 \tag{5.12}$$

Indeed, the formula for $\gamma_2 \cdot \theta_{m,n,k}$ follows from the following observation. For $k \in \mathbb{Z}_{r'|m}$, we have

$$\begin{aligned}
 &(\gamma_2^* \theta_{m,n,k})(x, y, z, t) \\
 &= e^{2\pi i r' m(z + \frac{\mu}{8} - \frac{\sqrt{v}}{2} y)} e^{int/\kappa'} \sum_{j \in \frac{k}{r'} + \mathbb{Z}} e^{-\pi r' |m|(x - \frac{\sqrt{v}}{2} + j\sqrt{v})^2} e^{\pi i r' j m(j\mu + 2\sqrt{v}y - \mu)} \\
 &= e^{2\pi i r' m z} e^{int/\kappa'} \sum_{j \in \frac{k}{r'} + \mathbb{Z}} e^{-\pi r' |m|(x + (j - \frac{1}{2})\sqrt{v})^2} e^{\pi i r' m(\frac{\mu}{4} - \sqrt{v}y)} e^{\pi i r' j m(j\mu + 2\sqrt{v}y - \mu)} \\
 &= e^{2\pi i r' m z} e^{int/\kappa'} \sum_{j \in \frac{k}{r'} + \mathbb{Z}} e^{-\pi r' |m|(x + (j - \frac{1}{2})\sqrt{v})^2} e^{\pi i r' (j - \frac{1}{2})m((j - \frac{1}{2})\mu + 2\sqrt{v}y)} \\
 &= e^{2\pi i r' m z} e^{int/\kappa'} \sum_{j \in \frac{k}{r'} + \mathbb{Z} + \frac{1}{2}} e^{-\pi r' |m|(x + j\sqrt{v})^2} e^{\pi i r' j m(j\mu + 2\sqrt{v}y)} \\
 &= \theta_{m,n,\tilde{k}}
 \end{aligned}$$

with $\tilde{k} := k + \frac{1}{2}r'|m| \pmod{r'|m|}$.

Consequently, $L^2(\Gamma \backslash \text{Osc}_1^M)^\Gamma$ decomposes into $(\mathcal{H}'_0)^\Gamma \oplus (\mathcal{H}'_1)^\Gamma$, where the subrepresentations $(\mathcal{H}'_0)^\Gamma$ and $(\mathcal{H}'_1)^\Gamma$ are given by

$$\begin{aligned}
 (\mathcal{H}'_0)^\Gamma &\cong \bigoplus_{n \in \mathbb{Z}} \mathbf{m}_{\mathcal{C}}(n) \mathcal{C}_{\frac{n}{2\pi\kappa'}} \oplus \bigoplus_{a \in \mathbb{A}(\mu, v)} \bigoplus_{K=0}^{\kappa'-1} \mathbf{m}_{\mathcal{S}}(a, K) \mathcal{S}_a^{K/\kappa'}, \\
 (\mathcal{H}'_1)^\Gamma &\cong \bigoplus_{m \in \mathbb{Z}_{\neq 0}} \bigoplus_{n \in \mathbb{Z}} \mathbf{m}_{\mathcal{F}}(m, n) \mathcal{F}_{r'm, \frac{n}{2\pi\kappa'}}
 \end{aligned}$$

where $\mathbf{m}_{\mathcal{C}}(n)$, $\mathbf{m}_{\mathcal{S}}(a, K)$ and $\mathbf{m}_{\mathcal{F}}(m, n)$ are given by (5.7), (5.8) and (5.9), respectively. Indeed, the formula for $\mathbf{m}_{\mathcal{C}}(n)$ follows from (5.10) and the one for $\mathbf{m}_{\mathcal{S}}(a, K)$ from (5.11). Finally, $\mathbf{m}_{\mathcal{F}}(m, n)$ is obtained from (5.12), where we used that $\varepsilon_3 = 0$ if r is odd.

In order to obtain $L^2(\Sigma(X))$, we have to pull back $(\mathcal{H}'_0)^\Gamma \oplus (\mathcal{H}'_1)^\Gamma$ by F_S and to tensor the result by Δ , i.e., to multiply by 4. By Table 1 we have

$$F_S^* \mathcal{C}_{\frac{n}{2\pi\kappa'}} = \mathcal{C}_{\frac{n}{2\pi\kappa'}}, \quad F_S^* \mathcal{S}_a^{K/\kappa'} = \mathcal{S}_{a/2}^{K/\kappa'}, \quad F_S^* \mathcal{F}_{r'm, \frac{n}{2\pi\kappa'}} = \mathcal{F}_{\frac{r'm}{4}, \frac{n}{2\pi\kappa'}}.$$

Finally, we replace r' by $2r$ and κ' by 2κ and obtain the assertion. ■

Remark 5.3. Actually, the proof shows more than we claimed in the theorem. It gives an explicit decomposition of the representation and not only an equivalence.

Example 5.4. Let us consider $X = L_r(\kappa, \mu, \nu) \backslash \text{Osc}_1$ endowed with the trivial spin structure, i.e., $\varepsilon_1 = \dots = \varepsilon_4 = 0$. Then $L^2(\Sigma(X)) \cong L^2(X) \otimes \mathbb{C}^4$. Indeed, Theorem 5.2 implies

$$\begin{aligned} \mathbf{m}_{\mathcal{E}}(n) &= \begin{cases} 1, & \text{if } n \text{ is even,} \\ 0, & \text{else,} \end{cases} \\ \mathbf{m}_{\mathcal{S}}(a, K) &= \begin{cases} \alpha(\mu, \nu, a/2), & \text{if } K \text{ is even,} \\ 0, & \text{else,} \end{cases} \\ \mathbf{m}_{\mathcal{F}}(m, n) &= \begin{cases} \frac{r|m|}{2}, & \text{if } m \text{ and } n \text{ are even,} \\ 0, & \text{else,} \end{cases} \end{aligned}$$

which coincides with the known formulas for the decomposition of $L^2(L_r(\kappa, \mu, \nu) \backslash \text{Osc}_1)$, see [7].

Example 5.5. In the case, where r is even and $\varepsilon_3 = 1$, we have $\mathbf{m}_{\mathcal{E}}(n) = 0$ for all $n \in \mathbb{Z}$, $\mathbf{m}_{\mathcal{S}}(a, K) = 0$ for all a and K , and

$$\mathbf{m}_{\mathcal{F}}(m, n) = \begin{cases} \frac{r|m|}{2}, & \text{if } m \text{ is odd and } \varepsilon_4 = n \in \mathbb{Z}_2, \\ 0, & \text{else.} \end{cases}$$

6. The spectrum

In this section, we compute the spectrum of (the closure of) the cubic Dirac operator $D^{1/3}$ on $X = L \backslash \text{Osc}_1$ for any basic lattice L of Osc_1 . We obtain that $\text{spec}(D^{1/3})$ consists only of the point spectrum and the continuous spectrum, which we will compute in Sections 6.2 and 6.3, respectively. It will turn out that $\text{spec}(D^{1/3}) = \mathbb{C}$. Finally we determine the spectrum of the Dirac operators D^t for all other t .

6.1. The spectrum of $-\Omega$

The operator $(D^{1/3})^2$ acts as $-\Omega$ in the first factor of the tensor product in (5.3) and trivially on the second one. On each irreducible representation, Ω acts by a scalar. Hence, Theorem 5.2 allows us to compute the spectrum of $-\Omega$ on quotients of the oscillator group by basic lattices.

If $L_r(\kappa, \mu, \nu)$ is a basic lattice, the volume of $X = L_r(\kappa, \mu, \nu) \backslash \text{Osc}_1$ only depends on the quotient of κ by r . For given $r, \kappa \in \mathbb{N}_{>0}$ we have

$$\beta := \frac{\pi r}{\kappa} = \frac{2\pi^2}{\text{vol}(X)}.$$

Furthermore, we define the set

$$\mathcal{A}_j(\mu, \nu) := \{\pi^2 a^2 \mid a \in \mathbb{R}_{>0}, \alpha_j(\mu, \nu, a) \neq 0\}, \quad j = 0, 1, 2.$$

where we have used the numbers $\alpha_j(\mu, \nu, a)$ introduced in (5.4). The following theorem describes the spectrum of $-\Omega$ in dependence of the spin structure on X . Recall that we consider only the case $(\varepsilon_1, \varepsilon_2) = (0, 1)$ if $\varepsilon_1 \neq \varepsilon_2$, see Remark 4.5.

Theorem 6.1. *Let $L = L_r(\kappa, \mu, \nu)$ be a basic lattice and $\varepsilon : L \rightarrow \mathbb{Z}_2$ be a homomorphism. Then the spectrum of $(D^{1/3})^2 = -\Omega$ on the spinor bundle of $X = L \setminus \text{Osc}_1$ corresponding to ε is given in Table 2.*

$(\varepsilon_1, \varepsilon_2)$	$(\varepsilon_3, \varepsilon_4 + \kappa)$			
	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$
$(0, 0)$	$\mathcal{A}_0(\mu, \nu) \cup 4\beta\mathbb{Z}$	$\mathcal{A}_0(\mu, \nu) \cup 2\beta\mathbb{Z}$	$2\beta\mathbb{Z}$	$\beta(2\mathbb{Z} + 1)$
$(0, 1)$	$\mathcal{A}_1(\mu, \nu) \cup 4\beta\mathbb{Z}$	$\mathcal{A}_1(\mu, \nu) \cup 2\beta\mathbb{Z}_{\neq 0}$	$2\beta\mathbb{Z}$	$\beta(2\mathbb{Z} + 1)$
$(1, 1)$	$\mathcal{A}_2(\mu, \nu) \cup 4\beta\mathbb{Z}$	$\mathcal{A}_2(\mu, \nu) \cup 2\beta\mathbb{Z}_{\neq 0}$	$2\beta\mathbb{Z}$	$\beta(2\mathbb{Z} + 1)$

Table 2. Spectrum of $-\Omega$.

Proof. The Casimir operator of a representation σ with respect to $\langle \cdot, \cdot \rangle$ equals

$$\Omega_\sigma = (\sigma_*(X))^2 + (\sigma_*(Y))^2 + 2(\sigma_*(Z))(\sigma_*(T)).$$

A straight forward computation yields $\Omega_{\mathcal{C}_d} = 0$, $\Omega_{\mathcal{S}_d^\varepsilon} = -4\pi^2 a^2$, $\Omega_{\mathcal{F}_{c,d}} = -2\pi c(4\pi d + 1)$ for $c > 0$ and $\Omega_{\mathcal{F}_{c,d}} = -2\pi c(4\pi d - 1)$ for $c < 0$. Therefore, $\Omega_{\mathcal{F}_{\frac{m}{2}, \frac{n}{4\pi\kappa}}} = -\beta m(n + \kappa)$ for $m > 0$ and $\Omega_{\mathcal{F}_{\frac{m}{2}, \frac{n}{4\pi\kappa}}} = -\beta m(n - \kappa)$ for $m < 0$.

A representation of the form \mathcal{C}_d for some d appears in $L^2(\Sigma(X))$ if and only if $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ by (5.7). A representation of the form $\mathcal{S}_{a/2}^{K/(2\kappa)}$ appears if and only if $\mathbf{m}_\mathcal{S}(a, K) \neq 0$. Furthermore, $\mathbf{m}_\mathcal{S}(a, K) \neq 0$ for some K if and only if $\varepsilon_3 = 0$ and

$$\begin{aligned} \alpha_0(\mu, \nu, a) &\neq 0, & \text{if } (\varepsilon_1, \varepsilon_2) &= (0, 0), \\ \alpha_1(\mu, \nu, a) &\neq 0, & \text{if } (\varepsilon_1, \varepsilon_2) &= (0, 1), \\ \alpha_2(\mu, \nu, a) &\neq 0, & \text{if } (\varepsilon_1, \varepsilon_2) &= (1, 1). \end{aligned}$$

Suppose that $(\varepsilon_3, \varepsilon_4 + \kappa) = (0, 0)$. Then $\mathbf{m}_\mathcal{F}(m, n) \neq 0$ if and only if $m \neq 0$ is even and $n + \kappa$ is also even. Thus the representations of the form $\mathcal{F}_{c,d}$ contribute the set $4\beta\mathbb{Z}$ to the spectrum of $-\Omega$. Next suppose that $(\varepsilon_3, \varepsilon_4 + \kappa) = (0, 1)$. Then $\mathbf{m}_\mathcal{F}(m, n) \neq 0$ if and only if $m \neq 0$ is even and $n + \kappa$ is odd. In this case we get the contribution $2\beta\mathbb{Z}_{\neq 0}$. Now assume that $(\varepsilon_3, \varepsilon_4 + \kappa) = (1, 0)$. Then $\mathbf{m}_\mathcal{F}(m, n) \neq 0$ if and only if m is odd and $n + \kappa$ is even. This gives the contribution $2\beta\mathbb{Z}$. Finally assume that $(\varepsilon_3, \varepsilon_4 + \kappa) = (1, 1)$. Then $\mathbf{m}_\mathcal{F}(m, n) \neq 0$ if and only if m and $n + \kappa$ are odd. In this case the representations of the form $\mathcal{F}_{c,d}$ contribute the set $\beta \cdot (2\mathbb{Z} + 1)$. ■

Example 6.2. As an example we consider the case $\nu = 1, \mu = 0$ and use the abbreviated notation $\|\cdot\| := \|\cdot\|_{\mu, \nu}$, i.e., $\|(k, l)\|^2 = k^2 + l^2$, $\alpha(a) := \alpha(0, 1, a)$ and $\mathbb{A} := \mathbb{A}(0, 1)$.

Then $\mathbb{A} = \{a \in \mathbb{R}_{>0} \mid \alpha(a) \neq 0\}$ contains exactly those a for which a^2 is an integer and the prime factors $q \equiv 3 \pmod 4$ of a^2 appear in even powers (Two Square Theorem), see for example [11, Theorem 366]. We consider the decomposition $\mathbb{A} = \mathbb{A}_0 \cup \mathbb{A}_1 \cup \mathbb{A}_2$, where

$$\begin{aligned} \mathbb{A}_0 &:= \{a \in \mathbb{A} \mid a^2 \equiv 0 \pmod 4\} = \{\sqrt{4}, \sqrt{8}, \sqrt{16}, \dots\}, \\ \mathbb{A}_1 &:= \{a \in \mathbb{A} \mid a^2 \equiv 1 \pmod 4\} = \{1, \sqrt{5}, \sqrt{9}, \sqrt{13}, \dots\}, \\ \mathbb{A}_2 &:= \{a \in \mathbb{A} \mid a^2 \equiv 2 \pmod 4\} = \{\sqrt{2}, \sqrt{10}, \sqrt{18}, \dots\}. \end{aligned}$$

Then $\mathcal{A}_j = \{\pi^2 a^2 \mid a \in \mathbb{A}_j\}$, $j = 0, 1, 2$. Notice that in this example the spectrum of the spin structures $(\varepsilon_1, \varepsilon_2) = (0, 1)$ and $(\varepsilon_1, \varepsilon_2) = (1, 0)$ coincide.

Proposition 6.3. *Let L be a basic lattice and consider a fixed spin structure corresponding to a homomorphism $\varepsilon : L \rightarrow \mathbb{Z}_2$. The spectrum of $(D^{1/3})^2 = -\Omega$ on $X = L \setminus \text{Osc}_1$ is symmetric if and only if ε restricted to $L \cap Z(H)$ is non-trivial. If $\text{spec}(-\Omega)$ is symmetric and contains 0, then it equals $\frac{4\pi^2}{\text{vol}(X)} \cdot \mathbb{Z}$. If it is symmetric but does not contain 0, then it is equal to $\frac{2\pi^2}{\text{vol}(X)} \cdot (2\mathbb{Z} + 1)$.*

Proof. Up to some inner automorphism, L is equal to some $L_r(\kappa, \mu, \nu)$ for some κ, r, μ, ν , which we do not know explicitly. The spin structure on $L_r(\kappa, \mu, \nu) \setminus \text{Osc}_1$ corresponding to the one on $L \setminus \text{Osc}_1$ under this automorphism is given by a quadruple $(\varepsilon_1, \dots, \varepsilon_4)$, which we also do not know explicitly. Only ε_3 can immediately be determined by ε . Indeed, $\varepsilon_3 = 0$, if ε restricted to $L \cap Z(H)$ is trivial and $\varepsilon_3 = 1$, if ε restricted to $L \cap Z(H)$ is non-trivial. Now it follows from Table 2 that the assertions of the proposition are equivalent to the condition that neither $\mathcal{A}_0, \mathcal{A}_1$ nor \mathcal{A}_2 is contained in $2\beta\mathbb{Z}$.

Since $\mathcal{A}_0 = \{\pi^2 \|(k, l)\|_{\mu, \nu}^2 \mid (k, l) \in \mathbb{Z}^2, k \text{ even}, l \text{ even}\}$, the set \mathcal{A}_0 contains the elements πs_j , $j = 1, 2, 3$ for

$$s_1 := \pi \|(0, 2)\|_{\mu, \nu}^2, \quad s_2 := \pi \|(2, 0)\|_{\mu, \nu}^2, \quad s_3 := \pi \|(2, 2)\|_{\mu, \nu}^2.$$

The condition $\mathcal{A}_0 \subset 2\beta\mathbb{Z}$ would imply that s_1, s_2 and s_3 are rational. Since $s_1 = 4\pi/\nu$, we get $\nu = \pi q$ for some $q \in \mathbb{Q}$. Furthermore, we have $s_3 - s_2 = s_1(1 - 2\mu)$, which implies that μ is rational. But then $s_2 = 4\pi(\nu + \frac{\mu^2}{\nu}) = 4(\pi^2 q + \frac{\mu^2}{q})$ would be irrational, which is a contradiction.

Similarly, $\mathcal{A}_1 \subset 2\beta\mathbb{Z}$ would imply that $\pi \|(0, 1)\|_{\mu, \nu}^2, \pi \|(2, 1)\|_{\mu, \nu}^2$, and $\pi \|(2, -1)\|_{\mu, \nu}^2$ are rational, which as above leads to a contradiction. Finally, if we assume that $\mathcal{A}_2 \subset 2\beta\mathbb{Z}$ we can use that $\pi \|(1, 1)\|_{\mu, \nu}^2, \pi \|(1, -1)\|_{\mu, \nu}^2$, and $\pi \|(1, 3)\|_{\mu, \nu}^2$ would be rational, which is also impossible. ■

6.2. Point spectrum and eigenspaces of $D^{1/3}$

In this subsection we compute the point spectrum of D^t on $X = L \setminus \text{Osc}_1$ for basic lattices L of Osc_1 . For the rest of the paper we use the convention $\sqrt{\lambda} := i\sqrt{|\lambda|}$ if $\lambda < 0$.

Proposition 6.4. *Let L be a basic lattice of Osc_1 and consider $X := L \setminus \text{Osc}_1$. Then $z \in \mathbb{C}$ belongs to the point spectrum of $D^{1/3}$ if and only if z^2 is in the spectrum of $-\Omega$.*

Proof. Let ψ be an eigenspinor of $-\Omega$ with eigenvalue λ . Then $\psi_{\pm} := D^{1/3}\psi \pm \sqrt{\lambda}\psi$ satisfy the equation $D^{1/3}\psi_{\pm} = \pm\sqrt{\lambda}\psi_{\pm}$. One of these two spinors has to be non-trivial. Thus $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an eigenvalue of $D^{1/3}$. Since on even-dimensional manifolds the point spectrum of the cubic Dirac operator is symmetric to zero, both $\pm\sqrt{\lambda}$ are eigenvalues of $D^{1/3}$. ■

The operator $D^{1/3}$ preserves every summand of the decomposition of $L^2(\Sigma(X))$ that we obtained in Theorem 5.2. In fact, derivations act only on the first factor in (5.1) by the right regular representation on functions and Clifford multiplication acts only on the second one. So next we want to decompose these summands into generalised eigenspaces of $D^{1/3}$. Each summand belongs to an eigenspace of $-\Omega$. Let λ be the corresponding eigenvalue. If $\lambda \neq 0$ the summand decomposes into eigenspaces of $D^{1/3}$ with eigenvalues $\pm\sqrt{\lambda}$. The projections to these subspaces are given by $P_{\lambda}^{\pm} := \frac{1}{2\sqrt{\lambda}}(\pm D^{1/3} + \sqrt{\lambda})$. For the summands with $\lambda = 0$, we will determine the kernel of $D^{1/3}$.

We will use the notation of Section 3.4 for the irreducible unitary representations of the oscillator group. Furthermore, we consider the complete orthonormal systems of the representation spaces introduced there. We also use the formulas for the Dirac operator with respect to the basis (u_1, \dots, u_4) of Δ introduced in Section 3.3.

Let $\mathcal{F}_{c,d} \otimes \Delta, c > 0$, be one of the summands in (5.6). Assume first that $\lambda \neq 0$. Then the eigenvalue of $-\Omega$ on $\mathcal{F}_{c,d}$ equals $\lambda = 2\pi c(4\pi d + 1)$. Using the projections P_{\pm} we calculate that the subspace of eigenvectors of $D^{1/3}$ with eigenvalue $\pm\sqrt{\lambda}$ is spanned by

$$\eta_0^{\pm} = \sqrt{2\pi c} \psi_0 \otimes u_1 \pm i\sqrt{2\pi d + \frac{1}{2}} \psi_0 \otimes u_2, \tag{6.1}$$

$$\eta_n^{\pm} = \sqrt{2\pi c} \psi_n \otimes u_1 \pm i\sqrt{2\pi d + \frac{1}{2}} \psi_n \otimes u_2 + \sqrt{n} \psi_{n-1} \otimes u_4, \quad n \geq 1, \tag{6.2}$$

$$\hat{\eta}_n^{\pm} = \sqrt{n+1} \psi_{n+1} \otimes u_2 - \sqrt{2\pi c} \psi_n \otimes u_3 \mp i\sqrt{2\pi d + \frac{1}{2}} \psi_n \otimes u_4, \quad n \geq 0. \tag{6.3}$$

If $\Omega = 0$ on $\mathcal{F}_{c,d}$ or equivalently if $4\pi d + 1 = 0$, then $\ker D^{1/3} = \text{im } D^{1/3}$ on $\mathcal{F}_{c,d}$. This space is spanned by

$$\begin{aligned} \eta_0 &= \sqrt{2\pi c} \psi_0 \otimes u_1, \\ \eta_n &= \sqrt{2\pi c} \psi_n \otimes u_1 + \sqrt{n} \psi_{n-1} \otimes u_4, \quad n \geq 1, \\ \hat{\eta}_n &= \sqrt{n+1} \psi_{n+1} \otimes u_2 - \sqrt{2\pi c} \psi_n \otimes u_3, \quad n \geq 0. \end{aligned}$$

Now let $\mathcal{F}_{c,d} \otimes \Delta, c < 0$, be one of the summands in (5.6). Then the eigenvalue of $-\Omega$ on $\mathcal{F}_{c,d}$ equals $\lambda = 2\pi c(4\pi d - 1)$. If $4\pi d - 1 > 0$, the subspace of eigenvectors of $D^{1/3}$ with eigenvalue $\pm\sqrt{\lambda}$ is spanned by

$$-\sqrt{2\pi|c|} \psi_0 \otimes u_3 \mp \sqrt{2\pi d - \frac{1}{2}} \psi_0 \otimes u_4, \tag{6.4}$$

$$\sqrt{2\pi|c|} \psi_n \otimes u_1 \pm \sqrt{2\pi d - \frac{1}{2}} \psi_n \otimes u_2 + \sqrt{n+1} \psi_{n+1} \otimes u_4, \quad n \geq 0, \tag{6.5}$$

$$\sqrt{n} \psi_{n-1} \otimes u_2 - \sqrt{2\pi|c|} \psi_n \otimes u_3 \mp \sqrt{2\pi d - \frac{1}{2}} \psi_n \otimes u_4, \quad n \geq 1. \tag{6.6}$$

If $4\pi d - 1 < 0$, the subspace of eigenvectors of $D^{1/3}$ with eigenvalue $\pm\sqrt{\lambda}$ is spanned by

$$-\sqrt{2\pi|c|} \psi_0 \otimes u_3 \pm i\sqrt{\frac{1}{2} - 2\pi d} \psi_0 \otimes u_4, \tag{6.7}$$

$$\sqrt{2\pi|c|} \psi_n \otimes u_1 \mp i\sqrt{\frac{1}{2} - 2\pi d} \psi_n \otimes u_2 + \sqrt{n+1} \psi_{n+1} \otimes u_4, \quad n \geq 0, \tag{6.8}$$

$$\sqrt{n} \psi_{n-1} \otimes u_2 - \sqrt{2\pi|c|} \psi_n \otimes u_3 \pm i\sqrt{\frac{1}{2} - 2\pi d} \psi_n \otimes u_4, \quad n \geq 1. \tag{6.9}$$

If $\Omega = 0$, then we obtain that $\ker D^{1/3} = \text{im } D^{1/3}$ on $\mathcal{F}_{c,d}$ is spanned by

$$\begin{aligned} &\sqrt{2\pi|c|} \psi_0 \otimes u_3, \\ &\sqrt{2\pi|c|} \psi_n \otimes u_1 + \sqrt{n+1} \psi_{n+1} \otimes u_4, \quad n \geq 0, \\ &\sqrt{n} \psi_{n-1} \otimes u_2 - \sqrt{2\pi|c|} \psi_n \otimes u_3, \quad n \geq 1. \end{aligned}$$

On summands of the form $\mathcal{S}_a^\tau \otimes \Delta$, $a > 0$, the operator $-\Omega$ has eigenvalue $4\pi^2 a^2$. Using again the projections P_\pm , we see that the subspaces of eigenvectors of $D^{1/3}$ with eigenvalue $\pm 2\pi a$ are spanned by

$$\phi_n \otimes u_2 \mp \phi_{n+1} \otimes u_4, \quad n \in \mathbb{Z}, \tag{6.10}$$

$$\pm \phi_n \otimes u_1 + \frac{i}{\sqrt{2\pi a}} \left(n + \tau + \frac{1}{2}\right) \phi_n \otimes u_2 + \phi_{n+1} \otimes u_3, \quad n \in \mathbb{Z}. \tag{6.11}$$

All summands of the form \mathcal{C}_d are in the kernel of $D^{1/3}$.

Let $\lambda \neq 0$ be an eigenvalue of $-\Omega$ on $L^2(\Sigma(X))$ and E_λ be the corresponding eigenspace. We have seen that the two projections $P_\lambda^\pm := \frac{1}{2\sqrt{\lambda}}(\pm D^{1/3} + \sqrt{\lambda}) : E_\lambda \rightarrow E_\lambda$ to the generalised eigenspaces of $D^{1/3}$ are non-trivial.

Proposition 6.5. *For any basic lattice L and any spin structure on X there exists an eigenvalue $\lambda \neq 0$ of $-\Omega$ such that the projections P_λ^\pm are not continuous.*

Proof. By Theorem 5.2, the decomposition of $L^2(\Sigma(X))$ contains a summand of the form $\mathcal{F}_{c,d} \otimes \Delta$, $c > 0$, for $\lambda = 2\pi c(4\pi d + 1) \neq 0$. By (6.2), the projections of the spinor $\zeta_n := 2i\sqrt{2\pi d + \frac{1}{2}} \psi_n \otimes u_2$ are equal to

$$P_\lambda^\pm \zeta_n = \pm\sqrt{2\pi c} \psi_n \otimes u_1 + i\sqrt{2\pi d + \frac{1}{2}} \psi_n \otimes u_2 \pm \sqrt{n} \psi_{n-1} \otimes u_4.$$

Since all ζ_n have the same length, this shows that the projections P_λ^\pm are unbounded. ■

6.3. The residual and the continuous spectrum of $D^{1/3}$

Proposition 6.6. *Let L be a basic lattice of Osc_1 . The residual spectrum of $D^{1/3}$ on $X = L \backslash \text{Osc}_1$ is empty for every spin structure on X . The continuous spectrum is equal to $\mathbb{C} \setminus \text{spec}_p(D^{1/3})$.*

Proof. To see that there is no residual spectrum, we use the symmetry properties of the spectrum. We already have seen in Section 2.3 that iD^t is essentially selfadjoint. Hence Fact 2.2 applies to this operator. We obtain that the residual spectrum $\text{spec}_r(D^t)$ is contained in the complex conjugate of $-\text{spec}_p(D^t)$. The point spectrum of $\text{spec}_p(D^{1/3})$ contains only real and purely imaginary values since the spectrum of the Casimir operator is real. Because X has even dimension, $\text{spec}_p(D^{1/3})$ is symmetric to zero. In particular, the point spectrum is invariant under complex conjugation. Thus $\text{spec}_r(D^{1/3}) \subset -\text{spec}_p(D^{1/3}) = \text{spec}_p(D^{1/3})$. Hence the residual spectrum is empty. We remark that the assumptions of [4, Satz 3.20] are satisfied, see Section 2.3. Application of item (4) of this theorem shows directly that $\text{spec}_r(D^{1/3})$ is empty.

Since the residual spectrum is empty, it suffices to show that the approximate spectrum of $D^{1/3}$ is equal to \mathbb{C} . Take $z \in \mathbb{C} \setminus \text{spec}_p(D^{1/3})$. We have to show that there is a sequence of spinors $(\Phi_j)_{j \in \mathbb{N}}$ with $\|\Phi_j\| = 1$ and $\|(D^{1/3} - z)\Phi_j\| \rightarrow 0$. Note first that for any $\lambda \in \text{spec}_p((D^{1/3})^2)$ and any corresponding unit eigenspinor Ψ of $(D^{1/3})^2$ the unit spinor

$$\Phi = \|(D^{1/3} + z)\Psi\|^{-1}(D^{1/3} + z)\Psi$$

satisfies $\|(D^{1/3} - z)\Phi\| = \|(D^{1/3} + z)\Psi\|^{-1}|\lambda - z^2|$. Therefore it suffices to find a sequence of unit eigenspinors Ψ_n of $(D^{1/3})^2$ such that $\|(D^{1/3} + z)(\Psi_n)\| \rightarrow \infty$. Recall that $L^2(\Sigma(X))$ contains summands equivalent to $\mathcal{F}_{c,d} \otimes \Delta$ for some $c > 0$, $d \in \mathbb{R}$ with $\lambda = 2\pi c(4\pi d + 1) \neq 0$, see Theorem 5.2. Therefore the existence of such a sequence follows from the unboundedness of the projections P_λ^\pm in this summand. ■

6.4. The spectrum of $D^{1/3}$

Here we want to summarise our results on the spectrum of $D^{1/3}$ from the preceding subsections.

Theorem 6.7. *Let $L \subset \text{Osc}_1$ be a basic lattice and $X := L \setminus \text{Osc}_1$ be the quotient space. We endow Osc_1 with its bi-invariant metric and fix a spin structure. Then the spectrum of the cubic Dirac operator $D^{1/3} : L^2(\Sigma(X)) \rightarrow L^2(\Sigma(X))$ is equal to \mathbb{C} . The point spectrum of $D^{1/3}$ depends on the spin structure. It consists of 0 if the kernel of Ω is non-trivial and the two roots of each of the non-zero eigenvalues of $-\Omega$. The residual spectrum is empty.*

In Theorem 6.1 we computed explicitly the point spectrum of $-\Omega$ depending on the spin structure. Combined with Theorem 6.7 we obtain in particular a complete description of the point spectrum of $D^{1/3}$.

6.5. A remark on the point spectrum of quotients by shifted lattices

In Sections 6.1 and 6.2 we proved that for basic lattices in Osc_1 the point spectrum of the cubic Dirac operator on the quotient space is discrete. Here we will show that there also exist (non-basic) lattices such that the point spectrum is not discrete. A similar statement for the wave operator has been proven in [7].

Example 6.8. Let $L = L_r(\kappa, \mu, \nu)$ be a basic lattice. We choose a real number u such that $\tilde{u} := 2\pi\kappa r u$ is irrational and consider $L' := T_u(L)$, where T_u is the automorphism defined in (3.1). We consider a spin structure for which $\varepsilon_3 = 0$ and $\varepsilon_4 = \kappa$. By (2.7), the right regular representation $L^2(\Sigma(L' \setminus \text{Osc}_1))$ is equivalent to $(T_u^{-1})^*(L^2(\Sigma(L \setminus \text{Osc}_1)))$. If $m \in \mathbb{N}_{>0}$ and $n \in \mathbb{Z}$ satisfy $(m, n) = (\varepsilon_3, \varepsilon_4) = (0, \kappa) \in \mathbb{Z}_2^2$, then $\mathcal{F}_{\frac{rm}{2}, \frac{n}{4\pi\kappa}}$ appears as a summand in $L^2(\Sigma(L \setminus \text{Osc}_1))$, see Theorem 5.2. Hence, in this case, $(T_u^{-1})^*(\mathcal{F}_{\frac{rm}{2}, \frac{n}{4\pi\kappa}}) = \mathcal{F}_{\frac{rm}{2}, \frac{n}{4\pi\kappa} - u\frac{rm}{2}}$ appears as a summand in $L^2(\Sigma(L' \setminus \text{Osc}_1))$. In particular, the point spectrum of Ω on $\Sigma(L' \setminus \text{Osc}_1)$ contains the eigenvalue of $\Omega_{\mathcal{F}_{\frac{rm}{2}, \frac{n}{4\pi\kappa} - u\frac{rm}{2}}}$, which equals $-\beta m(n + \kappa - \tilde{u}m)$. Therefore $\text{spec}_p(\Omega)$ contains the set

$$\begin{aligned} \mathcal{B} &:= \{ -\beta m(n + \kappa - \tilde{u}m) \mid m \in 2\mathbb{N}_{>0}, n + \kappa \in 2\mathbb{Z} \} \\ &= \{ -4\beta m'(n' - \tilde{u}m') \mid m' \in \mathbb{N}_{>0}, n' \in \mathbb{Z} \}. \end{aligned}$$

By Dirichlet’s approximation theorem, for every $N \in \mathbb{N}$, there exists a pair $(m', n') \in \mathbb{N} \times \mathbb{Z}$ with $0 < m' \leq N$ such that

$$|n' - \tilde{u}m'| < \frac{1}{N}.$$

This implies that the set $\{(n' - \tilde{u}m')m' \mid m' \in \mathbb{N}_{>0}, n' \in \mathbb{Z}\}$ contains infinitely many numbers in $[-1, 1]$ since \tilde{u} is irrational. Hence, \mathcal{B} and therefore also $\text{spec}_p(\Omega)$ contains an accumulation point in \mathbb{R} .

6.6. Application: the spectrum of D^t

In this subsection we consider the Dirac operator D^t for arbitrary $t \in \mathbb{R}$. Like $D^{1/3}$, D^t respects the decomposition of $L^2(\Sigma(X))$ into irreducible summands given in Theorem 5.2. In the following theorem, we consider again a basic lattice L . Remember, that in Section 4 we introduced the numbers $r, \kappa \in \mathbb{N}_{>0}$ associated with L and used them to define $\beta = \pi r/\kappa$.

Theorem 6.9. *Let $L \subset \text{Osc}_1$ be a basic lattice and $X := L \setminus \text{Osc}_1$ be the quotient space. We endow Osc_1 with its bi-invariant metric and fix a spin structure. Then the spectrum of the Dirac operator $D^t : L^2(\Sigma(X)) \rightarrow L^2(\Sigma(X))$ is equal to \mathbb{C} . The eigenvalues of D^t on the various summands of type $\mathcal{F}_{c,d} \otimes \Delta$, $\mathcal{S}_a \otimes \Delta$, and $\mathcal{C}_d \otimes \Delta$ are given in Table 3.*

rep	eigenvalues of $D^{1/3}$	eigenvalues of D^t
$\mathcal{F}_{\frac{rm}{2}, \frac{n}{4\pi\kappa}} \otimes \Delta, m > 0$	$\pm\sqrt{\lambda}, \lambda = \beta m(n + \kappa)$	$\pm(\lambda \pm \beta m\kappa(3t - 1))^{\frac{1}{2}}$
$\mathcal{F}_{\frac{rm}{2}, \frac{n}{4\pi\kappa}} \otimes \Delta, m < 0$	$\pm\sqrt{\lambda}, \lambda = \beta m(n - \kappa)$	
$\mathcal{S}_a \otimes \Delta$	$\pm 2\pi a$	$\pm 2\pi a$
$\mathcal{C}_d \otimes \Delta$	0	0

Table 3. Eigenvalues of D^t .

The residual spectrum of D^t is empty.

Proof. According to (2.2) and (3.6), we have

$$D^t = D^{1/3} + \frac{1}{2}(1 - 3t)\widehat{\gamma} = D^{1/3} + \frac{1}{2}i(3t - 1)\text{id} \otimes \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}. \tag{6.12}$$

Let us first consider the restriction of D^t to summands of type $\mathcal{F}_{c,d}$ for $c > 0$, $4\pi d + 1 \neq 0$. According to (5.6) we put $c = rm/2$ and $d = n/(4\pi\kappa)$. This gives $\lambda = 2\pi c(4\pi d + 1) = \beta m(n + \kappa)$. Recall that we already obtained eigenvectors of $D^{1/3}$ in (6.1)–(6.3). To not confuse the index used there with our n here, we will denote them by η_j^\pm and $\widehat{\eta}_j^\pm$ now. The subspaces $V_j := \text{span}\{\eta_j^+, \eta_j^-\}$ and $\widehat{V}_j := \text{span}\{\widehat{\eta}_j^+, \widehat{\eta}_j^-\}$, $j \geq 0$, are invariant under D^t and their sum spans $\mathcal{F}_{c,d}$. On these subspaces, D^t is given by the matrix

$$D^t = \begin{pmatrix} \sqrt{\lambda} + b & b \\ -b & -(\sqrt{\lambda} + b) \end{pmatrix} \tag{6.13}$$

with $b = \pm(3t - 1)(\frac{\pi c}{8\pi d + 2})^{1/2} = \pm(3t - 1)\frac{\kappa}{2}(\frac{\beta m}{n + \kappa})^{1/2}$, where the plus sign appears on V_j and the minus on \widehat{V}_j . Thus the eigenvalues given in Table 3 can be calculated. The formulas for $c < 0$ and $4\pi d - 1 \neq 0$ follow in the same way using the eigenvectors given in (6.4)–(6.9).

In the case $c > 0$, $4\pi d + 1 = 0$, $\mathcal{F}_{c,d}$ also splits into two-dimensional invariant subspaces of D^t . Indeed, D^t leaves invariant $\text{span}\{\eta_j, \zeta_j\}$ and $\text{span}\{\widehat{\eta}_j, \widehat{\zeta}_j\}$ with $\zeta_j = \psi_j \otimes u_2$ and $\widehat{\zeta}_j = \psi_j \otimes u_4$ for $j \geq 0$. It is given by

$$D^t = i\sqrt{\pi c} \begin{pmatrix} 0 & \mp 2 \\ 3t - 1 & 0 \end{pmatrix}. \tag{6.14}$$

with the minus sign on $\text{span}\{\eta_j, \zeta_j\}$ and the plus sign on $\text{span}\{\widehat{\eta}_j, \widehat{\zeta}_j\}$. For $c < 0$, $4\pi d - 1 = 0$, we obtain also a splitting into two-dimensional invariant subspaces on which D^t is again given by (6.14).

On summands of the form $\mathcal{S}_a^t \otimes \Delta$, $a > 0$, the vectors in (6.10) and (6.11) span an invariant subspace for every $n \in \mathbb{Z}$ with D^t given by

$$D^t = \pm \begin{pmatrix} 2\pi a & s \\ 0 & 2\pi a \end{pmatrix},$$

where $s = \frac{i\sqrt{2}}{2}(3t - 1)$.

On each summand of type \mathcal{C}_d the cubic Dirac operator acts trivially. Hence equation (6.12) implies that D^t is nilpotent on this summand. Thus its only eigenvalue is 0.

There is no residual spectrum for the same reasons as for $D^{1/3}$ because the point spectrum again contains only real and purely imaginary values.

The continuous spectrum is equal to $\mathbb{C} \setminus \text{spec}_p(D^t)$. To see this we use the same argument as for $D^{1/3}$. We can choose $c > 0$ and $d \in \mathbb{R}$ with $\lambda_t := 2\pi c(4\pi d + 3t) \neq 0$ such that $\mathcal{F}_{c,d} \otimes \Delta$ appears in the decomposition of $L^2(\Sigma(X))$. As in the proof of Proposition 6.6 we have only to show that the two projections $P_t^\pm := \frac{1}{2\sqrt{\lambda_t}}(\pm D^t + \sqrt{\lambda_t})$

from $\mathcal{F}_{c,d} \otimes \Delta$ to the generalised eigenspaces of D^t with eigenvalues $\pm\sqrt{\lambda_t}$ are not continuous. However, $\sqrt{\lambda_t}P_t^\pm$ differs from $\sqrt{\lambda_{1/3}}P_{1/3}^\pm$ only by a bounded operator and the assertion follows from Proposition 6.5. ■

Remark 6.10. From Theorem 6.9 we derive the following statements about the dependence of the point spectrum of D^t on t and thus on the torsion T^t .

Let us first have a look at the spectrum of $(D^t)^2$. Table 3 shows that for those spin structures for which $\varepsilon_3 = 0$, the point spectrum of $(D^{1/3})^2$ consists of two parts, namely $\mathcal{A}_j(\mu, \nu)$ and \mathcal{B}_j , where \mathcal{B}_j is one of the sets $4\beta\mathbb{Z}$, $2\beta\mathbb{Z}$, or $2\beta\mathbb{Z}_{\neq 0}$. If we now consider an arbitrary t , then the same $\mathcal{A}_j(\mu, \nu)$ also occur in the point spectrum of $(D^t)^2$, i.e., these parts of the point spectrum do not depend on t . The set of the remaining eigenvalues changes with t . The same picture arises for the point spectrum of D^t since the eigenvalues of D^t are the square roots of those of $(D^t)^2$. Since $\mathcal{A}_j(\mu, \nu)$ contains only positive numbers, the part of the point spectrum that does not depend on t is located on the real axis.

Now let the spin structure be such that its restriction to the Heisenberg group is non-trivial, i.e., such that $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \neq (0, 0, 0)$. Then the kernel of D^t is trivial if $3t\kappa$ is not an integer. Indeed, under the assumption on ε , the decomposition of $L^2(\Sigma(X))$ does not contain summands of type $\mathcal{C}_d \otimes \Delta$. So the kernel is spanned by elements of summands of type $\mathcal{F}_{c,d} \otimes \Delta$. However, Table 3 shows that under our assumption on t no summand of that kind contains non-trivial elements of the kernel.

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