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Pauli Hamiltonians with an Aharonov–Bohm flux

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Abstract. We study a two-dimensional Pauli operator describing a charged quantum particle with spin $1/2$ moving on a plane in presence of an orthogonal Aharonov–Bohm magnetic flux. We classify all the admissible self-adjoint realizations and give a complete picture of their spectral and scattering properties. Symmetries of the resulting Hamiltonians are also discussed, as well as their connection with the Dirac operator perturbed by an Aharonov–Bohm singularity.

1. Introduction

The motion of a charged particle with spin $1/2$ on a two-dimensional plane in presence of an external magnetic field perpendicular to it is described in non-relativistic quantum mechanics by the Pauli operator

$$H_{\mathrm{P}} = (\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A}))^2,$$

where, for any $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $\mathbf{A}$ is a vector potential such that $b = \mathrm{curl}\mathbf{A} = \nabla^{\perp} \cdot \mathbf{A}$, $\nabla^{\perp} := (-\partial_y, \partial_x)$, equals the magnetic field and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ is a matrix-valued vector whose components are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.1}$$

For regular magnetic fields, such an operator has already been studied in the literature and we refer to the recent works [6, 7] and references therein for further details. A similar analysis on non-simply connected domains is performed in [31], sharing some analogies with our purpose here (see also [21] for the case of the Dirac operator).

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We are indeed interested in discussing the properties of the Pauli operator when the magnetic field b is concentrated at a point, which we choose as the origin without loss of generality, i.e.,

$$b(\mathbf{x}) = 2\pi\alpha\delta(\mathbf{x}), \quad \alpha \in \mathbb{R}.$$

This ideally corresponds to an infinite solenoid of zero diameter passing through the origin. In the Coulomb gauge, the associated vector potential reads

$$\mathbf{A}(\mathbf{x}) := \alpha \frac{\mathbf{x}^{\perp}}{|\mathbf{x}|^2}, \tag{1.2}$$

where $\mathbf{x}^{\perp} := (-y, x)$. This setting matches the one of the famous Aharonov–Bohm (AB) effect [3], where however we are here taking also into account the spin degrees of freedom.

It is known that the singularity at the origin of the field affects the self-adjointness of the Hamiltonian operator and, accordingly, different self-adjoint extensions of the formal expression exist. The analogous subject for the Schrödinger operator, i.e., for a spinless charged particle, has been thoroughly analyzed in the literature (we refer to [1, 11–13, 16, 20, 35] for further details; see also [17, 18] and [24, 25] for results about the related radial operators) and our starting point is a similar classification of all self-adjoint realizations of the Pauli operator (1) (Section 2.1 and Section 2.2). Such a question has only been partially addressed in [26, 36, 37], also for more than one solenoid, yet focusing mainly on the Aharonov–Casher phenomenon about the existence and number of zero-energy modes. In this connection we also mention [19], where a distinguished self-adjoint extension is characterized for Pauli operators with measure-valued magnetic fields.

Our main goal is however a careful description of the spectral and scattering properties of the self-adjoint realizations of the Pauli operator (Section 2.3): with the exception of possible embedded eigenvalues, we provide a complete picture of the spectrum of each self-adjoint extension, including zero-energy resonances, and derive an explicit expression of the generalized eigenfunctions. Similar results for regular Pauli operators are contained in [9, 30] and [5, 15] (in the latters some applications to the Friedrichs realization of the AB Pauli operator are discussed). After proving the existence and completeness of the wave operators, this in turn allows us to obtain an expression for the scattering amplitude.

Inspired by the analysis in [22, 23] of the physical symmetries of the regular Pauli operator inherited from quantum field theory, we also investigate such symmetries (Appendix A) in presence of a singular magnetic field (Section 2.4). Furthermore, since the Pauli operator can be formally viewed as the square of the Dirac operator $D = \boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})$, we examine the interplay of this relation with the self-adjoint extensions mentioned above.

2. Main Results

As anticipated our first goal is to classify the self-adjoint realizations of the two-dimensional Pauli operator with an AB magnetic flux at the origin and characterize their spectral and scattering properties, starting from the formal expression given in (1). We use polar coordinates $(r, \vartheta) \in (0, +\infty) \times [0, 2\pi)$ and exploit the natural Hilbert space isomorphism $L^2(\mathbb{R}^2; \mathbb{C}^2) \simeq L^2(\mathbb{R}, r\,dr) \otimes L^2([0, 2\pi), \mathrm{d}\vartheta) \otimes \mathbb{C}^2$. Without loss of generality, we assume $\alpha \in (0, 1)$. The latter condition can indeed be realized by means of a unitary transformation: for any $\alpha \in \mathbb{R}$, let $\lfloor\alpha\rfloor \in \mathbb{Z}$ denote the corresponding integer part, then $(U\psi)(r, \vartheta) := e^{-i\lfloor\alpha\rfloor\vartheta}\psi(r, \vartheta)$ is a unitary operator on $L^2(\mathbb{R}^2; \mathbb{C}^2)$ fulfilling

$$U H_{\mathrm{P}} U^{-1} = (\boldsymbol{\sigma} \cdot (-i\nabla + \widetilde{\mathbf{A}}))^2, \quad \widetilde{\mathbf{A}} := (\alpha - \lfloor\alpha\rfloor)\frac{\mathbf{x}^{\perp}}{|\mathbf{x}|^2}.$$

Let us stress that U is indeed a unitary map, yet not a gauge transformation since the magnetic fluxes associated to H_{P} and $UH_{\mathrm{P}}U^{-1}$ are different.

The operator H_{P} is understood as a closable symmetric operator on the dense domain $C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\}; \mathbb{C}^2) = C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\}) \otimes \mathbb{C}^2$. However, it is also easy to verify that H_{P} is not essentially self-adjoint on such a domain, due to the singularity of $\mathbf{A}$ at the origin. At the algebraic level, the operator (1) can be written as

$$H_{\mathrm{P}} = \begin{pmatrix} \Pi_-\Pi_+ & 0 \\ 0 & \Pi_+\Pi_- \end{pmatrix}, \tag{2.1}$$

where $\Pi_\pm := (p_1 \pm i p_2)$ and p_j, $j = 1, 2$, are the components of the vector $\mathbf{p} := -i\nabla + \mathbf{A}$. It is readily seen that $\Pi_\pm$ are one the formal adjoint of the other. Passing to polar coordinates and noting that $\mathbf{A}(r, \vartheta) = \frac{\alpha}{r}(-\sin\vartheta, \cos\vartheta)$, $\partial_x = \cos\vartheta\,\partial_r - \frac{\sin\vartheta}{r}\partial_\vartheta$ and $\partial_y = \sin\vartheta\,\partial_r + \frac{\cos\vartheta}{r}\partial_\vartheta$, by a simple computation we get

$$\Pi_+ = e^{i\vartheta}\Big(-i\partial_r + \frac{1}{r}\partial_\vartheta + \frac{i\alpha}{r}\Big), \quad \Pi_- = e^{-i\vartheta}\Big(-i\partial_r - \frac{1}{r}\partial_\vartheta - \frac{i\alpha}{r}\Big).$$

Then, using the basic relation $[\partial_\vartheta, e^{\pm i\vartheta}] = \pm i e^{i\vartheta}$, we find

$$\begin{aligned} \Pi_-\Pi_+ &= \Big(-i\partial_r - \frac{1}{r}\partial_\vartheta - \frac{i(\alpha+1)}{r}\Big)\Big(-i\partial_r + \frac{1}{r}\partial_\vartheta + \frac{i\alpha}{r}\Big) \\ &= -\frac{1}{r}\partial_r(r\partial_r\cdot) + \frac{1}{r^2}(-i\partial_\vartheta + \alpha)^2, \\ \Pi_+\Pi_- &= \Big(-i\partial_r + \frac{1}{r}\partial_\vartheta + \frac{i(\alpha-1)}{r}\Big)\Big(-i\partial_r - \frac{1}{r}\partial_\vartheta - \frac{i\alpha}{r}\Big) \\ &= -\frac{1}{r}\partial_r(r\partial_r\cdot) + \frac{1}{r^2}(-i\partial_\vartheta + \alpha)^2. \end{aligned}$$

The above identities show that formally

$$\Pi_-\Pi_+ = \Pi_+\Pi_- = H_{\mathrm{S}}, \tag{2.2}$$

where $H_{\mathrm{S}} = (-i\nabla + \mathbf{A})^2$ is the AB Schrödinger operator acting on scalar functions.

2.1. Friedrichs extension

We start by discussing the properties of the most natural self-adjoint extension of H_{P}, i.e., the Friedrichs one. The simplest way to define it is to consider the quadratic form associated to H_{P}:

$$Q_{\mathrm{P}}[\boldsymbol{\psi}] := \sum_{s\in\{\uparrow,\downarrow\}} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, |(-i\nabla + \mathbf{A})\psi_s|^2, \tag{2.3}$$

making sense at least for

$$\boldsymbol{\psi} := \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \in C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\}; \mathbb{C}^2).$$

We use the following convention: spinors, i.e., functions from $\mathbb{R}^2 \to \mathbb{C}^2$, are denoted by italic bold letters (e.g., $\boldsymbol{\psi}, \boldsymbol{G}$, etc.), while regular vectors are denoted by bold letters (e.g., $\mathbf{x}, \mathbf{q}$, etc.). Making reference to the associated norm $\|\boldsymbol{\psi}\|_{Q_{\mathrm{P}}} := \|\boldsymbol{\psi}\|_2 + Q_{\mathrm{P}}[\boldsymbol{\psi}]$, we introduce the Friedrichs realization

$$\mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}] := \overline{C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\}, \mathbb{C}^2)}^{\|\cdot\|_{Q_{\mathrm{P}}}}, \qquad Q_{\mathrm{P}}^{(\mathrm{F})}[\boldsymbol{\psi}] := Q_{\mathrm{P}}[\boldsymbol{\psi}].$$

By a straightforward adaptation of [11, Proposition 1.1] (see also [12, Proposition 1.2]), we get the forthcoming Proposition 2.1. Here and in the sequel we refer to the angular average $\langle f\rangle : \mathbb{R}^+ \to \mathbb{C}$ of any scalar function $f : \mathbb{R}^2 \to \mathbb{C}$, given by

$$\langle f\rangle(r) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\vartheta\; f(r,\vartheta).$$

Proposition 2.1 (Friedrichs realization). *Let $\alpha \in (0,1)$. Then,*

(i) *the quadratic form $Q_{\mathrm{P}}^{(\mathrm{F})}$ is closed and non-negative on the domain*

$$\mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}] = \{\boldsymbol{\psi} \in H^1(\mathbb{R}^2;\mathbb{C}^2) \mid A_j\boldsymbol{\psi} \in L^2(\mathbb{R}^2;\mathbb{C}^2) \text{ for } j = 1,2\};$$

(ii) *for any $\boldsymbol{\psi} \in \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}]$ and for any $s \in \{\uparrow,\downarrow\}$,*

$$\lim_{r\to 0^+} \langle|\psi_s|^2\rangle(r) = 0, \qquad \lim_{r\to 0^+} r^2\langle|\partial_r\psi_s|^2\rangle(r) = 0; \tag{2.4}$$

(iii) *the unique self-adjoint operator* $H_{\mathrm{P}}^{(\mathrm{F})}$ *associated to* $Q_{\mathrm{P}}^{(\mathrm{F})}$ *is*

$$\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}) := \{\boldsymbol{\psi} \in \mathcal{D}[Q^{(\mathrm{F})}] \mid H_{\mathrm{P}}\boldsymbol{\psi} \in L^2(\mathbb{R}^2;\mathbb{C}^2)\},$$
$$H_{\mathrm{P}}^{(\mathrm{F})}\boldsymbol{\psi} := H_{\mathrm{P}}\boldsymbol{\psi}.$$

Remark 2.2 (Decomposition of $H_{\mathrm{P}}^{(\mathrm{F})}$). Due to the diagonal structure (2.1)–(2.2) of the Pauli operator H_{P}, from Proposition 2.1 it readily follows that

$$H_{\mathrm{P}}^{(\mathrm{F})} = \begin{pmatrix} H_{\mathrm{S}}^{(\mathrm{F})} & 0 \\ 0 & H_{\mathrm{S}}^{(\mathrm{F})} \end{pmatrix}, \tag{2.5}$$

where $H_{\mathrm{S}}^{(\mathrm{F})}$ is the Friedrichs realization of the AB Schrödinger operator (2.2) characterized in [11, Proposition 1.1].

2.2. Self-adjoint extensions

As anticipated, our first goal is to classify all the self-adjoint realizations of the operator H_{P}. To this purpose we will provide a family of quadratic forms inspired by those associated to the self-adjoint extensions of the AB Schrödinger operator H_{S} and show *a posteriori* that such forms are closed and bounded from below, as well as the fact that the associated operators exhausts all possible extensions of H_{P}.

Let then $g_\lambda^{(0)}, g_\lambda^{(-1)} \in L^2(\mathbb{R}^2;\mathbb{C})$ be the unique solutions to the AB Schödinger defect equation

$$(H_{\mathrm{S}} + \lambda^2) g_\lambda^{(\ell)} = 0 \quad \text{in } \mathbb{R}^2 \setminus \{\mathbf{0}\},$$

namely (see [34, Section 10.31]),

$$g_\lambda^{(\ell)}(r,\vartheta) = \lambda^{|\ell+\alpha|} K_{|\ell+\alpha|}(\lambda r) \frac{e^{i\ell\vartheta}}{\sqrt{2\pi}}, \qquad \ell \in \{0,-1\}. \tag{2.6}$$

We stress (see next (3.2)) that such functions have a local singularity at the origin proportional to $r^{-|\ell+\alpha|}$ and for this reason they do not belong to the domain of the Friedrichs realization $H_{\mathrm{S}}^{(\mathrm{F})}$. We construct then out these defect functions four independent solutions in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ of the formal equation $(H_{\mathrm{P}} + \lambda^2)\boldsymbol{G}_\lambda = 0$ in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, i.e.,

$$\boldsymbol{G}_{\lambda,s}^{(\ell)} := g_\lambda^{(\ell)} \begin{pmatrix} \delta_{s,\uparrow} \\ \delta_{s,\downarrow} \end{pmatrix}, \qquad s \in \{\uparrow,\downarrow\}, \ell \in \{0,-1\}. \tag{2.7}$$

By a heuristic evaluation of the expectation value $\langle \psi | H_{\mathrm{P}} | \psi \rangle$ for spinors of the form

$$\boldsymbol{\psi} = \boldsymbol{\phi}_\lambda + \sum_{s,\ell} q_s^{(\ell)} \boldsymbol{G}_{\lambda,s}^{(\ell)}$$

with $\boldsymbol{\phi}_\lambda \in \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}]$ and $\mathbf{q} := (q_\uparrow^{(0)}, q_\uparrow^{(-1)}, q_\downarrow^{(0)}, q_\downarrow^{(-1)}) \in \mathbb{C}^4$, we are lead to consider the quadratic form

$$Q_{\mathrm{P}}^{(\beta)}[\boldsymbol{\psi}] := Q_{\mathrm{P}}^{(\mathrm{F})}[\boldsymbol{\phi}_\lambda] - \lambda^2 \|\boldsymbol{\psi}\|_2^2 + \lambda^2 \|\boldsymbol{\phi}_\lambda\|_2^2 + \mathbf{q}^* \cdot [L(\lambda) + \beta]\mathbf{q}, \tag{2.8}$$

where $\beta = (\beta_{ss'}^{(\ell\ell')})_{s,s' \in \{\uparrow,\downarrow\}, \ell,\ell' \in \{0,-1\}} \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$ is any 4×4 Hermitian matrix labeling the extension and

$$L(\lambda) = \Big(\frac{\pi \lambda^{2|\ell+\alpha|}}{2\sin(\pi\alpha)} \delta_{ss'} \delta_{\ell\ell'} \Big)_{s,s' \in \{\uparrow,\downarrow\}, \ell,\ell' \in \{0,-1\}}. \tag{2.9}$$

Here and below we systematically write $\sum_{s,\ell}$ to indicate the double sum $\sum_{s \in \{\uparrow,\downarrow\}} \sum_{\ell \in \{0,-1\}}$. Moreover, we shall refer to the decomposition in angular harmonics, for fixed $s \in \{\uparrow, \downarrow\}$,

$$\psi_s(r,\vartheta) = \sum_{\ell \in \mathbb{Z}} \psi_s^{(\ell)}(r) \frac{e^{i\ell\vartheta}}{\sqrt{2\pi}}.$$

Theorem 2.3 (Self-adjoint extensions of H_{P}). *Let $\alpha \in (0,1)$ and $\lambda > 0$. Then, for any $\beta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$,*

(i) *the quadratic form $Q_{\mathrm{P}}^{(\beta)}$ is well defined on the domain*

$$\mathcal{D}[Q_{\mathrm{P}}^{(\beta)}] = \Big\{ \boldsymbol{\psi} = \boldsymbol{\phi}_\lambda + \sum_{s,\ell} q_s^{(\ell)} \boldsymbol{G}_{\lambda,s}^{(\ell)} \in L^2(\mathbb{R}^2;\mathbb{C}^2) \;|\; \boldsymbol{\phi}_\lambda \in \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}], \mathbf{q} \in \mathbb{C}^4 \Big\},$$

and it is independent of λ, closed and bounded from below;

(ii) *the unique self-adjoint operator $H_{\mathrm{P}}^{(\beta)}$ associated to $Q_{\mathrm{P}}^{(\beta)}$ is*

$$\mathcal{D}(H_{\mathrm{P}}^{(\beta)}) = \Big\{ \boldsymbol{\psi} = \boldsymbol{\phi}_\lambda + \sum_{s,\ell} q_s^{(\ell)} \boldsymbol{G}_{\lambda,s}^{(\ell)} \in \mathcal{D}[Q_{\mathrm{P}}^{(\beta)}] \;|\; \boldsymbol{\phi}_\lambda \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}),$$
$$[(L(\lambda) + \beta)\mathbf{q}]_s^{(\ell)} = 2^{|\ell+\alpha|-1} \Gamma(|\ell+\alpha|) \lim_{r \to 0^+} \frac{1}{r^{|\ell+\alpha|}} (|\ell+\alpha| + r\partial_r) \phi_{\lambda,s}^{(\ell)} \Big\}, \tag{2.10}$$

$$(H_{\mathrm{P}}^{(\beta)} + \lambda^2)\boldsymbol{\psi} := (H_{\mathrm{P}}^{(\mathrm{F})} + \lambda^2)\boldsymbol{\phi}_\lambda; \tag{2.11}$$

(iii) *the family $(H_{\mathrm{P}}^{(\beta)})_{\beta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})}$ exhausts all possible self-adjoint extensions of the symmetric operator H_{P} given in (1).*

Remark 2.4 (Friedrichs and Krein extensions). The Friedrichs Hamiltonian $H_{\mathrm{P}}^{(\mathrm{F})}$ is formally recovered taking $\beta = $ "∞"$\mathbb{1}$, i.e., setting all the charges $\mathbf{q}$ equal to zero. Another notable extension is the Krein's one, i.e., the smallest positive extension in form sense. Here, it is easier to identify it as the unique extension besides the

Friedrichs' one which is homogeneous of degree -2 under scaling, namely, the extension $H_{\mathcal{P}}^{(0)}$ with extension parameter $\beta = 0$.

Remark 2.5 (von Neumann parameterization). Given the solutions of the defect equation, it is possible to parameterize the self-adjoint extensions of H_{P} via the von Neumann theory. Denoting by $\bar{H}_{\mathrm{P}}$ the closure of the symmetric operator (1), there exists a 4-parameter family of self-adjoint extensions given, for any 4×4 unitary matrix $U \in \mathrm{M}_{4,\mathrm{Unit}}(\mathbb{C})$, by

$$\begin{aligned}\mathcal{D}(H_{\mathrm{P}}^{(U)}) &= \{\boldsymbol{\psi} = \boldsymbol{f} + \mathcal{G}_{+}\mathbf{c} + \mathcal{G}_{+}U\mathbf{c} \in L^2(\mathbb{R}^2;\mathbb{C}^2) \mid \boldsymbol{f} \in \mathcal{D}(\bar{H}_{\mathrm{P}}), \mathbf{c} \in \mathbb{C}^4\},\\ H_{\mathrm{P}}^{(U)}\boldsymbol{\psi} &= H_{\mathrm{P}}\boldsymbol{f} + i\mathcal{G}_{+}\mathbf{c} - i\mathcal{G}_{+}U\mathbf{c},\end{aligned}$$

where we put

$$\mathcal{G}_{\pm}\colon \mathbb{C}^4 \to L^2(\mathbb{R}^2,\mathbb{C}^2), \quad \mathcal{G}_{\pm}\boldsymbol{c} := \sum_{s,\ell} c_s^{(\ell)} \boldsymbol{G}_{\pm,s}^{(\ell)} / \|\boldsymbol{G}_{\pm,s}^{(\ell)}\|_2,$$

with

$$\boldsymbol{G}_{\pm,s}^{(\ell)} = g_{\pm}^{(\ell)} \begin{pmatrix} \delta_{s,\uparrow} \\ \delta_{s,\downarrow} \end{pmatrix}, \quad s \in \{\uparrow,\downarrow\}, \ell \in \{0,-1\},$$

$$g_{\pm}^{(\ell)}(r,\vartheta) = e^{\mp i\frac{\pi}{4}|\ell+\alpha|} \sqrt{\frac{4}{\pi}\cos\Big(\frac{\pi}{2}|\alpha+\ell|\Big)}\, \frac{e^{i\ell\vartheta}}{\sqrt{2\pi}}\, K_{|\ell+\alpha|}(e^{\mp i\pi/4} r),$$

for $\ell \in \{0,-1\}$. Of course, there is a one-to-one correspondence between such a family and the family of operators introduced in Theorem 2.3, which can be made explicit by deriving a relation $U = U(\beta)$ (see next proosition 2.8 and Remark 2.10).

2.3. Spectral and scattering properties

In order to investigate the spectral and scattering properties of the self-adjoint operators $H_{\mathrm{P}}^{(\beta)}$, we exploit general resolvent arguments using the Birman–Krein–Vishik theory of self-adjoint extensions to write the resolvent operator. We are going to refer to the general theory described in [38, 41].

Let us first notice that the Friedrichs Hamiltonian $H_{\mathrm{P}}^{(\mathrm{F})}$ is positive semi-definite and consider the associated resolvent operator

$$R_{\mathrm{P}}^{(\mathrm{F})}(z) := (H_{\mathrm{P}}^{(\mathrm{F})} - z)^{-1}\colon L^2(\mathbb{R}^2;\mathbb{C}^2) \to \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}), \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}^+.$$

Taking into account the diagonal structure (2.5) of the Friedrichs Hamiltonian $H_{\mathrm{P}}^{(\mathrm{F})}$ we readily get

$$R_{\mathrm{P}}^{(\mathrm{F})}(z) = \begin{pmatrix} R_{\mathrm{S}}^{(\mathrm{F})}(z) & 0 \\ 0 & R_{\mathrm{S}}^{(\mathrm{F})}(z) \end{pmatrix}, \tag{2.12}$$

where $R_{\mathrm{S}}^{(\mathrm{F})}(z) = (H_{\mathrm{S}}^{(\mathrm{F})} - z)^{-1}$ is the resolvent operator for the Friedrichs realization of the AB Schrödinger operator, acting as an integral operator with kernel [1, equation (3.2)]

$$R_{\mathrm{S}}^{(\mathrm{F})}(z;\mathbf{x},\mathbf{x}') = \sum_{\ell\in\mathbb{Z}} I_{|\ell+\alpha|}(-i\sqrt{z}(r\wedge r'))K_{|\ell+\alpha|}(-i\sqrt{z}(r\vee r'))\frac{e^{i\ell(\vartheta-\vartheta')}}{2\pi}. \quad (2.13)$$

Notice that the kernel in (2.13) is slightly different, compared to [1], as we write it using the modified Bessel functions of second kind I_ν, K_ν in place of the Bessel and Hankel functions J_ν, $H_\nu^{(1)}$ (this is obtained using the connection formulas [34, equations (10.27.6) and (10.27.8)]). Here and in the sequel we refer to the determination of the square root with $\Im\sqrt{z} > 0$ for all $z \in \mathbb{C}\setminus\mathbb{R}^+$, ensuring in particular that $\Re(-i\sqrt{z}) > 0$.

In view of the boundary conditions appearing in (2.10), we further introduce the trace operator

$$\tau = \bigoplus_{s,\ell} \tau_s^{(\ell)} : \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}) \to \mathbb{C}^4,$$
$$\tau_s^{(\ell)}\boldsymbol{\phi} := 2^{|\ell+\alpha|-1}\Gamma(|\ell+\alpha|)\lim_{r\to0^+}\frac{1}{r^{|\ell+\alpha|}}(|\ell+\alpha| + r\partial_r)\phi_s^{(\ell)}. \quad (2.14)$$

For any $z \in \mathbb{C}\setminus\mathbb{R}^+$, we put

$$\check{\mathcal{G}}(z) := \tau R_{\mathrm{P}}^{(\mathrm{F})}(z) \colon L^2(\mathbb{R}^2;\mathbb{C}^2) \to \mathbb{C}^4, \quad (2.15)$$

and define the corresponding single layer operator as

$$\mathcal{G}(z) := (\check{\mathcal{G}}(z^*))^* \colon \mathbb{C}^4 \to L^2(\mathbb{R}^2;\mathbb{C}^2). \quad (2.16)$$

The Weyl operator then reads

$$\Lambda(z) := \tau(\mathcal{G}(-1) - \mathcal{G}(z)) \colon \mathbb{C}^4 \to \mathbb{C}^4, \quad (2.17)$$

where we have chosen as a reference spectral point $z_0 = -1$. The first result we state is precisely about the resolvent of the self-adjoint realizations of H_{P}.

Theorem 2.6 (Resolvent of the self-adjoint extensions of H_{P}). *Let $\Theta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$. There exists a non-empty open set $\mathcal{Z} \subset \mathbb{C}\setminus\mathbb{R}$, such that, for any $z \in \mathcal{Z}$, the bounded operator*

$$R_{\mathrm{P}}^{(\Theta)}(z) := R_{\mathrm{P}}^{(\mathrm{F})}(z) + \mathcal{G}(z)[\Lambda(z)+\Theta]^{-1}\check{\mathcal{G}}(z) \quad (2.18)$$

is the resolvent of a self-adjoint operator $H_{\mathrm{P}}^{(\Theta)}$ coinciding with $H_{\mathrm{P}}^{(\mathrm{F})}$ on $\ker(\tau)$ *and defined by*

$$\mathcal{D}(H_{\mathrm{P}}^{(\Theta)}) := \{\boldsymbol{\psi} \in L^2(\mathbb{R}^2;\mathbb{C}^2) \mid \boldsymbol{\psi} = \boldsymbol{\varphi}_z + \mathcal{G}(z)\mathbf{q}, \boldsymbol{\varphi}_z \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}),$$
$$\mathbf{q} \in \mathbb{C}^4, \tau\boldsymbol{\varphi}_z = [\Lambda(z)+\Theta]\mathbf{q}\},$$
$$(H_{\mathrm{P}}^{(\Theta)} - z)\boldsymbol{\psi} = (H_{\mathrm{P}}^{(\mathrm{F})} - z)\boldsymbol{\varphi}_z. \quad (2.19)$$

Remark 2.7 (Range of validity of (2.18)). The set $\mathcal{Z} \subset \mathbb{C} \setminus \mathbb{R}^+$ consists of points z in the complex plane for which the 4×4 matrix $\Theta + \Lambda(z)$ is invertible, and it is not difficult to see that such a set is certainly non-empty (see next (3.8)). In fact, by [10, Theorem 2.19], the defining identity (2.18) extends to any $z \in \rho(H_{\mathrm{P}}^{(\mathrm{F})}) \cap \rho(H_{\mathrm{P}}^{(\Theta)})$, where $\rho(H_{\mathrm{P}}^{(\mathrm{F})})$ and $\rho(H_{\mathrm{P}}^{(\Theta)})$ are the resolvent sets of $H_{\mathrm{P}}^{(\mathrm{F})}$ and $H_{\mathrm{P}}^{(\Theta)}$, respectively.

Of course, the above Theorem 2.6 provides yet another parametrizion of the family of self-adjoint extensions of H_{P}. In fact, this parametrization comprises all such realizations by general arguments and therefore there must be a one-to-one correspondence with the family in Theorem 2.3.

Proposition 2.8 (Equivalence of parametrizations). *There is a one-to-one correspondence between the families* $\{H_{\mathrm{P}}^{(\Theta)}\}_{\Theta \in M_{4,\mathrm{Herm}}(\mathbb{C})}$ *and* $\{H_{\mathrm{P}}^{(\beta)}\}_{\beta \in M_{4,\mathrm{Herm}}(\mathbb{C})}$ *given by*

$$\Theta = \Theta(\beta) = L(1) + \beta, \tag{2.20}$$

where $L(\lambda)$ *is defined in* (2.9).

Remark 2.9 (Friedrichs and Krein extensions). Also in this case it appears that the Friedrichs Hamiltonian $H_{\mathrm{P}}^{(\mathrm{F})}$ is formally recovered for $\Theta = \text{“}\infty\text{”}\mathbb{1}$, while the Krein's one is simply identified by $\Theta = L(1)$.

Remark 2.10 (von Neumann and Krein parametrizations). The one-to-one correspondence between the von Neumann and Krein families of self-adjoint realizations is realized explicitly by (see, e.g., [39, Theorem 4.1 and Theorem 4.3] and [41, Theorem 3.1])

$$\Theta = \Theta(U) = -i\check{\mathcal{G}}(+i)(\hat{U} - \hat{U}_*)(\hat{U} + \hat{U}_*)^{-1}\mathcal{G}(-i),$$

where $\hat{U}$ is a unitary operator on the defect space $\operatorname{span}\{\mathcal{G}_+\mathbf{c}\}$, $\mathbf{c} \in \mathbb{C}^4$, acting as $\hat{U}\mathcal{G}_+\mathbf{c} := \mathcal{G}_- U\mathbf{c}$, and $\hat{U}_* := (H_{\mathrm{P}}^{(\mathrm{F})} - i)R_{\mathrm{P}}^{(\mathrm{F})}(-i)$ is the restriction of the Cayley transform of $H_{\mathrm{P}}^{(\mathrm{F})}$ to the same subspace. Of course, the family $H_{\mathrm{P}}^{(U)}$, $U \in \mathrm{M}_{4,\mathrm{Unit}}(\mathbb{C})$, comprises the Friedrichs and Krein realizations, which are respectively recovered for

$$U^{(\mathrm{F})} = -\mathbb{1}, \quad U^{(\mathrm{K})} = (-e^{i\pi|\alpha+\ell|}\delta_{ss'}\delta_{\ell\ell'}).$$

Let us continue the investigation of the spectral and scattering properties of the Pauli Hamiltonians characterized in Section 2.2 as distinct self-adjoint realizations in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ of the differential operator H_{P}. For convenience, we start by dealing with the Friedrichs extension and study its scattering properties with respect to the (self-adjoint) free Pauli operator

$$-\Delta_{\mathrm{P}} := \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix},$$

with domain $H^2(\mathbb{R}^2;\mathbb{C}^2)$. Notice that $\sigma(-\Delta_P) = \sigma_{ac}(-\Delta_P) = \mathbb{R}^+$, so the projector onto the subspace of absolute continuity of $-\Delta_P$ satisfies $P_{ac}(-\Delta_P) = \mathbb{1}$. More in general, for any $\Theta \in M_{4,\mathrm{Herm}}(\mathbb{C})$, we define the *wave operators*

$$\Omega_\pm^{(\Theta)} \equiv \Omega_\pm(H_P^{(\Theta)}, -\Delta_P) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_P^{(\Theta)}} e^{-it(-\Delta_P)}.$$

For later convenience, we refer here to the family $H_P^{(\Theta)}$, $\Theta \in M_{4,\mathrm{Herm}}(\mathbb{C})$, described in Theorem 2.6 (understanding $\Theta = \infty\mathbb{1}$ for the Friedrichs Hamiltonian $H_P^{(F)}$, see Theorem 2.9). We recall that, whenever they exist, the wave operators are said to be *complete* if [42, p. 19]

$$\operatorname{ran}\Omega_+^{(\Theta)} = \operatorname{ran}\Omega_-^{(\Theta)} = \operatorname{ran} P_{ac}(H_P^{(\Theta)}),$$

where $P_{ac}(H_P^{(\Theta)})$ is the spectral projector onto the absolute continuity subspace of $L^2(\mathbb{R}^2;\mathbb{C}^2)$ associated to $H_P^{(\Theta)}$. *Asymptotic completeness* further requires that one has $\sigma_{sc}(H_P^{(\Theta)}) = \varnothing$. Assuming that the wave operators exist, a fact we shall actually prove in the subsequent Theorem 2.11 and Theorem 2.15, we proceed to introduce the *scattering operator*

$$\mathrm{S}^{(\Theta)} := (\Omega_+^{(\Theta)})^* \Omega_-^{(\Theta)}.$$

Notice that $\mathrm{S}^{(\Theta)}$ is a unitary operator on $\operatorname{ran} P_{ac}(H_P^{(\Theta)})$ as soon as the wave operator $\Omega_\pm^{(\Theta)}$ are complete.

Finally, we introduce the following definition of zero-energy resonances of Pauli operators. We adopt an analogous definition for the zero-energy resonances of the Dirac operator (see next Proposition 2.23 and Proposition 2.25).

Definition 2.1 (Zero-energy resonance). A zero-energy resonance $\boldsymbol{\psi}$ of $H_P^{(\Theta)}$ is a distributional solution of the equation $H_P\boldsymbol{\psi} = 0$ in $L^2_{loc}(\mathbb{R}^2) \setminus L^2(\mathbb{R}^2)$, which fulfills the boundary condition at $\mathbf{x} = \mathbf{0}$ encoded in $\mathcal{D}(H_P^{(\Theta)})$ and remains bounded at infinity.

As anticipated, we start by analyzing the Friedrichs realization.

Theorem 2.11 (Scattering for $H_P^{(F)}$). *The wave operators $\Omega_\pm^{(F)}$ exist and are asymptotically complete. Moreover, the scattering operator $\mathrm{S}^{(F)}$ exists and is unitary on $L^2(\mathbb{R}^2;\mathbb{C}^2)$.*

A straightforward consequence of the above result is the spectral characterization of $H_P^{(F)}$.

Corollary 2.12 (Spectrum of $H_P^{(F)}$). *The spectrum of the Friedrichs Hamiltonian $H_P^{(F)}$ satisfies*

$$\sigma(H_P^{(F)}) = \sigma_{ac}(H_P^{(F)}) = [0, +\infty),$$

and, in particular, $\sigma_{pp}(H_P^{(F)}) = \sigma_{sc}(H_P^{(F)}) = \varnothing$.

To investigate further the scattering and spectrum of $H_{\mathrm{P}}^{(\mathrm{F})}$, we provide an explicit expression of the scattering matrix and amplitude, together with the scattering cross section. To proceed, let us refer to the plane waves (for $s \in \{\uparrow, \downarrow\}$ and $\mathbf{k} \in \mathbb{R}^2$)

$$\boldsymbol{\varphi}_{(s,\mathbf{k})}(\mathbf{x}) := \frac{1}{2\pi} e^{i\mathbf{k}\cdot\mathbf{x}} \begin{pmatrix} \delta_{s,\uparrow} \\ \delta_{s,\downarrow} \end{pmatrix} = \frac{1}{2\pi} \sum_{\ell} e^{i\ell(\vartheta-\omega)+i\frac{\pi}{2}|\ell|} J_{|\ell|}(kr) \begin{pmatrix} \delta_{s,\uparrow} \\ \delta_{s,\downarrow} \end{pmatrix},$$

where $\mathbf{x} = (r, \vartheta) \in \mathbb{R}^+ \times \mathbb{S}^1$, $\mathbf{k} = (k, \omega) \in \mathbb{R}^+ \times \mathbb{S}^1$ and we used [27, equation (8.511.4)] and [34, equation (10.4.1)]. Notice that, in the sense of distributions,

$$-\Delta_{\mathrm{P}} \boldsymbol{\varphi}_{(s,\mathbf{k})} = k^2 \boldsymbol{\varphi}_{(s,\mathbf{k})}.$$

Analogously, the generalized eigenfunctions $(\boldsymbol{\varphi}^{(\mathrm{F},\pm)}_{(s,\mathbf{k})})_{(s,\mathbf{k})\in\{\uparrow,\downarrow\}\times\mathbb{R}^2}$ corresponding to $\sigma_{\mathrm{ac}}(H_{\mathrm{P}}^{(\mathrm{F})})$ are the distributional solutions of the eigenvalue problem

$$H_{\mathrm{P}} \boldsymbol{\varphi}^{(\mathrm{F},\pm)}_{(s,\mathbf{k})} = k^2 \boldsymbol{\varphi}^{(\mathrm{F},\pm)}_{(s,\mathbf{k})},$$

fulfilling the local Friedrichs conditions $\nabla \boldsymbol{\varphi}^{(\mathrm{F},\pm)}_{(s,\mathbf{k})}, A_j \boldsymbol{\varphi}^{(\mathrm{F},\pm)}_{(s,\mathbf{k})} \in L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{C}^2)$ for $j = 1, 2$ (see Proposition 2.1), and the incoming (+) or outgoing (−) Sommerfeld radiation conditions

$$\lim_{r\to+\infty} r^{1/2} (\hat{\mathbf{x}} \cdot \nabla \pm ik) [\boldsymbol{\varphi}^{(\mathrm{F},\pm)}_{(s,\mathbf{k})}(\mathbf{x}) - \boldsymbol{\varphi}_{(s,\mathbf{k})}(\mathbf{x})] = 0. \tag{2.21}$$

Correspondingly, we introduce the Fourier transform

$$\mathfrak{F}: L^2(\mathbb{R}^2; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2)$$

defined as

$$(\mathfrak{F}\boldsymbol{\psi})_s(\mathbf{k}) := \sum_{s'\in\{\uparrow,\downarrow\}} \int_{\mathbb{R}^2} d\mathbf{x}\, (\boldsymbol{\varphi}_{(s,\mathbf{k})}(\mathbf{x}))^*_{s'} \psi_{s'}(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \psi_s(\mathbf{x}),$$

and the associated unitary map [8, Section 4.5.1]

$$F: L^2(\mathbb{R}^2; \mathbb{C}^2) \to \int_{\sigma(-\Delta_{\mathrm{P}})} d\lambda\, (L^2(\mathbb{R}^2; \mathbb{C}^2))_\lambda,$$

$$(F\boldsymbol{\psi})_{\lambda,s}(\omega) := (\mathfrak{F}\boldsymbol{\psi})_s(\sqrt{\lambda}, \omega) \in (L^2(\mathbb{R}^2; \mathbb{C}^2))_\lambda,$$

providing a direct integral decomposition of $L^2(\mathbb{R}^2; \mathbb{C}^2)$ with respect to the spectral measure of $-\Delta_{\mathrm{P}}$. Taking into account that $\mathrm{S}^{(\Theta)}$ commutes with the free Pauli operator $-\Delta_{\mathrm{P}}$ [42, p. 74], we proceed to define the *scattering matrix* as the fiber-wise restriction to $(L^2(\mathbb{R}^2; \mathbb{C}^2))_\lambda \equiv L^2(\mathbb{S}^1; \mathbb{C}^2)$ of the scattering operator $\mathrm{S}^{(\Theta)}$, namely,

$$\mathrm{S}^{(\Theta)}(\lambda) \boldsymbol{u}_\lambda = F \mathrm{S}^{(\Theta)} F^* \boldsymbol{u}_\lambda. \tag{2.22}$$

We shall typically refer to the associated integral kernel $\mathrm{S}^{(\mathrm{F})}_{ss'}(\lambda;\omega,\omega')$, fulfilling

$$(\mathrm{S}^{(\Theta)}(\lambda)\boldsymbol{u}_\lambda)_s(\omega) = \sum_{s'} \int_0^{2\pi} \mathrm{d}\omega' \mathrm{S}^{(\mathrm{F})}_{ss'}(\lambda;\omega,\omega') u_{\lambda,s'}(\omega')$$

for $s \in \{\uparrow,\downarrow\}$ and $\omega \in \mathbb{S}^1$. Following [29,43], we define the *scattering amplitude*

$$f^{(\Theta)}_{ss'}(\lambda;\omega,\omega') := \left(\frac{2\pi}{i\sqrt{\lambda}}\right)^{1/2} (\mathrm{S}^{(\Theta)}_{ss'}(\lambda;\omega,\omega') - \delta_{ss'}\delta(\omega-\omega')), \tag{2.23}$$

and the *differential cross section*

$$\frac{\mathrm{d}\sigma^{(\Theta)}_{ss'}}{\mathrm{d}\omega}(\lambda,\omega) := |f^{(\Theta)}_{ss'}(\lambda;\omega,0)|^2. \tag{2.24}$$

Theorem 2.13 (Generalized eigenfunctions and scattering matrix of $H^{(\mathrm{F})}_{\mathrm{P}}$). *The generalized eigenfunctions of $H^{(\mathrm{F})}_{\mathrm{P}}$ are*

$$\boldsymbol{\varphi}^{(\mathrm{F},\pm)}_{(s,\mathbf{k})}(\mathbf{x}) = \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}} e^{i\ell(\vartheta-\omega_\pm)\pm i\frac{\pi}{2}|\ell+\alpha|} J_{|\ell+\alpha|}(kr)\begin{pmatrix}\delta_{s,\uparrow}\\ \delta_{s,\downarrow}\end{pmatrix}, \tag{2.25}$$

where $\omega_+ = \omega$ and $\omega_- = \omega+\pi$. The integral kernel associated to the scattering matrix $\mathrm{S}^{(\mathrm{F})}(\lambda)$ is given by

$$\begin{aligned}\mathrm{S}^{(\mathrm{F})}_{ss'}(\lambda;\omega,\omega') &= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}} e^{i\pi(\ell-|\ell+\alpha|)+i\ell(\omega-\omega')}\delta_{ss'}\\ &= \left[\cos(\pi\alpha)\delta(\omega-\omega') + \frac{i}{\pi}\sin(\pi\alpha)\mathrm{p.v.}\left(\frac{1}{e^{i(\omega-\omega')}-1}\right)\right]\delta_{ss'},\end{aligned} \tag{2.26}$$

where $s,s' \in \{\uparrow,\downarrow\}$, $\omega,\omega' \in [0,2\pi)$ and p.v. *indicates the Cauchy principal value. Furthermore, the scattering amplitude is given by*

$$\begin{aligned}f^{(\mathrm{F})}_{ss'}(\lambda;\omega,\omega') = \left(\frac{2\pi}{i\sqrt{\lambda}}\right)^{1/2}&\Big[(\cos(\pi\alpha)-1)\delta(\omega-\omega')\\ &+\frac{i}{\pi}\sin(\pi\alpha)\mathrm{p.v.}\left(\frac{1}{e^{i(\omega-\omega')}-1}\right)\Big]\delta_{ss'},\end{aligned} \tag{2.27}$$

and the differential cross section for $\omega \neq 0$ is

$$\frac{\mathrm{d}\sigma^{(\mathrm{F})}_{ss'}}{\mathrm{d}\omega}(\lambda,\omega) = \frac{1}{2\pi\sqrt{\lambda}}\frac{\sin^2(\pi\alpha)}{\sin^2(\omega/2)}\delta_{ss'}. \tag{2.28}$$

Remark 2.14 (Pauli and AB Schrödinger operators). In accordance with the fact that $H^{(\mathrm{F})}_{\mathrm{P}}$ is just the direct sum of two copies of the Friedrichs AB Schrödinger Hamiltonian $H^{(\mathrm{F})}_{\mathrm{S}}$, see (2.5), the scattering operator and the scattering matrix also coincide with the direct sums of two copies of the analogous quantities related to the scalar case.

The analogue of Theorem 2.11 about the scattering for the self-adjoint extensions is the following.

Theorem 2.15 (Scattering for $H_{\mathrm{P}}^{(\Theta)}$). *For any $\Theta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$, the wave operators $\Omega_{\pm}^{(\Theta)}$ exist and are asymptotically complete. Moreover, the scattering operator $\mathrm{S}^{(\Theta)}$ exists and is unitary on $L^2(\mathbb{R}^2;\mathbb{C}^2)$.*

The spectrum of the self-adjoint extensions $H_{\mathrm{P}}^{(\Theta)}$ is on the other hand much richer than the one of $H_{\mathrm{P}}^{(\mathrm{F})}$.

Theorem 2.16 (Spectrum of $H_{\mathrm{P}}^{(\Theta)}$). *Let $\Theta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$. Then, the spectrum of the Hamiltonian $H_{\mathrm{P}}^{(\Theta)}$ is*

$$\sigma(H_{\mathrm{P}}^{(\Theta)}) = \sigma_{\mathrm{ac}}(H_{\mathrm{P}}^{(\Theta)}) \cup \sigma_{\mathrm{pp}}(H_{\mathrm{P}}^{(\Theta)}),$$

where

$$\sigma_{\mathrm{ac}}(H_{\mathrm{P}}^{(\Theta)}) = \mathbb{R}^+, \quad \{-\mu \in \mathbb{R}^- \mid \det[\Lambda(-\mu)+\Theta]=0\} \subseteq \sigma_{\mathrm{pp}}(H_{\mathrm{P}}^{(\Theta)}),$$

and, in particular, $\sigma_{\mathrm{sc}}(H_{\mathrm{P}}^{(\Theta)}) = \varnothing$. Furthermore, the eigenfunction associated to any negative eigenvalue $-\mu \in \sigma_{\mathrm{pp}}(H_{\mathrm{P}}^{(\Theta)})$ is given by $\mathscr{G}(-\mu)\mathbf{q}$ with $\mathbf{q} \in \ker[\Lambda(-\mu)+\Theta]$.

Remark 2.17 (Negative point spectrum of a specific extension). As an example, let us discuss the occurrence of negative eigenvalues for the extension with $\Theta = 0$. It can be checked by direct inspection that the condition $\det[\Lambda(-\mu)] = 0$, $\mu \in \mathbb{R}^+$, is fulfilled if and only if $\mu = 1$ (see the explicit expression (3.8) reported in the forthcoming Lemma 3.1 for $\Lambda(z)$). Moreover, $\Lambda(-1) = 0$ so that $\ker[\Lambda(-1)] = \mathbb{C}^4$. Accordingly, $-1 \in \sigma_{\mathrm{pp}}(H_{\mathrm{P}}^{(0)})$ is a fourthly degenerate eigenvalue of the Krein Hamiltonian and the associated eigenspace is spanned by

$$\boldsymbol{G}_{1,s}^{(\ell)} = K_{|\ell+\alpha|}(r)\frac{e^{i\ell\vartheta}}{\sqrt{2\pi}}\begin{pmatrix}\delta_{s,\uparrow}\\ \delta_{s,\downarrow}\end{pmatrix}, \quad s \in \{\uparrow,\downarrow\}, \ell \in \{0,-1\},$$

see (3.7) below, together with (2.7) and (2.6).

Remark 2.18 (Embedded eigenvalues). We do not discuss here the presence of eigenvalues embedded in the continuous spectrum. These would be exactly the exceptional points forming the set $\mathrm{e}_{+}^{(\Theta)}$ identified next in Proposition 3.3. Such a question has been investigated for regular magnetic fields in [5, 28].

We now proceed to characterize the (incoming and outgoing) generalized eigenfunctions $\boldsymbol{\varphi}_{(s,\mathbf{k})}^{(\Theta,\pm)}$ related to $\sigma_{\mathrm{ac}}(H_{\mathrm{P}}^{(\Theta)})$. The result below also allows to compute explicitly the scattering matrix and the differential cross-section but we omit the details for the sake of brevity.

Proposition 2.19 (Generalized eigenfunctions of $H_{\mathrm{P}}^{(\Theta)}$). *The generalized eigenfunctions of $H_{\mathrm{P}}^{(\Theta)}$ have the form*

$$\boldsymbol{\varphi}_{(s,\mathbf{k})}^{(\Theta,\pm)} = \boldsymbol{\varphi}_{(s,\mathbf{k})}^{(\mathrm{F},\pm)} + \mathscr{G}_{\pm}(k^2)[\Lambda_{\pm}(k^2) + \Theta]^{-1} \tau \boldsymbol{\varphi}_{(s,\mathbf{k})}^{(\mathrm{F},\pm)}, \tag{2.29}$$

where $\boldsymbol{\varphi}_{(s,\mathbf{k})}^{(\mathrm{F},\pm)}$ are the Friedrichs eigenfunctions (2.25), while $\Lambda_{\pm}(\lambda)$ and $\mathscr{G}_{\pm}(\lambda)$ are defined, respectively, as

$$\Lambda_{\pm}(\lambda) := \lim_{\varepsilon \to 0^+} \Lambda(\lambda \pm i\varepsilon), \quad \mathscr{G}_{\pm}(\lambda) := \lim_{\varepsilon \to 0^+} \mathscr{G}(\lambda \pm i\varepsilon).$$

Furthermore, the following asymptotics holds as $r \to +\infty$:

$$(\boldsymbol{\varphi}_{(s,\mathbf{k})}^{(\Theta,\pm)})_{s'}(\mathbf{x}) = \frac{1}{2\pi} e^{i\mathbf{k}\cdot\mathbf{x}} \delta_{ss'} + \frac{1}{r^{1/2}} f_{(s,s'),\mathbf{k}}^{(\Theta,\pm)} e^{\mp ikr} + \mathcal{O}\Big(\frac{1}{r^{3/2}}\Big),$$

where

$$\begin{aligned} f_{(s,s'),\mathbf{k}}^{(\Theta,\pm)} :=\ & \frac{e^{\pm i\frac{\pi}{4}}}{(2\pi)^{3/2}\sqrt{k}} \sum_{\ell\in\mathbb{Z}} (e^{\pm i\pi|\ell+\alpha|} - e^{\pm i\pi|\ell|}) e^{i\ell(\vartheta-\omega_\pm)} \delta_{ss'} \\ & + \frac{i\pi e^{\pm i\frac{\pi}{4}}}{(2\pi)^{3/2}\sqrt{k}} \sum_{\ell,\ell'\in\{0,-1\}} \{[\Lambda_{\pm}(k^2)+\Theta]^{-1}\}_{s',s}^{\ell',\ell} e^{i(\ell'\vartheta-\ell\omega_\pm)} (\pm ik)^{|\ell+\alpha|+|\ell'+\alpha|}. \end{aligned}$$

Zero-energy resonances are discussed in the next result.

Proposition 2.20 (Zero-energy resonances for $H_{\mathrm{P}}^{(\mathrm{F})}$ and $H_{\mathrm{P}}^{(\Theta)}$). *The Friedrichs Hamiltonian $H_{\mathrm{P}}^{(\mathrm{F})}$ has no zero-energy resonances. On the other hand, for any $\Theta \in$ $\mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$, the Hamiltonian $H_{\mathrm{P}}^{(\Theta)}$ has zero-energy resonances if and only if $\Lambda(0) +$ Θ is singular. More precisely, any zero-energy resonance $\boldsymbol{\psi}_0$ has the form*

$$\boldsymbol{\psi}_0 = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \quad \psi_s(r,\vartheta) = \sum_{\ell\in\{0,-1\}} q_s^{(\ell)} \frac{2^{|\ell+\alpha|-1}\Gamma(|\ell+\alpha|)}{r^{|\ell+\alpha|}} \frac{e^{i\ell\vartheta}}{\sqrt{2\pi}},$$

with $\mathbf{q} \in \ker[\Lambda(0) + \Theta]$.

Remark 2.21 (Alternative parametrization). Making reference to the quadratic form parametrization $H_{\mathrm{P}}^{(\beta)}$ of the Pauli Hamiltonian, using the bijection in Proposition 2.8 and noting that $L(1) = -\Lambda(0)$ (see (2.9) and (3.8)), we get

$$\Lambda(0) + \Theta = \beta.$$

According to Proposition 2.20, $H_{\mathrm{P}}^{(\beta)}$ possesses zero-energy resonances whenever $\det\beta = 0$.

2.4. Symmetries and connection with the Dirac operator

We investigate here the symmetries of the Pauli operator and of its self-adjoint realizations, as well as the connection with its "square root", i.e., the *Dirac operator*.

Let then $\mathcal{W}\colon L^2(\mathbb{R}^2;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2)$ be any unitary or anti-unitary operator. We say that $\mathcal{W}$ is a symmetry of the differential operator $H_{\mathrm{P}} \equiv H_{\mathrm{P}}(\mathbf{A})$ if

$$\mathcal{W} H_{\mathrm{P}}(\mathbf{A}) \mathcal{W}^{-1} = H_{\mathrm{P}}(\widetilde{\mathbf{A}}),$$

for some $\widetilde{\mathbf{A}}$ such that, in distributional sense,

$$\tilde{b} := \operatorname{curl} \widetilde{\mathbf{A}} = 2\pi\alpha\delta_{\mathbf{0}}.$$

Let us denote by σ_3 the third Pauli matrix, namely,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In view of the algebraic properties reported in Appendix A for Pauli operators with smooth vector potentials and since the differential operator of interest here is invariant under rotations around $\mathbf{x} = \mathbf{0}$, the only admissible symmetries of H_{P} are described by the following operators.

(i) *Linear transformations:*

$$\begin{aligned} &\mathcal{U}(S,T)\colon L^2(\mathbb{R}^2;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2), \\ &(\mathcal{U}\boldsymbol{\psi})(\mathbf{x}) = S(\mathbf{x})\boldsymbol{\psi}(T^{-1}\mathbf{x}), \end{aligned} \tag{2.30}$$

where (see (A.15))

$$\begin{cases} T \in \mathrm{SO}(2,\mathbb{R}), \\ S = e^{-i\eta_0\mathbb{1} - i\eta_3\sigma_3}, \quad \text{for some } \eta_0 \in C^1(\mathbb{R}^2), \eta_3 \in \mathbb{R}. \end{cases} \tag{2.31}$$

(ii) *Anti-linear transformations:*

$$\begin{aligned} &\mathcal{V}(S,T)\colon L^2(\mathbb{R}^2;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2), \\ &(\mathcal{V}\boldsymbol{\psi})(\mathbf{x}) = S(\mathbf{x})\boldsymbol{\psi}^*(T^{-1}\mathbf{x}), \end{aligned} \tag{2.32}$$

where (see (A.21))

$$\begin{cases} T \in O(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R}), \\ S = e^{-i\eta_0\mathbb{1} - i\eta_3\sigma_3}, \quad \text{for some } \eta_0 \in C^1(\mathbb{R}^2), \eta_3 \in \mathbb{R}. \end{cases}$$

By the above arguments, we deduce that the full symmetry group is

$$U_{\mathrm{em}}^{\mathrm{loc}}(1) \times U_{\mathrm{ax}}(1) \times U_{\mathrm{rot}}(1) \times \mathrm{CP}\,.$$

Among the symmetries described above, there is the very well-known $U_{\mathrm{em}}^{\mathrm{loc}}(1)$-*gauge symmetry* $\mathbf{A} \mapsto \mathbf{A} + \nabla\eta_0$ typical of the electromagnetic field, which is recovered when $T = \mathbb{1}$ and $\eta_3 = 0$ in (2.31). The two remaining continuous symmetries are given by the axial gauge and the rotational symmetries of the model. Lastly, the discrete CP-symmetry is the charge conjugation combined with parity.

We aim at investigating the extensions that are left invariant by the above symmetries (2.30) and (2.32). To that purpose, we keep track of the dependence on the magnetic potential and denote $H_{\mathrm{P}}^{(\beta)} \equiv H_{\mathrm{P}}^{(\beta)}(\mathbf{A})$. However, since we are interested in Pauli Hamiltonians with *fixed magnetic potential*, we only consider *global* gauge transformations, that is, we take $\eta_0 \in \mathbb{R}$ in (2.30) and (2.32). Notice that this means that the magnetic potential is unchanged, while we may act on the spinor in a non-trivial way. This corresponds to determining the matrices β in (2.10) such that

$$\mathcal{W} H_{\mathrm{P}}^{(\beta)}(\mathbf{A}) \mathcal{W}^{-1} = H_{\mathrm{P}}^{(\beta)}(\mathbf{A}),$$

or, equivalently,

$$\begin{aligned}
&\mathcal{W}\mathcal{D}(H_{\mathrm{P}}^{(\beta)}(\mathbf{A})) = \mathcal{D}(H_{\mathrm{P}}^{(\beta)}(\mathbf{A})),\\
&\mathcal{W}(H_{\mathrm{P}}^{(\beta)}(\mathbf{A}) + \lambda^2)\mathcal{W}^{-1}\widetilde{\boldsymbol{\psi}} = (H_{\mathrm{P}}^{(\beta)}(\mathbf{A}) + \lambda^2)\widetilde{\boldsymbol{\psi}},
\end{aligned}$$

For all $\widetilde{\boldsymbol{\psi}} \in \mathcal{W}\mathcal{D}(H_{\mathrm{P}}^{(\beta)}(\mathbf{A}))$, first for all the transformations $\mathcal{W} = \mathcal{U}$ of the form (2.30) and then for those $\mathcal{W} = \mathcal{V}$ of the form (2.32). Here and in what follows, we always assume $\lambda > 0$.

Proposition 2.22 (Symmetries). *Let $\eta_0 \in \mathbb{R}$. Then, transformations of the form* (2.30) *or* (2.32) *are symmetries of the self-adjoint extension $H_{\mathrm{P}}^{(\beta)}$ for any $\eta_3 \in \mathbb{R}$, if and only if β is a diagonal matrix.*

Concerning the connection with the Dirac operator, we first have to define it in presence of an AB magnetic potential, mostly referring to [37]. We start from the symmetric Dirac operator with domain $C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$, acting as

$$H_{\mathrm{D}}\boldsymbol{\psi} = \boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})\boldsymbol{\psi},$$

where $\mathbf{A} = \mathbf{A}(x)$ is the AB potential (1.2). Notice that, on $C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$, there holds

$$H_{\mathrm{P}} = H_{\mathrm{D}}^2.$$

We are going to determine the self-adjoint extensions of H_{D} using the von Neumann approach. To this avail, we first have to take the closure of $C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$ with respect to the graph norm $\boldsymbol{\psi} \mapsto \sqrt{\|\boldsymbol{\psi}\|^2 + \|H_{\mathrm{D}}\boldsymbol{\psi}\|^2}$. If $\boldsymbol{\psi} \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$, it is immediate to see that

$$\|H_{\mathrm{D}}\boldsymbol{\psi}\|^2 = \langle \boldsymbol{\psi} | H_{\mathrm{P}} | \boldsymbol{\psi} \rangle = Q_{\mathrm{P}}^{(\mathrm{F})}(\boldsymbol{\psi}),$$

so that

$$\mathcal{D}(\bar{H}_{\mathrm{D}}) = \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}].$$

Moreover, the operator $\bar{H}_{\mathrm{D}}$ has defect indices $(1, 1)$, and the defect spaces $\chi_\pm := \ker(\bar{H}_{\mathrm{D}}^* \mp i)$ are spanned by the functions

$$\boldsymbol{\xi}_\pm(r, \vartheta) = \begin{pmatrix} K_{1-\alpha}(r) e^{-i\vartheta} \\ \pm K_\alpha(r) \end{pmatrix}. \tag{2.33}$$

We thus have a one-parameter family of self-adjoint extensions of H_{D}, parameterized by the unitary operators mapping χ_+ into χ_-.

Proposition 2.23 (Self-adjoint extensions of H_{D}). *For any $\alpha \in (0, 1)$, the symmetric operator H_{D} admits a one-parameter family of self-adjoint extensions $H_{\mathrm{D}}^{(\gamma)}$, parameterized by $\gamma \in [0, 2\pi)$ and given by*

$$\mathcal{D}(H_{\mathrm{D}}^{(\gamma)}) = \{\boldsymbol{\psi} = \boldsymbol{\phi} + \mu(\boldsymbol{\xi}_+ + e^{i\gamma}\boldsymbol{\xi}_-) | \boldsymbol{\phi} \in \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}], \mu \in \mathbb{C}\},$$
$$H_{\mathrm{D}}^{(\gamma)}\boldsymbol{\psi} = H_{\mathrm{D}}\boldsymbol{\phi} + i\mu(\boldsymbol{\xi}_+ - i\boldsymbol{\xi}_-).$$

Alternatively, as illustrated in [37], the domain of the extensions $H_{\mathrm{D}}^{(\gamma)}$ can be described in terms of boundary conditions at the origin, as follows: given $\gamma \in [0, 2\pi)$, one can define the linear functionals $c_{-\alpha}^{\uparrow,\downarrow}$, $c_{\alpha-1}^{\uparrow,\downarrow}$ on $\mathcal{D}(H_{\mathrm{D}}^{(\gamma)})$ as

$$c_{-\alpha}^s(\boldsymbol{\psi}) = \lim_{r \to 0^+} r^\alpha \langle \psi_s \rangle = \frac{1}{2\pi} \lim_{r \to 0^+} r^\alpha \int_0^{2\pi} \mathrm{d}\vartheta\, \psi_s(r, \vartheta), \tag{2.34}$$
$$c_{\alpha-1}^s(\boldsymbol{\psi}) = \lim_{r \to 0^+} r^{1-\alpha} \langle e^{i\vartheta} \psi_s \rangle = \frac{1}{2\pi} \lim_{r \to 0^+} r^{1-\alpha} \int_0^{2\pi} \mathrm{d}\vartheta\, e^{i\vartheta} \psi_s(r, \vartheta),$$

for $s \in \{\uparrow, \downarrow\}$. Observe that, for a given $\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{(\gamma)})$, we have a decomposition $\boldsymbol{\psi} = \boldsymbol{\phi} + \mu(\boldsymbol{\xi}_+ + e^{i\gamma}\boldsymbol{\xi}_-)$ as in Proposition 2.23, so that $\boldsymbol{\phi}$ vanishes at the origin, thus having zero boundary value. The asymptotics of Bessel functions K_ν at the origin [27] then easily give

$$c_{-\alpha}^\uparrow(\boldsymbol{\psi}) = 0, \quad c_{\alpha-1}^\uparrow(\boldsymbol{\psi}) = \mu(1 + e^{i\gamma}) 2^{-\alpha} \Gamma(1 - \alpha),$$
$$c_{\alpha-1}^\downarrow(\boldsymbol{\psi}) = 0, \quad c_{-\alpha}^\downarrow(\boldsymbol{\psi}) = \mu(1 - e^{i\gamma}) 2^{-(1-\alpha)} \Gamma(\alpha),$$

where $\Gamma(z)$ is the Euler gamma function. We can then equivalently describe the domain of $H_{\mathrm{D}}^{(\gamma)}$ as

$$\mathcal{D}(H_{\mathrm{D}}^{(\gamma)}) = \Big\{ \boldsymbol{\psi} \in L^2(\mathbb{R}^2;\mathbb{C}^2) | H_{\mathrm{D}}\boldsymbol{\psi} \in L^2(\mathbb{R}^2;\mathbb{C}^2), \\ c^{\uparrow}_{\alpha-1}(\boldsymbol{\psi}) = i\cot(\gamma/2)\frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}c^{\downarrow}_{-\alpha}(\boldsymbol{\psi}), \\ c^{\uparrow}_{-\alpha}(\boldsymbol{\psi}) = 0,\ c^{\downarrow}_{\alpha-1}(\boldsymbol{\psi}) = 0 \Big\}. \tag{2.35}$$

We have already observed that the symmetric operators H_{P} and H_{D} defined on $C_c^\infty(\mathbb{R}^2 \setminus \{0\};\mathbb{C}^2)$, are such that $H_{\mathrm{D}}^2 = H_{\mathrm{P}}$. Now, we investigate whether the same property holds for the respective self-adjoint extensions. More precisely, we aim at determining which extension $H_{\mathrm{P}}^{(\beta)}$ is the square of an extension $H_{\mathrm{D}}^{(\gamma)}$, if any. It turns out that the answer to this question is actually trivial, since the square of all self-adjoint realizations of the Dirac operator coincides with the Friedrichs extension of H_{P}. As a byproduct of the proof, we are also able to classify all the extensions $H_{\mathrm{P}}^{(\beta)}$ such that $\mathcal{D}(H_{\mathrm{P}}^{(\beta)}) \subseteq \mathcal{D}(H_{\mathrm{D}}^{(\gamma)})$ for some $\gamma \in [0, 2\pi)$, i.e., both the first and second order operators are simultaneously well posed, although in general the latter is not the square of the former.

Proposition 2.24 (Dirac and Pauli operators). *For any $\gamma \in [0, 2\pi)$, there holds*

$$(H_{\mathrm{D}}^{(\gamma)})^2 = H_{\mathrm{P}}^{(\mathrm{F})}.$$

Furthermore, $\mathcal{D}(H_{\mathrm{P}}^{(\beta)}) \subseteq \mathcal{D}(H_{\mathrm{D}}^{(\gamma)})$, for some $\gamma \in [0, 2\pi)$, if and only if, for all $\lambda > 0$ large enough,

$$\begin{cases} \beta^{\ell,0}_{s,\uparrow} = \beta^{\ell,-1}_{s,\downarrow} = \infty, \\ [(L(\lambda)+\beta)^{-1}]^{-1,\ell}_{\uparrow,s} = i\cot(\gamma/2)[(L(\lambda)+\beta)^{-1}]^{0,\ell}_{\downarrow,s}, \end{cases}$$

for any $\ell \in \{-1, 0\}, s \in \{\uparrow, \downarrow\}$.

In passing, we point out the following result which has its own interest: unlike Pauli operators with an AB flux, any self-adjoint realization of the Dirac operator with an AB flux admits a zero-energy resonance.

Proposition 2.25 (Zero-energy resonances for H_{D}^{γ}). *For any $\gamma \in [0, 2\pi)$, the Dirac Hamiltonian H_{D}^{γ} has a zero-energy resonance $\boldsymbol{\psi}_0$ given by*

$$\boldsymbol{\psi}_0(r,\vartheta) = \begin{cases} \begin{pmatrix} i\cot(\gamma/2)\frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}e^{-i\vartheta}r^{-(1-\alpha)} \\ r^{-\alpha} \end{pmatrix} & \text{for } \gamma \neq 0, \\ \begin{pmatrix} e^{-i\vartheta}r^{-(1-\alpha)} \\ 0 \end{pmatrix} & \text{for } \gamma = 0. \end{cases}$$

Remark 2.26 (Resonances for Dirac and Pauli operators). The above result is consistent with the identity $(H_{\mathrm{D}}^{(\gamma)})^2 = H_{\mathrm{P}}^{(\mathrm{F})}$, stated in Proposition 2.24, and with the fact that $H_{\mathrm{P}}^{(\mathrm{F})}$ has no zero-energy resonance (see Proposition 2.20). Indeed, for any given $\gamma \in [0, 2\pi)$, even if the spinor $\boldsymbol{\psi}_0$ found in Proposition 2.25 does satisfy the distributional equation $H_{\mathrm{P}}\boldsymbol{\psi}_0 = (H_{\mathrm{D}}^{\gamma})^2\boldsymbol{\psi}_0 = 0$, it is not a resonance for the Friedrichs extension $H_{\mathrm{P}}^{(\mathrm{F})}$. This is due to the singular behavior at the origin, making it impossible to satisfy the boundary conditions encoded in $\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$ (see item *ii*) in Proposition 2.1 and Definition 2.1).

3. Proofs

This section is devoted to the proofs of the results previously stated. The first result we prove is the classification of self-adjoint extensions: we start by addressing the quadratic forms (2.8) in Section 3.1 and then apply Krein's theory in Section 3.2 to get an alternative parametrization of the family, through the expression of the resolvents. The latter are also going to be used to investigate spectral and scattering properties in Section 3.3. Finally, in Section 3.4 and Section 3.5 we study the symmetry properties of the extensions and the relation with the self-adjoint realization of the Dirac operator with an AB flux, respectively.

3.1. Quadratic forms

By studying the quadratic forms introduced in (2.8), we prove Theorem 2.3 (i) and (ii). Preliminarily, we observe that (see [27, equation (6.521.3)])

$$\|g_\lambda^{(\ell)}\|_2^2 = \frac{\pi|\ell+\alpha|}{2\sin(\pi\alpha)}\lambda^{2|\ell+\alpha|-2}, \tag{3.1}$$

and

$$g_\lambda^{(\ell)}(r,\vartheta) = \Big[\frac{\Gamma(|\ell+\alpha|)}{2^{1-|\ell+\alpha|}}\frac{1}{r^{|\ell+\alpha|}} + \frac{\Gamma(-|\ell+\alpha|)}{2^{1+|\ell+\alpha|}}\lambda^{2|\ell+\alpha|}r^{|\ell+\alpha|} + \mathcal{O}(r^{2-|\ell+\alpha|})\Big]\frac{e^{i\ell\vartheta}}{\sqrt{2\pi}}, \quad \text{for } r \to 0^+. \tag{3.2}$$

For later reference, let us also remark that (3.1) and (2.7) imply

$$\langle \boldsymbol{G}_{\lambda,s}^{(\ell)} \mid \boldsymbol{G}_{\lambda,s'}^{(\ell')}\rangle = \frac{\pi|\ell+\alpha|}{2\sin(\pi\alpha)}\lambda^{2|\ell+\alpha|-2}\delta_{ss'}\delta_{\ell\ell'}. \tag{3.3}$$

Proof of Theorem 2.3. (i) Let us first prove that the form $Q_{\mathrm{P}}^{(\beta)}$ is independent of the spectral parameter $\lambda > 0$. To this avail, let $\lambda_1 \neq \lambda_2$ and consider the two alternative representations

$$\boldsymbol{\psi} = \boldsymbol{\phi}_{\lambda_1} + \sum_{s,\ell} q_s^{(\ell)} \boldsymbol{G}_{\lambda_1,s}^{(\ell)}, \qquad \boldsymbol{\psi} = \boldsymbol{\phi}_{\lambda_2} + \sum_{s,\ell} q_s^{(\ell)} \boldsymbol{G}_{\lambda_2,s}^{(\ell)}.$$

In particular, noting that $\boldsymbol{G}_{\lambda_1,s}^{(\ell)} - \boldsymbol{G}_{\lambda_2,s}^{(\ell)} \in \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}]$ for any $s \in \{\uparrow, \downarrow\}$ and $\ell \in \{0, -1\}$, we have

$$\boldsymbol{\phi}_{\lambda_1} = \boldsymbol{\phi}_{\lambda_2} + \sum_{s,\ell} q_s^{(\ell)} (\boldsymbol{G}_{\lambda_2,s}^{(\ell)} - \boldsymbol{G}_{\lambda_1,s}^{(\ell)}).$$

Recalling the explicit expression (2.8) of the quadratic form, keeping in mind that $\boldsymbol{G}_{\lambda_1,s}^{(\ell)}, \boldsymbol{G}_{\lambda_2,s}^{(\ell)}$ are defect functions for the Pauli operator and using the identity (3.3), via some integrations by parts we infer

$$\begin{aligned}
& Q_{\mathrm{P}}^{(\beta)}\Big[\boldsymbol{\phi}_{\lambda_1} + \sum_{s,\ell} q_s^{(\ell)} \boldsymbol{G}_{\lambda_1,s}^{(\ell)}\Big] - Q_{\mathrm{P}}^{(\beta)}\Big[\boldsymbol{\phi}_{\lambda_2} + \sum_{s,\ell} q_s^{(\ell)} \boldsymbol{G}_{\lambda_2,s}^{(\ell)}\Big] \\
&\quad = -2\Re \sum_{s,\ell} q_s^{(\ell)} \lim_{r \to 0^+} \int_{\partial B_r(\mathbf{0})} \mathrm{d}\Sigma_r \boldsymbol{\phi}_{\lambda_2}^* \cdot \partial_r (\boldsymbol{G}_{\lambda_2,s}^{(\ell)} - \boldsymbol{G}_{\lambda_1,s}^{(\ell)}) \\
&\qquad - \sum_{s,\ell} |q_s^{(\ell)}|^2 \lim_{r \to 0^+} \int_{\partial B_r(\mathbf{0})} \mathrm{d}\Sigma_r (\boldsymbol{G}_{\lambda_2,s}^{(\ell)} - \boldsymbol{G}_{\lambda_1,s}^{(\ell)})^* \cdot \partial_r (\boldsymbol{G}_{\lambda_2,s}^{(\ell)} - \boldsymbol{G}_{\lambda_1,s}^{(\ell)}) \\
&\qquad + \sum_{s,\ell} |q_s^{(\ell)}|^2 \bigg[\frac{\pi}{2 \sin(\pi \alpha)} (\lambda_1^{2|\ell+\alpha|} - \lambda_2^{2|\ell+\alpha|}) \\
&\qquad\qquad + \lim_{r \to 0^+} \int_{\partial B_r(\mathbf{0})} \mathrm{d}\Sigma_r \big((\boldsymbol{G}_{\lambda_2,s}^{(\ell)})^* \cdot \partial_r \boldsymbol{G}_{\lambda_1,s}^{(\ell)} - (\partial_r \boldsymbol{G}_{\lambda_2,s}^{(\ell)})^* \cdot \boldsymbol{G}_{\lambda_1,s}^{(\ell)} \big) \bigg].
\end{aligned}$$

Using equation (2.4) for $\boldsymbol{\phi}_{\lambda_2} \in \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}]$ and the asymptotic expansion given in (3.2), by Cauchy–Schwarz inequality, we get

$$\bigg| \int_{\partial B_r(\mathbf{0})} \mathrm{d}\Sigma_r \boldsymbol{\phi}_{\lambda_2}^* \cdot \partial_r (\boldsymbol{G}_{\lambda_1,s}^{(\ell)} - \boldsymbol{G}_{\lambda_2,s}^{(\ell)}) \bigg| \leqslant C r^{|\ell+\alpha|} \sqrt{\langle |\boldsymbol{\phi}_{\lambda_2}|^2 \rangle} \xrightarrow[r \to 0^+]{} 0;$$

$$\bigg| \int_{\partial B_r(\mathbf{0})} \mathrm{d}\Sigma_r (\boldsymbol{G}_{\lambda_1,s}^{(\ell)} - \boldsymbol{G}_{\lambda_2,s}^{(\ell)})^* \cdot \partial_r (\boldsymbol{G}_{\lambda_1,s}^{(\ell)} - \boldsymbol{G}_{\lambda_2,s}^{(\ell)}) \bigg| \leqslant C r^{2|\ell+\alpha|} \xrightarrow[r \to 0^+]{} 0.$$

On the other hand, using again (3.2), we infer

$$(\boldsymbol{G}_{\lambda_2,s}^{(\ell)})^* \cdot \partial_r \boldsymbol{G}_{\lambda_1,s}^{(\ell)} - (\partial_r \boldsymbol{G}_{\lambda_2,s}^{(\ell)})^* \cdot \boldsymbol{G}_{\lambda_1,s}^{(\ell)} = \frac{\lambda_2^{2|\ell+\alpha|} - \lambda_1^{2|\ell+\alpha|}}{4 \sin(\pi \alpha) r} + \mathcal{O}(r^{1-2|\ell+\alpha|}),$$

which entails

$$\lim_{r\to 0^+}\int_{\partial B_r(\mathbf{0})} d\Sigma_r\big((\boldsymbol{G}^{(\ell)}_{\lambda_2,s})^*\cdot\partial_r\boldsymbol{G}^{(\ell)}_{\lambda_1,s}-(\partial_r\boldsymbol{G}^{(\ell)}_{\lambda_2,s})^*\cdot\boldsymbol{G}^{(\ell)}_{\lambda_1,s}\big)$$
$$=\frac{\pi}{2\sin(\pi\alpha)}\big(\lambda_2^{2|\ell+\alpha|}-\lambda_1^{2|\ell+\alpha|}\big).$$

Summing up, we obtain

$$Q^{(\beta)}_{\mathrm{P}}\Big[\boldsymbol{\phi}_{\lambda_1}+\sum_{s,\ell}q^{(\ell)}_s\boldsymbol{G}^{(\ell)}_{\lambda_1,s}\Big]-Q^{(\beta)}_{\mathrm{P}}\Big[\boldsymbol{\phi}_{\lambda_2}+\sum_{s,\ell}q^{(\ell)}_s\boldsymbol{G}^{(\ell)}_{\lambda_2,s}\Big]=0,$$

which proves that the form is independent of the spectral parameter.

Next, let us point out that, for $\lambda>0$ large enough, $L(\lambda)+\beta\geqslant c\lambda^{\min\{\alpha,1-\alpha\}}\mathbb{1}$, with $c>0$ and $\mathbb{1}$ being the identity operator, so that

$$Q^{(\beta)}_{\mathrm{P}}[\boldsymbol{\psi}]+\lambda^2\|\boldsymbol{\psi}\|_2^2\geqslant Q^{(\mathrm{F})}_{\mathrm{P}}[\boldsymbol{\phi}_\lambda]+\lambda^2\|\boldsymbol{\phi}_\lambda\|_2^2+c\lambda^{\min\{\alpha,1-\alpha\}}|\mathbf{q}|^2. \tag{3.4}$$

Since $Q^{(\mathrm{F})}_{\mathrm{P}}[\boldsymbol{\phi}_\lambda]$ is non-negative, the above relation ensures that $Q^{(\beta)}_{\mathrm{P}}$ is bounded from below. The closedness of $Q^{(\beta)}_{\mathrm{P}}$ can be then deduced by standard arguments starting from (3.4), as in [14, Theorem 2.4] (see also [45]).

(ii) Let us consider the sesquilinear form defined by polarization of (2.8), for

$$\boldsymbol{\psi}_1=\boldsymbol{\phi}_{1,\lambda}+\sum_{s,\ell}q^{(\ell)}_{1,s}\boldsymbol{G}^{(\ell)}_{\lambda,s}\quad\text{and}\quad\boldsymbol{\psi}_2=\boldsymbol{\phi}_{2,\lambda}+\sum_{s,\ell}q^{(\ell)}_{2,s}\boldsymbol{G}^{(\ell)}_{\lambda,s},$$

we get

$$Q^{(\beta)}_{\mathrm{P}}[\boldsymbol{\psi}_1,\boldsymbol{\psi}_2]=Q^{(\mathrm{F})}_{\mathrm{P}}[\boldsymbol{\phi}_{1,\lambda},\boldsymbol{\phi}_{2,\lambda}]-\lambda^2\langle\boldsymbol{\psi}_1\mid\boldsymbol{\psi}_2\rangle+\lambda^2\langle\boldsymbol{\phi}_{1,\lambda}\mid\boldsymbol{\phi}_{2,\lambda}\rangle$$
$$+\mathbf{q}_1^*\cdot(L(\lambda)+\beta)\mathbf{q}_2.$$

Fixing $\mathbf{q}_1=\mathbf{0}$, we get $Q^{(\beta)}_{\mathrm{P}}[\boldsymbol{\phi}_{1,\lambda},\boldsymbol{\psi}_2]=Q^{(\mathrm{F})}_{\mathrm{P}}[\boldsymbol{\phi}_{1,\lambda},\boldsymbol{\phi}_{2,\lambda}]-\lambda^2\langle\boldsymbol{\phi}_{1,\lambda}|\boldsymbol{\psi}_2-\boldsymbol{\phi}_{2,\lambda}\rangle$. Therefore, in order to have that $Q^{(\beta)}[\boldsymbol{\phi}_{1,\lambda},\boldsymbol{\psi}_2]=\langle\boldsymbol{\phi}_{1,\lambda}|\boldsymbol{\chi}\rangle$ for some $\boldsymbol{\chi}=:H^{(\beta)}_{\mathrm{P}}\boldsymbol{\psi}_2\in L^2(\mathbb{R}^2;\mathbb{C}^2)$, we must assume that $\boldsymbol{\phi}_{2,\lambda}\in\mathcal{D}(H^{(\mathrm{F})}_{\mathrm{P}})$ and set

$$\boldsymbol{\chi}=H^{(\mathrm{F})}_{\mathrm{P}}\boldsymbol{\phi}_{2,\lambda}-\lambda^2\sum_{s,\ell}q^{(\ell)}_{2,s}\boldsymbol{G}^{(\ell)}_{\lambda,s}\in L^2(\mathbb{R}^2;\mathbb{C}^2).$$

Taking this into account and imposing $Q^{(\beta)}_{\mathrm{P}}[\boldsymbol{\psi}_1,\boldsymbol{\psi}_2]=\langle\psi_1\mid H^{(\beta)}_{\mathrm{P}}\psi_2\rangle$ for $\mathbf{q}_1\neq 0$, via integration by parts, we deduce

$$\mathbf{q}_1^*\cdot(L(\lambda)+\beta)\mathbf{q}_2=\sum_{s,\ell}(q^{(\ell)}_{1,s})^*\langle\boldsymbol{G}^{(\ell)}_{\lambda,s}\mid(H^{(\mathrm{F})}_{\mathrm{P}}+\lambda^2)\boldsymbol{\phi}_{2,\lambda}\rangle$$
$$=\sum_{s,\ell}(q^{(\ell)}_{1,s})^*\lim_{r\to 0^+}\int_{\partial B_r(\mathbf{0})} d\Sigma_r\big((\boldsymbol{G}^{(\ell)}_{\lambda,s})^*\cdot\partial_r\boldsymbol{\phi}_{2,\lambda}-(\partial_r\boldsymbol{G}^{(\ell)}_{\lambda,s})^*\cdot\boldsymbol{\phi}_{2,\lambda}\big),$$

which, in view of (2.7), (3.2), and of the arbitrariness of $\mathbf{q}_1$, ultimately accounts for the boundary condition in (2.10). ■

3.2. Krein's theory

For later convenience, we first point out the following inequalities, valid for any $r, r' > 0$ and $\Im\sqrt{z} > 0$ (see [34, equations (10.32.2) and (10.32.8)]):

$$|I_{|\ell+\alpha|}(-i\sqrt{z}(r \wedge r'))| \leqslant \left(\frac{\sqrt{|z|}}{\Im\sqrt{z}}\right)^{|\ell+\alpha|} I_{|\ell+\alpha|}(\Im\sqrt{z}(r \wedge r')); \tag{3.5}$$

$$|K_{|\ell+\alpha|}(-i\sqrt{z}(r \vee r'))| \leqslant K_{|\ell+\alpha|}(\Im\sqrt{z}(r \vee r')). \tag{3.6}$$

We also recall the definition of $\Lambda(z)$ in (2.17), and notice that general arguments (see [38, Lemma 2.2] and following discussion) ensure that it is well defined and also entail the following identities, for all $z, w \in \mathbb{C} \setminus \mathbb{R}^+$:

$$\Lambda(z) - \Lambda(w) = (w - z)\check{\mathscr{G}}(z)\mathscr{G}(w), \quad (\Lambda(z))^* = \Lambda(z^*).$$

Proof of Theorem 2.6. Recalling once more that $H_N^{(\mathrm{F})}$ is positive semi-definite, the result follows from [38, Theorem 2.1] and [41, Theorem 3.1 and Corollary 3.2]. ■

Once the form of the resolvent operator for each of the self-adjoint extensions is known, as given by Krein's theory, it is natural to investigate the relations between those extensions and those obtained in (2.10) and (2.11). We start by proving the following.

Lemma 3.1. *For any $z \in \mathbb{C} \setminus \mathbb{R}^+$ and for all $\mathbf{q} \in \mathbb{C}^4$, there holds*

$$\mathscr{G}(z)\mathbf{q} = \sum_{s,\ell} \boldsymbol{G}^{(\ell)}_{-i\sqrt{z},s} q_s^{(\ell)}. \tag{3.7}$$

Moreover, for all $s, s' \in \{\uparrow, \downarrow\}$, $\ell, \ell' \in \{0, -1\}$ there holds

$$\Lambda^{(\ell\ell')}_{ss'}(z) = \frac{\pi}{2\sin(\pi\alpha)}[(-i\sqrt{z})^{2|\ell+\alpha|} - 1]\delta_{ss'}\delta_{\ell\ell'}. \tag{3.8}$$

Proof. To begin with, from (2.12), (2.13), and (2.14) (see also [34, Sections 10.27 and 10.29 (ii)]), we deduce that

$$\begin{aligned} &\tau_s^{(\ell)} R_{\mathrm{S}}^{(\mathrm{F})}(z^*)\boldsymbol{\psi} \\ &= \lim_{r\to 0^+} \frac{2^{|\ell+\alpha|-1}\Gamma(|\ell+\alpha|)}{r^{|\ell+\alpha|}}(|\ell+\alpha| + r\partial_r) \\ &\quad \times \int_0^\infty \mathrm{d}r' r' I_{|\ell+\alpha|}(-i\sqrt{z^*}(r \wedge r'))K_{|\ell+\alpha|}(-i\sqrt{z^*}(r \vee r'))\psi_s^{(\ell)}(r') \end{aligned}$$

$$= 2^{|\ell+\alpha|-1}\Gamma(|\ell+\alpha|)i\sqrt{z^*}$$
$$\times \lim_{r\to 0^+} r^{1-|\ell+\alpha|}\Big[K_{|\ell+\alpha|-1}(-i\sqrt{z^*}r)\int_0^r \mathrm{d}r' r' I_{|\ell+\alpha|}(-i\sqrt{z^*}r')\psi_s^{(\ell)}(r')$$
$$- I_{|\ell+\alpha|-1}(-i\sqrt{z^*}r)\int_r^\infty \mathrm{d}r' r' K_{|\ell+\alpha|}(-i\sqrt{z^*}r')\psi_s^{(\ell)}(r')\Big].$$

Exploiting the asymptotic behavior and normalization properties of the Bessel functions (see, e.g., [34, Section 10.30 (i)] and [27, equations (6.521.3)], together with the basic inequalities (3.5)–(3.6)), for $\ell \in \{0, -1\}$ and $r \to 0^+$, we infer

$$r^{1-|\ell+\alpha|}K_{|\ell+\alpha|-1}(-i\sqrt{z^*}r) = \frac{\Gamma(1-|\ell+\alpha|)}{2^{|\ell+\alpha|}(-i\sqrt{z^*})^{1-|\ell+\alpha|}} + \mathcal{O}(r^{2-2|\ell+\alpha|}),$$
$$r^{1-|\ell+\alpha|}I_{|\ell+\alpha|-1}(-i\sqrt{z^*}r) = \frac{2^{1-|\ell+\alpha|}}{\Gamma(|\ell+\alpha|)(-i\sqrt{z^*})^{1-|\ell+\alpha|}} + \mathcal{O}(r^2),$$

and

$$\Big|\int_0^r \mathrm{d}r' r' I_{|\ell+\alpha|}(-i\sqrt{z^*}r')\psi_s^{(\ell)}(r')\Big|$$
$$\leqslant \|\psi_s^{(\ell)}\|_{L^2(\mathbb{R}^+,rdr)}\Big(\int_0^r \mathrm{d}r' r' |I_{|\ell+\alpha|}(-i\sqrt{z^*}r')|^2\Big)^{1/2}$$
$$\leqslant \Big(\frac{\sqrt{|z|}}{\Im\sqrt{z^*}}\Big)^{|\ell+\alpha|}\|\psi_s^{(\ell)}\|_{L^2(\mathbb{R}^+,r\mathrm{d}r)}\Big(\int_0^r \mathrm{d}r' r' I^2_{|\ell+\alpha|}(\Im\sqrt{z^*}(r\wedge r'))\Big)^{1/2}$$
$$\leqslant C\|\boldsymbol{\psi}\|_2 r^{1+|\ell+\alpha|} \xrightarrow[r\to 0^+]{} 0,$$
$$\Big|\int_r^{+\infty} \mathrm{d}r' r' K_{|\ell+\alpha|}(-i\sqrt{z^*}r')\psi_s^{(\ell)}(r')\Big|$$
$$\leqslant \|\psi_s^{(\ell)}\|_{L^2(\mathbb{R}^+,r\mathrm{d}r)}\Big(\int_r^{+\infty} \mathrm{d}r' r' |K_{|\ell+\alpha|}(-i\sqrt{z^*}r')|^2\Big)^{1/2}$$
$$\leqslant \|\psi_s^{(\ell)}\|_{L^2(\mathbb{R}^+,r\mathrm{d}r)}\Big(\int_r^{+\infty} \mathrm{d}r' r' K^2_{|\ell+\alpha|}(\Im\sqrt{z^*}r')\Big)^{1/2}$$
$$\leqslant C\Big(\int_0^{+\infty} \mathrm{d}r' r' |K_{|\ell+\alpha|}(\Im\sqrt{z^*}r')|^2\Big)^{1/2}\|\boldsymbol{\psi}\|_2 < +\infty.$$

In view of the above considerations, by dominated convergence, we obtain

$$\tau_s^{(\ell)} R_{\mathrm{S}}^{(\mathrm{F})}(z^*)\boldsymbol{\psi} = \int_0^\infty \mathrm{d}r'r' \int_0^{2\pi} \mathrm{d}\vartheta' (-i\sqrt{z^*})^{|\ell+\alpha|} K_{|\ell+\alpha|}(-i\sqrt{z^*}r')\psi_s(r',\vartheta') \frac{e^{-i\ell\vartheta'}}{\sqrt{2\pi}}.$$

The identity (3.7) then follows by simple duality arguments, recalling the definition (2.16) of $\mathscr{G}(z)$ and the explicit expression for $\boldsymbol{G}_{\lambda,s}^{(\ell)}$ given by (2.6) and (2.7). Notice also that $(K_\nu(w))^* = K_\nu(w^*)$ for all $\nu > 0$, $w \in \mathbb{C}$ [34, equation (10.34.7)] and $(-i\sqrt{z^*})^* = -i\sqrt{z}$.

On the other side, exploiting again basic features of the Bessel functions, it can be checked that $\boldsymbol{G}_{1,s}^{(\ell)} - \boldsymbol{G}_{-i\sqrt{z},s}^{(\ell)} \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$. Then, a direct computation yields

$$\begin{aligned}
[\Lambda(z)\mathbf{q}]_s^{(\ell)} &= \sum_{s',\ell'} q_{s'}^{(\ell')} \tau_s^{(\ell)} (\boldsymbol{G}_{1,s'}^{(\ell')} - \boldsymbol{G}_{-i\sqrt{z},s'}^{(\ell')}) \\
&= \sum_{s',\ell'} q_{s'}^{(\ell')} 2^{|\ell+\alpha|-1}\Gamma(|\ell+\alpha|)\delta_{ss'}\delta_{\ell\ell'} \\
&\quad \times \lim_{r\to 0^+} \frac{1}{r^{|\ell+\alpha|}} (|\ell+\alpha| + r\partial_r)[\boldsymbol{G}_{1,s'}^{(\ell')} - \boldsymbol{G}_{-i\sqrt{z},s'}^{(\ell')}]_s^{(\ell)} \\
&= \frac{\pi}{2\sin(\pi|\ell+\alpha|)} [(-i\sqrt{z})^{2|\ell+\alpha|} - 1] q_s^{(\ell)},
\end{aligned}$$

which ultimately accounts for (3.8). ∎

Proof of Proposition 2.8. Fixing $z = -\lambda^2$, with $\lambda > 0$, a direct comparison makes evident that the self-adjoint extensions $H_{\mathrm{P}}^{(\beta)}$ and $H_{\mathrm{P}}^{(\Theta)}$ do indeed coincide if and only if $\boldsymbol{\phi}_\lambda = \boldsymbol{\varphi}_{-\lambda^2} \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$ and, accordingly,

$$[L(\lambda) + \beta]\mathbf{q} = [\Lambda(-\lambda^2) + \Theta]\mathbf{q}, \quad \text{for all } \mathbf{q} \in \mathbb{C}^4. \tag{3.9}$$

On account of the explicit expressions for $L(\lambda)$ and $\Lambda(-\lambda^2)$ reported respectively in (2.9) and (3.8), it appears that the above condition (3.9) is actually equivalent to (2.20). ∎

Finally, we can complete the proof of the classification of self-adjoint extensions.

Proof of Theorem 2.3 (iii). The exhaustiveness of Krein's classification, combined with the one-to-one correspondence provided by Proposition 2.8 yield the result. ∎

3.3. Spectral and scattering properties

Let us mention that many of the results on scattering described in the sequel rely on the *Limiting Absorption Principle* (LAP) for resolvent operators. In this connection, we refer to the weighted spaces

$$L^2_u(\mathbb{R}^2;\mathbb{C}^2) := L^2(\mathbb{R}^2,(1+|\mathbf{x}|^2)^{u/2}\mathrm{d}\mathbf{x})\otimes\mathbb{C}^2,$$

with $u\in\mathbb{R}$, and to the associated Banach spaces of bounded operators

$$\mathcal{B}(u,u') := \mathcal{B}(L^2_u(\mathbb{R}^2;\mathbb{C}^2);L^2_{u'}(\mathbb{R}^2;\mathbb{C}^2)).$$

Any resolvent $R^{(\Theta)}_{\mathrm{P}}(z)$ is said to *enjoy LAP* if the limits

$$R^{(\Theta)}_{\mathrm{P},\pm}(\lambda) := \lim_{\varepsilon\to 0+} R^{(\Theta)}_{\mathrm{P}}(\lambda\pm i\varepsilon)$$

exist in $\mathcal{B}(u,-u)$ for some $u>0$ and for all $\lambda\in\sigma_{\mathrm{ac}}(H^{(\Theta)}_{\mathrm{P}})\setminus \mathrm{e}_+(H^{(\Theta)}_{\mathrm{P}})$, where $\mathrm{e}_+(H^{(\Theta)}_{\mathrm{P}})$ is the (possibly empty) discrete set of eigenvalues embedded in the absolutely continuous spectrum.

We start by considering the Friedrichs Hamiltonian.

Proposition 3.2 (LAP for $H^{(\mathrm{F})}_{\mathrm{P}}$). *For any $\lambda\in\mathbb{R}^+$, the limits*

$$R^{(\mathrm{F})}_{\mathrm{P},\pm}(\lambda) := \lim_{\varepsilon\to 0+} R^{(\mathrm{F})}_{\mathrm{P}}(\lambda\pm i\varepsilon)$$

exist in $\mathcal{B}(u,-u)$ for any $u>1$ and the convergence is locally uniform.

Proof. The thesis is a straightforward consequence of [29, Proposition 7.3] and of the diagonal structure of the Friedrichs Hamiltonian, see (2.5). ∎

The action of $R^{(\mathrm{F})}_{\mathrm{P},\pm}(\lambda)$ can be deduced from the explicit representation (2.13) for the corresponding Schrödinger resolvent operator and reads, for $s\in\{\uparrow,\downarrow\}$,

$$\begin{aligned}
&(R^{(\mathrm{F})}_{\mathrm{P},\pm}(\lambda)\boldsymbol{\psi})_s(r,\vartheta)\\
&= \int_0^\infty \mathrm{d}r'r' \int_0^{2\pi}\mathrm{d}\vartheta' \sum_{\ell\in\mathbb{Z}} I_{|\ell+\alpha|}(\mp i\sqrt{\lambda}(r\wedge r'))\\
&\quad\times K_{|\ell+\alpha|}(\mp i\sqrt{\lambda}(r\vee r'))\psi_s(r',\vartheta')\frac{e^{i\ell(\vartheta-\vartheta')}}{2\pi}\\
&= \frac{i}{4}\int_0^\infty \mathrm{d}r'r' \int_0^{2\pi}\mathrm{d}\vartheta' \sum_{\ell\in\mathbb{Z}} J_{|\ell+\alpha|}(\pm\sqrt{\lambda}(r\wedge r'))H^{(1)}_{|\ell+\alpha|}(\pm\sqrt{\lambda}(r\vee r'))\\
&\quad\times\psi_s(r',\vartheta')e^{i\ell(\vartheta-\vartheta')}.
\end{aligned}$$

In the second line we have used the connection formulas reported in [34, equations (10.27.6) and (10.27.8)].

Proof of Theorem 2.11. Also in this case, the thesis follows from classical results on the Schrödinger Hamiltonian, see [43] and [29, Proposition 7.4]. The absence of singular continuous spectrum is ensured by the LAP established in Proposition 3.2 for $R_{\mathrm{P}}^{(\mathrm{F})}(z)$ (see, e.g., [2, Theorem 6.1] and [33, Corollary 4.7]). The stated properties of the scattering operator ultimately follow by standard arguments. ∎

Proof of Corollary 2.12. Since the Friedrichs quadratic form is non-negative (see equation (2.3)), it appears that $\sigma(H_{\mathrm{P}}^{(\mathrm{F})}) \subseteq \mathbb{R}^+$. We already noticed that Proposition 3.2 ensures the absence of singular continuous spectrum. Moreover, the existence and completeness of the wave operators established in Theorem 2.11 grants that $\sigma_{\mathrm{ac}}(H_{\mathrm{P}}^{(\mathrm{F})}) = \sigma_{\mathrm{ac}}(-\Delta_{\mathrm{P}}) = \mathbb{R}^+$. Finally, it can be checked by direct inspection that $H_{\mathrm{P}}^{(\mathrm{F})}$ has no eigenvalue (see also [15, Theorem 3.3]). ∎

We address the explicit form of the generalized eigenfunction of the Friedrichs Hamiltonian.

Proof of Theorem 2.13. By decomposition in angular harmonics and some explicit computations, one obtains formula (2.25) (see also [4, 43, 44]). Notice that, despite solving the radial eigenvalue problem, the Bessel functions $Y_{|\ell+\alpha|}$ do not appear in (2.25) since they do not satisfy the proper local behavior close to $\mathbf{x} = \mathbf{0}$. Let us further remark that the coefficients in the expansion (2.25) have been fixed so as to fulfill the radiation conditions (2.21). Indeed, using the known asymptotic expansion of the Bessel functions [34, equation (10.7.8)], it can be checked that there exist $f_{\mathbf{k}}^{(\mathrm{F},\pm)} \in L^2(\mathbb{S}^1; \mathbb{C})$ such that

$$\boldsymbol{\varphi}_{(s,\mathbf{k})}^{(\mathrm{F},\pm)}(\mathbf{x}) = \left[\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{2\pi} + f_{\mathbf{k}}^{(\mathrm{F},\pm)} \frac{e^{\mp i|\mathbf{k}||\mathbf{x}|}}{|\mathbf{x}|^{1/2}} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{3/2}}\right)\right]\begin{pmatrix}\delta_{s,\uparrow}\\ \delta_{s,\downarrow}\end{pmatrix}, \quad \text{for } |\mathbf{x}| \to +\infty. \tag{3.10}$$

More precisely, one has

$$f_{\mathbf{k}}^{(\mathrm{F},\pm)}(\vartheta) = \frac{e^{\pm i\frac{\pi}{4}}}{(2\pi)^{3/2}\sqrt{k}} \sum_{\ell\in\mathbb{Z}} (e^{\pm i\pi|\ell+\alpha|} - e^{\pm i\pi|\ell|}) e^{i\ell(\vartheta-\omega_\pm)}. \tag{3.11}$$

Incidentally, we notice that the above expansion shows that $\boldsymbol{\varphi}_{(s,\mathbf{k})}^{(\mathrm{F},\pm)} \in L^2_{-u}(\mathbb{R}^2; \mathbb{C}^2)$ for any $u > 2$.

Keeping in mind that $H_{\mathrm{P}}^{(\mathrm{F})}$ has purely absolutely continuous spectrum (see Corollary 2.12), we define the modified Fourier transforms

$$\begin{aligned}
&\mathfrak{F}_{\pm}^{(\mathrm{F})} \colon L^2(\mathbb{R}^2; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2),\\
&(\mathfrak{F}_{\pm}^{(\mathrm{F})} \boldsymbol{\psi})_s(\mathbf{k}) := \sum_{s'\in\{\uparrow,\downarrow\}} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, \overline{(\varphi_{(s,\mathbf{k})}^{(\mathrm{F},\pm)}(\mathbf{x}))_{s'}}\, \psi_{s'}(\mathbf{x}).
\end{aligned} \tag{3.12}$$

Let us now return to the wave operators $\Omega_{\pm}^{(\mathrm{F})}$, whose existence and asymptotic completeness have been established in Theorem 2.11. These are known to fulfill the identity (see, e.g., [33, Theorem 5.5])

$$\Omega_{\pm}^{(\mathrm{F})} = (\mathfrak{F}_{\pm}^{(\mathrm{F})})^* \mathfrak{F}.$$

Accordingly, the unitary scattering operator is given by

$$\mathrm{S}^{(\mathrm{F})} = \mathfrak{F}^* \mathfrak{F}_{+}^{(\mathrm{F})} (\mathfrak{F}_{-}^{(\mathrm{F})})^* \mathfrak{F}. \tag{3.13}$$

Recalling the definition (2.22) of the scattering matrix $\mathrm{S}^{(\mathrm{F})}(\lambda)$ and making reference to (2.25), (3.12), and (3.13), we get (2.26) by distributional computations (see [43, equation (4.8)] and [29, p. 315]). In particular, let us mention that an elementary calculation yields

$$\begin{aligned}
&\sum_{s''} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, (\varphi_{(s,\mathbf{k})}^{(\mathrm{F},+)})_{s''}^*(\mathbf{x}) (\varphi_{(s',\mathbf{k}')}^{(\mathrm{F},-)})_{s''}(\mathbf{x}) \\
&\quad = \frac{1}{2\pi} \sum_{\ell\in\mathbb{Z}} e^{i\ell(\omega-\omega'+\pi)-i\pi|\ell+\alpha|} \int_0^\infty \mathrm{d}r\, r\, J_{|\ell+\alpha|}(kr) J_{|\ell+\alpha|}(k'r) \delta_{ss'}.
\end{aligned}$$

By means of [27, equation (6.541)] and [34, equations (10.40.1) and (10.40.2)], it can be checked that in the sense of distributions there holds

$$\begin{aligned}
&\int_0^\infty \mathrm{d}r\, r\, J_{|\ell+\alpha|}(kr) J_{|\ell+\alpha|}(k'r) \\
&\quad = \lim_{\zeta\to+\infty} \int_0^\infty \mathrm{d}r\, \frac{\zeta^2 r}{\zeta^2 + r^2} J_{|\ell+\alpha|}(kr) J_{|\ell+\alpha|}(k'r) \\
&\quad = \lim_{\zeta\to+\infty} \zeta^2 I_{|\ell+\alpha|}(\zeta(k \wedge k')) K_{|\ell+\alpha|}(\zeta(k \vee k')) \\
&\quad = \frac{1}{2k} \delta(k' - k) = \delta((k')^2 - k^2).
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
&\sum_{s''} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, (\varphi_{(s,\mathbf{k})}^{(\mathrm{F},+)}(\mathbf{x}))_{s''}^* (\varphi_{(s',\mathbf{k}')}^{(\mathrm{F},-)}(\mathbf{x}))_{s''} \\
&\quad = \frac{1}{2\pi} \sum_{\ell\in\mathbb{Z}} e^{i\ell(\omega-\omega')+i\pi(\ell-|\ell+\alpha|)} \delta((k')^2 - k^2) \delta_{ss'},
\end{aligned}$$

which is a key ingredient for the derivation of the first equality in (2.26). For the second equality we refer to [29, equation (4.8)]. Finally, on account of (2.23), (2.24), and (2.26) one readily infers (2.27) and (2.28). ∎

We now deal with the other self-adjoint realizations, starting with the following analogue of Proposition 3.2.

Proposition 3.3 (LAP for $H_{\mathrm{P}}^{(\Theta)}$). *Let $\Theta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$. Then, there exists a (possibly empty) discrete set $\mathrm{e}_{+}^{(\Theta)}$, with $\#[\mathrm{e}_{+}^{(\Theta)}] \leqslant 4$, such that for any $\lambda \in \mathbb{R}^{+} \setminus \mathrm{e}_{+}^{(\Theta)}$ the limits*

$$R_{\mathrm{P},\pm}^{(\Theta)}(\lambda) := \lim_{\varepsilon \to 0^{+}} R_{\mathrm{P}}^{(\Theta)}(\lambda \pm i\varepsilon)$$

exist in $\mathcal{B}(u,-u)$ for any $u > 1$ and the convergence is locally uniform. Moreover, there holds

$$R_{\mathrm{P},\pm}^{(\Theta)}(\lambda) = R_{\mathrm{P},\pm}^{(\mathrm{F})}(\lambda) + \mathcal{G}_{\pm}(\lambda)[\Lambda_{\pm}(\lambda) + \Theta]^{-1}\check{\mathcal{G}}_{\pm}(\lambda),$$

where

$$\begin{aligned} \Lambda_{\pm}(\lambda) &:= \lim_{\varepsilon \to 0^{+}} \Lambda(\lambda \pm i\varepsilon) \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C}), \\ \mathcal{G}_{\pm}(\lambda) &:= \lim_{\varepsilon \to 0^{+}} \mathcal{G}(\lambda \pm i\varepsilon) \in \mathcal{B}(\mathbb{C}^4; L^2_{-u}(\mathbb{R}^2;\mathbb{C}^2)); \\ \check{\mathcal{G}}_{\pm}(\lambda) &:= \lim_{\varepsilon \to 0^{+}} \tau R_{\mathrm{P}}^{(\mathrm{F})}(\lambda \mp i\varepsilon) \in \mathcal{B}(L^2_{u}(\mathbb{R}^2;\mathbb{C}^2);\mathbb{C}^4). \end{aligned} \tag{3.14}$$

Proof. Let us refer to the Krein formula (2.18) for the resolvent operator $R_{\mathrm{P}}^{(\Theta)}(z)$. We firstly recall that the Friedrichs resolvent $R_{\mathrm{P}}^{(\mathrm{F})}(z)$ enjoys LAP in $\mathcal{B}(u,-u)$ for any $u > 1$, see Proposition 3.2. On the other hand, using the explicit expression (3.8) for $\Lambda(z)$, we obtain

$$\begin{aligned} (\Lambda_{\pm}(\lambda))_{ss'}^{(\ell\ell')} &= \lim_{\varepsilon \to 0^{+}} \Lambda_{ss'}^{(\ell\ell')}(\lambda \pm i\varepsilon) \\ &= \frac{\pi}{2\sin(\pi|\ell+\alpha|)}[e^{\mp i\pi|\ell+\alpha|}\lambda^{|\ell+\alpha|} - 1]\delta_{ss'}\delta_{\ell\ell'}. \end{aligned} \tag{3.15}$$

From here we deduce that, depending on the specific choice of Θ, the matrices $\Lambda_{\pm}(\lambda) + \Theta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$ can indeed become singular for suitable values of $\lambda \in \mathbb{R}^{+}$. We indicate with $\mathrm{e}_{+}(H_{\mathrm{P}}^{(\Theta)})$ the collection of such exceptional points and notice that its cardinality is at most 4. It is evident that the convergence in (3.15) is uniform on any compact subset of $\mathbb{R}^{+} \setminus \mathrm{e}_{+}(H_{\mathrm{P}}^{(\Theta)})$, so the same can be said for the inverses $[\Lambda_{\pm}(\lambda) + \Theta]^{-1}$.

To say more, using (2.6), (2.7), and (3.7), together with the Bessel connection formula [34, equation (10.27.8)], for any $\mathbf{q} \in \mathbb{C}^4$, we infer

$$\begin{aligned} (\mathcal{G}_{\pm}(\lambda)\mathbf{q})_s(r,\vartheta) &= \lim_{\varepsilon \to 0^{+}} (\mathcal{G}(\lambda \pm i\varepsilon)\mathbf{q})_s(r,\vartheta) \\ &= \frac{i\pi}{2}\sum_{\ell} q_s^{(\ell)}(\mp\sqrt{\lambda})^{|\ell+\alpha|} H^{(1)}_{|\ell+\alpha|}(\mp\sqrt{\lambda}r)\frac{e^{i\ell\vartheta}}{\sqrt{2\pi}}. \end{aligned} \tag{3.16}$$

Taking into account the regularity of the Hankel function $H_\nu^{(1)}$ and its asymptotic expansions for small and large arguments [34, equations (10.7.7) and (10.17.5)], by an elementary change of variable it can be checked that

$$\|\mathcal{G}_\pm(\lambda)\mathbf{q}\|^2_{L^2_{-u}(\mathbb{R}^2;\mathbb{C}^2)} \leqslant C\lambda^{|\ell+\alpha|-1}\sum_{\ell\in\{0,-1\}}\left(\int_0^1 \mathrm{d}\rho\frac{1}{\rho^{2|\ell+\alpha|-1}} + \lambda^{u/2}\int_1^\infty \mathrm{d}\rho\frac{1}{\rho^u}\right)|\mathbf{q}|^2.$$

The above estimate shows that $\mathcal{G}_\pm(\lambda)$, with $\lambda > 0$, are bounded operators from $\mathbb{C}^4$ into $L^2_{-u}(\mathbb{R}^2;\mathbb{C}^2)$ for any $u > 1$. On top of that, using a known integral representation for the Bessel functions [27, equation (8.421.9)] it can be checked that the limit in (3.16) is attained uniformly on any compact subset of $\mathbb{R}^+$. Since $\check{\mathcal{G}}(z)$ is the adjoint of $\mathcal{G}(z^*)$, see (2.15) and (2.16), by elementary duality considerations the above arguments also prove that the limits $\check{\mathcal{G}}_\pm(\lambda)$ defined in (3.14) identify a pair of bounded operators from $(L^2_{-u}(\mathbb{R}^2;\mathbb{C}^2))' \simeq L^2_u(\mathbb{R}^2;\mathbb{C}^2)$ to $(\mathbb{C}^4)' \simeq \mathbb{C}^4$ for any $u > 1$. ∎

Proof of Theorem 2.15. Let us first remark that the resolvent operator associated to $H_\mathrm{P}^{(\Theta)}$ is a finite rank perturbation of the resolvent related to the Friedrichs Hamiltonian $H_\mathrm{P}^{(\mathrm{F})}$. This suffices to infer that the wave operators $\Omega_\pm(H_\mathrm{P}^{(\Theta)}, H_\mathrm{P}^{(\mathrm{F})})$ exist and are complete [42, Theorem XI.9]. On the other hand, recall that existence and completeness of the wave operators $\Omega_\pm(H_\mathrm{P}^{(\mathrm{F})}, -\Delta_\mathrm{P})$ has already been established in Theorem 2.11. Then, existence and completeness of $\Omega_\pm^{(\Theta)}$ follows readily from the chain rule for wave operators [42, Chapter XI, p. 18, Proposition 2]. Finally, we deduce asymptotic completeness noting that the LAP established in Proposition 3.3 for $R_\mathrm{P}^{(\Theta)}(z)$ ensures the absence of singular continuous spectrum. ∎

Proof of Theorem 2.16. On one side, the existence and asymptotic completeness of the wave operators $\Omega_\pm(H_\mathrm{P}^{(\Theta)}, -\Delta_\mathrm{P})$ ensure that $\sigma_\mathrm{ac}(H_\mathrm{P}^{(\Theta)}) = \sigma_\mathrm{ac}(-\Delta_\mathrm{P}) = \mathbb{R}^+$ and the absence of singular continuous spectrum. On the other side, from [40, Theorem 3.4] it follows that the map $\mathbf{q} \mapsto \mathcal{G}(-\mu)\mathbf{q}$ is a bijection from $\ker[\Lambda(-\mu) + \Theta]$ to $\ker(H_\mathrm{P}^{(\Theta)} + \mu)$, which proves the part of the thesis regarding $\sigma_\mathrm{pp}(H_\mathrm{P}^{(\Theta)})$. ∎

Proof of Proposition 2.19. We deduce the expression in (2.29) by a straightforward adaptation of [33, Theorem 5.1]. Using (2.25) and the Bessel function asymptotics [34, equation (10.7.3)] we get that

$$\begin{aligned}\tau_{s'}^{(\ell)}\boldsymbol{\varphi}^{(\mathrm{F},\pm)}_{(s,\mathbf{k})} &= 2^{|\ell+\alpha|-1}\Gamma(|\ell+\alpha|)\lim_{r\to0+}\frac{1}{r^{|\ell+\alpha|}}(|\ell+\alpha| + r\partial_r)(\varphi^{(\mathrm{F},\pm)}_{(s,\mathbf{k})})^{(\ell)}_{s'} \\ &= \delta_{ss'}\frac{2^{|\ell+\alpha|-1}}{\sqrt{2\pi}}\Gamma(|\ell+\alpha|)e^{\pm i\frac{\pi}{2}|\ell+\alpha|-i\ell\omega_\pm} \\ &\quad\times\lim_{r\to0+}\frac{1}{r^{|\ell+\alpha|}}(|\ell+\alpha| + r\partial_r)J_{|\ell+\alpha|}(kr) = (\pm ik)^{|\ell+\alpha|}\frac{e^{-i\ell\omega_\pm}}{\sqrt{2\pi}}\delta_{ss'}.\end{aligned}$$

Recalling the explicit expressions (3.15) and (3.16), we obtain

$$(\boldsymbol{\varphi}^{(\Theta,\pm)}_{(s,\mathbf{k})})_{s'}(r,\vartheta) = \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}} e^{i\ell(\vartheta-\omega_\pm)\pm i\frac{\pi}{2}|\ell+\alpha|} J_{|\ell+\alpha|}(kr)\delta_{ss'} + \frac{i}{4}\sum_{\ell,\ell'\in\{0,-1\}}\{[\Lambda_\pm(k^2)+\Theta]^{-1}\}^{\ell',\ell}_{s',s} e^{i(\ell'\vartheta-\ell\omega_\pm)} \times (\pm ik)^{|\ell+\alpha|}(\mp k)^{|\ell'+\alpha|} H^{(1)}_{|\ell'+\alpha|}(\mp kr).$$

To say more, in view of (3.10) and (3.11), by means of [34, equation (10.17.5)] we deduce the following, for $|\mathbf{x}|\to+\infty$,

$$f^{(\Theta,\pm)}_{(s,s'),\mathbf{k}} = \frac{e^{\pm i\frac{\pi}{4}}}{(2\pi)^{3/2}\sqrt{k}}\sum_{\ell\in\mathbb{Z}}(e^{\pm i\pi|\ell+\alpha|}-e^{\pm i\pi|\ell|})e^{i\ell(\vartheta-\omega_\pm)}\delta_{ss'} + \frac{i\pi e^{\pm i\frac{\pi}{4}}}{(2\pi)^{3/2}\sqrt{k}}\sum_{\ell,\ell'\in\{0,-1\}}\{[\Lambda_\pm(k^2)+\Theta]^{-1}\}^{\ell',\ell}_{s',s}e^{i(\ell'\vartheta-\ell\omega_\pm)}(\pm ik)^{|\ell+\alpha|+|\ell'+\alpha|}.$$

This confirms that $\boldsymbol{\varphi}^{(\Theta,+)}_{(s,\mathbf{k})}$ and $\boldsymbol{\varphi}^{(\Theta,-)}_{(s,\mathbf{k})}$ fulfill, respectively, the incoming and outgoing Sommerfeld radiation conditions. ∎

Proof of Proposition 2.20. By decomposition in angular harmonics and an explicit calculation, it can be checked that the only distributional solutions of the zero-energy equation $H_{\mathrm{P}}\boldsymbol{\psi}=0$ are of the form

$$\psi_s(r,\vartheta) = \sum_{\ell\in\mathbb{Z}}\frac{e^{i\ell\vartheta}}{\sqrt{2\pi}}[c^{(\ell)}_s r^{-|\ell+\alpha|} + d^{(\ell)}_s r^{|\ell+\alpha|}],$$

with suitable coefficients $\mathbf{c},\mathbf{d}\in\mathbb{C}^4$ and for $s\in\{\uparrow,\downarrow\}$. The condition of uniform boundedness at infinity forces us to fix $\mathbf{d}=0$. On one hand, the local Friedrichs conditions $\nabla\boldsymbol{\phi}_0, A_j\boldsymbol{\phi}_0\in L^2_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{C}^2)$ demand that $\mathbf{c}=0$. On the other hand, for a general extension, to exhibit the proper singular behavior at $\mathbf{x}=\mathbf{0}$ encoded in $\mathcal{D}(H^{(\Theta)}_{\mathrm{P}})$, the above solutions must be locally of the form $\boldsymbol{\psi}=\boldsymbol{\phi}_0+\mathcal{G}(0)\mathbf{q}$ with $\tau\boldsymbol{\phi}_0=[\Lambda(0)+\Theta]\mathbf{q}$, see (2.19). Given that there is no regular part, namely $\boldsymbol{\phi}_0=\mathbf{0}$, this requirement can be fulfilled only if

$$c^{(\ell)}_s = \begin{cases}\dfrac{\Gamma(|\ell+\alpha|)}{2^{1-|\ell+\alpha|}}q^{(\ell)}_s & \text{for } \ell\in\{0,-1\},\\ 0 & \text{for } \ell\in\mathbb{Z}\setminus\{0,-1\},\end{cases} \qquad \text{for some } \mathbf{q}\in\ker[\Lambda(0)+\Theta],$$

which yields the thesis. ∎

3.4. Symmetries

Proof of Proposition 2.22. Consider the transformations (2.30), as the case (2.32) is completely analogous. Then,

$$(\mathcal{U}\boldsymbol{\psi})(\mathbf{x}) = e^{-i\eta_0 - i\eta_3\sigma_3}\boldsymbol{\psi}(T^{-1}\mathbf{x}), \quad T \in \mathrm{SO}(2,\mathbb{R}), \eta_0, \eta_3 \in \mathbb{R}.$$

Concerning the Friedrichs realization, the computations in Appendix A show that

$$\mathcal{U}H_{\mathrm{P}}^{(\mathrm{F})}(\mathbf{A})\mathcal{U}^{-1} = H_{\mathrm{P}}^{(\mathrm{F})}(\mathbf{A}). \tag{3.17}$$

We now consider a generic self-adjoint extension $H_{\mathrm{P}}^{(\beta)}$, $\beta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$, belonging to the family characterized in Theorem 2.3. Notice that, in order to prove the invariance of the domain, it suffices to prove the inclusion

$$\mathcal{U}\mathcal{D}(H_{\mathrm{P}}^{(\beta)}(\mathbf{A})) \subseteq \mathcal{D}(H_{\mathrm{P}}^{(\beta)}(\mathbf{A})).$$

Given $\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{P}}^{(\beta)}(\mathbf{A}))$, such an element decomposes as in (2.10) and then

$$\mathcal{U}\boldsymbol{\psi} = \mathcal{U}\boldsymbol{\phi}_\lambda + \sum_{s,\ell} q^{(\ell)}{}_s \mathcal{U}\boldsymbol{G}_{\lambda,s}^{(\ell)}.$$

Thus, one needs to rewrite the last terms in the above formula as in (2.10), defining a new charge $\tilde{\mathbf{q}}$, and then proceed to check the boundary conditions. To this aim, since the multiplicative factor $e^{-i\eta_0}$ drops in the latter, in order to simplify the computations we can take $\eta_0 = 0$, so that we actually consider

$$(\mathcal{U}\boldsymbol{\psi})(\mathbf{x}) = e^{-i\eta_3\sigma_3}\boldsymbol{\psi}(T^{-1}\mathbf{x}).$$

Writing the rotation matrix T as

$$T = \begin{pmatrix} \cos\varsigma & -\sin\varsigma \\ \sin\varsigma & \cos\varsigma \end{pmatrix}, \quad \varsigma \in [0, 2\pi),$$

and using (2.6) and (2.7) one finds

$$\mathcal{U}\boldsymbol{G}_{\lambda,\uparrow}^{(\ell)} = e^{-i(\eta_3+\varsigma\ell)}\boldsymbol{G}_{\lambda,\uparrow}^{(\ell)}, \quad \mathcal{U}\boldsymbol{G}_{\lambda,\downarrow}^{(\ell)} = e^{i(\eta_3-\varsigma\ell)}\boldsymbol{G}_{\lambda,\downarrow}^{(\ell)}.$$

Then, $\tilde{\boldsymbol{\psi}} := \mathcal{U}\boldsymbol{\psi}$ can be rewritten as

$$\tilde{\boldsymbol{\psi}} = \tilde{\boldsymbol{\phi}}_\lambda + \sum_{s,\ell} \tilde{q}_s^{(\ell)} \boldsymbol{G}_{\lambda,s}^{(\ell)},$$

with $\tilde{\boldsymbol{\phi}}_\lambda := \mathcal{U}\boldsymbol{\phi}_\lambda$ and the new charges

$$\tilde{q}_\uparrow^{(\ell)} = e^{-i(\eta_3+\varsigma\ell)}q_\uparrow^{(\ell)}, \quad \tilde{q}_\downarrow^{(\ell)} = e^{i(\eta_3-\varsigma\ell)}q_\downarrow^{(\ell)}.$$

Notice that

$$\tilde{\boldsymbol{\phi}}_\lambda(x) = (\mathcal{U}\boldsymbol{\phi}_\lambda)(x) = \begin{pmatrix} e^{-i\eta_3}\phi_{\lambda,\uparrow}(T^{-1}x) \\ e^{i\eta_3}\phi_{\lambda,\downarrow}(T^{-1}x) \end{pmatrix}.$$

Simple calculations give the following results for the Fourier coefficients involved

$$\tilde{\phi}^{(\ell)}_{\lambda,\uparrow} = e^{-i(\eta_3+\varsigma\ell)}\phi^{(\ell)}_{\lambda,\uparrow}, \quad \tilde{\phi}^{(\ell)}_{\lambda,\downarrow} = e^{i(\eta_3-\varsigma\ell)}\phi^{(\ell)}_{\lambda,\downarrow}.$$

Let us now examine the boundary conditions in (2.10). After some long but straightforward algebraic computations one sees that the following conditions must be fulfilled:

$$\begin{aligned}
&(L(\lambda)+\beta)^{(\ell\ell')}_{\uparrow\uparrow}(e^{i\varsigma(\ell-\ell')}-1) = 0,\\
&(L(\lambda)+\beta)^{(\ell\ell')}_{\uparrow\downarrow}(e^{2i\eta_3+i\varsigma(\ell-\ell')}-1) = 0,\\
&(L(\lambda)+\beta)^{(\ell\ell')}_{\downarrow\uparrow}(e^{-2i\eta_3+i\varsigma(\ell-\ell')}-1) = 0,\\
&(L(\lambda)+\beta)^{(\ell\ell')}_{\downarrow\downarrow}(e^{i\varsigma(\ell-\ell')}-1) = 0.
\end{aligned}$$

Notice that the second and third equations are indeed equivalent, since $L(\lambda)$, β are Hermitian. Therefore, for generic values of η_3, ς, since $L(\lambda)$ is diagonal, β must be a diagonal matrix too.

Concerning the action of the operator, thanks to (3.17) there holds

$$\begin{aligned}
\mathcal{U}(H_{\mathrm{P}}^{(\beta)}+\lambda^2)\mathcal{U}^{-1}(\mathcal{U}\boldsymbol{\psi}) &= \mathcal{U}(H_{\mathrm{P}}^{(\beta)}+\lambda^2)\boldsymbol{\psi} = \mathcal{U}(H_{\mathrm{P}}^{(\mathrm{F})}+\lambda^2)\boldsymbol{\psi}\\
&= \mathcal{U}(H_{\mathrm{P}}^{(\mathrm{F})}+\lambda^2)\mathcal{U}^{-1}(\mathcal{U}\boldsymbol{\psi}) = (H_{\mathrm{P}}^{(\mathrm{F})}+\lambda^2)(\mathcal{U}\boldsymbol{\psi})\\
&= (H_{\mathrm{P}}^{(\beta)}+\lambda^2)(\mathcal{U}\boldsymbol{\psi}).
\end{aligned}$$ ∎

3.5. Comparison with the Dirac operator

Proof of Proposition 2.24. Consider any of the self-adjoint extensions $H_{\mathrm{P}}^{(\beta)}$, and take $\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{P}}^{(\beta)})$ as in (2.10). We first prove that if we additionally require that $\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{(\gamma)})^2)$, for some $\gamma \in [0, 2\pi)$, then we must have $\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$. In other words, our aim is to prove that

$$\mathcal{D}(H_{\mathrm{P}}^{(\beta)}) \cap \mathcal{D}((H_{\mathrm{D}}^{(\gamma)})^2) \subseteq \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}).$$

By inspection of the proof below, we actually prove a stronger result (see (3.20) and (3.23)), that is,

$$\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}) \subseteq \mathcal{D}((H_{\mathrm{D}}^{(\gamma)})^2), \quad \text{for any } \gamma \in [0, 2\pi). \tag{3.18}$$

To this aim, we need to show that $\mathbf{q} = \mathbf{0}$, referring to the decomposition in (2.10).

This will be achieved by imposing

$$\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{(\gamma)}), \qquad H_{\mathrm{D}}^{(\gamma)} \boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{(\gamma)}). \tag{3.19}$$

We fix the first condition exploiting the characterization of the domain through boundary conditions at the origin, as in (2.35). Recalling (2.6) and (2.7), we observe that, since $\boldsymbol{\phi}_\lambda \in \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}]$, the spinor vanishes at the origin, so that, for $s \in \{\uparrow, \downarrow\}$,

$$c^s_{-\alpha}(\boldsymbol{\phi}_\lambda) = 0, \quad c^s_{\alpha-1}(\boldsymbol{\phi}_\lambda) = 0. \tag{3.20}$$

Then, thanks to the asymptotics (3.2), it is not hard to see that

$$0 = c^{\uparrow}_{-\alpha}(\boldsymbol{\psi}) = 2^{-(1-\alpha)}\Gamma(\alpha) q_{\uparrow}^{(0)} \implies q_{\uparrow}^{(0)} = 0.$$

Similarly, there holds

$$0 = c^{\downarrow}_{\alpha-1}(\boldsymbol{\psi}) = 2^{-\alpha}\Gamma(1-\alpha) q_{\downarrow}^{(-1)} \implies q_{\downarrow}^{(-1)} = 0.$$

Moreover, imposing

$$c^{\uparrow}_{\alpha-1}(\boldsymbol{\psi}) = i\cot(\gamma/2) \frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)} c^{\downarrow}_{-\alpha}(\boldsymbol{\psi}),$$

and using again (3.2), we get

$$q_{\uparrow}^{(-1)} = i\cot(\gamma/2) q_{\downarrow}^{(0)}. \tag{3.21}$$

We now need to impose the second condition in (3.19), namely $H_{\mathrm{D}}^{(\gamma)} \boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{(\gamma)})$. We preliminarily observe that ϕ_λ can be rewritten as

$$\boldsymbol{\phi}_\lambda = R^{(\mathrm{F})}(-\lambda^2) \boldsymbol{f}, \quad \text{for some } \boldsymbol{f} = \begin{pmatrix} f^{\uparrow} \\ f^{\downarrow} \end{pmatrix} \in L^2(\mathbb{R}^2; \mathbb{C}^2), \tag{3.22}$$

where $R^{(\mathrm{F})}(-\lambda^2)$ is the resolvent of the Friedrichs extension $H_{\mathrm{P}}^{(\mathrm{F})}$. Building on this fact and exploiting (2.13), we want to prove that we also have

$$c^s_{-\alpha}(H_{\mathrm{D}} \boldsymbol{\phi}_\lambda) = 0, \quad c^s_{\alpha-1}(H_{\mathrm{D}} \boldsymbol{\phi}_\lambda) = 0, \tag{3.23}$$

and thus we only need to impose boundary conditions on the singular part of the wave function, in order to find further restrictions on the charges $q_{\uparrow}^{(-1)}, q_{\downarrow}^{(0)}$. Recall that, in polar coordinates, H_{D} reads

$$H_{\mathrm{D}} = \begin{pmatrix} 0 & e^{-i\vartheta}(-i\partial_r - \frac{\partial_\vartheta + i\alpha}{r}) \\ e^{i\vartheta}(-i\partial_r + \frac{\partial_\vartheta + i\alpha}{r}) & 0 \end{pmatrix}.$$

Moreover, by (2.13) and (3.22), we get

$$\begin{aligned}(H_{\mathrm{D}}\boldsymbol{\phi}_\lambda)_\uparrow = &\sum_{k\in\mathbb{Z}}\Big[\Big(-i\partial_r - i\,\frac{(k+\alpha)}{r}\Big)K_{|k+\alpha|}(\lambda r)\Big] \\ &\times \int_0^r \mathrm{d}r'r' I_{|k+\alpha|}(\lambda r') f_\downarrow^k(r')\,\frac{e^{i(k-1)\vartheta}}{\sqrt{2\pi}} \\ &+ \sum_{k\in\mathbb{Z}}\Big[\Big(-i\partial_r - i\,\frac{(k+\alpha)}{r}\Big)I_{|k+\alpha|}(\lambda r)\Big] \\ &\times \int_r^{+\infty} \mathrm{d}r'r' K_{|k+\alpha|}(\lambda r') f_\downarrow^k(r')\frac{e^{i(k-1)\vartheta}}{\sqrt{2\pi}},\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}(H_{\mathrm{D}}\boldsymbol{\phi}_\lambda)_\downarrow = &\sum_{k\in\mathbb{Z}}\Big[\Big(-i\partial_r + i\,\frac{(k+\alpha)}{r}\Big)K_{|k+\alpha|}(\lambda r)\Big] \\ &\times \int_0^r \mathrm{d}r'r' I_{|k+\alpha|}(\lambda r') f_\uparrow^k(r')\,\frac{e^{i(k+1)\vartheta}}{\sqrt{2\pi}} \\ &+ \sum_{k\in\mathbb{Z}}\Big[\Big(-i\partial_r + i\,\frac{(k+\alpha)}{r}\Big)I_{|k+\alpha|}(\lambda r)\Big] \\ &\times \int_r^{+\infty} \mathrm{d}r'r' K_{|k+\alpha|}(\lambda r') f_\uparrow^k(r')\,\frac{e^{i(k+1)\vartheta}}{\sqrt{2\pi}}.\end{aligned} \tag{3.25}$$

Notice that in the above formula boundary terms coming from the r-dependent extremes of integration cancel out. Recall that the verification of the boundary conditions require the evaluation of the linear functionals (2.34). Let us prove that

$$c_{-\alpha}^{\uparrow}(H_{\mathrm{D}}\boldsymbol{\phi}_\lambda) = 0, \tag{3.26}$$

the other conditions in (3.23) being treated similarly. Taking the angular average and using basic properties of Bessel functions, we find

$$\begin{aligned}\langle (H_{\mathrm{D}}\boldsymbol{\phi}_\lambda)_\uparrow\rangle(r) = i\lambda\Big(&K_\alpha(r)\int_0^r \mathrm{d}r'r' I_{1+\alpha}(\lambda r') f_\downarrow^1(r') \\ &- I_\alpha(r)\int_r^{+\infty}\mathrm{d}r'r' K_{1+\alpha}(\lambda r') f_\downarrow^1(r')\Big).\end{aligned}$$

We are now led to analyze the asymptotic behavior of the last term on the right-hand side of the above formula, as $r \to 0^+$. To this aim, recall the asymptotics of Bessel functions $I_\nu(t) \sim t^\nu$ and $K_\nu(t) \sim t^{-\nu}$ as $t \to 0^+$. Then, there holds

$$\left| \int_0^r \mathrm{d}r' r' I_{1+\alpha}(\lambda r') f_\downarrow^1(r') \right| \leqslant \|f_\downarrow\|_2 \left(\int_0^r \mathrm{d}r' r' (I_{1+\alpha}(\lambda r'))^2 \right)^{1/2} \leqslant C r^{2+\alpha},$$

where $C > 0$ is a constant independent of r. Moreover, we get

$$\left| \int_r^{+\infty} \mathrm{d}r' r' K_{1+\alpha}(\lambda r') f_\downarrow^1(r') \right| \leqslant \|f_\downarrow\|_2 \left(\int_r^{+\infty} \mathrm{d}r' r' (K_{1+\alpha}(\lambda r'))^2 \right)^{1/2} \leqslant C r^{-\alpha}.$$

Combining the above observations, we find

$$\langle (H_{\mathrm{D}} \boldsymbol{\phi}_\lambda)_\uparrow \rangle (r) = O(1), \quad \text{as } r \to 0^+,$$

and then

$$c_{-\alpha}^{\uparrow}(H_{\mathrm{D}} \boldsymbol{\phi}_\lambda) = \lim_{r \to 0^+} r^\alpha \langle (H_{\mathrm{D}} \boldsymbol{\phi}_\lambda)_\uparrow \rangle (r) = 0.$$

As already remarked, the other conditions in (3.23) follow by similar arguments.

Now, observe that

$$\begin{aligned} H_{\mathrm{D}} \boldsymbol{\psi} &= H_{\mathrm{D}} \boldsymbol{\phi}_\lambda + q_\uparrow^{(-1)} H_{\mathrm{D}} \boldsymbol{G}_{\lambda,\uparrow}^{(-1)} + q_\downarrow^{(0)} H_{\mathrm{D}} \boldsymbol{G}_{\lambda,\downarrow}^{(0)} \\ &= H_{\mathrm{D}} \boldsymbol{\phi}_\lambda + i\lambda (q_\uparrow^{(-1)} \boldsymbol{G}_{\lambda,\downarrow}^{(0)} + q_\downarrow^{(0)} \boldsymbol{G}_{\lambda,\uparrow}^{(-1)}), \end{aligned}$$

as we have

$$\boldsymbol{G}_{\lambda,\uparrow}^{(0)}(r) = \frac{\lambda^{1-\alpha}}{2\sqrt{2\pi}} (\boldsymbol{\xi}_+(\lambda r) + \boldsymbol{\xi}_-(\lambda r)), \quad \boldsymbol{G}_{\lambda,\downarrow}^{(0)}(r) = \frac{\lambda^{1-\alpha}}{2\sqrt{2\pi}} (\boldsymbol{\xi}_+(\lambda r) - \boldsymbol{\xi}_-(\lambda r)),$$

with the $\boldsymbol{\xi}_\pm$ as in (2.33). By the previous remarks, checking that $H_{\mathrm{D}}^{(\gamma)} \boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{(\gamma)})$ leads to

$$c_{-\alpha}^{\uparrow}(H_{\mathrm{D}}^{(\gamma)} \boldsymbol{\psi}) = 0, \quad c_{\alpha-1}^{\downarrow}(H_{\mathrm{D}}^{(\gamma)} \boldsymbol{\psi}) = 0,$$

$$c_{\alpha-1}^{\uparrow}(\boldsymbol{\psi}) = i \cot(\gamma/2) \frac{2^{1-2\alpha} \Gamma(1-\alpha)}{\Gamma(\alpha)} c_{-\alpha}^{\downarrow}(\boldsymbol{\psi}) \implies q_\uparrow^{(-1)} = -i \tan(\gamma/2) q_\downarrow^{(0)}.$$

Combining the above conditions with (3.21), we get

$$q_\uparrow^{(-1)} = q_\downarrow^{(0)} = 0.$$

Thus, we conclude that $\boldsymbol{\psi} = \boldsymbol{\phi}_\lambda \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$.

On the other hand, as already remarked, the above arguments actually yields the inclusion (3.18), so that in order to conclude proof we only need to show that

$$(H_{\mathrm{D}}^{(\gamma)})^2|_{\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})} = H_{\mathrm{P}}^{(\mathrm{F})},$$

and thus since $H_{\mathrm{P}}^{(\mathrm{F})}$ and $(H_{\mathrm{D}}^{(\gamma)})^2$ are both self-adjoint, they must coincide. To this aim, it is useful to recall the following properties of Bessel functions [27, equation (8.486)]

$$\Big(\frac{\mathrm{d}}{\mathrm{d}r} \pm \frac{\nu}{r}\Big) I_\nu(r) = I_{\nu\mp1}(r), \qquad \Big(\frac{\mathrm{d}}{\mathrm{d}r} \pm \frac{\nu}{r}\Big) K_\nu(r) = -K_{\nu\mp1}(r).$$

Indeed, using those properties in (3.24) and (3.25), and arguing as in the proof of (3.26) it is not hard to see that, given $\boldsymbol{\phi} \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$, $H_{\mathrm{D}}\boldsymbol{\phi} = O(1)$, as $r \to 0^+$, i.e., it is regular at the origin. Then we conclude that

$$(H_{\mathrm{D}}^{(\gamma)})^2\boldsymbol{\phi} = H_{\mathrm{D}}^2\boldsymbol{\phi} = H_{\mathrm{P}}\boldsymbol{\phi} = H_{\mathrm{P}}^{(\mathrm{F})}\boldsymbol{\phi},$$

where we have used the fact that, as differential operators, $H_{\mathrm{D}}^2 = H_{\mathrm{P}}$, and the claim is proved. ■

Proof of Proposition 2.25. Proceeding as in the proof of Proposition 2.20, it can be checked that the only distributional solutions of the zero-energy equation $H_{\mathrm{D}}\boldsymbol{\psi} = 0$ are of the form

$$\boldsymbol{\psi}(r,\vartheta) = \begin{pmatrix} \sum_{\ell\in\mathbb{Z}} d_\uparrow^{(\ell)} \frac{e^{i\ell\vartheta}}{\sqrt{2\pi}} r^{\ell+\alpha} \\ \sum_{\ell\in\mathbb{Z}} d_\downarrow^{(\ell)} \frac{e^{i\ell\vartheta}}{\sqrt{2\pi}} r^{-(\ell+\alpha)} \end{pmatrix},$$

with suitable coefficients $\mathbf{d} \in \mathbb{C}^4$. The condition of uniform boundedness at infinity forces us to fix $d_\uparrow^{(\ell)} = 0$ for $\ell \geqslant 0$ and $d_\downarrow^{(\ell)} = 0$ for $\ell \leqslant -1$. On the other hand, the condition $\boldsymbol{\psi} \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ entails $d_\uparrow^{(\ell)} = 0$ for $\ell \leqslant -2$ and $d_\downarrow^{(\ell)} = 0$ for $\ell \geqslant 1$. Summing up, the only admissible zero-energy resonances have the form

$$\boldsymbol{\psi}(r,\vartheta) = \begin{pmatrix} d_\uparrow^{(-1)} \frac{e^{-i\vartheta}}{\sqrt{2\pi}} \frac{1}{r^{1-\alpha}} \\ d_\downarrow^{(0)} \frac{1}{\sqrt{2\pi}} \frac{1}{r^{\alpha}} \end{pmatrix}.$$

Then, it can be easily checked that the boundary conditions associated to $\mathcal{D}(H_{\mathrm{D}}^{(\gamma)})$, see (2.35), are verified if and only if

$$d_\uparrow^{(-1)} = i\cot(\gamma/2)\frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)} d_\downarrow^{(0)},$$

which concludes the proof. ■

A. Symmetries of 2D Pauli and Dirac operators

We consider here generic Pauli and Dirac operators of the form

$$H_{\mathrm{P}}(\mathbf{A}) := (\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A}))^2 = \sigma_j \sigma_\ell (-i\partial_j + A_j)(-i\partial_\ell + A_\ell), \tag{A.1}$$

$$H_{\mathrm{D}}(\mathbf{A}) := \boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A}) = \sigma_j(-i\partial_j + A_j), \tag{A.2}$$

where $\mathbf{A} = (A_1, A_2)$ is any vector-valued distribution in $\mathbb{R}^2$ and we are using Einstein's convention to sum over the repeated indices $j, \ell \in \{1, 2\}$. Let us stress that here we are uniquely concerned with the algebraic features of $H_{\mathrm{P}}(\mathbf{A})$ and $H_{\mathrm{D}}(\mathbf{A})$. Accordingly, we neglect all domain and self-adjointness issues.

We henceforth proceed to classify the transformations which leave the structure of $H_{\mathrm{P}}(\mathbf{A})$ and $H_{\mathrm{D}}(\mathbf{A})$ invariant. More precisely, we consider (anti-)linear transformations involving both spin and coordinate degrees of freedom of the form

$$\begin{aligned} &\mathcal{U}(S, T, \mathbf{x}_0)\colon L^2(\mathbb{R}^2; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2), \\ &(\mathcal{U}\psi)(\mathbf{x}) = S(\mathbf{x})\psi(T^{-1}\mathbf{x} - \mathbf{x}_0); \end{aligned} \tag{A.3}$$

$$\begin{aligned} &\mathcal{V}(S, T, \mathbf{x}_0)\colon L^2(\mathbb{R}^2; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2), \\ &(\mathcal{V}\psi)(\mathbf{x}) = S(\mathbf{x})\psi^*(T^{-1}\mathbf{x} - \mathbf{x}_0). \end{aligned} \tag{A.4}$$

Here $S(\mathbf{x})\colon \mathbb{R}^2 \to \mathrm{GL}(2, \mathbb{C})$ is any smooth section of the trivial fiber bundle $\mathbb{R}^2 \times \mathrm{GL}(2, \mathbb{C})$, $T \in \mathrm{GL}(2, \mathbb{R})$ is any constant matrix, and $\mathbf{x}_0 \in \mathbb{R}^2$ is any fixed vector. In the forthcoming Sections A.1 and A.2 we identify all (anti-)unitary operators $\mathcal{W}$ of the form (A.3) or (A.4) fulfilling the following identities, for some suitable vector-valued distribution $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2)$:

$$\mathcal{W} H_{\mathrm{P}}(\mathbf{A}) \mathcal{W}^{-1} = H_{\mathrm{P}}(\widetilde{\mathbf{A}}), \tag{A.5}$$

$$\mathcal{W} H_{\mathrm{D}}(\mathbf{A}) \mathcal{W}^{-1} = H_{\mathrm{D}}(\widetilde{\mathbf{A}}). \tag{A.6}$$

Note that local coordinates transformations always modify the structure of the differential operators $H_{\mathrm{P}}(\mathbf{A})$ and $H_{\mathrm{D}}(\mathbf{A})$. In fact, non-affine transformations always introduce non-trivial curvature contributions. For this reason we restrict the attention to transformations which are affine in the space coordinates and local in the spin degree of freedom.

Remark A.1 (Symmetries). We say that a (anti-)unitary operator $\mathcal{W}$ is a (physical) *symmetry* of the differential operators $H_{\mathrm{P}}(\mathbf{A})$ and $H_{\mathrm{D}}(\mathbf{A})$ if the associated magnetic field $b := \mathrm{curl}\mathbf{A} = \partial_1 A_2 - \partial_2 A_1$ remains invariant, namely,

$$\tilde{b} = \mathrm{curl}\widetilde{\mathbf{A}} = \mathrm{curl}\mathbf{A} = b.$$

A.1. Symmetries of the Pauli Hamiltonian

Let us first focus on the Pauli operator (A.1) and classify all transformations of the form (A.3) and (A.4) fulfilling (A.5).

A.1.1. Linear transformations. For any map $\mathcal{U}$ of the form (A.3), by a direct calculation we obtain

$$\begin{aligned}\mathcal{U}H_{\mathrm{P}}(\mathbf{A})\mathcal{U}^{-1} = {} & S(\mathbf{x})\sigma_j\sigma_\ell S^{-1}(\mathbf{x})T_{hj}T_{m\ell} \\ & \times(-i\partial_h - iS(\mathbf{x})\partial_h S^{-1}(\mathbf{x}) + T^{-1}_{kh}A_k(T^{-1}\mathbf{x}-\mathbf{x}_0)) \\ & \times(-i\partial_m - iS(\mathbf{x})\partial_m S^{-1}(\mathbf{x}) + T^{-1}_{mn}A_m(T^{-1}\mathbf{x}-\mathbf{x}_0)).\end{aligned}$$

This shows that $\mathcal{U}H_{\mathrm{P}}(\mathbf{A})\mathcal{U}^{-1}$ is itself a Pauli operator of the form (A.1), if and only if the following two conditions are simultaneously fulfilled, for some suitable vector potential $\widetilde{\mathbf{A}}$:

$$S(\mathbf{x})\sigma_j\sigma_\ell S^{-1}(\mathbf{x})T_{hj}T_{m\ell} = \sigma_h\sigma_m; \tag{A.7}$$

$$T^{-1}_{kh}A_k(T^{-1}\mathbf{x}-\mathbf{x}_0)\mathbb{1} - iS(\mathbf{x})\partial_h S^{-1}(\mathbf{x}) = \widetilde{A}_h(\mathbf{x})\mathbb{1}. \tag{A.8}$$

Using the basic algebraic identities $\{\sigma_h,\sigma_m\} = 2\delta_{hm}\mathbb{1}$ and $[\sigma_h,\sigma_m] = 2i\varepsilon_{hmr}\sigma_r$, from (A.7) we deduce $T_{hj}T_{mj} = \delta_{hm}$ and $S(\mathbf{x})\sigma_{\mathrm{P}}S^{-1}(\mathbf{x})\varepsilon_{j\ell p}T_{hj}T_{m\ell} = \varepsilon_{hmr}\sigma_r$. These are equivalent to

$$TT^t = \mathbb{1}, \tag{A.9}$$

$$S(\mathbf{x})\sigma_3 S^{-1}(\mathbf{x}) = (\det T^{-1})\sigma_3. \tag{A.10}$$

Recalling that we are assuming $T \in \mathrm{GL}(2,\mathbb{R})$, (A.9) clearly entails $T \in O(2,\mathbb{R})$. As a consequence, we see that $\mathcal{U}$ is unitary in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ only if $S = S(\mathbf{x})$ is a smooth section of $\mathbb{R}^2 \times \mathrm{U}(2,\mathbb{C})$. Notice that by a trivial application of Stone's theorem we have $\mathrm{U}(2,\mathbb{C}) = \{e^{-i(\eta_0\mathbb{1}+\eta_1\sigma_1+\eta_2\sigma_2+\eta_3\sigma_3)} \mid \eta_0,\eta_1,\eta_2,\eta_3 \in \mathbb{R}\}$. Keeping in mind that $\det T^{-1} = \det T = \pm 1$ and using elementary algebraic properties of the Pauli matrices, we then infer that (A.10) can be fulfilled only if either of the following two alternatives occurs:

$$T \in \mathrm{SO}(2,\mathbb{R}), \qquad S = e^{-i\eta_0\mathbb{1}-i\eta_3\sigma_3}, \tag{A.11}$$

for some $\eta_0,\eta_3 \in C^\infty(\mathbb{R}^2)$, or

$$T \in O(2,\mathbb{R})\setminus\mathrm{SO}(2,\mathbb{R}), \qquad S = e^{-i\eta_0\mathbb{1}-i\eta_1\sigma_1-i\eta_2\sigma_2}, \tag{A.12}$$

for some $\eta_0,\eta_1,\eta_2 \in C^\infty(\mathbb{R}^2)$, such that

$$\sqrt{\eta_1^2(\mathbf{x})+\eta_2^2(\mathbf{x})} = \frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ldots. \tag{A.13}$$

On the other hand, matching the condition (A.8) requires that

$$S(\mathbf{x})\partial_h S^{-1}(\mathbf{x}) = s_h(\mathbf{x})\mathbb{1}, \tag{A.14}$$

for some scalar function $s_h \in C^\infty(\mathbb{R}^2)$. Taking (A.11) and (A.12) into account, in can be checked that (A.14) can be fulfilled only if either η_3 or η_1, η_2 are constant, respectively. Summing up, the only transformations U of the form (A.3) satisfying (A.5), for some suitable $\widetilde{\mathbf{A}}$, correspond to

$$T \in \mathrm{SO}(2,\mathbb{R}), \quad S = e^{-i\eta_0\mathbb{1} - i\eta_3\sigma_3}, \tag{A.15}$$

for some $\eta_0 \in C^\infty(\mathbb{R}^2)$, $\eta_3 \in \mathbb{R}$, or

$$T \in O(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R}), \quad S = e^{-i\eta_0\mathbb{1} - i\eta_1\sigma_1 - i\eta_2\sigma_2}, \tag{A.16}$$

for some $\eta_0 \in C^\infty(\mathbb{R}^2)$ and $\eta_1, \eta_2 \in \mathbb{R}$ satisfying (A.13).

Remark A.2 (Special symmetries). In both cases (A.15) and (A.16), identity (A.8) reduces to

$$\widetilde{\mathbf{A}}(\mathbf{x}) = T\mathbf{A}(T^{-1}\mathbf{x} - \mathbf{x}_0) + \nabla\eta_0(\mathbf{x}),$$

which implies, in turn,

$$\tilde{b}(\mathbf{x}) = (\det T) b(T^{-1}\mathbf{x} - \mathbf{x}_0).$$

This shows that, in general, $\mathcal{U}$ is a symmetry of $H_{\mathrm{P}}(\mathbf{A})$ only if $T = \mathbb{1}$, $\mathbf{x}_0 = \mathbf{0}$ and, in compliance with (A.15), $S = e^{-i\eta_0\mathbb{1} - i\eta_3\sigma_3}$ for some $\eta_0 \in C^\infty(\mathbb{R}^2)$, $\eta_3 \in \mathbb{R}$. Of course, whenever the magnetic field exhibits specific features, the class of symmetries of the model could comprise additional transformations. For instance, it appears that a uniform magnetic field $b = \text{const.}$ is left invariant by any transformation $\mathcal{U}$ of the form (A.3) with T, S as in (A.15) and $\mathbf{x}_0 \in \mathbb{R}^2$.

Example A.3. For $S = e^{-i\eta_0}$ with $\eta_0 \in C^\infty(\mathbb{R}^2)$, $T = \mathbb{1}$ and $\mathbf{x}_0 = \mathbf{0}$, $\mathcal{U}$ is the standard (local) U(1) electromagnetic gauge transformation. In this case, we have $\widetilde{\mathbf{A}} = \mathbf{A} + \nabla\eta_0$.

Example A.4. For $S = e^{-i\eta_3\sigma_3}$ with $\eta_3 \in \mathbb{R}$, $T = \mathbb{1}$ and $\mathbf{x}_0 = \mathbf{0}$, $\mathcal{U}$ is the U(1) (global) axial gauge transformation. In this case, we have $\widetilde{\mathbf{A}} = \mathbf{A}$.

Example A.5. For $S = \mathbb{1}$, $T \in \mathrm{SO}(2,\mathbb{R})$ and $\mathbf{x}_0 = \mathbf{0}$, $\mathcal{U}$ is a simple rotation of the coordinate system. In this case, we have $\widetilde{\mathbf{A}}(\mathbf{x}) = T\mathbf{A}(T^{-1}\mathbf{x} - \mathbf{x}_0)$ and $\tilde{b}(\mathbf{x}) = b(T^{-1}\mathbf{x})$. Of course, any such transformation is a symmetry of the Hamiltonian $H_{\mathrm{P}}(\mathbf{A})$ whenever the magnetic field is radial, i.e., $b(\mathbf{x}) = b(|\mathbf{x}|)$.

A.1.2. Anti-linear transformations. For any map $\mathcal{V}$ as in (A.4), taking into account that both the vector potential $\mathbf{A} = (A_1, A_2)$ and the matrix elements T_{jh} are real, we get

$$\begin{aligned}\mathcal{V} H_{\mathrm{P}}(\mathbf{A}) \mathcal{V}^{-1} = S(\mathbf{x}) \sigma_j{}^* \sigma_\ell{}^* S^{-1}(\mathbf{x}) T_{hj} T_{m\ell} \times \\ \times \left(-i \partial_h - i S(\mathbf{x}) \partial_h S^{-1}(\mathbf{x}) - T^{-1}_{kh} A_k(T^{-1}\mathbf{x} - \mathbf{x}_0)\right) \times \\ \times \left(-i \partial_m - i S(\mathbf{x}) \partial_m S^{-1}(\mathbf{x}) - T^{-1}_{mn} A_m(T^{-1}\mathbf{x} - \mathbf{x}_0)\right).\end{aligned}$$

Therefore, for $\mathcal{V} H_{\mathrm{P}}(\mathbf{A}) \mathcal{V}^{-1}$ to fulfill (A.5), it is necessary and sufficient that the following conditions are both verified, for some vector-valued distribution $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2)$:

$$S(\mathbf{x}) \sigma_j^* \sigma_\ell^* \mathrm{S}^{-1}(\mathbf{x}) T_{hj} T_{m\ell} = \sigma_h \sigma_m, \tag{A.17}$$

$$T^{-1}_{kh} A_k(T^{-1}\mathbf{x} - \mathbf{x}_0) \mathbb{1} + i S(\mathbf{x}) \partial_h S^{-1}(\mathbf{x}) = -\widetilde{A}_h(\mathbf{x}) \mathbb{1}. \tag{A.18}$$

Using again basic commutation relations for the Pauli matrices and noting that $\sigma_3 = \sigma_3^*$, from (A.17) we deduce

$$\begin{aligned} T T^t &= \mathbb{1}, \\ S(\mathbf{x}) \sigma_3 S^{-1}(\mathbf{x}) &= -(\det T^{-1}) \sigma_3. \end{aligned} \tag{A.19}$$

In particular, we have $T \in O(2, \mathbb{R})$. So, $|\det T| = 1$ and $\mathcal{V}$ is anti-unitary in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ only if $S = S(\mathbf{x})$ is a smooth section of $\mathbb{R}^2 \times \mathrm{U}(2, \mathbb{C})$. Taking this into account, by arguments similar to those outlined in the previous subsection, we deduce that (A.19) can be fulfilled if and only if either one of the following two alternatives happens:

$$T \in \mathrm{SO}(2, \mathbb{R}), \qquad S = e^{-i\eta_0 \mathbb{1} - i\eta_1 \sigma_1 - i\eta_2 \sigma_2}, \tag{A.20}$$

for some $\eta_0 \in C^\infty(\mathbb{R}^2)$, $\eta_1, \eta_2 \in \mathbb{R}$, such that (A.13) holds, or

$$T \in O(2, \mathbb{R}) \setminus \mathrm{SO}(2, \mathbb{R}), \qquad S = e^{-i\eta_0 \mathbb{1} - i\eta_3 \sigma_3}, \tag{A.21}$$

for some $\eta_0 \in C^\infty(\mathbb{R}^2)$, $\eta_3 \in \mathbb{R}$.

Remark A.6 (Special symmetries). In both cases (A.20) and (A.21), condition (A.18) reduces to

$$\widetilde{\mathbf{A}}(\mathbf{x}) = -T\mathbf{A}(T^{-1}\mathbf{x} - \mathbf{x}_0) + \nabla \eta_0(\mathbf{x}),$$

which yields

$$\tilde{b}(\mathbf{x}) = -(\det T) b(T^{-1}\mathbf{x} - \mathbf{x}_0).$$

This makes evident that, in general, none of the admissible anti-unitary transformations (A.4) leaves the magnetic field invariant. Nevertheless, $\mathcal{V}$ can be a symmetry if b possesses specific features. For example, if the magnetic field is radial, namely $b(\mathbf{x}) = b(|\mathbf{x}|)$, then any transformation $\mathcal{V}$ of the form (A.4) with T, S as in (A.21) and $\mathbf{x}_0 = \mathbf{0}$ is indeed a symmetry.

Example A.7. For $S = \sigma_1$, $T = \mathbb{1}$, and $\mathbf{x}_0 = \mathbf{0}$, $\mathcal{V}$ is the so-called *spin-flip transformation*. In this case, $\widetilde{\mathbf{A}}(\mathbf{x}) = -\mathbf{A}(\mathbf{x})$ and $\tilde{b}(\mathbf{x}) = -b(\mathbf{x})$, showing that $\mathcal{V}$ coincides with the charge conjugation as well.

Example A.8. For $S = \sigma_2$, $T = -\mathbb{1} \in \mathrm{SO}(2,\mathbb{R})$ (notice that this choice of T describes a rotation of an angle π in the plane), and $\mathbf{x}_0 = \mathbf{0}$, $\mathcal{V}$ is the so-called *Kramers map* (see, e.g., [32]). In this case, $\widetilde{\mathbf{A}}(\mathbf{x}) = -\mathbf{A}(-\mathbf{x})$ and $\tilde{b}(\mathbf{x}) = -b(-\mathbf{x})$.

Example A.9. For $S = \mathbb{1}$, $T = P \in O(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$ with $P(x,y) = (-x,y)$, and $\mathbf{x}_0 = \mathbf{0}$, $\mathcal{V}$ is the CP-transformation. In this case, $\widetilde{\mathbf{A}}(\mathbf{x}) = (A_1(-x,y), -A_2(-x,y))$ and $\tilde{b}(x,y) = b(-x,y)$.

A.2. Symmetries of Dirac Hamiltonians

We now consider the Dirac operator (A.2) and characterize all transformations of the form (A.3) and (A.4) fulfilling (A.6). Since $[H_{\mathrm{D}}(\mathbf{A})]^2 = H_{\mathrm{P}}(\mathbf{A})$ at the pure algebraic level, it appears that any admissible symmetry of $H_{\mathrm{D}}(\mathbf{A})$ must also be a symmetry of $H_{\mathrm{P}}(\mathbf{A})$. Accordingly, in the sequel we shall restrict the attention to the family of transformations classified before in (A.15)–(A.16) and (A.20)–(A.21). For later reference let us recall the well-known Rodrigues' rotation formula, holding for arbitrary $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ with $\eta := |\boldsymbol{\eta}|$:

$$e^{-i\boldsymbol{\eta}\cdot\boldsymbol{\sigma}}\boldsymbol{\sigma}e^{i\boldsymbol{\eta}\cdot\boldsymbol{\sigma}} = \cos(2\eta)\boldsymbol{\sigma} - \frac{1}{\eta}\sin(2\eta)\boldsymbol{\eta}\wedge\boldsymbol{\sigma} + \frac{1}{\eta^2}(1-\cos(2\eta))(\boldsymbol{\eta}\cdot\boldsymbol{\sigma})\boldsymbol{\eta}. \quad \text{(A.22)}$$

A.2.1. Linear transformations. We first notice that, for any map $\mathcal{U}$ of the form (A.3), there holds

$$\mathcal{U}H_{\mathrm{D}}(\mathbf{A})\mathcal{U}^{-1} = S(\mathbf{x})\sigma_j S^{-1}(\mathbf{x})T_{hj}(-i\partial_h - iS(\mathbf{x})\partial_h S^{-1}(\mathbf{x}) + T^{-1}_{kh}A_k(T^{-1}\mathbf{x}-\mathbf{x}_0)).$$

Therefore, $\mathcal{U}H_{\mathrm{D}}(\mathbf{A})\mathcal{U}^{-1}$ is itself a Dirac Hamiltonian of the form (A.2) if and only if, for some suitable $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2)$, (A.8) is verified and

$$S(\mathbf{x})\sigma_j S^{-1}(\mathbf{x})T_{hj} = \sigma_h. \quad \text{(A.23)}$$

Let us stress that (A.23) implies (A.7). We also recall that the only admissible maps $\mathcal{U}$ fulfilling (A.5) certainly belong to either of the two families characterized in (A.15) and (A.16).

On one side, consider any choice of T and S as in (A.15). Making reference to the parametrization

$$T = \begin{pmatrix} \cos\varsigma & -\sin\varsigma \\ \sin\varsigma & \cos\varsigma \end{pmatrix}, \quad \text{with } \varsigma \in [0, 2\pi),$$

and using (A.22), by a trivial relabeling of the indexes from (A.23) we infer, for $j \in \{1, 2\}$,

$$\cos(2\eta_3)\sigma_j + \sin(2\eta_3)\varepsilon_{3j\ell}\sigma_\ell = e^{-i\eta_3\sigma_3}\sigma_j e^{i\eta_3\sigma_3} = T^t_{j\ell}\sigma_\ell$$
$$= (\cos\varsigma)\sigma_j + (\sin\varsigma)\varepsilon_{3j\ell}\sigma_\ell.$$

Clearly, this chain of identities can be fulfilled only if $\varsigma = 2\eta_3 \mod 2\pi$.

On the other side, let us fix T and S as in (A.16). Using the parametrization

$$T = \begin{pmatrix} \cos\varsigma & \sin\varsigma \\ \sin\varsigma & -\cos\varsigma \end{pmatrix}, \quad \text{with } \varsigma \in [0, 2\pi),$$

by arguments similar to those outlined above, from (A.23) we deduce

$$-\sigma_j + \frac{2\eta_j\eta_\ell}{\eta^2}\sigma_\ell = e^{-i(\eta_1\sigma_1+\eta_2\sigma_2)}\sigma_j e^{i(\eta_1\sigma_1+\eta_2\sigma_2)} = T^t_{j\ell}\sigma_\ell$$
$$= (\sin\varsigma)\sigma_j + (\cos\varsigma)\varepsilon_{3j\ell}\sigma_\ell.$$

Keeping in mind that σ_1 is real and σ_2 is imaginary, it can be checked by direct inspection that there is no admissible choice of ς and η_1, η_2 compatible with (A.16) fulfilling the latter chain of identities.

Summing up, the only linear transformations $\mathcal{U}$ of the form (A.3) fulfilling (A.6), for some suitable $\widetilde{\mathbf{A}}$, are given by

$$T = \begin{pmatrix} \cos(2\eta_3) & -\sin(2\eta_3) \\ \sin(2\eta_3) & \cos(2\eta_3) \end{pmatrix}, \quad S = e^{-i\eta_0\mathbb{1} - i\eta_3\sigma_3},$$

for some $\eta_0 \in C^\infty(\mathbb{R}^2)$, $\eta_3 \in \mathbb{R}$. Notice that these maps describe simultaneous rotations of the space coordinates in the plane $\mathbb{R}^2$ and of the spin degree of freedom, together with the usual U(1) electromagnetic local gauge transformation.

A.2.2. Anti-linear transformation. We first notice that, for any map $\mathcal{V}$ of the form (A.4), there holds

$$\mathcal{V} H_{\mathrm{D}}(\mathbf{A})\mathcal{V}^{-1} = -S(\mathbf{x})\sigma_j^* S^{-1}(\mathbf{x})T_{hj}(-i\partial_h - iS(\mathbf{x})\partial_h S^{-1}(\mathbf{x}) - T^{-1}_{kh}A_k(T^{-1}\mathbf{x} - \mathbf{x}_0)).$$

This makes evident that $\mathcal{V} H_{\mathrm{D}}(\mathbf{A})\mathcal{V}^{-1}$ is itself a Dirac Hamiltonian of the form (A.2) (cf. (A.6)) if and only if, for some suitable $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2)$, (A.18) is verified and

$$S(\mathbf{x})\sigma_j^* S^{-1}(\mathbf{x})T_{hj} = -\sigma_h. \tag{A.24}$$

Notice that (A.17) is indeed a consequence of (A.24). To proceed, let us point out that (A.24) can be equivalently rephrased as

$$S^{-1}(\mathbf{x})\sigma_h S(\mathbf{x}) = -T_{hj}\sigma_j^*,$$

and notice that $S^{-1}(\mathbf{x}) = e^{i(\eta_0 \mathbb{1} + \boldsymbol{\eta}\cdot\boldsymbol{\sigma})}$ for all $S(\mathbf{x}) = e^{-i(\eta_0 \mathbb{1} + \boldsymbol{\eta}\cdot\boldsymbol{\sigma})}$. Moreover, recall once more that σ_1 is real and σ_2 is imaginary.

On one hand, by arguments similar to those described in Section A.2.1 we deduce that the only transformations $\mathcal{V}$ of the form (A.4) with T and S as in (A.20), fulfilling (A.6) for some $\widetilde{\mathbf{A}}$, are

$$T = \begin{pmatrix} \frac{\eta_2^2-\eta_1^2}{\eta_1^2+\eta_2^2} & \frac{2\eta_1\eta_2}{\eta_1^2+\eta_2^2} \\ -\frac{2\eta_1\eta_2}{\eta_1^2+\eta_2^2} & \frac{\eta_2^2-\eta_1^2}{\eta_1^2+\eta_2^2} \end{pmatrix}, \qquad S = e^{-i\eta_0\mathbb{1} - i\eta_1\sigma_1 - i\eta_2\sigma_2},$$

for some $\eta_0 \in C^\infty(\mathbb{R}^2)$ and $\eta_1, \eta_2 \in \mathbb{R}$, such that (A.13) holds. On the other hand, the only maps $\mathcal{V}$ of the form (A.4) with T and S as in (A.21), fulfilling (A.6) for some $\widetilde{\mathbf{A}}$, are given by

$$T = \begin{pmatrix} \cos(\frac{\pi}{4} + k\pi) & \sin(\frac{\pi}{4} + k\pi) \\ \sin(\frac{\pi}{4} + k\pi) & -\cos(\frac{\pi}{4} + k\pi) \end{pmatrix}, \qquad S = e^{-i\eta_0\mathbb{1} - i(\frac{5\pi}{8} + \frac{\pi k}{2})\sigma_3},$$

for some $\eta_0 \in C^\infty(\mathbb{R}^2), k \in \mathbb{Z}$.

Summarizing and changing slightly the parametrization, we infer that the only anti-linear transformations $\mathcal{V}$ of the form (A.4) fulfilling (A.6), for some suitable $\widetilde{\mathbf{A}}$, are given by

$$T = \begin{pmatrix} \cos(2\eta) & -\sin(2\eta) \\ \sin(2\eta) & \cos(2\eta) \end{pmatrix}, \qquad S = e^{-i\eta_0\mathbb{1}}(\sigma_1 \sin\eta - \sigma_2 \cos\eta),$$

or

$$T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, \qquad S = \pm e^{-i\eta_0\mathbb{1} - i\frac{5\pi}{8}\sigma_3},$$

or

$$T = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \qquad S = \pm e^{-i\eta_0\mathbb{1} - i\frac{\pi}{8}\sigma_3},$$

with $\eta_0 \in C^\infty(\mathbb{R}^2)$ and $\eta \in [0, 2\pi)$.

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