J. Eur. Math. Soc. 26, 3685–3737 (2024) DOI 10.4171/JEMS/1412

David Schwein

Formal degree of regular supercuspidals

Received January 20, 2021; revised November 1, 2022

Abstract. Supercuspidal representations are usually infinite-dimensional, so the size of such a representation cannot be measured by its dimension; the formal degree is a better alternative. Hiraga, Ichino, and Ikeda conjectured a formula for the formal degree of a supercuspidal in terms of its L-parameter only. Our first main result is to compute the formal degrees of the supercuspidal representations constructed by Yu. Our second result, using the first, is to verify that Kaletha's regular supercuspidal L-packets satisfy the conjecture.

Keywords: formal degree, local Langlands correspondence, reductive p-adic groups.

Contents

David Schwein: Department of Mathematics, University of Bonn, 53115 Bonn, Germany; schwein@math.uni-bonn.de

Mathematics Subject Classification 2020: 22E50 (primary); 11S37 (secondary).

Introduction

Let G be a reductive algebraic group over a nonarchimedean local field k. In the study of the representation theory of the topological group $G(k)$, the supercuspidal representations are fundamental: there is a precise sense in which all irreducible (smooth or unitary) representations can be constructed from supercuspidals. Much recent work has thus focused on the construction and study of supercuspidal representations.

In 2001, Yu [58], building on earlier work of Howe [34] and Adler [1], gave a general construction of supercuspidal representations when G splits over a tamely ramified extension of k. Six years later, Kim [40] proved that Yu's construction is exhaustive when k has characteristic zero and the residue characteristic p of k is larger than some ineffective bound depending on G. Recently, Fintzen [18] improved Kim's result to show that Yu's construction produces all supercuspidals when p does not divide the cardinality of the absolute Weyl group. Hence Yu's construction produces all supercuspidals for many reductive groups, though not all of them. Moreover, the explicit nature of Yu's construction makes his supercuspidals amenable to close study.

The collection of irreducible unitary representations of $G(k)$ carries a natural topology, the Fell topology, and a natural Borel measure, the Plancherel measure. The Fell topology is canonical but the Plancherel measure depends on a choice of Haar measure on $G(k)$. When G is semisimple, every supercuspidal representation π is unitary and thus appears as a point of the unitary dual. Since supercuspidal representations of semisimple groups are discrete series, this point is isolated. We may thus ask for the measure of the point, an interesting numerical invariant of π called the *formal degree*.

When G is not semisimple it is no longer necessarily the case that all supercuspidal representations are unitary. Nonetheless, we can define the formal degree of an arbitrary supercuspidal representation in a way that generalizes the formal degree of a unitary supercuspidal. The definition, given in Section 3.2, makes no reference to the unitary dual, so we may forget about the unitary representations of $G(k)$ and focus our attention on the supercuspidal ones.

Our first main result (Theorem A) is to compute the formal degrees of Yu's supercuspidal representations. The formula uses some notation that we must briefly recall for its statement to be intelligible. Yu's construction takes as input a 5-tuple Ψ. The first member of Ψ is an increasing sequence $(G^i)_{0\le i\le d}$ of twisted Levi subgroups; let R_i denote the absolute root system of G^i. The second member is a certain point y in the Bruhat–Tits building $\mathscr{B}(G)$. The third member is an increasing sequence $(r_i)_{0\le i\le d}$ of nonnegative real numbers. The fourth member is a certain irreducible representation ρ of the stabilizer $G^0(k)_{[y]}$ of the image $[y]$ of y in $\mathscr{B}^{\mathrm{red}}(G)$. We compute the formal degree with respect to a certain Haar measure μ constructed by Gan and Gross and discussed later in the introduction. Gan and Gross's measure depends on a choice of additive character and in the formula we choose a level-zero character. For simplicity of exposition we assume in our discussion of the formula that G is semisimple, though this assumption is relaxed in the paper proper. Finally, let $\exp_q(t) := q^t$.

Theorem A. *Let G be a semisimple k-group and let Ψ be a generic cuspidal G-datum with associated supercuspidal representation π. Then the formal degree of π with respect to μ is*

$$\frac{\dim\rho}{[G^0(k)_{[y]} : G^0(k)_{y,0+}]}\exp_q\Big(\tfrac{1}{2}\dim G + \tfrac{1}{2}\dim G^0_{y,0:0+} + \tfrac{1}{2}\sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\Big).$$

The proof of Theorem A boils down, after several reductions, to computations in Bruhat –Tits theory. Yu's supercuspidals are compactly induced from a certain finite-dimensional irreducible representation τ of a certain compact-open subgroup K. There is a general formula for the formal degree of a compact induction which specializes, in this case, to the ratio $\dim\tau/\mathrm{vol}(\mu, K)$. The main difficulty is to compute the volume of K. We first situate K as a finite-index subgroup of a group of known measure; this step reduces the problem to computing the index. Using the theory of the Moy–Prasad filtration, we can translate the computation of this index into the computation of the length of a certain subquotient of the Lie algebra. The length computation, Theorem 3.24, is our key technical result in the proof of the formal-degree formula.

We can thus compute the formal degree of a broad class of supercuspidal representations. The other main result of the paper synthesizes this computation with Langlands's arithmetic parameterization of supercuspidals.

It is expected that the set $\Pi(G)$ of smooth irreducible representations of $G(k)$ is classified by certain homomorphisms $\varphi : W_k' \to {}^LG$, called *L-parameters*. Here W_k is the Weil group of k, $W_k' := \mathrm{SL}_2(\mathbb{C}) \times W_k$ is the Weil–Deligne group of k, and ${}^LG := \widehat{G} \rtimes W_k$ is the (Weil form of the) L-group of G. The expected classification consists in a partition

$$\Pi(G) = \bigsqcup_{\varphi} \Pi_\varphi(G)$$

of $\Pi(G)$ into finite subsets $\Pi_\varphi(G)$, called *L-packets*, indexed by (equivalence classes of) L-parameters φ. The sets $\Pi_\varphi(G)$ are supposed to satisfy many compatibility conditions, the simplest of which are summarized in Borel's Corvallis article [6, Section 10.3]. The resulting partition of $\Pi(G)$ is called a *local Langlands correspondence*. Although a local Langlands correspondence has been constructed for many classes of groups and representations, the full correspondence remains a conjecture. Even after fixing the group G, it is usually quite difficult to establish the correspondence for the entirety of $\Pi(G)$. Recent work has thus focused on constructing the L-packets of particular L-parameters.

Refining the outline of the correspondence, Langlands [42, Chapitre IV] suggested that the elements of the L-packet $\Pi_\varphi(G)$ are parameterized by representations of a certain finite group attached to φ. Refining Langlands's proposal, Vogan [57, Section 9] enhanced L-parameters to pairs (φ, ρ) consisting of an L-parameter φ and an irreducible representation ρ of the finite group $\pi_0(S_\varphi)$, where S_φ is the preimage in $\widehat{G}_{\mathrm{sc}}$ of the centralizer in $\widehat{G}$ of φ. Unlike ordinary L-parameters, these enhanced parameters keep track of the inner class of G: one imposes an additional condition [32, Section 1] on the central character of ρ, roughly, that it correspond to the inner class of G via Kottwitz's classification of inner forms. In this formulation, it is expected [4, Section 1.2] that the local Langlands correspondence becomes a bijection, in other words, that the irreducible representations ρ of S_φ satisfying the central character condition parameterize the L-packet $\Pi_\varphi(G)$.

Using Vogan's enhanced L-parameters, Hiraga, Ichino, and Ikeda [32, 33] predicted that the formal degree of an essentially discrete series representation can be computed in terms of its L-parameter. They verified the conjecture in many cases, in particular, for real reductive groups and for inner forms of SL_n and GL_n. We will state their conjecture in a moment after reviewing two of its inputs.

First, in a paper attaching motives to reductive groups, Gross [23, Section 4] constructed a certain Haar measure $\mu = \mu_\psi$ on $G(k)$ depending on an additive character ψ of k. Two years later, he and Gan [25, Section 5] constructed a closely related measure that conjecturally agrees with the original one. Hiraga, Ichino, and Ikeda originally predicted that Gross's measure was the right one for their conjecture, but realized later [33] that one should use Gross and Gan's measure instead.

Second, one can attach to the parameter φ and additional data – a finite-dimensional representation r of ${}^L G$ and a nontrivial additive character ψ of k – a meromorphic function $\gamma(s, \varphi, r, \psi)$ of the complex variable s, called a γ-factor of φ. The γ-factor is a product of L- and ε-factors; Section 4.1 recalls the precise formula. For the representation r of ${}^L G$ we choose the adjoint representation Ad of ${}^L G$ on $\hat{\mathfrak{g}}/\hat{\mathfrak{z}}^{\Gamma_k}$, where $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{z}}$ are the Lie algebras of $\widehat{G}$ and of its center and where Γ_k is the absolute Galois group of k. Division by $\hat{\mathfrak{z}}^{\Gamma_k}$ ensures that the γ-factor is defined at $s = 0$. We call the factor $\gamma(0, \varphi, \mathrm{Ad}, \psi)$ appearing in the conjecture the *adjoint γ-factor* of φ.

We can now state the conjecture of Hiraga, Ichino, and Ikeda [32, Conjecture 1.4] on the formal degree, referred to in this paper as the "formal degree conjecture" for the sake of brevity. Let $S_\varphi^\natural$ denote the centralizer of φ in $\widehat{G^{\mathrm{a}}}$, where $G^{\mathrm{a}} := G/A$ with A the maximal split central torus of G.

Conjecture 0.1. *Let π be an essentially discrete series representation of $G(k)$, let (φ, ρ) be the enhanced parameter of π, let ψ be an additive character of k, and let μ_ψ be the Gross–Gan measure on $G(k)$ attached to ψ. Then*

$$\deg(\pi, \mu_\psi) = \frac{\dim \rho}{|\pi_0(S_\varphi^\natural)|} \cdot |\gamma(0, \varphi, \mathrm{Ad}, \psi)|.$$

Hiraga, Ichino, and Ikeda verified their conjecture in several cases, building on work of many others: for an archimedean base field, using Harish-Chandra's theory of discrete series [29]; for inner forms of GL_n and SL_n, using work of Silberger and Zink [54, 61]; for some Steinberg representations, using work of Kottwitz [41] and Gross [23, 24]; for some unipotent discrete series of adjoint split exceptional groups, using work of Reeder [49]; and for some depth-zero supercuspidals of pure inner forms of unramified groups, using work of DeBacker and Reeder [11]. In the years following the announcement of the conjecture, it was shown to hold for U_3, Sp_4, and GSp_4 by Gan and Ichino [20]; for epipelagic supercuspidals by Reeder and Yu [50] and Kaletha [36]; for simple supercuspidals by Gross and Reeder [26]; for odd special orthogonal and metaplectic groups by Ichino, Lapid, and Mao [35]; for unitary groups by Beuzart-Plessis [5]; and for unipotent representations by Feng, Opdam, and Solleveld [15, 16].

The formal degree conjecture is a "meta-conjecture" in the sense that it depends itself on a conjecture, the local Langlands correspondence. In order to verify the formal degree conjecture one must first have access to a candidate local Langlands correspondence, or at least, to candidate L-packets. Strictly speaking, the previous sentence is not entirely true because some groups admit an analytic construction of the γ-factor that bypasses the local Langlands correspondence, though the two are expected to be compatible. The main example is Godement–Jacquet's [22] construction of the L- and ε-factors for representations of the general linear group, generalizing Tate's thesis. Their construction explains how Hiraga–Ichino–Ikeda were able to verify the formal degree conjecture for the general linear group using work that predated the Henniart [31] and Harris–Taylor [30] constructions of the local Langlands correspondence. Nonetheless, for the representations we consider in this paper, an analytic theory of the γ-factor is not yet available, and so we work with L-packets.

Recently, Kaletha [37] has organized into L-packets most of Yu's supercuspidal representations, the "regular supercuspidal representations". His construction passes through a pair (S, θ) consisting of an elliptic maximal torus S of G and a character θ of $S(k)$. On the Galois side, one uses the Langlands–Shelstad theory of χ-data and extensions of L-embeddings [43] to construct an L-parameter for G from (S, θ). On the automorphic side, one uses the pair (S, θ) to produce an input to Yu's construction, hence a supercuspidal representation π of $G(k)$. We can thus interpret π as a functorial lift of θ with respect to the embedding $S \hookrightarrow G$. The L-packet of φ consists, roughly, of all π produced in this way as we pass through the various conjugacy classes of embeddings of S in G; Section 2 reviews the construction in more detail.

Since we can compute the formal degree of Yu's representations, and Kaletha's L-packets consist of such representations, it is natural to ask whether the L-packets satisfy the formal degree conjecture. Our second main result, proved in the body of the paper as Theorem B, is that they do.

Theorem B. *Kaletha's regular L-packets satisfy the formal degree conjecture.*

To prove Theorem B, we start by computing the adjoint representation attached to a regular supercuspidal parameter: it is a direct sum of the complexified character lattice of S and some monomial representations constructed from θ and the root system of S. The γ-factor of the character lattice has already been computed in the literature. As for the monomial representations, computing their γ-factors amounts to computing the depths of the inducing characters. The inducing characters are very close to certain characters naturally constructed from θ, and whose depth is usually easy to understand; the difficulty in the proof is to quantify the difference between the two characters. To quantify it, we prove that χ-data satisfy a natural base change formula, Theorem 4.11, and that the inducing characters are ramified, Lemma 4.21.

A refinement of the formal degree conjecture due to Gross and Reeder [26, Conjecture 8.3] predicts the root number of the adjoint representation. In future work, I hope to determine whether Kaletha's regular L-packets also satisfy this refined conjecture.

The structure of this paper mirrors the formal degree conjecture. After two preliminary sections that fix notation and review the Langlands correspondence for regular supercuspidals, we compute the formal degree of a Yu representation in Section 3, we compute the Galois side of the formal degree conjecture in Section 4, and we compare the two in the brief Section 5.

1. Notation

1.1. Sets

Let $|X|$ denote the cardinality of the set X. Given a subset $Y \subseteq X$, let $\mathbb{1}_Y : X \to \{0,1\}$ denote the indicator function of Y.

Many operations on sets are expressed by superscripts or subscripts. When we have several operations denoted this way, say $X \mapsto X_a$ and $X \mapsto X_b$, we use a comma to denote the concatenation: $X \mapsto X_a \mapsto X_{a,b}$. This expression is notationally simpler than the longer form $(X_a)_b$ and should cause no confusion.

1.2. Filtrations

Suppose I is a totally ordered index set and $(X_i)_{i\in I}$ is a decreasing, I-indexed filtration of the set X. For $i \in I$, define

$$X_{i+} := \bigcup_{j>i} X_j. \tag{1.1}$$

Let $\widetilde{I} := I \cup \{i+ : i \in I\} \cup \{\infty\}$ denote Bruhat and Tits's extension of I [8, Section 6.4.1]; their definition is for $I = \mathbb{R}$ only, but it is clear how to extend it to arbitrary I. Equation (1.1) together with the convention

$$X_\infty := \bigcap_i X_i$$

defines an extension of the given filtration to an $\widetilde{I}$-indexed filtration. If in addition $X = G$ is a group and each $X_i = G_i$ is a subgroup of G then define, for $i < j$ in $\widetilde{I}$,

$$G_{i:j} := G_i/G_j.$$

We apply this formalism to the Moy–Prasad filtration on a p-adic group $G(k)$ and its Lie algebra $\mathfrak{g}$ in Section 1.6, and to the Weil group W_k in Section 1.3. The filtrations on $G(k)$ and W_k are indexed by $\mathbb{R}_{\geq 0}$, and the filtrations on $\mathfrak{g}$ are indexed by $\mathbb{R}$. In Section 2.1 we consider a filtration (of a root system) that is increasing, not decreasing. When the filtration on X is increasing, its extension to $\widetilde{I}$ is defined by

$$X_{i+} := \bigcap_{j>i} X_j, \quad X_\infty := \bigcup_i X_i.$$

1.3. Fields

Let k be a nonarchimedean local field of odd residue characteristic p, let $\mathcal{O}$ denote the ring of integers of k, and let κ denote the residue field of $\mathcal{O}$. Given a finite algebraic extension ℓ of k, let $e_{\ell/k}$ denote the ramification degree and $f_{\ell/k}$ the residue degree.

Remark 1.2. Many of the works this paper is built on, for instance, Kaletha's construction of regular supercuspidal L-packets [37], assume that p is odd. For this reason we also assume for the rest of the paper that p is odd.

Let $\mathrm{ord}_k : k^\times \to \mathbb{Z}$ denote the unique discrete valuation on k with value group $\mathbb{Z}$. We extend ord_k to a valuation on the separable closure $\bar{k}$ and denote the extension by ord_k as well. Hence the value group for a finite extension ℓ is $\mathrm{ord}_k(\ell^\times) = e_{\ell/k}^{-1}\mathbb{Z}$.

Remark 1.3. Aesthetic reasons might lead one to consider a more general value group for k than $\mathbb{Z}$. Indeed, per our convention, the value group for an extension of k is generally larger than $\mathbb{Z}$. Most of the computations in this paper can be modified to accommodate a different choice of value group because their defining objects inherit that choice. Many of the depth computations of Section 3 could be modified to carry through because the Moy–Prasad filtration inherits its jumps from the value group. Similarly, the Artin conductor computations of Section 4.1 could be modified to carry through because the upper numbering filtration inherits its jumps from the value group. However, this modification would break the connection between the Artin conductor and the Artin representation. Moreover, since the ε-factor is defined independent of the value group, the relationship between the Artin conductor and the ε-factor, in (4.1), holds only for value group $\mathbb{Z}$.

Given a finite $\mathcal{O}$-module M, let $\mathrm{len}\, M$ denote the length of M. When k has positive characteristic the module M is a κ-vector space and its length is its dimension, but when k has mixed characteristic the module M is *not* a vector space, and we must work instead with its length.

Let $q := |\kappa|$, a power of p, let $\exp_q(t) := q^t$, and let $\log_q$ be the functional inverse of $\exp_q$. The function $\exp_q$ is related to the length by the equation

$$\exp_q \mathrm{len}\, M = |M|.$$

Let W_k denote the Weil group of k, let $I_k \subseteq W_k$ denote the inertia subgroup of W_k, let $P_k \subseteq I_k$ denote the wild inertia subgroup, and for $r \geq 0$ let $W_k^r \subseteq W_k$ denote the rth subgroup of W_k in its upper numbering filtration, computed with respect to the valuation ord. A *representation of the Weil group* is a continuous, finite-dimensional, complex representation π of W_k. Given a finite extension ℓ of k, let $\pi|_\ell := \pi|_{W_\ell}$. The *depth* of π is defined as

$$\operatorname{depth} \pi := \inf\{r \in \mathbb{R} : \pi(W_k^{r+}) = 1\}.$$

In order to make the filtration on the Weil group compatible with the Moy–Prasad filtration, we need to modify the upper numbering filtration on W_ℓ for a finite extension ℓ of k by using the valuation ord_k to define it instead of the valuation ord_ℓ. When the extension is tame, as is usually the case, this modification has the effect of scaling the indices of the filtration by $e_{\ell/k}^{-1}$. To make the dependence on k clear, let depth_k denote the depth of a representation of W_ℓ where its filtration is computed using ord_k. The distinction is crucial for the proof of Lemma 4.5.

1.4. Groups

Let G be a reductive k-group, let Z be the center of G, and let A be the maximal split subtorus of Z. We reserve the symbols S and T for tori, often maximal tori of G. Let $\mathfrak{g}$ and $\mathfrak{z}$ denote the Lie algebras of G and Z, respectively.

Remark 1.4. There are three exceptions to the notational convention that G denotes a reductive group and S and T denote tori.

First, we sometimes need to work with κ-groups instead of k-groups. This practice cannot be avoided, but it is so rare that we did not see the need to introduce a separate notational convention for κ-groups. So G, S, and T denote κ-groups in this setting, which takes place in small portions of Sections 2.2 and 3.9.

Second, for reasons of notational clarity, in Section 3 the symbol $\underline{G}$ denotes a k-group and the symbol G denotes the topological group of its rational points, as discussed in Section 3.1.

Third, in Section 3.2, where we discuss the formal degree, G denotes an arbitrary locally profinite group, the proper setting for that theory.

Given a subgroup H of G, let $H^{\mathrm{a}} := G/A$. The letter "a" abbreviates "anisotropic". The notation hides the dependence on the ambient group G, but the meaning should be clear from context: typically H is a maximal torus or (twisted) Levi subgroup of G.

Let $\widehat{G}$ denote the Langlands dual group of G, a complex reductive group, and let ${}^L G = \widehat{G} \rtimes W$ denote the Weil form of the dual group.

Given an extension ℓ of k and an ℓ-group H, let $\operatorname{Res}_{\ell/k} H$ denote the Weil restriction of H from ℓ to k.

1.5. Root systems

Given a reductive k-group G and a maximal torus T of G, let $R(G, T)$ be the absolute root system of G with respect to T, that is, the root system of $G_{\bar{k}}$ with respect to $T_{\bar{k}}$, together with its natural Galois action. Let $\underline{R}(G,T)$ denote the set of Galois orbits of $R(G,T)$. I prefer to think of $\underline{R}(G,T)$ as the “functor of roots” in the sense of SGA 3 [14, Section XIX.3], an equivalent but more elaborate perspective. Reserve the letters $\alpha, \beta, \gamma, \ldots$ for elements of R, and the underlines $\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \ldots$ for elements of $\underline{R}$. Given $\underline{\alpha} \in \underline{R}$, let $\underline{\alpha}(\bar{k})$ denote the elements of $\underline{\alpha}$, a subset of R.

Given $\alpha \in R$, let Γ_α denote the stabilizer of α in Γ_k and let $k_\alpha := \bar{k}^{\Gamma_\alpha}$ denote the fixed field of Γ_α. Given $\underline{\alpha} \in R$, let $k_{\underline{\alpha}}$ denote a fixed field extension of k that is isomorphic, for some $\alpha \in \underline{\alpha}(\bar{k})$, to the extension k_α, and let $\kappa_{\underline{\alpha}}$ denote the residue field of $k_{\underline{\alpha}}$. We can define $k_{\underline{\alpha}}$ canonically as the inverse limit of the groupoid of extensions k_α with $\alpha \in \underline{\alpha}(\bar{k})$, in the style of Deligne and Lusztig [13, Section 1.1], but since Γ_k is nonabelian, it is impossible to canonically identify this limit with any one k_α. The notation $k_{\underline{\alpha}}$ helps us avoid choosing such an α, and is reserved for expressions that depend only on the isomorphism class of the extension, such as the degree $[k_{\underline{\alpha}} : k]$.

1.6. Bruhat–Tits theory

Given a reductive k-group G, let $\mathcal{B}(G)$ and $\mathcal{B}^{\mathrm{red}}(G)$ be the extended and reduced Bruhat–Tits buildings of G. Let $[x]$ denote the image of x under the canonical reduction map $\mathcal{B}(G) \to \mathcal{B}^{\mathrm{red}}(G)$. Given a maximal split torus T of G, let $\mathcal{A}(G,T)$ and $\mathcal{A}^{\mathrm{red}}(G,T)$ denote the extended and reduced apartments of T. The apartment $\mathcal{A}(G,T)$ is noncanonically isomorphic to the building $\mathcal{B}(T)$. For us, the words “building” and “apartment” refer to the extended forms.

The apartment of a maximal split torus is a classical construction, defined in Bruhat and Tits’s original papers on buildings [8, 9]. More recently, in a paper establishing tame descent for buildings [48], Prasad and Yu showed for any tame maximal torus S of G how to embed the building of S into the building of G. Although the embedding is not canonical, the image of the embedding is canonical. We denote this image by $\mathcal{A}(G,S)$ and call it the *apartment* of G in S, though that terminology is typically reserved for maximal split tori only.

Given a point x of $\mathcal{B}(G)$ or $\mathcal{B}^{\mathrm{red}}(G)$, let $G(k)_x$ denote the stabilizer of x in $G(k)$. When the center of G is anisotropic the parahoric group $G(k)_{x,0}$ is of finite index in the stabilizer $G(k)_x$; in general, $A(k)_0 G(k)_{x,0}$ is of finite index in $G(k)_x$.

For each $x \in \mathcal{B}(G)$, Moy and Prasad [45, Sections 2 and 3] defined a canonical decreasing $\mathbb{R}_{\geq 0}$-indexed filtration of $G(k)_x$ and a canonical decreasing $\mathbb{R}$-indexed filtration of $\mathfrak{g}$, denoted by $G(k)_{x,r}$ and $\mathfrak{g}_{x,r}$ and called the *Moy–Prasad filtrations*. When $G = S$ is a torus, the point x is irrelevant and we suppress it from the notation. Although the filtrations are defined for every G, they are particularly well-behaved when G splits over a tame extension. In this case, for instance, there is a canonical isomorphism

$$G_{x,r:r+} \simeq \mathfrak{g}_{x,r:r+}$$

for every $r > 0$, called the *Moy–Prasad isomorphism*. Sections 3.3 to 3.5 discuss in much more detail these filtrations and a generalization of them due to Yu.

The Moy–Prasad filtration is compatible with our chosen discrete valuation on k in the sense that for $r > 0$,

$$k_r^\times := \{a \in k^\times : \mathrm{ord}_k(a-1) \geq r\}.$$

Given a finite separable extension ℓ of k and a reductive ℓ-group H, we compute the Moy–Prasad filtration on $H(\ell)$ with respect to the norm ord_k, not ord_ℓ. This convention implies that the Moy–Prasad filtration is a topological invariant independent of the base field in the following sense: the Moy–Prasad filtration on the group $G(\ell)$ agrees with the Moy–Prasad filtration on the identical group $(\mathrm{Res}_{\ell/k} G)(k)$.

The *depth* of an irreducible admissible representation π of $G(k)$, denoted by $\mathrm{depth}_k \pi$, is the smallest real number r such that for some $x \in \mathcal{B}(G)$, the representation π has a nonzero vector fixed by $G(k)_{x,r+}$. The subscript k is a reminder that the depth, via the Moy–Prasad filtration by which it is defined, depends on the base field k. It is not a priori clear that this minimum is attained, but Moy and Prasad [46, Theorem 3.5] showed that the depth is a nonnegative rational number. This result makes the depth an indispensable tool in the representation theory of p-adic groups.

Remark 1.5. Most of Bruhat–Tits theory carries through when the field k is assumed only to be Henselian. For example, the results of Sections 3.3 to 3.6 hold at this level of generality. But once representation theory enters the picture, we must assume k is a local field.

1.7. Base change for groups and vector spaces

Given a scheme X over a base scheme Y and a morphism $Z \to Y$, let X_Z denote the base change of X from Y to Z, that is, the pullback $X \times_Y Z$. When $Z = \mathrm{Spec}\, A$ is the spectrum of a field A, we write X_A for X_Z. The schemes X that we base change are in practice always group schemes.

Similar notation can be used for base change of modules. Given a B-algebra A and an A-module M, let $M_B := M \otimes_A B$.

We take the position that an algebraic group carries the information of its base scheme. This forces our terminology to differ slightly from common practice in the literature where a k-group is thought of as a $\bar{k}$-group with a k-rational structure. In this common language one can speak, given two k-groups G and H, of morphisms $G \to H$ that are not defined over k. For us, a morphism $G \to H$ is automatically defined over k. To speak of a morphism “not defined over k” in this common sense, we would speak of a morphism $G_\ell \to H_\ell$ where ℓ is an extension of k, especially $\ell = \bar{k}$.

1.8. Base change for characters

Let S be a k-torus, let ℓ be a finite separable extension of k, and let $T := \mathrm{Res}_{\ell/k} S_\ell$. Since $X^*(T) = \mathrm{Ind}_{\ell/k} \mathrm{Res}_{\ell/k} X^*(S)$, there is a canonical map $X^*(S) \to X^*(T)$ of Galois lattices, the unit of the adjoint pair $(\mathrm{Res}_{\ell/k}, \mathrm{Ind}_{\ell/k})$. The dual of this unit is a canonical map $N_{\ell/k} : T \to S$, called the *norm map*; we use the same name and notation for the map

$T(k) = S(\ell) \to S(k)$ on rational points. Given a character $\theta : S(k) \to \mathbb{C}^\times$, define the character $\theta_{\ell/k} : S(\ell) \to \mathbb{C}^\times$ by precomposition with the norm:

$$\theta_{\ell/k} := \theta \circ N_{\ell/k},$$

We call $\theta_{\ell/k}$ the *base change* of θ from k to ℓ. In contrast to the usual notation for base change of schemes, the notation for base change of characters must include k, not just ℓ, because the base field k cannot be recovered from the topological group $S(k)$.

The base change operation for characters realizes base change in the local Langlands correspondence. For tori, this correspondence is a bijection between the complex character group of $S(k)$ and the Galois cohomology group $\mathrm{H}^1(W_k, \widehat{S})$. On the Galois side, we can restrict the L-parameter $\hat{\theta}$ of a character θ to the Weil group W_ℓ of a finite separable extension. But by the local Langlands correspondence for S_ℓ, this parameter $\hat{\theta}|_\ell$ corresponds to a character of $S(\ell)$. It is well known, and a formal consequence of the properties of the local Langlands correspondence for tori, that this character is precisely the base-changed character just defined: symbolically,

$$\hat{\theta}|_\ell = \widehat{\theta_{\ell/k}}.$$

Yu's Ottawa article [59] nicely summarizes the local Langlands correspondence for tori, and proves that for tame tori, the local Langlands correspondence preserves depth [59, Section 7.10]. He does not discuss base change, however.

We also need to understand how base change affects depth.

Lemma 1.6. *Let ℓ be a finite separable extension of k, let S be a k-torus, and let $\theta : S(k) \to \mathbb{C}^\times$ be a character. If either*

(i) *ℓ/k is unramified and* $\operatorname{depth}_k \theta \geq 0$, *or*

(ii) *ℓ/k is tamely ramified and* $\operatorname{depth}_k \theta > 0$,

then $\operatorname{depth}_k \theta = \operatorname{depth}_k \theta_{\ell/k}$.

Proof. This follows from Yu's depth-preservation theorem and the assertion above that the local Langlands correspondence intertwines base change with restriction. ∎

2. Langlands correspondence for regular supercuspidals

In this section we review the Langlands correspondence for regular supercuspidal representations, following Kaletha's article [37]. Many of the definitions, for instance, regularity of L-parameters, are rather technical, and instead of restating them, we point to their definitions in the literature. The description of regular representations and the construction of their L-parameters passes through a pair (S, θ) consisting of an elliptic maximal torus S of G and a character θ of $S(k)$ satisfying certain regularity conditions reviewed in Section 2.1. The primary goal of this section, then, is to understand, to the extent needed to verify the formal degree conjecture, how these pairs interface with both sides of the Langlands correspondence.

On the automorphic side, the pair (S, θ) produces an input to Yu's construction [58] of supercuspidals; we explain how this works in Section 2.2. In this way we produce a supercuspidal representation $\pi_{(S,\theta)}$ of $G(k)$. When (S, θ) is "tame elliptic regular", the representations that arise this way are precisely the regular supercuspidal representations.

On the Galois side, one can define a certain class of "regular supercuspidal parameters" and show that each arises from a pair (S, θ) as the composition

$$W_k \xrightarrow{{}^L\theta} {}^LS \xrightarrow{{}^Lj_\chi} {}^LG.$$

Here the first map corresponds to θ under the local Langlands correspondence for tori and the second map is an extension of a given Galois-stable embedding $\hat{j} : \widehat{S} \to \widehat{G}$. There is a general procedure, reviewed in Section 2.3, for extending $\hat{j}$ to ${}^Lj_\chi$ using a certain object χ called a set of χ-data. In our setting one canonically constructs such χ-data from the pair (S, θ), producing a canonical extension ${}^Lj_\chi$ and thus a canonical L-parameter. Using pairs (S, θ), we organize regular supercuspidal parameters into L-packets in Section 2.4. To first approximation a regular supercuspidal L-packet consists of the regular supercuspidal representations $\pi_{(jS,\theta\circ j^{-1})}$ as j ranges over $G(k)$-conjugacy classes of admissible embeddings $j : S \hookrightarrow G$; in reality, however, we must slightly modify the character $\theta \circ j^{-1}$.

2.1. *Tame elliptic regular pairs*

Pairs (S, θ), consisting of a k-torus S and a character $\theta : S(k) \to \mathbb{C}^\times$ subject to certain conditions, mediate the local Langlands correspondence for regular supercuspidals. This subsection reviews these conditions, following the discussion in Kaletha's article [37, Sections 3.6 and 3.7], and uses them to compute the depths of certain auxiliary characters that arise in Section 4.5.

The simplest condition is tameness: the pair (S, θ) is *tame* if S is tame, that is, if S splits over a tamely ramified extension of k. In this paper S is always assumed to be tame, though we sometimes repeat the hypothesis for emphasis.

All other conditions on our pair require that S be embedded as a maximal torus of a reductive group G. This requirement is extremely natural on the automorphic side, but on the Galois side, we must reinterpret it carefully since the embedding is allowed to vary.

So assume in the rest of this subsection that S is a maximal torus of G. The pair (S, θ) is *elliptic* if S is elliptic, that is, if the torus S/Z (where Z is the center of G) is anisotropic.

For the final condition, regularity, we need to probe more deeply the relationship between (S, θ) and G. Let $R = R(G, S)$ and let ℓ be the splitting field of S. Given a real number $r > 0$, consider the set of roots

$$R^r := \{\alpha \in R : (\theta_{\ell/k} \circ \alpha^\vee)(\ell_r^\times) = 1\}.$$

The assignment $r \mapsto R^r$ is an increasing, Galois-stable, $\mathbb{R}$-indexed filtration of R. Let $R^{r+} := \bigcup_{s>r} R^s$, let $r_{d-1} > \cdots > r_0 > 0$ be the breaks of this filtration, and let $r_{-1} = 0$

and $r_d = \operatorname{depth}_k \theta$. For each $0 \le i \le d$, let $R_i := R^{(r_{i-1})+}$. It turns out [37, Lemma 3.6.1] that R_i is a Levi subsystem of R. Let G^i be the connected reductive subgroup of G containing S whose root system with respect to S is R_i; the group G^i can be constructed by Galois descent, for example. Let $G^{-1} := S$.

Definition 2.1 ([37, Definition 3.7.5]). A tame elliptic pair is *regular* if

(i) the action of the inertia group on R_0 preserves a set of positive roots,

(ii) the stabilizer of the action of the group $N(G^0, S)(k)/S(k)$ on $\theta|_{S(k)_0}$ is trivial.

It is *extra regular* if, in addition,

(ii′) the stabilizer of the action of the group $\Omega(G^0, S)(k)$ on $\theta|_{S(k)_0}$ is trivial.

Here $N(G^0, S)$ is the normalizer of G^0 in S and $\Omega(G^0, S)$ is the Weyl group.

When we compute in Section 4 the formal degree of the regular parameter attached to (S, θ), half of the problem (the “root summand” of Section 4.5) boils down to knowing for each coroot $\alpha^\vee$ the depth of the character

$$\theta_{k_\alpha/k} \circ \alpha^\vee;$$

here $\alpha^\vee$ is interpreted as a homomorphism $k_\alpha^\times \to S(k_\alpha)$. Therefore, our main goal in this subsection is to compute the depth of this character. To carry out the computation, we systematically decompose θ as a product of characters of known depth using Kaletha’s notion of a Howe factorization, after reviewing an important component of that definition, due to Yu.

The definition of a Howe factorization relies on a definition of Yu [58, Section 9] for a character $\phi : H(k) \to \mathbb{C}^\times$ of a twisted Levi subgroup H of G to be *G-generic of depth r*. We need not concern ourselves with the precise definition of G-genericity, but we do need one of its consequences, which approximates the full definition.

Lemma 2.2. *Let G be a reductive k-group, let H be a tame twisted Levi subgroup of G, let $S \subseteq H$ be a tame maximal torus, and let $\phi : H(k) \to \mathbb{C}^\times$ be a character of positive depth r whose restriction to $H_{\mathrm{sc}}(k)$ is trivial. Then ϕ is G-generic if and only if for every root $\alpha \in R(G, S) \setminus R(H, S)$ and every finite tame extension ℓ of k_α, the character $\phi_{\ell/k} \circ \alpha_\ell^\vee$ of $\ell^\times$ has depth r.*

Proof. Kaletha [37, Lemma 3.6.8] proved this lemma in the case where ℓ is a fixed splitting field of S. We can deduce our result in the case where $\ell = k_\alpha$ from his result using the naturality of the norm map, and from there, we can deduce the result in general using naturality and Lemma 1.6. ■

Corollary 2.3. *In the setting of Lemma 2.2,* $\operatorname{depth}_k \phi \in \operatorname{ord}(k_\alpha^\times)$ *for each $\alpha \in R(G, S) \setminus R(H, S)$.*

Regularity is much less restrictive than genericity, but we need to know something about genericity in order to understand the depths of various auxiliary characters constructed from θ in Section 4.5. Roughly speaking, any character, regular or not, can be decomposed as a product of generic characters related to the filtration of the root system. This decomposition is called a Howe factorization.

Definition 2.4. A *Howe factorization* of (S, θ) is a sequence $(\phi_i : G^i(k) \to \mathbb{C}^\times)_{i=-1}^d$ of characters satisfying the following properties:

(i) $\theta = \prod_{i=-1}^d \phi_i|_{S(k)}$.

(ii) The character ϕ_i is trivial on $G^i_{\mathrm{sc}}(k)$ for $0 \leq i \leq d$.

(iii) The character ϕ_i is G^{i+1}-generic of depth r_i for $0 \leq i \leq d-1$; the character ϕ_d is trivial if $r_d = r_{d-1}$ and has depth r_d otherwise; and the character ϕ_{-1} is trivial if $G^0 = S$ and otherwise satisfies $\phi_{-1}|_{S(k)_{0+}} = 1$.

It turns out [37, Proposition 3.6.7] that every tame pair admits a Howe factorization. When $\alpha \notin R_0$, we can use this factorization to compute the depth of $\theta_{k_\alpha/k} \circ \alpha^\vee$.

Lemma 2.5. *Let (S, θ) be a tame pair and let $\alpha \in R_i$, where $1 \leq i \leq d$. Then the character $\theta_{k_\alpha/k} \circ \alpha^\vee : k_\alpha^\times \to \mathbb{C}^\times$ has depth r_{i-1}.*

Proof. Let $(\phi_j : G^j(k) \to \mathbb{C}^\times)_{j=-1}^d$ be a Howe factorization of (S, θ). Then

$$\theta_{k_\alpha/k} \circ \alpha^\vee = \prod_{j=-1}^{d} \phi_{j,k_\alpha/k} \circ \alpha^\vee.$$

Since $\alpha^\vee$ factors through G^i, condition (ii) of a Howe factorization implies that the factors of this product are trivial for $j \geq i$. By Lemma 2.2, the jth remaining factor has depth r_j, and since the sequence $j \mapsto r_j$ is strictly increasing, the product has depth r_{i-1}. ∎

Lemma 2.5 conspicuously omits the case where $\alpha \in R_0$. We have more to say about this in Section 4.5, especially in Lemma 4.21 and Remark 4.22.

2.2. *Regular representations*

Yu's seminal paper [58] constructs a broad class of supercuspidal representations starting from a certain triple $(\vec{G}, \pi_{-1}, \vec{\phi})$, which we call, for reference, a *Yu datum*. This subsection reviews these triples and explains how a tame elliptic regular pair gives rise to a Yu datum. Later, Section 3.7 explains in more detail how to construct supercuspidal representations from Yu data.

There are three stages of representations used in Yu's construction, each informing the previous one: representations of finite groups of Lie type; depth-zero supercuspidal representations; and supercuspidal representations of arbitrary depth. Since the definition of regular supercuspidal passes through each of these stages, we start by reviewing each stage in turn.

The first stage is finite groups of Lie type. In this paragraph only, let G be a reductive κ-group. Representations of $G(\kappa)$ are well understood through the work of Deligne and Lusztig [13]. They attached to an elliptic pair (S, θ) over κ (that is, S is an elliptic maximal torus of G) a virtual representation $\mathrm{R}_{(S,\theta)}$, the Deligne–Lusztig induction. If the character θ of the κ-torus S is *regular* [37, Definition 3.4.16] then $\pm \mathrm{R}_{(S,\theta)}$ is an irreducible representation. We say that a representation ρ of $G(\kappa)$ is *regular* if it is isomorphic to such a Deligne–Lusztig representation.

Passing to the second stage, depth-zero supercuspidals, Morris [44] was the first to realize that for a general reductive group, depth-zero supercuspidal representations could be constructed from cuspidal representations of finite groups of Lie type. A more precise version of Morris's result is the following classification theorem [46, Proposition 6.8] of Moy and Prasad: for every depth-zero supercuspidal representation π of $G(k)$, there is a vertex $x \in \mathcal{B}^{\mathrm{red}}(G)$ such that $\pi|_{G(k)_{x,0}}$ contains the inflation to $G(k)_{x,0}$ of an irreducible cuspidal representation $\widetilde{\rho}$ of $G(k)_{x,0:0+}$. Furthermore, we can recover π by compact induction: there is some representation ρ of $G(k)_x$ such that $\rho|_{G(k)_{x,0}}$ contains the inflation of $\widetilde{\rho}$ and such that $\pi = \text{c-Ind}_{G(k)_x}^{G(k)} \rho$. The depth-zero supercuspidal representation π is *regular* if the representation ρ of $G(k)_{x,0:0+}$ is regular. Regular depth-zero supercuspidals π enjoy two pleasant properties.

First, there is a canonical bijection between regular depth-zero supercuspidals and conjugacy classes of regular tame elliptic pairs (S, θ) of depth zero in which the torus S is *maximally unramified* in G [37, Definition 3.4.1]. In particular, we can recover S from π.

Second, it turns out [37, Sections 3.4.4 and 3.4.5] that $\pi_{(S,\theta)}$ can be compactly induced from a representation $\eta_{(S,\theta)}$ of the group $S(k)G(k)_{x,0}$. This is an improvement over Moy and Prasad's theorem, which uses the larger stabilizer group $G(k)_x$ instead; generally $S(k)G(k)_{x,0}$ is easier to understand than $G(k)_x$. The fact that π is compactly induced from this smaller group plays a crucial role in our final computation, in Section 3.9, of the formal degree of a regular supercuspidal.

We can now discuss the third stage, Yu's general construction of supercuspidals.

To start, we recall the definition of a Yu datum. A subgroup H of G is a *twisted Levi* subgroup if there is a finite separable extension ℓ of k splitting G such that H_ℓ is a Levi subgroup of G_ℓ, and H is *tame* if ℓ can be taken to be a tame extension of k. A *twisted Levi sequence* in G is an increasing sequence

$$\vec{G} = (G^0 \subsetneq G^1 \subsetneq \cdots \subsetneq G^d)$$

of twisted Levi subgroups of G; it is *tame* if each of its members is tame. The first component $\vec{G}$ of a Yu datum is a tame twisted Levi sequence; the second component π_{-1} of a Yu datum is a depth-zero supercuspidal representation of $G^0(k)$; and the third component $\vec{\phi}$ of a Yu datum is a sequence of characters

$$\vec{\phi} = (\phi_i : G^i(k) \to \mathbb{C}^\times)_{i=0}^d.$$

These three objects are required to satisfy certain conditions that Section 3.7 spells out in detail. In fact, in that section we work with a certain five-tuple instead of a Yu datum, but the two objects are closely related [27, Section 3.1].

To simplify the following definition, we assume in the rest of this subsection that p does not divide the order of the fundamental group of G. Kaletha defined regularity in general using z-extensions [37, Section 3.7.4], but we have no need to understand how this works.

Definition 2.6. A Yu datum $(\vec{G}, \pi_{-1}, \vec{\phi})$ is *regular* if π_{-1} is a regular depth-zero supercuspidal representation. A supercuspidal representation is *regular* if it is isomorphic to a supercuspidal representation constructed from a regular Yu datum.

We have thus defined the supercuspidal representations of interest to us; the next matter is to connect them to torus-character pairs.

Given a Yu datum $(\vec{G}, \pi_{-1}, \vec{\theta})$, we can find a maximally unramified torus S of G^0 and a regular depth-zero character ϕ_{-1} of $S(k)$ such that $\pi_{-1} = \pi_{(S,\phi_{-1})}$. Setting

$$\theta = \prod_{i=-1}^{d} \phi_i|_{S(k)}$$

then produces a tame elliptic regular pair (S, θ). Conversely, given such a pair (S, θ), with Howe factorization $(\phi_i : G^i(k) \to \mathbb{C}^\times)_{i=-1}^{d}$, the triple

$$(\vec{G} = (G^i)_{0\leq i\leq d}, \pi_{-1} = \pi_{(S,\phi_{-1})}, \vec{\phi} = (\phi_i)_{i=0}^{d})$$

is a Yu datum. It turns out [37, Proposition 3.7.8] that these assignments are bijections modulo the appropriate equivalences. In this way, we can form a regular supercuspidal representation $\pi_{(S,\theta)}$ from a tame elliptic regular pair (S, θ).

Warning 2.7. Fintzen [17, Section 4], and independently Spice, realized that in the construction of supercuspidals Yu's paper omits a certain sign. This omission traces back to a computation in a classical paper of Gérardin [21, Theorem 2.4 (b)]. The omission invalidates several intermediate results in Yu's work, though not the final conclusion that the representations he constructs are supercuspidal. The corrected signs have several further applications in the study of such representations [19]. Consequently, in our discussion of Yu's work we may safely ignore these correction terms.

2.3. L-embeddings

In this subsection we explain and study a formalism of Langlands and Shelstad for extending an embedding $\hat{\jmath} : \widehat{S} \to \widehat{G}$ with Galois-stable $\widehat{G}$-conjugacy class to an L-embedding ${}^L j : {}^L S \to {}^L G$. Fix a Γ_k-pinning of $\widehat{G}$ with maximal torus $\widehat{T}$.

The first difficulty in extending $\hat{\jmath}$ to ${}^L j$ is to reconcile the Galois actions on $\widehat{S}$ and $\widehat{G}$. Specifically, let τ_G denote the action (homomorphism) of W_k on $\widehat{T}$ through its action on $\widehat{G}$; let τ_S denote the action of W_k on $\widehat{T}$ through its action on $\widehat{S}$, transferred using $\hat{\jmath}$; let N be the normalizer of $\widehat{T}$ in $\widehat{G}$; and let $\Omega = \Omega(\widehat{G}, \widehat{T})$ be the Weyl group. Given a Weil element $w \in W_k$, thought of as an element of ${}^L S$, its image under an extension ${}^L j$ has the form nw where $n \in N$ lifts the Weyl element

$$\omega_{S,G}(w) := \tau_S(w)\tau_G(w)^{-1} \in \Omega,$$

so that nw acts on $\widehat{T}$ by the $\widehat{S}$-action. The lift exists precisely because the $\widehat{G}$-conjugacy class of $\hat{\jmath}$ is Galois-stable. To define the extension ${}^L j$, then, we need only choose a specific lift n of $\omega_{S,G}(w)$ to N.

For many reductive groups, in particular the special linear group, the projection map $N \to \Omega$ does not admit a homomorphic section. Nonetheless, by finding a good way to lift fundamental reflections, Tits [56] defined a canonical set-theoretic section $n : \Omega \to N$, which we call the *Tits lift*. Its precise definition [43, Section 2.1] is not so important for us.

Since the Tits lift is not a homomorphism, the candidate formula

$$w \mapsto n(\omega_{S,G}(w))w$$

for ${}^L j|_{W_k}$ does not define a homomorphism $W_k \to {}^L G$. To get around this problem, Langlands and Shelstad studied the failure of this formula to define a homomorphism as measured by the function $t : W_k \times W_k \to \widehat{T}$ given by

$$t(w_1, w_2) := (n(w_1)w_1)(n(w_2)w_2)(n(w_1w_2)w_1w_2)^{-1}.$$

Using an object χ called a set of χ-data, whose definition we review momentarily, they constructed a function $r_\chi : W_k \to \widehat{S}$ that negates this failure in the sense that $\partial(\hat{j} \circ r_\chi) = t^{-1}$, where ∂ is the coboundary operator. Hence the modified formula

$$w \mapsto \hat{j}(r_\chi(w))n(\omega_{S,G}(w))w$$

does define a homomorphism $W_k \to {}^L G$, and in total, the extension ${}^L j_\chi$ of $\hat{j}$ is given by the formula

$${}^L j_\chi(sw) = \hat{j}(s\, r_\chi(w))n(\omega_{S,G}(w))w.$$

This concludes our outline of the Langlands–Shelstad procedure to extend an embedding of a torus to an L-embedding. Later, in Section 4.2, we recall the definitions of χ-data and the function r_χ.

2.4. *Regular parameters*

This subsection is largely an expository account of Sections 5.2 and 6.1 of Kaletha's article [37]. Our goal is to describe the regular supercuspidal L-parameters and their construction from tame elliptic extra regular pairs.

The definition of regularity for L-parameters [37, Definition 5.2.3] is not important for us, so we omit the precise statement: roughly speaking, a parameter φ is regular if it takes the wild subgroup to a torus and the centralizer of the inertia subgroup is abelian (in fact, this is the definition of a "strongly regular" parameter). Consequently, the groups S_φ and $S_\varphi^\natural$ appearing in the statement of the formal degree conjecture are abelian, so that their irreducible representations are one-dimensional. This means that we can ignore the factor $\dim \rho$ appearing in the Galois side of the formal degree conjecture, since it equals 1.

However, we should explain the relationship between regularity and torus-character pairs. To classify regular parameters, Kaletha introduced an auxiliary category of *regular supercuspidal L-packet data* whose objects are quadruples $(S, \hat{j}, \chi, \theta)$ consisting of

- a tame torus S of dimension the absolute rank of G,

- an embedding $\hat{j} : \widehat{S} \to \widehat{G}$ of complex reductive groups whose $\widehat{G}$-conjugacy class is Galois stable,
- a minimally ramified set of χ-data for $R(G, S)$,
- a character $\theta : S(k) \to \mathbb{C}^\times$.

For the meaning of *minimally ramified χ-data*, see Definitions 4.6 and 4.15 in Section 4.2. These objects are required to satisfy additional conditions [37, Definition 5.2.4] that do not concern us here. One can also define a morphism of such data, organizing them into a category in which all morphisms are isomorphisms.

In general, the $\widehat{G}$-conjugacy class of $\hat{j}$ gives rise to a Galois-stable $G(\bar{k})$-conjugacy class of embeddings $S_{\bar{k}} \hookrightarrow G_{\bar{k}}$ whose elements are called *admissible* (with respect to $\hat{j}$) [37, Section 5.1]. Since S is elliptic, this $G(\bar{k})$-conjugacy class contains embeddings defined over k. That is, S can be embedded as a maximal torus of G. However, neither the embedding nor its $G(k)$-conjugacy class is canonical, and the definition of regular supercuspidal L-packet data makes no choice of embedding. This failure is related to the need to organize supercuspidal representations into L-packets.

The key property of the category of regular supercuspidal L-packet data is that the isomorphism classes of its objects are in natural bijection with equivalence classes of regular supercuspidal parameters. Given a regular supercuspidal L-packet datum $(S, \hat{j}, \theta, \chi)$, its parameter is the composition

$$^L j_\chi \circ {}^L\theta,$$

where $^L j_\chi$ is the L-embedding of Section 2.3; this is the direction of the correspondence that we need to understand when we compute, in Section 4, the absolute value of the adjoint γ-factor.

Let (S, θ) be a tame elliptic extra regular pair. Assume that there is at least one admissible embedding j of S as a maximal torus of G. Instead of just pulling back the character θ to jS, we need to modify it slightly: define

$$j\theta' := \theta \circ j^{-1} \cdot \varepsilon$$

where $\varepsilon = \varepsilon_{f,\mathrm{ram}} \cdot \varepsilon^{\mathrm{ram}}$ is a certain tamely ramified Weyl-invariant character of jS [37, Section 5.3]. Kaletha used the character formula of Adler, DeBacker, and Spice [2, 3, 12] to construct from (S, θ, j) a certain minimally ramified set χ of χ-data, which appears, for one, in the definition of ε. Then the L-packet corresponding to the datum $(S, \hat{j}, \chi, \theta)$ consists of the set of regular supercuspidal representations

$$\pi_{(jS, j\theta')}$$

where $j : S \hookrightarrow G$ ranges over the $G(k)$-conjugacy classes of admissible embeddings.

3. Automorphic side

In Section 2.2, we outlined Yu's construction of supercuspidal representations. In this section we calculate the formal degree of such a representation. This result is of independent interest, and could be used to verify the formal degree conjecture for broader classes of supercuspidal representations than those considered in this paper.

The basic idea of the computation is quite simple, but various technical complications arise in the process. As Section 3.7 explains, Yu's representations are obtained by compact induction of a finite-dimensional representation of a compact-open (or really, compact-mod-center) subgroup of $G(k)$. There is a general formula for the formal degree of such a representation in terms of the dimension of the starting representation and the volume of the subgroup. Section 3.2 explains this formula and reviews the notion of formal degree. To compute the formal degree of a Yu representation, then, one need only compute two numbers, a dimension and a volume.

The dimension comes from Deligne–Lusztig theory and is straightforward to compute in our case. We work it out in Section 3.9, where we specialize the formal degree computation to the case of a regular supercuspidal representation.

The volume comes from Bruhat–Tits theory, and is much more difficult to compute. Still, the basic idea is clear. Computing the volume of a compact-open subgroup amounts to computing its index in a larger group of known volume, so the volume computation boils down to an index computation. Using the Moy–Prasad isomorphism, that index computation, in turn, boils down to a computation of the subquotients in the Moy–Prasad filtration on the Lie algebra.

The groups used in Yu's construction generalize the subgroups of the Moy–Prasad filtration. In Sections 3.3 to 3.5 we review their construction and explain various ways in which Yu's theory is an elaboration of the theory of Moy and Prasad [45, 46], or going back even farther, the theory of Bruhat and Tits [8, 9]. Our goal in that lengthy section is to generalize the Moy–Prasad isomorphism to Yu's groups, and to understand the extent to which the Lie algebra of one of Yu's groups decomposes as a direct sum of root lines. After these preliminaries, we compute the dimension of such a Lie algebra in Section 3.6. The Moy–Prasad isomorphism then translates this dimension into the subgroup index that we need to compute. At this point the main steps are in place, and in Section 3.8 we walk up the staircase and finish the computation.

To make statements like Lemma 3.34 easier to read, in Section 3.1 we introduce notation particular to Section 3 for reductive groups and their topological groups of rational points.

3.1. Notation

It is important for many reasons to distinguish between a linear algebraic k-group and the group of its rational points. The former is a k-scheme and the latter is just an abstract group, a topological group if k has a topology. Many constructions are easier on the level of schemes, but for the kind of representation theory we consider in this paper, one can work only with the group of rational points.

It is conventional in algebraic geometry to denote a k-group by G and the group of its rational points by $G(k)$. Following the convention in this section, however, would create a confusing proliferation of "(k)" suffixes. In Section 3 only, therefore, we underline, denoting k-groups by $\underline{G}, \underline{H}, \ldots$ and their groups of rational points by $G, H, \ldots$:

$$G := \underline{G}(k).$$

In particular, we write $\vec{\underline{G}}$ for a twisted Levi sequence and $\vec{G}$ for the sequence of topological groups obtained from it by taking k-points. The convention extends to $\mathcal{O}$- and κ-schemes as well. For instance, we write $\underline{G}_{x,r}$ for the smooth $\mathcal{O}$-group, constructed by Yu [60, Section 8], whose group of $\mathcal{O}$-points is the Moy–Prasad group $G_{x,r}$. Similarly, $\underline{G}_{x,0:0+}$ denotes the maximal reductive quotient of the special fiber of $\underline{G}_x$, a κ-group.

3.2. Formal degree

Here we review the formal degree, following the relevant section of Renard's monograph on representations of p-adic groups [51, Section IV.3], then calculate the formal degree of a compactly induced representation. In contrast to the conventions of the rest of the paper, in this subsection only, let G be a unimodular locally profinite group, let Z be the center of G, and let (π, V) be a smooth irreducible representation of G. Let A be a closed subgroup of Z such that Z/A is compact, and let μ be a Haar measure on G/A. Several of our results use A in the statement but are independent of the choice of A.

The *matrix coefficient* of π with respect to $v \in V$ and $v^\vee \in V^\vee$ (where $V^\vee$ is the smooth dual of V) is the function $\pi_{v,v^\vee} : G \to \mathbb{C}$ defined by $\pi_{v,v^\vee}(x) = \langle \pi(x)v, v^\vee \rangle$. Since π is irreducible, there is a character χ of Z, called the *central character* of π, such that $\pi(z) = \chi(z)$ for all $z \in Z$. Assume that the central character is unitary. In this case, the function $x \mapsto |\pi_{v,v^\vee}(x)|^2$ is constant on cosets of A, and hence defines a function on G/A. We write $|\pi_{v,v^\vee}|^2_{L^2(G/A,\mu)}$ for the integral of this function, and say that (π, V) is *discrete series* (with respect to A) if $|\pi_{v,v^\vee}|_{L^2(G/A,\mu)} < \infty$ for all $v \in V$ and $v^\vee \in V^\vee$. This condition is independent of A, and also of the choice of Haar measure on G/A.

In practice, it is useful to slightly weaken the definition of a discrete series representation, and to define a representation to be *essentially discrete series* if it becomes discrete series after twisting by some character of the group.

It can be shown that every discrete series representation is *unitary* in the sense that it admits a positive-definite G-invariant Hermitian product.[1] The resulting isomorphism between $\overline{V}$ and $V^\vee$ defines a matrix coefficient $\pi_{v,w}$ for $v, w \in V$. Set $\pi_v := \pi_{v,v}$.

For a discrete series representation (π, V), one would hope for a relationship between the norm of a vector and the L^2-norm of its matrix coefficient. Although these norms are not equal in general, it turns out that they differ by a multiplicative constant depending only on π and the Haar measure on G/A, not on the vector. This constant is called the *formal degree*.

[1] Our notion of unitary is almost the same as the analogous notion of Hilbert space theory, with the caveat that smooth unitary representations generally do not form Hilbert spaces because they are incomplete for the induced norm.

Definition 3.1 ([51, Section IV.3.3]). Let (π, V) be an essentially discrete series representation and let μ be a Haar measure on G/A. If π is in addition a discrete series representation then there exists a positive real constant $\deg(\pi, \mu)$, called the *formal degree* of π, such that for all $v \in V$,

$$|\pi_v|^2_{L^2(G/A,\mu)} = \frac{|v|^2}{\deg(\pi, \mu)}.$$

In general, we define the formal degree of π as the formal degree of any discrete series representation of G obtained from π by twisting by a character of G.

According to Harish-Chandra [28, p. 4] the definition is due to Mackey, though he did not publish it. Evidently the formal degree scales inversely with the Haar measure used to define it:

$$\deg(\pi, c\mu) = c^{-1} \deg(\pi, \mu).$$

Remark 3.2. Assume Z is compact in this remark for simplicity. In harmonic analysis, one studies the set of irreducible unitary representations of G by endowing it with a certain topology and measure, called the *Plancherel measure*. The measure, though not the topology, depends on a choice of Haar measure μ on G. An irreducible unitary representation (π, V) is discrete series if and only if it is an isolated point of positive measure, and that measure is the formal degree of π with respect to μ. In particular, if G is compact and we choose the Haar measure on G giving it volume 1 then the formal degree of (π, V) is just the dimension of V.

Discrete series representations are closely related to supercuspidal representations, but neither notion implies the other. Recall that a smooth irreducible representation is *supercuspidal* if its matrix coefficients have compact support modulo the center, or equivalently, modulo A. If a supercuspidal representation has unitary central character then it is certainly discrete series, since compactly supported functions are square-integrable; but since not every compactly supported function is square-integrable, in general, there are discrete series representations that are not supercuspidal. At the same time, since supercuspidal representations need not have unitary central character, not all supercuspidal representations are discrete series. However, when G is a p-adic reductive group, every supercuspidal representation of G is essentially discrete series, and we may thus speak of the representation's formal degree.

Since Yu's supercuspidal representations are compactly induced, we compute their formal degree using a general formula for the formal degree of a compactly induced representation. As a preliminary step, we define a natural Hermitian product on a compactly induced representation.

Let K be an open, compact-mod-A subgroup of G (this condition is independent of A), let (ρ, W) be a smooth irreducible unitary representation of K, and let (π, V) be the representation of G compactly induced from (ρ, W). That is, V is the space of smooth functions $f : G \to W$ whose support is compact-mod-K and that satisfy $f(hx) = \rho(h) f(x)$ for all $h \in K$ and $x \in G$. The representation (π, V) is unitary; in fact, an invariant scalar product is given by the formula

$$\langle f_1, f_2\rangle = \int_{K\backslash G} \langle f_1(x), f_2(x)\rangle \,\mathrm{d}\mu_{K\backslash G}(x), \tag{3.3}$$

where $\mu_{K\backslash G}$ is any positive G-invariant Radon measure on $K\backslash G$, for instance, the counting measure.

Lemma 3.4. *Let (ρ, W) be a finite-dimensional unitary representation of K, let (π, V) be the compact induction of W to G, and let μ be a Haar measure on G/A. If π is irreducible then*

$$\deg(\pi, \mu) = \frac{\dim \rho}{\mathrm{vol}(K/(K \cap A), \mu)}.$$

Proof. We start by defining an isometric embedding $W \hookrightarrow V$. Given a vector $w \in W$, define $\dot{w} \in V$ by

$$\dot{w}(x) = \mathbb{1}_K(x)\rho(x)w.$$

The space $K\backslash G$ is discrete because K is open, so we can take the measure $\mu_{K\backslash G}$ in (3.3) to be the counting measure. With this choice, the map $W \to V$ defined by $w \mapsto \dot{w}$ is an isometric embedding. It follows that the matrix coefficient of $\dot{w}$ is the extension by zero of the matrix coefficient of w, and that their L^2-norms coincide provided that we take the Haar measure on $K/(K \cap A)$ to be the restriction of μ, denoted also by μ. Then for any nonzero $w \in W$,

$$\deg(\pi, \mu) = |\dot{w}|^2 \cdot |\pi_{\dot{w}}|^{-2}_{L^2(G/A,\mu)} = |w|^2 \cdot |\rho_w|^{-2}_{L^2(K/(K\cap A),\mu)} = \deg(\rho, \mu).$$

Finally, since the formal degree scales inversely to the Haar measure used to define it,

$$\deg(\rho, \mu) = \mathrm{vol}(K/(K \cap A), \mu_0)^{-1} \deg(\rho, \mu_0)$$

where μ_0 is the measure on $K/(K \cap A)$ assigning it total volume 1. The degree of ρ with respect to this measure is just the dimension of W. ■

3.3. Concave-function subgroups: Split case

Suppose $\underline{G}$ is split with split maximal torus $\underline{T}$ and root system $R = R(\underline{G}, \underline{T})$, so that $R = \underline{R}$. In this setting, Bruhat and Tits [8, Section 6.4] showed how to construct from a function $f : R \cup \{0\} \to \widetilde{\mathbb{R}}$ and point $x \in \mathcal{A}(\underline{G}, \underline{T})$ a subgroup $G_{x,f}$ of G and a subgroup $\mathfrak{g}_{x,f}$ in $\mathfrak{g}$. In this subsection we review Bruhat and Tits's construction. The eventual goal, in later subsections, is to explain how their construction generalizes to the construction of subgroups that appear in Yu's construction of supercuspidals, and to then study Yu's subgroups.

The definitions of $G_{x,f}$ and $\mathfrak{g}_{x,f}$ are quite natural. The point x defines, or is, depending on one's point of view, a family of additive valuations $(v_x^\alpha : \underline{U}^\alpha(k) \to \mathbb{R})_{\alpha\in R}$, where $\underline{U}^\alpha$ is the root group of α. Since $\underline{U}^\alpha$ is canonically isomorphic to the root line $\mathfrak{g}^\alpha$, we may also think of v_x^α as a valuation $\mathfrak{g}^\alpha \to \mathbb{R}$. Now let

$$U^\alpha_{x,r} := \{u \in U^\alpha : v_x^\alpha(u) \geq r\}, \quad \mathfrak{g}^\alpha_{x,r} := \{X \in \mathfrak{g}^\alpha : v_x^\alpha(X) \geq r\}.$$

As for the point $\alpha = 0$, we can think of $\underline{T}$ as the root group $\underline{U}^0$ and its Lie algebra $\mathfrak{t}$ as the root space $\mathfrak{g}^0$. These objects carry their own filtrations: let

$$T_r := \{t \in T : \forall \chi \in X^*(T),\ \mathrm{ord}(\chi(t) - 1) \geq r\}$$

and

$$\mathfrak{t}_r := \{X \in T : \forall \chi \in X^*(T),\ \mathrm{ord}(\mathrm{d}\chi(X)) \geq r\}.$$

The objects T_r and $\mathfrak{t}_r$ do not depend on x, but we reserve the right to denote them by $T_{x,r}$ and $\mathfrak{t}_{x,r}$ for uniformity of notation.

Warning 3.5. The group G^0 written here is unrelated to the zeroth group in a twisted Levi sequence, even though the notation for the two is the same. The two notations never appear in the same subsection, however, so there is little risk of confusion.

Given a function $f : R \cup \{0\} \to \widetilde{\mathbb{R}}$, let

$$U^\alpha_{x,f} := U^\alpha_{x,f(\alpha)}, \quad \mathfrak{g}^\alpha_{x,f} := \mathfrak{g}^\alpha_{x,f(\alpha)}$$

for any $\alpha \in R \cup \{0\}$. The group $G_{x,f}$ is then defined as the subgroup of G generated by the subgroups $U^\alpha_{x,f}$ with $\alpha \in R \cup \{0\}$, and the lattice $\mathfrak{g}_{x,f}$ is defined as the subgroup of $\mathfrak{g}$ spanned by the subgroups $\mathfrak{g}^\alpha_{x,f}$ with $\alpha \in R \cup \{0\}$.

Remark 3.6. When $r = \infty$ the group $U^\alpha_{x,\infty}$ is trivial, and when in addition $\alpha \neq 0$ we can recover the filtrations on the root groups and root lines as $\mathfrak{g}^\alpha_{x,r} = \mathfrak{g}_{x,f}$ and $U^\alpha_{x,r} = G_{x,f}$ where

$$f(\beta) = \begin{cases} r & \text{if } \beta = \alpha, \\ \infty & \text{if not.} \end{cases}$$

In order for the construction of $G_{x,f}$ to behave nicely[2] we must assume that f is nonnegative and *concave*, that is, for all finite families $(\alpha_i)_{i \in I}$ of elements of $R \cup \{0\}$,

$$f\Big(\sum_{i \in I} \alpha_i\Big) \leq \sum_{i \in I} f(\alpha_i)$$

whenever $\sum_{i \in I} \alpha_i \in R \cup \{0\}$. We can define $\mathfrak{g}_{x,f}$ for any f whatsoever, but when f is concave, $\mathfrak{g}_{x,f}$ is a Lie subalgebra of $\mathfrak{g}$.

This completes our discussion of the split case. We next generalize the split case to the tame case, a simple exercise in Galois descent.

Remark 3.7. The exact nature and history of these definitions is rather complex. When $f(0) = 0$, Bruhat and Tits defined concave-function subgroups not just for split reductive groups but for any reductive group with a chosen maximal split torus. In fact, they gave the construction in even greater generality, for a certain object called a *valuation of a root datum*. The construction appeared in their first paper on buildings [8]. In their second [9],

[2]By "behave nicely" we mean at least that the root subgroups of $G_{x,f}$ are not larger than expected, in other words, that $G_{x,f} \cap U^\alpha = U^\alpha_{x,f}$.

they showed that a reductive group gives rise to a valuation of a root datum, thereby completing the construction of the concave-function subgroups of reductive groups.

Bruhat and Tits did not define a filtration on tori, however, and in its absence, they were unable to handle the case $f(0) > 0$. Prasad and Raghunathan [47, Section 2] overcame this difficulty, defining a filtration on any parahoric subgroup of a simply-connected, absolutely quasi-simple group. Ten years later, Moy and Prasad [45, Section 2] realized that it was more natural to vary the indexing of the filtration, though not the underlying subgroups, using points of the building. Soon thereafter, Schneider and Stuhler [52, Chapter I] generalized the filtration to an arbitrary reductive group over a local field. The common name for this filtration on parahoric subgroups, the *Moy–Prasad filtration*, refers to only two of these authors.

Finally, motivated by his construction of supercuspidals, Yu [60, Section 8] showed how to combine the Moy–Prasad filtration with Bruhat and Tits's concave-function subgroups, thereby handling the case where $f(0) > 0$.

3.4. *Concave-function subgroups: Tame case*

We no longer assume that $\underline{G}$ is split, only that it is split over a tamely ramified extension. Hence we must distinguish between $\underline{R} := \underline{R}(\underline{G}, \underline{T})$ and $R := R(\underline{G}, \underline{T})$. In this setting, Yu constructed for each concave function $f : \underline{R} \cup \{0\} \to \widetilde{\mathbb{R}}$ and point $x \in \mathcal{A}(\underline{G}, \underline{T})$ a subgroup $G_{x,f}$, generalizing the construction of the previous subsection.

Let $\ell \supseteq k$ be some fixed tame Galois extension of k splitting the maximal torus $\underline{T}$ and let $\Gamma_{\ell/k}$ be the Galois group of ℓ over k. A function $f : \underline{R} \cup \{0\} \to \widetilde{\mathbb{R}}$ can be interpreted as a Galois-invariant function $R \cup \{0\} \to \widetilde{\mathbb{R}}$, and we say that f is *concave* if the associated Galois-invariant function is concave.

Since f is Galois-invariant, the subgroups $(\mathfrak{g}_\ell)_{x,f}$ and $\underline{G}(\ell)_{x,f}$ are Galois-invariant and we define

$$\mathfrak{g}_{x,f} := (\mathfrak{g}_\ell)_{x,f}^{\Gamma_{\ell/k}}, \quad G_{x,f} := \underline{G}(\ell)_{x,f}^{\Gamma_{\ell/k}}.$$

These groups do not depend on the choice of ℓ. We need not assume that f is concave to define these objects, although they are best-behaved in that case. We can also construct for each $\underline{\alpha}$ in $\underline{R} \cup \{0\}$ a root space

$$\mathfrak{g}^{\underline{\alpha}} := \Big(\bigoplus_{\alpha \in \underline{\alpha}(\bar{k})} \mathfrak{g}_\ell^{\alpha}\Big)^{\Gamma_{\ell/k}}.$$

When $\alpha = 0$, the root space $\mathfrak{g}^0$ is just the Lie algebra of $\underline{T}$. Combining this construction with Remark 3.6, we see that each root space $\mathfrak{g}^{\underline{\alpha}}$ admits a natural filtration: define

$$\mathfrak{g}^{\underline{\alpha}}_{x,f} = \Big(\bigoplus_{\alpha \in \underline{\alpha}(\bar{k})} (\mathfrak{g}_\ell^{\alpha})_{x,f}\Big)^{\Gamma_{\ell/k}}.$$

It follows immediately from the definitions that

$$\mathfrak{g}_{x,f} = \bigoplus_{\underline{\alpha} \in \underline{R} \cup \{0\}} \mathfrak{g}^{\underline{\alpha}}_{x,f}.$$

More generally, given a subset $\underline{R}' \subseteq \underline{R} \cup \{0\}$, let

$$\mathfrak{g}^{\underline{R}'} := \bigoplus_{\underline{\alpha} \in \underline{R}'} \mathfrak{g}^{\underline{\alpha}}, \quad \mathfrak{g}^{\underline{R}'}_{x,f} := \bigoplus_{\underline{\alpha} \in \underline{R}'} \mathfrak{g}^{\underline{\alpha}}_{x,f}.$$

Warning 3.8. In contrast to the function of Remark 3.6, the function $f : \underline{R} \cup \{0\} \to \widetilde{\mathbb{R}}$ defined, for a fixed $\underline{\alpha} \in \underline{R}$, by

$$f(\underline{\beta}) = \begin{cases} r & \text{if } \underline{\beta} = \underline{\alpha}, \\ \infty & \text{if not,} \end{cases}$$

is generally not concave. Failure of concavity relates to the fact that when $\underline{T}$ is not split, there is generally no "root group" (or even "root variety") $\underline{U}^{\underline{\alpha}}$ whose Lie algebra is $\mathfrak{g}^{\underline{\alpha}}$.

Remark 3.9. If we assume that f is finite, or without loss of generality, that f takes values in $\mathbb{R}$, then the group $G_{x,f}$ can be interpreted as the integral points of an $\mathcal{O}$-group: that is, there is a canonical $\mathcal{O}$-group $\underline{G}_{x,f}$, an integral model of $\underline{G}$, such that

$$\underline{G}_{x,f}(\mathcal{O}) = G_{x,f}$$

as subgroups of G. The construction is due to Yu [60, Section 8].

Having defined the group $G_{x,f}$ and Lie algebra $\mathfrak{g}_{x,f}$, we now study them. There are three areas of interest for our later applications.

First, in our calculation of the formal degree, it greatly simplifies notation to reduce to the case where G has anisotropic center. We record here the lemma effecting this reduction.

Lemma 3.10. *Let $\underline{G}$ be a reductive k-group, let $\underline{T}$ be a tame maximal torus of $\underline{G}$, let $f : \underline{R}(\underline{G}, \underline{T}) \to \widetilde{\mathbb{R}}$ be a positive concave function, let $\underline{A} \subseteq \underline{T}$ be a split central torus of $\underline{G}$, and let $y \in \mathcal{A}(\underline{T}, \underline{G})$. Then the groups $G_{y,f}/(A \cap G_{y,f})$ and $(G/A)_{y,f}$ are identical subgroups of G/A.*

Proof. This follows in the split case using Hilbert's Theorem 90, and the general case follows immediately from the split case by taking Galois invariants. ∎

Second, when we compute certain subgroup indices in Section 3.8, we need to understand how the groups of the form $G_{x,f}$ intersect for fixed x.

Lemma 3.11. *Let $f, g : \underline{R} \cup \{0\} \to \widetilde{\mathbb{R}}$ be positive concave functions. Then*

$$G_{x,f} \cap G_{x,g} = G_{x,\max(f,g)}.$$

Proof. In the case where G is split, a classical result of Bruhat and Tits [8, Proposition 6.4.48] can be used to show [60, Section 8.3.1] that the natural multiplication map

$$\prod_{\alpha \in R \cup \{0\}} U^{\alpha}_{x,f(\alpha)} \to G_{x,f}$$

is a bijection for a certain ordering of the factors. Using tame descent, this observation reduces the proof to the obvious fact (still in the split case) that

$$U^{\alpha}_{x,r} \cap U^{\alpha}_{x,s} = U^{\alpha}_{x,\max(r,s)}. \qquad \blacksquare$$

Third and lastly, we compare subgroup indices between G and $\mathfrak{g}$, generalizing the Moy–Prasad isomorphism. In Section 3.8, this comparison reduces a volume computation to a length computation, which we carry out in Section 3.6. Given functions $f, g : \underline{R} \cup \{0\} \to \widetilde{\mathbb{R}}$ with $f \leq g$, let

$$\mathfrak{g}_{x,f:g} := \mathfrak{g}_{x,f}/\mathfrak{g}_{x,g}.$$

Lemma 3.12. *Let $f, g : \underline{R} \cup \{0\} \to \widetilde{\mathbb{R}}$ be positive concave functions such that $f \leq g$. Assume in addition that is satisfied:*

$$g(a) \leq \sum_{i \in I} f(a_i) + \sum_{j \in J} f(b_j)$$

for all nonempty finite sequences $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ of elements of $\underline{R} \cup \{0\}$ such that $a := \sum_{i\in I} a_i + \sum_{j\in J} b_j \in \underline{R} \cup \{0\}$. Then

(i) *$G_{x,f}$ normalizes $G_{x,g}$, so that the quotient $G_{x,f:g} := G_{x,f}/G_{x,g}$ is a group,*

(ii) *$[G_{x,f}, G_{x,f}] \subseteq G_{x,g}$, so that the group $G_{x,f:g}$ is abelian,*

(iii) *there is a canonical isomorphism $\mathfrak{g}_{x,f:g} \simeq G_{x,f:g}$ of abelian groups.*

Proof. This follows from [8, Propositions 6.4.43, 6.4.44, 6.4.48] in the split case and [60, Section 2] in general. ∎

Remark 3.13. In Lemma 3.12, it is tempting to instead impose the simpler condition that $g(a) \leq \sum_{i\in I} f(a_i)$ for all nonempty finite sequences $(a_i)_{i\in I}$ such that $\sum_{i\in I} a_i = a$. However, this stronger condition would significantly weaken the conclusion: the condition implies, by taking the constant sequence, that $g \leq f$, so that $g = f$.

Corollary 3.14. *Let $f, g : \underline{R} \cup \{0\} \to \mathbb{R}$ be positive concave functions such that $f \leq g$, and suppose there is a chain of concave functions*

$$f = f_0 \leq f_1 \leq \cdots \leq f_n = g$$

such that for each i with $1 \leq i \leq n$, the pair (f_{i-1}, f_i) satisfies the conditions of Lemma 3.12. Then

$$|G_{x,f:g}| = |\mathfrak{g}_{x,f:g}|.$$

3.5. *Yu's groups*

In our application, it is enough to work with the subgroups constructed from a certain restricted class of concave functions, those constructed from admissible sequences and tame twisted Levi sequences. The construction specializes that of Section 3.4, so we work in the same setting. After reviewing Yu's construction of these groups, we specialize the theory of Section 3.4 to show that they admit a Moy–Prasad isomorphism.

A sequence $\vec{r} = (r_i)_{i=0}^d$ in $\widetilde{\mathbb{R}}$ is *admissible* if there is some j with $0 \le j \le d$ such that $0 \le r_0 = \cdots = r_j$ and $\frac{1}{2} r_j \le r_{j+1} \le \cdots \le r_d$. The admissible sequence is *weakly increasing* if, in addition, $r_i \le r_{i+1}$ for all i with $0 \le i \le d-1$.

Recall the definition of a (tame) twisted Levi sequence $\vec{\underline{G}}$ from Section 2.2. We assume that the tame maximal torus $\underline{T}$ of $\underline{G}$ is contained in $\underline{G}^0$, so that $x \in \mathcal{B}(\underline{G}^i)$ for each i. For each i with $0 \le i \le d$, let $\underline{R}_i := \underline{R}(\underline{G}^i, \underline{T})$.

It is sometimes necessary to work with truncated twisted Levi sequences. Given integers a and b such that $0 \le a \le b \le d$, let

$$\vec{\underline{G}}^{[a,b]} := (\underline{G}^a \subsetneq \cdots \subsetneq \underline{G}^b);$$

given an integer i with $0 \le i \le d$, define $\vec{\underline{G}}^{(i)} := \vec{\underline{G}}^{[0,i]}$.

Given an admissible sequence $\vec{r}$ and a twisted Levi sequence $\vec{\underline{G}}$ of the same length d, define the function $f_{\vec{r}} : \underline{R} \to \widetilde{\mathbb{R}}$ by

$$f_{\vec{r}}(\underline{\alpha}) := \begin{cases} r_0 & \text{if } \underline{\alpha} \in \underline{R}_0 \cup \{0\}, \\ r_i & \text{if } \underline{\alpha} \in \underline{R}_i \setminus \underline{R}_{i-1},\ 1 \le i \le d. \end{cases}$$

Since $\vec{r}$ is admissible, the function $f_{\vec{r}}$ is concave [58, Lemma 1.2]. Hence we may define

$$\vec{G}_{x,\vec{r}} := G_{x,f_{\vec{r}}}, \quad \vec{\mathfrak{g}}_{x,\vec{r}} := \mathfrak{g}_{x,f_{\vec{r}}}.$$

Given a second admissible sequence $\vec{s}$ of length d with $r_i \le s_i$ for all i, define

$$\vec{G}_{x,\vec{r}:\vec{s}} := G_{x,f_{\vec{r}}:f_{\vec{s}}}, \quad \vec{\mathfrak{g}}_{x,\vec{r}:\vec{s}} := \mathfrak{g}_{x,f_{\vec{r}}:f_{\vec{s}}}.$$

Remark 3.15. The group $\vec{G}_{x,\vec{r}}$ depends only on $\vec{\underline{G}}$, x, and $\vec{r}$; in particular, it is independent of the choice of the torus T provided that $x \in \mathcal{B}(T)$ [58, Section 1]. The same cannot necessarily be said for a general group of the form $G_{x,f}$, however, because the domain of definition of f knows something about the torus T, namely, the Galois action on the root system. It may be possible to formulate an independence result in this more general setting, but one would have to find an object, like Yu's twisted Levi sequence, common to the various elliptic maximal tori whose buildings are a given fixed subset of $\mathcal{B}(G)$.

In this setting, Yu [58, Lemma 1.3 and Corollary 2.4] generalized the Moy–Prasad isomorphism.

Lemma 3.16. *Let $\vec{\underline{G}}$ be a tame Levi sequence of length d and let $\vec{r}$ and $\vec{s}$ be admissible sequences of length d such that for all i,*

$$0 < r_i \le s_i \le \min(r_i, \ldots, r_d) + \min(\vec{r}). \tag{3.17}$$

Then $\vec{G}_{x,\vec{r}:\vec{s}}$ is an abelian group canonically isomorphic to $\vec{\mathfrak{g}}_{x,\vec{r}:\vec{s}}$.

Proof. Condition (3.17) implies that $(f_{\vec{r}}, f_{\vec{s}})$ satisfies the conditions of Lemma 3.12. ∎

Corollary 3.18. *Let $\underline{\vec{G}}$ be a tame Levi sequence of length d and let $\vec{r}$ and $\vec{s}$ be weakly increasing admissible sequences of length d such that $0 < r_i \leq s_i < \infty$ for all i. Then*

$$|\vec{G}_{x,\vec{r}:\vec{s}}| = |\vec{\mathfrak{g}}_{x,\vec{r}:\vec{s}}|.$$

Proof. Since $\vec{r}$ is weakly increasing, condition (3.17) simplifies to

$$s_i \in [r_i, r_i + r_0] \quad \text{for all } i. \tag{3.19}$$

It is now an elementary but tedious exercise to construct a chain $(\vec{s}^{\,(j)})_{0\leq j\leq N}$ of weakly increasing admissible sequences $\vec{s}^{\,(j)} = (s_0^{(j)} \leq \cdots \leq s_d^{(j)})$, where $N \gg 0$, such that $s^{(0)} = \vec{r}$ and $s^{(d)} = \vec{s}$, such that $s_i^{(j)} \leq s_i^{(j+1)}$ for all i and j with $0 \leq j \leq N-1$, and such that each pair $(\vec{s}^{\,(j-1)}, \vec{s}^{\,(j)})$ satisfies condition (3.17) of Lemma 3.16. After completing this exercise, we invoke Corollary 3.14. ∎

We need the corollary in the following special case only.

Corollary 3.20. *Let $\underline{\vec{G}}$ be a tame Levi sequence of length d and let $\vec{r}$ be a weakly increasing admissible sequence of length d. Then*

$$|\vec{G}_{x,0+:\vec{r}}| = |\vec{\mathfrak{g}}_{x,0+:\vec{r}}|.$$

3.6. Length computation

Retaining the notation of Section 3.4, let $f : \underline{R} \to \mathbb{R}$ be a positive function and let $\underline{R}' \subseteq \underline{R}$ be a subset. Our goal in this subsection is to compute the length of the $\mathcal{O}$-module $\mathfrak{g}_{x,0+:f}^{\underline{R}'}$, culminating in Theorem 3.24.

We start by studying the jumps in the filtration on $\mathfrak{g}_x^{\underline{\alpha}}$. For $\underline{\alpha} \in \underline{R}$, consider the set

$$\operatorname{ord}_x \underline{\alpha} := \{t \in \mathbb{R} : \mathfrak{g}_{x,t:t+}^{\underline{\alpha}} \neq 0\}$$

of jumps in the Moy–Prasad filtration of $\mathfrak{g}^{\underline{\alpha}}$, defined and studied by DeBacker and Spice [12, Definition 3.6]. A full description of $\operatorname{ord}_x \underline{\alpha}$ requires an understanding of the point x, and thus the way in which $\mathcal{B}(\underline{T})$ embeds in $\mathcal{B}(\underline{G})$. This is quite difficult in general. But for us it is enough to know several weak properties of these sets.

Lemma 3.21. *Let $\underline{T}$ be a tame maximal torus of $\underline{G}$, let $x \in \mathcal{A}(\underline{G}, \underline{T})$, and let $\underline{\alpha} \in \underline{R}$. Then*

(i) $\operatorname{ord}_x(-\underline{\alpha}) = -\operatorname{ord}_x \underline{\alpha}$,
(ii) $\operatorname{ord}_x \underline{\alpha}$ *is an* $\operatorname{ord}_k(k_{\underline{\alpha}}^{\times})$*-torsor,*
(iii) $\operatorname{len} \mathfrak{g}_{x,t:t+}^{\underline{\alpha}} = [\kappa_{\underline{\alpha}} : \kappa] \mathbb{1}_{\operatorname{ord}_x \underline{\alpha}}(t)$.

Proof. Property (i) follows from a PGL_2 calculation [12, Corollary 3.11]. As for the other properties, earlier we mentioned that $\mathfrak{g}^{\underline{\alpha}}$ was isomorphic to $k_{\underline{\alpha}}$, though not canonically. Choose one such isomorphism $\phi : k_{\underline{\alpha}} \to \mathfrak{g}_{\underline{\alpha}}$. This isomorphism is compatible with the Moy–Prasad filtration in the following sense: there is a real number r_0 such that for all $r \in \mathbb{R}$, the isomorphism ϕ restricts to an isomorphism $k_{\underline{\alpha},r+r_0} \simeq \mathfrak{g}_{\underline{\alpha},r}$ of $\mathcal{O}$-modules. Properties (ii) and (iii) now follow immediately. ∎

Corollary 3.22. *Let $\underline{T}$ be a tame maximal torus of $\underline{G}$, let $x \in \mathcal{A}(\underline{G}, \underline{T})$, let $f : \underline{R} \cup \{0\} \to \mathbb{R}$ be a positive function, and let $\underline{R}' \subseteq \underline{R}$. Then*

$$\operatorname{len} \mathfrak{g}_{x,0+:f}^{\underline{R}'} = \sum_{\underline{\alpha} \in \underline{R}'} \sum_{0<t<f(\underline{\alpha})} [\kappa_{\underline{\alpha}} : \kappa] \mathbb{1}_{\operatorname{ord}_x \underline{\alpha}}(t).$$

Proof. It suffices to prove the corollary in the case where $\underline{R}' = \{\underline{\alpha}\}$. Then both sides equal

$$\sum_{\substack{t \in \operatorname{ord}_{\underline{\alpha}} x \\ 0<t<f(\underline{\alpha})}} \operatorname{len} \mathfrak{g}_{x,t:t+}^{\underline{\alpha}}$$

by Lemma 3.21 (iii). ∎

To simplify the sum in Corollary 3.22 we prove a more general result about summation of discretely supported functions $h : \mathbb{R} \to \mathbb{N}$. Given a lattice $\Lambda \subset \mathbb{R}$, that is, a nontrivial cyclic subgroup, say that h is Λ-*periodic* if $h(t + \lambda) = h(t)$ for all $t \in \mathbb{R}$ and $\lambda \in \Lambda$. Given a bounded interval $I \subset \mathbb{R}$ with endpoints $a < b$, define

$$\sum_I{}' h = \sum_{t \in I}{}' h(t) = \sum_{a<t<b} h(t) + \tfrac{1}{2}[a \in I]h(a) + \tfrac{1}{2}[b \in I]h(b),$$

where $[\cdot]$ is the Iverson bracket. Because of the normalization at the endpoints, this summation operator enjoys the property that

$$\sum_I{}' h + \sum_J{}' h = \sum_{I \cup J}{}' h$$

whenever I, J, and $I \cup J$ are compact intervals.

Lemma 3.23. *Let $\Lambda \subset \mathbb{R}$ be a free abelian group of rank* 1*, let $\lambda_0 := \min(\Lambda \cap \mathbb{R}_{>0})$, let $h : \mathbb{R} \to \mathbb{N}$ be a discretely supported function, and let $H(s) := \sum'_{0 \le t \le s} h(t)$ for $s > 0$. Suppose h is even and Λ-periodic. Then for all $s \in \frac{1}{2}\Lambda \cap \mathbb{R}_{>0}$,*

$$H(s) = \frac{s}{\lambda_0} H(\lambda_0).$$

Proof. Since $H(s + \lambda) = H(s) + H(\lambda)$, induction reduces the proof to the case where $s = \lambda_0$ or $s = \frac{1}{2}\lambda_0$. The first case is obvious; for the second, use $h(t) = h(\lambda_0 - t)$. ∎

We can now compute a certain sum that appears in the formal degree.

Theorem 3.24. *Let T be a tame maximal torus of G, let $x \in \mathcal{A}(\underline{G}, \underline{T})$, let $f : \underline{R} \cup \{0\} \to \mathbb{R}$ be a positive even function, and let $\underline{R}' \subseteq \underline{R}$ be a subset closed under negation. Suppose that $f(\underline{\alpha}) \in \frac{1}{2} \operatorname{ord}(k_{\underline{\alpha}}^\times)$ for all $\underline{\alpha} \in \underline{R}$. Then*

$$\operatorname{len}(\mathfrak{g}_{x,0+:f}^{\underline{R}'}) + \tfrac{1}{2} \operatorname{len}(\mathfrak{g}_{x,0:0+}^{\underline{R}'}) + \tfrac{1}{2} \operatorname{len}(\mathfrak{g}_{x,f:f+}^{\underline{R}'}) = \sum_{\underline{\alpha} \in \underline{R}'} [k_{\underline{\alpha}} : k] f(\underline{\alpha}).$$

Proof. By Corollary 3.22, the left-hand side of the formula is

$$\operatorname{len}(\mathfrak{g}^{\underline{R}'}_{x,0+:f}) + \tfrac{1}{2}\operatorname{len}(\mathfrak{g}^{\underline{R}'}_{x,0:0+}) + \tfrac{1}{2}\operatorname{len}(\mathfrak{g}^{\underline{R}'}_{x,f:f+}) = \sum_{\underline{\alpha}\in\underline{R}'}\sideset{}{'}\sum_{0\le t\le f(\underline{\alpha})}[\kappa_{\underline{\alpha}}:\kappa]\mathbb{1}_{\operatorname{ord}_x\underline{\alpha}}(t).$$

Since f is even and $\kappa_{\underline{\alpha}} = \kappa_{-\underline{\alpha}}$, the right-hand side above is

$$\sum_{\underline{\alpha}\in\underline{R}'}\sideset{}{'}\sum_{0\le t\le f(\underline{\alpha})}\tfrac{1}{2}[\kappa_{\underline{\alpha}}:\kappa]\big(\mathbb{1}_{\operatorname{ord}_x\underline{\alpha}}(t) + \mathbb{1}_{\operatorname{ord}_x(-\underline{\alpha})}(t)\big).$$

By Lemma 3.21, the function $\mathbb{1}_{\operatorname{ord}_x\underline{\alpha}} + \mathbb{1}_{\operatorname{ord}_x(-\underline{\alpha})}$ is even and $\operatorname{ord}(k_{\underline{\alpha}}^\times)$-periodic. Hence we may apply Lemma 3.23 to conclude that

$$\sideset{}{'}\sum_{0\le t\le f(\underline{\alpha})}\tfrac{1}{2}[\kappa_{\underline{\alpha}}:\kappa]\big(\mathbb{1}_{\operatorname{ord}_x\underline{\alpha}}(t) + \mathbb{1}_{\operatorname{ord}_x(-\underline{\alpha})}(t)\big) = [k_{\underline{\alpha}}:k]f(\underline{\alpha}),$$

using the fact that

$$\sideset{}{'}\sum_{0\le t\le 1}[\kappa_{\underline{\alpha}}:\kappa]\mathbb{1}_{\operatorname{ord}_x\underline{\alpha}}(t) = \sum_{0\le t<1}[\kappa_{\underline{\alpha}}:\kappa]\mathbb{1}_{\operatorname{ord}_x\underline{\alpha}}(t) = [\kappa_{\underline{\alpha}}:\kappa]\operatorname{ord}_k(k_{\underline{\alpha}}^\times) = [k_{\underline{\alpha}}:k]. \quad \blacksquare$$

3.7. Yu's construction

In this subsection we describe Yu's supercuspidal representations, following Hakim and Murnaghan's expanded exposition [27, Section 3] of Yu's original paper [58]. Yu's full construction is quite elaborate, but fortunately, it is enough for us to understand only the parts of the construction needed to calculate the formal degree.

A *cuspidal $\underline{G}$-datum* is a 5-tuple $\Psi = (\vec{\underline{G}}, y, \vec{r}, \rho, \vec{\phi})$ consisting of

- a tame twisted Levi sequence $\vec{\underline{G}}$ such that $\underline{Z}^0/\underline{Z}$ is anisotropic, where $\underline{Z}$ is the center of $\underline{G}$ and $\underline{Z}^0$ is the center of $\underline{G}^0$,
- a point y in the apartment of a tame maximal torus of $\underline{G}^0$,
- an increasing sequence $\vec{r} = (0 < r_0 < r_1 < \cdots < r_{d-1} \le r_d)$ of real numbers (if $d = 0$ then we only require that $0 \le r_0$),
- an irreducible representation ρ of $G^0_{[y]}$ whose restriction to $G^0_{y,0+}$ is 1-isotypic and for which the compact induction $\text{c-Ind}_{K^0}^{G^0}\rho$ is irreducible (hence supercuspidal),
- a sequence $\vec{\phi} = (\phi_0, \ldots, \phi_d)$ of characters, with ϕ_i a character of G^i, such that $\phi_d = 1$ if $r_d = r_{d-1}$ and otherwise ϕ_i has depth r_i for all i.

The datum is *generic* if for each $i \ne d$ the character ϕ_i of G^i is $\underline{G}^{i+1}$-generic in the sense of Section 2.1, and the datum is *regular* if, in addition, the depth-zero supercuspidal representation $\text{c-Ind}_{K^0}^{G^0}\rho$ is regular.

Many of the objects used in Yu's construction and built from a cuspidal $\underline{G}$-datum do not depend on the representations ρ and $\vec{\phi}$. To make this independence explicit, we define a *cuspidal $\underline{G}$-datum without representations* to be a 3-tuple $(\vec{\underline{G}}, y, \vec{r})$ consisting of the first three components of a cuspidal $\underline{G}$-datum.

Let $\Psi = (\vec{\underline{G}}, \vec{r}, y)$ be a cuspidal $\underline{G}$-datum without representations. From Ψ we can construct the following subgroups:

$$\begin{aligned} K^0 &= G^0_{[y]}, & K^0_+ &= G^0_{y,0+}, \\ K^{i+1} &= K^0 \vec{G}^{(i+1)}_{y,(0+,s_0,\ldots,s_i)}, & K^{i+1}_+ &= \vec{G}^{(i+1)}_{y,(0+,s_0+,\ldots,s_i+)}, \\ J^{i+1} &= (G^i, G^{i+1})_{y,(r_i,s_i)}, & J^{i+1}_+ &= (G^i, G^{i+1})_{y,(r_i,s_i+)}, \\ K &:= K^d := K^{d+1}, & K_+ &:= K^d_+ := K^{d+1}_+. \end{aligned}$$

Here $0 \le i \le d-1$ and $s_i := r_i/2$. Generally the dependence of these objects on Ψ is implicit, but if we wish to make the dependence explicit we indicate it with a subscript, for instance, $K = K_\Psi$. When Ψ is regular, an additional group can be constructed: the maximal torus S of Section 2.2, maximally unramified in G^0. The groups K^{i+1} and K^{i+1}_+ are particularly important; later on, we need to express them in the following alternative ways:

$$K^{i+1} = K^0 \vec{G}^{[1,i+1]}_{y,s_0,\ldots,s_i}, \quad K^{i+1}_+ = K^0_+ \vec{G}^{[1,i+1]}_{y,s_0+,\ldots,s_i+}, \tag{3.25}$$

$$K^{i+1} = K^i J^{i+1}, \qquad K^{i+1}_+ = K^i_+ J^{i+1}. \tag{3.26}$$

We can now outline Yu's construction. Let $\Psi = (\vec{\underline{G}}, y, \vec{r}, \rho, \vec{\phi})$ be a generic cuspidal $\underline{G}$-datum. For each $0 \le i \le d-1$ there is a certain finite-dimensional irreducible representation[3] ρ_i of K^{i+1} constructed from ϕ_i. To specify ρ_i precisely one uses the theory of the Weil–Heisenberg representation. For our purposes, it is enough to know that the quotient $W_i := J^{i+1}/J^{i+1}_+$ is a finite-dimensional symplectic vector space over $\mathbb{F}_p$ and the underlying complex vector space of ρ_i can be realized as the complex-valued functions on a maximal isotropic subspace W_i^0 of W_i. Since $\dim_{\mathbb{F}_p}(W_i^0) = \frac{1}{2}\dim_{\mathbb{F}_p}(W_i)$, so that $|W_i^0| = |W_i|^{1/2}$, it follows that

$$\dim \rho_i = [J^{i+1} : J^{i+1}_+]^{1/2}.$$

The representation ρ_i is then inflated to a representation[4] τ_i of K. Section 3.9 explains in more detail how the inflation procedure works, but at the moment, it is enough to know that the inflation procedure preserves dimension. In the edge case $i = -1$ take τ_{-1} to be the inflation (by the same procedure) of ρ to K, and in the edge case $i = d$ take τ_d to be the restriction of ϕ_d to K; we could also handle the case $i = -1$ in the same way as the case $0 \le i \le d-1$ by defining $\rho_{-1} := \rho$. Finally, define the supercuspidal representation π attached to Ψ as the compact induction

$$\pi = \text{c-Ind}_K^G \tau, \quad \tau := \tau_{-1} \otimes \tau_0 \otimes \cdots \otimes \tau_d.$$

[3] Our ρ_i is Hakim and Murnaghan's ϕ_i'.

[4] Our τ_i is Hakim and Murnaghan's κ_i.

In summary, then, by Lemma 3.4 the formal degree of the supercuspidal representation π attached to a cuspidal generic Yu datum Ψ has formal degree

$$\deg(\pi_\Psi, \mu) = \frac{\dim \rho \cdot \prod_{i=0}^{d-1} [J^{i+1} : J_+^{i+1}]^{1/2}}{\mathrm{vol}(K/A, \mu)} \tag{3.27}$$

where $\underline{A}$ is the maximal split central subtorus of $\underline{G}$ and μ is a Haar measure on G/A.

It greatly simplifies the notation in our computation of the formal degree to reduce to the case where the center of $\underline{G}$ is anisotropic, that is, $\underline{A} = 1$. Given a generic cuspidal $\underline{G}$-datum without representations $\Psi = (\vec{\underline{G}}, y, \vec{r})$, let $\overline{\Psi} = (\overrightarrow{\underline{G}^{\mathrm{a}}}, y, \vec{r}\,)$ denote the reduction of Ψ modulo $\underline{A}$, that is, $\underline{G}^{\mathrm{a},i} := \underline{G}^{i,\mathrm{a}} = \underline{G}^i/\underline{A}$.

Lemma 3.28. *Let Ψ be a generic cuspidal $\underline{G}$-datum without representations and let $\overline{\Psi}$ be the reduction of Ψ modulo $\underline{A}$. Then*

(i) *the groups $K_{\overline{\Psi}}$ and K_Ψ/A are identical as subgroups of G/A,*

(ii) $[J_\Psi^{i+1} : J_{\Psi,+}^{i+1}] = [J_{\overline{\Psi}}^{i+1} : J_{\overline{\Psi},+}^{i+1}]$ *for all* $0 \le i \le d-1$.

Proof. For (i), since

$$\frac{K_\Psi}{A} = \frac{G^0_{[y]}}{A} \cdot \frac{A\vec{G}^{(d)}_{y,(0+,\vec{s})}}{\vec{G}^{(d)}_{y,(0+,\vec{s})}} = \frac{G^0_{[y]}}{A} \cdot \frac{\vec{G}^{(d)}_{y,(0+,\vec{s})}}{\vec{G}^{(d)}_{y,(0+,\vec{s})} \cap A},$$

it suffices by Lemma 3.10 to show that

$$G^0_{[y]}/A = (G^0/A)_{[y]},$$

and this follows immediately from the surjectivity of the map $\underline{G}(k) \to (\underline{G}/\underline{A})(k)$ and the natural identification of the reduced buildings of $\underline{G}$ and $\underline{G}/\underline{A}$.

(ii) follows by an argument similar to the proof of Lemma 3.10. ■

3.8. *Degree computation*

In this subsection we compute the formal degree of Yu's supercuspidal representation. Let $\underline{G}$ be a reductive k-group, let $\Psi = (\vec{\underline{G}}, y, \vec{r}, \rho, \vec{\phi})$ be a cuspidal $\underline{G}$-datum, let $\underline{A}$ be the maximal split central subtorus of $\underline{G}$, and let μ be the Haar measure on G/A attached by Gan and Gross [25, 33] to a level-zero additive character of k.

Starting from (3.27), we will reduce the problem of computing the formal degree to the problem of computing certain subgroup indices. We will then be in a position to apply Theorem 3.24, finishing the calculation. By Lemma 3.28, we may assume for now that the center of G is anisotropic, though of course this restriction will have to be relaxed in the final formula.

We will start by simplifying the volume of K. To begin,

$$\mathrm{vol}(K, \mu)^{-1} = \mathrm{vol}(G_{y,0+}, \mu)^{-1} \frac{[KG_{y,0+} : K]}{[KG_{y,0+} : G_{y,0+}]}. \tag{3.29}$$

Lemma 3.30. *Let G be a group, let H be a subgroup of G, and let $N \subseteq M$ be subgroups of G normalized by H.*[5] *Then*

$$[MH : NH] = \frac{[M : N]}{[M \cap H : N \cap H]}$$

provided that all three indices in the expression are finite.

Proof. The inclusion $M \cap H \hookrightarrow M$ induces an injective map $M \cap H/N \cap H \hookrightarrow M/N$ which we can use to interpret the former as a subgroup of the latter. The group M/N acts transitively by left multiplication on the coset space MH/NH. Consider the stabilizer of the identity coset under this action. Clearly $M \cap H/N \cap H$ lies in the stabilizer, and conversely, it is easy to see that any representative of an element of the stabilizer can be translated by an element of N to lie in $M \cap H$. So $M \cap H/N \cap H$ is the stabilizer of the identity element, and the orbit-stabilizer theorem concludes the proof. ■

By (3.25) and Lemmas 3.11 and 3.30,

$$[KG_{y,0+} : K] = [K^0 G_{y,0+} : K^0 \vec{G}^{[1,d]}_{y,(s_0,\ldots,s_{d-1})}] = \frac{[G_{y,0+} : \vec{G}^{[1,d]}_{y,(s_0,\ldots,s_{d-1})}]}{[G^0_{y,0+} : G^0_{y,s_0}]}. \tag{3.31}$$

It follows from Lemma 3.11 again that $[KG_{y,0+} : G_{y,0+}] = [K^0 : K^0_+]$. Combining this calculation with (3.29) and (3.31) yields

$$\mathrm{vol}(K,\mu)^{-1} = \mathrm{vol}(G_{y,0+},\mu)^{-1} \frac{[G_{y,0+} : \vec{G}^{[1,d]}_{y,(s_0,\ldots,s_{d-1})}]}{[G^0_{[y]} : G^0_{y,s_0}]}.$$

The volume of $G_{y,0+}$ has been computed in the literature.

Lemma 3.32. *Let $x \in \mathcal{B}(G)$. If G is tame then* $\mathrm{vol}(G_{x,0+},\mu)^{-1} = q^{(\dim G)/2} |\mathfrak{g}_{x,0:0+}|^{1/2}$.

Proof. DeBacker and Reeder [11, Section 5.1] defined a Haar measure ν on G such that for any $x \in \mathcal{B}(G)$,

$$\mathrm{vol}(G(k)_{x,0},\nu) = \frac{|G_{x,0:0+}|}{|\mathfrak{g}_{x,0:0+}|^{1/2}};$$

in particular, ν does not depend on x. Kaletha [36, Lemma 5.15] used the tameness of G to show that $\nu = q^{(\dim G)/2}\mu$. Combining these results proves the lemma. ■

At this point we know that

$$\begin{aligned}\deg(\pi,\mu) = {} & \frac{\dim \rho}{[G^0_{[y]} : G^0_{y,0+}]} q^{(\dim G)/2} |\mathfrak{g}_{y,0:0+}|^{1/2} \\ & \times \frac{[G_{y,0+} : \vec{G}^{[1,d]}_{y,(s_0,\ldots,s_{d-1})}]}{[G^0_{y,0+} : G^0_{y,s_0}]} \prod_{i=0}^{d-1} [J^{i+1} : J^{i+1}_+]^{1/2},\end{aligned} \tag{3.33}$$

and we can use our earlier results on concave functions to simplify the expression.

[5] This condition implies that the product sets MH and NH are groups.

Lemma 3.34.

$$|\mathfrak{g}_{y,0:0+}|^{1/2}\frac{[G_{y,0+}:\vec{G}^{[1,d]}_{y,(s_0,\ldots,s_{d-1})}]}{[G^0_{y,0+}:G^0_{y,s_0}]}\prod_{i=0}^{d-1}[J^{i+1}:J^{i+1}_+]^{1/2}$$
$$=\exp_q\Big(\tfrac{1}{2}\operatorname{len}\mathfrak{g}^0_{y,0:0+}+\tfrac{1}{2}\sum_{i=0}^{d-1}r_i(|R_{i+1}|-|R_i|)\Big).$$

Proof. Let $f=f_{(s_0,\ldots,s_{d-1})}$ for the twisted Levi sequence $\underline{\vec{G}}^{[1,d]}$. By Lemma 3.16,

$$[J^{i+1}:J^{i+1}_+]=|(\mathfrak{g}^i,\mathfrak{g}^{i+1})_{y,(r_i,s_i):(r_i,s_i+)}|$$
$$=\big|(\mathfrak{g}^{i+1})^{\underline{R}_{i+1}\setminus\underline{R}_i}_{y,s_i:s_i+}\big|=\exp_q\Big(\sum_{\underline{\alpha}\in\underline{R}_{i+1}\setminus\underline{R}_i}\operatorname{len}\mathfrak{g}^{\underline{\alpha}}_{y,f:f+}\Big).$$

By Corollary 3.20,

$$[G_{y,0+}:\vec{G}^{[1,d]}_{y,(s_0,\ldots,s_{d-1})}]=|\vec{G}^{[1,d]}_{y,(0+,\ldots,0+):(s_0,\ldots,s_{d-1})}|$$
$$=\exp_q\Big(\operatorname{len}\mathfrak{g}^0_{y,0:s_0}+\sum_{i=0}^{d-1}\sum_{\underline{R}_{i+1}\setminus\underline{R}_i}\operatorname{len}\mathfrak{g}^{\underline{\alpha}}_{y,0:f}\Big)$$

and $[G^0_{y,0+}:G^0_{y,s_0}]=\exp_q\operatorname{len}\mathfrak{g}^0_{y,0:s_0}$, so that

$$\frac{[G_{y,0+}:\vec{G}^{[1,d]}_{y,(s_0,\ldots,s_{d-1})}]}{[G^0_{y,0+}:G^0_{y,s_0}]}=\exp_q\Big(\sum_{i=0}^{d-1}\sum_{\underline{R}_{i+1}\setminus\underline{R}_i}\operatorname{len}\mathfrak{g}^{\underline{\alpha}}_{y,0:f}\Big)=\exp_q\Big(\sum_{\underline{\alpha}\in\underline{R}\setminus\underline{R}_0}\operatorname{len}\mathfrak{g}^{\underline{\alpha}}_{y,0:f}\Big).$$

We now recognize that

$$\frac{|\mathfrak{g}_{y,0:0+}|^{1/2}}{|\mathfrak{g}^0_{y,0:0+}|^{1/2}}\cdot\frac{[G_{y,0+}:\vec{G}^{[1,d]}_{y,(s_0,\ldots,s_{d-1})}]}{[G^0_{y,0+}:G^0_{y,s_0}]}\prod_{i=0}^{d-1}[J^{i+1}:J^{i+1}_+]^{1/2}$$

equals $\exp_q$ of

$$\sum_{\underline{\alpha}\in\underline{R}\setminus\underline{R}_0}\;\sideset{}{'}\sum_{0\le t\le f(\underline{\alpha})}\operatorname{len}\mathfrak{g}^{\underline{\alpha}}_{y,t:t+}.$$

By Corollary 2.3 the hypotheses of Theorem 3.24 are satisfied, and the expression above becomes

$$\tfrac{1}{2}\sum_{i=0}^{d-1}r_i(|R_{i+1}|-|R_i|).$$

The proof is finished by recalling that $|\mathfrak{g}^0_{y,0:0+}|^{1/2}=\exp_q(\tfrac{1}{2}\operatorname{len}(\mathfrak{g}^0_{y,0:0+}))$. ■

We now make the reduction promised in Lemma 3.28. Recall the notation for $\overline{\Psi}$ defined immediately before Lemma 3.28.

Theorem A. *Let $\underline{G}$ be a reductive k-group and let Ψ be a generic cuspidal $\underline{G}$-datum with associated supercuspidal representation π. Then*

$$\deg(\pi,\mu) = \frac{\dim\rho}{[G^{\mathrm{a},0}_{[y]} : G^{\mathrm{a},0}_{y,0+}]} \exp_q\left(\tfrac{1}{2}\dim\underline{G}^{\mathrm{a}} + \tfrac{1}{2}\dim\underline{G}^{\mathrm{a},0}_{y,0:0+} + \tfrac{1}{2}\sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\right).$$

Proof. When $\underline{A} = 1$, the formula follows from (3.33), Lemma 3.34, and the fact that $\mathfrak{g}^0_{y,0:0+}$ is the Lie algebra of $\underline{G}^{\mathrm{a},0}_{y,0:0+}$. In general, according to (3.27), with the exception of the factor $\dim\rho$, the formal degree depends only on the underlying G-datum without representations $\Psi = (\vec{\underline{G}}, y, \vec{r})$. It now suffices to observe that by Lemma 3.28, the expression

$$\frac{\prod_{i=0}^{d-1}[J^{i+1} : J^{i+1}_+]^{1/2}}{\mathrm{vol}(K/A,\mu)}$$

is the same for both Ψ and the reduced datum $\overline{\Psi} := (\overrightarrow{\underline{G}^{\mathrm{a}}}, y, \vec{r}\,)$. ■

3.9. Regular supercuspidals

In the special case where the supercuspidal representation is regular, we can further simplify the expression for its formal degree. We have already seen, in Section 2.2, the source of this simplification: an arbitrary depth-zero supercuspidal π is compactly induced from a finite-dimensional representation ρ of the group $G_{[y]}$, but when the supercuspidal π is regular, we can recover a maximal torus S from π, and π is induced from a finite-dimensional representation η of the smaller group $SG_{y,0}$. In fact, the former representation is induced from the latter:

$$\rho = \mathrm{Ind}_{SG_{y,0}}^{G_{[y]}} \eta. \tag{3.35}$$

In the depth-zero case, replacing ρ by η in the formula for the formal degree thus multiplies the rest of the formula by the index $[G_{[y]} : SG_{y,0}]$. And since η is an extension of a Deligne–Lusztig representation, the literature provides a formula for its dimension. These observations simplify the formal degree for depth-zero regular supercuspidals.

When the regular supercuspidal has positive depth, however, there is a slight complication: Equation (3.35) must be propagated from $G_{[y]} = K^0$ to K. And to propagate the formula, we need to understand the inflation procedure mentioned in passing in Section 3.7. Nonetheless, inflation is compatible with induction in the most straightforward way, and in the end, the effect on the formula for the formal degree is the same.

Yu's inflation procedure is quite simple; we explain it following Hakim and Murnaghan [27, Section 3.4]. Recall equation (3.26), that $K^{i+1} = K^i J^{i+1}$. Suppose we are given a representation ρ of K^i satisfying the following condition:

$$\text{the restriction of } \rho \text{ to } K^i \cap J^{i+1} \text{ is 1-isotypic.} \tag{3.36}$$

Ultimately we will apply the following analysis to the representation ρ_i, which satisfies condition (3.36). Hence ρ may be interpreted as (the inflation of) a representation of the quotient group $K^i/(K^i \cap J^{i+1})$. Since K^i normalizes J^{i+1}, the decomposition of

(3.26) becomes a semidirect product after dividing by $K^i \cap J^{i+1}$. We can now inflate this representation of the quotient first to the semidirect product $K^{i+1}/(K^i \cap J^{i+1})$, then to the full group K^{i+1}. Let

$$\mathrm{Inf}_{K^i}^{K^{i+1}} \rho$$

denote the resulting representation. Since this representation is 1-isotypic on J^{i+1}, and since $K^{i+1} \cap J^{i+2} \subseteq J^{i+1}$, the representation satisfies condition (3.36) with i replaced by $i+1$. By induction, we can therefore define for any $j \geq i+1$, and in particular for $j = d$ (if $i < d$), the inflated representation

$$\mathrm{Inf}_{K_i}^{K^j} \rho := \mathrm{Inf}_{K^{j-1}}^{K^j} \cdots \mathrm{Inf}_{K^i}^{K^{i+1}} \rho.$$

Lemma 3.37. *Let Ψ be a cuspidal $\underline{G}$-datum; recall the notations of Section 3.7 for the various objects attached to Ψ. Let $H^i := SG^0_{y,0}K^i_+$ and let $H := H^d$. Suppose that there is a (necessarily irreducible) representation η of H^0 such that $\rho := \mathrm{Ind}_{H^0}^{K^0} \eta$. Then $\sigma_{-1} := \mathrm{Inf}_{H^0}^{H} \eta$ is defined and there is a canonical identification*

$$\tau = \mathrm{Ind}_H^K(\sigma_{-1} \otimes \tau_0|_H \otimes \cdots \otimes \tau_d|_H)$$

of representations of K.

Proof. The notation $\mathrm{Inf}_{H^i}^{H^j}$ mimics $\mathrm{Inf}_{K^i}^{K^j}$: the same construction works if the symbol K is replaced everywhere by H. Since ρ is 1-isotypic on J^1, so is η; hence $\sigma_{-1} := \mathrm{inf}_{H^0}^{H} \eta$ is defined. There is a canonical identification

$$\mathrm{Ind}_H^K \mathrm{Inf}_{H^0}^H \eta = \mathrm{Inf}_{K^0}^K \mathrm{Ind}_{H^0}^{K^0} \eta,$$

so that $\tau = (\mathrm{Ind}_H^K \sigma_{-1}) \otimes \tau_0 \otimes \cdots \otimes \tau_d$. The result now follows from the well-known formula $(\mathrm{Ind}_H^K \pi_1) \otimes \pi_2 = \mathrm{Ind}_H^K(\pi_1 \otimes (\pi_2|_H))$, where π_1 is a representation of H and π_2 is a representation of K [51, Section III.2.11]. ■

Kaletha [37, Section 3.4.4, proof of Lemma 3.4.20] showed that for a regular Yu datum, the finite-dimensional representation ρ has the property that $\rho = \mathrm{Ind}_{SG^0_{y,0}}^{K^0} \eta$ where η extends the inflation to $G^0_{y,0}$ of a Deligne–Lusztig induced representation of the group $G^0_{y,0:0+}$. We can thus compute $\dim \eta$ using Deligne–Lusztig theory.

Let us briefly recall the dimension formula for a Deligne–Lusztig induction. In this paragraph only, let $\underline{G}$ be a reductive κ-group, let $\underline{S}$ be a maximal elliptic torus of $\underline{G}$, and let $\theta : S \to \mathbb{C}^\times$ be a character. Deligne and Lusztig computed the dimension of the virtual representation $\mathrm{R}_{(S,\theta)}$ in their original paper [13, Corollary 7.2]; it is

$$\dim \mathrm{R}_{(S,\theta)} = \frac{[G:S]}{\dim \mathrm{St}_G}$$

where St_G is the Steinberg representation of G. In his classical book on representations of finite groups of Lie type, Carter [10, Corollary 6.4.3] computed the dimension of the Steinberg representation; it is

$$\log_q \dim \mathrm{St}_G = \tfrac{1}{2}(\dim \underline{G} - \dim \underline{S}).$$

We can now assemble these results to give our final formula. Recall the notation of Theorem A.

Corollary 3.38. *Let Ψ be a regular generic cuspidal $\underline{G}$-datum with resulting supercuspidal representation π and let $\underline{S}$ be the maximally unramified maximal torus of $\underline{G}^0$ resulting from Ψ, as explained in Section 2.2. Then*

$$\deg(\pi,\mu) = |S^{\mathrm{a}}_{0:0+}|^{-1} \exp_q\Big(\tfrac{1}{2}\dim \underline{G}^{\mathrm{a}} + \tfrac{1}{2}\operatorname{rank} \underline{G}^{\mathrm{a},0}_{y,0:0+} + \sum_{i=0}^{d-1} s_i(|R_{i+1}| - |R_i|)\Big). \tag{3.39}$$

Proof. By Lemma 3.37, Lemma 3.4, and Kaletha's description of ρ, the formula of Theorem A remains true if $\dim\rho$ is replaced by

$$[K^0 : SG^0_{y,0}]\dim\eta = [G^{\mathrm{a}}_{[y]} : S^{\mathrm{a}}G^{\mathrm{a},0}_{y,0}]\dim\eta.$$

The dimension formula for η discussed in the paragraph above shows that

$$\begin{aligned}\dim\eta &= [G^0_{y,0} : S_0 G^0_{y,0+}]\exp_q\big(\tfrac{1}{2}(\dim \underline{G}^0_{y,0:0+} - \dim \underline{S}_{0:0+})\big)^{-1} \\ &= [G^{\mathrm{a},0}_{y,0} : S^{\mathrm{a}}_0 G^{\mathrm{a},0}_{y,0+}]\exp_q\big(\tfrac{1}{2}(\dim \underline{G}^{\mathrm{a},0}_{y,0:0+} - \operatorname{rank} \underline{G}^{\mathrm{a},0}_{y,0:0+})\big)^{-1}.\end{aligned}$$

The formula now follows. ∎

4. Galois side

Let (S,θ) be a tame elliptic pair. We saw in Section 2.4 that when θ is extra regular, such a pair can be extended to a regular supercuspidal L-packet datum $(S,\hat{j},\chi,\theta)$, and that the resulting set of χ-data can then be used to form the regular supercuspidal parameter

$$\varphi_{(S,\theta)} := {}^L j_\chi \circ {}^L\theta.$$

Moreover, every regular supercuspidal parameter arises in this way. Our goal in this section is to compute the Galois side of the formal degree conjecture for the parameter $\varphi = \varphi_{(S,\theta)}$. As we mentioned in Section 2.4, the group $S^\natural_\varphi$ is abelian and thus has only one-dimensional irreducible representations, so that the Galois side of the conjecture simplifies to

$$\frac{|\gamma(0,\varphi,\mathrm{Ad},\psi)|}{|\pi_0(S^\natural_\varphi)|}.$$

Moreover, the factor $|\pi_0(S^\natural_\varphi)|$ has been computed in the literature. So our task is to compute the absolute value $|\gamma(0,\varphi,\mathrm{Ad},\psi)|$ of the adjoint γ-factor.

We start by reviewing in Section 4.1 the general definition of the γ-factor. As a second preliminary step, we work out in Section 4.2 how to base change the function r_χ used to solve the extension problem of Section 2.3.

To compute the adjoint γ-factor, we give an explicit description of the adjoint representation attached to the L-parameter φ. It turns out that this representation decomposes as a direct sum of two representations, one coming from the maximal torus of the dual group and the other from its root system. We can thus compute the adjoint γ-factors separately, in Sections 4.4 and 4.5, and multiply them together for the final answer, in Section 4.6. Beginning in Section 4.3, the start of the γ-factor computation proper, we must fix $\hat{\jmath}$ and χ in the L-packet datum $(S, \hat{\jmath}, \chi, \theta)$ extending (S, θ).

4.1. Review of L-, ε- and γ-factors

The γ-factor of a representation (π, V) of the Weil group W_k is defined by the formula

$$\gamma(s, \pi, \psi, \mu) := \varepsilon(s, \pi, \psi, \mu)\frac{L(1-s, \pi^\vee)}{L(s, \pi)}$$

where ψ is a nontrivial additive character of k and μ is an additive Haar measure on k. Hence the γ-factor is built from two quantities, the L-factor and the ε-factor.

In this subsection we recall the definitions of the L-factor and the ε-factor, following Tate's Corvallis notes [55]. Roughly speaking, the L-factor carries information about the unramified part of the representation and the ε-factor carries information about the ramified part. Since computing the absolute value of an ε-factor amounts to computing an Artin conductor, we also explain how to compute this quantity in our application, following Serre [53, Chapter VI].

The *L-factor* of π is the meromorphic function

$$L(s, \pi) := \det\big(1 - q^{-s}\pi(\mathrm{Frob}) \bigm| V^{I_k}\big)^{-1}$$

where $I_k \subset W_k$ is the inertia group and $\mathrm{Frob} \in W_k$ is a Frobenius element. Later, we use the fact that the L-factor is *inductive*: if $\ell \supseteq k$ is a field extension of k and (π, V) is a finite-dimensional complex representation of W_k then

$$L(s, \mathrm{Ind}_{\ell/k}(\pi)) = L(s, \pi), \quad \mathrm{Ind}_{\ell/k} := \mathrm{Ind}_{W_\ell}^{W_k}.$$

The ε-factor is more subtle to define than the L-factor, and most of the subtlety resides in its complex argument. Fortunately, since we are interested in only the absolute value of the γ-factor, not its complex argument, we content ourselves with a description of the absolute value of the ε-factor instead.

Changing s, ψ, or μ scales the ε-factor by a known quantity. We may thus define with no loss of information the simplified ε-factor

$$\varepsilon(\pi) := \varepsilon(0, \pi, \psi, \mu)$$

where ψ has level zero, that is, $\max\{n : \psi(\pi^{-n}\mathcal{O}) = 1\} = 0$, and where the Haar measure μ is self-dual with respect to ψ. With these conventions, the absolute value of the ε-factor is

$$|\varepsilon(\pi)|^2 = q^{\mathrm{cond}\,\pi} \tag{4.1}$$

where $\operatorname{cond}\pi$ is the Artin conductor of π. So computing the absolute value of the ε-factor amounts to computing the Artin conductor.

The Artin conductor is defined by the following procedure. Given a Galois representation (π, V), choose a Galois extension ℓ of k such that $\pi|_{W_\ell}$ is trivial, and let $\Gamma_{\ell/k}$ be the Galois group of ℓ over k. Then the Artin conductor satisfies the formula

$$\operatorname{cond}\pi = \sum_{i\geq 0} \frac{\operatorname{codim}(V^{\Gamma_{\ell/k,i}})}{[\Gamma_{\ell/k,0} : \Gamma_{\ell/k,i}]}$$

where $i \mapsto \Gamma_{\ell/k,i}$ is the lower numbering filtration. This formula is independent of the choice of ℓ. We can now extend the definition to all complex representations of the Weil group, not necessarily those of Galois type, by stipulating that the Artin conductor be unchanged by unramified twists. In particular, $\operatorname{cond}\pi = 0$ if and only if π is unramified, and

$$\operatorname{cond}\pi = \operatorname{codim} V^{I_k} \tag{4.2}$$

if π is tamely ramified. Heuristically, the numerical invariant $\operatorname{cond}\pi$ is an enhancement of equation (4.2) that takes wild ramification into account and measures the extent to which π ramifies.

When π is irreducible, this heuristic is made precise by the formula

$$\operatorname{cond}\pi = (\dim\pi)(1 + \operatorname{depth}_k \pi). \tag{4.3}$$

In particular, (4.3) holds if π is a character. For our application, we need only understand how to compute the Artin conductor of a tamely induced representation. Unlike the L-factor, the Artin conductor is not invariant under induction. The best we can say in general is that given a finite extension ℓ of k and a representation π of W_ℓ, the induced representation has conductor

$$\operatorname{cond}\operatorname{Ind}_{\ell/k}\pi = \operatorname{ord}_k(\operatorname{disc}_{\ell/k})\dim\pi + f_{\ell/k}\operatorname{cond}\pi \tag{4.4}$$

where $\operatorname{disc}_{\ell/k}$ is the discriminant of ℓ over k. But when ℓ is tamely ramified over k and $\pi = \chi$ is a character, the formula simplifies considerably, even if, unlike in (4.3), the induced representation is reducible.

Lemma 4.5. *Let $\ell \supseteq k$ be a finite tame extension and let $\chi : W_\ell \to \mathbb{C}^\times$ be a character. Then*

$$\operatorname{cond}\operatorname{Ind}_{\ell/k}\chi = [\ell : k](1 + \operatorname{depth}_k \chi).$$

Proof. Check, using the tameness of the extension, that $\operatorname{ord}_k \operatorname{disc}_{\ell/k} = [\ell : k] - f_{\ell/k}$. This computation together with (4.3) and (4.4) yields

$$\operatorname{cond}\operatorname{Ind}_{\ell/k}\chi = [\ell : k] + f_{\ell/k}\operatorname{depth}_\ell \chi.$$

Now use the fact that

$$\operatorname{depth}_\ell \chi = e_{\ell/k}\operatorname{depth}_k \chi. \qquad \blacksquare$$

The L-factor, ε-factor, and Artin conductor are additive in the sense that

$$L(s,\pi) = L(s,\pi_1)L(s,\pi_2), \quad \varepsilon(\pi) = \varepsilon(\pi_1)\varepsilon(\pi_2), \quad \operatorname{cond}\pi = \operatorname{cond}\pi_1 + \operatorname{cond}\pi_2$$

where $\pi = \pi_1 \oplus \pi_2$. Hence the γ-factor is additive as well. This simple but crucial fact allows us to restrict our attention to summands of the adjoint representation.

4.2. Base change for χ-data

The main goal of this subsection is to determine how χ-data behave under base change, that is, restriction to the Weil group of a finite separable extension of k. Once we understand the effect of base change for arbitrary χ-data, we study its effect on minimally ramified χ-data.

Most of the definitions of this subsection are due to Langlands and Shelstad [43, Section 2.5], but our treatment is also influenced by Kaletha's recent reinterpretation of Langlands and Shelstad's formalism [38].

Let ℓ be a separable quadratic extension of k. Local class field theory shows that the quotient $k^\times/N_{\ell/k}(\ell^\times)$ is cyclic of order 2. The *quadratic sign character* of the extension $\ell \supseteq k$ is the character $k^\times \to \{\pm 1\}$ given by projection onto this quotient.

A root $\alpha \in R(G,S)$ is *symmetric* if it is Galois-conjugate to $-\alpha$, and is *asymmetric* otherwise. Letting $k_{\pm\alpha}$ denote the fixed field of the stabilizer in Γ_k of $\{\pm\alpha\}$, the extension $k_{\pm\alpha} \subseteq k_\alpha$ has degree 2 if α is symmetric and degree 1 if α is asymmetric. A symmetric root α is *unramified* if the quadratic extension $k_\alpha \supset k_{\pm\alpha}$ is unramified, and is *ramified* otherwise.

Definition 4.6. Let $R = R(G,S)$. A *set of χ-data* for (S,G) (or just S if G is understood) is a collection $\chi = (\chi_\alpha : k_\alpha^\times \to \mathbb{C}^\times)_{\alpha\in R}$ of characters satisfying the following properties:

(i) $\chi_{-\alpha} = \chi_\alpha^{-1}$.

(ii) $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$ for all $\sigma \in \Gamma_k$.

(iii) If α is symmetric then χ_α extends the quadratic sign character of $k_\alpha \supset k_{\pm\alpha}$.

Kaletha [38, Section 3] has interpreted a set of χ-data as giving rise to a character of a certain double cover of a torus, and the function r_χ as the L-parameter of this character. In light of that interpretation, we would expect that restricting r_χ to an extension of k corresponds to composing the χ-data with the norm map, in analogy with the discussion from Section 1.8. This turns out to be the case, as Theorem 4.11 shows.

Definition 4.7. Let χ be a set of χ-data and let ℓ be a finite separable extension of k. The *base change* of χ to ℓ is the χ-datum χ_ℓ defined by $\chi_{\ell,\alpha} := \chi_{\alpha,\ell_\alpha/k_\alpha}$.

The definition of base change makes sense only if the formula for χ_ℓ defines a χ-datum. We should immediately check this.

Lemma 4.8. *The function χ_ℓ of Definition 4.7 is a set of χ-data.*

Proof. Negation equivariance is clear. Compatibility with the Galois group follows from the easily verified formula

$$\sigma^{-1} \circ N_{\ell_{\sigma\alpha}/k_{\sigma\alpha}} = N_{\ell_\alpha/k_\alpha} \circ \sigma^{-1}.$$

As for the third property, if α is symmetric over ℓ then it is also symmetric over k and the canonical map $\Gamma_{\ell_\alpha/\ell_{\pm\alpha}} \to \Gamma_{k_\alpha/k_{\pm\alpha}}$ is an isomorphism. Now recall [55, (1.2.2)] that the local class field theory homomorphism $W_k \to k^\times$ (whose abelianization is the Artin reciprocity isomorphism) intertwines the inclusion $W_\ell \hookrightarrow W_k$ with the norm map $\ell^\times \to k^\times$. It follows that the canonical map

$$N_{\ell_\alpha/k_\alpha}(\ell_\alpha^\times)/N_{\ell_\alpha/k_\alpha}(\ell_{\pm\alpha}^\times) \to k_\alpha^\times/k_{\pm\alpha}^\times$$

is an isomorphism, and hence $\chi_{\ell,\alpha}$ extends the quadratic sign character of $\ell_\alpha \supset \ell_{\pm\alpha}$. ∎

It is time to start defining the function r_χ. The definition requires a brief preliminary discussion of abstract group theory. Let G be a group and K a subgroup. A section $u : K\backslash G \to G$ of the projection $G \to K\backslash G$ – in other words, a choice of coset representatives – gives rise to a $K\backslash G$-indexed family of set maps $u_x : G \to K$ for $x \in K\backslash G$. To define the maps u_x, we write down the element $u(x)g$ and decompose it as a product of an element of K followed by its coset representative in $K\backslash G$; that is, u_x is defined by the equation

$$u_x(g)u(xg) = u(x)g.$$

Before defining r_χ, a word is in order on the exact nature of the object we are defining, since that object depends on many arbitrary choices. A *gauge* is a function $p : R \to \{\pm 1\}$ such that $p(-\alpha) = -p(\alpha)$ for all $\alpha \in R$. Our construction produces for each gauge p a cohomology class $r_{\chi,p}$ of 1-cochains $W_k \to \widehat{S}$. As the gauge p varies, the cohomology classes $r_{\chi,p}$ are not identical. However, there is a canonical means of relating one class to the other. For any two gauges p and q, Langlands and Shelstad constructed a canonical 1-cochain $s_{q/p}$, depending only on $\underline{R}$ and not on the choice of χ-data. By definition, the cohomology classes $r_{\chi,p}$ are related by the equation

$$r_{\chi,q} = s_{q/p} r_{\chi,p}, \tag{4.9}$$

and the 1-cochains $s_{q/p}$ satisfy the right compatibility conditions to make these equations consistent [43, Corollary 2.4.B]. In the construction that follows we therefore define $r_{\chi,p}$ for a particular choice of gauge, and equation (4.9) then defines $r_{\chi,q}$ for any other q.

We can now write down the formula defining r_χ, making several arbitrary choices along the way. First, choose

(i) a section $[\underline{\alpha}] \mapsto \alpha$ of the orbit map $R \to \underline{R}/\{\pm 1\}$.

Each $[\underline{\alpha}] \in \underline{R}/\{\pm 1\}$ thus gives rise to two subgroups W_α and $W_{\pm\alpha}$, the stabilizers of α and $\{\pm\alpha\}$ in W_k. For each $[\underline{\alpha}]$, choose in addition

(ii) a section $u^\alpha : W_{\pm\alpha}\backslash W_k \to W_k$,

(iii) a section $v^\alpha : W_\alpha\backslash W_{\pm\alpha} \to W_{\pm\alpha}$.

What we call "choosing a section" is more commonly called "choosing coset representatives". Using these choices, define the element $r_\chi(w)$ of $\widehat{S} = X^*(S)_{\mathbb{C}}$ by

$$r_\chi(w) = \prod_{[\alpha],x} \chi_\alpha(v_0^\alpha(u_x^\alpha(w)))^{u^\alpha(x)^{-1}\alpha}, \quad [\underline{\alpha}] \in \underline{R}/\{\pm 1\},\ x \in W_{\pm\alpha}\backslash W_k. \tag{4.10}$$

We still have to explain the dependence on the gauge. Use choices (i) and (ii) above to define the gauge $p : R \to \{\pm 1\}$ by setting $p(\beta) = 1$ if and only if $\beta = u^\alpha(x)^{-1}\alpha$ for some $x \in W_{\pm\alpha}\backslash W_k$. Then (4.10) defines $r_{\chi,p} := r_\chi$. Now use (4.9) to extend the definition to all gauges.

Theorem 4.11. *Let $p : R \to \{\pm 1\}$ be a gauge, let χ be a set of χ-data for S, and let ℓ be a finite separable extension of k. Then*

$$r_{\chi,p}|_\ell = r_{\chi_\ell,p}$$

for some set of auxiliary choices ((i)–(iii) *above*) *in the definition of r_χ and r_{χ_ℓ}.*

Proof. First, some preliminary notation on group actions. Given a right G-set X and elements $x, y \in X$ that lie in the same G-orbit, let $x^{-1}y$, the transporter from x to y, denote the set of elements of G taking x to y. If $y = gx$ then $x^{-1}y = G_x g$ where G_x is the stabilizer of x in G.

Let $W := W_k$ and $W' := W_\ell$. It suffices to consider the case where R is a transitive $(\Gamma_k \times \{\pm 1\})$-set. The plan of the proof is to make choices (i)–(iii) for r_χ and for r_{χ_ℓ} so that the equation $r_{\chi,p}|_\ell = r_{\chi_\ell,p}$ holds on the nose, not up to cohomology. To get equality, not just cohomology, some of our choices depend on other choices.

Fix $\alpha \in R$ (choice (i) for r_χ). The double cosets $z \in W_{\pm\alpha}\backslash W/W'$ index the W'-orbits of $R/\{\pm 1\}$. Choose a section $c : W_{\pm\alpha}\backslash W/W' \to W$, and for each double coset z let $\alpha_z := c(z)^{-1}\alpha$ (choice (i) for r_{χ_ℓ}), so that the α_z form a set of representatives for the W'-orbits of $R/\{\pm 1\}$. Choose sections $u^z : W'_{\pm\alpha}\backslash W' \to W'$ (choice (ii) for r_{χ_ℓ}) and $v : W_\alpha\backslash W_{\pm\alpha} \to W_{\pm\alpha}$ (choice (iii) for r_χ), and use them to define sections $v^z : W'_{\alpha_z}\backslash W'_{\pm\alpha_z} \to W'_{\pm\alpha_z}$ (choice (iii) for r_{χ_ℓ}) by the formula

$$v^z(y) = c(z)^{-1}v(c(z)yc(z)^{-1})c(z).$$

Define the section $u : W_{\pm\alpha}\backslash W \to W$ (choice (ii) for r_χ) by $u(x) := c(z)u^z(y)$ where $z = xW'$ and $y = (Kc(z))^{-1}x$.

We have now made all necessary choices to define r_χ and r_{χ_ℓ}. It remains to check that these choices define the same gauge p and that $r_{\chi,p}|_\ell = r_{\chi_\ell,p}$.

To check that the gauges agree, first check that the assignment $x \mapsto (z, y)$ is a bijection from $W_{\pm\alpha}\backslash W$ to the set of pairs (z, y) with $z \in W_{\pm\alpha}\backslash W/W'$ and $y \in W'_{\pm\alpha_z}\backslash W'$. In the rest of the proof, we assume that x is related to (y, z) by this bijection. Hence a root is of the form $u(x)^{-1}\alpha$ with $x \in W_{\pm\alpha}\backslash W$ if and only if it is of the form $u^z(y)^{-1}\alpha_z$ with $z \in W_{\pm\alpha}\backslash W/W'$ and $y \in W'_{\pm\alpha_z}\backslash W'$.

Recall that for each $x \in W_{\pm\alpha}\backslash W$, there is a function $u_x : W \to W_{\pm\alpha}$ obtained from u by the equation

$$u(x)w = u_x(w)u(xw);$$

similarly, for each $z \in W_{\pm\alpha}\backslash W/W'$ and and $y \in W'_{\pm\alpha_z}\backslash W'$, there is a function $u^z_y : W' \to W'_{\pm\alpha_z}$ obtained from u^z by the equation

$$u^z(y)w' = u^z_y(w')u^z(yw').$$

These two constructions are related in the following way.

Claim 4.12. *Let $w \in W'$. Then $u_x(w') = c(z)u^z_y(w')c(z)^{-1}$.*

Proof. Let $x' = xw'$, let $z' = x'W'$, and let $y' = (Kc(z'))^{-1}x'$, so that (z', y') is obtained from x' in the same way as (z, y) was obtained from x. Expand the defining equation of $u_x(w')$:

$$c(z)u^z(y)w' = u_x(w')c(z')u^{z'}(y').$$

Since $z = z'$ and $y' = yw'$,

$$c(z)^{-1}u_x(y)c(z)u^z(yw') = u^z(y)w'.$$ ■

Use the sections u and v to compute the L-parameter of χ:

$$r_\chi(w) = \prod_x \chi_\alpha(v_0(u_x(w)))^{u(w)^{-1}\alpha}, \quad x \in W_{\pm\alpha}\backslash W.$$

Assume now that $w = w' \in W'$. Use Claim 4.12 to simplify the expression to

$$\begin{aligned} r_\chi(w') &= \prod_{z,y} \chi_\alpha(v_0(c(z)u^z_y(w')c(z)^{-1}))^{u(w')^{-1}\alpha} \quad (z \in W_{\pm\alpha}\backslash W/W',\ y \in W'_{\pm\alpha_z}\backslash W') \\ &= \prod_{z,y} \chi_\alpha(c(z)^{-1}v^z_0(u^z_y(w'))c(z))^{u(w')^{-1}\alpha} = \prod_{z,y} \chi_{\alpha_z}(v^z_0(u^z_y(w')))^{u^z(w')^{-1}\alpha_z}. \end{aligned}$$

To complete the argument, use the property cited above that the local class field theory homomorphism intertwines inclusion with the norm map. ■

Remark 4.13. Unlike most of the other parts of this paper, Theorem 4.11 and the definitions preceding it do not require the torus S to split over a tamely ramified extension; they hold for any torus whatsoever.

We can use the base change formula to bound the ramification of r_χ.

Corollary 4.14. *Let χ be a set of χ-data for a tamely ramified torus S. If each χ_α is tamely ramified of finite order then r_χ is tamely ramified of finite order.*

To be clear, since r_χ is not a character, by "order" we mean order as a cochain and by "tamely ramified" we mean trivial on P_k.

Proof. By hypothesis, there is a finite tamely ramified extension ℓ of k such that for each α the character $\chi_{\alpha,\ell}$ is trivial, so that χ_ℓ is trivial. Then r_χ restricts trivially to ℓ by Theorem 4.11 and is therefore tamely ramified of finite order. ■

To conclude this subsection we define minimally ramified χ-data following Kaletha [37, Definition 4.6.1]. The definition is relevant to this subsection because a minimally ramified set of χ-data satisfies the hypotheses, and thus the conclusion, of Corollary 4.14; we use this observation in the proof of Lemma 4.17.

Definition 4.15. A χ-datum χ for S is *minimally ramified* if S is tame and in addition χ_α is trivial for asymmetric α, unramified for unramified symmetric α, and tamely ramified for ramified symmetric α.

4.3. *Adjoint representation*

Our goal in this subsection is to describe the adjoint representation of a regular parameter, specifically, its decomposition into a "toral summand" coming from the torus S and a "root summand" coming from the root system $R(G,S)$. In subsequent subsections, we compute the ε-factor of both summands.

Recall from Section 2.3 that the regular parameter $\varphi_{(S,\theta)}$ is given by the formula

$$\varphi_{(S,\theta)}(w) = \hat{j}\big(\hat{\theta}(w) r_\chi(w)\big) n(\omega_{S,G}(w)) w$$

where $\hat{j} : \widehat{S} \to \widehat{G}$ is an admissible embedding with image a Galois-stable maximal torus $\widehat{T}$ and χ is a certain (carefully chosen) set of minimally ramified χ-data. We obtain the adjoint representation from $\varphi = \varphi_{(S,\theta)}$ by composing it with the adjoint homomorphism ${}^L G \to \mathrm{GL}(V)$ where $V := \hat{\mathfrak{g}}/\hat{\mathfrak{z}}^{\Gamma_k}$. The representation decomposes as a direct sum

$$V = V_{\mathrm{toral}} \oplus V_{\mathrm{root}}$$

where $V_{\mathrm{toral}} := \hat{\mathfrak{t}}/\hat{\mathfrak{z}}^{\Gamma_k}$ and

$$V_{\mathrm{root}} := \bigoplus_{\alpha \in R(G,S)} \hat{\mathfrak{g}}_\alpha.$$

Here $\hat{\mathfrak{g}}_\alpha$ is the usual $\alpha^\vee$-eigenline for the action of $\widehat{S}$ on $\hat{\mathfrak{g}}$, where $\alpha^\vee$ is interpreted, via $\hat{j}$ and the canonical identification $X^*(\widehat{T}) = X_*(T)$, as a root of $X^*(\widehat{T})$. We call V_{toral} the *toral summand* and V_{root} the *root summand*. From our formula for φ we can work out the adjoint Weil actions on these summands.

4.4. *Toral summand*

For the toral summand, it is useful to momentarily consider the vector space $\widetilde{V}_{\mathrm{toral}} := \mathrm{Lie}(\widehat{T})$ equipped with the adjoint Weil action of φ, so that the projection $\widetilde{V}_{\mathrm{toral}} \to V_{\mathrm{toral}}$ is Weil-equivariant. In general, for any complex torus T the natural inclusion $X_*(T) \hookrightarrow \mathrm{Lie}(T)$ gives rise to a canonical identification $X_*(T)_{\mathbb{C}} \simeq \mathrm{Lie}(T)$. The representation $\widetilde{V}_{\mathrm{toral}}$ of the Weil group W_k is therefore the complexification of the lattice $\Lambda = X_*(\widehat{T})$, isomorphic to

$X^*(S)$ by $X^*(\hat{\jmath})$ and the canonical identification $X_*(\widehat{T}) = X^*(T)$. The Galois action on the lattice Λ is transferred via this chain of identifications from the Galois action on $X^*(S)$ arising from the structure of S as a torus over k. To summarize, there is an identification of representations

$$\widetilde{V}_{\text{toral}} \simeq X^*(S)_{\mathbb{C}}.$$

Although $X^*(S^{\mathrm{a}})$ is a sublattice of $X^*(S)$, not a quotient, since $X^*(S^{\mathrm{a}}) = X^*(S)^{\Gamma_k}$ the smaller lattice becomes a canonical quotient of the larger after complexifying both. We thus have a second identification

$$V_{\text{toral}} \simeq X^*(S^{\mathrm{a}})_{\mathbb{C}}.$$

We can now compute the toral γ-factor using the lattice

$$M := X^*(S^{\mathrm{a}})^{I_k},$$

whose complexification is the vector space used to compute the L-factor of V_{toral}.

Lemma 4.16. $|\gamma(0, V_{\text{toral}})| = \exp_q\left(\frac{1}{2}(\dim S^{\mathrm{a}} + \dim M)\right)\dfrac{|M_{\text{Frob}}|}{|(\overline{\kappa}^\times \otimes M^\vee)^{\text{Frob}}|}.$

Proof. We omit several details because the calculation closely follows Kaletha's earlier work on epipelagic representations [36, Section 5.4].

It is easy to dispense with the L-factor at $s = 0$:

$$L(0, V_{\text{toral}})^{-1} = \det(1 - \text{Frob} \mid M_{\mathbb{C}}) = |M_{\text{Frob}}|,$$

where M_{Frob} denotes the coinvariants of Frobenius. The L-factor at $s = 1$ is

$$L(1, V_{\text{toral}})^{-1} = \det(1 - q^{-1}\text{Frob} \mid M_{\mathbb{C}}) = (-q)^{-\dim M} \det(1 - q\text{Frob}^{-1} \mid M_{\mathbb{C}}).$$

The determinantal factor in the last equation can be rewritten as

$$\det(1 - q\text{Frob}^{-1} \mid M_{\mathbb{C}}) = \det(1 - q\text{Frob} \mid M_{\mathbb{C}}^\vee) = |(\overline{\kappa}^\times \otimes M^\vee)^{\text{Frob}}|,$$

meaning that $L(1, V_{\text{toral}})^{-1} = q^{-\dim M} \cdot |(\overline{\kappa}^\times \otimes M^\vee)^{\text{Frob}}|$. Collecting the two L-factors gives

$$\left|\frac{L(1, V_{\text{toral}})}{L(0, V_{\text{toral}})}\right| = q^{\dim M} \frac{|M_{\text{Frob}}|}{|(\overline{\kappa}^\times \otimes M^\vee)^{\text{Frob}}|}.$$

Since S is tamely ramified, (4.2) shows that the Artin conductor of the toral summand is just

$$\text{cond}\, V_{\text{toral}} = \dim(V_{\text{toral}}/V_{\text{toral}}^{I_k}) = \dim S^{\mathrm{a}} - \dim M.$$

By our formula (4.1) relating the Artin conductor and the ε-factor,

$$|\varepsilon(V_{\text{toral}})| = \exp_q\left(\tfrac{1}{2}(\dim S^{\mathrm{a}} - \dim M)\right). \qquad \blacksquare$$

4.5. Root summand

The root summand is a direct sum of representations induced from characters of closed, finite-index subgroups of W_k. An element $w \in W_k$ acts on V_{root} through φ as follows. First, the action of W_k on $X^*(\widehat{S})$ $(= X_*(S))$ induces an action on the root system $R = R(G, S)$, and the element w permutes the root lines by this action. Second, the toral element

$$t_w := \hat{\jmath}(\hat{\theta}(w) r_\chi(w)) \in \widehat{T}$$

scales each root line $\hat{\mathfrak{g}}_\alpha$ by $\alpha^\vee(t_w)$, where $\alpha^\vee \in R^\vee(G, S)$ is interpreted as a character of $\widehat{T}$ using $\hat{\jmath}$. It follows that V_{root} is a direct sum of monomial representations. That is, for each Galois orbit $\underline{\alpha} \in \underline{R}(G, S)$ the subrepresentation

$$V_{\underline{\alpha}} := \bigoplus_{\alpha \in \underline{\alpha}(\bar{k})} \hat{\mathfrak{g}}_\alpha$$

is monomial and V_{root} is the direct sum (over $\underline{R}(G, S)$) of these representations. Further, after choosing a representative $\alpha \in \underline{\alpha}(\bar{k})$, we can identify V_α with the representation induced from a certain character ψ_α by which W_α acts on $\hat{\mathfrak{g}}_\alpha$. The essential matter, then, is to understand these characters ψ_α, and specifically, as it turns out, their depths.

Although the factor $n(\omega_{S,G}(w))w$ stabilizes the line $\hat{\mathfrak{g}}_\alpha$, it may fail to centralize it: instead, the factor scales the line by a certain sign $d_\alpha(w) \in \{\pm 1\}$. It follows that ψ_α is the product of two characters:

$$\psi_\alpha(w) = \big(d_\alpha(w)\langle \alpha^\vee, \hat{\jmath}(r_\chi(w))\rangle\big) \cdot \langle \alpha^\vee, (\hat{\jmath} \circ \hat{\theta})(w)\rangle$$

where $\langle -, - \rangle$ denotes the evaluation pairing $X^*(\widehat{T}) \otimes \widehat{T} \to \mathbb{C}^\times$. Although neither factor of the first character above is a character, their product is.

There are two essential cases in the analysis of the character ψ_α, depending on whether or not the character $\langle \alpha^\vee, (\hat{\jmath} \circ \hat{\theta})\rangle|_{W_\alpha}$ has positive depth. By the local Langlands correspondence the depth of this character is the same as the depth of the character $\theta_{k_\alpha/k} \circ \alpha^\vee : k_\alpha^\times \to \mathbb{C}^\times$, and we know something about these depths from Section 2.1.

Lemma 4.17. *The character $(d_\alpha \cdot \langle \alpha^\vee, \hat{\jmath} \circ r_\chi\rangle)|_{W_\alpha}$ of W_α is tamely ramified.*

Proof. The function d_α takes values in $\{\pm 1\}$. Since p is odd (see Remark 1.2), d_α is trivial on P_k. By Corollary 4.14, the character $d_\alpha \cdot \langle \alpha^\vee, \hat{\jmath} \circ r_\chi\rangle$ is trivial on P_k and has finite order. ∎

From this we can immediately deduce the following corollary.

Corollary 4.18. *If* $\operatorname{depth}(\theta_{k_\alpha/k} \circ \alpha^\vee) > 0$ *then* $\operatorname{depth}(\theta_{k_\alpha/k} \circ \alpha^\vee) = \operatorname{depth}(\psi_\alpha)$.

It remains to analyze the case where the depth of $\theta_{k_\alpha/k} \circ \alpha^\vee$ is not positive. We first assume that S is maximally unramified in G, then remove this assumption.

Lemma 4.19. *If S is maximally unramified and* $\operatorname{depth}(\theta_{k_\alpha/k} \circ \alpha^\vee) \le 0$ *then* $\operatorname{depth} \psi_\alpha = 0$.

Proof. It is clear from Lemma 4.17 that depth $\psi_\alpha \le 0$, so we need only show that ψ_α is ramified. Using the assumption that θ is extra regular, Kaletha [37, Proposition 5.2.7] proved that the parameter $\varphi = {}^L j_\chi \circ {}^L\theta$ is *regular* [37, Definition 5.2.3], meaning in particular that the connected centralizer of the inertia subgroup I_k in $\widehat{G}$ is abelian. So although the full centralizer of inertia may not be abelian, it does at least have the property that all of its elements are semisimple. Our proof proceeds by contradiction: assuming that ψ_α is unramified, we show that the centralizer of inertia contains a nontrivial unipotent element, a contradiction.

Since θ is regular (Definition 2.1), the roots α with $\operatorname{depth}(\theta_{k_\alpha/k} \circ \alpha^\vee) \le 0$ form a root subsystem R_0 of $R = R(\widehat{T}, \widehat{G})$, and the action of inertia on R_0 preserves a set R_0^+ of positive roots. Let $H := \operatorname{Ad}(\varphi(I_k))$, let $H\alpha$ denote the H-orbit of $\alpha \in R$, and let $U_\alpha \subset \widehat{G}$ be the root group for $\alpha \in R$. Since $I_k \cap W_\alpha$ is the inertia group of k_α and $\bigoplus_{\beta \in H\alpha} \hat{\mathfrak{g}}_\beta$ is a monomial representation of I_k induced from ψ_α, the character ψ_α is unramified if and only if the following three groups coincide: the stabilizer of U_α in H, the centralizer of U_α in H, and the centralizer of α in H. Moreover, ψ_α is unramified if and only if ψ_β is unramified for each $\beta \in H\alpha$. Assume $\alpha \in R_0^+$ satisfies these equivalent properties and has maximal length among all such roots. The proof works just as well if $\alpha \in R_0^-$, so we focus on the positive roots.

First, suppose the roots in the H-orbit $H\alpha$ of α are pairwise orthogonal. Choose a nontrivial element $u_\alpha \in U_\alpha$. For each $\beta \in H\alpha$, choose $x \in H$ such that $\beta = x\alpha$, and let $u_\beta := xu_\alpha$. The element u_β depends only on u_α and β and not on x. Consider the product

$$u = \prod_{\beta \in H\alpha} u_\beta.$$

Then u is invariant under H, hence centralizes inertia. But at the same time u is not semisimple because the H-orbit of α consists of positive roots, contradicting regularity.

In the remaining case, when the roots in the H-orbit of α are not pairwise orthogonal, a slight elaboration of the previous argument yields a contradiction. In this case the H-orbit of α admits an involution $\beta \mapsto \bar{\beta}$ such that $\langle \beta, \gamma \rangle \ne 0$ (for $\beta, \gamma \in H\alpha$) if and only if $\gamma \in \{\beta, \bar{\beta}\}$. From each pair $\{\beta, \bar{\beta}\}$ with $\beta \in H\alpha$ choose one element, including the element α, and let $(H\alpha)_+$ be the resulting set of orbit representatives, so that $H\alpha = (H\alpha)_+ \sqcup \overline{(H\alpha)_+}$. Choose a nontrivial element $u_\alpha \in U_\alpha$, choose $x \in H$ such that $x\alpha = \bar{\alpha}$, and define the element $u_{\bar{\alpha}} := xu_\alpha$, independent of the choice of x. The commutator subgroup $U_{\alpha+\bar{\alpha}} = [U_\alpha, U_{\bar{\alpha}}]$ is stabilized by x, and since we assumed that α had maximal length among the possible counterexamples to our theorem, it is not centralized by x. (In fact, x must act by inversion on this group because $xu_{\bar{\alpha}} = u_\alpha$.) Hence there is an element $u_{\alpha+\bar{\alpha}} \in U_{\alpha+\bar{\alpha}}$ with

$$u_{\alpha+\bar{\alpha}}^{-1} \cdot xu_{\alpha+\bar{\alpha}} = [u_\alpha, u_{\bar{\alpha}}], \quad \text{that is,} \quad u_{\alpha+\bar{\alpha}} u_\alpha u_{\bar{\alpha}} = x(u_{\alpha+\bar{\alpha}} u_\alpha u_{\bar{\alpha}}).$$

For each $\beta \in (H\alpha)_+$ choose $x \in H$ such that $\beta = x\alpha$, let $u_\beta := xu_\alpha$, and let $u_{\bar{\beta}} := xu_{\bar{\alpha}}$; these elements are independent of the choice of x. As before, define the element

$$u = \prod_{\beta \in (H\alpha)_+} u_{\beta+\bar{\beta}} u_\beta u_{\bar{\beta}}.$$

Since the action $\mathrm{Ad} \circ \varphi$ of wild inertia on $\widehat{G}^0$ is trivial, the group H acts on the factors of u through some abelian quotient. Hence u centralizes inertia but is not semisimple, contradicting regularity. ■

Remark 4.20. Kaletha [39] defines an L-parameter to be *torally wild* if it takes wild inertia to a maximal torus of $\widehat{G}$, and shows that torally wild L-parameters factor through the L-group of a tame maximal torus. The proof of Lemma 4.19 shows that this larger class of parameters satisfies the conclusion of the lemma.

Lemma 4.21. *If* $\mathrm{depth}(\theta_{k_\alpha/k} \circ \alpha^\vee) \leq 0$ *then* $\mathrm{depth}\, \psi_\alpha = 0$.

Proof. Recall from Section 2.1 that there is a twisted Levi subgroup G^0 of G such that $\alpha \in R(S, G^0)$ if and only if $\mathrm{depth}(\theta_{k_\alpha/k} \circ \alpha^\vee) \leq 0$, and that S is maximally unramified in G. Lemma 4.19 handles the case where $G = G^0$, so we assume that $G^0 \subsetneq G$.

To deal with the general case we factor the L-embedding ${}^LS \to {}^LG$ through ${}^LG^0$. Kaletha [37, Lemmas 5.2.9, 5.2.8] showed that there is an L-embedding ${}^Lj_{G^0,G} : {}^LG^0 \to {}^LG$ with the following property: the composite parameter $W_k \to {}^LS \to {}^LG$ is given by the formula

$$w \mapsto \hat{j}\big(\hat{\theta}_b(w)\hat{\theta}(w)r_\chi(w)\big)n(\omega_{S,G}(w))w$$

where $\theta_b : S(k) \to \mathbb{C}^\times$ is tamely ramified and $\Omega(S, G^0)^{\Gamma_k}$-invariant. Furthermore, from the construction of ${}^Lj_{G^0,G}$ it is clear that the embedding $\hat{\mathfrak{g}}^0 \hookrightarrow \hat{\mathfrak{g}}$ is ${}^Lj_{G^0,G}$-equivariant. In this way we reduce to the previous case of $G = G^0$ but with θ replaced by $\theta \cdot \theta_b^{-1}$. This replacement does not affect the validity of the reduction: since θ_b is $\Omega(S, G^0)^{\Gamma_k}$-invariant, the character $\theta' = \theta \cdot \theta_b^{-1}$ is still regular [37, Fact 3.7.6], and since θ_b is tamely ramified, we still have $\mathrm{depth}(\theta'_{k_\alpha/k} \circ \alpha^\vee) \leq 0$. ■

Remark 4.22. Unlike Corollary 4.18, Lemma 4.21 does not assert that ψ_α and $\theta_{k_\alpha/k} \circ \alpha^\vee$ have the same depth when the former has depth zero. I expect this stronger assertion to be true. It would be enough to prove that if θ is extra regular then $\mathrm{depth}(\theta_{k_\alpha/k} \circ \alpha^\vee) \geq 0$. But I was unable to prove the stronger assertion and a weaker statement sufficed.

In summary, the root summand decomposes as a direct sum

$$V_{\mathrm{root}} = \bigoplus_{\underline{\alpha} \in \underline{R}(G,S)} V_{\underline{\alpha}}$$

where, for any $\alpha \in \underline{\alpha}(\bar{k})$, the representation $V_{\underline{\alpha}}$ is induced from a character ψ_α of W_α with known depth. We can now easily compute the γ-factor. Recall the Galois sets R_i and depths $r_i \geq 0$ of Section 2.1.

Lemma 4.23. $|\gamma(0, V_{\mathrm{root}})| = \exp_q\Big(\frac{1}{2}|R| + \frac{1}{2}\sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\Big)$.

Proof. Suppose that $\alpha \in R_{i+1}$ for $0 \leq i \leq d-1$. Lemmas 2.5 and 4.21 and Corollary 4.18 show that $\mathrm{depth}_k\, \psi_\alpha = r_i$. Since L-factors are inductive, $L(s, V_{\underline{\alpha}}) = L(s, \psi_\alpha)$, and since

ψ_α is ramified, its L-factor is trivial. As for the absolute value of the ε-factor, since the extension $k_\alpha \supseteq k$ is tame, Lemma 4.5 shows that cond $V_{\underline{\alpha}} = (1 + r_i)|\underline{\alpha}(\bar{k})|$, so that

$$\varepsilon(V_{\underline{\alpha}}) = \exp_q\left(\tfrac{1}{2}(1 + r_i)|\underline{\alpha}(\bar{k})|\right).$$

Summing over $\underline{\alpha} \in \underline{R}(G, S)$ finishes the proof. ∎

4.6. Summary

Let A be the maximal split central subtorus of G, let $G^{\mathrm{a}} := G/A$, let $S^{\mathrm{a}} := S/A$, and let $M := X^*(S^{\mathrm{a}})^{I_k}$. Lemmas 4.16 and 4.23 show that the absolute value of the adjoint γ-factor is

$$|\gamma(0, V)| = \frac{|M_{\mathrm{Frob}}|}{|(\overline{\kappa}^\times \otimes M^\vee)^{\mathrm{Frob}}|} \exp_q\Big(\tfrac{1}{2}\dim G^{\mathrm{a}} + \tfrac{1}{2}\dim M + \tfrac{1}{2}\sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\Big).$$

Finally, since $|\pi_0(S_\varphi^\natural)| = |X_*(S^{\mathrm{a}})_{\Gamma_k}|$ by [36, Lemma 5.13], the Galois side of the formal degree conjecture is

$$\frac{|M_{\mathrm{Frob}}|}{|X_*(S^{\mathrm{a}})_{\Gamma_k}| \cdot |(\overline{\kappa}^\times \otimes M^\vee)^{\mathrm{Frob}}|} \exp_q\Big(\tfrac{1}{2}\dim G^{\mathrm{a}} + \tfrac{1}{2}\dim M + \tfrac{1}{2}\sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\Big). \tag{4.24}$$

5. Comparison

In this short final section we combine our work from Sections 3 and 4 with several results from the literature to show that the automorphic and Galois sides of the formal degree conjecture are equal.

Theorem B. *Kaletha's regular L-packets satisfy the formal degree conjecture.*

Let (S, θ) be a tame elliptic regular pair and let $\varphi = \varphi_{(S,\theta)}$ be the L-parameter attached to this pair by the constructions of Section 2.4. The Galois side of the formal degree conjecture for φ is expressed in (4.24). Recall the notation of Section 4.6.

The supercuspidal representations in the L-packet for φ are of the form $\pi_{(jS,j\theta')}$ as described in Section 2.4, where j ranges over conjugacy classes of admissible embeddings $j : S \hookrightarrow G$. Since $j\theta'$ and $\theta \circ j^{-1}$ differ by a tamely ramified character, the formal degrees of $\pi_{(S,\theta)}$ and $\pi_{(jS,j\theta')}$, as expressed in (3.39), agree. So on the automorphic side, we can assume for the purpose of computing the formal degree that the relevant pair is (S, θ), even though it is actually $(jS, j\theta')$.

To compute the dimension of the lattice $M := X^*(S^{\mathrm{a}})^{I_k}$ from Section 4.6, we prove an analogue for tori of the Néron–Ogg–Shafarevich criterion for abelian varieties.

Lemma 5.1. *Let k be a Henselian, discretely-valued field with residue field κ and let T be a tame k-torus. Then there is a canonical identification $X^*(T)^{I_k} = X^*(T_{0:0+})$.*

Here $T_{0:0+}$ denotes the maximal reductive quotient of T_0, that is, the reductive κ-group whose $\bar{\kappa}$ points are $T(k^{\mathrm{nr}})_{0:0+}$ where k^{nr} is the maximal unramified extension of k.

Proof. Since $X^*(T)^{I_k} = X^*(T)_{k^{\mathrm{nr}}}$ we can use étale descent for the Moy–Prasad filtration [60, Section 9.1] to reduce the proof to the case where κ is separably closed. Let $T^{\mathrm{s}} \subseteq T$ be the maximal split subtorus, so that $X^*(T)^{I_k} = X^*(T^{\mathrm{s}}) = X^*(T^{\mathrm{s}}_{0:0+})$ since now $I_k = \Gamma_k$.

It suffices to prove that the canonical inclusion $T^{\mathrm{s}}_{0:0+} \hookrightarrow T_{0:0+}$ is an isomorphism. The proof rests on two facts from SGA 3. Since S_0 is smooth and affine, the moduli space of its maximal tori is represented by a smooth $\mathcal{O}$-scheme [14, Exposé XII, Corollaire 1.10]. By Hensel's Lemma [14, Exposé XI, Corollaire 1.11], every κ-point of this moduli space lifts to an $\mathcal{O}$-point. ∎

At this point, we know that the $\exp_q$ factors in (3.39) and (4.24) are equal.

Lemma 5.2 ([36, Lemma 5.17]). $[S^{\mathrm{a}}(k) : S^{\mathrm{a}}(k)_{0+}] = |X_*(S^{\mathrm{a}})^{\mathrm{Frob}}_{I_k}| \cdot |(\overline{\kappa}^{\times} \otimes M^{\vee})^{\mathrm{Frob}}|$.

Let us now compare the remaining factors outside of $\exp_q$. On the automorphic side we have $[S^{\mathrm{a}}(k) : S^{\mathrm{a}}(k)_{0+}]^{-1}$; on the Galois side we have

$$\frac{|M_{\mathrm{Frob}}|}{|X_*(S^{\mathrm{a}})_{\Gamma_k}| \cdot |(\overline{\kappa}^{\times} \otimes M^{\vee})^{\mathrm{Frob}}|}.$$

The ratio of one to the other is

$$\frac{|X_*(S^{\mathrm{a}})^{I_k}_{\mathrm{Frob}}| \cdot |X_*(S^{\mathrm{a}})^{\mathrm{Frob}}_{I_k}|}{|X_*(S^{\mathrm{a}})_{\Gamma_k}|},$$

using the fact that $M = X_*(S^{\mathrm{a}})^{I_k}$. This ratio equals 1 [36, Lemma 5.18].

Acknowledgments. I would like to thank my advisor Tasho Kaletha for proposing the problem of calculating the formal degree, sharing his wealth of knowledge on the Langlands program, and encouraging me when I felt hopelessly stuck. I am also grateful to Atsushi Ichino for notifying me of other cases where the formal degree conjecture has been proved; to Stephen DeBacker for explaining how to compute the apartment of an elliptic maximal torus; to Karol Koziol for discussing the problem and encouraging me to work toward the proof of Lemma 4.21; to Andy O'Desky for catching several typos throughout the paper; to Maarten Solleveld for catching a typo in equation 3.33 and clarifying several inaccurate references; and to the referee for suggesting improvements.

Funding. This work was supported by the National Science Foundation RTG grant DMS 1840234.

References

[1] Adler, J. D.: Refined anisotropic K-types and supercuspidal representations. Pacific J. Math. **185**, 1–32 (1998) Zbl 0924.22015 MR 1653184

[2] Adler, J. D., Spice, L.: Good product expansions for tame elements of p-adic groups. Int. Math. Res. Papers **2008**, art. rpn003, 95 pp. Zbl 1153.22010 MR 2431235

[3] Adler, J. D., Spice, L.: Supercuspidal characters of reductive p-adic groups. Amer. J. Math. **131**, 1137–1210 (2009) Zbl 1173.22012 MR 2543925

[4] Aubert, A.-M., Baum, P., Plymen, R., Solleveld, M.: Conjectures about p-adic groups and their noncommutative geometry. In: Around Langlands correspondences, Contemporary Mathematics 691, American Mathematical Society, Providence, RI, 15–51 (2017) Zbl 1468.22020 MR 3666049

[5] Beuzart-Plessis, R.: Plancherel formula for $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(E)$ and applications to the Ichino–Ikeda and formal degree conjectures for unitary groups. Invent. Math. **225**, 159–297 (2021) Zbl 1482.22018 MR 4270666

[6] Borel, A.: Automorphic L-functions. In: Automorphic forms, representations, and L-functions (Corvallis, OR, 1977), Part 2, Proceedings of Symposia in Pure Mathematics 33, American Mathematical Society, Providence, RI, 27–61 (1979) Zbl 0412.10017 MR 546608

[7] Borel, A., Casselman, W. (eds.): Automorphic forms, representations, and L-functions (Corvallis, OR, 1977), Proceedings of Symposia in Pure Mathematics 33, Part 2, American Mathematical Society, Providence, RI (1979) MR 546606

[8] Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. Inst. Hautes Études Sci. Publ. Math. **41**, 5–251 (1972) Zbl 0254.14017 MR 327923

[9] Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. Inst. Hautes Études Sci. Publ. Math. **60**, 197–376 (1984) Zbl 0597.14041 MR 756316

[10] Carter, R. W.: Finite groups of Lie type. Pure and Applied Mathematics (New York), Wiley, New York (1985) Zbl 0567.20023 MR 794307

[11] DeBacker, S., Reeder, M.: Depth-zero supercuspidal L-packets and their stability. Ann. of Math. (2) **169**, 795–901 (2009) Zbl 1193.11111 MR 2480618

[12] DeBacker, S., Spice, L.: Stability of character sums for positive-depth, supercuspidal representations. J. Reine Angew. Math. **742**, 47–78 (2018) Zbl 1397.22015 MR 3849622

[13] Deligne, P., Lusztig, G.: Representations of reductive groups over finite fields. Ann. of Math. (2) **103**, 103–161 (1976) Zbl 0336.20029 MR 393266

[14] Demazure, M., Grothendieck, A.: Séminaire de géométrie algébrique du Bois Marie – 1962-64 – Schémas en groupes – (SGA 3). Revised and annotated edition of the 1970 French original, https://webusers.imj-prg.fr/~patrick.polo/SGA3/ (2011)

[15] Feng, Y., Opdam, E., Solleveld, M.: Supercuspidal unipotent representations: L-packets and formal degrees. J. École Polytech. Math. **7**, 1133–1193 (2020) Zbl 07628517 MR 4167790

[16] Feng, Y., Opdam, E., Solleveld, M.: On formal degrees of unipotent representations. J. Inst. Math. Jussieu **21**, 1947–1999 (2022) Zbl 07628517 MR 4515286

[17] Fintzen, J.: On the construction of tame supercuspidal representations. Compos. Math. **157**, 2733–2746 (2021) Zbl 1495.22009 MR 4357723

[18] Fintzen, J.: Types for tame p-adic groups. Ann. of Math. (2) **193**, 303–346 (2021) Zbl 1495.22009 MR 4199732

[19] Fintzen, J., Kaletha, T., Spice, L.: A twisted Yu construction, Harish-Chandra characters, and endoscopy. Duke Math. J. **172**, 2241–2301 (2023) MR 4654051

[20] Gan, W. T., Ichino, A.: Formal degrees and local theta correspondence. Invent. Math. **195**, 509–672 (2014) Zbl 1297.22017 MR 3166215

[21] Gérardin, P.: Weil representations associated to finite fields. J. Algebra **46**, 54–101 (1977) Zbl 0359.20008 MR 460477

[22] Godement, R., Jacquet, H.: Zeta functions of simple algebras. Lecture Notes in Mathematics 260, Springer, Berlin (1972) Zbl 0244.12011 MR 342495

[23] Gross, B. H.: On the motive of a reductive group. Invent. Math. **130**, 287–313 (1997) Zbl 0904.11014 MR 1474159

[24] Gross, B. H.: On the motive of G and the principal homomorphism $\mathrm{SL}_2 \to \hat{G}$. Asian J. Math. **1**, 208–213 (1997) Zbl 0942.20031 MR 1480995

[25] Gross, B. H., Gan, W. T.: Haar measure and the Artin conductor. Trans. Amer. Math. Soc. **351**, 1691–1704 (1999) Zbl 0991.20033 MR 1458303

[26] Gross, B. H., Reeder, M.: Arithmetic invariants of discrete Langlands parameters. Duke Math. J. **154**, 431–508 (2010) Zbl 1207.11111 MR 2730575

[27] Hakim, J., Murnaghan, F.: Distinguished tame supercuspidal representations. Int. Math. Res. Papers **2008**, art. rpn005, 166 pp. Zbl 1160.22008 MR 2431732

[28] Harish-Chandra: Harmonic analysis on reductive p-adic groups. Lecture Notes in Mathematics 162, Springer, Berlin (1970) Zbl 0202.41101 MR 414797

[29] Harish-Chandra: Harmonic analysis on real reductive groups. I. The theory of the constant term. J. Funct. Anal. **19**, 104–204 (1975) Zbl 0315.43002 MR 399356

[30] Harris, M., Taylor, R.: The geometry and cohomology of some simple Shimura varieties. Annals of Mathematics Studies 151, Princeton University Press, Princeton, NJ (2001) Zbl 1036.11027 MR 1876802

[31] Henniart, G.: Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. Invent. Math. **139**, 439–455 (2000) Zbl 1048.11092 MR 1738446

[32] Hiraga, K., Ichino, A., Ikeda, T.: Formal degrees and adjoint γ-factors. J. Amer. Math. Soc. **21**, 283–304 (2008) Zbl 1131.22014 MR 2350057

[33] Hiraga, K., Ichino, A., Ikeda, T.: Correction to: "Formal degrees and adjoint γ-factors". J. Amer. Math. Soc. **21**, 1211–1213 (2008) Zbl 1131.22014 MR 2425185

[34] Howe, R. E.: Tamely ramified supercuspidal representations of Gl_n. Pacific J. Math. **73**, 437–460 (1977) Zbl 0404.22019 MR 492087

[35] Ichino, A., Lapid, E., Mao, Z.: On the formal degrees of square-integrable representations of odd special orthogonal and metaplectic groups. Duke Math. J. **166**, 1301–1348 (2017) Zbl 1378.11061 MR 3649356

[36] Kaletha, T.: Epipelagic L-packets and rectifying characters. Invent. Math. **202**, 1–89 (2015) Zbl 1428.11201 MR 3402796

[37] Kaletha, T.: Regular supercuspidal representations. J. Amer. Math. Soc. **32**, 1071–1170 (2019) Zbl 1473.22012 MR 4013740

[38] Kaletha, T.: On L-embeddings and double covers of tori over local fields. arXiv:1907.05173 (2019)

[39] Kaletha, T.: Supercuspidal L-packets. arXiv:1912.03274 (2019)

[40] Kim, J.-L.: Supercuspidal representations: an exhaustion theorem. J. Amer. Math. Soc. **20**, 273–320 (2007) Zbl 1111.22015 MR 2276772

[41] Kottwitz, R. E.: Tamagawa numbers. Ann. of Math. (2) **127**, 629–646 (1988) Zbl 0678.22012 MR 942522

[42] Langlands, R. P.: Les débuts d'une formule des traces stable. Publications Mathématiques de l'Université Paris VII 13, Paris (1983) Zbl 0532.22017 MR 697567

[43] Langlands, R. P., Shelstad, D.: On the definition of transfer factors. Math. Ann. **278**, 219–271 (1987) Zbl 0644.22005 MR 909227

[44] Morris, L.: Tamely ramified intertwining algebras. Invent. Math. **114**, 1–54 (1993) Zbl 0854.22022 MR 1235019

[45] Moy, A., Prasad, G.: Unrefined minimal K-types for p-adic groups. Invent. Math. **116**, 393–408 (1994) Zbl 0804.22008 MR 1253198

[46] Moy, A., Prasad, G.: Jacquet functors and unrefined minimal K-types. Comment. Math. Helv. **71**, 98–121 (1996) Zbl 0860.22006 MR 1371680

[47] Prasad, G., Raghunathan, M. S.: Topological central extensions of semisimple groups over local fields. Ann. of Math. (2) **119**, 143–201 (1984) Zbl 0552.20025 MR 736564

[48] Prasad, G., Yu, J.-K.: On finite group actions on reductive groups and buildings. Invent. Math. **147**, 545–560 (2002) Zbl 1020.22003 MR 1893005

[49] Reeder, M.: Formal degrees and L-packets of unipotent discrete series representations of exceptional p-adic groups. J. Reine Angew. Math. **520**, 37–93 (2000) Zbl 0947.20026 MR 1748271

[50] Reeder, M., Yu, J.-K.: Epipelagic representations and invariant theory. J. Amer. Math. Soc. **27**, 437–477 (2014) MR 3164986

[51] Renard, D.: Représentations des groupes réductifs p-adiques. Cours Spécialisés 17, Société Mathématique de France, Paris (2010) Zbl 1186.22020 MR 2567785

[52] Schneider, P., Stuhler, U.: Representation theory and sheaves on the Bruhat–Tits building. Inst. Hautes Études Sci. Publ. Math. **85**, 97–191 (1997) Zbl 0892.22012 MR 1471867

[53] Serre, J.-P.: Local fields. Graduate Texts in Mathematics 67, Springer, New York (1979) Zbl 0423.12016 MR 554237

[54] Silberger, A. J., Zink, E.-W.: The formal degree of discrete series representations of central simple algebras over p-adic fields. Preprint 154, Max Planck Institute for Mathematics (1996)

[55] Tate, J.: Number theoretic background. In: Automorphic forms, representations, and L-functions (Corvallis, OR, 1977), Part 2, Proceedings of Symposia in Pure Mathematics 33, American Mathematical Society, Providence, RI, 3–26 (1979) Zbl 0422.12007 MR 546607

[56] Tits, J.: Normalisateurs de tores. I. Groupes de Coxeter étendus. J. Algebra **4**, 96–116 (1966) Zbl 0145.24703 MR 206117

[57] Vogan, D. A., Jr.: The local Langlands conjecture. In: Representation theory of groups and algebras, Contemporary Mathematics 145, American Mathematical Society, Providence, RI, 305–379 (1993) MR 1216197

[58] Yu, J.-K.: Construction of tame supercuspidal representations. J. Amer. Math. Soc. **14**, 579–622 (2001) Zbl 0971.22012 MR 1824988

[59] Yu, J.-K.: On the local Langlands correspondence for tori. In: Ottawa lectures on admissible representations of reductive p-adic groups, Fields Institute Monographs 26, American Mathematical Society, Providence, RI, 177–183 (2009) Zbl 1187.11045 MR 2508725

[60] Yu, J.-K.: Smooth models associated to concave functions in Bruhat–Tits theory. In: Autour des schémas en groupes. Vol. III, Panoramas et Synthèses 47, Société Mathématique de France, Paris, 227–258 (2015) Zbl 1356.20018 MR 3525846

[61] Zink, E.-W.: Comparison of GL_N and division algebra representations II. Preprint 49, Max Planck Institute for Mathematics (1993)