

# Asymptotic analysis and optimization of an elastic body surrounded by thin layers

Mustapha El Jarroudi, Jamal El Amrani, Mhamed El Merzguioui,  
Mustapha Er-Riani, and Adel Settati

**Abstract.** We consider an elastic body surrounded by thin elastic layers along a part of its boundary. We study the asymptotic behavior of the structure as the maximum thickness of the layers tends to zero. We derive an effective boundary integral energy involving a matrix of Borel measures not charging polar sets and having the same support contained in the boundary. We characterize this matrix for three special cases: periodic layers, layers which are determined by a given nonnegative function  $h$ , and layers with abrupt changes along self similar fractals. We then consider an optimal control problem, which consists in determining the shape of the best material distribution around the elastic body, under the maximal work of external loads, and characterize the optimal zones on its boundary where possible elastic layers could take place.

## 1. Introduction

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^3$  with Lipschitz continuous boundary  $\partial\Omega = \Gamma_1 \cup \Gamma_2$ , such that  $\Gamma_1 \cap \Gamma_2 = \emptyset$  and  $|\Gamma_1|, |\Gamma_2| > 0$ , where  $|\Gamma_i|$ ;  $i = 1, 2$ , denotes the Lebesgue measure of  $\Gamma_i$ . Let  $\Sigma_\varepsilon$  be an arbitrary layer of maximum thickness  $\varepsilon > 0$  extending  $\Omega$  near  $\Gamma_1$  (see Figure 1). Without loss of generality, we may suppose that, for small parameter  $\varepsilon \in (0, 1)$ ,

$$\Sigma_\varepsilon = \{s + tn(s); s \in \Gamma_1, 0 < t < \varepsilon h_\varepsilon(s)\},$$

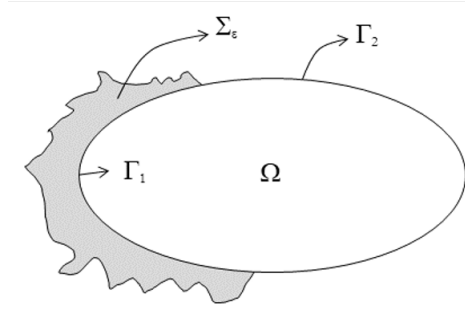
where  $n(s)$  is the outward unit normal on  $s \in \Gamma_1$  and  $h_\varepsilon$  is a positive locally Lipschitz continuous function satisfying

$$\sup_\varepsilon \|h_\varepsilon\|_{L^\infty(\Gamma_1)} \leq 1.$$

---

*Mathematics Subject Classification 2020:* 35B40 (primary); 49J45, 35Q93 (secondary).

*Keywords:* elastic material, thin elastic layers, asymptotic analysis, boundary integral energy, optimal control problem.



**Figure 1.** A layer  $\Sigma_\varepsilon$  extending the set  $\Omega$  along the part  $\Gamma_1$  of its boundary.

We set

$$\begin{aligned}\Omega_\varepsilon &= \Omega \cup \Gamma_1 \cup \Sigma_\varepsilon, \\ \Gamma_{1,\varepsilon} &= \partial\Sigma_\varepsilon \setminus \partial\Omega.\end{aligned}$$

We suppose that  $\Omega$  is the reference configuration of a linear elastic material. This means that the deformation tensor  $e(u) = (e_{ij}(u))_{i,j=1,2,3}$ , with

$$e_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

for some displacement  $u$ , is linked to the stress tensor  $\sigma(u) = (\sigma_{ij}(u))_{i,j=1,2,3}$  by Hooke's law,

$$\sigma_{ij}(u) = a_{ijkl} e_{kl}(u); \quad i, j = 1, 2, 3, \quad (1.1)$$

where  $a_{ijkl}$ ;  $i, j, k, l = 1, 2, 3$ , are material coefficients and where the summation convention with respect to repeated indices has been used and will be used in the sequel. We suppose that

$$a_{ijkl}(x) = a_{jikl}(x) = a_{ljki}(x), \quad \forall i, j, k, l = 1, 2, 3, \forall x \in \Omega, \quad (1.2a)$$

$$c_1 \xi_{ij} \xi_{ij} \leq a_{ijkl}(x) \xi_{ij} \xi_{ij} \leq c_2 \xi_{ij} \xi_{ij}, \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^{3 \times 3}, \quad (1.2b)$$

where  $c_1$  and  $c_2$  are positive constants. We suppose that  $\Sigma_\varepsilon$  is the reference configuration of a linear elastic material with material coefficients  $\varepsilon a_{ijkl}$ ;  $i, j, k, l = 1, 2, 3$ . We suppose that a perfect adhesion occurs between  $\Omega$  and  $\Sigma_\varepsilon$  along their common interface  $\Gamma_1$ . We suppose that the material in  $\Omega_\varepsilon$  is submitted to volumic forces with density  $f \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ . We define the sequence of functionals  $(\mathcal{F}_\varepsilon)_\varepsilon$  on  $L^2(\mathbb{R}^3, \mathbb{R}^3)$  by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int_\Omega \sigma_{ij}(u) e_{ij}(u) dx + \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(u) e_{ij}(u) dx & \text{if } u \in H_0^1(\Omega_\varepsilon, \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.3)$$

The equilibrium state of the elastic material in  $\Omega_\varepsilon$  is described by the minimization problem

$$\min_{u \in L^2(\Omega_\varepsilon, \mathbb{R}^3)} \left\{ \mathcal{F}_\varepsilon(u) - 2 \int_\Omega f \cdot u dx \right\}. \quad (1.4)$$

Using  $\Gamma$ -convergence methods (see, for instance, [16] and [18]), we prove that the effective potential energy of the material turns out to be of the form

$$\mathcal{F}_0(u) = \begin{cases} \int_\Omega \sigma_{ij}(u) e_{ij}(u) dx + \int_{\Gamma_1} u_i u_j d\mu_{ij} & \text{if } u \in H_{\mu, \Gamma_2}(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \quad (1.5)$$

where  $\mu = (\mu_{ij})_{i,j=1,2,3}$  is a symmetric matrix of Borel measures  $\mu_{ij}$  not charging polar sets (sets of capacity zero), having the same support contained in  $\Gamma_1$ , and satisfying  $\mu_{ij}(B)\zeta_i\zeta_j \geq 0$ , for every Borel set  $B \subset \mathbb{R}^3$  and every  $\zeta \in \mathbb{R}^3$ , and where

$$H_{\mu, \Gamma_2}(\Omega, \mathbb{R}^3) = H_{\Gamma_2}^1(\Omega, \mathbb{R}^3) \cap L_\mu^2(\Gamma_1, \mathbb{R}^3), \quad (1.6)$$

with

$$H_{\Gamma_2}^1(\Omega_\varepsilon, \mathbb{R}^3) = \{v \in H^1(\Omega_\varepsilon, \mathbb{R}^3); v = 0 \text{ on } \Gamma_2\},$$

and

$$L_\mu^2(\Gamma_1, \mathbb{R}^3) = \left\{ v : \Gamma_1 \rightarrow \mathbb{R}^3; \int_{\Gamma_1} v_i v_j d\mu_{ij} < +\infty \right\}.$$

The solution  $u^0$  of the limit problem, stated in Corollary 15, satisfies the following Robin type boundary condition:

$$\sigma_{ij}(u^0)n_j + \mu_{ij}u_i^0 = 0 \quad \text{on } \Gamma_1,$$

where  $n$  is the outward unit normal on  $s \in \Gamma_1$ . We then consider some special cases. We first consider the case where the thickness of  $\Sigma_\varepsilon$  varies periodically along  $\Gamma_1$ . The problem becomes invariant by translation and the measure  $\mu_{ij}$ ;  $i, j = 1, 2, 3$ , is the Haar measure on  $\Gamma_1$  with  $\mu_{ij} = K_{ij} ds$ , where  $ds$  is the surface measure on  $\Gamma_1$  given by the Riemannian metric and  $K_{ij}$ ;  $i, j = 1, 2, 3$ , are constants in  $\mathbb{R}$  satisfying  $K_{ij}\zeta_i\zeta_j \geq 0$ ,  $\forall \zeta \in \mathbb{R}^3$ . We identify the constants  $K_{ij}$ ;  $i, j = 1, 2, 3$ , by constructing appropriate local problems. We secondly suppose that  $\Sigma_\varepsilon$  has thickness of the form  $\varepsilon h(s)$ . We prove in this case that  $\mu_{ij} = \kappa_i(s) \frac{ds}{h(s)} \delta_{ij}$ , where  $\delta_{ij}$  denotes Kronecker's symbol and  $\kappa_i(s)$ ;  $i = 1, 2, 3$ , are material coefficients. We then suppose that  $\Gamma_1$  is contained in the plane  $\{x_3 = 0\}$  and consider a thin layer  $\Sigma_\varepsilon$  with abrupt changes along a self similar fractal  $\Lambda$  with similarity dimension  $d$ . We prove that

$$\mu_{ij} = \left( \kappa_i(x_1, x_2) dx_1 dx_2 |_{\Gamma_1} + \frac{2\pi(c-1/2)\kappa_i(s)}{\mathcal{H}^d(\Lambda)} d\mathcal{H}^d(s) |_\Lambda \right) \delta_{ij},$$

where  $c$  is a positive constant given in Theorem 19,  $d\mathcal{H}^d$  is the  $d$ -dimensional Hausdorff measure, and  $\kappa_i(s)$ ;  $i = 1, 2, 3$ , are material coefficients.

The asymptotic behavior of the scalar version of  $\Omega$  surrounded by arbitrary layers  $\Sigma_\varepsilon$  of maximum thickness  $\varepsilon$  was studied in [11]. A general integral on the boundary  $\partial\Omega$  written as  $\int_{\partial\Omega} u^2 d\mu$  where  $\mu$  is a nonnegative Borel measure on  $\mathbb{R}^3$  not charging polar sets (but possibly  $+\infty$  on large subsets of  $\mathbb{R}^3$ ) was obtained at the limit. The characterization of the measure  $\mu$  was given in terms of suitable asymptotic capacities associated with  $\Omega_\varepsilon \setminus \Omega$ . The asymptotic behavior for an incompressible viscous flow in  $\Omega$  surrounded by arbitrary thin layers  $\Sigma_\varepsilon$  has been addressed in [19]. A general Navier wall law was obtained, with the proportionality coefficient being a symmetric matrix of Borel measures, having their supports contained in the solid boundary of  $\Omega$ . Several papers, among which [1, 2, 5–7, 12, 14], and [26], have studied the asymptotic behavior of elliptic operators in domains surrounded by thin layers of periodically varying thickness  $\varepsilon h(s)$  or with general smooth thickness  $h^\varepsilon \leq \varepsilon$ .

An important field to which this work is closely related is the asymptotic behavior of a biological body surrounded by thin layers of soft growing tissues resulting from the proliferation of tumor cells. Constitutive models combining the stress-strain relation of linear elasticity with a growth term of avascular tumors have been developed in several papers (see, for instance, [3, 4, 24, 29]). This analysis provides an asymptotic description of stresses in soft tissues growing around a biological body.

This problem has also some implications for modeling the behavior of elastic bodies reinforced with flexible or soft thin elastic layers such as rubber and textile. Some of these materials are indeed known for their nonlinearity in the stress-strain relationship. However, as the reinforced body is modeled as linear elastic with small strains, linear elasticity is assumed here for the material within the layers. Note that materials reinforced with flexible materials as textile are used for the construction of durable and more sustainable elastic structures (see, for instance, [27] and [28]). For two-dimensional plastic layers under longitudinal shear obeying a hardening stress-strain law with a functional energy given by

$$\varepsilon^{p-1} \int_{\Sigma_\varepsilon} |\nabla w(x, y)|^p dx dy,$$

where  $p$  is the power hardening parameter and  $w$  is the only non-vanishing component of displacement (assumed to be the  $z$ -component), we can prove, using the present work and the integral representation theorem of [15], that the effective energy is given by  $\int_{\Gamma_1} |w(x, y)|^p d\mu$  where  $\mu$  is a nonnegative Borel measure on  $\mathbb{R}^3$  not charging polar sets.

In Section 6, we consider an optimal control problem which consists in determining the shape of the best material distribution around  $\Omega$  under the maximal work of external loads. We prove that, for a given quantity  $\eta > 0$  of material, there exists an

optimal diagonal matrix  $\mathbf{h} = \text{Diag}(h_i)_{i=1,2,3}$  of  $\Gamma_1$ -measurable functions  $h_i : \Gamma_1 \rightarrow [0, +\infty)$ , such that

$$h_i(u^\eta) = \eta \frac{|u_i^\eta|}{\int_{\Gamma_1} |u_i^\eta| ds},$$

where  $u^\eta = (u_1^\eta, u_2^\eta, u_3^\eta)$  is the solution of problem (6.4). We then study the best way to reinforce an elastic body by a flexible elastic layer as  $\eta$  tends to zero. For a biological body, this last study allows to characterize the zones on its boundary where possible soft tissues will grow.

We recall that the scalar version of this problem was investigated as part of the shape optimization of optimal thermal insulators by several authors (see, for instance, [9, 10, 20], and recently [8] and [22]). The problem of optimizing the distribution of material, surrounding a homogeneous elastic plate, which minimizes the energy has been studied in [13].

## 2. Functional framework

We define the capacity  $\text{Cap}$  of every compact subset  $K \subset \mathbb{R}^3$  as

$$\text{Cap}(K) := \inf \left\{ \int_{\mathbb{R}^N} |\nabla \varphi|^2 dx + \int_{\mathbb{R}^N} |\varphi|^2 dx; \varphi \in C_c^\infty(\mathbb{R}^3), \varphi \geq 1 \text{ on } K \right\}.$$

For every open subset  $U \subset \mathbb{R}^3$ , we set

$$\text{Cap}(U) := \sup \{ \text{Cap}(K); K \subset U, K \text{ compact} \}.$$

For every Lebesgue measurable subset  $B \subset \mathbb{R}^3$ , we define

$$\text{Cap}(B) := \inf \{ \text{Cap}(U); B \subset U, U \text{ open} \}.$$

Let  $\mathcal{B}(\mathbb{R}^3)$  be the  $\sigma$ -field of Lebesgue measurable subsets of  $\mathbb{R}^3$ . A property is said to be true quasi-everywhere (q.e.) on  $B \in \mathcal{B}(\mathbb{R}^3)$  if it is true except on a subset of  $B$  of capacity  $\text{Cap}$  equal to 0. A function  $u : B \rightarrow \overline{\mathbb{R}}; B \in \mathcal{B}(\mathbb{R}^3)$ , is quasi-continuous on  $B$  if, for every  $\varepsilon > 0$ , there exists an open subset  $U_\varepsilon \subset B$  with  $\text{Cap}(U_\varepsilon) < \varepsilon$ , such that the restriction of  $u$  to  $B \setminus U_\varepsilon$  is continuous. Every function  $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$  has a quasi-continuous representant  $\tilde{u}$ , which is unique quasi-everywhere in  $\Omega$ , (see, for instance, [30, Theorem 3.1.4]);  $\tilde{u}$  is given by

$$\tilde{u}(x) = \lim_{r \rightarrow 0^+} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy \quad \text{for q.e. } x \in \mathbb{R}^3,$$

where  $B(x, r)$  is the ball centered at  $x$  and of radius  $r > 0$ . We now define some notions concerning families of subsets of  $\mathbb{R}^3$  (see, for instance, [16, Chapter 14]) and a class of functionals of  $\mathbb{R}^3$  (see, for instance, [17]).

- Definition 1.** (1) A subset  $\mathcal{D} \subset \mathcal{B}(\mathbb{R}^3)$  is a dense family in  $\mathcal{B}(\mathbb{R}^3)$  if, for every  $A, B \in \mathcal{B}(\mathbb{R}^3)$  with  $\bar{A} \subset \overset{\circ}{B}$ , there exists  $D \in \mathcal{D}$  such that:  $\bar{A} \subset \overset{\circ}{D} \subset \bar{D} \subset \overset{\circ}{B}$ , where  $\overset{\circ}{A}$  (resp.  $\bar{A}$ ) denotes the interior (resp. the closure) of  $A$ .
- (2) A subset  $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$  is a rich family if, for every family  $(A_t)_{t \in ]0, 1[} \subset \mathcal{B}(\mathbb{R}^3)$  such that  $\bar{A}_s \subset \overset{\circ}{A}_t$ , for every  $s < t$ , the set  $\{t \in ]0, 1[; A_t \notin \mathcal{R}\}$  is at most countable.
- (3) Let  $\mathcal{O}(\mathbb{R}^3)$  be the set of all open subsets of  $\Omega$ . We consider the class  $\mathbb{F}$  of functionals  $F$  from  $H^1(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{O}(\mathbb{R}^3)$  to  $[0, +\infty]$  satisfying:
- (a) (*lower semi-continuity*): for every open subset  $\omega \in \mathcal{O}(\mathbb{R}^3)$ , the functional  $u \mapsto F(u, \omega)$  is lower semi-continuous with respect to the strong topology of the space  $H^1(\mathbb{R}^3; \mathbb{R}^3)$ ,
- (b) (*measure property*): for every  $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ ,  $\omega \mapsto F(u, \omega)$  is the restriction to  $\mathcal{O}(\mathbb{R}^3)$  of a nonnegative Borel measure still denoted  $F(u, \omega)$ ,
- (c) (*localization*): for every  $\omega \in \mathcal{O}(\mathbb{R}^3)$  and every  $u, v \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ :  $u|_\omega = v|_\omega \Rightarrow F(u, \omega) = F(v, \omega)$ ,
- (d) ( $C^1$ -convexity): for every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ , the functional  $u \mapsto F(u, \omega)$  is convex on  $H^1(\mathbb{R}^3, \mathbb{R}^3)$  and for every  $\varphi \in C^1(\mathbb{R}^3)$  with  $0 \leq \varphi \leq 1$ ,

$$F(\varphi u + (1 - \varphi)v, \omega) \leq F(u, \omega) + F(v, \omega).$$

**Example 2.** We consider the functional  $F_\varepsilon$  defined on  $H^1(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{O}(\mathbb{R}^3)$  by

$$F_\varepsilon(u, \omega) = \begin{cases} 0 & \text{if } \tilde{u} = 0, \text{ q.e. on } \Gamma_{1,\varepsilon} \cap \omega, \\ +\infty & \text{otherwise,} \end{cases} \quad (2.1)$$

then  $F_\varepsilon \in \mathbb{F}$ .

Let us set the following definitions.

**Definition 3.** (1) A Borel measure  $\lambda$  is absolutely continuous with respect to the capacity  $\text{Cap}$  if

$$\forall B \in \mathcal{B}(\mathbb{R}^3) : \text{Cap}(B) = 0 \Rightarrow \lambda(B) = 0,$$

(2)  $\mathcal{M}_0(\mathbb{R}^3)$  is the set of nonnegative Borel measures which are absolutely continuous with respect to the capacity  $\text{Cap}$ .

**Example 4.** For every  $E \subset \Omega$  such that  $\text{Cap}(E) > 0$ , we define the measure  $\infty_E$  by

$$\infty_E(B) = \begin{cases} 0 & \text{if } \text{Cap}(B \cap E) = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

for every  $B \in \mathcal{B}(\mathbb{R}^3)$ . Then  $\infty_E$  belongs to  $\mathcal{M}_0(\Omega)$ .

Note that, for every  $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$  and every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ , the functional  $F_\varepsilon$  defined in (2.1) can be written as

$$F_\varepsilon(u, \omega) = \int_\omega |\tilde{u}|^2 d\infty_{\Gamma_{1,\varepsilon}} = \int_\omega |u|^2 d\infty_{\Gamma_{1,\varepsilon}}. \quad (2.2)$$

We have the following integral representation of functionals of  $\mathbb{F}$ .

**Theorem 5** (See [17, Theorem 7.5]). *For every  $F \in \mathbb{F}$ , there exist a finite measure  $\mu \in \mathcal{M}_0(\mathbb{R}^3)$ , a nonnegative Borel measure  $\nu$ , and a Borel function  $g : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow [0, +\infty]$  with  $\zeta \mapsto g(x, \zeta)$  convex and lower semi-continuous on  $\mathbb{R}^3$ , such that, for every  $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$  and every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ ,*

$$F(u, \omega) = \int_\omega g(x, \tilde{u}(x)) d\mu + \nu(\omega).$$

Moreover, if  $F$  is quadratic then the following corollary (see [17, Corollary 8.4]) holds.

**Corollary 6.** *Let  $F \in \mathbb{F}$ . Assume that  $F(\cdot, \omega)$  is quadratic for every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ . Then, there exist*

- (1) *a finite measure  $\mu \in \mathcal{M}_0(\mathbb{R}^3)$ ,*
- (2) *a symmetric matrix  $(a_{ij})_{i,j=1,2,3}$ , of Borel functions from  $\mathbb{R}^3$  to  $\mathbb{R}$  satisfying  $a_{ij}(x)\zeta_i\zeta_j \geq 0$  for every  $\zeta \in \mathbb{R}^3$  and for q.e.  $x \in \mathbb{R}^3$ ,*
- (3) *for every  $x \in \mathbb{R}^3$ , a linear subspace  $V(x)$  of  $\mathbb{R}^3$ , such that, for every  $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$  and every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ ,*
  - (a) *if  $F(u, \omega) < +\infty$  then  $u(x) \in V(x)$  for q.e.  $x \in \mathbb{R}^3$ ,*
  - (b) *if  $u(x) \in V(x)$ , for q.e.  $x \in \mathbb{R}^3$ , then*

$$F(u, \omega) = \int_\omega a_{ij}(x)u_i(x)u_j(x) d\mu(x).$$

**Remark 7.** Let  $F \in \mathbb{F}$  such that  $F(\cdot, \omega)$  is quadratic for every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ ,  $\mu \in \mathcal{M}_0(\mathbb{R}^3)$  be the associated measure by the above corollary,

$$\Theta = \bigcup_{\omega \in \mathcal{S}(F)} \omega, \quad (2.3)$$

where

$$\mathcal{S}(F) = \{\omega \in \mathcal{O}(\mathbb{R}^N); F(\cdot, \omega) < +\infty \text{ q.e. in } \omega\},$$

and  $V(x)$  be the linear subspace of  $\mathbb{R}^3$  defined by

$$V(x) = \begin{cases} \mathbb{R}^3 & \text{if } x \in \Theta, \\ \{0\} & \text{if } x \in \mathbb{R}^3 \setminus \Theta. \end{cases}$$

We define the  $3 \times 3$  matrix of measures

$$\begin{aligned}\boldsymbol{\mu} &= (\mu_{ij})_{i,j=1,2,3} \\ &= (a_{ij}\mu)_{i,j=1,2,3} + \infty_{\mathbb{R}^3 \setminus \Theta} \text{Id}_3.\end{aligned}$$

Then, for every  $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$  and every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ ,

$$\int_{\omega} u_i u_j d\mu_{ij} = \begin{cases} \int_{\omega} a_{ij} u_i u_j d\mu & \text{if } u(x) \in V(x) \text{ for q.e. } x \in \omega, \\ +\infty & \text{otherwise,} \end{cases}$$

and the functional  $F$  can be written as

$$F(u, \omega) = \int_{\omega} u_i u_j d\mu_{ij} = \langle \boldsymbol{\mu} u, u \rangle.$$

### 3. A priori estimates

We define the sequence of functionals  $(\Phi_{\varepsilon})_{\varepsilon}$  on  $L^2(\mathbb{R}^3, \mathbb{R}^3)$  by

$$\Phi_{\varepsilon}(u) = \begin{cases} \int_{\Omega} \sigma_{ij}(u) e_{ij}(u) dx + \varepsilon \int_{\Sigma_{\varepsilon}} \sigma_{ij}(u) e_{ij}(u) dx & \text{if } u \in H_{\Gamma_2}^1(\Omega_{\varepsilon}, \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.1)$$

We have the following results.

**Proposition 8.** *Let  $u^{\varepsilon} \in H_{\Gamma_2}^1(\Omega_{\varepsilon}, \mathbb{R}^3)$  such that  $\sup_{\varepsilon} \Phi_{\varepsilon}(u^{\varepsilon}) < +\infty$ . Then*

- (1)  $\sup_{\varepsilon} (\int_{\Omega} |\nabla u^{\varepsilon}|^2 dx + \varepsilon \int_{\Sigma_{\varepsilon}} |\nabla u^{\varepsilon}|^2 dx) < +\infty$ ,
- (2)  $\sup_{\varepsilon} \int_{\Omega} |u^{\varepsilon}|^2 dx < +\infty$  and  $\sup_{\varepsilon} \frac{1}{\varepsilon} \int_{\Sigma_{\varepsilon}} |u^{\varepsilon}|^2 dx < +\infty$ .

*Proof.* (1) Let  $s + tn(s) \in \Sigma_{\varepsilon}$ . We have

$$\begin{aligned}(u_i^{\varepsilon}(s + tn(s)) - u_i^{\varepsilon}(s))^2 &= \left| \int_0^t \nabla u_i^{\varepsilon}(s + \zeta n(s)) \cdot n(s) d\zeta \right|^2 \\ &\leq \varepsilon \int_0^{\varepsilon h_{\varepsilon}(s)} |\nabla u_i^{\varepsilon}(s + \zeta n(s))|^2 d\zeta,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{\Gamma_1} \int_0^{\varepsilon h_{\varepsilon}(s)} |u_i^{\varepsilon}(s + \zeta n(s))|^2 d\zeta ds \\ &\leq C \left( \varepsilon \int_{\Gamma_1} (u_i^{\varepsilon}(s))^2 ds \right. \\ &\quad \left. + \varepsilon^2 \int_{\Gamma_1} \int_0^{\varepsilon h_{\varepsilon}(s)} |\nabla u_i^{\varepsilon}(s + \zeta n(s))|^2 (1 + \zeta \chi(s, t)) d\zeta ds \right), \quad (3.2)\end{aligned}$$



where  $\varkappa$  is the curvature of  $\Gamma_1$  and  $C$  is a positive constant independent of  $\varepsilon$ . As  $u^\varepsilon = 0$  on  $\Gamma_2$ , we have, using the Korn inequality in  $\Omega$  and  $\Omega_\varepsilon$ , respectively, that

$$\begin{aligned} \int_{\Omega} |\nabla u^\varepsilon|^2 dx &\leq C \int_{\Omega} \sigma_{ij}(u^\varepsilon) e_{ij}(u^\varepsilon) dx \\ &\leq C \Phi_\varepsilon(u^\varepsilon), \\ \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx &\leq C \varepsilon \int_{\Omega_\varepsilon} \sigma_{ij}(u^\varepsilon) e_{ij}(u^\varepsilon) dx \\ &\leq C \Phi_\varepsilon(u^\varepsilon), \end{aligned} \quad (3.3)$$

from which we deduce that

$$\sup_{\varepsilon} \left( \int_{\Omega} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \right) < +\infty. \quad (3.4)$$

(2) Using the Poincaré inequality and the trace theorem, we deduce from (3.3) that

$$\begin{aligned} \int_{\Omega} |u^\varepsilon|^2 dx &\leq C \int_{\Omega} |\nabla u^\varepsilon|^2 dx \\ &\leq C \Phi_\varepsilon(u^\varepsilon), \\ \int_{\Gamma_1} |u^\varepsilon(s)|^2 ds &\leq C \int_{\Omega} |\nabla u^\varepsilon|^2 dx \\ &\leq C \Phi_\varepsilon(u^\varepsilon), \end{aligned} \quad (3.5)$$

and, using (3.2)–(3.5), we deduce that

$$\int_{\Sigma_\varepsilon} |u^\varepsilon|^2 dx \leq C \varepsilon \Phi_\varepsilon(u^\varepsilon). \quad (3.6)$$

We obtain from (3.5) and (3.6) that

$$\sup_{\varepsilon} \int_{\Omega} |u^\varepsilon|^2 dx < +\infty \quad \text{and} \quad \sup_{\varepsilon} \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} |u^\varepsilon|^2 dx < +\infty. \quad \blacksquare$$

**Remark 9.** According to [25, pages 354–355], as  $\partial\Omega_\varepsilon$  is locally Lipschitz, it is also uniformly Lipschitz. Then, using [25, Theorem 12.15], we infer that every  $v \in H_{\Gamma_2}^1(\Omega_\varepsilon, \mathbb{R}^3)$  has an extension  $\mathcal{E}v \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  verifying, in particular,  $\mathcal{E}v(x) = v(x)$  for almost every (a.e.)  $x \in \Omega_\varepsilon$ , and

$$\|\mathcal{E}v\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)} \leq (1 + M) \|v\|_{L^2(\Omega_\varepsilon, \mathbb{R}^3)}, \quad (3.7)$$

where  $M$  is a positive integer. We deduce from Proposition 8 and inequality (3.7) that, for every sequence  $(u^\varepsilon)_\varepsilon$ , such that  $u^\varepsilon \in H_{\Gamma_2}^1(\Omega_\varepsilon, \mathbb{R}^3)$  and  $\sup_\varepsilon \Phi_\varepsilon(u^\varepsilon) < +\infty$ , there exists a subsequence, still denoted  $(u^\varepsilon)_\varepsilon$ , such that  $(u^\varepsilon)_\varepsilon$  weakly converges in  $H^1(\Omega, \mathbb{R}^3)$ , as  $\varepsilon$  tends to 0, to some  $u \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$ , and its extension  $(\mathcal{E}u^\varepsilon)_\varepsilon$  strongly converges in  $L^2(\mathbb{R}^3, \mathbb{R}^3)$  to  $\mathcal{E}u$ .

## 4. Convergence

Let  $v^\varepsilon \in H_{\Gamma_2}^1(\Omega_\varepsilon, \mathbb{R}^3)$  and  $v \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$ . We denote in the same way their extensions to  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ . According to Proposition 8 and Remark 9, we introduce the following topology  $\tau$ .

**Definition 10.** A sequence  $(u^\varepsilon)_\varepsilon$ ;  $u^\varepsilon \in H_{\Gamma_2}^1(\Omega_\varepsilon, \mathbb{R}^3)$ ,  $\tau$ -converges to  $u$ , as  $\varepsilon$  tends to 0, if

- (1)  $u^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u$   $H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$ -weak,
- (2)  $u^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u$   $L^2(\mathbb{R}^3, \mathbb{R}^3)$ -strong.

We have the following result.

**Lemma 11.** Let  $(\Phi_\varepsilon)_\varepsilon$  be the sequence of functionals defined in (3.1). Then  $(\Phi_\varepsilon)_\varepsilon$   $\Gamma$ -converges, with respect to the topology  $\tau$ , to the functional  $\Phi$  defined on  $L^2(\mathbb{R}^3, \mathbb{R}^3)$  by

$$\Phi(u) = \begin{cases} \int_{\Omega} \sigma_{ij}(u) e_{ij}(u) dx & \text{if } u \in W_{\Gamma_2}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.1)$$

where

$$W_{\Gamma_2} = H_{\Gamma_2}^1(\Omega, \mathbb{R}^3) \cap L^2(\mathbb{R}^3, \mathbb{R}^3).$$

*Proof.* Let  $u \in W_{\Gamma_2}$ . We consider the set  $\Omega_{0,\varepsilon} = \overline{\Omega} \cup \Sigma_{0,\varepsilon}$ , where  $\Sigma_{0,\varepsilon}$  is a layer of thickness  $\varepsilon$  surrounding  $\Omega$  defined by

$$\Sigma_{0,\varepsilon} = \{x \in \mathbb{R}^3; 0 < d(x, \partial\Omega) < \varepsilon\},$$

where  $d(x, \partial\Omega)$  is the Euclidean distance from  $x$  to  $\partial\Omega$ . Let  $v^{0,\varepsilon} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$  such that

$$\|u - v^{0,\varepsilon}\|_{L^2(\mathbb{R}^3 \setminus \Omega_{0,\varepsilon}, \mathbb{R}^3)} < \varepsilon.$$

We define the function  $\widetilde{v^{0,\varepsilon}}$  by

$$\widetilde{v^{0,\varepsilon}} = \begin{cases} v^{0,\varepsilon} & \text{in } \mathbb{R}^3 \setminus \Omega_{0,\varepsilon}, \\ 0 & \text{in } \Omega_{0,\varepsilon}. \end{cases}$$

We consider a mollifier  $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^3)$  with support in the ball  $B(0, \varepsilon)$  of radius  $\varepsilon$  centered at the origin such that  $\int_{\mathbb{R}^N} \rho_\varepsilon(x) dx = 1$ . We then define the sequence  $(w^{0,\varepsilon})_\varepsilon$  by  $w^{0,\varepsilon} = \rho_\varepsilon * \widetilde{v^{0,\varepsilon}}$  and the sequence  $(u^{0,\varepsilon})_\varepsilon$  of test-functions by

$$u^{0,\varepsilon} = \begin{cases} w^{0,\varepsilon} & \text{in } \mathbb{R}^3 \setminus \Omega_{0,\varepsilon}, \\ u \frac{(\varepsilon - d(x, \partial\Omega))}{\varepsilon} & \text{in } \Sigma_{0,\varepsilon}, \\ u & \text{in } \overline{\Omega}. \end{cases}$$

We can easily check that  $u^{0,\varepsilon} \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ ,  $(u^{0,\varepsilon})_\varepsilon$   $\tau$ -converges to  $u$  as  $\varepsilon$  tends to 0, and

$$\limsup_{\varepsilon \rightarrow 0} \Phi_\varepsilon(u^{0,\varepsilon}) \leq \int_{\Omega} \sigma_{ij}(u) e_{ij}(u) dx = \Phi(u). \quad (4.2)$$

Let  $(u^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $(u^\varepsilon)_\varepsilon$   $\tau$ -converges to some  $u$  as  $\varepsilon$  tends to 0. Then, since  $\Phi(u^\varepsilon) \leq \Phi_\varepsilon(u^\varepsilon)$ , we infer that

$$\Phi(u) \leq \liminf_{\varepsilon \rightarrow 0} \Phi(u^\varepsilon) \leq \liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(u^\varepsilon). \quad (4.3)$$

We deduce from (4.2) and (4.3) that  $(\Phi_\varepsilon)_\varepsilon$   $\Gamma$ -converges to  $\Phi$  with respect to the topology  $\tau$ . ■

We introduce the functional  $G_\varepsilon$  defined on  $L^2(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{B}(\mathbb{R}^3)$  by

$$G_\varepsilon(u, B) = \begin{cases} \Phi_\varepsilon(u) + F_\varepsilon(u, B) & \text{if } u \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \quad (4.4)$$

where  $F_\varepsilon$  is defined in (2.1).

Our main result in this section reads as follows.

**Theorem 12.** *There exist a rich family  $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$  and a symmetric matrix  $\mu = (\mu_{ij})_{i,j=1,2,3}$  of Borel measures  $\mu_{ij}$ , having the same support contained in  $\Gamma_1$ , which are absolutely continuous with respect to the capacity  $\text{Cap}$ , and satisfying*

$$\mu_{ij}(B) \zeta_i \zeta_j \geq 0, \quad \forall \zeta \in \mathbb{R}^3, \forall B \in \mathcal{B}(\mathbb{R}^3),$$

such that, for every  $u \in W_{\Gamma_2}$  and every  $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbb{R}^3)$

$$(\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} G_\varepsilon)(u, \omega) = \Phi(u) + \int_{\Gamma_1 \cap \omega} u_i u_j d\mu_{ij},$$

where the  $\Gamma$ -limit is taken with respect to the topology  $\tau$  and  $\Phi$  is the functional defined in (4.1).

*Proof.* The upper and lower  $\Gamma$ -limits, with respect to the topology  $\tau$ , exist and are respectively defined on  $W_{\Gamma_2} \times \mathcal{B}(\mathbb{R}^3)$  by

$$G^S(u, B) = \inf_{\varepsilon \rightarrow 0} \left\{ \limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u^\varepsilon, B); u^\varepsilon \xrightarrow{\tau} u \right\},$$

$$G^I(u, B) = \inf_{\varepsilon \rightarrow 0} \left\{ \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u^\varepsilon, B); u^\varepsilon \xrightarrow{\tau} u \right\}.$$

We see that, for every  $B \in \mathcal{B}(\mathbb{R}^3)$ , we have  $G^s(\cdot, B) \geq \Phi(\cdot)$  and  $G^i(\cdot, B) \geq \Phi(\cdot)$ . Let us introduce the nonnegative functionals  $F^s$  and  $F^i$  defined, for every  $B \in \mathcal{B}(\mathbb{R}^3)$ , by

$$F^a(u, B) = \begin{cases} G^a(u, B) - \Phi(u) & \text{if } u \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \quad (4.5)$$

where  $a = s, i$ . Let  $u \in W_{\Gamma_2}$  and  $u^\varepsilon \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  be such that  $(u^\varepsilon)_\varepsilon$   $\tau$ -converges to  $u$ . Denoting in the same way the extension of  $u$  to the space  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ , we set  $z^\varepsilon = u^\varepsilon - u$ . We can easily check that  $(z^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ ,  $(z^\varepsilon)_\varepsilon$   $\tau$ -converges to 0, and, using (4.5),

$$F^s(u, B) = \inf \left\{ \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, B)); z^\varepsilon \xrightarrow{\tau} 0 \right\}, \quad (4.6a)$$

$$F^i(u, B) = \inf \left\{ \liminf_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, B)); z^\varepsilon \xrightarrow{\tau} 0 \right\}, \quad (4.6b)$$

where, for any  $v \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ ,

$$\Phi_{0,\varepsilon}(v) = \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(v) e_{ij}(v) dx. \quad (4.7)$$

The functionals  $F^s$  and  $F^i$  satisfy the following properties.

**Lemma 13.** (1) Let  $u \in W_{\Gamma_2}$  and  $A, B \in \mathcal{O}(\mathbb{R}^3)$ . Then

$$F^s(u, A \cup B) \leq F^s(u, A) + F^s(u, B).$$

(2) Let  $u \in W_{\Gamma_2}$  and  $A, B \in \mathcal{O}(\mathbb{R}^3)$  such that  $A \cap B = \emptyset$ . Let  $A', B' \in \mathcal{O}(\mathbb{R}^3)$  such that  $\overline{A'} \subset A$  and  $\overline{B'} \subset B$ . Then

$$F^i(u, A \cup B) \geq F^i(u, A') + F^i(u, B').$$

(3) For every open subset  $\omega \in \mathcal{O}(\mathbb{R}^3)$ ,  $F^s(\cdot, \omega)$  is lower semi-continuous for the strong topology of  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ .

(4) Let  $\omega \in \mathcal{O}(\mathbb{R}^3)$  and  $u, v \in W_{\Gamma_2}$  such that  $u|_\omega = v|_\omega$ . Then  $F^s(u, \omega) = F^s(v, \omega)$ .

(5) For every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ , the functional  $u \mapsto F^s(u, \omega)$  is convex and  $C^1$ -convex on  $W_{\Gamma_2}$ .

*Proof.* (1) Let  $u \in W_{\Gamma_2}$ . Let  $\omega \in \mathcal{O}(\mathbb{R}^3)$  such that  $\overline{\omega} \subset B \setminus \overline{A}$ . Then, owing to (4.6a), there exist two sequences  $(z^{1,\varepsilon})_\varepsilon$  and  $(z^{2,\varepsilon})_\varepsilon$  in  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$   $\tau$ -converging to 0 such that

$$\begin{aligned} F^s(u, A) &= \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^{1,\varepsilon}) + F_\varepsilon(z^{1,\varepsilon} + u, A)), \\ F^s(u, \omega) &= \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^{2,\varepsilon}) + F_\varepsilon(z^{2,\varepsilon} + u, \omega)). \end{aligned} \quad (4.8)$$

Let  $\varphi \in C_c^\infty(\mathbb{R}^3)$  such that  $0 \leq \varphi \leq 1$  in  $\mathbb{R}^3$ ,  $\varphi = 0$  in  $A$  and  $\varphi = 1$  in  $\omega$ . We define the sequence  $(z^\varepsilon)_\varepsilon$  by

$$z^\varepsilon = (1 - \varphi)z^{1,\varepsilon} + \varphi z^{2,\varepsilon}.$$

Then  $z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0$  and, according to (4.6a),

$$F^s(u, A \cup \omega) \leq \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, A \cup \omega)). \quad (4.9)$$

Let us denote  $\mathbb{R}_{\text{sym}}^9$  the set of  $3 \times 3$ -real symmetric matrices. We define the quadratic form  $Q$  by

$$Q(\tau) = a_{ijkl} \tau_{kl} \tau_{ij}, \quad \forall \tau \in \mathbb{R}_{\text{sym}}^9. \quad (4.10)$$

Using (4.10), we have, according to (1.1), that, for every  $\eta \in (0, 1)$ ,

$$\begin{aligned} \Phi_{0,\varepsilon}(z^\varepsilon) &= \varepsilon \int_{\Sigma_\varepsilon} Q(e(z^\varepsilon)) dx \\ &= \varepsilon \int_{\Sigma_\varepsilon} Q((1 - \varphi)e(z^{1,\varepsilon}) + \varphi e(z^{2,\varepsilon}) + (z^{2,\varepsilon} - z^{1,\varepsilon}) \otimes \nabla \varphi) dx \\ &= \varepsilon \int_{\Sigma_\varepsilon} Q\left((1 - \eta) \frac{(1 - \varphi)e(z^{1,\varepsilon}) + \varphi e(z^{2,\varepsilon})}{1 - \eta} + \eta \frac{(z^{2,\varepsilon} - z^{1,\varepsilon}) \otimes \nabla \varphi}{\eta}\right). \end{aligned} \quad (4.11)$$

Using the convexity of  $Q$ , we have, for every  $\eta \in (0, 1)$ ,

$$\begin{aligned} \Phi_{0,\varepsilon}(z^\varepsilon) &\leq \frac{\varepsilon}{1 - \eta} \int_{\Sigma_\varepsilon} Q((1 - \varphi)e(z^{1,\varepsilon}) + \varphi e(z^{2,\varepsilon}) + (z^{2,\varepsilon} - z^{1,\varepsilon}) \otimes \nabla \varphi) dx \\ &\quad + \frac{\varepsilon}{\eta} \int_{\Sigma_\varepsilon} Q((z^{2,\varepsilon} - z^{1,\varepsilon}) \otimes \nabla \varphi) dx, \end{aligned}$$

and, since  $0 \leq \varphi \leq 1$ , we have, using once again the convexity of  $Q$  and the coercivity property (1.2b), that

$$\begin{aligned} \Phi_{0,\varepsilon}(z^\varepsilon) &\leq \frac{\varepsilon}{1 - \eta} \int_{\Sigma_\varepsilon} (1 - \varphi) Q(e(z^{1,\varepsilon})) dx \\ &\quad + \frac{\varepsilon}{1 - \eta} \int_{\Sigma_\varepsilon} \varphi Q(e(z^{2,\varepsilon})) dx \\ &\quad + \frac{\varepsilon c_2}{\eta} \int_{\Sigma_\varepsilon} |z^{2,\varepsilon} - z^{1,\varepsilon}|^2 |\nabla \varphi|^2 dx, \end{aligned}$$

where  $c_2$  is the constant appearing in (1.2b). Then, taking into account the properties of  $\varphi$ , we deduce that

$$\begin{aligned} \Phi_{0,\varepsilon}(w^\varepsilon) &\leq \frac{1}{1 - \eta} (\Phi_{0,\varepsilon}(z^{1,\varepsilon}) + \Phi_{0,\varepsilon}(z^{2,\varepsilon})) \\ &\quad + \frac{\varepsilon C}{\eta} \int_{\Sigma_\varepsilon} |z^{2,\varepsilon} - z^{1,\varepsilon}|^2 dx, \end{aligned} \quad (4.12)$$

where  $C$  is a positive constant independent of  $\varepsilon$  and  $\eta$ . On the other hand, as  $F_\varepsilon$  is  $C^1$ -convex and  $F_\varepsilon(z^{m,\varepsilon} + u, \cdot)$ ;  $m = 1, 2$ , is the restriction to  $\mathcal{O}(\mathbb{R}^3)$  of a nonnegative Borel measure, we have

$$\begin{aligned} F_\varepsilon(z^\varepsilon + u, A \cup \omega) &= F_\varepsilon((1 - \varphi)(z^{1,\varepsilon} + u) + \varphi(z^{2,\varepsilon} + u), A \cup \omega) \\ &\leq F_\varepsilon(z^{1,\varepsilon} + u, A) + F_\varepsilon(z^{2,\varepsilon} + u, \omega). \end{aligned} \quad (4.13)$$

We deduce from (4.9), using (4.12) and (4.13), that

$$\begin{aligned} F^S(u, A \cup \omega) &\leq \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, A \cup \omega)) \\ &\leq \frac{1}{1 - \eta} \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^{1,\varepsilon}) + F_\varepsilon(z^{1,\varepsilon} + u, A)) \\ &\quad + \frac{1}{1 - \eta} \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^{2,\varepsilon}) + F_\varepsilon(z^{2,\varepsilon} + u, \omega)) \\ &\quad + \frac{C}{\eta} \limsup_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon} |z^{2,\varepsilon} - z^{1,\varepsilon}|^2 dx. \end{aligned} \quad (4.14)$$

As  $\limsup_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon} |z^{2,\varepsilon} - z^{1,\varepsilon}|^2 dx = 0$ , we deduce from (4.14) that

$$\begin{aligned} F^S(u, A \cup \omega) &\leq \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, A \cup \omega)) \\ &\leq \frac{1}{1 - \eta} \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^{1,\varepsilon}) + F_\varepsilon(z^{1,\varepsilon} + u, A)) \\ &\quad + \frac{1}{1 - \eta} \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^{2,\varepsilon}) + F_\varepsilon(z^{2,\varepsilon} + u, \omega)), \end{aligned}$$

thus, letting  $\eta$  tend to 0, we have, according to (4.8), that

$$F^S(u, A \cup \omega) \leq F^S(u, A) + F^S(u, \omega),$$

and, letting  $\omega$  increase to  $B$ , we conclude that

$$F^S(u, A \cup B) \leq F^S(u, A) + F^S(u, B).$$

(2) We deduce from (4.6b) that there exists a sequence  $(z^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0$  and

$$F^i(u, A \cup B) = \liminf_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, A \cup B)).$$

Let  $A', B' \in \mathcal{O}(\mathbb{R}^3)$  such that  $\overline{A'} \subset A$  and  $\overline{B'} \subset B$ . Let  $\varphi_\varepsilon^1 \in C_c^\infty(\mathbb{R}^3)$  such that  $0 \leq \varphi_\varepsilon^1 \leq 1$ ,  $\varphi_\varepsilon^1 = 0$  in  $\mathbb{R}^3 \setminus A \cap \Sigma_\varepsilon$  and  $\varphi_\varepsilon^1 = 1$  in  $\overline{A'} \cap \Sigma_\varepsilon$ . Let  $\varphi_\varepsilon^2 \in C_c^\infty(\mathbb{R}^3)$  such

that  $0 \leq \varphi_\varepsilon^2 \leq 1$ ,  $\varphi_\varepsilon^2 = 0$  in  $\mathbb{R}^3 \setminus B \cap \Sigma_\varepsilon$  and  $\varphi_\varepsilon^2 = 1$  in  $\overline{B'} \cap \Sigma_\varepsilon$ . Let  $z^{m,\varepsilon} = \varphi_\varepsilon^m z^\varepsilon$ ;  $m = 1, 2$ . Let us set

$$\Sigma_\varepsilon^1 = A \cap \Sigma_\varepsilon \quad \text{and} \quad \Sigma_\varepsilon^2 = B \cap \Sigma_\varepsilon.$$

Then, using a convexity argument, we deduce that, for every  $\eta \in (0, 1)$ ,

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon}) dx &= \varepsilon \int_{\Sigma_\varepsilon^m} Q(e(\varphi_\varepsilon^m z^\varepsilon)) dx \\ &= \varepsilon \int_{\Sigma_\varepsilon^m} Q(\varphi_\varepsilon^m e(z^\varepsilon) + z^\varepsilon \otimes \nabla \varphi_\varepsilon^m) dx \\ &\leq \frac{\varepsilon}{1-\eta} \int_{\Sigma_\varepsilon^m} Q(e(z^\varepsilon)) dx + \frac{\varepsilon c_2}{\eta} \int_{\Sigma_\varepsilon^m} |z^\varepsilon|^2 |\nabla \varphi_\varepsilon^m|^2 dx, \end{aligned} \quad (4.15)$$

where  $c_2$  is the constant appearing in (1.2b). Observing that the diameters of  $A \cap \Sigma_\varepsilon \setminus \overline{A'} \cap \Sigma_\varepsilon$  and  $B \cap \Sigma_\varepsilon \setminus \overline{B'} \cap \Sigma_\varepsilon$  are independent of  $\varepsilon$ , we infer that  $|\nabla \varphi_\varepsilon^1|$  and  $|\nabla \varphi_\varepsilon^2|$  on  $A \cap \Sigma_\varepsilon \setminus \overline{A'} \cap \Sigma_\varepsilon$  and on  $B \cap \Sigma_\varepsilon \setminus \overline{B'} \cap \Sigma_\varepsilon$  respectively are uniformly bounded by a positive constant independent of  $\varepsilon$ . Then, using the fact that  $z^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$   $L^2(\mathbb{R}^3, \mathbb{R}^3)$ -strong, we deduce that

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon c_2}{\eta} \int_{\Sigma_\varepsilon^m} |z^\varepsilon|^2 |\nabla \varphi_\varepsilon^m|^2 dx = 0,$$

hence, passing to the lower limit in (4.15), we get

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon}) dx \leq \liminf_{\varepsilon \rightarrow 0} \frac{\varepsilon}{1-\eta} \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^\varepsilon) e_{ij}(z^\varepsilon) dx,$$

and, letting  $\eta$  tend to 0,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon}) dx \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^\varepsilon) e_{ij}(z^\varepsilon) dx. \quad (4.16)$$

Observing that  $\int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon}) dx = \Phi_{0,\varepsilon}(z^{m,\varepsilon})$ ;  $m = 1, 2$ , we deduce, using the fact that  $A \cap B = \emptyset$ , the inequality (4.16), and the measure property of  $F_\varepsilon(z^\varepsilon + u, \cdot)$ , that

$$\begin{aligned} F^i(u, A \cup B) &= \liminf_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, A \cup B)) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \left( \varepsilon \sum_{m=1,2} \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^\varepsilon) e_{ij}(z^\varepsilon) dx + F_\varepsilon(z^\varepsilon + u, A' \cup B') \right) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \left( \varepsilon \int_{\Sigma_\varepsilon^1} \sigma_{ij}(z^{1,\varepsilon}) e_{ij}(z^{1,\varepsilon}) dx + F_\varepsilon(z^{1,\varepsilon} + u, A') \right) \\ &\quad + \liminf_{\varepsilon \rightarrow 0} \left( \varepsilon \int_{\Sigma_\varepsilon^2} \sigma_{ij}(z^{2,\varepsilon}) e_{ij}(z^{2,\varepsilon}) dx + F_\varepsilon(z^{2,\varepsilon} + u, B') \right) \end{aligned}$$

$$\begin{aligned}
&= \liminf_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^{1,\varepsilon}) + F_\varepsilon(z^{1,\varepsilon} + u, A')) \\
&\quad + \liminf_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^{2,\varepsilon}) + F_\varepsilon(z^{2,\varepsilon} + u, A')) \\
&\geq F^i(u, A') + F^i(u, B').
\end{aligned}$$

(3) Let  $(u_k)_k \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  converging to some  $u$  in the strong topology of  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ . Then  $(u_k)_k$   $\tau$ -converges to  $u$  and, since  $F^s$  is lower semi-continuous as an upper  $\Gamma$ -limit of a sequence of lower semi-continuous functionals, we have

$$\liminf_{k \rightarrow \infty} F^s(u_k, \omega) \geq F^s(u, \omega),$$

for every  $\omega \in \mathcal{O}(\mathbb{R}^3)$ .

(4) Let  $\omega \in \mathcal{O}(\mathbb{R}^3)$  and  $u, v \in W_{\Gamma_2}$  such that  $u|_\omega = v|_\omega$ . Then

$$F_\varepsilon(z^\varepsilon + u, \omega) = F_\varepsilon(z^\varepsilon + v, \omega),$$

for every sequence  $(z^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$   $\tau$ -converging to 0. This implies that  $F^s(u, \omega) = F^s(v, \omega)$ .

(5) Let  $\varphi \in C^1(\mathbb{R}^3)$  such that  $0 \leq \varphi \leq 1$ . Let  $u, v \in W_{\Gamma_2}$ , and  $\omega \in \mathcal{O}(\mathbb{R}^3)$ . Then, as  $F_\varepsilon$  is  $C^1$ -convex, we have, for every sequence  $(z^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$   $\tau$ -converging to 0,

$$\begin{aligned}
F_\varepsilon(z^\varepsilon + \varphi u + (1 - \varphi)v, \omega) &= F_\varepsilon(\varphi(z^\varepsilon + u) + (1 - \varphi)(z^\varepsilon + v), \omega) \\
&\leq F_\varepsilon(z^\varepsilon + u, \omega) + F_\varepsilon(z^\varepsilon + v, \omega). \tag{4.17}
\end{aligned}$$

As  $\Phi_{0,\varepsilon}$  is nonnegative, we deduce from (4.17) that, for every  $\varepsilon > 0$ ,

$$\begin{aligned}
&\limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + \varphi u + (1 - \varphi)v, \omega)) \\
&\leq \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, \omega) + \Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + v, \omega)) \\
&\leq \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, \omega)) + \limsup_{\varepsilon \rightarrow 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + v, \omega)).
\end{aligned}$$

Taking the infimum over all sequences  $(z^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$   $\tau$ -converging to 0, we infer that

$$F^s(\varphi u + (1 - \varphi)v, \omega) \leq F^s(u, \omega) + F^s(v, \omega).$$

We can prove in a similar way that  $F^s$  is convex. Thus  $F^s$  is  $C^1$ -convex.  $\blacksquare$

Now, according to the compactness theorem of [18], there exist a subsequence  $(\varepsilon_k)_k$  and a countable dense family  $\mathcal{D} \subset \mathcal{B}(\mathbb{R}^3)$  such that, for every  $u \in W_{\Gamma_2}$  and every  $B \in \mathcal{D}$ , we have the following  $\Gamma$ -limit taken with respect to the topology  $\tau$ :

$$(\Gamma\text{-}\lim_{k \rightarrow +\infty} G_{\varepsilon_k})(u, B) = G(u, B), \tag{4.18}$$



where  $G_\varepsilon$  is the functional defined in (4.4). Defining for any  $B \in \mathcal{D}$  the functional  $F$  by

$$F(u, B) = \begin{cases} G(u, B) - \Phi(u) & \text{if } u \in W_{\Gamma_2}, \\ +\infty & \text{otherwise,} \end{cases}$$

we deduce from (4.18) that  $F = F^s = F^i$  on  $W_{\Gamma_2} \times \mathcal{D}$ . We extend  $F$  on  $W_{\Gamma_2} \times \mathcal{B}(\mathbb{R}^3)$  by setting

$$F(u, B) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} F^s(u, D) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} F^i(u, D). \quad (4.19)$$

We define the family  $\mathcal{R}(F)$  by

$$\begin{aligned} \mathcal{R}(F) &= \{B \in \mathcal{B}(\mathbb{R}^3); \forall u \in W_{\Gamma_2}, F_+(u, B) = \sup_{D \in \mathcal{D}, \overline{D} \subset \overset{\circ}{B}} F^s(u, D) \\ &= \inf_{D \in \mathcal{D}, \overline{B} \subset \overset{\circ}{D}} F^s(u, D) = F_-(u, B)\}. \end{aligned}$$

Then (see, for instance, [16, Proposition 14.14])  $\mathcal{R}(F)$  is a rich family in  $\mathcal{B}(\mathbb{R}^3)$  and  $F = F^s = F_+^s = F_-^s = F_+^i = F_-^i = F^i$  on  $\mathcal{R}(F)$ . We deduce that, for every  $u \in W_{\Gamma_2}$  and every  $B \in \mathcal{R}(F)$ ,

$$\begin{aligned} F(u, B) &= \inf \left\{ \limsup_{k \rightarrow +\infty} (\Phi_{0, \varepsilon_k}(z^{\varepsilon_k}) + F_\varepsilon(z^{\varepsilon_k} + u, B)); z^{\varepsilon_k} \xrightarrow[k \rightarrow \infty]{\tau} 0 \right\} \\ &= \inf \left\{ \liminf_{k \rightarrow +\infty} (\Phi_{0, \varepsilon_k}(z^{\varepsilon_k}) + F_\varepsilon(z^{\varepsilon_k} + u, B)); z^{\varepsilon_k} \xrightarrow[k \rightarrow \infty]{\tau} 0 \right\}. \end{aligned} \quad (4.20)$$

Let  $\varepsilon'$  denote any subsequence of  $\varepsilon$ . Repeating the above arguments, we deduce that there exist a subsequence  $(\varepsilon'_k)_k$ , a functional  $F^*$ , and a rich family  $\mathcal{R}(F^*)$ , such that, for every  $u \in W_{\Gamma_2}$  and every  $B \in \mathcal{R}(F^*)$

$$\begin{aligned} F^*(u, B) &= \inf \left\{ \limsup_{k \rightarrow +\infty} (\Phi_{0, \varepsilon'_k}(z^{\varepsilon'_k}) + F_{\varepsilon'_k}(z^{\varepsilon'_k} + u, B)); z^{\varepsilon'_k} \xrightarrow[k \rightarrow +\infty]{\tau} 0 \right\} \\ &= \inf \left\{ \liminf_{k \rightarrow +\infty} (\Phi_{0, \varepsilon'_k}(z^{\varepsilon'_k}) + F_{\varepsilon'_k}(z^{\varepsilon'_k} + u, B)); z^{\varepsilon'_k} \xrightarrow[k \rightarrow +\infty]{\tau} 0 \right\}. \end{aligned} \quad (4.21)$$

As  $\mathcal{R}(F) \cap \mathcal{R}(F^*)$  is still a rich family, we deduce that, for every  $u \in W_{\Gamma_2}$ ,

$$F(u, \cdot) = F^*(u, \cdot) \text{ on } \mathcal{R}(F) \cap \mathcal{R}(F^*). \quad (4.22)$$

Since a countable intersection of rich families is also a rich family, we can repeat the above process and deduce that there exists a rich family  $\mathcal{R}$  in  $\mathcal{B}(\mathbb{R}^3)$  on which the above limits coincide. We thus obtain that, with respect to the topology  $\tau$ , for every  $u \in W_{\Gamma_2}$  and every  $B \in \mathcal{R}$ ,

$$(\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} G_\varepsilon)(u, B) = G(u, B) = \Phi(u) + F(u, B).$$

We now prove the following.

**Lemma 14.** *The functional  $F$  belongs to the class  $\mathbb{F}$ .*

*Proof.* Let  $\omega$  be any element of  $\mathcal{R}(F) \cap \mathcal{O}(\mathbb{R}^3)$ . As  $F^s(\cdot, \omega)$  is lower semi-continuous with respect to the strong topology of  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  by virtue of Lemma 13 (3), we have, according to (4.19), that the functional  $u \mapsto F(u, \omega)$  is lower semi-continuous with respect to the strong topology of  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ . Owing to Lemma 13 (4), to the fact that  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3) \subset W_{\Gamma_2}$ , and to (4.19), we have that  $F(u, \omega) = F(v, \omega)$ , for every  $\omega \in \mathcal{O}(\mathbb{R}^3)$  and every  $u, v \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $u|_{\omega} = v|_{\omega}$ . According to Lemma 13 (5), the functional  $u \mapsto F(u, \omega)$  is convex and  $C^1$ -convex on  $W_{\Gamma_2}$  and therefore on  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ . Let us now prove that, for every  $u \in W_{\Gamma_2}$ ,  $\omega \mapsto F(u, \omega)$  is the restriction to  $\mathcal{O}(\mathbb{R}^3)$  of a nonnegative Borel measure still denoted  $F(u, \omega)$ .

Let  $u \in W_{\Gamma_2}$ . Let  $\omega_1, \omega_2 \in \mathcal{O}(\mathbb{R}^3)$  such that  $\omega_1 \cap \omega_2 = \emptyset$ . Let  $B \in \mathcal{B}(\mathbb{R}^3)$  such that  $\overline{B} \subset \omega_1 \cup \omega_2$ . We have

$$B = (B \cap \omega_1) \cup (B \cap \omega_2) \quad \text{and} \quad \overline{B \cap \omega_i} = \overline{B} \cap \omega_i; \quad i = 1, 2.$$

Then, using Lemma 13 (1), we get

$$\begin{aligned} F^s(u, B) &= F^s(u, (B \cap \omega_1) \cup (B \cap \omega_2)) \\ &\leq F^s(u, B \cap \omega_1) + F^s(u, B \cap \omega_2) \\ &\leq F(u, \omega_1) + F(u, \omega_2), \end{aligned}$$

thus

$$\sup_{\overline{B} \subset \omega_1 \cup \omega_2} F^s(u, B) \leq F(u, \omega_1) + F(u, \omega_2),$$

from which we deduce that

$$F(u, \omega_1 \cup \omega_2) \leq F(u, \omega_1) + F(u, \omega_2). \quad (4.23)$$

Let  $u \in W_{\Gamma_2}$ . Let  $\omega_1, \omega_2 \in \mathcal{O}(\mathbb{R}^3)$  such that  $\omega_1 \cap \omega_2 = \emptyset$ . Let  $B_1 \in \mathcal{B}(\mathbb{R}^3)$ ,  $B_2 \in \mathcal{B}(\mathbb{R}^3)$  and  $B \in \mathcal{O}(\mathbb{R}^3)$  such that  $\overline{B_1} \cup \overline{B_2} \subset B \subset \overline{B} \subset \omega_1 \cup \omega_2$ . We have that

$$\begin{aligned} B &= (B \cap \omega_1) \cup (B \cap \omega_2), \\ (B \cap \omega_1) \cap (B \cap \omega_2) &= \emptyset, \\ \overline{B_i} \subset B \cap \omega_i \subset \overline{B} \cap \omega_i &= \overline{B \cap \omega_i} \subset \omega_i; \quad i = 1, 2. \end{aligned}$$

Then, using Lemma 13 (2), we get

$$\begin{aligned} F(u, \omega_1 \cup \omega_2) &\geq F^i(u, B \cap \omega_1) + F^i(u, B \cap \omega_2) \\ &\geq F^i(u, B_1) + F^i(u, B_2), \end{aligned}$$

thus

$$F(u, \omega_1 \cup \omega_2) \geq \sup_{\overline{B_1} \subset \omega_1} F^i(u, B_1) + \sup_{\overline{B_2} \subset \omega_2} F^i(u, B_2),$$

from which we deduce that

$$F(u, \omega_1 \cup \omega_2) \geq F(u, \omega_1) + F(u, \omega_2). \quad (4.24)$$

We deduce from (4.23) and (4.24) that  $F$  is additive on  $\mathcal{O}(\mathbb{R}^3)$ . Let  $(\omega_k)_k$  be any non-decreasing sequence of open sets in  $\mathcal{R}(F) \cap \mathcal{O}(\mathbb{R}^3)$  and set  $\omega = \bigcup_k \omega_k$ . We have, for every  $k$ ,  $F(u, \omega_k) \leq F(u, \omega)$ , from which we deduce that

$$\limsup_{k \rightarrow \infty} F(u, \omega_k) \leq F(u, \omega). \quad (4.25)$$

On the other hand, using the Borel–Lebesgue theorem, we deduce that, for every  $\overline{B} \subset \omega$ , there exists  $k_0$  such that  $\overline{B} \subset \omega_{k_0}$ . Thus

$$F^s(u, B) \leq F(u, \omega_{k_0}) \leq \limsup_{k \rightarrow \infty} F(u, \omega_k),$$

from which we deduce that

$$\sup_{\overline{B} \subset \omega} F^s(u, B) \leq \limsup_{k \rightarrow \infty} F(u, \omega_k),$$

which yields

$$F(u, \omega) \leq \limsup_{k \rightarrow \infty} F(u, \omega_k). \quad (4.26)$$

Therefore, according to (4.25) and (4.26),  $F(u, \omega) = \limsup_{k \rightarrow \infty} F(u, \omega_k)$ . Hence,  $F$  is  $\sigma$ -additive on  $\mathcal{O}(\mathbb{R}^3)$ . Consequently, for every  $u \in W_{\Gamma_2}$  (and particularly for every  $u \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ ),  $F(u, \cdot)$  is a positive Borel measure, which is outer regular by definition. ■

Since  $\Phi_\varepsilon$  and  $F_\varepsilon$  are quadratic, we deduce from (4.20) that  $F(\cdot, \omega)$  is quadratic for every  $\omega \in \mathcal{O}(\mathbb{R}^3) \cap \mathcal{R}$ . Then, owing to Lemma 14, we deduce, applying Corollary 6, that there exist a finite measure  $\mu \in \mathcal{M}_0(\mathbb{R}^3)$ , a symmetric matrix  $(a_{ij})_{i,j=1,2,3}$  of Borel functions from  $\mathbb{R}^3$  to  $\mathbb{R}$  satisfying  $a_{ij}(x)\zeta_i\zeta_j \geq 0$  for every  $\zeta \in \mathbb{R}^3$  and for q.e.  $x \in \mathbb{R}^3$ , such that, for every  $u \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  and every  $\omega \in \mathcal{O}(\mathbb{R}^3) \cap \mathcal{R}$ ,

$$\begin{aligned} F(u, \omega) &= \int_{\omega} a_{ij}(x)u_i(x)u_j(x)d\mu(x) \\ &= \langle \mu u, u \rangle \\ &= \int_{\omega} u_i u_j d\mu_{ij}, \end{aligned}$$

where, according to Remark 7,  $\mu = (\mu_{ij})_{i,j=1,2,3}$  with  $\mu_{ij} = a_{ij}\mu + \infty_{\mathbb{R}^N \setminus \Theta} \delta_{ij}$ ;  $i, j = 1, 2, 3$ ,  $\Theta$  being the set defined in (2.3). Let us now define

$$d_\varepsilon = \sup\{d(x, \Omega); x \in \text{spt } \infty_{\Gamma_{1,\varepsilon}}\},$$

where  $d(x, \Omega)$  is the Euclidean distance from  $x$  to  $\Omega$  and  $\text{spt } \infty_{\Gamma_{1,\varepsilon}}$  denotes the support of the measure  $\infty_{\Gamma_{1,\varepsilon}}$ . Let  $u, v \in W_{\Gamma_2}$  with  $u = v$  a.e. in  $\Omega$ . We denote in the same way the extensions of  $u$  and  $v$  to  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ . Let  $(u^\varepsilon)_\varepsilon \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $(u^\varepsilon)_\varepsilon$   $\tau$ -converges to  $u$ . We suppose that

$$\begin{aligned} G(u, \mathbb{R}^3) &= \Phi(u) + F(u, \mathbb{R}^3) \\ &= \lim_{\varepsilon \rightarrow 0} G_\varepsilon(u^\varepsilon, \mathbb{R}^3) \\ &= \lim_{\varepsilon \rightarrow 0} (\Phi_\varepsilon(u^\varepsilon) + F_\varepsilon(u^\varepsilon, \mathbb{R}^3)). \end{aligned} \quad (4.27)$$

Let  $(v^\varepsilon)_\varepsilon \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $(v^\varepsilon)_\varepsilon$   $\tau$ -converges to  $v$ . We suppose that

$$\lim_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(v^\varepsilon) = 0. \quad (4.28)$$

Let us define the sequence  $(w^\varepsilon)_\varepsilon$  by  $w^\varepsilon = \varphi_\varepsilon u^\varepsilon + (1 - \varphi_\varepsilon)v^\varepsilon$ , where  $(\varphi_\varepsilon)_\varepsilon$  is some sequence in  $C_0^1(\mathbb{R}^3)$  such that  $0 \leq \varphi_\varepsilon \leq 1$  in  $\mathbb{R}^3$ ,  $\varphi_\varepsilon = 1$  in  $\overline{\Omega} \cap \text{spt } \infty_{\Gamma_{1,\varepsilon}}$ ,  $\varepsilon |\nabla \varphi_\varepsilon|^2 \leq 2\sqrt{\varepsilon}$ , and  $\varphi_\varepsilon = 0$  whenever  $d(x, \Omega) > d_\varepsilon + \varepsilon^{1/4}$ . Then  $(w^\varepsilon)_\varepsilon$   $\tau$ -converges to  $v$  and, using  $\Gamma$ -convergence properties, we deduce that

$$\Phi(v) + F(v, \mathbb{R}^3) \leq \liminf_{\varepsilon \rightarrow 0} (\Phi_\varepsilon(w^\varepsilon) + F_\varepsilon(w^\varepsilon, \mathbb{R}^3)). \quad (4.29)$$

Since  $w^\varepsilon = u^\varepsilon$  q.e. on  $\text{spt } \infty_{\Gamma_{1,\varepsilon}}$ , we have, according to (2.2),

$$F_\varepsilon(u^\varepsilon, \mathbb{R}^3) = F_\varepsilon(w^\varepsilon, \mathbb{R}^3). \quad (4.30)$$

Using the form (4.11) and the convexity of  $Q$ , we have, for every  $\eta \in (0, 1)$ ,

$$\begin{aligned} \Phi_\varepsilon(w^\varepsilon) &\leq \frac{1}{1-\eta} \int_{\Omega} Q(\varphi_\varepsilon e(u^\varepsilon) + (1-\varphi_\varepsilon)e(v^\varepsilon)) dx \\ &\quad + \frac{\varepsilon}{1-\eta} \int_{\Sigma_\varepsilon} Q(\varphi_\varepsilon e(u^\varepsilon) + (1-\varphi_\varepsilon)e(v^\varepsilon)) dx \\ &\quad + \frac{1}{\eta} \int_{\Omega} Q((u^\varepsilon - v^\varepsilon) \otimes \nabla \varphi_\varepsilon) dx \\ &\quad + \frac{\varepsilon}{\eta} \int_{\Sigma_\varepsilon} Q((u^\varepsilon - v^\varepsilon) \otimes \nabla \varphi_\varepsilon) dx, \end{aligned}$$

and, since  $0 \leq \varphi_\varepsilon \leq 1$ , we have, using the convexity of  $Q$  and the coercivity property (1.2b), that

$$\begin{aligned}
 \Phi_\varepsilon(w^\varepsilon) &\leq \frac{1}{1-\eta} \int_{\Omega} \varphi_\varepsilon Q(e(u^\varepsilon)) dx \\
 &\quad + \frac{\varepsilon}{1-\eta} \int_{\Sigma_\varepsilon} \varphi_\varepsilon Q(e(u^\varepsilon)) dx \\
 &\quad + \frac{1}{1-\eta} \int_{\Omega} (1-\varphi_\varepsilon) Q(e(v^\varepsilon)) dx \\
 &\quad + \frac{\varepsilon}{1-\eta} \int_{\Sigma_\varepsilon} (1-\varphi_\varepsilon) Q(e(v^\varepsilon)) dx \\
 &\quad + \frac{c_2}{\eta} \int_{\Omega} |u^\varepsilon - v^\varepsilon|^2 |\nabla \varphi_\varepsilon|^2 dx \\
 &\quad + \frac{\varepsilon c_2}{\eta} \int_{\Sigma_\varepsilon} |u^\varepsilon - v^\varepsilon|^2 |\nabla \varphi_\varepsilon|^2 dx,
 \end{aligned}$$

where  $c_2$  is the constant appearing in (1.2b). Then, taking into account the properties of  $\varphi_\varepsilon$ , we deduce that

$$\begin{aligned}
 \Phi_\varepsilon(w^\varepsilon) &\leq \frac{1}{1-\eta} \Phi_\varepsilon(u^\varepsilon) \\
 &\quad + \frac{\varepsilon}{1-\eta} \int_{\Sigma_\varepsilon} \sigma_{ij}(v^\varepsilon) e_{ij}(v^\varepsilon) dx \\
 &\quad + \frac{\varepsilon^{1/2} C}{\eta} \int_{\Sigma_\varepsilon} |u^\varepsilon - v^\varepsilon|^2 dx,
 \end{aligned} \tag{4.31}$$

where  $C$  is a positive constant independent of  $\varepsilon$  and  $\eta$ . From (4.27)–(4.31) it follows that

$$\Phi(v) + F(v, \mathbb{R}^3) \leq \frac{1}{1-\eta} (\Phi(u) + F(u, \mathbb{R}^3)).$$

As  $\Phi(u) = \Phi(v)$ , we obtain, as  $\eta \rightarrow 0$ , that  $F(v, \mathbb{R}^3) \leq F(u, \mathbb{R}^3)$  and, changing the role of  $u$  and  $v$ , that  $F(u, \mathbb{R}^3) \leq F(v, \mathbb{R}^3)$ . Thus

$$F(u, \mathbb{R}^3) = F(v, \mathbb{R}^3) = \int_{\mathbb{R}^3} v_i v_j d\mu_{ij},$$

which implies that  $\text{spt } \mu_{ij} \subset \overline{\Omega}$ ,  $\forall i, j = 1, 2, 3$ .

Let  $u \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$ . We have the following inequality:

$$0 \leq \Phi(u) + \int_{\mathbb{R}^N} u_i u_j d\mu_{ij} \leq \liminf_{\varepsilon \rightarrow 0} (\Phi_\varepsilon(u) + F_\varepsilon(u, \mathbb{R}^3)). \tag{4.32}$$

Then, taking  $u \in H_0^1(\Omega, \mathbb{R}^3)$ , we have  $F_\varepsilon(u, \mathbb{R}^3) = 0$ , for any  $\varepsilon > 0$ , and

$$\liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(u) = \Phi(u).$$

We deduce, using (4.32), that

$$\int_{\Omega \cup \Gamma_2} u_i u_j d\mu_{ij} = 0, \quad \forall i, j = 1, 2, 3.$$

Therefore,  $\text{spt } \mu_{ij} \subset \Gamma_1$ ,  $\Theta \subset \Gamma_1$  and  $\mu_{ij} = a_{ij}\mu + \infty_{\Gamma_1 \setminus \Theta} \delta_{ij}$ ,  $\forall i, j = 1, 2, 3$ . ■

Let us write the associated limit problem obtained as  $\varepsilon \rightarrow 0$ .

**Corollary 15.** *Problem (1.4) admits a unique solution  $u^\varepsilon$  which  $\tau$ -converges to  $u^0 \in H_{\mu, \Gamma_2}(\Omega, \mathbb{R}^3)$  which is the unique solution of the minimization problem*

$$\min_{u \in L^2(\mathbb{R}^3, \mathbb{R}^3)} \left\{ \mathcal{F}_0(u) - 2 \int_{\Omega} f \cdot u dx \right\},$$

where  $H_{\mu, \Gamma_2}(\Omega, \mathbb{R}^3)$  is the space of admissible displacements defined in (1.6) and  $\mathcal{F}_0$  is the functional defined in (1.5). This solution coincides with the unique solution of the elasticity system

$$\begin{cases} -\sigma_{ij,j}(u) = f_i & \text{in } \Omega; i = 1, 2, 3, \\ \sigma_{ij}(u)n_j + \mu_{ij}u_j = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2. \end{cases} \quad (4.33)$$

Moreover, we have  $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^\varepsilon) = \mathcal{F}_0(u^0)$ .

*Proof.* Observe that the functional  $\mathcal{F}_\varepsilon$  can be written as  $\mathcal{F}_\varepsilon(u) = G_\varepsilon(u, \mathbb{R}^3)$ , for every  $u \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ . As the Dirichlet condition  $u = 0$  on  $\Gamma_{1,\varepsilon}$  is prescribed in the capacity sense, we can prove (see, for instance, [30]) that, using the classical Poincaré and Korn inequalities, problem (1.4) has a unique solution  $u^\varepsilon \in H_0^1(\Omega_\varepsilon, \mathbb{R}^3)$ . Using Proposition 8 and Theorem 12, we deduce, according to [16, Theorem 7.8], that the whole sequence  $(u^\varepsilon)_\varepsilon$   $\tau$ -converges to the unique solution  $u^0 \in H_{\mu, \Gamma_2}(\Omega, \mathbb{R}^3)$  of the problem

$$\min_{u \in L^2(\mathbb{R}^3, \mathbb{R}^3)} \left\{ \mathcal{F}_0(u) - 2 \int_{\Omega} f \cdot u dx \right\} = \lim_{\varepsilon \rightarrow 0} \min_{u \in L^2(\Omega_\varepsilon, \mathbb{R}^3)} \left\{ \mathcal{F}_\varepsilon(u) - 2 \int_{\Omega} f \cdot u dx \right\},$$

$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^\varepsilon) = \mathcal{F}_0(u^0)$ , and  $u^0$  coincides with the unique solution of problem (4.33). ■

## 5. Special cases

### 5.1. Periodic case

Let  $V \subset \mathbb{R}^2$  and  $p : V \rightarrow \mathbb{R}^3$  be a parameterization of  $\Gamma_1$  such that  $p$  is a one-to-one mapping of class  $C^2$  and the rank of  $\nabla p(\vartheta)$  is 2 for each point  $\vartheta = (\vartheta_1, \vartheta_2) \in V$ . For

the sake of simplification, we suppose that  $p(V) = \Gamma_1$ . Let  $Y = (0, 1)^2$  and  $k \in \mathbb{Z}^2$ . We set

$$\begin{aligned} Y_\varepsilon^k &= k\varepsilon + \varepsilon Y, \\ I_\varepsilon &= \{k \in \mathbb{Z}^2; Y_\varepsilon^k \subset V\}. \end{aligned} \quad (5.1)$$

According to (5.1), we see that the measure  $|V \setminus V_\varepsilon|$  of the set  $V \setminus V_\varepsilon$ , where  $V_\varepsilon = \bigcup_{k \in I_\varepsilon} Y_\varepsilon^k$ , tends to 0 as  $\varepsilon \rightarrow 0$  so that  $|\Gamma_1 \setminus \bigcup_{k \in I_\varepsilon} p(Y_\varepsilon^k)|$  tends to 0 as  $\varepsilon \rightarrow 0$  and  $\Gamma_1 \approx \bigcup_{k \in I_\varepsilon} p(Y_\varepsilon^k)$ . Let us consider a positive 1-periodic function  $h \in C^2(Y)$ . We consider here layers  $\Sigma_\varepsilon$  of the form

$$\Sigma_\varepsilon = \left\{ x = s + tn(s); s = p(\vartheta), 0 < t < \varepsilon h\left(\frac{\vartheta}{\varepsilon}\right), \vartheta \in V_\varepsilon \right\}.$$

Then, according to Theorem 12, there exist a rich family  $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$  and a symmetric matrix  $\boldsymbol{\mu} = (\mu_{ij})_{i,j=1,2,3}$  of Borel measures  $\mu_{ij}$ , which are absolutely continuous with respect to the capacity  $\text{Cap}$ , having the same support contained in  $\Gamma_1$ , and satisfying  $\mu_{ij}(B)\zeta_i\zeta_j \geq 0, \forall \zeta \in \mathbb{R}^3, \forall B \in \mathcal{B}(\mathbb{R}^3)$ , such that, for every  $u \in W_{\Gamma_2}$  and every  $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbb{R}^3)$ , we have the following equality:

$$\begin{aligned} & \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^\varepsilon) : z^\varepsilon + u = 0 \text{ on } \omega \cap \left\{ t = \varepsilon h\left(\frac{\vartheta}{\varepsilon}\right) \right\}; \vartheta \in V_\varepsilon, z^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \right\} \\ &= \int_{\Gamma_1 \cap \omega} u_i u_j d\mu_{ij}, \end{aligned} \quad (5.2)$$

where  $\Phi_{0,\varepsilon}$  is defined in (4.7). Since  $\Sigma_\varepsilon$  has a periodic structure, the problem

$$\inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^\varepsilon) : z^\varepsilon + u = 0 \text{ on } \omega \cap \left\{ t = \varepsilon h\left(\frac{\vartheta}{\varepsilon}\right) \right\}; \vartheta \in V_\varepsilon, z^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \right\},$$

is invariant by translation on  $V_\varepsilon$  and the measure  $\mu_{ij}; i, j = 1, 2, 3$ , is the Haar measure on  $\Gamma_1$ . Then

$$\mu_{ij} = K_{ij} d\rho, \quad (5.3)$$

on  $\Gamma_1$ , where  $d\rho$  is the surface measure on  $\Gamma_1$  which is given by the Riemannian metric and  $K_{ij}; i, j = 1, 2, 3$ , are constants in  $\overline{\mathbb{R}}$  satisfying  $K_{ij}\zeta_i\zeta_j \geq 0, \forall \zeta \in \mathbb{R}^3$ . Our purpose is to identify the constants  $K_{ij}$ . Let us denote  $Z_h$  the set defined by

$$Z_h = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3; y' = (y_1, y_2) \in Y, y_3 \in (0, h(y'))\}.$$

We consider, for  $m = 1, 2, 3$ , the following problem:

$$\left\{ \begin{array}{ll} \text{div}(w^m) = 0 & \text{in } Z_h, \\ w^m = e^m & \text{on } \{y_3 = h(y')\}, \\ w^m = 0 & \text{on } \{y_3 = 0\}, \\ w^m \text{ is } Y\text{-periodic,} & \end{array} \right.$$

where  $e^m = (\delta_{1m}, \delta_{2m}, \delta_{3m})$ ;  $m = 1, 2, 3$ . Let  $t_h = \max_{y \in Y} h(y)$  and  $H > t_h$  be a fixed number. Let us now denote  $Z_H$  the set defined by

$$Z_H = \{y \in \mathbb{R}^3; y' \in Y, y_3 \in (h(y'), H)\}.$$

We consider, for  $m = 1, 2, 3$ , the following problem:

$$\begin{cases} \operatorname{div}(\tilde{w}^m) = 0 & \text{in } Z_H, \\ \tilde{w}^m = e^m & \text{on } \{y_3 = h(y')\}, \\ \tilde{w}^m = 0 & \text{on } \{y_3 = H\}, \\ \tilde{w}^m \text{ is } Y\text{-periodic.} \end{cases} \quad (5.4)$$

Let us define the layer  $\Sigma_{H,\varepsilon}$  by

$$\Sigma_{H,\varepsilon} = \left\{x = s + tn(s); s = p(\vartheta), \varepsilon h\left(\frac{\vartheta}{\varepsilon}\right) < t < \varepsilon H; \vartheta \in V_\varepsilon\right\},$$

and the sequence  $(z_0^{m,\varepsilon})_\varepsilon$ ;  $m = 1, 2, 3$ , of test-functions, by

$$z_0^{m,\varepsilon}(x) = \begin{cases} w^m\left(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}\right) & \text{if } x = s + tn(s) \in \Sigma_\varepsilon, \\ \tilde{w}^m\left(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}\right) & \text{if } x = s + tn(s) \in \Sigma_{H,\varepsilon}. \end{cases} \quad (5.5)$$

The properties of the sequence  $(z_0^{m,\varepsilon})_\varepsilon$ ;  $m = 1, 2, 3$ , are stated in the following.

**Lemma 16.** *We have*

- (1)  $(z_0^{m,\varepsilon})_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  and  $(z_0^{m,\varepsilon})_\varepsilon$   $\tau$ -converges to 0.
- (2)  $\lim_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_0^{m,\varepsilon}) = C_m |\Gamma_1|$ , where

$$C_m = \int_{Z_h} \sigma_{ij}(w^m) e_{ij}(w^m) dy dt.$$

*Proof.* (1) Observing that  $z_0^{m,\varepsilon} = e^m$  on  $\{t = \varepsilon h(\frac{p^{-1}(s)}{\varepsilon})\}$ , and  $z_0^{m,\varepsilon} = 0$  on  $\{t = 0\} \cup \{t = \varepsilon H\}$ , we deduce that  $(z_0^{m,\varepsilon})_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$ , and, since  $z_0^{m,\varepsilon} = 0$  in  $\Omega$ ,  $z_0^{m,\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$   $H^1(\Omega, \mathbb{R}^3)$ -weak. On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^3} |z_0^{m,\varepsilon}|^2 dx &= \int_{\Sigma_\varepsilon \cup \Sigma_{H,\varepsilon}} |z_0^{m,\varepsilon}|^2 dx \\ &= \sum_{k \in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_0^{\varepsilon H} |z_0^{m,\varepsilon}|^2 (1 + t\chi(s,t)) dt ds \\ &= \sum_{k \in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_0^{\varepsilon h(\frac{p^{-1}(s)}{\varepsilon})} \left| w^m\left(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}\right) \right|^2 (1 + t\chi(s,t)) dt ds \\ &\quad + \sum_{k \in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_{\varepsilon h(\frac{p^{-1}(s)}{\varepsilon})}^{\varepsilon H} \left| \tilde{w}^m\left(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}\right) \right|^2 (1 + t\chi(s,t)) dt ds \end{aligned}$$



$$\begin{aligned}
 &= \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} \int_0^{\varepsilon h(\frac{\vartheta}{\varepsilon})} \left| w^m \left( \frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon} \right) \right|^2 (1 + t\chi(p(\vartheta), t)) J dt d\vartheta \\
 &+ \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} \int_{\varepsilon h(\frac{\vartheta}{\varepsilon})}^{\varepsilon H} \left| \tilde{w}^m \left( \frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon} \right) \right|^2 (1 + t\chi(p(\vartheta), t)) J dt d\vartheta, \quad (5.6)
 \end{aligned}$$

where  $\chi$  is the curvature of  $\Gamma_1$  and  $J = |\det(r_1 r_2 r_3)|$  with  $(r_1 r_2) = \nabla p$  and  $r_3 = n(p^{-1}(s))$ . Observing that  $J$  and  $\chi$  are uniformly bounded by some positive constant independent of  $\varepsilon$ , we deduce from (5.6) that

$$\begin{aligned}
 \int_{\mathbb{R}^3} |z_0^{m,\varepsilon}|^2 dx &\leq C \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} \int_0^{\varepsilon h(\frac{\vartheta}{\varepsilon})} \left| w^m \left( \frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon} \right) \right|^2 (1 + t\chi(p(\vartheta), t)) dt d\vartheta \\
 &+ C \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} \int_{\varepsilon h(\frac{\vartheta}{\varepsilon})}^{\varepsilon H} \left| \tilde{w}^m \left( \frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon} \right) \right|^2 (1 + t\chi(p(\vartheta), t)) dt d\vartheta \\
 &\leq C \sum_{k \in I_\varepsilon} \varepsilon^3 \int_{Z_h} |w^m(y, t)|^2 (1 + \varepsilon t) dy dt \\
 &+ C \sum_{k \in I_\varepsilon} \varepsilon^3 \int_{Z_H} |\tilde{w}^m(y, t)|^2 (1 + \varepsilon t) dy dt,
 \end{aligned}$$

from which we deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} |z_0^{m,\varepsilon}|^2 dx = 0.$$

(2) We have

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_0^{m,\varepsilon}) &= \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_0^{m,\varepsilon}) e_{ij}(z_0^{m,\varepsilon}) dx \\
 &= \lim_{\varepsilon \rightarrow 0} \varepsilon \sum_{k \in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_0^{\varepsilon h(\frac{p^{-1}(s)}{\varepsilon})} A_{ij}^{m,\varepsilon} B_{ij}^{m,\varepsilon} (1 + t\chi(s, t)) dt ds
 \end{aligned} \quad (5.7)$$

where

$$\begin{aligned}
 A_{ij}^{m,\varepsilon} &= \check{\sigma}_{ij} \left( w^m \left( \frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon} \right) \right), \\
 B_{ij}^{m,\varepsilon} &= \check{e}_{ij} \left( w^m \left( \frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon} \right) \right),
 \end{aligned}$$

where  $\check{\sigma}_{ij}$  and  $\check{\epsilon}_{ij}$ ;  $i, j = 1, 2, 3$ , are, respectively, the components of stress and deformation tensors in the local basis  $r_1, r_2, r_3$ , with

$$\begin{aligned}\check{\sigma}_{ij}(u) &= a_{ijkl}\check{\epsilon}_{kl}(u), \\ \check{\epsilon}_{ij}(u) &= \frac{1}{2}(\nabla_j u_i + \nabla_i u_j),\end{aligned}\tag{5.8}$$

where  $\nabla_j u_i = \frac{\partial u_i}{\partial s^j} - u_l \Gamma_{ij}^l$ ;  $\Gamma_{ij}^l = \Gamma_{ji}^l = \frac{\partial s^l}{\partial x_k} \frac{\partial^2 x_k}{\partial s^i \partial s^j}$  being the Christoffel symbol of the second kind with

$$x(s) = s + tn = s^1 r_1 + s^2 r_2 + t r_3.$$

Let us set  $p^{-1}(s) = \vartheta = \varepsilon y' + \varepsilon k$ ;  $k \in \mathbb{Z}^2$ , and  $y_3 = \frac{t}{\varepsilon}$ . Then, for every  $\vartheta \in Y_\varepsilon^k$ ;  $k \in \mathbb{Z}^2$ , we have that  $J(\vartheta) = \det \nabla p(\varepsilon k) + O(\varepsilon)$  and

$$\begin{aligned}& \sum_{k \in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_0^{\varepsilon h(\frac{p^{-1}(s)}{\varepsilon})} A_{ij}^{m,\varepsilon} B_{ij}^{m,\varepsilon} (1 + t\chi(s, t)) dt ds \\ &= \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} \int_0^{\varepsilon h(\frac{\vartheta}{\varepsilon})} C_{ij}^{m,\varepsilon} D_{ij}^{m,\varepsilon} (1 + t\chi) J(\vartheta) dt d\vartheta \\ &= \sum_{k \in I_\varepsilon} \varepsilon^3 \int_{Z_h} \sigma_{ij}(w^m) e_{ij}(w^m) (1 + \varepsilon y_3 \chi) J(\varepsilon y' + \varepsilon k) dy \\ &= \varepsilon |\det \nabla p(\varepsilon k)| C_m + O(\varepsilon)\varepsilon,\end{aligned}\tag{5.9}$$

where

$$\begin{aligned}C_{ij}^{m,\varepsilon} &= \sigma_{ij} \left( w^m \left( \frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon} \right) \right), \\ D_{ij}^{m,\varepsilon} &= e_{ij} \left( w^m \left( \frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon} \right) \right).\end{aligned}$$

Thus, combining (5.7) and (5.9), we deduce that

$$\lim_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_0^{m,\varepsilon}) = \lim_{\varepsilon \rightarrow 0} \sum_{k \in I_\varepsilon} \varepsilon^2 |\det \nabla p(\varepsilon k)| C_m = C_m |\Gamma_1|. \quad \blacksquare$$

The constants  $K_{ij}$ ;  $i, j = 1, 2, 3$ , in equality (5.3) are identified in the following.

**Theorem 17.** *We have*

$$K_{lm} = \left( \int_{Z_h} \sigma_{ij}(w^l) e_{ij}(w^m) dy dt \right) \delta_{lm}.$$

*Proof.* According to (5.2) and (5.3), we have

$$\begin{aligned} & \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^\varepsilon); z^\varepsilon + u = 0 \text{ on } \left\{ t = \varepsilon h \left( \frac{\vartheta}{\varepsilon} \right) \right\}, \forall \vartheta \in V_\varepsilon, z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\} \\ & = K_{ij} \int_{\Gamma_1} u_i u_j d\rho. \end{aligned} \quad (5.10)$$

Let  $u = -e^m$  on  $\bar{\Sigma}_\varepsilon$ . We deduce from (5.10) that

$$K_{mm}|\Gamma_1| = \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^\varepsilon); z^\varepsilon = e^m \text{ on } \left\{ t = \varepsilon h \left( \frac{\vartheta}{\varepsilon} \right) \right\}, \forall \vartheta \in V_\varepsilon, z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\},$$

from which we deduce, using Lemma 16, that

$$K_{mm}|\Gamma_1| \leq C_m|\Gamma_1|,$$

hence

$$K_{mm} \leq C_m. \quad (5.11)$$

Let  $(z^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $z^\varepsilon = e^m$  on  $\{t = \varepsilon h(\frac{\vartheta}{\varepsilon})\}$  and  $(z^\varepsilon)_\varepsilon$   $\tau$ -converges to 0. We have from the definition of the subdifferentiability of convex functionals that

$$\Phi_{0,\varepsilon}(z^\varepsilon) \geq \Phi_{0,\varepsilon}(z_0^\varepsilon) + 2\varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_0^{m,\varepsilon}) e_{ij}(z^\varepsilon - z_0^{m,\varepsilon}) dx. \quad (5.12)$$

Using Green's formula, we deduce that

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_0^{m,\varepsilon}) e_{ij}(z^\varepsilon - z_0^{m,\varepsilon}) dx &= -\varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij,j}(z_0^{m,\varepsilon})(z_i^\varepsilon - z_{0,i}^{m,\varepsilon}) dx \\ &+ \varepsilon \int_{\Gamma_1 \cup \Gamma_{1,\varepsilon}} \sigma_{ij}(z_0^{m,\varepsilon}) n_j (z_i^\varepsilon - z_{0,i}^{m,\varepsilon}) ds. \end{aligned} \quad (5.13)$$

As  $z^\varepsilon - z_0^{m,\varepsilon} = 0$  on  $\Gamma_{1,\varepsilon}$  and  $z_0^{m,\varepsilon} = 0$  on  $\Gamma_1$ , we have that

$$\varepsilon \int_{\Gamma_1 \cup \Gamma_{1,\varepsilon}} \sigma_{ij}(z_0^{m,\varepsilon}) n_j (z_i^\varepsilon - z_{0,i}^{m,\varepsilon}) ds = \varepsilon \int_{\Gamma_1} \sigma_{ij}(z_0^{m,\varepsilon}) n_j z_i^\varepsilon ds. \quad (5.14)$$

Then, using the expression (5.5) of  $z_0^{m,\varepsilon}$  and the trace theorem, we infer that

$$\begin{aligned} \left| \varepsilon \int_{\Gamma_1} \sigma_{ij}(z_0^{m,\varepsilon}) n_j z_i^\varepsilon dx \right| &\leq \left( \varepsilon \int_{\Gamma_1} (\sigma_{ij}(z_0^{m,\varepsilon}) n_j)^2 ds \right)^{1/2} \left( \varepsilon \int_{\Gamma_1} (z_i^{m,\varepsilon})^2 ds \right)^{1/2} \\ &\leq C \left( \int_Y (\sigma_{ij}(w^m) n_j)^2 dy \right)^{1/2} \left( \varepsilon \int_\Omega |\nabla z_i^{m,\varepsilon}|^2 dx \right)^{1/2}, \end{aligned}$$

from which we deduce that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Gamma_1 \cup \Gamma_{1,\varepsilon}} \sigma_{ij}(z_0^{m,\varepsilon}) n_j (z_i^\varepsilon - z_{0,i}^{m,\varepsilon}) ds = 0. \quad (5.15)$$

Besides, an easy computation implies that

$$\begin{aligned} \varepsilon \sigma_{ij,j}(z_0^{m,\varepsilon}) &\xrightarrow{\varepsilon \rightarrow 0} \mathbf{1}_{\Gamma_1} \int_{Z_h} \sigma_{ij,j}(w^m) dy dt \\ &= 0, \end{aligned} \quad (5.16)$$

in  $L^2(\mathbb{R}^3)$ -weak, where  $\mathbf{1}_{\Gamma_1}$  is the characteristic function of  $\Gamma_1$ . Thus, combining (5.13), (5.15), and (5.16), we infer that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_0^{m,\varepsilon}) e_{ij}(z^\varepsilon - z_0^{m,\varepsilon}) dx = 0,$$

and, passing to the lower limit in (5.12), we deduce that

$$\liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_0^\varepsilon) = C_m |\Gamma_1|. \quad (5.17)$$

Now, taking in (5.17) the infimum over all sequences  $(z^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $z^\varepsilon = e^m$  on  $\{t = \varepsilon h(\frac{\vartheta}{\varepsilon})\}$  and  $z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0$ , we deduce that

$$K_{mm} |\Gamma_1| \geq C_m |\Gamma_1|,$$

hence

$$K_{mm} \geq C_m. \quad (5.18)$$

We conclude from (5.11) and (5.18) that  $K_{mm} = C_m$ . Taking  $u = -(e^1 + e^2)$  on  $\overline{\Sigma}_\varepsilon$ , we deduce from (5.10) that

$$\begin{aligned} (K_{11} + 2K_{12} + K_{22}) |\Gamma_1| &= \inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^\varepsilon); z^\varepsilon = e^1 + e^2 \text{ on } \left\{ t = \varepsilon h\left(\frac{\vartheta}{\varepsilon}\right) \right\}, \right. \\ &\quad \left. \forall \vartheta \in V_\varepsilon, \text{ and } z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0 \right\} \\ &\leq \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_0^{1,\varepsilon} + z_0^{2,\varepsilon}). \end{aligned} \quad (5.19)$$

As

$$\int_{Z_h} \sigma_{ij}(w^m) e_{ij}(w^l) dy dt = 0 \quad \text{for } m \neq l,$$

we have

$$\liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_0^{1,\varepsilon} + z_0^{2,\varepsilon}) = |\Gamma_1| (C_1 + C_2),$$

hence, according to (5.19),

$$K_{12} \leq 0. \quad (5.20)$$

Let  $(z^\varepsilon)_\varepsilon \subset H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $z^\varepsilon = e^1 + e^2$  on  $\{t = \varepsilon h(\frac{\vartheta}{\varepsilon})\}$  and  $z^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$ . We have from the definition of the subdifferentiability of convex functionals that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^\varepsilon) &\geq \liminf_{\varepsilon \rightarrow 0} \left( \Phi_{0,\varepsilon}(z_0^{1,\varepsilon} + z_0^{2,\varepsilon}) \right. \\ &\quad \left. + 2\varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_0^{1,\varepsilon} + z_0^{2,\varepsilon}) e_{ij}(z^\varepsilon - (z_0^{1,\varepsilon} + z_0^{2,\varepsilon})) dx \right) \\ &\geq |\Gamma_1|(C_1 + C_2), \end{aligned}$$

from which we deduce that

$$K_{12} \geq 0. \quad (5.21)$$

Thus, combining (5.20) and (5.21), we deduce that  $K_{12} = 0$ . Doing the same for  $K_{23}$ , we deduce that  $K_{lm} = 0$  for  $l \neq m$ . ■

## 5.2. Case of thickness $\varepsilon h(s)$

We suppose here that

$$\Sigma_\varepsilon = \{s + tn(s); s \in \Gamma_1, 0 < t < \varepsilon h(s)\},$$

where  $h$  is a positive continuous function on  $\Gamma_1$ . Then, according to Theorem 12, there exist a rich family  $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$  and a symmetric matrix  $\boldsymbol{\mu} = (\mu_{ij})_{i,j=1,2,3}$  of Borel measures  $\mu_{ij}$ , which are absolutely continuous with respect to the capacity  $\text{Cap}$ , having the same support contained in  $\Gamma_1$ , and satisfying  $\mu_{ij}(\mathcal{B})\zeta_i\zeta_j \geq 0$ ,  $\forall \zeta \in \mathbb{R}^3$ ,  $\forall \mathcal{B} \in \mathcal{B}(\mathbb{R}^3)$ , such that, for every  $u \in W_{\Gamma_2}$  and every  $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$ , we have

$$\inf \left\{ \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^\varepsilon); z^\varepsilon + u = 0 \text{ on } \omega \times \{t = \varepsilon h(s)\}, z^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \right\} = \int_\omega u_i u_j d\mu_{ij}. \quad (5.22)$$

The main result in this subsection reads as follows.

**Theorem 18.** *The matrix of measures  $(\mu_{ij})_{i,j=1,2,3}$  is given by*

$$\mu_{ij} = \kappa_i(s) \frac{ds}{h(s)} \delta_{ij}; \quad i, j = 1, 2, 3.$$

*For homogeneous and isotropic materials*

$$\kappa_i = \begin{cases} \frac{E}{(1+\nu)} & \text{for } i = 1, 2, \\ \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \text{for } i = 3. \end{cases}$$

*Proof.* We consider the sequence  $(w^{m,\varepsilon})_\varepsilon; m = 1, 2, 3$ , defined, for  $x = s + tn(s) \in \Sigma_\varepsilon$ , by  $w^{m,\varepsilon}(x) = e^m \frac{t}{\varepsilon h(s)}$ , and the sequence  $(\tilde{w}^{m,\varepsilon})_\varepsilon; m = 1, 2, 3$ , defined, for  $x =$

$s + tn(s) \in \Sigma_{\varepsilon, H}$ , by  $\tilde{w}^{m, \varepsilon}(x) = e^m \frac{t - \varepsilon H}{\varepsilon(h(s) - H)}$ , where  $H$  is a positive number such that  $H > \sup_{s \in \Gamma_1} h(s)$  and

$$\Sigma_{\varepsilon, H} = \{s + tn(s); s \in \Gamma_1, \varepsilon h(s) < t < \varepsilon H\}.$$

Let  $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$ . Let  $\delta > 0$  be a small parameter. We define the open set  $\omega_\delta \subset \Gamma_1$  by

$$\omega_\delta = \{s \in \Gamma_1; d(s, \omega) < \delta\},$$

where  $d(s, \omega)$  is the distance between  $s$  and  $\omega$  in curvilinear coordinates on  $\Gamma_1$ . We define the auxiliary layers

$$\begin{aligned} \Sigma_{\varepsilon, \omega} &= \{s + tn(s); s \in \omega, 0 < t < \varepsilon h(s)\}, \\ \Sigma_{\varepsilon, \omega, H} &= \{s + tn(s); s \in \omega, \varepsilon h(s) < t < \varepsilon H\}, \\ \Sigma_{\varepsilon, \omega_\delta} &= \{s + tn(s); s \in \omega_\delta, 0 < t < \varepsilon h(s)\}, \\ \Sigma_{\varepsilon, \omega_\delta, H} &= \{s + tn(s); s \in \omega_\delta, \varepsilon h(s) < t < \varepsilon H\}. \end{aligned}$$

Let  $\varphi_{\delta, \varepsilon}$  be a smooth function such that  $0 \leq \varphi_{\delta, \varepsilon} \leq 1$  in  $\mathbb{R}^3$  and

$$\varphi_{\delta, \varepsilon} = \begin{cases} 1 & \text{in } \Sigma_{\varepsilon, \omega} \cup \Gamma_{1, \varepsilon, \omega} \cup \Sigma_{\varepsilon, \omega, H}, \\ 0 & \text{in } \Sigma_\varepsilon \setminus \overline{\Sigma_{\varepsilon, \omega_\delta} \cup \Sigma_{\varepsilon, \omega_\delta, H}}, \end{cases}$$

where  $\Gamma_{1, \varepsilon, \omega} = \partial \Sigma_{\varepsilon, \omega} \cap \partial \Sigma_{\varepsilon, \omega, H}$ . Let us define the sequence  $(z_{0, \omega_\delta}^{m, \varepsilon})_\varepsilon; m = 1, 2, 3$ , by

$$z_{0, \omega_\delta}^{m, \varepsilon} = \begin{cases} \varphi_{\delta, \varepsilon} w^{m, \varepsilon} & \text{in } \Sigma_{\varepsilon, \omega_\delta}, \\ \varphi_{\delta, \varepsilon} \tilde{w}^{m, \varepsilon} & \text{in } \Sigma_{\varepsilon, \omega_\delta, H}, \end{cases}$$

so that  $z_{0, \omega_\delta}^{m, \varepsilon} \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  and  $z_{0, \omega}^{m, \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$ . We then compute

$$\lim_{\varepsilon \rightarrow 0} \Phi_{0, \varepsilon}(z_{0, \omega_\delta}^{m, \varepsilon}) = \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_{\varepsilon, \omega_\delta}} \sigma_{ij}(z_{0, \omega_\delta}^{m, \varepsilon}) e_{ij}(z_{0, \omega_\delta}^{m, \varepsilon}) dx \quad (5.23a)$$

$$= \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_{\varepsilon, \omega}} \sigma_{ij}(w^{m, \varepsilon}) e_{ij}(w^{m, \varepsilon}) dx \quad (5.23b)$$

$$+ \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_{\varepsilon, \omega_\delta} \setminus \Sigma_{\varepsilon, \omega}} \sigma_{ij}(\varphi_{\delta, \varepsilon} w^{m, \varepsilon}) e_{ij}(\varphi_{\delta, \varepsilon} w^{m, \varepsilon}) dx. \quad (5.23c)$$

We have that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_{\varepsilon, \omega}} \sigma_{ij}(w^{m, \varepsilon}) e_{ij}(w^{m, \varepsilon}) dx \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\omega} \int_0^{\varepsilon h(s)} \check{\sigma}_{ij}(w^{m, \varepsilon}) \check{e}_{ij}(w^{m, \varepsilon}) (1 + t\chi) dt ds \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\omega} \kappa_m(s) \int_0^{\varepsilon h(s)} \left| \frac{\partial w_m^{m, \varepsilon}}{\partial t} \right|^2 (1 + t\chi) dt ds \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_{\omega} \kappa_m(s) \int_0^{h(s)} \frac{1}{\varepsilon^2 h^2(s)} (1 + \varepsilon t \varkappa) dt ds \\
 &= \int_{\omega} \frac{\kappa_m(s)}{h(s)} ds,
 \end{aligned} \tag{5.24}$$

where  $\check{\sigma}_{ij}$  and  $\check{\varepsilon}_{ij}$  are defined in (5.8),  $\varkappa$  is the curvature of  $\Gamma_1$ , and  $\kappa_m(s)$ ;  $m = 1, 2, 3$ , are material coefficients. On the other hand, using a convexity argument and the coercivity property (1.2b), we infer that, for every  $\eta \in (0, 1)$ ,

$$\begin{aligned}
 &\varepsilon \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} \sigma_{ij}(\varphi_{\delta, \varepsilon} w^{m, \varepsilon}) e_{ij}(\varphi_{\delta, \varepsilon} w^{m, \varepsilon}) dx \\
 &\leq \frac{\varepsilon}{1 - \eta} \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} \sigma_{ij}(w^{m, \varepsilon}) e_{ij}(w^{m, \varepsilon}) dx \\
 &\quad + \frac{\varepsilon C}{\eta} \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} |w^{m, \varepsilon}|^2 |\nabla \varphi_{\delta, \varepsilon}|^2 dx \\
 &\leq \frac{\varepsilon}{1 - \eta} \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} \sigma_{ij}(w^{m, \varepsilon}) e_{ij}(w^{m, \varepsilon}) dx \\
 &\quad + \frac{\varepsilon C}{\eta \delta^2} \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} |w^{m, \varepsilon}|^2 dx,
 \end{aligned}$$

where  $C$  is a positive constant independent of  $\varepsilon$ . Then, taking  $\delta = \sqrt{\varepsilon}$ , we deduce, using (5.24) and the fact that  $|\omega_{\delta} \setminus \omega| \rightarrow 0$  as  $\delta \rightarrow 0$ , that

$$\begin{aligned}
 &\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{1 - \eta} \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} \sigma_{ij}(w^{m, \varepsilon}) e_{ij}(w^{m, \varepsilon}) dx \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{1 - \eta} \int_{\omega_{\delta} \setminus \omega} \int_0^{\varepsilon h(s)} \check{\sigma}_{ij}(w^{m, \varepsilon}) \check{\varepsilon}_{ij}(w^{m, \varepsilon}) (1 + t \varkappa) dt ds \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{1 - \eta} \int_{\omega_{\delta} \setminus \omega} \kappa_m(s) \int_0^{\varepsilon h(s)} \left| \frac{\partial w_m^{m, \varepsilon}}{\partial t} \right|^2 (1 + t \varkappa) dt ds \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^2}{1 - \eta} \int_{\omega_{\delta} \setminus \omega} \kappa_m(s) \int_0^{h(s)} \frac{1}{\varepsilon^2 h^2(s)} (1 + \varepsilon t \varkappa) dt ds \\
 &= 0,
 \end{aligned}$$

and, since  $w^{m, \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$   $L^2(\mathbb{R}^3, \mathbb{R}^3)$ -strong,

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon C}{\eta \delta^2} \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} |w^{m, \varepsilon}|^2 dx = \lim_{\varepsilon \rightarrow 0} \frac{C}{\eta} \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} |w^{m, \varepsilon}|^2 dx = 0,$$

hence

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_{\varepsilon, \omega \delta} \setminus \Sigma_{\varepsilon, \omega}} \sigma_{ij}(\varphi_{\delta, \varepsilon} w^{m, \varepsilon}) e_{ij}(\varphi_{\delta, \varepsilon} w^{m, \varepsilon}) dx = 0. \tag{5.25}$$

Thus, replacing in (5.23b) and (5.23c) by the limits obtained in (5.24) and (5.25) respectively, we get

$$\lim_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_{0,\omega_\delta}^{m,\varepsilon}) = \int_{\omega} \frac{\kappa_m(s)}{h(s)} ds.$$

According to (5.22), we have, using to the above equality, that, for every  $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$ ,

$$\mu_{mm}(\omega) \leq \lim_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_{0,\omega_\delta}^{m,\varepsilon}) = \int_{\omega} \frac{\kappa_m(s)}{h(s)} ds. \quad (5.26)$$

Let  $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$ . Let  $(z^{m,\varepsilon})_\varepsilon$ ;  $m = 1, 2, 3$ , be any sequence in  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $z^{m,\varepsilon} = e^m$  on  $\omega \times \{t = \varepsilon h(s)\}$  and  $z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\tau} 0$ . Let us consider the subdifferential inequality

$$\Phi_{0,\varepsilon}(z^{m,\varepsilon}) \geq \Phi_{0,\varepsilon}(z_{0,\omega_\delta}^{m,\varepsilon}) + 2\varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_{0,\omega_\delta}^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon} - z_{0,\omega_\delta}^{m,\varepsilon}) dx. \quad (5.27)$$

By calculations similar to those carried out in (5.13)–(5.16) (in the previous subsection) we deduce that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_{0,\omega_\delta}^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon} - z_{0,\omega_\delta}^{m,\varepsilon}) dx = 0,$$

which implies, passing to the lower limit in (5.27), that

$$\liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z^{m,\varepsilon}) \geq \liminf_{\varepsilon \rightarrow 0} \Phi_{0,\varepsilon}(z_{0,\omega_\delta}^{m,\varepsilon}),$$

thus, taking the infimum over all sequences  $(z^{m,\varepsilon})_\varepsilon$  satisfying the above properties, we get

$$\mu_{mm}(\omega) \geq \int_{\omega} \frac{\kappa_m(s)}{h(s)} ds. \quad (5.28)$$

Therefore, according to (5.26) and (5.28), we have that

$$\mu_{mm}(\omega) = \int_{\omega} \frac{\kappa_m(s)}{h(s)} ds.$$

Moreover, replacing  $z_{0,\omega_\delta}^{m,\varepsilon}$  by  $z_{0,\omega_\delta}^{m,\varepsilon} + z_{0,\omega_\delta}^{l,\varepsilon}$ ;  $l \neq m$ , we prove as in the previous subsection, that  $\mu_{ml}(\omega) = \mu_{lm}(\omega) = 0$ . Thus

$$\mu_{ij} = \kappa_i(s) \frac{ds}{h(s)} \delta_{ij}, \quad \forall i, j = 1, 2, 3.$$

If the material is homogeneous and isotropic with

$$a_{ijkl} = \frac{E}{2(1+\nu)} \left\{ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} + \frac{2\nu}{(1-2\nu)} \delta_{ij} \delta_{kl} \right\},$$

where  $E > 0$  is the Young modulus and  $\nu \in (0, 1/2)$  is the Poisson ratio, then we obtain, after some computations, that  $\kappa_m = \frac{E}{(1+\nu)}$  for  $m = 1, 2$ , and  $\kappa_m = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$  for  $m = 3$ . ■



### 5.3. Fractal layers

Let  $\mathbb{R}^{3-}$  be the halfspace defined by

$$\mathbb{R}^{3-} = \{x = (x_1, x_3, x_3) \in \mathbb{R}^3; x_3 < 0\}.$$

Let  $\Omega \subset \mathbb{R}^{3-}$  be a bounded open subset with Lipschitz continuous boundary  $\partial\Omega$ . We set

$$\begin{aligned}\Gamma_1 &= \partial\Omega \cap \{x_3 = 0\}, \\ \Gamma_2 &= \partial\Omega \setminus \Gamma_1.\end{aligned}$$

We suppose that  $\Gamma_1 \neq \emptyset$ . Let  $N$  be a positive integer and  $\psi_1, \dots, \psi_N$  be a finite family of contractive similitudes on  $\mathbb{R}^2$  with ratio  $0 < \rho < 1$ . There exists a unique compact subset  $\Lambda \subset \mathbb{R}^2$ , such that

$$\Lambda = \bigcup_{i=1}^N \psi_i(\Lambda).$$

We suppose that the family  $(\psi_i)_{i=1, \dots, N}$  satisfies the following open set condition: there exists a bounded open subset  $U \subset \mathbb{R}^2$  such that

$$\begin{aligned}\psi_i(U) &\subset U \quad \forall i = 1, \dots, N, \\ \psi_i(U) \cap \psi_j(U) &= \emptyset \quad \text{if } i \neq j.\end{aligned}$$

This condition prevents distinct copies  $\psi_i(\Lambda)$  from having overlapping interiors. The real number  $d = -\ln N / \ln \rho$  is the similarity dimension of  $\Lambda$ . Moreover, there exists a unique Borel regular measure  $\varpi$  with unit mass which is invariant for  $\{\psi_1, \dots, \psi_N\}$ , that is

$$\int_{\Lambda} \varphi d\varpi = \sum_{i=1}^N \rho^{-d} \int_{\Lambda} \varphi \circ \psi_i d\varpi, \quad (5.29)$$

for every integrable  $\varphi : \Lambda \rightarrow \mathbb{R}$ , and  $\varpi$  is supported on  $\Lambda$ . Indeed, the measure  $\varpi$  is given by

$$\varpi = \frac{\mathcal{H}^d \llcorner \Lambda}{\mathcal{H}^d(\Lambda)},$$

where  $\mathcal{H}^d$  is the  $d$ -dimensional Hausdorff measure. For the definitions of the self-similar fractals, their dimensions and their Hausdorff measures, we refer to [23]. We suppose here that  $d > 1$  and  $\Lambda \subset \Gamma_1$ .

Let us denote  $x' = (x_1, x_2)$  and  $D(x', R)$  the disk of radius  $R > 0$  centered at  $x'$ . Let  $x'_0 \in \Lambda$  be a fixed point, we define, for every positive integer  $k$  and every indices  $i_1, \dots, i_k \in \{1, 2, \dots, N\}$ ,

$$\begin{aligned}\psi_{i_1, \dots, i_k} &= \psi_{i_1} \circ \dots \circ \psi_{i_k}, \\ x'_{i_1, \dots, i_k} &= \psi_{i_1, \dots, i_k}(x'_0), \\ D_{i_1, \dots, i_k} &= D(x'_{i_1, \dots, i_k}, \rho^{dk}),\end{aligned}\tag{5.30}$$

and set

$$D_k = \bigcup_{i_1, \dots, i_k \in \{1, 2, \dots, N\}} D_{i_1, \dots, i_k}.$$

Let  $\zeta \in C^1([0, 1])$  such that

$$\zeta > 0, \quad \zeta(1) = 1, \quad \text{and} \quad \int_0^1 \frac{t}{\zeta(t)} dt \geq 1/2.$$

For every  $k \in \mathbb{N}$ , we define the function  $h_k$  on  $\Gamma_1$  by

$$h_k(x') = \begin{cases} \sum_{i_1, \dots, i_k \in \{1, \dots, N\}} \zeta\left(\frac{|x' - x'_{i_1, \dots, i_k}|}{\sqrt{\rho^{dk}}}\right) \mathbf{1}_{D_{i_1, \dots, i_k}} & \text{if } x' \in D_k, \\ 1 & \text{if } x' \in \Gamma_1 \setminus D_k. \end{cases}$$

We define the layer

$$\Sigma_k = \{x \in \mathbb{R}^3; x' \in \Gamma_1, 0 < x_3 < \varepsilon_k h_k(x')\},$$

where  $\varepsilon_k = \rho^k$ . Then, according to Theorem 12, there exist a rich family  $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$  and a symmetric matrix  $\mu = (\mu_{ij})_{i,j=1,2,3}$  of Borel measures  $\mu_{ij}$ , which are absolutely continuous with respect to the capacity  $\text{Cap}$ , having the same support contained in  $\Gamma_1$ , and satisfying  $\mu_{ij}(B)\zeta_i \zeta_j \geq 0$ ,  $\forall \zeta \in \mathbb{R}^3$ ,  $\forall B \in \mathcal{B}(\mathbb{R}^3)$ , such that, for every  $u \in W_{\Gamma_2}$  and every  $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$ , we have

$$\begin{aligned}& \inf \left\{ \liminf_{k \rightarrow \infty} \Phi_{0, \varepsilon_k}(z^k); z^k + u = 0 \text{ on } \omega \times \{x_3 = \varepsilon_k h_k(x')\}, z^k \xrightarrow[k \rightarrow \infty]{\tau} 0 \right\} \\ &= \int_{\omega} u_i u_j d\mu_{ij},\end{aligned}\tag{5.31}$$

where

$$\Phi_{0, \varepsilon_k}(z^k) = \varepsilon_k \int_{\Sigma_k} \sigma_{ij}(z^k) e_{ij}(z^k) dx.$$

The main result in this subsection is stated in the following.

**Theorem 19.** *The matrix of measures  $(\mu_{ij})_{i,j=1,2,3}$  is given by*

$$\mu_{ij} = \left( \kappa_i(x') dx' |_{\Gamma_1} + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)} \kappa_i(s) d\mathcal{H}^d(s) |_{\Lambda} \right) \delta_{ij}; \quad i, j = 1, 2, 3,$$

where  $c = \int_0^1 \frac{r}{\xi(r)} dr$  and  $\kappa_i(s)$ ;  $i = 1, 2, 3$ , are material coefficients. For homogeneous and isotropic materials,

$$\kappa_i = \begin{cases} \frac{E}{(1+\nu)} & \text{for } i = 1, 2, \\ \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \text{for } i = 3. \end{cases}$$

*Proof.* We consider the sequence  $(w^{m,k})_k$ ;  $m = 1, 2, 3$ , defined, for every  $x \in \Sigma_k$ , by  $w^{m,k}(x) = e^m \frac{x_3}{\varepsilon_k h_k(x')}$ , and the sequence  $(\tilde{w}^{m,k})_k$ ;  $m = 1, 2, 3$ , defined, for every  $x \in \Sigma_{k,H}$ , by  $\tilde{w}^{m,k}(x) = e^m \frac{x_3 - \varepsilon_k H}{\varepsilon_k (h_k(x') - H)}$ , where  $H$  is a positive number such that  $H > \sup_{s \in [0,1]} \zeta(s)$  and

$$\Sigma_{k,H} = \{x \in \mathbb{R}^3; x' \in \Gamma_1, \varepsilon_k h_k(x') < x_3 < \varepsilon_k H\}.$$

Let  $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$ . Let  $(\delta_k)_k$  be a sequence of small positive numbers such that  $\lim_{k \rightarrow \infty} \delta_k = 0$ . We define, for every  $k \in \mathbb{N}$ , the open set  $\omega_{\delta_k} \subset \Gamma_1$  by

$$\omega_{\delta_k} = \{x' \in \Gamma_1; d(x', \omega) < \delta_k\},$$

We define the layers

$$\begin{aligned} \Sigma_{k,\omega} &= \{x; x' \in \omega, 0 < x_3 < \varepsilon_k h(s)\}, \\ \Sigma_{k,\omega,H} &= \{x; x' \in \omega, \varepsilon_k h_k(x') < x_3 < \varepsilon_k H\}, \\ \Sigma_{k,\omega_{\delta_k}} &= \{x; x' \in \omega_{\delta_k}, 0 < x_3 < \varepsilon_k h(s)\}, \\ \Sigma_{k,\omega_{\delta_k},H} &= \{x; x' \in \omega_{\delta_k}, \varepsilon_k h_k(x') < x_3 < \varepsilon_k H\}. \end{aligned}$$

Let  $\varphi_k$  be a smooth function such that  $0 \leq \varphi_k \leq 1$  in  $\mathbb{R}^3$  and

$$\varphi_k = \begin{cases} 1 & \text{in } \Sigma_{k,\omega} \cup \Gamma_{1,\varepsilon_k,\omega} \cup \Sigma_{k,\omega,H}, \\ 0 & \text{in } \Sigma_\varepsilon \setminus \overline{\Sigma_{k,\omega_{\delta_k}} \cup \Sigma_{k,\omega_{\delta_k},H}}, \end{cases}$$

where  $\Gamma_{1,\varepsilon_k,\omega} = \partial \Sigma_{k,\omega} \cap \partial \Sigma_{k,\omega,H}$ . Let us define, for  $m = 1, 2, 3$ , the sequence  $(z_0^{m,k})_k$  by

$$z_0^{m,k} = \begin{cases} \varphi_k w^{m,k} & \text{in } \Sigma_{k,\omega_{\delta_k}}, \\ \varphi_k \tilde{w}^{m,k} & \text{in } \Sigma_{k,\omega_{\delta_k},H}, \end{cases}$$

so that  $z_0^{m,k} \in H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  and  $z_0^{m,k} \xrightarrow[k \rightarrow \infty]{\tau} 0$ . We then compute

$$\begin{aligned} & \lim_{k \rightarrow \infty} \varepsilon_k \int_{\Sigma_k} \sigma_{ij}(z_0^{m,k}) e_{ij}(z_0^{m,k}) dx \\ &= \lim_{k \rightarrow \infty} \varepsilon_k \int_{\omega \cap D_k} \int_0^{\varepsilon_k h_k(x')} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k}) dx \\ &+ \lim_{k \rightarrow \infty} \varepsilon_k \int_{\omega \setminus D_k} \int_0^{\varepsilon_k} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k}) dx \\ &+ \lim_{k \rightarrow \infty} \varepsilon_k \int_{\omega_{\delta_k} \setminus \omega} \int_0^{\varepsilon_k} \sigma_{ij}(\varphi_k w^{m,k}) e_{ij}(\varphi_k w^{m,k}) dx. \end{aligned}$$

As in the previous subsection, we can easily prove that, for  $\delta_k = \sqrt{\varepsilon_k}$ ,

$$\lim_{k \rightarrow \infty} \varepsilon_k \int_{\omega_{\delta_k} \setminus \omega} \int_0^{\varepsilon_k} \sigma_{ij}(\varphi_k w^{m,k}) e_{ij}(\varphi_k w^{m,k}) dx = 0.$$

On the other hand, passing in polar coordinates

$$r = \frac{|x' - x'_{i_1, \dots, i_k}|}{\rho^{dk}} \quad \text{and} \quad \theta \in [0, 2\pi],$$

we obtain, using the fact that  $\rho^{dk} = \frac{1}{N^k}$ , that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \varepsilon_k \int_{\omega \cap D_k} \int_0^{\varepsilon_k h_k(x')} \sigma_{ij}(z_0^{m,k}) e_{ij}(z_0^{m,k}) dx \\ &= \lim_{k \rightarrow \infty} \sum_{i_1, \dots, i_k \in \{1, 2, \dots, N\}} \int_{\omega \cap D_{i_1, \dots, i_k}} \frac{\kappa_m(x')}{\zeta\left(\frac{|x' - x'_{i_1, \dots, i_k}|}{\sqrt{\rho^{dk}}}\right)} dx' \\ &= \lim_{k \rightarrow \infty} \sum_{\substack{i_1, \dots, i_k \in \{1, 2, \dots, N\}, \\ D_{i_1, \dots, i_k} \subset \Lambda \cap \omega}} 2\pi \rho^{dk} \kappa_m(x'_{i_1, \dots, i_k}) \int_0^1 \frac{r}{\zeta(r)} dr \\ &= 2\pi \int_0^1 \frac{r}{\zeta(r)} dr \lim_{k \rightarrow \infty} \sum_{\substack{i_1, \dots, i_k \in \{1, 2, \dots, N\}, \\ D_{i_1, \dots, i_k} \subset \Lambda \cap \omega}} \frac{\kappa_m(x'_{i_1, \dots, i_k})}{N^k}. \end{aligned}$$

Then, thanks to property (5.29) which states that the measure  $\frac{\mathcal{H}^d|_{\Lambda}}{\mathcal{H}^d(\Lambda)}$  is invariant for  $\{\psi_1, \dots, \psi_N\}$ , we deduce from the ergodic theorem of [21, Theorem 6.1], using

notations (5.30), that

$$\begin{aligned}
 & 2\pi \int_0^1 \frac{r}{\zeta(r)} dr \lim_{k \rightarrow \infty} \sum_{\substack{i_1, \dots, i_k \in \{1, 2, \dots, N\}, \\ D_{i_1, \dots, i_k} \subset \Lambda \cap \omega}} \frac{\kappa_m(x'_{i_1, \dots, i_k})}{N^k} \\
 &= 2\pi \int_0^1 \frac{r}{\zeta(r)} dr \lim_{k \rightarrow \infty} \sum_{\substack{i_1, \dots, i_k \in \{1, 2, \dots, N\}, \\ D_{i_1, \dots, i_k} \subset \Lambda \cap \omega}} \frac{\kappa_m(\psi_{i_1, \dots, i_k}(x'_0))}{N^k} \\
 &= \frac{2\pi}{\mathcal{H}^d(\Lambda)} \int_0^1 \frac{r}{\zeta(r)} dr \int_{\Lambda \cap \omega} \kappa_m(s) d\mathcal{H}^d(s). \tag{5.32}
 \end{aligned}$$

We obtain in similar way that

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} \varepsilon_k \int_{\omega \setminus D_k} \int_0^{\varepsilon_k} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k}) dx \\
 &= \lim_{k \rightarrow \infty} \varepsilon_k \int_{\omega} \int_0^{\varepsilon_k} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k}) dx \\
 &\quad - \lim_{k \rightarrow \infty} \varepsilon_k \int_{\omega \cap D_k} \int_0^{\varepsilon_k} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k}) dx \\
 &= \int_{\omega} \kappa_m(x') dx' - \frac{2\pi}{\mathcal{H}^d(\Lambda)} \int_{\Lambda \cap \omega} \kappa_m(s) d\mathcal{H}^d(s). \tag{5.33}
 \end{aligned}$$

Now, according to (5.31), (5.32), and (5.33), we have, for every  $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$ ,

$$\begin{aligned}
 \mu_{mm}(\omega) &\leq \lim_{k \rightarrow \infty} \Phi_{0, \varepsilon_k}(z_0^{m,k}) \\
 &= \int_{\omega} \kappa_m(x') dx' + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)} \kappa_i(s) \int_{\omega \cap \Lambda} \kappa_m(s) d\mathcal{H}^d(s). \tag{5.34}
 \end{aligned}$$

Let  $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$ . Let  $(z^{m,k})_k$ ;  $m = 1, 2, 3$ , be any sequence in  $H_{\Gamma_2}^1(\mathbb{R}^3, \mathbb{R}^3)$  such that  $z^{m,k} = e^m$  on  $\omega \times \{x_3 = \varepsilon_k h_k(x')\}$  and  $z^{m,k} \xrightarrow[k \rightarrow \infty]{\tau} 0$ . Let us consider the subdifferential inequality

$$\Phi_{0, \varepsilon_k}(z^{m,k}) \geq \Phi_{0, \varepsilon_k}(z_0^{m,k}) + 2\varepsilon_k \int_{\Sigma_k} \sigma_{ij}(z_0^{m,k}) e_{ij}(z^{m,k} - z_0^{m,k}) dx. \tag{5.35}$$

Using (5.13)–(5.16) in the penultimate subsection, we deduce, passing to the lower limit in (5.35), that

$$\liminf_{k \rightarrow \infty} \Phi_{0, \varepsilon_k}(z^{m,k}) \geq \liminf_{k \rightarrow \infty} \Phi_{0, \varepsilon_k}(z_0^{m,k}),$$

hence, taking the infimum over all sequences  $(z^{m,k})_k$  satisfying the above properties, we obtain

$$\mu_{mm}(\omega) \geq \int_{\omega} \kappa_m(x') dx' + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)} \kappa_i(s) \int_{\omega \cap \Lambda} \kappa_m(s) d\mathcal{H}^d(s). \tag{5.36}$$

Thus, according to (5.34) and (5.36), we conclude that

$$\mu_{mm}(\omega) = \int_{\omega} \kappa_m(x') dx' + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)} \kappa_i(s) \int_{\omega \cap \Lambda} \kappa_m(s) d\mathcal{H}^d(s).$$

Moreover, replacing  $z_0^{m,k}$  by  $z_0^{m,k} + z_0^{l,k}$ ;  $l \neq m$ , we prove as in the last two subsections, that  $\mu_{ml}(\omega) = \mu_{lm}(\omega) = 0$ . Therefore

$$\begin{aligned} \mu_{ml} = & \left( \int_{\omega} \kappa_m(x') dx' \right. \\ & \left. + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)} \kappa_i(s) \int_{\omega \cap \Lambda} \kappa_m(s) d\mathcal{H}^d(s) \right) \delta_{ml}, \quad \forall m, l = 1, 2, 3. \quad \blacksquare \end{aligned}$$

## 6. Optimization problems

Let  $\eta > 0$ . We suppose that  $\boldsymbol{\mu} = \mathbf{h}ds$ , where  $\mathbf{h}$  is a diagonal matrix  $\text{Diag}(h_i)_{i=1,2,3}$  of  $\Gamma_1$ -measurable functions  $h_i : \Gamma_1 \rightarrow (0, +\infty)$  such that

$$\int_{\Gamma_1} h_i(s) ds = \eta, \quad \forall i = 1, 2, 3.$$

Let  $\mathcal{D}_\eta$  denote the set of all these matrices. Let us consider the following problem:

$$\begin{cases} -\text{div } \sigma(u^h) = f & \text{in } \Omega, \\ \mathbf{h}\sigma(u^h)n + u^h = 0 & \text{on } \Gamma_1, \\ u^h = 0 & \text{on } \Gamma_2, \end{cases} \quad (6.1)$$

which has a unique solution  $u^h \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$ . We define the functional  $F(\mathbf{h}, \cdot)$  by

$$F(\mathbf{h}, u) = \begin{cases} \frac{1}{2} \int_{\Omega} \sigma_{ij}(u) e_{ij}(u) dx + \frac{1}{2} \sum_{i=1}^3 \int_{\Gamma_1} \frac{u_i^2}{h_i} ds \\ - \int_{\Omega} f \cdot u dx \\ +\infty \end{cases} \quad \begin{array}{l} \text{if } u \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3), \\ \text{otherwise.} \end{array}$$

We can easily check that

$$F(\mathbf{h}, u^h) = -\frac{1}{2} \int_{\Omega} f \cdot u^h dx. \quad (6.2)$$

We consider the following optimal control problem:

$$\min_{\mathbf{h} \in \mathcal{D}_\eta} \min_{u \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)} F(\mathbf{h}, u). \quad (6.3)$$

According to (6.2), the minimization of  $F$ , with respect to  $u$ , is equivalent to the maximization of the work of the external loads on  $\Omega$ . We have the following result.

**Theorem 20.** *Problem (6.3) admits a unique solution  $u^\eta \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$  which satisfies*

$$\begin{cases} -\operatorname{div} \sigma(u) = f & \text{in } \Omega, \\ \sigma_{ij}(u)n_j + \left( \int_{\Gamma_1} |u_i| ds \right) \frac{\operatorname{sign}(u_i)}{\eta} = 0 & \text{on } \Gamma_1, \end{cases} \quad (6.4)$$

where  $\operatorname{sign}$  is defined by

$$\operatorname{sign}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

*Proof.* Let us consider the following equivalent problem to (6.3),

$$\min_{u \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)} \min_{\mathbf{h} \in \mathcal{D}_\eta} F(\mathbf{h}, u). \quad (6.5)$$

Let  $u \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$  and  $\mathbf{h}^\eta(u) \in \mathcal{D}_\eta$  be the unique solution of the following problem:

$$\min_{\mathbf{h} \in \mathcal{D}_\eta} F(\mathbf{h}, u). \quad (6.6)$$

Using Hölder's inequality, we get

$$\left( \int_{\Gamma_1} |u_i| ds \right)^2 \leq \left( \int_{\Gamma_1} h_i ds \right) \left( \int_{\Gamma_1} \frac{u_i^2}{h_i} ds \right),$$

for every  $i = 1, 2, 3$ . The minimum of problem (6.6), with respect to  $h_i$ , is reached when

$$\left( \int_{\Gamma_1} |u_i| ds \right)^2 = \left( \int_{\Gamma_1} h_i ds \right) \left( \int_{\Gamma_1} \frac{u_i^2}{h_i} ds \right),$$

which occurs if and only if  $h_i$  has the form

$$h_i^\eta(u) = \eta \frac{|u_i^\eta|}{\int_{\Gamma_1} |u_i^\eta| ds},$$

for every  $i = 1, 2, 3$ . Let us define

$$\begin{aligned} G_\eta(u) &= F(\mathbf{h}^\eta(u), u) \\ &= \frac{1}{2} \int_{\Omega} \sigma_{ij}(u) e_{ij}(u) dx + \frac{1}{2\eta} \sum_{i=1}^3 \left( \int_{\Gamma_1} |u_i| ds \right)^2 - \int_{\Omega} f \cdot u dx. \end{aligned}$$

Then it is possible to see that problem (6.5) becomes

$$\min_{u \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)} G_\eta(u). \quad (6.7)$$

Then, observing that  $G_\eta$  is strictly convex, coercive and lower semi-continuous with respect to the weak topology of  $H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$ , we deduce that problem (6.7) has a unique solution  $u^\eta \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$  which is the solution of problem (6.4). ■

**Example 21.** Let us consider a homogeneous isotropic material in the unit ball  $\Omega = D(0, 1)$  centered at the origin. We suppose that a uniform pressure of intensity  $P_0$  acts on the sphere  $\Gamma_1 = \partial D(0, 1)$ . Then, for  $f \equiv 0$ , the radial displacement  $u^\eta$  given by

$$\begin{cases} u_r^\eta(r) = \frac{P_0\eta}{4\pi} - P_0 \frac{(1+\nu)(1-2\nu)}{E}(r-R), \\ u_\theta^\eta(r) = 0, \end{cases}$$

satisfies equations (6.4) with  $\mathbf{h}^\eta(u^\eta) = \text{Diag}(\frac{\eta}{4\pi}, 0, 0)$ .

Let  $\Omega$  be a smoothly bounded open subset of  $\mathbb{R}^3$  (we suppose that at least  $\Omega$  is of a class  $C^{1,\alpha}$  with  $0 < \alpha < 1$ ) and  $f \in C(\overline{\Omega}, \mathbb{R}^3)$ . There exists a unique solution  $u^* \in H^2(\Omega, \mathbb{R}^3) \cap H_0^1(\Omega, \mathbb{R}^3)$  to the following problem:

$$\begin{cases} -\text{div } \sigma(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which corresponds to problem (6.7) for  $\eta = 0$ . Let  $I_\eta$  be the functional defined on  $H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$  by

$$I_\eta(v) = \frac{\eta}{2} \int_{\Omega} \sigma_{ij}(v) e_{ij}(v) dx + \frac{1}{2} \sum_{i=1}^3 \left( \int_{\Gamma_1} |v_i| ds \right)^2 + \int_{\Gamma_1} \sigma_{ij}(u^*) n_j v_i ds.$$

Then,  $v^\eta = \frac{u^h - u^*}{\eta}$ , where  $u^h$  is the solution of problem (6.1), is the unique minimizer of  $I_\eta$ . Let  $\psi \in H^{1/2}(\Gamma_1, \mathbb{R}^3)$ . There exists a unique function  $v_\psi$  such that

$$\int_{\Omega} \sigma_{ij}(v_\psi) e_{ij}(v_\psi) dx = \inf_{\{w \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3); w|_{\Gamma_1} = \psi\}} \int_{\Omega} \sigma_{ij}(w) e_{ij}(w) dx. \quad (6.8)$$

Let us denote by  $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$  the space of vectorial Radon measures on  $\Gamma_1$ . We consider the functional  $J_\eta$  defined on  $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$  by

$$J_\eta(\psi) = \begin{cases} \frac{\eta}{2} \int_{\Omega} \sigma_{ij}(v_\psi) e_{ij}(v_\psi) dx + \frac{1}{2} \sum_{i=1}^3 \left( \int_{\Gamma_1} |\psi_i| ds \right)^2 \\ \quad + \int_{\Gamma_1} \sigma_{ij}(u^*) n_j \psi_i ds & \text{if } \psi \in H^{1/2}(\Gamma_1, \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, according to (6.8),  $v^\eta|_{\Gamma_1} = \frac{u^h - u^*}{\eta}|_{\Gamma_1} = \frac{u^h}{\eta}|_{\Gamma_1}$  is the unique minimizer of  $J_\eta$ . This implies that the problem  $\min_{v \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)} I_\eta(v)$  is equivalent to the minimization problem  $\min_{\psi \in H^{1/2}(\Gamma_1, \mathbb{R}^3)} J_\eta(\psi)$ .



We have the following compactness and  $\Gamma$ -convergence results for  $J_\eta$ .

**Theorem 22.** (1)  $\sup_\eta \sum_{i=1}^3 \int_{\Gamma_1} |v_i^\eta| ds < +\infty$ .

(2) *The sequence  $(J_\eta)_\eta$   $\Gamma$ -converges, as  $\eta$  tends to zero, with respect to the weak\* topology of  $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$ , to the functional  $J$  defined on  $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$  by*

$$J(\lambda) = \frac{1}{2} \sum_{i=1}^3 |\lambda_i|^2(\Gamma_1) + \int_{\Gamma_1} \sigma_{ij}(u^*) n_j d\lambda_i,$$

where  $|\lambda_i|$  is the total variation of the measure  $\lambda_i$ ;  $i = 1, 2, 3$ .

*Proof.* (1) Using the smoothness of  $\Omega$ ,  $f$ , and  $u^*$ , we infer that there exists a positive constant  $C$  such that, for every  $i = 1, 2, 3$ ,

$$\sup_{\Gamma_1} |\sigma_{ij}(u^*) n_j| \leq C,$$

from which we deduce that

$$J_\eta(v^\eta|_{\Gamma_1}) \geq \frac{1}{2} \sum_{i=1}^3 \left( \int_{\Gamma_1} |v_i^\eta| ds \right)^2 - C \sum_{i=1}^3 \int_{\Gamma_1} |v_i^\eta| ds. \quad (6.9)$$

Now, observing that  $J_\eta(v^\eta|_{\Gamma_1}) \leq J_\eta(0)$ , we deduce that

$$\sup_\eta J_\eta(v^\eta|_{\Gamma_1}) \leq 0,$$

and, using (6.9), we get

$$\sup_\eta \sum_{i=1}^3 \int_{\Gamma_1} |v_i^\eta| ds \leq C.$$

We deduce from the above uniform boundedness that, up to some subsequence,

$$v^\eta|_{\Gamma_1} \xrightarrow{\eta \rightarrow 0} \lambda \quad \mathcal{M}(\Gamma_1, \mathbb{R}^3)\text{-weak*}. \quad (6.10)$$

(2) The scalar version of this assertion was proved in [20, Theorem 3.5].

(a) *Lower limit inequality.* Let  $(\lambda^\eta)_\eta \subset H^{1/2}(\Gamma_1, \mathbb{R}^3)$  such that  $\lambda^\eta \xrightarrow{\eta \rightarrow 0} \lambda$   $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$ -weak\*. As the functional  $v \mapsto |v|$ , where  $|v|$  is the total variation of the measure  $v$ , is lower semi-continuous on  $\mathcal{M}(\Gamma_1)$ , we have that, for every  $i = 1, 2, 3$ ,

$$\liminf_{\eta \rightarrow 0} \int_{\Gamma_1} |\lambda_i^\eta| ds \geq |\lambda_i|(\Gamma_1),$$

from which we deduce that

$$\liminf_{\eta \rightarrow 0} J_\eta(\lambda^\eta) \geq J(\lambda). \quad (6.11)$$

(b) *Upper limit inequality.* Without loss of generality, we suppose that  $\Omega \subset \{x_3 > 0\}$  and that  $\partial\Omega \cap \{x_3 = 0\} = \Gamma_1$ . Let us set  $x' = (x_1, x_2)$  and define, for  $\varepsilon > 0$ , the mollifier  $\zeta_\varepsilon$  by

$$\zeta_\varepsilon(x') = \begin{cases} \frac{C_0}{\varepsilon^2} \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |x'|^2}\right) & \text{if } |x'| < \varepsilon, \\ 0 & \text{if } |x'| \geq \varepsilon, \end{cases} \quad (6.12)$$

where  $C_0 = (\int_{B(0,1)} \exp(-\frac{1}{1-|y|^2}) dy)^{-1}$ ;  $B(0, 1)$  being the unit ball of  $\mathbb{R}^2$  centered at the origin. Let  $(\omega_{[1/\varepsilon]})_\varepsilon$ , where  $[1/\varepsilon]$  is the integer part of  $1/\varepsilon$ , be a sequence of open sets such that

$$\begin{cases} \omega_1 \subset \omega_2 \subset \dots \subset \omega_{[1/\varepsilon]} \subset \dots \subset \Gamma_1, \\ \bigcup_{\varepsilon > 0} \omega_{[1/\varepsilon]} = \Gamma_1, \\ d(\omega_{[1/\varepsilon]}, \partial\Gamma_1) = \varepsilon. \end{cases} \quad (6.13)$$

Observing that, for  $\varepsilon \in (0, 1)$ ,  $[1/\varepsilon] - 1 = [1/\varepsilon - 1]$ , we define the partition of unity  $(\varphi_\varepsilon)_\varepsilon$  by

$$\begin{cases} \varphi_\varepsilon \in C_c^\infty(\omega_{[1/\varepsilon]}), \\ \varphi_\varepsilon(x') = 1 & \text{in } \omega_{[1/\varepsilon]-1}, \\ 0 \leq \varphi_\varepsilon(x') \leq 1 & \text{in } \Gamma_1. \end{cases} \quad (6.14)$$

Let  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathcal{M}(\Gamma_1, \mathbb{R}^3)$ . Using (6.12)–(6.14), we define the sequence  $(\lambda^\varepsilon)_\varepsilon$  by  $\lambda^\varepsilon = (\lambda * \zeta_\varepsilon)\varphi_\varepsilon$ . Then  $\lambda^\varepsilon \in C_c^\infty(\Gamma_1, \mathbb{R}^3)$ ,  $|\nabla \lambda^\varepsilon(x')| \leq C/\varepsilon^3$ , for every  $x' \in \Gamma_1$ , and

$$\lambda^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \lambda \quad \mathcal{M}(\Gamma_1, \mathbb{R}^3)\text{-weak}^*. \quad (6.15)$$

Let us define the sequence of function  $(w^\varepsilon)_\varepsilon$  from  $\Omega$  to  $\mathbb{R}^3$  by

$$w_i^\varepsilon(x) = \frac{\varepsilon - x_3}{\varepsilon} \lambda_i^\varepsilon(x'), \quad \forall i = 1, 2, 3.$$

Then  $w^\varepsilon \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$ . Let us now set

$$\begin{aligned} \varepsilon &= \eta^{1/16}, \\ w^\eta &= w^{\eta^{1/16}}, \\ \lambda^\eta &= \lambda^{\eta^{1/16}}. \end{aligned} \quad (6.16)$$

Using (6.15) and (6.16), we deduce that

$$\eta \int_{\Omega} \sigma_{ij}(w^\eta) e_{ij}(w^\eta) dx \leq C \sqrt{\eta},$$

and

$$\begin{aligned} \limsup_{\eta \rightarrow 0} I_\eta(w^\eta) &\leq \limsup_{\eta \rightarrow 0} \frac{1}{2} \sum_{i=1}^3 \left( \int_{\Gamma_1} |w_i^\eta| ds \right)^2 + \limsup_{\eta \rightarrow 0} \int_{\Gamma_1} \sigma_{ij}(u^*) n_j w_i^\eta ds \\ &\leq \frac{1}{2} \sum_{i=1}^3 |\lambda_i|^2 (\Gamma_1) + \int_{\Gamma_1} \sigma_{ij}(u^*) n_j d\lambda_i, \end{aligned}$$

from which we deduce, since  $w^\eta|_{\Gamma_1} = \lambda^\eta$ , that

$$\limsup_{\eta \rightarrow 0} J_\eta(\lambda^\eta) \leq J(\lambda). \quad (6.17)$$

The two inequalities (6.11) and (6.17) imply the second assertion of the theorem. ■

Let us set

$$\begin{aligned} M_i &= \sup_{\Gamma_1} |\sigma_{ij}(u^*) n_j|, \\ K_i^\pm &= \{s \in \Gamma_1; \sigma_{ij}(u^*) n_j(s) = \pm M_i\}. \end{aligned} \quad (6.18)$$

We can now state our result concerning the optimal location where possible elastic layers could take place.

**Theorem 23.** *We have*

- (1) *The sequence  $(\frac{u^\eta}{\eta}|_{\Gamma_1})_\eta$ , where  $u^\eta \in H_{\Gamma_2}^1(\Omega, \mathbb{R}^3)$  is the solution of problem (6.7), converges in  $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$ -weak\*, as  $\eta$  tends to 0, to a measure  $\lambda = (\lambda_i)_{i=1,2,3}$  such that  $\text{spt } \lambda_i \subset K_i^+ \cup K_i^-$ , with  $\lambda_i$  positive in  $K_i^-$  and negative in  $K_i^+$ , for every  $i = 1, 2, 3$ .*
- (2) *For every  $i = 1, 2, 3$ ,  $\int_{\Gamma_1} \sigma_{ij}(u^*) n_j d\lambda_i = -M_i$ .*
- (3) *For every  $i = 1, 2, 3$ ,*

$$\lim_{\eta \rightarrow 0} \int_{\Gamma_1} \left| \frac{u_i^\eta}{\eta} \right| ds = |\lambda_i|(\Gamma_1) = M_i$$

*and the sequence  $(\frac{h^\eta}{\eta}|_{\Gamma_1})_\eta$  converges in  $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$ -weak\*, as  $\eta$  tends to 0, to the measure  $\mathbf{v} = (v_i)_{i=1,2,3}$  given by  $v_i M_i = \lambda_i$ .*

*Proof.* (1) Firstly, we deduce from (6.10) that the sequence  $(\frac{u^\eta}{\eta}|_{\Gamma_1})_\eta$  converges in  $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$ -weak\*, as  $\eta$  tends to 0, to a measure  $\lambda = (\lambda_i)_{i=1,2,3}$  such that

$$J(\lambda) = \min_{v \in \mathcal{M}(\Gamma_1, \mathbb{R}^3)} J(v).$$

(2) Let us set

$$\mathcal{M}_1(\Gamma_1, \mathbb{R}^3) = \{v \in \mathcal{M}(\Gamma_1, \mathbb{R}^3); |v_i|(\Gamma_1) = 1, i = 1, 2, 3\},$$

and introduce the functional  $\tilde{J}$  defined from  $[0, +\infty)^3 \times \mathcal{M}_1(\Gamma_1, \mathbb{R}^3)$  to  $\mathbb{R}$  by

$$\begin{aligned} \tilde{J}(t_1, t_2, t_3, v_1, v_2, v_3) &= J(t_1 v_1, t_2 v_2, t_3 v_3) \\ &= \frac{1}{2} \sum_{i=1}^3 t_i^2 + \sum_{i=1}^3 t_i \int_{\Gamma_1} \sigma_{ij}(u^*) n_j d v_i. \end{aligned}$$

We have

$$\min_{v \in \mathcal{M}(\Gamma_1, \mathbb{R}^3)} J(v) = \min_{v \in \mathcal{M}_1(\Gamma_1, \mathbb{R}^3)} \min_{\substack{t_i > 0, \\ i=1,2,3}} \tilde{J}(t_1, t_2, t_3, v_1, v_2, v_3). \quad (6.19)$$

Let us denote  $(\mathbf{t}, \mathbf{v})$ ;  $\mathbf{t} = (t_i)_{i=1,2,3}$  and  $\mathbf{v} = (v_i)_{i=1,2,3}$ , the minimizer of the right-hand side of (6.19). One can easily check that if  $\int_{\Gamma_1} \sigma_{ij}(u^*) n_j d \mu_i \geq 0$ , for every  $i = 1, 2, 3$ , then  $t_i = 0$  and

$$\min_{\substack{t_i > 0, \\ i=1,2,3}} \tilde{J}(t_1, t_2, t_3, v_1, v_2, v_3) = 0,$$

and if  $\int_{\Gamma_1} \sigma_{ij}(u^*) n_j d v_i < 0$  for every  $i = 1, 2, 3$ , then  $t_i = -\int_{\Gamma_1} \sigma_{ij}(u^*) n_j d v_i$ ,

$$\min_{\substack{t_i > 0, \\ i=1,2,3}} \tilde{J}(t_1, t_2, t_3, v_1, v_2, v_3) = -\frac{1}{2} \sum_{i=1}^3 \left( \int_{\Gamma_1} \sigma_{ij}(u^*) n_j d v_i \right)^2,$$

and  $\mathbf{v} = (v_i)_{i=1,2,3}$  minimizes  $(\int_{\Gamma_1} \sigma_{ij}(u^*) n_j d v_i)_{i=1,2,3}$ .

For every  $i = 1, 2, 3$ , we have that  $\int_{\Gamma_1} \sigma_{ij}(u^*) n_j d v_i \geq -M_i$  and the equality holds if and only if  $\text{spt } v_i \subset K_i^+ \cup K_i^-$ ,  $v_i$  is positive in  $K_i^-$  and negative in  $K_i^+$ . We deduce, according to (6.18), that  $\lambda_i = M_i v_i$ .

(3) As  $(\frac{u^\eta}{\eta}|_{\Gamma_1})_\eta$  converges in  $\mathcal{M}(\Gamma_1, \mathbb{R}^N)$ -weak\* to  $\lambda = (M_i v_i)_{i=1,2,3}$ , we have that

$$\lim_{\eta \rightarrow 0} \int_{\Gamma_1} \left| \frac{u_i^\eta}{\eta} \right| ds = |\lambda_i|(\Gamma_1) = M_i v_i(\Gamma_1) = M_i,$$

from which we deduce that, for every  $i = 1, 2, 3$ , the sequence  $(\frac{|u_i^\eta|}{\int_{\Gamma_1} |u_i^\eta| ds})_h$  converges in  $\mathcal{M}(\Gamma_1)$ -weak\* to  $v_i$ , which means that the sequence  $(\frac{h^\eta}{\eta}|_{\Gamma_1})_\eta$  converges in  $\mathcal{M}(\Gamma_1)$ -weak\* to  $v_i$ . ■

For a biological body, this last theorem provides a tool allowing a characterization of the zones where soft tissues are likely to grow. For a reinforced material this theorem shows that we can have an optimal reinforcement, as  $\eta$  is infinitesimal, if we introduce a material in the points where the tractions  $|\sigma_{ij}(u^*) n_j|$  are maximal.

## References

- [1] E. Acerbi and G. Buttazzo, [Limit problems for plates surrounded by soft material](#). *Arch. Rational Mech. Anal.* **92** (1986), no. 4, 355–370 Zbl [0624.73021](#) MR [0823123](#)
- [2] E. Acerbi and G. Buttazzo, [Reinforcement problems in the calculus of variations](#). *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3** (1986), no. 4, 273–284 Zbl [0607.73018](#) MR [0853383](#)
- [3] R. P. Araujo and D. L. S. McElwain, [A linear-elastic model of anisotropic tumour growth](#). *European J. Appl. Math.* **15** (2004), no. 3, 365–384 Zbl [1057.92034](#) MR [2092919](#)
- [4] R. P. Araujo and D. L. S. McElwain, [The nature of the stresses induced during tissue growth](#). *Appl. Math. Lett.* **18** (2005), no. 10, 1081–1088 Zbl [1079.74515](#) MR [2161871](#)
- [5] M. Boutkrida, N. Grenon, J. Mossino, and G. Moussa, [Limit behaviour of thin insulating layers around multiconnected domains](#). *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **19** (2002), no. 1, 13–40 Zbl [1152.35311](#) MR [1902544](#)
- [6] M. Boutkrida, J. Mossino, and G. Moussa, [On nonhomogeneous reinforcements of varying shape and different exponents](#). *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **2** (1999), no. 3, 517–536 Zbl [0935.49005](#) MR [1719574](#)
- [7] H. Brézis, L. A. Caffarelli, and A. Friedman, [Reinforcement problems for elliptic equations and variational inequalities](#). *Ann. Mat. Pura Appl. (4)* **123** (1980), 219–246 Zbl [0434.35079](#) MR [0581931](#)
- [8] D. Bucur, G. Buttazzo, and C. Nitsch, [Symmetry breaking for a problem in optimal insulation](#). *J. Math. Pures Appl. (9)* **107** (2017), no. 4, 451–463 MR [3623640](#)
- [9] G. Buttazzo, [An optimization problem for thin insulating layers around a conducting medium](#). In *Boundary control and boundary variations (Nice, 1986)*, pp. 91–95, Lecture Notes in Comput. Sci. 100, Springer, Berlin, 1988 Zbl [0685.73043](#) MR [0942449](#)
- [10] G. Buttazzo, [Thin insulating layers: the optimization point of view](#). In *Material instabilities in continuum mechanics (Edinburgh, 1985–1986)*, pp. 11–19, Oxford Sci. Publ., Oxford University Press, New York, 1988 Zbl [0645.73015](#) MR [0970514](#)
- [11] G. Buttazzo, G. Dal Maso, and U. Mosco, [Asymptotic behaviour for Dirichlet problems in domains bounded by thin layers](#). In *Partial differential equations and the calculus of variations, Vol. I*, pp. 193–249, Progr. Nonlinear Differential Equations Appl. 1, Birkhäuser Boston, Boston, MA, 1989 Zbl [0699.35062](#) MR [1034006](#)
- [12] G. Buttazzo and R. V. Kohn, [Reinforcement by a thin layer with oscillating thickness](#). *Appl. Math. Optim.* **16** (1987), no. 3, 247–261 MR [0901816](#)
- [13] G. Buttazzo and O. M. Zeine, [Un problème d’optimisation de plaques](#). *RAIRO Modél. Math. Anal. Numér.* **31** (1997), no. 2, 167–184 Zbl [0867.73051](#) MR [1437119](#)
- [14] L. A. Caffarelli and A. Friedman, [Reinforcement problems in elastoplasticity](#). *Rocky Mountain J. Math.* **10** (1980), no. 1, 155–184 Zbl [0452.73030](#) MR [0573869](#)
- [15] G. Dal Maso, [On the integral representation of certain local functionals](#). *Ricerche Mat.* **32** (1983), no. 1, 85–113 Zbl [0543.49001](#) MR [0740203](#)
- [16] G. Dal Maso, [An introduction to  \$\Gamma\$ -convergence](#). *Prog. Nonlinear Differ. Equ. Appl.* 8, Birkhäuser Boston, Inc., Boston, MA, 1993 Zbl [0816.49001](#) MR [1201152](#)

- [17] G. Dal Maso, A. Defranceschi, and E. Vitali, Integral representation for a class of  $C^1$ -convex functionals. *J. Math. Pures Appl. (9)* **73** (1994), no. 1, 1–46 Zbl [0853.49013](#) MR [1260599](#)
- [18] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* **58** (1975), no. 6, 842–850 MR [0448194](#)
- [19] M. El Jarroudi and A. Brillard, [Boundary asymptotic analysis for an incompressible viscous flow: Navier wall laws](#). *Appl. Math. Optim.* **57** (2008), no. 3, 371–400 Zbl [1172.35053](#) MR [2407318](#)
- [20] P. Esposito and G. Riey, Asymptotic behaviour of a thin insulation problem. *J. Convex Anal.* **10** (2003), no. 2, 379–388 Zbl [1119.35309](#) MR [2043863](#)
- [21] K. Falconer, *Techniques in fractal geometry*. John Wiley & Sons, Ltd., Chichester, 1997 Zbl [0869.28003](#) MR [1449135](#)
- [22] Y. Huang, Q. Li, and Q. Li, [Stability analysis on two thermal insulation problems](#). *J. Math. Pures Appl. (9)* **168** (2022), 168–191 Zbl [1502.49032](#) MR [4515257](#)
- [23] J. E. Hutchinson, [Fractals and self-similarity](#). *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747 Zbl [0598.28011](#) MR [0625600](#)
- [24] A. F. Jones, H. M. Byrne, J. S. Gibson, and J. W. Dold, [A mathematical model of the stress induced during avascular tumour growth](#). *J. Math. Biol.* **40** (2000), no. 6, 473–499 Zbl [0964.92025](#) MR [1770937](#)
- [25] G. Leoni, *A first course in Sobolev spaces*. Grad. Stud. Math. 105, American Mathematical Society, Providence, RI, 2009 Zbl [1180.46001](#) MR [2527916](#)
- [26] J. Mossino and M. Vanninathan, Torsion problem in multiconnected reinforced structures. *Asymptot. Anal.* **31** (2002), no. 3-4, 247–263 Zbl [1028.35019](#) MR [1937839](#)
- [27] D. Q. Ngo and H. C. Nguyen, [Experimental and numerical investigations of textile-reinforced concrete thin-wall panel bolted connections](#). *Case Stud. Constr. Mate.* **19** (2023), article no. e02229
- [28] A. Scholzen, R. Chudoba, and J. Hegger, [Thin-walled shell structures made of textile-reinforced concrete](#). *Struct. Concr.* **16** (2015), no. 1, 106–114.
- [29] M. A. Shannon and B. Rubinsky, The effect of tumour growth on the stress distribution in tissue. *Adv. Biol. Heat & Mass Transfer* **231** (1992), 35–38 Zbl [1151.78364](#)
- [30] W. P. Ziemer, *Weakly differentiable functions*. Grad. Texts in Math. 120, Springer, New York, 1989 Zbl [0692.46022](#) MR [1014685](#)

Received 1 November 2023; revised 19 May 2024.

### Mustapha El Jarroudi

Department of Mathematics, Abdelmalek Essaâdi University, FST Tanger, B.P. 416, 90000 Tangier, Morocco; [eljarroudi@hotmail.com](mailto:eljarroudi@hotmail.com), [meljarroudi@uae.ac.ma](mailto:meljarroudi@uae.ac.ma)

### Jamal El Amrani

Department of Mathematics, Abdelmalek Essaâdi University, B.P. 416, 90000 Tangier, Morocco; [jjelamrani@gmail.com](mailto:jjelamrani@gmail.com)

**Mhamed El Merzguioui**

Department of Mathematics, Abdelmalek Essaâdi University, B.P. 416, 90000 Tangier, Morocco; [mh.elmerzguioui@gmail.com](mailto:mh.elmerzguioui@gmail.com)

**Mustapha Er-Riani**

Department of Mathematics, Abdelmalek Essaâdi University, B.P. 416, 90000 Tangier, Morocco; [erriani@hotmail.com](mailto:erriani@hotmail.com)

**Adel Settati**

Department of Mathematics, Abdelmalek Essaâdi University, B.P. 416, 90000 Tangier, Morocco; [settati\\_adel@yahoo.fr](mailto:settati_adel@yahoo.fr)