A combinatorial higher-rank hyperbolicity condition

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Abstract. We investigate a coarse version of a $2(n + 1)$-point inequality characterizing metric spaces of combinatorial dimension at most n due to Dress. This condition, experimentally called (n, δ)-hyperbolicity, reduces to Gromov's quadruple definition of δ-hyperbolicity in case $n = 1$. The l_∞-product of n δ-hyperbolic spaces is (n, δ)-hyperbolic. Every (n, δ)-hyperbolic metric space, without any further assumptions, possesses a slim $(n + 1)$-simplex property analogous to the slimness of quasi-geodesic triangles in Gromov hyperbolic spaces. In connection with recent work in geometric group theory, we show that every Helly group and every hierarchically hyperbolic group of (asymptotic) rank n acts geometrically on some (n, δ)-hyperbolic space.

1. Introduction

Generalizations and variations of Gromov hyperbolicity [19] belong to the most present themes in geometric group theory today (see, for example, the introduction in [25] for a comprehensive list of these developments). Here we continue the investigation of higher-rank hyperbolicity phenomena from [20, Section 6.B$_2$] and [17, 30, 37]. These results show in particular that most of the characteristic properties of Gromov hyperbolic spaces, regarding the shape of triangles, quasi-geodesics, and isoperimetric inequalities, among others, have adequate rank n analogues ($n \geq 2$) in a context of generalized global non-positive curvature. We refer to the paragraph preceding Theorem 1.3 below for a sample result. The focus in the present paper is on a more foundational, partly combinatorial aspect. We explore a coarse $2(n + 1)$-point inequality for general metric spaces that reduces to Gromov's quadruple definition of δ-hyperbolicity in the case $n = 1$ and, if $\delta = 0$, to an inequality characterizing metric spaces of combinatorial dimension at most n due to Dress [13]. The latter concept measures the combinatorial complexity of the induced metric on finite subsets in terms of the dimension of the (polyhedral) injective hull of these sets (see below and Section 3). Throughout the paper, the unified condition is referred to briefly as (n, δ)-*hyperbolicity*. This notion turns out to possess a variety of remarkable properties,

tying up higher-rank hyperbolicity with (coarsely) injective metric spaces, injective hulls, and some recent developments in geometric group theory.

We now proceed to the actual definition. The common origin of the two afore-mentioned special cases $\delta = 0$ and $n = 1$ is the well-known observation that a metric space X admits an isometric embedding into a metric ($\mathbb{R}$-)tree if and only if every quadruple (x, x', y, y') of points in X satisfies the inequality

$$d(x, x') + d(y, y') \leq \max\{d(x, y) + d(x', y'), d(x, y') + d(x', y)\}. \quad (1.1)$$

The smallest complete such tree is provided by the injective hull $E(X)$ of X; see [28] and [13, pp. 322, 326, 329] (where the injective hull is referred to as the *tight span* T_X of X). A metric space Y is *injective*, as an object in the metric category with 1-Lipschitz maps as morphisms, if every such map $\phi: A \to Y$ defined on a subset of a metric space B extends to a 1-Lipschitz map $\bar{\phi}: B \to Y$. The injective hull $E(X)$ is characterized by the universal property that every isometric embedding of X into some injective metric space Y factors through $E(X)$. On the one hand, [13, Theorem 9, p. 327] generalizes the above observation to metric spaces with an injective hull of dimension at most n or, more precisely, with the property that the topological dimension of $E(V)$ is less than or equal to n for every finite set $V \subset X$. The respective $2(n + 1)$-point condition is precisely what we call $(n, 0)$-hyperbolicity in Definition 1.1 below (and we will give another proof of Dress' theorem in Corollary 4.4). On the other hand, [19, Definition 1.1.C] is equivalent to a relaxed form of inequality (1.1): X is δ-*hyperbolic*, for $\delta \geq 0$, if and only if

$$d(x, x') + d(y, y') \leq \max\{d(x, y) + d(x', y'), d(x, y') + d(x', y)\} + 2\delta \quad (1.2)$$

for every quadruple (x, x', y, y') of points in X. In the case $n = 1$, the following condition is indeed a (somewhat inefficient) reformulation of this inequality (see Proposition 2.2 for the details).

Definition 1.1. Let $n \geq 0$ be an integer, and let $I = I_n$ denote the $2(n + 1)$-element set $\{\pm 1, \pm 2, \ldots, \pm(n + 1)\}$ with the canonical involution $-\mathrm{id}$. A metric space X is called (n, δ)-*hyperbolic*, for some $\delta \geq 0$, if for every family $(x_i)_{i \in I}$ of points in X, there exists a permutation $\alpha \neq -\mathrm{id}$ of I such that

$$\sum_{i \in I} d(x_i, x_{-i}) \leq \sum_{i \in I} d\left(x_i, x_{\alpha(i)}\right) + 2\delta. \quad (1.3)$$

We say that X is $(n, *)$-*hyperbolic* if X is (n, δ)-hyperbolic for some δ.

To emphasize the analogy with (1.2), we could choose α in (1.3) so as to maximize the sum on the right. Note, however, that for $n = 1$, there are $|I| = 4$ summands on either side. If $n = 0$, then $\alpha = \mathrm{id}$ is the only permutation of $I = \{1, -1\}$ distinct from $-\mathrm{id}$, thus a metric space X is $(0, \delta)$-hyperbolic if and only if the diameter

diam(X) is less than or equal to δ. An (n, δ)-hyperbolic metric space is (n', δ')-hyperbolic for all $n' \geq n$ and $\delta' \geq \delta$ (see Lemma 2.3).

We briefly summarize some further basic properties. The l_∞-product of (n_i, δ)-hyperbolic spaces X_i, $i = 1, 2$, is $(n_1 + n_2, \delta)$-hyperbolic. In particular, the l_∞-product of n δ-hyperbolic spaces is (n, δ)-hyperbolic (Proposition 2.4). In general, $(n, *)$-hyperbolicity is preserved under rough isometries, that is, $(1, c)$-quasi-isometries for $c \geq 0$ (Lemma 2.5). Anticipating Theorem 1.4, we mention that quasi-isometry invariance is granted for the class of coarsely injective metric spaces. (It should be noted that for general metric spaces, quasi-isometry invariance fails also in the case $n = 1$; see, for example, [7, Remark 4.1.3 (2)].) The asymptotic rank asrk(X) of an (n, δ)-hyperbolic space X is at most n, and every asymptotic cone of a sequence of pointed (n, δ)-hyperbolic spaces is $(n, 0)$-hyperbolic (Proposition 2.7).

Next, we relate the notion of (n, δ)-hyperbolicity to injective hulls. Remarkably, Gromov's δ-inequality (1.2) passes on from X to $E(X)$. This provides a most efficient way of embedding a general δ-hyperbolic metric space into a complete, contractible, geodesic δ-hyperbolic space with some more features reminiscent of global non-positive curvature (see [15, Section 4.4] and [32, Propositions 1.2, 1.3, 3.8]). Likewise, the injective hull of an (n, δ)-hyperbolic metric space X is (n, δ)-hyperbolic (Proposition 4.1). We then prove the following characterization. The key step is the implication (4) $\Rightarrow$ (1), which is shown in a quantitative form in Proposition 4.3.

Theorem 1.2. *For every metric space X and every integer $n \geq 0$, the following assertions are equivalent:*

(1) *X is $(n, *)$-hyperbolic;*

(2) *the injective hull $E(X)$ is $(n, *)$-hyperbolic;*

(3) *asrk$(E(X)) \leq n$;*

(4) *there is a constant r_0 such that $E(X)$ contains no isometric copy of the l_∞-ball $B(0, r) = [-r, r]^{n+1} \subset l_\infty^{n+1} := (\mathbb{R}^{n+1}, \|\cdot\|_\infty)$ for $r > r_0$.*

We turn to a more geometric higher-rank hyperbolicity condition, analogous to the slimness of quasi-geodesic triangles in Gromov hyperbolic spaces. We say that a metric space X has the *slim simplex property* (SS$_n$) if for all $\lambda \geq 1$ and $c \geq 0$ there exists a constant $D \geq 0$ such that if Δ is a Euclidean $(n + 1)$-simplex and $\phi: \partial\Delta \to X$ is a map whose restriction to every facet is a (λ, c)-quasi-isometric embedding, then the image of every facet is within distance D of the union of the images of the remaining ones. This property was first established in [30, Theorem 1.1] for spaces of asymptotic rank at most n in a setup reminiscent of non-positive curvature, including in particular all proper metric spaces with a conical geodesic bicombing (as defined in [11]). A stronger uniform statement has been given in [17, Theorem 7.2], and by virtue of

the properties of the injective hull we can deduce a completely general result in the present context.

Theorem 1.3. *Every (n, δ)-hyperbolic metric space X satisfies the slim simplex property (SS_n) with a constant D depending only on n, δ, λ, c.*

In fact, the argument yields a constant of the form

$$D = (1 + c) \cdot D'(n, \delta, \lambda)$$

(see Theorem 5.1). In [17, Proposition 7.4], the authors show in turn that every metric space X satisfying (SS_n) with $D = (1 + c) \cdot D'(\lambda)$ has asymptotic rank at most n. However, it is not true in general that a metric space satisfying (SS_n) is $(n, *)$-hyperbolic. For example, l_∞^n is $(n, 0)$-hyperbolic, and hence has property (SS_n), whereas the Euclidean $\mathbb{R}^n$ (being quasi-isometric to l_∞^n) satisfies (SS_n) but fails to be $(n, *)$-hyperbolic for $n \geq 2$ (see Proposition 2.6). On the positive side, it follows easily from the implication $(4) \Rightarrow (1)$ in Theorem 1.2 that every injective metric space X with property (SS_n) is $(n, *)$-hyperbolic. Since $(n, *)$-hyperbolicity is preserved under rough isometries, it is thus natural to seek a generalization to the following class of metric spaces, recently considered in [8, 22, 23]. We call a metric space X *coarsely injective* if there is a constant $c \geq 0$ such that every 1-Lipschitz map $\phi: A \to X$ defined on a subset of a metric space B has an extension $\bar{\phi}: B \to X$ satisfying

$$d\left(\bar{\phi}(b), \bar{\phi}(b')\right) \leq d(b, b') + c$$

for all $b, b' \in B$. This is equivalent to $E(X)$ being within finite distance of the image of the canonical embedding $e: X \to E(X)$ and also to X being roughly isometric to an injective metric space; see Proposition 5.2. It was shown in [31] (see also [9, 32]) that every geodesic Gromov hyperbolic space is coarsely injective. Thus the following result generalizes various known characterizations of hyperbolicity to higher rank.

Theorem 1.4. *Let X be a coarsely injective metric space. For every $n \geq 0$, the following properties are equivalent:*

(1) *X is $(n, *)$-hyperbolic;*

(2) *X satisfies the slim simplex property (SS_n);*

(3) *$\mathrm{asrk}(X) \leq n$;*

(4) *every asymptotic cone of X is $(n, 0)$-hyperbolic;*

(5) *for all $c > 0$ there exists r_0 such that there is no $(1, c)$-quasi-isometric embedding of $B(0, r) \subset l_\infty^{n+1}$ into X for $r > r_0$.*

If X is in addition proper and cocompact, then the asymptotic rank of X is finite, so all properties hold for $n = \mathrm{asrk}(X)$.

Note that since conditions (2) and (3) are preserved under quasi-isometries, the theorem also shows that $(n, *)$-hyperbolicity is a quasi-isometry invariant for coarsely injective spaces.

We now discuss a few applications of the above results in connection with some recent developments in geometric group theory.

A first corollary pertains to the class of Helly groups introduced in [8] and further explored in [21, 24, 26, 33]. A connected locally finite graph is called a *Helly graph* if its vertex set V, endowed with the natural integer valued metric, has the property that every family of pairwise intersecting balls has non-empty intersection. Then V is coarsely injective, and the injective hull $E(V)$ is proper and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in l_∞^n, for every $n \geq 1$. Furthermore, if the graph has uniformly bounded degrees, then $E(V)$ is finite-dimensional. See (the proofs of) [32, Theorem 1.1] and [8, Theorem 6.3]. A group G is called a *Helly group* if G acts geometrically (that is, properly discontinuously and cocompactly by isometries) on the vertex set V of a Helly graph and, hence, geometrically on $E(V)$. The following corollary of Theorems 1.2 and 1.3 applies more generally to groups acting geometrically on an injective metric space. Recent examples in [27] show that not every such group is Helly (see Corollary D therein). Recall also that a group with a geometric action on a proper geodesic metric space X is finitely generated and, with respect to any word metric, quasi-isometric to X (see, for example, [16, Theorem 8.37]).

Corollary 1.5. *Suppose that G is a group acting geometrically on a proper injective metric space X, and endow G with any word metric. Then $n := \mathrm{asrk}(X) = \mathrm{asrk}(G)$ is finite and agrees with*

(1) *the minimal integer n_1 such that X is $(n_1, *)$-hyperbolic;*

(2) *the minimal integer n_2 such that G satisfies (SS_{n_2});*

(3) *the maximal integer n_3 such that there is a quasi-isometric embedding of $\mathbb{R}^{n_3}$ into G;*

(4) *the maximal integer n_4 such that X contains an isometric copy of $l_\infty^{n_4}$.*

Furthermore, G has no free abelian subgroup of rank $n + 1$.

Next we relate $(n, *)$-hyperbolicity to hierarchically hyperbolic spaces or groups, as defined in [3] and [4]. Every hierarchically hyperbolic space (X, d) has a finite rank ν (see [5, Definition 1.10]). In general,

$$\nu \leq \bar{\nu} \leq \mathrm{asrk}(X),$$

where $\bar{\nu}$ denotes the supremum of all integers k such that there exist constants λ, c and (λ, c)-quasi-isometric embeddings of $B(0, r) \subset \mathbb{R}^k$ into X for all $r > 0$ (a quasi-

isometry invariant). However, according to the discussion in [5, Section 1.1.3], the equality $\nu = \bar{\nu}$ holds for all natural examples of hierarchically hyperbolic spaces and in particular for all hierarchically hyperbolic groups. For the latter, $\bar{\nu}$ equals the maximal dimension of a quasi-flat in the group. In the recent paper [23, Theorem A], it is shown that every hierarchically hyperbolic space (X, d) admits a coarsely injective metric ϱ quasi-isometric to d. In the case of a hierarchically hyperbolic group $X = G$, the metric ϱ can be chosen so that G acts geometrically on the (proper) coarsely injective space (X, ϱ). Combining this result with Theorem 1.4, we get the following corollary.

Corollary 1.6. *Let (X, d) be a hierarchically hyperbolic group $X = G$ of rank ν or, more generally, a hierarchically hyperbolic space with $\nu = \bar{\nu}$, and let ϱ be a coarsely injective metric on X quasi-isometric to d (see above). Then*

$$\nu = \mathrm{asrk}(X, \varrho) = \mathrm{asrk}(X, d),$$

and this is the minimal integer such that (X, ϱ) is $(\nu, *)$-hyperbolic. In particular, X satisfies the slim simplex property (SS_ν) with respect to either ϱ or d.

Lastly, we consider Riemannian symmetric spaces of non-compact type. Spaces of rank 1 are Gromov hyperbolic and thus $(1, *)$-hyperbolic in the above terminology. Spaces of rank $n \geq 2$ have asymptotic rank n and satisfy the slim simplex property (SS_n) by [30]; however, with respect to the Riemannian metric, they are not $(n, *)$-hyperbolic. The only n-dimensional $(n, *)$-hyperbolic normed space is l_∞^n, up to isometry (see Proposition 2.6), so the question is whether a given non-compact symmetric space $X = G/K$ of rank $n \geq 2$ admits an $(n, *)$-hyperbolic, G-invariant distance function such that the maximal flats are isometric to l_∞^n with respect to the induced metric. A result in [34] (see also [35]) shows that the G-invariant distance functions on X corresponding to norms on the maximal flats are in bijection with the G-invariant Finsler metrics (of class C^0) on X and also with the norms on $T_p F$ invariant under the Weyl group, for a base point p of X and any maximal flat F through p. Thus the rank n symmetric spaces in question are those whose Weyl group preserves an n-cube, and it remains to see that the resulting metric is indeed $(n, *)$-hyperbolic. The recent paper [22] shows that for $X = \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$, as well as for every classical irreducible symmetric space of non-compact type associated with the automorphism group G of a non-degenerate bilinear or sesquilinear form, there is a coarsely injective, G-invariant metric d on X making the maximal flats isometric to l_∞^n; furthermore, the injective hull of (X, d) is proper. The following immediate consequence of Theorem 1.4 thus applies to all classical groups G *not of type* SL, as defined in [22].

Corollary 1.7. *If a symmetric space $X = G/K$ of non-compact type and rank $n \geq 2$ is equipped with a G-invariant coarsely injective Finsler metric d, then (X, d) is $(n, *)$-hyperbolic.*

The rest of the paper is organized as follows. Section 2 records the basic properties of $(n, *)$-hyperbolic spaces. In Section 3, we first review the construction of the injective hull and the definition of the combinatorial dimension, then we prove some auxiliary results. Section 4 discusses injective hulls of $(n, *)$-hyperbolic spaces and establishes Theorem 1.2. In the concluding Section 5, we turn to the slim simplex property and prove the remaining result stated above.

2. Basic properties

We begin with some elementary observations regarding Definition 1.1. First we notice that in case $n \geq 1$ the permutation $\alpha \neq -\mathrm{id}$ can always be taken to be fixed point free. Recall that we put $I_n = \{\pm 1, \ldots, \pm(n + 1)\}$.

Lemma 2.1. *Let X be a metric space, and let $x_i \in X$ for $i \in I = I_n$, where $n \geq 1$. Then for every permutation $\alpha \neq -\mathrm{id}$ of I there is a permutation $\bar{\alpha} \neq -\mathrm{id}$ of I without fixed points such that $S(\alpha) := \sum_{i \in I} d(x_i, x_{\alpha(i)}) \leq S(\bar{\alpha})$.*

Proof. This holds trivially for $\alpha = \mathrm{id}$, as $S(\mathrm{id}) = 0$. On the other hand, if $\alpha \neq \mathrm{id}$ and $\alpha(i) = i$ for some $i \in I$, then α has a cycle involving a pair $j \neq k$ with $\alpha(j) = k$, and there is an $\alpha' \neq -\mathrm{id}$ with $\alpha'(j) = i$ and $\alpha'(i) = k$ such that $S(\alpha') \geq S(\alpha)$ by the triangle inequality. Eliminating all fixed points in this way, one gets a permutation $\bar{\alpha}$, as desired. ∎

We now give the details of the characterization for $n = 1$.

Proposition 2.2. *A metric space X is $(1, \delta)$-hyperbolic if and only if X is Gromov δ-hyperbolic.*

Proof. If X is δ-hyperbolic, then by adding the term $L := d(x, x') + d(y, y')$ to each of the three sums in (1.2) and by substituting $(x_1, x_{-1}, x_2, x_{-2}) := (x, x', y, y')$, one gets (1.3) for some cyclic permutation $\alpha \neq -\mathrm{id}$ of $I = \{\pm 1, \pm 2\}$.

Conversely, suppose that (1.3) holds for some $\alpha \neq -\mathrm{id}$. By Lemma 2.1 we can assume that α has no fixed points. We put $(x, x', y, y') := (x_1, x_{-1}, x_2, x_{-2})$ and consider three cases. In the first case, α is cyclic, and the respective sum $S(\alpha)$ equals L plus either $d(x, y) + d(x', y')$ or $d(x, y') + d(x', y)$. Then (1.2) follows upon subtracting L on both sides. In the second case, α is still cyclic, but
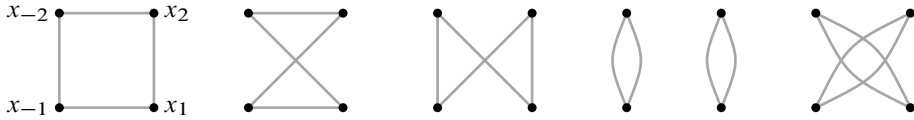
$$S(\alpha) = d(x, y) + d(y, x') + d(x', y') + d(y', x).$$

Figure 1. Illustration of the sums $S(\alpha)$ for the fixed point free permutations $\alpha \neq -$id of $\{\pm 1, \pm 2\}$.

Then there is an involution $\hat{\alpha} \neq -$id such that $S(\alpha) \leq S(\hat{\alpha})$, the latter sum being equal to $2(d(x, y) + d(x', y'))$ or $2(d(x, y') + d(x', y))$. This reduces the second case to the remaining case, where α is an involution distinct from $-$id. Then, dividing (1.3) by 2, one obtains (1.2) with δ in place of 2δ. See Figure 1 for an illustration. ∎

For completeness we record the obvious monotonicity properties.

Lemma 2.3. *Let X be a metric space.*

(1) *If X is (n, δ)-hyperbolic, then X is (n', δ)-hyperbolic for all $n' > n$;*

(2) *X is (n, δ)-hyperbolic if and only if X is (n, δ')-hyperbolic for all $\delta' > \delta$.*

Proof. For (1), given $n' > n$ and a family of points x_i with $i \in I_{n'}$, there is a permutation $\alpha \neq -$id of I_n such that (1.3) holds for the corresponding subfamily, and we can simply add the terms $d(x_i, x_{-i})$ for $i \in I_{n'} \setminus I_n$ on both sides and extend α accordingly.

To prove the 'if' direction in (2), let $x_i \in X$ for $i \in I = I_n$. For every $\delta' > \delta$ there is a permutation $\alpha \neq -$id of I such that (1.3) holds with δ' in place of δ, and for a suitable sequence $\delta'_k \to \delta$ the corresponding permutations all agree, so that (1.3) holds in the limit. ∎

We turn to products. The l_∞-*product* of an l-tuple of metric spaces (X_k, d_k), $k = 1, \ldots, l$, is the set $X = \prod_{k=1}^l X_k$ with the metric d defined by

$$d(x, y) := \max\{d_k(x_k, y_k) : k = 1, \ldots, l\}$$

for all pairs of points $x = (x_1, \ldots, x_l)$ and $y = (y_1, \ldots, y_l)$ in X. The following proposition is a direct adaptation of the result for $\delta = 0$ given in [13, (5.15)].

Proposition 2.4. *If (X, d) is the l_∞-product of an l-tuple of metric spaces as above, and (X_k, d_k) is (n_k, δ)-hyperbolic, then (X, d) is (n, δ)-hyperbolic for $n := \sum_{k=1}^l n_k$. In particular, the l_∞-product of n δ-hyperbolic metric spaces is (n, δ)-hyperbolic.*

Proof. Let $x_i = (x_{i,1}, \ldots, x_{i,l}) \in X$ for $i \in I = I_n$. For $k = 1, \ldots, l$, define

$$I(k) := \{i \in I : d(x_i, x_{-i}) = d_k(x_{i,k}, x_{-i,k})\}.$$

Note that $I = \bigcup_{k=1}^{l} I(k)$ and $I(k) = -I(k)$, in particular, $|I(k)|$ is even. Since $|I| = 2(n+1) > 2n$, there is an index k with $|I(k)| > 2n_k$, and hence $|I(k)| \geq 2(n_k + 1)$. As (X_k, d_k) is (n_k, δ)-hyperbolic, there exists a permutation $\alpha \neq -\mathrm{id}$ of $I(k)$ such that

$$\sum_{i \in I(k)} d_k(x_{i,k}, x_{-i,k}) \leq \sum_{i \in I(k)} d_k(x_{i,k}, x_{\alpha(i),k}) + 2\delta$$

(compare the first part of Lemma 2.3 in case $|I(k)| > 2(n_k + 1)$). Using the definition of $I(k)$ and the inequality $d_k(x_{i,k}, x_{\alpha(i),k}) \leq d(x_i, x_{\alpha(i)})$, and extending α by $-\mathrm{id}$ on $I \setminus I(k)$, we get a permutation $\alpha \neq -\mathrm{id}$ of I such that (1.3) holds. ∎

Let $X = (X, d)$ and $Y = (Y, d)$ be two metric spaces. For constants $\lambda, c \geq 0$, a map $\phi \colon X \to Y$ will be called (λ, c)-*Lipschitz* if

$$d\big(\phi(x), \phi(x')\big) \leq \lambda\, d(x, x') + c$$

for all $x, x' \in X$. If, in addition, $\lambda \geq 1$ and

$$d\big(\phi(x), \phi(x')\big) \geq \lambda^{-1} d(x, x') - c$$

for all $x, x' \in X$, then ϕ is a (λ, c)-*quasi-isometric embedding*; in the case $\lambda = 1$, we call ϕ a *roughly isometric embedding*. A *quasi-isometry* or *rough isometry* $\phi \colon X \to Y$ is a quasi-isometric or roughly isometric embedding, respectively, such that Y is within finite distance from the image $\phi(X)$.

Lemma 2.5. *If Y is an (n, δ)-hyperbolic space and $f \colon X \to Y$ is a $(1, \varepsilon)$-quasi-isometric embedding, then X is $(n, \delta + 2(n+1)\varepsilon)$-hyperbolic. In particular, $(n, *)$-hyperbolicity is preserved under rough isometries.*

Proof. Let $x_i \in X$ for $i \in I = I_n$. Then

$$\sum_{i \in I} d(x_i, x_{-i}) \leq \sum_{i \in I} d\big(f(x_i), f(x_{-i})\big) + 2(n+1)\varepsilon$$

$$\leq \sum_{i \in I} d\big(f(x_i), f(x_{\alpha(i)})\big) + 2\delta + 2(n+1)\varepsilon$$

$$\leq \sum_{i \in I} d\big(x_i, x_{\alpha(i)}\big) + 2\delta + 4(n+1)\varepsilon$$

for some permutation $\alpha \neq -\mathrm{id}$ of I. ∎

Evidently every $(n, *)$-hyperbolic normed space (and, more generally, every metric space admitting dilations) is in fact $(n, 0)$-hyperbolic. The following classification follows from some well-known results about injective hulls of normed spaces together with Dress' characterization of the combinatorial dimension, but can also be proved more directly.

Proposition 2.6. *A normed space is $(n, 0)$-hyperbolic if and only if it is finite-dimensional with a polyhedral norm, in which case the minimal such n equals the number of pairs of opposite facets of the unit ball. In particular, every $(n, 0)$-hyperbolic normed space has dimension at most n, and equality occurs if and only if it is isometric to l_∞^n.*

Proof. Let $(X, \|\cdot\|)$ be a finite-dimensional normed space with a polyhedral norm, such that the unit ball B has n pairs $\pm F_1, \ldots, \pm F_n$ of opposite facets. For each of these pairs, let $f_i \in X^*$ be the linear functional that is ± 1 on $\pm F_i$. Then

$$f = (f_1, \ldots, f_n) \colon X \to l_\infty^n$$

is a linear isometric embedding, and since l_∞^n is $(n, 0)$-hyperbolic by Proposition 2.4, so is X. To see that n is minimal with this property, choose a relatively interior point x_i in each F_i, and put $x_{-i} := -x_i$. This gives a set $\{x_i : i \in I_{n-1}\} \subset \partial B$ of cardinality $2n$ such that no two distinct elements are connected by a line segment in ∂B. Then

$$\|x_i - x_j\| = 2\left\|\frac{1}{2}(x_i + x_{-j})\right\| < 2$$

whenever $j \neq -i$, and so

$$\sum_{i \in I_{n-1}} \|x_i - x_{\alpha(i)}\| < \sum_{i \in I_{n-1}} \|x_i - x_{-i}\|$$

for every permutation $\alpha \neq -\mathrm{id}$ of I_{n-1}. Thus X is not $(n-1, 0)$-hyperbolic. Clearly, n is greater than or equal to the dimension of X, with equality if and only if B is a parallelotope and X is isometric to l_∞^n via the above f.

Suppose now that the unit ball B is not polyhedral, whereas, without loss of generality, X is still finite-dimensional. Choose a convex set $C_1 \subset \partial B$ that is maximal with respect to inclusion (a singleton if B is strictly convex), and a point x_1 in the interior C_1^0 relative to the affine hull of C_1. Then no point in $\partial B \setminus C_1$ is connected to x_1 by a line segment in ∂B. Recursively, for $n \geq 2$, if C_{n-1} and $x_{n-1} \in C_{n-1}^0$ are chosen, pick a maximal convex set $C_n \subset \partial B$ such that C_n^0 is disjoint from $D_{n-1} = \bigcup_{k=1}^{n-1}(C_k \cup -C_k)$, and a point $x_n \in C_n^0$. Note that $D_{n-1} \neq \emptyset$ for all n, because B is not polyhedral. Thus, for arbitrarily large n, we find a set $\{\pm x_1, \ldots, \pm x_n\} \subset \partial B$ such that no two distinct elements are connected by a line segment in ∂B, and it follows as above that X is not $(n-1, 0)$-hyperbolic. ∎

Next we relate $(n, *)$-hyperbolicity to the notion of asymptotic rank, which originates from [20] and was further discussed in [10, 29, 37].

Given a sequence $(X_k)_{k \in \mathbb{N}}$ of metric spaces $X_k = (X_k, d_k)$, we call a compact metric space $Z = (Z, d_Z)$ an *asymptotic subset* of (X_k) if there exist a sequence $0 < r_k \to \infty$ and subsets $Z_k \subset X_k$ such that the rescaled sets $(Z_k, r_k^{-1} d_k)$ converge in

the Gromov–Hausdorff topology to Z; equivalently, there exist sequences of positive numbers $r_k \to \infty$, $\varepsilon_k \to 0$, and $(1, \varepsilon_k)$-quasi-isometric embeddings

$$\phi_k \colon Z \to (X_k, r_k^{-1} d_k).$$

Every asymptotic subset admits an isometric embedding into some asymptotic cone X_ω of (X_k) with the same scale factors (where ω is any non-principal ultrafilter on $\mathbb{N}$) and, conversely, every compact subset of an asymptotic cone X_ω of (X_k) is an asymptotic subset of some subsequence of (X_k) (see [16, Section 10.6] for a discussion of asymptotic cones). We define the *asymptotic rank* of the sequence (X_k) as the supremum of all integers $m \geq 0$ for which there exists an asymptotic subset of (X_k) bi-Lipschitz homeomorphic to a compact subset of $\mathbb{R}^m$ with positive Lebesgue measure. It can be shown by a metric differentiation argument that if such an asymptotic subset exists, then there is a norm on $\mathbb{R}^m$ whose unit ball is an asymptotic subset of some subsequence of (X_k) (see [37, Corollary 2.2 and Proposition 3.1]). For a single metric space $X = (X, d)$, the *asymptotic rank* $\mathrm{asrk}(X)$ of X is defined as the asymptotic rank of the constant sequence $(X_k, d_k) = (X, d)$.

Proposition 2.7. *Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of (n, δ)-hyperbolic metric spaces $X_k = (X_k, d_k)$. Then every asymptotic subset and every asymptotic cone of (X_k) is $(n, 0)$-hyperbolic, and the asymptotic rank of (X_k) is at most n. In particular, $\mathrm{asrk}(X) \leq n$ for any $(n, *)$-hyperbolic space X.*

Proof. Let Z be an asymptotic subset of (X_k). There are sequences $r_k \to \infty$ and $\varepsilon_k \to 0$ such that, for every k, there exists a $(1, \varepsilon_k)$-quasi-isometric embedding of Z into the $(n, r_k^{-1} \delta)$-hyperbolic space $(X_k, r_k^{-1} d_k)$. Thus Z is $(n, r_k^{-1} \delta + 2(n + 1)\varepsilon_k)$-hyperbolic for all k (Lemma 2.5) and hence $(n, 0)$-hyperbolic (Lemma 2.3).

If X_ω is an asymptotic cone of (X_k), then every finite set $Z \subset X_\omega$ is an asymptotic subset of some subsequence of (X_k), hence Z is $(n, 0)$-hyperbolic, and so is X_ω.

For the assertions about the asymptotic rank, suppose that Z is an asymptotic subset of (X_k) bi-Lipschitz homeomorphic to a compact subset of $\mathbb{R}^m$ with positive Lebesgue measure. Then, as mentioned above, there is a norm $\| \cdot \|$ on $\mathbb{R}^m$ whose unit ball B is an asymptotic subset of some subsequence of (X_k). By the first part of the proof, B is $(n, 0)$-hyperbolic, and so $m \leq n$ by Proposition 2.6. This shows that the asymptotic rank of (X_k) is at most n. ∎

3. Injective hulls and combinatorial dimension

In this section we first review the definitions of the injective hull and the combinatorial dimension, then we state some auxiliary results.

Recall that a metric space (Y, d) is *injective* if partially defined 1-Lipschitz maps into Y can always be extended while preserving the Lipschitz constant. The most basic examples of injective metric spaces are $\mathbb{R}$, $l_\infty(S)$ for any index set S, complete $\mathbb{R}$-trees, and l_∞-products thereof. Injective metric spaces are complete, geodesic, contractible, and share some more properties with spaces of non-positive curvature (see [32]). Isbell [28] showed that every metric space has an *injective hull* $(e, E(X))$, that is, $E(X)$ is an injective metric space, $e \colon X \to E(X)$ is an isometric embedding, and for every isometric embedding of X into some injective metric space Y there is an isometric embedding $E(X) \to Y$ such that the diagram

$$X \xrightarrow{\ e\ } E(X)$$
$$\searrow \qquad \downarrow$$
$$Y$$

commutes. If (i, Y) is another injective hull of X, then there exists a unique isometry $j \colon E(X) \to Y$ such that $j \circ e = i$. Isbell's construction was rediscovered and further investigated by Dress [13] who called $E(X)$ the *tight span* T_X of X. We briefly review the explicit construction of $E(X)$.

Let $\Delta(X)$ denote the set of all functions $f \colon X \to \mathbb{R}$ satisfying

$$f(x) + f(y) \geq d(x, y)$$

for all $x, y \in X$ (in [13], $\Delta(X)$ is denoted P_X). The subset $E(X) \subset \Delta(X)$ of *extremal functions* (in the terminology of [28]) consists of all minimal elements of the partially ordered set $(\Delta(X), \leq)$. If $f \in \Delta(X)$, then

$$f^*(x) := \sup_{z \in X} \big(d(x, z) - f(z)\big) \leq f(x) \tag{3.1}$$

for all $x \in X$, and $q(f) := \frac{1}{2}(f + f^*) \leq f$ belongs to $\Delta(X)$. Hence, if $f \in E(X)$, then $f^* = f$, and, conversely, every function $f \colon X \to \mathbb{R}$ with $f^* = f$ is extremal. Moreover, by iterating the transformation $q \colon \Delta(X) \to \Delta(X)$ and by passing to point-wise limits, one obtains a canonical map

$$p \colon \Delta(X) \to E(X)$$

such that $p(f) \leq f$ for all $f \in \Delta(X)$ and $p(f \circ \gamma) = p(f) \circ \gamma$ for all isometries γ of X (see [13, p. 332] or [32, Proposition 3.1]). For every $y \in X$, the distance function $d_y := d(\cdot, y)$ is extremal. By plugging the inequality $d(x, z) \leq d(x, y) + d(y, z)$ into the definition in (3.1) one sees that f^* is 1-Lipschitz, thus every $f \in E(X)$ is 1-Lipschitz and satisfies $d_y - f(y) \leq f \leq d_y + f(y)$ and

$$\|f - d_y\|_\infty = \sup_{x \in X} |f(x) - d_y(x)| = f(y) \tag{3.2}$$

for all $y \in X$. It follows that $\|f - g\|_\infty < \infty$ for every pair of functions $f, g \in E(X)$. This provides $E(X)$ with a metric, and the map

$$e \colon X \to E(X), \quad x \mapsto d_x,$$

is a canonical isometric embedding. The retraction $p \colon \Delta(X) \to E(X)$ is 1-Lipschitz with respect to the (possibly infinite) l_∞-distance on $\Delta(X)$. We refer to [28] and [32] for two different proofs that $(e, E(X))$ is indeed an injective hull of X.

Let $f, g \in E(X)$. It follows from (3.2) and the triangle inequality that

$$f(x) + \|f - g\|_\infty + g(y) \geq \|d_x - d_y\|_\infty = d(x, y) \tag{3.3}$$

for all $x, y \in X$. The next lemma shows that the points x, y can be chosen such that equality holds up to an arbitrarily small error (compare [13, Theorem 3 (iii)]). In particular, if X is compact, then there exists a pair x, y such that f, g lie on a geodesic from d_x to d_y.

Lemma 3.1. *For $f, g \in E(X)$ and $\varepsilon > 0$, there exist $x, y \in X$ such that*

$$f(x) + \|f - g\|_\infty + g(y) < d(x, y) + \varepsilon.$$

Proof. Note that $\|f - g\|_\infty$ is either less than $f(y) - g(y) + \frac{\varepsilon}{2}$ for some $y \in X$ or less than $g(x) - f(x) + \frac{\varepsilon}{2}$ for some $x \in X$. Furthermore, in the first case there is an $x \in X$ such that $f(y) < d(x, y) - f(x) + \frac{\varepsilon}{2}$, and in the second case there is a $y \in X$ such that $g(x) < d(x, y) - g(y) + \frac{\varepsilon}{2}$. In either case, this gives the desired inequality. ∎

We remark further that if X is compact, then so is $E(X)$, as a direct consequence of the Arzelà–Ascoli theorem.

Suppose now, for the moment, that X is finite. Then $E(X)$ is a subcomplex of the boundary of the unbounded polyhedral set $\Delta(X) \subset \mathbb{R}^X$. The faces of $\Delta(X)$ that belong to $E(X)$ are exactly those whose affine hull is determined by a system of equations of the form

$$f(x_i) + f(x_j) = d(x_i, x_j)$$

such that every point of X occurs at least once in the system. These are precisely the bounded faces of $\Delta(X)$. Given $f \in E(X)$, there is a unique minimal face P containing f in its relative interior. The dimension of P can be read off from the *equality graph* of f on the set X with edges $\{x_i, x_j\}$ corresponding to the above equations: f is uniquely determined on all connected components with a cycle of odd length or a loop $\{x, x\}$ (occurring only if $f = d_x$), whereas f has one degree of freedom on each of the remaining components. Thus $\dim(P) = n$ is the number of bipartite connected components. If $x_1, \ldots, x_n \in X$ are such that there is one x_i in each

of them, then the map sending $g \in P$ to $(g(x_1), \ldots, g(x_n))$ is an isometry from P onto a polytope in l_∞^n (see [32, Lemma 4.1 and Theorem 4.3]). In particular, for finite X, $E(X)$ has the structure of a finite polyhedral complex of dimension at most $\frac{1}{2}|X|$ with l_∞-metrics on the cells. See [14, p. 93] for the possible shapes of $E(X)$ for generic metric spaces up to cardinality 5.

The *combinatorial dimension* $\dim_{\mathrm{comb}}(X)$ of a metric space X equals the supremum of $\dim(E(V))$ over all finite subsets $V \subset X$. This notion was introduced by Dress. In particular, [13, Theorem 9′, p. 380] provides a variety of characterizations, whereas Theorem 9 in the introduction of loc. cit. highlights the equivalence of X being $(n, 0)$-hyperbolic, in our terminology, and the inequality $\dim_{\mathrm{comb}}(X) \leq n$. This equivalence will also follow from the results in the next section in combination with the following characterization (see Corollary 4.4).

Proposition 3.2. *For every metric space X and $n \geq 1$, the following are equivalent:*

(1) $\dim_{\mathrm{comb}}(X) \geq n$;

(2) $E(X)$ *contains an isometric copy of a non-empty open subset of* l_∞^n;

(3) $E(X)$ *contains an isometric copy of* $\{\pm s e_i : i = 1, \ldots, n\} \subset l_\infty^n = (\mathbb{R}^n, \|\cdot\|_\infty)$ *for some $s > 0$ (where $e_1, \ldots, e_n$ denote the canonical basis vectors of $\mathbb{R}^n$ as usual).*

Proof. If (1) holds, then there is a finite set $V \subset X$ with $\dim(E(V)) \geq n$, and $E(V)$ embeds isometrically into $E(X)$, thus $E(X)$ contains a copy of an n-dimensional cell of $E(V)$, and (2) follows. Evidently (2) implies (3).

We show that (3) implies (1). Let $I := \{\pm 1, \ldots, \pm n\}$, and let $\{f_i : i \in I\}$ be a subset of $E(X)$ isometric to $\{\pm s e_i : i = 1, \ldots, n\} \subset l_\infty^n$ for some $s > 0$, so that $\|f_i - f_{-i}\|_\infty = 2s$ for all $i \in I$, and $\|f_i - f_j\|_\infty = s$ whenever $j \in I \setminus \{i, -i\}$. Fix an $\varepsilon \in (0, \frac{s}{2})$. Lemma 3.1, together with (3.3), shows that for every $i \in \{1, \ldots, n\}$ there exist $x_i, x_{-i} \in X$ such that

$$d(x_i, x_{-i}) \leq f_i(x_i) + 2s + f_{-i}(x_{-i}) < d(x_i, x_{-i}) + \varepsilon.$$

Note that $d(x_i, x_{-i}) > 2s - \varepsilon > 0$. Furthermore, if $j \in I \setminus \{i, -i\}$, then

$$d(x_i, x_j) \leq f_i(x_i) + s + f_j(x_j);$$

assuming without loss of generality that $f_i(x_i) \geq f_j(x_j)$, we infer that

$$\begin{aligned}
d(x_i, x_j) &\geq d(x_i, x_{-i}) - d(x_j, x_{-i}) \\
&> \big(f_i(x_i) + 2s + f_{-i}(x_{-i}) - \varepsilon\big) - \big(f_j(x_j) + s + f_{-i}(x_{-i})\big) \\
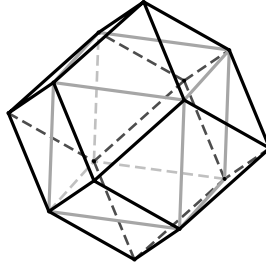&\geq s - \varepsilon > 0.
\end{aligned}$$

Figure 2. The injective hull of $\{\pm se_i\} \subset l_\infty^3$ is a rhombic dodecahedron whose extra vertices span a cube of edge length s.

Thus the set $V := \{x_i : i \in I\}$ has cardinality $2n$. By putting $h(x_i) := f_i(x_i) + s$ for all $i \in I$, we get a function $h \in \Delta(V)$. Let $g := p(h) \in E(V)$, and recall that $g \leq h$. For every $i \in I$, we have

$$h(x_i) + h(x_{-i}) - \varepsilon < d(x_i, x_{-i}) \leq g(x_i) + g(x_{-i}),$$

hence $g(x_i) > h(x_i) - \varepsilon$, and if $j \in I \setminus \{i, -i\}$, then

$$g(x_i) + g(x_j) > h(x_i) + h(x_j) - 2\varepsilon \geq d(x_i, x_j) + s - 2\varepsilon > d(x_i, x_j)$$

by the choice of ε. Since $g \in E(V)$, it follows that $g(x_i) + g(x_{-i}) = d(x_i, x_{-i})$ for $i = 1, \ldots, n$. Thus the equality graph of g on V has just n pairwise disjoint edges, hence n bipartite components, and so the minimal cell of $E(V)$ containing g has dimension n, as discussed earlier. Since $\dim(E(V)) \leq \frac{1}{2}|v| = n$, equality holds, and (1) follows. ∎

We conclude this section with a quantitative version of the above implication $(3) \Rightarrow (2)$. If an injective metric space Y contains an isometric copy of a set $\{\pm se_i\} \subset l_\infty^n$ as in (3), then Y also contains an isometric copy of the injective hull $E(\{\pm se_i\})$. The latter is isometric to a convex polytope in l_∞^n which in turn contains the ball $B(0, \frac{s}{2}) = [-\frac{s}{2}, \frac{s}{2}]^n \subset l_\infty^n$. In case $n = 3$, this polytope is the rhombic dodecahedron shown in Figure 2 (compare [18]).

For our purposes, the following result will suffice.

Lemma 3.3. *Let Y be an injective metric space. If Y contains an isometric copy of $\{\pm se_i\} \subset l_\infty^n$ for some $s > 0$ and $n \geq 1$, then there is also an isometric embedding of the ball $B(0, \frac{s}{2}) = [-\frac{s}{2}, \frac{s}{2}]^n \subset l_\infty^n$ into Y.*

Proof. Since Y is injective, every isometric embedding $\phi: \{\pm se_i\} \to Y$ extends to a 1-Lipschitz map $\phi: \{\pm se_i\} \cup l[-\frac{s}{2}, \frac{s}{2}]^n \to Y$. Let $x, y \in [-\frac{s}{2}, \frac{s}{2}]^n$. After possibly

interchanging x and y, we have $\|x - y\|_\infty = x_i - y_i$ for some i. Then

$$
\begin{aligned}
d\big(\phi(se_i), \phi(-se_i)\big) &\le d\big(\phi(se_i), \phi(x)\big) + d\big(\phi(x), \phi(y)\big) + d\big(\phi(y), \phi(-se_i)\big) \\
&\le \|se_i - x\|_\infty + \|x - y\|_\infty + \|y + se_i\|_\infty \\
&= (s - x_i) + (x_i - y_i) + (y_i + s),
\end{aligned}
$$

and since $d(\phi(se_i), \phi(-se_i)) = 2s$, the equality $d(\phi(x), \phi(y)) = \|x - y\|_\infty$ holds. $\blacksquare$

4. The injective hull of an (n, δ)-hyperbolic space

We now relate (n, δ)-hyperbolicity to injective hulls. The following proposition generalizes the known result for $n = 1$ (compare [15, Section 4.4], [14, Chapter 5], or [32, Proposition 1.3]).

Proposition 4.1. *A metric space X is (n, δ)-hyperbolic if and only if its injective hull $E(X)$ is (n, δ)-hyperbolic.*

Proof. If $E(X)$ is (n, δ)-hyperbolic, then so is $e(X) \subset E(X)$ and hence X.

Conversely, suppose that X is (n, δ)-hyperbolic, and let $f_i \in E(X)$ for $i \in I = I_n$. Fix $\varepsilon > 0$ for the moment. Lemma 3.1 shows that for every $i \in \{1, \ldots, n + 1\}$ there exist $x_i, x_{-i} \in X$ such that

$$
\|f_i - f_{-i}\|_\infty < d(x_i, x_{-i}) - f_i(x_i) - f_{-i}(x_{-i}) + \varepsilon.
$$

Setting $S := \frac{1}{2} \sum_{i \in I} (f_i(x_i) + f_{-i}(x_{-i})) = \sum_{i \in I} f_i(x_i)$, we get that

$$
\sum_{i \in I} \|f_i - f_{-i}\|_\infty < \sum_{i \in I} d(x_i, x_{-i}) - 2S + 2(n + 1)\varepsilon.
$$

By the assumption on X there exists a permutation $\alpha \ne -\mathrm{id}$ of I with

$$
\sum_{i \in I} d(x_i, x_{-i}) \le \sum_{i \in I} d(x_i, x_{\alpha(i)}) + 2\delta.
$$

Using the inequalities $d(x_i, x_{\alpha(i)}) \le f_i(x_i) + f_i(x_{\alpha(i)})$ we deduce that

$$
\begin{aligned}
\sum_{i \in I} \|f_i - f_{-i}\|_\infty &< \sum_{i \in I} \big(f_i(x_i) + f_i(x_{\alpha(i)})\big) - 2S + 2\delta + 2(n + 1)\varepsilon \\
&= \sum_{i \in I} \big(f_i(x_{\alpha(i)}) - f_{\alpha(i)}(x_{\alpha(i)})\big) + 2\delta + 2(n + 1)\varepsilon \\
&\le \sum_{i \in I} \|f_i - f_{\alpha(i)}\|_\infty + 2\delta + 2(n + 1)\varepsilon.
\end{aligned}
$$

Thus $E(X)$ is $(n, \delta + (n+1)\varepsilon)$-hyperbolic for all $\varepsilon > 0$, and hence (n, δ)-hyperbolic (Lemma 2.3). ∎

Our next goal is to show that a metric space X is $(n, *)$-hyperbolic if and only if, intuitively, its injective hull has no large $(n+1)$-dimensional subsets. To measure the size, we will use the sets $\{\pm se_i : i = 1, \ldots, n+1\} \subset l_\infty^{n+1}$ for $s > 0$ (compare Lemma 3.3). In preparation for the actual result, Proposition 4.3 below, we state the following criterion.

Lemma 4.2. *Let $n \geq 1$, and let $V = \{x_i : i \in I = I_n\}$ be a metric space of cardinality $2(n+1)$. Let $\mathcal{A}$ denote the set of the $n+1$ pairs $\{x_i, x_{-i}\}$, and let $\mathcal{A}^c$ denote the set of all pairs $\{x, y\} \notin \mathcal{A}$ of two distinct points in V. Suppose that there is a function $f : V \to \mathbb{R}$ such that $f(x_i) + f(x_{-i}) = d(x_i, x_{-i})$ for all pairs $\{x_i, x_{-i}\} \in \mathcal{A}$ and*

$$s := \min\{f(x) + f(y) - d(x, y) : \{x, y\} \in \mathcal{A}^c\} > 0.$$

Then $f \in E(V)$, and there is an isometric embedding of $\{0\} \cup \{\pm se_i\} \subset l_\infty^{n+1}$ into $E(V)$ mapping 0 to f.

Proof. To show that $f \in E(V)$ it only remains to verify that $f \geq 0$. For every $y \in V$ there is an edge $\{x_i, x_{-i}\} \in \mathcal{A}$ not containing y, so that $\{x_i, y\}, \{x_{-i}, y\} \in \mathcal{A}^c$, hence

$$f(x_i) + f(x_{-i}) = d(x_i, x_{-i})$$
$$\leq d(x_i, y) + d(x_{-i}, y)$$
$$\leq f(x_i) + f(y) - s + f(x_{-i}) + f(y) - s,$$

and therefore $f(y) \geq s > 0$. Now, for every $i \in I$, we define $f_i : V \to \mathbb{R}$ such that $f_i(x_{\pm i}) = f(x_{\pm i}) \pm s$ and $f_i(x_j) = f(x_j)$ for all $j \in I \setminus \{i, -i\}$. Then

$$f_i(x_j) + f_i(x_{-j}) = d(x_j, x_{-j})$$

for all $\{x_j, x_{-j}\} \in \mathcal{A}$, and

$$f_i(x) + f_i(y) \geq d(x, y)$$

whenever $x, y \in V$; thus $f_i \in E(V)$. Note that $\|f - f_i\|_\infty = s$ and $\|f_i - f_{-i}\|_\infty = 2s$ for all $i \in I$, and $\|f_i - f_j\|_\infty = s$ whenever $j \notin \{i, -i\}$. This yields an isometric embedding, as desired. ∎

We now have the following key result.

Proposition 4.3. *Let $n \geq 1$. If X is (n, δ)-hyperbolic, then the injective hull $E(X)$ contains no isometric copy of $\{\pm se_i\} \subset l_\infty^{n+1}$ for $s > \delta$. Conversely, if $E(X)$ contains no isometric copy of $\{\pm se_i\} \subset l_\infty^{n+1}$ for $s > \delta$, then X is $(n, n\delta)$-hyperbolic.*

For $n = 1$, this reduces to the well-known fact that X is δ-hyperbolic if and only if $E(X)$ contains no isometric copy of $\{\pm se_i\} \subset l_\infty^2$, or, equivalently, of $[0, s]^2 \subset \ell_1^2$, for $s > \delta$ (compare [13, p. 335f.], the introduction in [2], and the discussion at the end of [12, Section 3]).

Proof. Suppose that X is (n, δ)-hyperbolic, and for some $s > 0$ there is a subset $\{f_i : i \in I = I_n\}$ of $E(X)$ isometric to $\{\pm se_i\} \subset l_\infty^{n+1}$, so that $\|f_i - f_{-i}\|_\infty = 2s$ for all $i \in I$ and $\|f_i - f_j\|_\infty = s$ whenever $j \notin \{i, -i\}$. By Proposition 4.1, $E(X)$ is itself (n, δ)-hyperbolic, thus

$$\sum_{i \in I} \|f_i - f_{-i}\|_\infty \le \sum_{i \in I} \|f_i - f_{\alpha(i)}\|_\infty + 2\delta$$

for some permutation $\alpha \ne -\mathrm{id}$ of I. Note that all summands in the first sum are equal to the maximal distance $2s$, whereas at least two terms in the second sum are $\le s$. It follows that $s \le \delta$.

We prove the second part. If $|X| < 2(n + 1)$, then X is $(n, 0)$-hyperbolic. Suppose now that $V = \{x_i : i \in I = I_n\} \subset X$ is a set of cardinality $2(n + 1)$. Define $\mathcal{A}$ and $\mathcal{A}^c$ as in Lemma 4.2, and consider the set $\mathcal{F}$ of all functions $f : V \to \mathbb{R}$ such that

$$f(x_i) + f(x_{-i}) = d(x_i, x_{-i})$$

for all $\{x_i, x_{-i}\} \in \mathcal{A}$. For $f \in \mathcal{F}$, put

$$s_f := \min\{f(x) + f(y) - d(x, y) : \{x, y\} \in \mathcal{A}^c\},$$
$$\mathcal{B}_f := \{\{x, y\} \in \mathcal{A}^c : f(x) + f(y) - d(x, y) = s_f\}.$$

Note that $s_f \le \mathrm{diam}(V)$, because there is a pair $\{x_i, x_j\} \in \mathcal{A}^c$ such that $f(x_i) \le \frac{1}{2}d(x_i, x_{-i})$ and $f(x_j) \le \frac{1}{2}d(x_j, x_{-j})$. We now fix $f \in \mathcal{F}$ for the rest of the proof such that

$$s_f = \bar{s} := \sup\{s_g : g \in \mathcal{F}\}$$

and $|\mathcal{B}_f| \le |\mathcal{B}_g|$ for all $g \in \mathcal{F}$ with $s_g = \bar{s}$. The elements of $\mathcal{A}$ and $\mathcal{B} := \mathcal{B}_f$ will be called $\mathcal{A}$-edges and $\mathcal{B}$-edges. We claim that for every $\mathcal{A}$-edge, either both vertices belong also to a $\mathcal{B}$-edge, or neither of the two vertices has this property. Suppose to the contrary that for some index $i \in I$, the point x_i is in $\bigcup \mathcal{B}$, whereas x_{-i} is not. Then, for some sufficiently small $\varepsilon > 0$, the function $g \in \mathcal{F}$ defined by $g(x_{\pm i}) := f(x_{\pm i}) \pm \varepsilon$ and $g(y) := f(y)$ otherwise would satisfy $s_g = \bar{s}$ and $\mathcal{B}_g \subsetneq \mathcal{B} = \mathcal{B}_f$, in contradiction to the choice of f. Since $\mathcal{B} \ne \emptyset$, it follows from this claim that there is a non-empty connected subgraph of $(V, \mathcal{A} \cup \mathcal{B})$ such that each of its vertices belongs to a unique $\mathcal{A}$-edge and at least one $\mathcal{B}$-edge. Among all such subgraphs we select one with the least number of edges and call it G. There are two possible types, as described next.
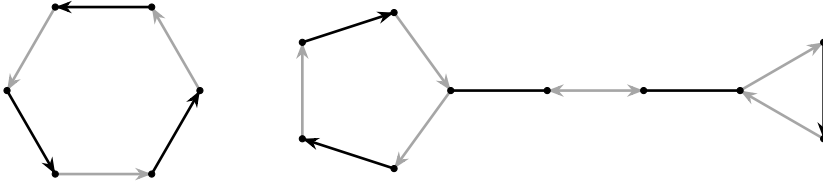
Figure 3. The two possible types of the (undirected) graph G, with $\mathcal{A}$-edges shown in black, $\mathcal{B}$-edges in gray. The arrows indicate the effect of the permutation $x_i \mapsto x_{\alpha(i)}$.

The first possibility is that G is simply a cycle graph with an even number of edges alternating between $\mathcal{A}$ and $\mathcal{B}$. In this case we choose an orientation of G and define the permutation $\alpha\colon I \to I$ such that the map $x_i \mapsto x_{\alpha(i)}$ sends each vertex of G to the following one and every other point x_i in V to x_{-i}. In the remaining case, when G is not an alternating cycle, by minimality G contains no such cycle as a proper subgraph either. Then, starting with an oriented $\mathcal{A}$-edge of G, we follow an alternating path in G, stopping at the first vertex v that was visited already earlier. Since the $\mathcal{A}$-edges are pairwise disjoint, the last edge belongs to $\mathcal{B}$. As there is no alternating cycle, by deleting the initial subpath up to the first occurrence of v we get an alternating loop based at v that starts and ends with a $\mathcal{B}$-edge. Proceeding with the oriented $\mathcal{A}$-edge issuing from v, we choose another alternating path, ending at the first vertex w occurring already earlier in the whole construction. Again, the last edge is in $\mathcal{B}$, and since there is no alternating cycle it follows that w cannot be part of the loop based at v. We conclude that in the second case, G consists of two disjoint alternating loops based at v and w, respectively, each starting and ending with a $\mathcal{B}$-edge, and an alternating path from v to w (possibly of length one), starting and ending with an $\mathcal{A}$-edge. Then we define the permutation $\alpha\colon I \to I$ such that the map $x_i \mapsto x_{\alpha(i)}$ cyclically permutes each of the two loops, moving every vertex to the next one, and interchanges the two vertices of every $\mathcal{B}$-edge in the path from v to w. Furthermore, as in the first case, $x_i \mapsto x_{\alpha(i)} = x_{-i}$ on the remaining part of V. See Figure 3.

Now, in either case,

$$\sum_{i \in I} d(x_i, x_{-i}) = \sum_{i \in I} \big(f(x_i) + f(x_{-i}) \big)$$

$$= \sum_{i \in I} \big(f(x_i) + f(x_{\alpha(i)}) \big)$$

$$= \sum_{i \in I} d(x_i, x_{\alpha(i)}) + k\overline{s},$$

where k is the number of indices $i \in I$ corresponding to a $\mathcal{B}$-edge $\{x_i, x_{\alpha(i)}\}$ of G. We conclude that if $\overline{s} \leq 0$, then X is $(n, 0)$-hyperbolic. It remains to consider the case $\overline{s} > 0$. Note that if G is of the first type, then the cycle G has at most $|V| = 2(n + 1)$ vertices, so $k \leq n + 1$. If G is of the second type, then k equals the total number of $\mathcal{B}$-edges in the two loops plus twice the number of $\mathcal{B}$-edges in the path from v to w. This is equal to the total number of $\mathcal{A}$-edges in the two loops plus twice the number of $\mathcal{A}$-edges in the path from v to w. Since each of the loops contains at least one $\mathcal{A}$-edge, which is counted only once, it follows that $k \leq |V| - 2 = 2n$. By Lemma 4.2, there is an isometric copy of $\{\pm\overline{s}e_i\} \subset l_\infty^{n+1}$ in $E(V)$, and hence also in $E(X)$. Then $\overline{s} \leq \delta$ by assumption, so $k\overline{s} \leq 2n\delta$, and therefore X is $(n, n\delta)$-hyperbolic. ∎

Proposition 4.3 may be viewed as a stable version of [13, Theorem 9, p. 327], which follows as a corollary.

Corollary 4.4. *A metric space X is $(n, 0)$-hyperbolic if and only if $\dim_{comb}(X) \leq n$.*

Proof. Let $n \geq 1$. By Proposition 4.3, X is $(n, 0)$-hyperbolic if and only if $E(X)$ contains no isometric copy of $\{\pm se_i\} \subset l_\infty^{n+1}$ for $s > 0$, and by Proposition 3.2 this holds if and only if $\dim_{comb}(X) \leq n$. ∎

We now prove Theorem 1.2 stated in the introduction, which subsumes some of the results obtained so far.

Proof of Theorem 1.2. If X is (n, δ)-hyperbolic, then $E(X)$ is (n, δ)-hyperbolic (Proposition 4.1), and has therefore asymptotic rank $\leq n$ (Proposition 2.7). Evidently (3) implies (4). Lastly, suppose that (4) holds. If $n = 0$, then $E(X)$ and X are bounded and so X is $(0, *)$-hyperbolic. If $n \geq 1$, then $E(X)$ contains no isometric copy of $\{\pm se_i\} \subset l_\infty^{n+1}$ for $s > 2r_0$ (Lemma 3.3), and hence X is $(n, 2r_0 n)$-hyperbolic (Proposition 4.3). ∎

5. The slim simplex property and coarse injectivity

We turn to the slim simplex property (SS_n) stated in the introduction. By a *Euclidean $(n + 1)$-simplex* Δ we mean the convex hull of $n + 2$ points in $\mathbb{R}^{n+1}$ such that the interior of Δ is non-empty, and a *facet* of Δ is the convex hull of $n + 1$ of these vertices. We restate Theorem 1.3 in a slightly stronger form. The proof uses Proposition 4.1 and Proposition 2.7 to apply a result from [17], which shows that (SS_n) holds uniformly for certain classes of proper metric spaces.

Theorem 5.1. *Let X be an (n, δ)-hyperbolic metric space. Let Δ be a Euclidean $(n + 1)$-simplex, and let $\phi: \partial\Delta \to X$ be a map such that for some constants $\lambda \geq 1$*

and $c \geq 0$, the restriction of ϕ to each facet of Δ is a (λ, c)-quasi-isometric embedding. Then, for every facet F, the image $\phi(F)$ is contained in the closed $(1 + c)D$-neighborhood of $\phi(\overline{\partial\Delta \setminus F})$ for some constant D depending only on n, δ, λ (and not on X).

Proof. If $n = 0$, then $\text{diam}(X) \leq \delta$, and the result holds. Let now $n \geq 1$.

We consider X as a subset of its injective hull $E(X)$ and write d also for the metric of $E(X)$. First we approximate $\phi \colon \partial\Delta \to X \subset E(X)$ by a piecewise Lipschitz map as follows. Let β denote the induced inner metric on $\partial\Delta$. Since every shortest curve connecting two points in $\partial\Delta$ meets each of the $n + 2$ facets in at most one (possibly degenerate) subsegment, ϕ is $(\lambda, (n + 2)c)$-Lipschitz with respect to β. Let $Z \subset \partial\Delta$ be a maximal set subject to the condition that $\beta(z, z') \geq (n + 2)c\lambda^{-1}$ whenever $z, z' \in Z$ are distinct. For any such z, z',

$$d\big(\phi(z), \phi(z')\big) \leq \lambda\, \beta(z, z') + (n + 2)c \leq 2\lambda\, \beta(z, z').$$

Since $E(X)$ is injective, $\phi|_Z$ extends to a 2λ-Lipschitz map $\phi' \colon \partial\Delta \to E(X)$ with respect to β. For every $x \in \partial\Delta$ there exists a $z \in Z$ with $\beta(x, z) \leq (n + 2)c\lambda^{-1}$, hence

$$\begin{aligned} d\big(\phi'(x), \phi(x)\big) &\leq d\big(\phi'(x), \phi'(z)\big) + d\big(\phi(z), \phi(x)\big) \\ &\leq 2\lambda\, \beta(x, z) + \lambda\, \beta(x, z) + (n + 2)c \\ &\leq 4(n + 2)c. \end{aligned}$$

Furthermore, if x, y are two points in the same facet, then

$$d\big(\phi'(x), \phi'(y)\big) \geq d\big(\phi(x), \phi(y)\big) - 8(n + 2)c \geq \lambda^{-1}\|x - y\| - c'$$

for $c' := (8n + 17)c$, and $d(\phi'(x), \phi'(y)) \leq 2\lambda\, \beta(x, y) = 2\lambda\, \|x - y\|$.

By Proposition 4.1, $E(X)$ is (n, δ)-hyperbolic. Since $\phi'(\partial\Delta) \subset E(X)$ is compact, so is its injective hull, and hence there exists a compact injective subspace $Y \subset E(X)$ containing $\phi'(\partial\Delta)$. We now apply [17, Theorem 7.2] for the class $\mathcal{X}$ of all compact, injective, (n, δ)-hyperbolic spaces Y (see also the concluding remark in its proof for a simplification). There are two assumptions on the class $\mathcal{X}$ of metric spaces in this theorem. The first is that all members of $\mathcal{X}$ satisfy certain coning inequalities in dimensions $\leq n$ with a uniform constant. Since every injective metric space has a conical geodesic bicombing, this holds with constant 1 (see [32, Proposition 3.8], [36, Proposition 2.10], and [30, Section 2.7]). The second assumption is that every sequence $(Y_k)_{k \in \mathbb{N}}$ in $\mathcal{X}$ has asymptotic rank at most n, which is satisfied by Proposition 2.7. The conclusion is that there is a constant $D' = D'(\mathcal{X}, n, \lambda)$, hence depending only on n, δ, λ, such that for every facet F of Δ, the image $\phi'(F)$ is contained in the

closed $(1 + c')D'$-neighborhood of the union of the images of the remaining facets. Since ϕ and ϕ' are uniformly close to each other, this gives the result. ∎

Recall from the introduction that a metric space X is *coarsely injective* if there exists a constant $c \geq 0$ such that every 1-Lipschitz map $\phi: A \to X$ defined on a subset of metric space B has a $(1, c)$-Lipschitz extension $\bar{\phi}: B \to X$. To make the constant explicit, we say that X is *c-coarsely injective*. This property implies, more generally, that every (λ, ε)-Lipschitz map $\phi: A \to X$ on $A \subset B$ has a $(\lambda, \varepsilon + c)$-Lipschitz extension $\bar{\phi}: B \to X$, because such a map ϕ is 1-Lipschitz with respect to the metric $d_{\lambda,\varepsilon}$ on B satisfying

$$d_{\lambda,\varepsilon}(b, b') = \lambda \, d(b, b') + \varepsilon$$

for every pair of distinct points $b, b' \in B$.

The following result generalizes the well-known fact that a metric space X is injective if and only if X is hyperconvex (see [1]). We call X *coarsely hyperconvex* if, for some constant $c \geq 0$, whenever $\{(x_s, r_s)\}_{s \in S}$ is a family in $X \times \mathbb{R}$ satisfying

$$r_s + r_t \geq d(x_s, x_t)$$

for all pairs of indices $s, t \in S$, then $\bigcap_{s \in S} B(x_s, r_s + c) \neq \emptyset$. To make the constant explicit, we say that X is *c-coarsely hyperconvex*. For a geodesic metric space X, this can be reformulated as the following *coarse Helly property* (compare [8, Section 3.3]): any family $\{B(x_s, r_s)\}_{s \in S}$ of pairwise intersecting closed balls in X satisfies $\bigcap_{s \in S} B(x_s, r_s + c) \neq \emptyset$.

Proposition 5.2. *For every metric space X, the following are equivalent:*

(1) *X is coarsely injective;*

(2) *X is coarsely hyperconvex;*

(3) *$E(X)$ is within finite distance from the image of the canonical embedding $e: X \to E(X)$;*

(4) *X is roughly isometric to an injective metric space Y.*

In view of (4) it is clear that all of these properties are preserved under rough isometries. As the proof will show, all implications are quantitative.

For a geodesic Gromov hyperbolic space X, (1), (2), and (3) were established individually in [9, 31, 32], respectively. However, the equivalence of these properties was observed only recently; see [8, Proposition 3.12] for (2) and (3).

Proof. To show that (1) implies (2), let $\{(x_s, r_s)\}_{s \in S} \subset X \times \mathbb{R}$ be a family such that $r_s + r_t \geq d(x_s, x_t)$ for all $s, t \in S$. Consider the corresponding set $A := \{x_s : s \in S\}$

and put $r(a) := \inf\{r_s + d(a, x_s) : s \in S\}$ for all $a \in A$. For $a, a' \in A$, the triangle inequalities

$$|r(a) - r(a')| \leq d(a, a') \leq r(a) + r(a')$$

hold, thus there is a metric extension $B := A \cup \{b\}$ of A with $d(a, b) = r(a)$ for all $a \in A$. Now if X is c-coarsely injective, then the inclusion map $A \to X$ extends to a $(1, c)$-Lipschitz map on B, and the image point y of b satisfies

$$d(a, y) \leq d(a, b) + c$$

for all $a \in A$, hence $d(x_s, y) \leq r_s + c$ for all $s \in S$.

We show that (2) implies (3). Suppose that X is c-coarsely hyperconvex, and let $f \in E(X)$. Since $f(x) + f(x') \geq d(x, x')$ for all $x, x' \in X$, it follows that there exists a point $y \in X$ with $d(x, y) \leq f(x) + c$ for all $x \in X$. Since f is extremal,

$$f(y) = \sup_{x \in X} \big(d(x, y) - f(x)\big),$$

thus (by (3.2)) $\|f - d_y\|_\infty = f(y) \leq c$.

It is clear that if (3) holds, then $e : X \to E(X)$ is a rough isometry between X and $E(X)$.

It remains to show that (4) implies (1). Suppose that $i : X \to Y$ is a $(1, \varepsilon)$-Lipschitz map into an injective metric space Y, and $j : Y \to X$ is a $(1, \varepsilon)$-Lipschitz map such that $d(x, j \circ i(x)) \leq \varepsilon$ for all $x \in X$. Let $\phi : A \to X$ be a 1-Lipschitz map defined on $A \subset B$. Then $i \circ \phi : A \to Y$ is $(1, \varepsilon)$-Lipschitz and extends to a $(1, \varepsilon)$-Lipschitz map $\psi : B \to Y$, furthermore $j \circ \psi : B \to X$ is $(1, 2\varepsilon)$-Lipschitz, and

$$d\big(\phi(a), j \circ \psi(a)\big) = d\big(\phi(a), j \circ i(\phi(a))\big) \leq \varepsilon$$

for all $a \in A$. Hence, the map $\bar\phi : B \to X$ defined by $\bar\phi(a) := \phi(a)$ for all $a \in A$ and $\bar\phi(b) := j \circ \psi(b)$ for all $b \in B \setminus A$ is a $(1, 3\varepsilon)$-Lipschitz extension of ϕ. ∎

We now prove our main result regarding coarsely injective spaces.

Proof of Theorem 1.4. Theorem 1.3 shows that (1) implies (2), and Proposition 2.7 shows that (1) implies (3) as well as (4).

We show by contraposition that each of the conditions (2), (3), (4) implies (5). Suppose that there exist an $\varepsilon > 0$ and $(1, \varepsilon)$-quasi-isometric embeddings of $B(0, k) \subset l_\infty^{n+1}$ into X for all integers $k \geq 1$. Then one finds a Euclidean $(n + 1)$-simplex Δ and a sequence of maps $\phi_k : \partial(k\Delta) \to X$ violating (SS_n). Furthermore, the unit ball in l_∞^{n+1} is an asymptotic subset of the constant sequence $X_k = X$, and hence admits an isometric embedding into some asymptotic cone X_ω of X, thus $\mathrm{asrk}(X) \geq n + 1$, and X_ω fails to be $(n, 0)$-hyperbolic.

For the proof of the implication (5) $\Rightarrow$ (1) and the last assertion of the theorem, note that since X is coarsely injective, $E(X)$ is within finite distance from $e(X)$, so there exist a $c > 0$ and a $(1, c)$-quasi-isometric embedding $E(X) \to X$. Hence, if (5) holds, then $E(X)$ cannot contain isometric copies of too large balls in l_∞^{n+1}, and Theorem 1.2 then shows that X is $(n, *)$-hyperbolic. Similarly, if X is proper and cocompact, then there is an n such that every set $V \subset X$ of distinct points at mutual distance $\geq c$ and with diameter $\leq 3c$ has less than 2^{n+1} elements, thus $E(X)$ contains no isometric copy of $\{-c, c\}^{n+1} \subset l_\infty^{n+1}$. Then Theorem 1.2 shows that $\mathrm{asrk}(E(X)) \leq n$, and thus $\mathrm{asrk}(X) \leq n$. ∎

We conclude with the proofs of the three corollaries stated in the introduction.

Proof of Corollary 1.5. Since X is proper and cocompact, n_4 is finite and equal to the maximal integer for which there exist isometric embeddings of $B(0, r) \subset l_\infty^{n_4}$ into X for all $r > 0$. As an injective metric space, X is isometric to $E(X)$, so Theorem 1.2 shows that $n = n_1 = n_4$. Since G is quasi-isometric to X, Theorem 1.3 shows further that $n_2 \leq n_1$, and evidently $n_4 \leq n_3 \leq n_2$.

For the last assertion, suppose to the contrary that G has a free abelian subgroup of rank $n + 1$. By [32, Proposition 3.8], X possesses an equivariant conical geodesic bicombing, and it follows from [12, Proposition 4.4 and Lemma 6.1] that there is an isometric embedding of $\mathbb{Z}^{n+1}$ into X with respect to the metric on $\mathbb{Z}^{n+1}$ induced by some norm on $\mathbb{R}^{n+1}$. Thus there is a quasi-isometric embedding of $\mathbb{R}^{n+1}$ into X. Alternatively, by the Algebraic Flat Torus theorem for semihyperbolic groups stated in [6, p. 475], every monomorphism of $\mathbb{Z}^{n+1}$ into G is a quasi-isometric embedding. ∎

Proof of Corollary 1.6. Since the rank ν of X equals the 'quasi-ball rank' $\bar{\nu}$, for all λ, c there is a radius r_0 such that there is no (λ, c)-quasi-isometric embedding of $B(0, r) \subset \mathbb{R}^{\nu+1}$ into X for $r > r_0$. It follows that property (5) of Theorem 1.4 holds with ν in place of n and with respect to the coarsely injective metric ϱ. Thus (X, ϱ) is $(\nu, *)$-hyperbolic, and X satisfies (SS_n) and has asymptotic rank at most ν with respect to either ϱ or the original metric d. Furthermore, every metric space X with quasi-ball rank $\bar{\nu}$ admits an asymptotic subset bi-Lipschitz homeomorphic to the unit ball in $\mathbb{R}^{\bar{\nu}}$, so $\bar{\nu} \leq \mathrm{asrk}(X)$ (in general the inequality may be strict; for example, $X = \{k^2 : k \in \mathbb{N}\} \subset \mathbb{R}$ satisfies $\bar{\nu} = 0$ and $\mathrm{asrk}(X) = 1$). We conclude that $\nu = \mathrm{asrk}(X)$, and ν is the least integer such that (X, ϱ) is $(\nu, *)$-hyperbolic. ∎

Proof of Corollary 1.7. Every such Finsler metric d is bi-Lipschitz equivalent to the Riemannian metric, so (X, d) has asymptotic rank n and is therefore $(n, *)$-hyperbolic by Theorem 1.4. ∎

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