The quasi-periods of the Weierstrass zeta-function

Mario BONK

Abstract. We study the ratio $p = \eta_1/\eta_2$ of the quasi-periods of the Weierstrass ζ-function in dependence of the ratio $\tau = \omega_1/\omega_2$ of the generators of the underlying rank-2 lattice. We will give an explicit geometric description of the map $\tau \mapsto p(\tau)$. As a consequence, we obtain an explanation of a theorem by Heins who showed that p attains every value in the Riemann sphere infinitely often. Our main result is implicit in the classical literature, but it seems not to be very well known.

Essentially, this is an expository paper. We hope that it is easily accessible and may serve as an introduction to these classical themes.

1. Introduction

Throughout this paper, we assume that $\omega_1, \omega_2 \neq 0$ are two numbers in the complex plane $\mathbb{C}$ that are linearly independent over the field of real numbers $\mathbb{R}$. We define

$$\tau := \omega_1/\omega_2$$

and assume that

$$\mathrm{Im}(\tau) > 0.$$

We also consider the associated rank-2 lattice

$$(1.1) \qquad \Gamma = \{k\omega_1 + n\omega_2 : n, k \in \mathbb{Z}\}$$

generated by ω_1 and ω_2.

The *Weierstrass ζ-function* associated with this lattice Γ is given by the series

$$(1.2) \qquad \zeta(u; \Gamma) = \frac{1}{u} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{u - \gamma} + \frac{1}{\gamma} + \frac{u}{\gamma^2} \right).$$

We simply write $\zeta(u)$ if the underlying lattice Γ is understood. It is well known that the series (1.2) converges absolutely and locally uniformly for $u \in \mathbb{C} \setminus \Gamma$. Moreover, ζ is an odd meromorphic function with poles of first order precisely at the lattice points. It has the periodicity property

$$(1.3) \qquad\qquad \zeta(u + \omega_k) = \zeta(u) + \eta_k$$

for some constants $\eta_k \in \mathbb{C}$, $k = 1, 2$ (for a discussion of all these facts, see [4, Chapter 4]). Throughout this paper, we assume that u represents a variable in $\mathbb{C}$, and so (1.3) and similar formulas are identities valid for all $u \in \mathbb{C}$.

We call η_1 and η_2 the *quasi-periods* of ζ associated with the given generators ω_1 and ω_2 of Γ. Evaluating (1.3) at $u = -\omega_k/2$ and using that ζ is an odd function, we see that

$$\eta_k = 2\zeta\left(\frac{1}{2}\omega_k\right)$$

for $k = 1, 2$. It also follows from (1.3) that

$$\zeta(u + k\omega_1 + n\omega_2) = \zeta(u) + k\eta_1 + n\eta_2$$

for $k, n \in \mathbb{Z}$.

Since there are no doubly-periodic meromorphic functions on $\mathbb{C}$ with period lattice Γ and only first-order poles at the lattice points (see [4, Corollary, p. 23]), η_1 and η_2 cannot both vanish. This means that

$$p := \eta_1/\eta_2 \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$

is well defined. Moreover, it follows from the formulas above that η_1 and η_2 are homogeneous of degree -1 when considered as functions of the pair (ω_1, ω_2). This means that if we make the substitution

$$(\omega_1, \omega_2) \mapsto (t\omega_1, t\omega_2)$$

with $t \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, then we get a corresponding substitution

$$(\eta_1, \eta_2) \mapsto (t^{-1}\eta_1, t^{-1}\eta_2).$$

This implies that the function p is homogeneous of degree 0, and hence, we can consider $p = p(\tau)$ as function of $\tau = \omega_1/\omega_2$.

Let $\mathbb{H} := \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ be the open upper halfplane. The function $\tau \in \mathbb{H} \mapsto p(\tau) \in \widehat{\mathbb{C}}$ can be described explicitly, as the main result of this paper shows. In order to state this, we first introduce some terminology.

A *closed Jordan region X* in the Riemann sphere $\widehat{\mathbb{C}}$ is a compact set homeomorphic to a closed disk. Then, its boundary ∂X is a Jordan curve and the set of interior points $\text{inte}(X) := X \setminus \partial X$ a simply connected open region.

A *circular arc triangle T* is a closed Jordan region in $\widehat{\mathbb{C}}$ whose boundary is decomposed into three non-overlapping circular arcs considered as the *sides* of T. The three endpoints of these arcs are the *vertices* of T.

We say that f is a *conformal map* between two circular arc triangles X and Y in $\widehat{\mathbb{C}}$ if f is a homeomorphism between X and Y that sends the vertices of X to the vertices of Y and is a biholomorphism between $\text{inte}(X)$ and $\text{inte}(Y)$. We say that f is an *anti-conformal map* between X and Y if $z \in X \mapsto \overline{f(z)} \in \overline{Y}$ is a conformal map between X and the complex conjugate $\overline{Y} := \{\bar{w} : w \in Y\}$ of Y.

We can now state our result.

Theorem 1.1. *The function $\tau \mapsto p(\tau)$ is meromorphic on $\mathbb{H}$. It maps the circular arc triangle*

$$(1.4) \qquad T_0 := \{\tau \in \mathbb{C} : 0 \leq \text{Re}(\tau) \leq 1/2, \text{Im}(\tau) > 0, |\tau| \geq 1\} \cup \{\infty\}$$

conformally onto the circular arc triangle

$$(1.5) \quad T_1 := \{\tau \in \mathbb{C} : 0 \leq \text{Re}(\tau) \leq 1/2\} \setminus \{\tau \in \mathbb{C} : \text{Im}(\tau) < 0, |\tau| > 1\} \cup \{\infty\}.$$

Here, the vertices of the triangles correspond under the map p in the following way:

$$p(i) = -i, \quad p\left(\frac{1}{2}(1 + i\sqrt{3})\right) = \frac{1}{2}(1 - i\sqrt{3}), \quad p(\infty) = \infty.$$

Of course, the last relation has to be understood in a limiting sense: if $\tau \in T_0 \cap \mathbb{H} \to \infty$, then $p(\tau) \in T_1 \cap \mathbb{H} \to \infty$. In the following, we always think of p (and similar functions) as extended to the point ∞ in this way.

In order to give a more intuitive description of the map p based on Theorem 1.1, we proceed as follows. Note that the triangle T_0 has angles $0, \pi/2, \pi/3$ at its vertices $\infty, i, \rho := \frac{1}{2}(1 + i\sqrt{3})$, respectively. Let $T \subseteq \widehat{\mathbb{C}}$ be an arbitrary circular arc triangle in the Riemann sphere with angles equal to $0, \pi/2, \pi/3$. Each such triangle is Möbius equivalent to T_0 or its complex conjugate $\overline{T_0}$ (see Lemma 3.6 below). This implies that the triangle T is contained in the intersection $D_1 \cap D_2$ of two closed disks D_1 and D_2 in $\widehat{\mathbb{C}}$ each bounded by a circle in $\widehat{\mathbb{C}}$ containing one of the sides of T and such that these boundaries touch at the vertex of T with angle 0. Then, the closure T' of the set $(D_1 \cap D_2) \setminus T$ is a circular arc triangle that we call *complementary* to T, since $D_1 \cap D_2 = T \cup T'$. Note that T' has the same vertices as T, is bounded by arcs of the same three circles as T, and has angles $0, \pi/2, 2\pi/3$ (see Figure 1). This relation
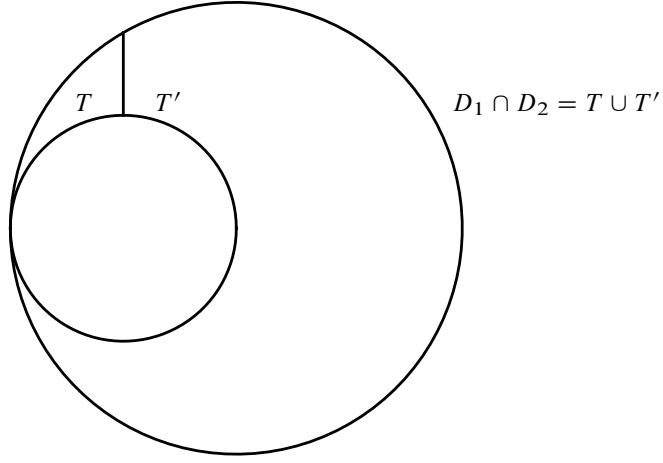
FIGURE 1
A circular arc triangle T with angles 0, $\pi/2$, $\pi/3$ and its complementary triangle T'.

of T and its complementary triangle T' is invariant under Möbius transformations and complex conjugation.

Note that $\overline{T_1}$ is the complementary triangle to T_0 in this sense (see Figure 2). In particular, the complex conjugate $\tau \mapsto \bar{p}(\tau) := \overline{p(\tau)}$ of p maps T_0 anti-conformally onto its complementary triangle

$$T_0' = \overline{T_1} = \{\infty\} \cup \{\tau \in \mathbb{C} : 0 \le \mathrm{Re}(\tau) \le 1/2\} \backslash \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0, |\tau| > 1\}$$

such that the common vertices i, ρ, and ∞ are fixed under $\bar{p}$.

Now, it is well known that by successive reflections in the sides of T_0 we generate a tessellation $\mathcal{T}$ of the set

$$\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$$

by circular arc triangles with angles 0, $\pi/2$, $\pi/3$ (see [11, Section 2.2] and Figure 3 for an illustration).

Let G be the group of conformal or anti-conformal homeomorphisms of $\widehat{\mathbb{C}}$ generated by the reflections in the sides of T_0. By applying the Schwarz Reflection Principle repeatedly and the invariance of the relation between a triangle and its complementary triangle under this procedure, we immediately see that $\bar{p} \circ S = S \circ \bar{p}$ for all $S \in G$. This gives the following consequence.

Corollary 1.2. *Let T be any triangle in the tessellation $\mathcal{T}$ of $\mathbb{H}^*$ obtained by successive reflections in the sides of T_0, and let T' be the complementary triangle of T as defined above. Then, $\bar{p}$ is an anti-conformal map of T onto T' that fixes the vertices of T.*
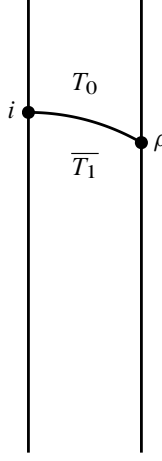
FIGURE 2
The circular arc triangle T_0 with its complement $\overline{T_1}$.

The Möbius transformations of the form

(1.6)
$$\tau \mapsto S(\tau) = \frac{a\tau + b}{c\tau + d},$$

where $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ form the *modular group* $\mathrm{PSL}_2(\mathbb{Z})$. This is a subgroup of index 2 of the group G defined above.

It follows from our previous considerations that

(1.7)
$$p \circ S = S \circ p$$

for all $S \in \mathrm{PSL}_2(\mathbb{Z})$. Meromorphic functions f with this type of transformation behavior ($f \circ S = \tilde{S} \circ f$, where S runs through a Fuchsian group Γ and $\tilde{S}$ is a Möbius transformation associated with S) are called *polymorphic* in the classical literature. We will refer to property (1.7) of p as its $\mathrm{PSL}_2(\mathbb{Z})$-*equivariance*.

We will soon see that this equivariant behavior of p can easily be derived analytically (see Proposition 2.2). Theorem 1.1 and Corollary 1.2 essentially explain this behavior from a geometric perspective.

The circular arc triangle

(1.8)
$$V_0 = \left\{ \tau \in \mathbb{H} : 0 \leq \mathrm{Re}(\tau) \leq 1, |\tau - 1/2| \geq 1/2 \right\} \cup \{\infty\}$$

has all its angles equal to 0. It consists of a union of six triangles from the tessellation $\mathcal{T}$ (see Figure 4). It is easy to see from Corollary 1.2 that the continuous extension of
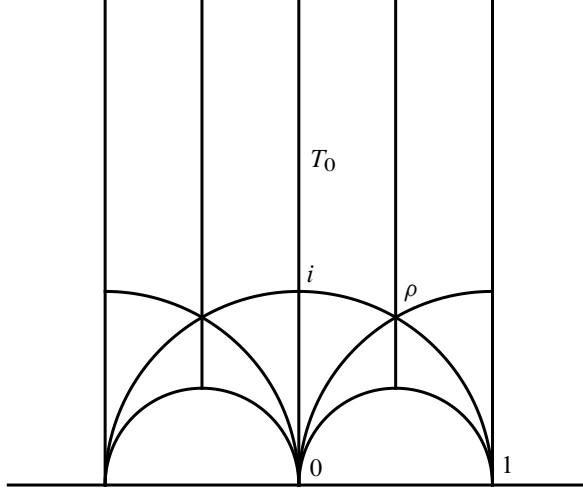
FIGURE 3
Some circular arc triangles in the tessellation $\mathcal{T}$.

$\bar{p}$ to V_0 attains every value in $\widehat{\mathbb{C}}$ once, twice, or thrice. This is illustrated in Figure 4, where the level of darkness indicates how often the regions are covered (the darker the gray, the more often the region is covered; note that some relevant circles are drawn in black which does not correspond to how often the points in these circles are attained). In particular, $\bar{p}(V_0 \cap \mathbb{H}) = \widehat{\mathbb{C}} = p(V_0 \cap \mathbb{H})$.

Now, if we successively reflect in the sides of V_0, we obtain another tessellation $\mathcal{T}'$ of $\mathbb{H}^*$ by circular triangles whose angles are all equal to 0. Each of the triangles V in this tessellation is a union of six triangles in $\mathcal{T}$. Then, Corollary 1.2 and the previous discussion imply that $p(V \cap \mathbb{H}) = \widehat{\mathbb{C}}$. We obtain the following consequence.

Corollary 1.3. *The map p attains every value in $\widehat{\mathbb{C}}$ infinitely often on $\mathbb{H}$.*

This was the main result in the papers [8, 9] by Heins. He proved it by a purely function-theoretic argument without an explicit description of the map p.

We will give two proofs of Theorem 1.1. The first one is along classical lines. We suitably normalize the quasi-periods η_1 and η_2 and show that these normalized periods form a fundamental system of solutions of a hypergeometric differential equation with a certain elliptic modular function, the *absolute invariant J*, as the variable (see (3.21)). This will show that $p = p(J)$ is a Schwarz triangle map sending the closed lower J-halfplane considered as a circular arc triangle with vertices 0, 1, ∞ to a circular arc triangle with angles $2\pi/3$, $\pi/2$, 0. On the other hand, it is known that J is a conformal
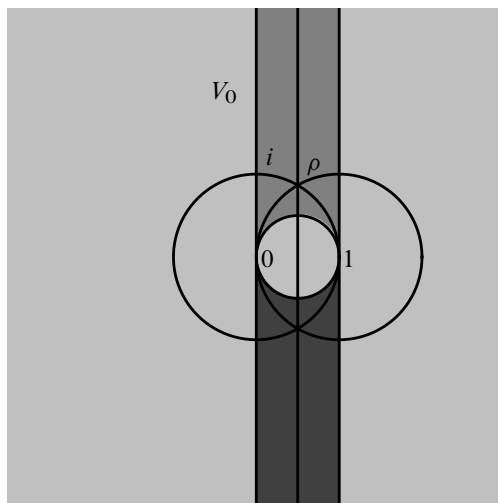
FIGURE 4
Sets covered by the map $\bar{p}$ restricted to V_0.

map of the circular arc triangle T_0 onto the lower J-halfplane. This will lead to the explicit description of the map $\tau \mapsto p(\tau)$.

Our second approach to Theorem 1.1 is more direct: we will show that p is a homeomorphism from ∂T_0 onto ∂T_1. The statement then essentially follows from the Argument Principle (see Proposition 4.1).

We should emphasize that none of our results are really new. The hypergeometric ODE (3.21) for the suitably normalized quasi-periods η_1 and η_2 goes back to at least as far as Fricke–Klein. The quasi-periods of ζ are really the periods of the elliptic integral of the second kind in Weierstrass normalization (see [5, p. 152 ff. and p. 198 ff.] for a thorough discussion). The ODE valid for them (in suitable normalization) is an example of what is now called a Picard–Fuchs equation. They are satisfied by periods of other elliptic and more general integrals (see [16] for a gentle introduction). Fricke–Klein were also well aware of the connection to Schwarz triangle maps, but they seem not to have stated Theorem 1.1 explicitly even though it easily follows from their considerations.

In the more recent literature, the function p (in the form (2.7) below) was considered by various authors (see, for example, [10, 12, 14]). In [12], the connection to hypergeometric ODEs was realized and an explicit formula for the Schwarzian derivative $\{p, \tau\}$ was obtained (see Section 5 for more discussion), but the explicit geometric description of p was not pointed out. In [14], some mapping properties of p were studied without the full realization of the statement of Theorem 1.1. Actually,

the authors of [14] explicitly ask whether p is a conformal map on the fundamental region of the modular group given by the union of T_0 and its reflection in the imaginary axis. This immediately follows from Corollary 1.2. In [10], the authors do prove a biholomorphism property of p that is equivalent to Theorem 1.1. Their method is somewhat ad hoc, based on similar considerations as our second argument for the proof of Theorem 1.1.

This paper is of an expository nature giving an introduction to this classical subject. An interesting topic for further investigation is whether mapping properties similar to Theorem 1.1 can also be obtained for ratios of periods of other naturally appearing elliptic integrals. It is to be hoped that this paper will be a starting point for further studies along these lines.

2. First facts about the quasi-periods

We first compute some special values of p and then derive a representation of p in terms of a Fourier series. All of this is well known from the classical literature (see [5] for example). A modern account is given in [4].

If $\omega_1 = i$ and $\omega_2 = 1$, then $\Gamma = \{ki + n : k, n \in \mathbb{Z}\}$, and so $i\Gamma = \Gamma$. This implies that, for $\zeta(u) = \zeta(u; \Gamma)$, we have

$$\zeta(iu; \Gamma) = \zeta(iu; i\Gamma) = -i\zeta(u; \Gamma).$$

Hence,

$$\eta_1 = 2\zeta(i/2) = -2i\zeta(1/2) = -i\eta_2,$$

and so,

(2.1) $$p(i) = \eta_1/\eta_2 = -i.$$

Throughout this paper, we use the notation

$$\rho = \frac{1}{2}(1 + i\sqrt{3}).$$

Now, let $\omega_1 = \rho$, $\omega_2 = 1$, $\Gamma = \{k\rho + n : k, n \in \mathbb{Z}\}$. Then,

$$\rho\Gamma = \Gamma$$

which implies

$$\zeta(\rho u) = \zeta(\rho u; \Gamma) = \zeta(\rho u; \rho\Gamma) = \rho^{-1}\zeta(u; \Gamma) = \rho^{-1}\zeta(u) = \bar{\rho}\zeta(u).$$

Hence,

$$\eta_1 = 2\zeta(\rho/2) = 2\bar{\rho}\zeta(1/2) = \bar{\rho}\eta_2$$

and

(2.2)
$$p(\rho) = \eta_1/\eta_2 = \bar{\rho}.$$

Now, let ω_1 and ω_2 again be arbitrary, Γ be the associated rank-2 lattice as in (1.1), and $\zeta(u) = \zeta(u; \Gamma)$. If we integrate ζ over the contour given by the parallelogram Q with vertices $\pm\omega_1/2 \pm \omega_2/2$, in positive orientation, then the residue theorem gives us the *Legendre relation*

(2.3)
$$2\pi i = \omega_1\eta_2 - \omega_2\eta_1.$$

See [4, pp. 50–51] for more details.

We will now prove a much deeper fact that connects the quasi-periods with the theory of modular forms which we will review in Section 3. We first set up some notation. As before, $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$ will denote a variable in the open upper halfplane $\mathbb{H}$. Throughout this paper, we set

$$q := e^{2\pi i\tau}.$$

Note that $|q| < 1$ for $\tau \in \mathbb{H}$. We define

$$\sigma_k(n) := \sum_{m=1, m|n}^{n} m^k$$

for $k, n \in \mathbb{N}$. So, $\sigma_k(n)$ is the sum of all kth-powers m^k of natural numbers m that divide n. Finally, we define

(2.4)
$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n.$$

It is immediate that this series (as a function of $\tau \in \mathbb{H}$) converges locally uniformly on $\mathbb{H}$. Hence, E_2 is a holomorphic function on $\mathbb{H}$.

Proposition 2.1. *For given ω_1, ω_2, let η_1, η_2 be the quasi-periods of the associated ζ-function. Then,*

(2.5)
$$\eta_1 = -\frac{2\pi i}{\omega_2} + \frac{\pi^2\omega_1}{3\omega_2^2}E_2(\tau),$$

(2.6)
$$\eta_2 = \frac{\pi^2}{3\omega_2}E_2(\tau).$$

In particular,

(2.7)
$$p(\tau) = \eta_1/\eta_2 = \tau - \frac{6i}{\pi E_2(\tau)}.$$

Since $E_2(\tau) \to 1$ as $\mathrm{Im}(\tau) \to +\infty$, it follows from (2.7) that

$$(2.8) \qquad\qquad p(\tau) \to \infty \text{ as } \mathrm{Im}(\tau) \to +\infty.$$

Formulas (2.5) and (2.6) are well known; see, for example, [5, p. 311]. Since they are the basis for our further considerations, we will give a proof for the sake of completeness. We will follow an argument outlined in [5, pp. 262–263].

Proof. It is enough to show (2.6). Then, (2.5) follows from this and the Legendre relation (2.3), and (2.7) is an immediate consequence of (2.5) and (2.6).

Our starting point is formula (1.2). For ease of notation, we use the abbreviation $(k, n) = k\omega_1 + n\omega_2$ and set

$$\Sigma_k(u) := \sum_{n \in \mathbb{Z}} \left(\frac{1}{u - (k, n)} + \frac{1}{(k, n)} + \frac{u}{(k, n)^2} \right)$$

for $k \in \mathbb{Z} \setminus \{0\}$. Moreover, we set

$$\Sigma_0(u) := \frac{1}{u} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\frac{1}{u - (0, n)} + \frac{1}{(0, n)} + \frac{u}{(0, n)^2} \right).$$

Since the series (1.2) converges absolutely and uniformly in $u \in \mathbb{C} \setminus \Gamma$, we can rearrange the order of summation; so

$$\zeta(u) = \sum_{k \in \mathbb{Z}} \Sigma_k(u).$$

The idea now is to express $\eta_2 = \zeta(u + \omega_2) - \zeta(u)$ in terms of the differences of the functions Σ_k. First, note that

$$\Sigma_0(u + \omega_2) - \Sigma_0(u) = \frac{2}{\omega_2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3\omega_2}.$$

For the other differences, we use the standard fact that

$$(2.9) \qquad\qquad \frac{\pi^2}{\sin^2(\pi u)} = \sum_{n \in \mathbb{Z}} \frac{1}{(u - n)^2}$$

for $u \in \mathbb{C} \setminus \mathbb{Z}$. By considering partial sums, we see that, for $k \in \mathbb{Z} \setminus \{0\}$, we have

$$\Sigma_k(u + \omega_2) - \Sigma_k(u) = \sum_{n \in \mathbb{Z}} \frac{\omega_2}{(k, n)^2} = \frac{1}{\omega_2} \sum_{n \in \mathbb{Z}} \frac{1}{(k\tau + n)^2}$$

$$= \frac{\pi^2}{\omega_2 \sin^2(|k|\pi\tau)} = \frac{-4\pi^2 q^{|k|}}{\omega_2 (1 - q^{|k|})^2}$$

$$= -\frac{4\pi^2}{\omega_2} \sum_{n=1}^{\infty} n q^{n|k|}.$$

Putting this all together, we see that

$$\eta_2 = \zeta(u + \omega_2) - \zeta(u) = \sum_{k \in \mathbb{Z}}(\Sigma_k(u + \omega_2) - \Sigma_k(u))$$

$$= \frac{\pi^2}{3\omega_2} - \frac{8\pi^2}{\omega_2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}nq^{nk} = \frac{\pi^2}{3\omega_2} - \frac{8\pi^2}{\omega_2}\sum_{n=1}^{\infty}\sigma_1(n)q^n = \frac{\pi^2}{3\omega_2}E_2(\tau),$$

as desired. ∎

We now give an analytic argument for the equivariant nature of p under the modular group and a closely related transformation behavior of the function E_2.

Proposition 2.2. *Let $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. Then,*

$$(2.10) \qquad p\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{ap(\tau) + b}{cp(\tau) + d}$$

and

$$(2.11) \qquad E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi}c(c\tau + d).$$

Proof. We define $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$. Then, ω_1', ω_2' generate the same lattice Γ as ω_1, ω_2, and so give rise to the same ζ-function. The first quasi-period η_1' associated with the pair ω_1', ω_2' can be computed from the following identity in u:

$$\eta_1' = \zeta(u + \omega_1') - \zeta(u) = \zeta(u + a\omega_1 + b\omega_2) - \zeta(u) = a\eta_1 + b\eta_2.$$

Similarly, for the second quasi-period η_2' associated with ω_1', ω_2', we have $\eta_2' = c\eta_1 + d\eta_2$. Equation (2.10) then immediately follows by passing to quotients.

By (2.6), we have

$$E_2(\tau) = 3\eta_2\omega_2/\pi^2.$$

So, if we use the previous notation and write $\tau' := \omega_1'/\omega_2' = (a\tau + b)/(c\tau + d)$, then we have

$$E_2(\tau') = 3\eta_2'\omega_2'/\pi^2 = 3(c\eta_1 + d\eta_2)(c\omega_1 + d\omega_2)/\pi^2$$

$$= (cp(\tau) + d)(c\tau + d)E_2(\tau) = (c(\tau - 6i/(\pi E_2(\tau))) + d)(c\tau + d)E_2(\tau)$$

$$= (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi}c(c\tau + d). \qquad ∎$$

A curious consequence of Corollary 1.3 is that there are pairs (ω_1, ω_2) for which η_1 or η_2 vanishes (they cannot both be zero for a given pair (ω_1, ω_2)). In this case, ζ is

a periodic, but not doubly-periodic function. This seems to have been first noticed by Watson [13] and was also pointed out by Heins in [8].

Note that (2.10) implies that $\eta_1 = 0$ for a pair (ω_1, ω_2) if and only if $\eta_2 = 0$ for the pair $(-\omega_2, \omega_1)$. On the other hand, by (2.6), we have $\eta_2 = 0$ if and only if $E_2(\tau) = 0$. So, studying values (ω_1, ω_2) where η_1 or η_2 vanishes amounts to the same as finding all zeros of E_2. As Heins alluded to in [9], it is easy to see that E_2 has a zero on the positive imaginary axis. Watson [13] had actually computed this zero with great accuracy. The location of all the zeros of E_2 in $\mathbb{H}$ was studied in depth in [10, 14].

3. Modular forms

In order to proceed, we need some facts about *modular forms*. This is all standard material going back to Fricke–Klein [6] and beyond. A modern account can be found in [11]. The survey by Zagier [15] gives a fresh perspective with connections to many other areas. All we need are some basic fundamentals of the theory. Many of them are discussed in [4] and all of them in [11, Chapters 1–3], for example. We start with a definition.

Definition 3.1. A holomorphic function f on $\mathbb{H}$ is called a *(holomorphic and inhomogeneous) modular form of weight $k \in \mathbb{N}$* if it satisfies

$$(3.1) \qquad f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all $\tau \in \mathbb{H}$ and $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. Moreover, we require that

$$(3.2) \qquad f(\tau) = O(1) \text{ as } \operatorname{Im}(\tau) \to +\infty.$$

Note that a modular form is 1-periodic: $f(\tau + 1) = f(\tau)$. So, it will have a Fourier series expansion

$$(3.3) \qquad f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \tau} = \sum_{n \in \mathbb{Z}} a_n q^n$$

that converges for all $q \in \mathbb{C}$ with $|q| < 1$. Condition (3.1) ensures that no terms with negative powers of q occur in this series, and so,

$$(3.4) \qquad f(\tau) = \sum_{n \in \mathbb{N}_0} a_n q^n.$$

If f is a function on $\mathbb{H}$, $k \in \mathbb{N}$, and $S \in \operatorname{PSL}_2(\mathbb{Z})$ is as in (1.6), then we can define an operation $f|_k S$ by setting $(f|_k S)(\tau) = f(S(\tau))(c\tau + d)^{-k}$. Condition (3.1) then simply becomes $(f|_k S)(\tau) = f(\tau)$.

One can easily check that if $S, T \in \mathrm{PSL}_2(\mathbb{Z})$, then

$$f|_k(S \circ T) = (f|_k S)|_k T.$$

This implies that in order to verify (3.1) it is enough to do this for generators of $\mathrm{PSL}_2(\mathbb{Z})$, for example for $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$. We see that condition (3.1) is equivalent to the identities

$$f(\tau + 1) = f(\tau) \quad \text{and} \quad f(-1/\tau) = \tau^k f(\tau).$$

We say that f is a *modular function* if it is meromorphic on $\mathbb{H}$ and satisfies $f \circ S = f$ for all $S \in \mathrm{PSL}_2(\mathbb{Z})$. Moreover, we require that, for some $N \in \mathbb{N}_0$, we have

(3.5) $$f(\tau) = O(q^{-N}) \quad \text{as } \mathrm{Im}(\tau) \to +\infty.$$

A modular function f is 1-periodic and has a Fourier expansion as in (3.3) converging if $|q| > 0$ is small enough. Condition (3.5) ensures that in this Fourier expansion only finitely many non-zero terms with negative n occur.

Note that if g and $h \neq 0$ are modular forms of the same weight, then $f = g/h$ is a modular function. If g and h are modular forms of weights k and l, respectively, then gh is a modular form of weight $k + l$.

Let f be a modular form of weight k. Then, we can pass to an associated *homogeneous modular form* $\tilde{f}$ depending on two variables ω_1 and ω_2 (satisfying our standing assumptions) by setting

$$\tilde{f}(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_2}\right)^k f(\omega_1/\omega_2) = \left(\frac{2\pi}{\omega_2}\right)^k f(\tau).$$

The incorporation of the factor 2π here has some advantages. Then, $\tilde{f}$ is a homogeneous function of degree $-k$ in the sense that

$$\tilde{f}(t\omega_1, t\omega_2) = t^{-k} \tilde{f}(\omega_1, \omega_2)$$

for $t \in \mathbb{C}^*$. Moreover, (3.1) is equivalent to

$$\tilde{f}(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) = \tilde{f}(\omega_1, \omega_2).$$

So, the transformation behavior of a homogeneous modular form becomes more transparent at the cost of having to deal with a function of two variables.

The (inhomogeneous and holomorphic) modular forms of a given weight $k \in \mathbb{N}$ obviously form a vector space $\mathcal{M}_k$ over $\mathbb{C}$. It immediately follows from the definitions that there are no non-trivial modular forms of odd weight (indeed, change the signs of a, b, c, d in (3.1)). If k is even, then the following well-known fact gives the dimension of $\mathcal{M}_k$ for even k (see [11, Theorem 18, p. 47]). The fact that $\mathcal{M}_k$ is finite-dimensional is probably the single most important fact in the theory of modular forms.

Proposition 3.2. *For even $k \in \mathbb{N}$, we have*

$$\dim \mathcal{M}_k = \begin{cases} \lfloor k/12 \rfloor & \text{for } k \equiv 2 \bmod 12, \\ \lfloor k/12 \rfloor + 1 & \text{for } k \not\equiv 2 \bmod 12. \end{cases}$$

In particular, there are no non-trivial modular forms of weight 2. Moreover, the spaces of modular forms of weight 4, 6, and 8 are all one-dimensional and hence spanned by any non-trivial form in these spaces. The latter fact is all we need in the following.

To obtain non-trivial modular forms in $\mathcal{M}_4$ and $\mathcal{M}_6$, we consider the *Eisenstein series G_k* defined for even $k \geq 4$ as

$$G_k(\tau) = \sideset{}{'}\sum_{m,n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}.$$

Here, the prime on the sum means that the term with $m = n = 0$ should be omitted from the summation.

The series representing the function G_k, $k \geq 4$, converges absolutely and locally uniformly for $\tau \in \mathbb{H}$; so G_k is a holomorphic function of $\tau \in \mathbb{H}$. Actually, G_k is a modular form of weight k. Indeed, it is immediate to see that G_k has the right transformation behavior as in (3.1). Moreover, one can explicitly obtain a Fourier expansion of G_k as in (3.4). This can be seen by a computation similar to the proof of Proposition 2.1 where one uses an identity obtained from (2.9) by differentiating $(k-2)$-times. This is standard and the details can be found in [11, Chapter 3], for example. We only need the result for $k = 4$ and $k = 6$. Namely, we have

$$G_4 = \frac{\pi^4}{45} E_4, \quad G_6 = \frac{2\pi^6}{945} E_6,$$

where

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6 = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

In particular, G_4, or equivalently E_4, is a modular form spanning $\mathcal{M}_4$, and G_6 or E_6 are modular forms each spanning $\mathcal{M}_6$. The space $\mathcal{M}_8$ is spanned by E_4^2.

While E_2 (as defined in (2.4)) is not a modular form, it has a transformation behavior that is closely related. Indeed, each function E_k, $k = 2, 4, 6$, is 1-periodic, that is, for $\tau \in \mathbb{H}$ it satisfies

$$E_k(\tau + 1) = E_k(\tau);$$

moreover, we have

$$(3.6) \qquad E_2(-1/\tau) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau,$$

$$(3.7) \qquad E_4(-1/\tau) = \tau^4 E_4(\tau),$$

$$(3.8) \qquad E_6(-1/\tau) = \tau^6 E_6(\tau).$$

The first equation follows from (2.11), while the last two equations follow from the fact that E_4 and E_6 are modular forms of weight 4 and weight 6, respectively. Due to its simple transformation behavior, E_2 is called a *quasi-modular form of weight* 2.

If, as before, $\rho = \frac{1}{2}(1 + i\sqrt{3})$, then

$$-1/\rho = -\frac{1}{2}(1 - i\sqrt{3}) = \rho - 1,$$

and so,

$$\rho^4 E_4(\rho) = E_4(-1/\rho) = E_4(\rho - 1) = E_4(\rho).$$

Hence,

$$(3.9) \qquad E_4(\rho) = 0.$$

It follows from the *valence formula* for modular forms (see [11, Theorem 13, p. 41]) that ρ is the only zero of E_4 on $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ up to equivalence under $\mathrm{PSL}_2(\mathbb{Z})$. More precisely, we have

$$(3.10) \quad E_4(\tau) = 0 \text{ for } \tau \in \mathbb{H}^* \text{ if and only if } \tau = S(\rho) \text{ for some } S \in \mathrm{PSL}_2(\mathbb{Z}).$$

If we insert $\tau = i$ in the transformation formula (3.8) for E_6, we see that $E_6(i) = -E_6(i)$, and so,

$$(3.11) \qquad E_6(i) = 0.$$

Again, i is the only zero of E_6 up to equivalence under $\mathrm{PSL}_2(\mathbb{Z})$, but we will not need this fact.

We will derive expressions for the derivatives

$$E_k' = dE_k/d\tau.$$

These formulas are often attributed to Ramanujan, but they can be traced back (in different notation) to Fricke–Klein and beyond. To absorb a factor $2\pi i$ that appears in these formulas, we introduce the abbreviation

$$DF = \frac{1}{2\pi i}\frac{dF(\tau)}{d\tau}$$

for ease of notation.

Proposition 3.3. *The following identities are valid:*

$$(3.12) \qquad\qquad DE_2 = \frac{1}{12}(E_2^2 - E_4),$$

$$(3.13) \qquad\qquad DE_4 = \frac{1}{3}(E_2 E_4 - E_6),$$

$$(3.14) \qquad\qquad DE_6 = \frac{1}{2}(E_2 E_6 - E_4^2).$$

Proof. To prove the first formula, we consider the function $f = DE_2 - \frac{1}{12}E_2^2$ on $\mathbb{H}$. It is 1-periodic and has a Fourier expansion of the form (3.4) as follows from computations with the underlying q-power series for E_2. Moreover, the transformation behavior of E_2 gives

$$\tau^{-2}E_2'(-1/\tau) = \frac{d}{d\tau}E_2(-1/\tau) = \frac{d}{d\tau}\left(\tau^2 E_2(\tau) - \frac{6i}{\pi}\tau\right)$$
$$= 2\tau E_2(\tau) + \tau^2 E_2'(\tau) - \frac{6i}{\pi},$$

and so,

$$DE_2(-1/\tau) = \tau^4 DE_2(\tau) - \frac{i}{\pi}\tau^3 E_2(\tau) - \frac{3}{\pi^2}\tau^2.$$

Hence,

$$f(-1/\tau) = DE_2(-1/\tau) - \frac{1}{12}E_2(-1/\tau)^2$$
$$= \tau^4 DE_2(\tau) - \frac{i}{\pi}\tau^3 E_2(\tau) - \frac{3}{\pi^2}\tau^2 - \frac{1}{12}\left(\tau^2 E_2(\tau) - \frac{6i}{\pi}\tau\right)^2$$
$$= \tau^4 f(\tau).$$

It follows that f is a modular form of weight 4. Since the space of these forms has dimension 1 and is spanned by E_4, there exists a constant $c \in \mathbb{C}$ such that $f = cE_4$. To determine this constant, we consider the leading coefficients of the q-expansions of f and E_4. We find that

$$f(\tau) = -\frac{1}{12} + O(q), \quad E_4 = 1 + O(q),$$

and so, $c = -\frac{1}{12}$. The claim follows.

The other identities follow from similar considerations: one shows that $DE_4 - \frac{1}{3}E_2 E_4$ is a modular form of weight -6 and $DE_6 - \frac{1}{2}E_2 E_6$ is a modular form of weight -8. Moreover, the spaces of these forms are spanned by E_6 and E_4^2, respectively. Again, one determines the proportionality constants by considering q-expansions. ∎

To put the preceding argument into a more general perspective: if f is a modular form of weight $k = 4, 6, \ldots$, then one can show by computations similar to the ones in the proof of Proposition 3.3 that its *Serre derivative* (see [15, p. 48])

$$\vartheta_k f := Df - \frac{k}{12} E_2 f$$

is a modular form of weight $k + 2$.

We need two more auxiliary functions. The first one is

$$\Delta := \frac{1}{1728}(E_4^3 - E_6^2).$$

This is a modular form of weight 12 (note that the notation in [4, 11] is different from ours). From Proposition 3.3 it follows that

$$
\begin{aligned}
D\Delta &= \frac{1}{1728}(D(E_4^3) - D(E_6^2)) = \frac{1}{1728}(3E_4^2 DE_4 - 2E_6 DE_6) \\
&= \frac{1}{1728}(E_4^3 - E_6^2)E_2 = \Delta E_2.
\end{aligned}
$$

Based on this and the expansion of E_2 as a q-series, it is easy to derive the well-known formula

(3.15) $$\Delta = q \prod_{n=1}^{\infty}(1 - q^n)^{24}.$$

Indeed, both sides in this identity are holomorphic functions on $\mathbb{H}$ with the same logarithmic derivatives, namely, $2\pi i E_2$; so they must represent the same function up to a multiplicative constant. This constant is equal to 1 as a comparison of the leading terms of the q-expansions shows.

It immediately follows from (3.15) that Δ has no zeros on $\mathbb{H}$ and that it takes positive real values for τ on the positive imaginary axis. In the following, we will consider various roots $\Delta^{1/k}$ of Δ for $k \in \mathbb{N}$. Since Δ has no zeros on $\mathbb{H}$, these are holomorphic functions on $\mathbb{H}$. A priori, $\Delta^{1/k}$ is only defined up to multiplication by a kth root of unity. We fix this ambiguity so that $\Delta^{1/k}$ attains positive real values on the positive imaginary axis.

We need one more auxiliary function, namely, the *J-invariant* given as

(3.16) $$J = E_4^3/(E_4^3 - E_6^2) = \frac{1}{1728}E_4^3/\Delta.$$

This is a modular function. It is well known that J maps the circular arc triangle T_0 as in (1.4) conformally to the closed lower halfplane, considered as a circular arc triangle with vertices 0, 1, ∞ (see [4, Chapter 6; Theorem 5, p. 90]).

At least the correspondence of vertices is easy to see: the definition of J immediately gives

$$J(\tau) = \frac{1}{1728} q^{-1} + O(1) \quad \text{as } \operatorname{Im}(\tau) \to +\infty.$$

Hence, $J(\infty) = \infty$ (understood in a limiting sense). Moreover, the definition of J in combination with (3.9) and (3.11) gives

$$J(\rho) = 0 \quad \text{and} \quad J(i) = 1.$$

By (3.16), we obtain a uniquely determined holomorphic third root of J by setting

$$J^{1/3} := \frac{1}{12} E_4 / \Delta^{1/3}.$$

Similarly, by (3.16), we have

$$J - 1 = \frac{1}{1728} E_6^2 / \Delta,$$

which allows us to define the holomorphic function

$$(J-1)^{1/2} := \frac{1}{24\sqrt{3}} E_6 / \Delta^{1/2}.$$

Note that these definitions fix the ambiguity of $J^{1/3}$ and $(J-1)^{1/2}$ in such a way that these functions take positive real values with τ on the positive imaginary axis with $\operatorname{Im}(\tau)$ large.

The previous formulas can be rewritten as follows:

$$E_4 = 12 J^{1/3} \Delta^{1/3}, \quad E_6 = 24\sqrt{3}(J-1)^{1/2} \Delta^{1/2}.$$

We also have

$$(3.17) \qquad DJ = \frac{1}{1728} \left(\frac{3 E_4^2 D(E_4)}{\Delta} - \frac{E_4^3 D(\Delta)}{\Delta^2} \right)$$

$$= -\frac{1}{1728} E_4^2 E_6 / \Delta = -2\sqrt{3} J^{2/3}(J-1)^{1/2} \Delta^{1/6}.$$

We want to introduce J as a new variable instead of τ for some functions of τ. The map $\tau \in \mathbb{H} \mapsto J(\tau)$ is locally injective at all points not equivalent to i or ρ under the modular group. The latter points are exactly those where J takes on the values 1 or 0. Moreover, $J(\tau)$ attains all values in $\widehat{\mathbb{C}}$ except ∞. This means that $\tau = \tau(J)$ is locally well defined away from the points 0, 1, ∞ in the J-plane. Once we fix such a local branch $\tau = \tau(J)$, we can analytically continue it along any path in $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. In

this way, we obtain a multi-valued function on $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$, whose branches differ by postcomposition with elements in the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

We will now fix such a local branch $\tau = \tau(J)$ and introduce the following functions, considered as depending on the variable J:

$$(3.18) \qquad \Omega_1 = \tau \Delta^{1/12}, \quad \Omega_2 = \Delta^{1/12},$$

$$(3.19) \qquad H_1 = \frac{1}{\Delta^{1/12}}\left(\tau E_2 - \frac{6i}{\pi}\right), \quad H_2 = \frac{E_2}{\Delta^{1/12}}.$$

Note that by (2.7), we have

$$(3.20) \qquad H_1(\tau)/H_2(\tau) = \tau - \frac{6i}{\pi E_2(\tau)} = p(\tau).$$

The meaning of the expressions (3.18) and (3.19) becomes clearer if one transitions to homogeneous functions of the variables ω_1 and ω_2. See Section 5 for more details.

Lemma 3.4. *For $k = 1, 2$ we have the following identities:*

$$\frac{d\Omega_k}{dJ} = -\frac{1}{24\sqrt{3}} J^{-2/3}(J-1)^{-1/2} H_k,$$

$$\frac{dH_k}{dJ} = \frac{1}{2\sqrt{3}} J^{-1/3}(J-1)^{-1/2} \Omega_k.$$

Proof. We have

$$D\Omega_1 = \frac{1}{12}\Delta^{1/12}\tau E_2 - \frac{i}{2\pi}\Delta^{1/12}, \quad D\Omega_2 = \frac{1}{12}\Delta^{1/12}E_2.$$

Hence,

$$\frac{d\Omega_2}{dJ} = D\Omega_2/DJ$$

$$= -\frac{1}{24\sqrt{3}} E_2 J^{-2/3}(J-1)^{-1/2}/\Delta^{1/12}$$

$$= -\frac{1}{24\sqrt{3}} J^{-2/3}(J-1)^{-1/2} H_2.$$

Similarly,

$$\frac{d\Omega_1}{dJ} = D\Omega_1/DJ = \frac{\frac{1}{12}\Delta^{1/12}\tau E_2 - \frac{i}{2\pi}\Delta^{1/12}}{-2\sqrt{3}J^{2/3}(J-1)^{1/2}\Delta^{1/6}}$$

$$= -\frac{1}{24\sqrt{3}} J^{-2/3}(J-1)^{-1/2} H_1.$$

The first identity follows.

For the second identity, note that

$$DH_2 = D(E_2/\Delta^{1/12}) = DE_2/\Delta^{1/12} - \frac{1}{12}E_2^2/\Delta^{1/12}$$

$$= -\frac{1}{12}E_4/\Delta^{1/12} = -J^{1/3}\Delta^{1/4},$$

and so,

$$\frac{dH_2}{dJ} = DH_2/DJ = \frac{-J^{1/3}\Delta^{1/4}}{-2\sqrt{3}J^{2/3}(J-1)^{1/2}\Delta^{1/6}} = \frac{1}{2\sqrt{3}}J^{-1/3}(J-1)^{-1/2}\Omega_2.$$

Similarly,

$$DH_1 = D\left(\tau E_2 - \frac{6i}{\pi}\right)\Big/\Delta^{1/12} - \frac{1}{12}\left(\tau E_2 - \frac{6i}{\pi}\right)E_2\Big/\Delta^{1/12}$$

$$= \left(-\frac{i}{2\pi}E_2 + \frac{1}{12}\tau(E_2^2 - E_4)\right)\Big/\Delta^{1/12} - \frac{1}{12}\left(\tau E_2 - \frac{6i}{\pi}\right)E_2\Big/\Delta^{1/12}$$

$$= -\frac{1}{12}\tau E_4/\Delta^{1/12} = -\tau J^{1/3}\Delta^{1/4} = \tau DH_2.$$

It follows that

$$\frac{dH_1}{dJ} = DH_1/DJ = \tau DH_2/DJ = \frac{1}{2\sqrt{3}}J^{-1/3}(J-1)^{-1/2}\tau\Omega_2$$

$$= \frac{1}{2\sqrt{3}}J^{-1/3}(J-1)^{-1/2}\Omega_1. \qquad\blacksquare$$

Proposition 3.5. *The functions H_1 and H_2 considered as functions of J form a fundamental system of solutions of the hypergeometric differential equation*

$$(3.21)\qquad w''(z) + \frac{5z-2}{6z(z-1)}w'(z) + \frac{1}{144z(z-1)}w(z) = 0.$$

As mentioned in the introduction, this is not new. It follows from the methods of Fricke–Klein and was explicitly stated in [5, formula (9), p. 326]. To derive (3.21), Fricke used homogeneous modular forms and an associated differentiation process in contrast to our approach based on inhomogeneous modular forms (see Section 5 for more discussion).

Proof. We use the system of equations from Lemma 3.4. For simplicity, we write $H = H_k$ and $\Omega = \Omega_k$ for $k = 1, 2$. Then,

$$\frac{d^2H}{dJ^2} = \frac{d}{dJ}\left(\frac{1}{2\sqrt{3}}J^{-1/3}(J-1)^{-1/2}\Omega\right)$$

$$= \left(-\frac{1}{3J} - \frac{1}{2(J-1)}\right)\frac{dH}{dJ} + \frac{1}{2\sqrt{3}}J^{-1/3}(J-1)^{-1/2}\frac{d\Omega}{dJ}$$

$$= -\frac{5J-2}{6J(J-1)}\frac{dH}{dJ} - \frac{1}{144J(J-1)}H.$$

It follows that H_1 and H_2 are solutions of the above differential equation. Since the ratio $p = H_1/H_2$ of these functions is non-constant by (2.7), they are linearly independent and form a fundamental system of solutions.   ∎

The general form of the hypergeometric differential equation (see [3, Part 7, Chapter 2] or [1, Chapter 8]) is

$$(3.22) \qquad w''(z) + \frac{z(1 + \alpha + \beta) - \gamma}{z(z-1)} w'(z) + \frac{\alpha\beta}{z(z-1)} w(z) = 0.$$

So, the parameters in (3.21) are $\alpha = \beta = -1/12$ and $\gamma = 1/3$. This implies that the ratio $p = H_1/H_2$ is a conformal map of the closed upper or lower J-halfplane considered as circular arc triangles with vertices $0, 1, \infty$ onto a circular arc triangle with angles $\lambda\pi, \mu\pi, \nu\pi$, respectively, where

$$\lambda = 1 - \gamma = 2/3, \quad \mu = \gamma - \alpha - \beta = 1/2, \quad \nu = \alpha - \beta = 0.$$

This is thoroughly explained in [3, Part 7, Chapter 2] or [1, pp. 252–254].

Before we proceed, we record a simple geometric fact.

Lemma 3.6. *Let $T \subseteq \widehat{\mathbb{C}}$ be a circular arc triangle. If T has angles $0, \pi/2, \pi/3$, then it is Möbius equivalent to T_0 or $\overline{T_0}$. If T has angles $0, \pi/2, 2\pi/3$, then it is Möbius equivalent to T_1 or $\overline{T_1}$.*

Here, we say that two circular arc triangles X and Y are *Möbius equivalent* if there exists a Möbius transformation S (i.e., a biholomorphism on $\widehat{\mathbb{C}}$) that gives a conformal map between X and Y (as defined in the introduction). Recall that T_0 and T_1 are defined in (1.4) and (1.5), respectively.

Proof. The first part is true in even greater generality and its proof is fairly standard (see, for example, [3, Section 394 and Figure 83]); so we will only give an outline of the argument.

Suppose $T \subseteq \widehat{\mathbb{C}}$ has angles $0, \pi/2, \pi/3$ at its vertices $a, b, c \in \partial T$, respectively. By applying an auxiliary Möbius transformation to T, we may assume that $a = \infty$. Then, the side of T containing b and c must be an arc on a circle C and cannot be a segment on a line (in the latter case the angles at b and c would have to add up to π which they do not). By applying another Möbius transformation (namely, a Euclidean similarity), we may also assume that C is the unit circle and that the two sides of T containing $a = \infty, b$ and a, c, respectively, are rays parallel to the positive imaginary axis. It is then easy to see that the only possibilities are $b = i$, $c = \rho$, in which case $T = T_0$, or $b = i$, $c = -\bar{\rho}$ in which case T is equal to the reflection image of T_0 in the imaginary axis, and hence Möbius equivalent to $\overline{T_0}$.

If $T \subseteq \hat{\mathbb{C}}$ has angles $0, \pi/2, 2\pi/3$ at its vertices $a, b, c \in \partial T$, respectively, then the quickest way to verify the statement is to reduce it to the first part. For this we consider the unique circular arc triangle $\widetilde{T}$ with the same vertices a, b, c and angles $0, \pi/2, \pi/3$, respectively, so that T is complementary to $\widetilde{T}$ in the sense discussed in the introduction. Then, $\widetilde{T} \cup T$ forms a lune as in Figure 1. By what we have seen, we can find a Möbius transformation S that gives a conformal map of $\widetilde{T}$ onto T_0 or $\overline{T_0}$. Then, S is a conformal map of T onto $\overline{T_1}$ or T_1 and the statement follows.   ∎

The different cases in the previous lemma can easily be distinguished by taking boundary orientation into account. Let us assume the circular arc triangle T has angles $0, \pi/2, \pi/3$ at its vertices $a, b, c \in \partial T$, respectively. Then, we orient ∂T so that we traverse a, b, c in this cyclic order if we run through ∂T according to the orientation. Now, suppose T lies on the left of ∂T with this orientation. Since Möbius transformations are orientation-preserving, it then follows that T cannot be Möbius equivalent to $\overline{T_0}$, and so, it must be Möbius equivalent to T_0. If T lies on the right of ∂T, then T is Möbius equivalent to $\overline{T_0}$. A similar analysis applies to circular arc triangles that are Möbius equivalent to T_1 and $\overline{T_1}$.

The proof of our main result is now easy.

Proof of Theorem 1.1. We know that $\tau \mapsto J(\tau)$ sends T_0 conformally to the circular arc triangle given by the closed lower halfplane. Here, the vertex correspondence under the map is $\infty \mapsto \infty, i \mapsto 1, \rho \mapsto 0$. On the other hand, we know that $p = H_1/H_2$, now considered as a function of J, sends the lower J-halfplane conformally onto some circular arc triangle T with vertices $a, b, c \in \hat{\mathbb{C}}$ where the angles are $0, \pi/2, 2\pi/3$, respectively. Here, the vertex correspondence under $J \mapsto H_1/H_2$ is given by $0 \mapsto c$, $1 \mapsto b, \infty \mapsto a$.

If we compose $\tau \mapsto J$ with $J \mapsto H_1/H_2$, then we obtain the map

$$\tau \mapsto H_1(\tau)/H_2(\tau) = p(\tau).$$

In particular, p is a conformal map between the circular arc triangles T_0 and T such that $p(\infty) = a, p(i) = b, p(\rho) = c$. On the other hand, we know the values that p obtains at these locations (see (2.1), (2.2), and (2.8)). It follows that $a = p(\infty) = \infty$, $b = p(i) = -i, c = p(\rho) = \bar{\rho}$.

If we orient ∂T_0 so that T_0 lies on the left of ∂T_0, then the vertices of T_0 are in cyclic order ∞, i, ρ. Since the conformal map p of T_0 onto T preserves orientation, T lies on the left of ∂T if ∂T carries the induced orientation under p. This corresponds to the cyclic order $p(\infty) = \infty, p(i) = -i, p(\rho) = \bar{\rho}$ of the vertices of T.

Lemma 3.6 and the subsequent discussion after this lemma now imply that T is Möbius equivalent to T_1 (and not to $\overline{T_1}$). Since T and T_1 have the same vertices, it follows that $T = T_1$ and the statement follows.   ∎

4. An alternative approach

Our proof of Theorem 1.1 is in a sense the "classical" proof. It may be interesting to point out a different and simple direct argument that avoids the theory of hypergeometric ODEs. We require some preparation.

Proposition 4.1. *Suppose $U \subseteq \hat{\mathbb{C}}$ is a closed Jordan region and $f : \text{inte}(U) \to \mathbb{C}$ is a holomorphic function that has a continuous extension (as a map into $\hat{\mathbb{C}}$) to the boundary $\partial U \subseteq \hat{\mathbb{C}}$. Suppose this extension maps ∂U homeomorphically to the boundary $\partial V \subseteq \hat{\mathbb{C}}$ of a closed Jordan region $V \subseteq \hat{\mathbb{C}}$ with $\text{inte}(V) \subseteq \mathbb{C}$.*

If $\infty \in \partial V$ we assume in addition that if the Jordan curve ∂U is oriented so that U lies on the left of ∂U and if ∂V carries the orientation induced by f, then V lies on the left of ∂V.

Under these assumptions, f is a homeomorphism of U onto V that is a biholomorphism between $\text{inte}(U)$ and $\text{inte}(V)$.

In our application of this statement, ∞ will be on the boundary of U and V. This is the reason why the formulation is somewhat technical and we cannot simply assume $U, V \subseteq \mathbb{C}$.

Proof. The Argument Principle implies that on $\text{inte}(U)$ the function f attains each value in $\text{inte}(V)$ once and no other values. The statement easily follows. As this reasoning is fairly standard, we skip the details (see [2, pp. 310–311] for some background and [2, Exercise 9.17 (i)] for a closely related statement). ∎

We need the following estimate for E_2 (see [10, Lemma 2.3]).

Lemma 4.2. *We have*
$$|E_2(\tau) - 1| \le \frac{24|q|}{(1 - |q|)^3}$$
for $\tau \in \mathbb{H}$. In particular,
$$E_2(\tau) \ne 0 \quad \text{for } \text{Im}(\tau) \ge \sqrt{3}/2.$$

Proof. We have
$$|E_2(\tau) - 1| \le 24 \sum_{n=1}^{\infty} \frac{n|q|^n}{1 - |q|^n} \le \frac{24}{1 - |q|} \sum_{n=1}^{\infty} n|q|^n = \frac{24|q|}{(1 - |q|)^3}.$$

Since $|q| = e^{-2\pi \text{Im}(\tau)}$, the last expression is monotonically decreasing as a function of $\text{Im}(\tau)$. It follows that for $\text{Im}(\tau) \ge \sqrt{3}/2$, we have
$$|E_2(\tau) - 1| \le \frac{24e^{-2\pi\sqrt{3}/2}}{(1 - e^{-2\pi\sqrt{3}/2})^3} = 0.105 \cdots .$$

The statement follows. ∎

Lemma 4.3. *The function p is holomorphic near each point $\tau \in T_0 \cap \mathbb{H}$. Moreover, $p'(\tau) = 0$ for $\tau \in T_0 \cap \mathbb{H}$ if and only if $\tau = \rho$.*

So, the only critical point of p on $T_0 \cap \mathbb{H}$ is at $\tau = \rho$.

Proof. The first part follows from (2.7), because E_2 does not take the value 0 on $T_0 \cap \mathbb{H}$ by Lemma 4.2. By (2.7) and (3.12), we also have

$$(4.1) \qquad p'(\tau) = 1 + \frac{6i E_2'(\tau)}{\pi E_2^2(\tau)} = \frac{E_4(\tau)}{E_2^2(\tau)}.$$

Now, by (3.10) all zeros of E_4 are given by $S(\rho)$, $S \in \mathrm{PSL}_2(\mathbb{Z})$. No point $\tau \in T_0 \setminus \{\rho\}$ is equivalent to ρ under the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}^*$. It follows that $\tau = \rho$ is the only zero of E_4 and hence of p' on T_0. ∎

Alternative proof of Theorem 1.1. We denote by

$$\begin{aligned}
A &:= \{it : t \geq 1\} \cup \{\infty\}, \\
B &:= \{e^{it} : \pi/3 \leq t \leq \pi/2\}, \\
C &:= \left\{\frac{1}{2} + it : t \geq \sqrt{3}/2\right\} \cup \{\infty\}
\end{aligned}$$

the three sides of T_0 and by

$$\begin{aligned}
A' &:= \{it : t \geq -1\} \cup \{\infty\}, \\
B' &:= \{e^{it} : -\pi/2 \leq t \leq -\pi/3\}, \\
C' &:= \left\{\frac{1}{2} + it : t \geq -\sqrt{3}/2\right\} \cup \{\infty\}
\end{aligned}$$

the three sides of T_1.

It follows directly from the definition (see (2.4)) that $E_2(\tau) \in \mathbb{R}$ when $\tau \in \mathbb{H}$ with $\mathrm{Re}(\tau) = 0$ or $\mathrm{Re}(\tau) = 1/2$. Here, $E_2(\tau) > 0$ if in addition $\mathrm{Im}(\tau)$ is large enough.

As a consequence, $\mathrm{Re}(p(\tau)) = 0$ if $\mathrm{Re}(\tau) = 0$ (see (2.7)). In particular, p sends $A \setminus \{\infty\}$ into the imaginary axis. Moreover, since $p'(\tau) \neq 0$ on A, we move strictly monotonically from $p(i) = -i$ to $p(\infty) = \infty$ as $\tau \in A$ travels from i to ∞. Here, $\mathrm{Im}(p(\tau))$ must be strictly increasing (and not decreasing) with increasing $\mathrm{Im}(\tau)$ for $\tau \in A$ as $\mathrm{Im}(p(\tau))$ is positive for $\tau \in A$ with large $\mathrm{Im}(\tau)$. This implies that p sends A homeomorphically onto A'.

Similarly, $\mathrm{Re}(p(\tau)) = 1/2$ if $\mathrm{Re}(\tau) = 1/2$. Hence, p sends $C \setminus \{\infty\}$ into the line $L := \{\tau \in \mathbb{C} : \mathrm{Re}(\tau) = 1/2\}$. Since $p'(\tau) \neq 0$ for interior points of C, we move strictly monotonically from $p(\rho) = \bar{\rho}$ to $p(\infty) = \infty$ on the line L as $\tau \in C$ moves from ρ to ∞ on C. Again, $\mathrm{Im}(p(\tau)) > 0$ for $\tau \in C$ with large $\mathrm{Im}(\tau)$, and so, we must have a strict increase here. This implies that p sends C homeomorphically onto C'.

Since p sends A into A', by the Schwarz Reflection Principle, we must have

$$p(-\bar{\tau}) = -\overline{p(\tau)}$$

for all $\tau \in \mathbb{H}$. If $\tau \in B$, then τ lies on the unit circle, and so, $\bar{\tau} = 1/\tau$. Then, the transformation behavior of p implies that

$$\overline{p(\tau)} = -p(-\bar{\tau}) = -p(-1/\tau) = 1/p(\tau),$$

and so, $|p(\tau)| = 1$. We see that p sends B into the unit circle. Since p' does not vanish at any interior point of B, we move strictly monotonically along the unit circle from $p(i) = -i$ to $p(\rho) = \bar{\rho}$ as τ moves from i to ρ along B. This implies that p sends B homeomorphically onto B' or the complementary arc B'' of B' on the unit circle with the same endpoints as B'.

Here, we actually must have $p(B) = B'$ as follows by considering orientations. Indeed, $p'(i) \neq 0$, and so p is a conformal and hence orientation-preserving map near i. We know that p sends A oriented in a positive direction from i to ∞ into the arc A' oriented from $-i$ to ∞. Since B lies on the right of the oriented arc A locally near i, the image $p(B)$ must lie on the right of A' near $p(i) = -i$. This only leaves the possibility that p sends B homeomorphically onto B' and not onto B''.

We conclude that p sends the three sides A, B, C of T_0 homeomorphically onto the three sides A', B', C' of T_1. It follows that p is a homeomorphism of ∂T_0 onto ∂T_1. Since p is holomorphic on the interior of T_0, Proposition 4.1 now implies that p is a conformal map of the circular arc triangle T_0 onto the circular arc triangle T_1. Here, the vertex correspondence is as in the statement and the claim follows. ∎

We want to point out a consequence of our considerations.

Corollary 4.4. *The critical points of p on $\mathbb{H}$ are precisely the points that are equivalent to ρ under the action of $\mathrm{PSL}_2(\mathbb{Z})$.*

Proof. It follows from Corollary 1.2 and the Schwarz Reflection Principle that if T is a triangle from the tessellation $\mathcal{T}$ (as defined in the introduction), then p is a conformal map near each point $\tau \in T \cap \mathbb{H}$ distinct from the vertex τ_T of T where the angle is $\pi/3$. There the angle is doubled by p, and so τ_T is a critical point (of first order). Therefore, p has precisely the critical points τ_T, $T \in \mathcal{T}$. These are precisely the points that are equivalent to ρ under the action of $\mathrm{PSL}_2(\mathbb{Z})$. ∎

As the proof shows, all critical points of p are of first order, meaning that the local degree of the map is 2 at these points. Corollary 4.4 can also be derived from (4.1) and (3.10).

5. Remarks

(1) Our route to derive Proposition 3.5 is essentially due to Fricke–Klein. In contrast to our approach, Fricke–Klein usually prefer to work with homogeneous modular forms. If $\tilde{f}(\omega_1, \omega_2)$ is such a form, then they use an associated form

$$D_\eta \tilde{f} = \eta_1 \frac{\partial \tilde{f}}{\partial \omega_1} + \eta_2 \frac{\partial \tilde{f}}{\partial \omega_2}.$$

If $\tilde{f}$ is a homogeneous modular form of degree $-k$, then it is straightforward to see that $\tilde{g} = D_\eta \tilde{f}$ is a homogeneous modular form of degree $-k - 2$. The differentiation process D_η occurs implicitly in Fricke–Klein [6] and was systematically used by Fricke in [5] (see also [7]).

Let us consider the associated inhomogeneous forms f and g so that

$$\tilde{f}(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_2}\right)^k f(\omega_1/\omega_2) \quad \text{and} \quad \tilde{g}(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_2}\right)^{k+2} g(\omega_1/\omega_2).$$

Then,

$$\begin{aligned}
\tilde{g} = D_\eta \tilde{f} &= \eta_1 \frac{\partial \tilde{f}}{\partial \omega_1} + \eta_2 \frac{\partial \tilde{f}}{\partial \omega_2} \\
&= (2\pi)^k (\eta_1 \omega_2^{-k-1} - \eta_2 \omega_1 \omega_2^{-k-2}) f'(\tau) - k(2\pi)^k \eta_2 \omega_2^{-k-1} f(\tau) \\
&= -(2\pi)^{k+1} i \omega_2^{-k-2} f'(\tau) - \frac{k}{12} (2\pi)^{k+2} \omega_2^{-k-2} E_2(\tau) f(\tau).
\end{aligned}$$

So, passing to the inhomogeneous version g of $\tilde{g}$, we have

$$g(\tau) = \frac{1}{(2\pi)^{k+2}} \tilde{g}(\tau, 1) = \frac{1}{2\pi i} f'(\tau) - \frac{k}{12} E_2(\tau) f(\tau),$$

and so,

$$g = Df - \frac{k}{12} E_2 f = \vartheta_k f.$$

This shows that the Serre derivative $\vartheta_k f$ of a modular form f of weight k corresponds to the homogeneous form $D_\eta \tilde{f}$. So, Serre derivative and the differentiation process D_η are essentially the same, but only differ whether one considers inhomogeneous or homogeneous modular forms.

(2) The meaning of the functions in (3.18) and (3.19) becomes more transparent if one writes them using the homogeneous function

$$\tilde{\Delta}(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_2}\right)^{12} \Delta(\omega_1/\omega_2)$$

and the associated root

$$\widetilde{\Delta}^{1/12}(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_2}\right)\Delta^{1/12}(\tau).$$

Then,

$$\Omega_1 = \frac{\omega_1}{2\pi}\widetilde{\Delta}^{1/12}(\omega_1, \omega_2), \quad \Omega_2 = \frac{\omega_2}{2\pi}\widetilde{\Delta}^{1/12}(\omega_1, \omega_2).$$

So, Ω_1 and Ω_2 are essentially just the periods ω_1 and ω_2 renormalized to make them homogeneous of degree 0. Similarly, using (2.5) and (2.6), one can see that

$$H_1 = \frac{6\eta_1}{\pi}\widetilde{\Delta}^{-1/12}(\omega_1, \omega_2), \quad H_2 = \frac{6\eta_2}{\pi}\widetilde{\Delta}^{-1/12}(\omega_1, \omega_2).$$

Again, here H_1 and H_2 are homogeneous of degree 0 as functions of the pair (ω_1, ω_2) and hence functions of $\tau = \omega_1/\omega_2$ alone; so the transition from (η_1, η_2) to (H_1, H_2) is also a normalization procedure.

(3) We can use Lemma 3.4 also to derive a hypergeometric differential equation with a pair of fundamental solutions given by Ω_1 and Ω_2. Indeed, if we use notation as in the proof of Proposition 3.5, then we have

$$\begin{aligned}
\frac{d^2\Omega}{dJ^2} &= \frac{d}{dJ}\left(-\frac{1}{24\sqrt{3}}J^{-2/3}(J-1)^{-1/2}H\right) \\
&= \left(-\frac{2}{3J} - \frac{1}{2(J-1)}\right)\frac{d\Omega}{dJ} - \frac{1}{24\sqrt{3}}J^{-2/3}(J-1)^{-1/2}\frac{dH}{dJ} \\
&= -\frac{7J-4}{6J(J-1)}\frac{d\Omega}{dJ} - \frac{1}{144J(J-1)}\Omega.
\end{aligned}$$

This is a hypergeometric differential equation with parameters $\alpha = \beta = 1/12$ and $\gamma = 2/3$. We conclude that $\tau = \Omega_1/\Omega_2$ as a function of J maps the closed upper and lower J-halfplanes onto circular arc triangles with angles $\lambda\pi$, $\mu\pi$, $\nu\pi$, where

$$\lambda = 1 - \gamma = 1/3, \quad \mu = \gamma - \alpha - \beta = 1/2, \quad \nu = \alpha - \beta = 0.$$

Indeed, this is in agreement with the fact, pointed out earlier, that $\tau \in T_0 \mapsto J(\tau)$ is a conformal map of T_0 onto the closed lower halfplane.

(4) The *Schwarzian derivative* of a meromorphic function $f(z)$ depending on a complex variable z is defined as

$$\begin{aligned}
\{f, z\} &:= \frac{d^2}{dz^2}\log f'(z) - \frac{1}{2}\left(\frac{d}{dz}\log f'(z)\right)^2 \\
&= \frac{2f'(z)f'''(z) - 3f''(z)^2}{2f'(z)^2}.
\end{aligned}$$

It is well known (see [1, p. 253]) that the ratio $f = w_1/w_2$ of two fundamental solutions of the hypergeometric differential equation has a Schwarzian derivative given by

$$\{f, z\} = \frac{1 - \lambda^2}{2z^2} + \frac{1 - \mu^2}{2(1 - z)^2} + \frac{1 - \lambda^2 - \mu^2 + \nu^2}{2z(1 - z)},$$

where

$$\lambda = 1 - \gamma, \quad \mu = \gamma - \alpha - \beta, \quad \nu = \alpha - \beta.$$

For our function $p = H_1/H_2$, we have $\lambda = 2/3$, $\mu = 1/2$, $\nu = 0$, and so,

$$\{p, J\} = \frac{5}{18J^2} + \frac{3}{8(1 - J)^2} + \frac{11}{72J(1 - J)}.$$

Similarly,

$$\{\tau, J\} = \frac{4}{9J^2} + \frac{3}{8(1 - J)^2} + \frac{23}{72J(1 - J)}.$$

The Schwarzian derivative of a function f is invariant under post-composition with a Möbius transformations S, namely,

$$\{S \circ f, z\} = \{f, z\}.$$

Moreover, we have the following chain rule for the Schwarzian derivative:

$$\{f \circ g, z\} = \{f, g\}\left(\frac{dg}{dz}\right)^2 + \{g, z\}.$$

If $S \in \mathrm{PSL}_2(\mathbb{Z})$ is arbitrary, then by $\mathrm{PSL}_2(\mathbb{Z})$-equivariance of our function p we have $S \circ p = p \circ S$. It follows that

$$f(\tau) := \{p, \tau\} = \{S \circ p, \tau\} = \{p \circ S, \tau\} = f(S(\tau))S'(\tau)^2.$$

This implies that f has the same transformation behavior as a modular form of weight 4.

This can be seen more explicitly as follows:

$$\{p, \tau\} = \{p, J\}J'(\tau)^2 + \{J, \tau\} = (\{p, J\} - \{\tau, J\})J'(\tau)^2$$
$$= -\frac{J'^2}{6}\left(\frac{1}{J^2} + \frac{1}{J(1 - J)}\right) = \frac{J'^2}{6J^2(J - 1)}$$
$$= -8\pi^2\Delta^{1/3}/J^{2/3} = -1152\pi^2\Delta/E_4^2.$$

This last expression for $\{p, \tau\}$ was also recorded in [12, Proposition 6.2]. Since Δ has weight 12 and E_4 has weight 4, it clearly transforms as a modular form of weight 4. Note that $\{p, \tau\}$ is not a modular form according to our definition as this function has

poles, namely, exactly at the points equivalent to ρ under $\mathrm{PSL}_2(\mathbb{Z})$. These points are precisely the critical points of p as we know from Corollary 4.4.

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Mario Bᴏɴᴋ, Department of Mathematics, University of California, Los Angeles, CA 90055, USA; *e-mail:* mbonk@math.ucla.edu