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Quantitative asymptotic regularity of the VAM iteration with error terms for m-accretive operators in Banach spaces

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Abstract. In this paper, we obtain, by using proof mining methods, quantitative results on the asymptotic regularity of the viscosity approximation method (VAM) with error terms for m-accretive operators in Banach spaces. For concrete instances of the parameter sequences, linear rates are computed by applying a lemma due to Sabach and Shtern.

1. Introduction

Let X be a normed space, $A : X \to 2^X$ an accretive operator with a nonempty set of zeros, and $C \subseteq X$ a nonempty closed convex subset of X such that $\overline{\text{dom}\, A} \subseteq C \subseteq \text{ran}(\text{Id} + \gamma A)$ for all $\gamma > 0$. Xu et al. [26] studied recently the following iteration:

$$\text{VAM} \quad x_0 = x, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J^A_{\lambda_n} x_n, \tag{1.1}$$

where $x \in C$, $f : C \to C$ is an α-contraction for $\alpha \in [0, 1)$, (λ_n) is a sequence in $(0, \infty)$, (α_n) is a sequence in $[0, 1]$, and, for every $n \in \mathbb{N}$, $J^A_{\lambda_n}$ is the resolvent of order λ_n of A.

The VAM iteration is an instance of the viscosity approximation method applied to resolvents of accretive operators in Banach spaces (see, for example, [2,16,19,21,22,25]). If one takes $f(x) = u \in X$ in (1.1), one gets the Halpern-type proximal point algorithm (HPPA), introduced by Kamimura and Takahashi [10] and Xu [24], a modification of the proximal point algorithm that was studied in a series of papers in recent years. Thus, VAM is a viscosity version of the HPPA.

Xu et al. [26] proved, in the setting of uniformly convex and/or uniformly Gâteaux differentiable Banach spaces, strong convergence results for the VAM iteration towards a zero of A, extending results for the HPPA obtained by Aoyama and Toyoda [1].

As it is the case with numerous convergence proofs, an intermediate step is to obtain the asymptotic regularity of the iteration. Asymptotic regularity was defined by Browder and Petryshyn [5] for the Picard iteration and extended to general iterations by Borwein,

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Reich, and Shafrir [3]. By inspecting the proofs from [26], one can see that asymptotic regularity of the VAM iteration holds, under some hypotheses on (α_n), (λ_n), in the more general setting of Banach spaces.

In this paper, we prove quantitative asymptotic regularity results for the VAMe iteration for m-accretive operators, defined by adding error terms to the VAM iteration (see (4.1)). These quantitative results provide uniform rates of asymptotic regularity, $(J^A_{\lambda_n})$-asymptotic regularity and, for all $m \in \mathbb{N}$, $J^A_{\lambda_m}$-asymptotic regularity for VAMe. We compute such linear rates for concrete instances of the parameter sequences (α_n), (λ_n) as an application of a lemma of Sabach and Shtern [23]. As VAM and HPPA for m-accretive operators are particular cases of our VAMe iteration, we obtain rates for these iterations, too. Furthermore, as an immediate consequence of our quantitative results, we obtain qualitative asymptotic regularity results for the VAMe iteration.

The results from the paper are obtained by applying methods of proof mining, a research program concerned with the extraction, by using proof-theoretic techniques, of new information from mathematical proofs. We refer to Kohlenbach's textbook [12] for details on proof mining and to [13, 14] for surveys of recent applications in nonlinear analysis and optimization. Finally, let us remark that proof mining was applied recently by Kohlenbach and Pinto [15] to obtain quantitative results, providing rates of metastability, for viscosity approximation methods in W-hyperbolic spaces.

2. Preliminaries

Let X be a normed space and $A : X \to 2^X$ a set-valued operator on X. As usual, we identify the operator A with its graph $\text{gra}\, A = \{(x, y) \in X \times X \mid y \in Ax\}$.

Let $\text{dom}\, A = \{x \in X \mid Ax \neq \emptyset\}$ be the domain of A and $\text{ran}\, A = \bigcup_{x \in X} Ax$ the range of A. Furthermore, we denote by $\text{zer}\, A$ the set of zeros of A, that is,

$$\text{zer}\, A = \{x \in X \mid 0 \in Ax\}.$$

The definition of the inverse A^{-1} of A is given through its graph:

$$\text{gra}\, A^{-1} = \{(y, x) \in X \times X \mid (x, y) \in \text{gra}\, A\} = \{(y, x) \in X \times X \mid y \in Ax\}.$$

If $\lambda \in \mathbb{R}$ and B is another set-valued operator on X, then $\lambda A = \{(x, \lambda y) \mid x \in X, y \in Ax\}$ and $A + B = \{(x, y + z) \mid x \in X, y \in Ax, z \in Bx\}$. For every $\gamma > 0$, the resolvent J^A_γ of order γ of A is defined by $J^A_\gamma = (\text{Id} + \gamma A)^{-1}$, where Id is the identity operator on X. One can easily verify that $\text{dom}\, J^A_\gamma = \text{ran}(\text{Id} + \gamma A)$ and $\text{ran}\, J^A_\gamma = \text{dom}\, A$.

Let us recall that if $\emptyset \neq C \subseteq X$ and $T : C \to X$ is a mapping, we denote by $\text{Fix}(T)$ the set of fixed points of T and T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in X$.

An operator A is said to be *accretive* [4, 11] if for all $x, y \in \text{dom}\, A$, $u \in Ax$, $v \in Ay$, and $\gamma > 0$,

$$\|x - y + \gamma(u - v)\| \geq \|x - y\|.$$

It is well known that, for any accretive operator A and for all $\gamma > 0$,

$$J_\gamma^A : \operatorname{ran}(\mathrm{Id} + \gamma A) \to \operatorname{dom} A$$

is a nonexpansive mapping such that $\mathrm{Fix}(J_\gamma^A) = \operatorname{zer} A$ (see, for example, [9, Corollary 3.4.1] and [9, Proposition 6.7.1] for proofs).

Lemma 2.1. *Assume that A is an accretive operator. Let $\lambda, \gamma > 0$.*

(i) *If $x \in \operatorname{ran}(\mathrm{Id} + \lambda A)$, then $\frac{\gamma}{\lambda}x + (1 - \frac{\gamma}{\lambda})J_\lambda^A x \in \operatorname{ran}(\mathrm{Id} + \gamma A)$ and*

$$J_\lambda^A x = J_\gamma^A\Big(\frac{\gamma}{\lambda}x + \Big(1 - \frac{\gamma}{\lambda}\Big)J_\lambda^A x\Big). \tag{2.1}$$

(ii) *For all $x \in \operatorname{ran}(\mathrm{Id} + \lambda A) \cap \operatorname{ran}(\mathrm{Id} + \gamma A)$,*

$$\|J_\gamma^A x - J_\lambda^A x\| \le \Big|1 - \frac{\gamma}{\lambda}\Big| \big\|J_\lambda^A x - x\big\|. \tag{2.2}$$

Proof. For the proof of (2.1), see [9, Proposition 3.4.1]. Inequality (2.2) follows immediately from (2.1) and the fact that J_γ^A is nonexpansive:

$$\begin{aligned}\|J_\gamma^A x - J_\lambda^A x\| &= \Big\|J_\gamma^A x - J_\gamma^A\Big(\frac{\gamma}{\lambda}x + \Big(1 - \frac{\gamma}{\lambda}\Big)J_\lambda^A x\Big)\Big\| \\ &\le \Big\|x - \frac{\gamma}{\lambda}x - \Big(1 - \frac{\gamma}{\lambda}\Big)J_\lambda^A x\Big\| = \Big|1 - \frac{\gamma}{\lambda}\Big| \big\|x - J_\lambda^A x\big\|. \end{aligned}$$ ∎

An *m-accretive* operator is an accretive operator A that satisfies $\operatorname{ran}(\mathrm{Id} + \gamma A) = X$ for all $\gamma > 0$. It follows that, for an m-accretive operator A, (2.1) and (2.2) hold for all $x \in X$.

3. Quantitative notions and lemmas

Let us recall the main quantitative notions that will be used in this paper. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X, d). A mapping $\varphi : \mathbb{N} \to \mathbb{N}$ is said to be

(i) a Cauchy modulus of (a_n) if, for all $k \in \mathbb{N}$ and all $n \ge \varphi(k)$,

$$d(a_{n+p}, a_n) \le \frac{1}{k+1} \quad \text{holds for all } p \in \mathbb{N},$$

(ii) a rate of convergence of (a_n) (towards $a \in X$) if, for all $k \in \mathbb{N}$ and all $n \ge \varphi(k)$,

$$d(a_n, a) \le \frac{1}{k+1}.$$

Obviously, (a_n) is Cauchy iff (a_n) has a Cauchy modulus, and $\lim_{n\to\infty} a_n = a$ iff (a_n) has a rate of convergence towards a.

Assume that $\sum_{n=0}^{\infty} b_n$ is a series of nonnegative real numbers and $(\tilde{b}_n = \sum_{i=0}^{n} b_i)$ is the sequence of partial sums. Then, a Cauchy modulus of the series is a Cauchy modulus of $(\tilde{b}_n)$. A rate of divergence of the series is a mapping $\theta : \mathbb{N} \to \mathbb{N}$ satisfying $\sum_{i=0}^{\theta(n)} b_i \geq n$ for all $n \in \mathbb{N}$. It is clear that $\sum_{n=0}^{\infty} b_n$ diverges iff it has a rate of divergence.

Let (y_n) be a sequence in a metric space (X, d), $\Phi : \mathbb{N} \to \mathbb{N}$, C be a nonempty subset of X, $T : C \to C$, and $(T_n : C \to C)_{n\in\mathbb{N}}$ be a countable family of mappings. We say that

(i) (y_n) is asymptotically regular with rate Φ (or Φ is a rate of asymptotic regularity of (y_n)) if $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ with rate of convergence Φ;

(ii) (y_n) is T-asymptotically regular with rate Φ (or Φ is a rate of T-asymptotic regularity of (y_n)) if $\lim_{n\to\infty} d(y_n, Ty_n) = 0$ with rate of convergence Φ;

(iii) (y_n) is (T_n)-asymptotically regular with rate Φ (or Φ is a rate of (T_n)-asymptotic regularity of (y_n)) if $\lim_{n\to\infty} d(y_n, T_n y_n) = 0$ with rate of convergence Φ.

3.1. Useful lemmas on sequences of real numbers

Lemma 3.1. *If (b_n) is a sequence in $[0, 1]$ and θ is a rate of divergence for $\sum_{n=0}^{\infty} b_n$, then $\theta(n) \geq n - 2$ for all $n \in \mathbb{N}$.*

Proof. Assume that $\theta(n) < n - 2$ for some $n \in \mathbb{N}$. It follows that $\sum_{i=0}^{\theta(n)} b_i \leq \sum_{i=0}^{n-2} b_i \leq n - 1 < n$, which is a contradiction. ■

Lemma 3.2. *Let (a_n), (b_n) be sequences of nonnegative real numbers, $p, q \in \mathbb{N}$, and $c_n = pa_n + qb_n$ for all $n \in \mathbb{N}$. Assume that (a_n) is Cauchy with Cauchy modulus φ_1 and (b_n) is Cauchy with Cauchy modulus φ_2. Then, (c_n) is Cauchy with Cauchy modulus*

$$\varphi(k) = \max\{\varphi_1(2p(k + 1) - 1), \varphi_2(2q(k + 1) - 1)\}.$$

Proof. Let $k \in \mathbb{N}$. We get that, for all $n \geq \varphi(k)$ and all $p \in \mathbb{N}$,

$$\begin{aligned} c_{n+p} - c_n &= p(a_{n+p} - a_n) + q(b_{n+p} - b_n) \\ &\leq p \cdot \frac{1}{2p(k + 1)} + q \cdot \frac{1}{2q(k + 1)} = \frac{1}{k + 1}. \end{aligned}$$ ■

The following result is [18, Proposition 2.7], which is a reformulation of [8, Lemma 2.9 (1)], obtained by taking $\frac{1}{k+1}$ instead of ε. It is a quantitative version of a particular case of a very useful lemma on sequences of real numbers due to Xu [24].

Proposition 3.3. *Let (a_n) be a sequence in $[0, 1]$ and (c_n), (s_n) sequences of nonnegative real numbers such that for all $n \in \mathbb{N}$,*

$$s_{n+1} \leq (1 - a_n)s_n + c_n. \tag{3.1}$$

Assume that $L \in \mathbb{N}^*$ is an upper bound on (s_n), $\sum_{n=0}^{\infty} a_n$ diverges with rate of divergence θ, and $\sum_{n=0}^{\infty} c_n$ converges with Cauchy modulus χ.

Then, $\lim_{n\to\infty} s_n = 0$ with rate of convergence Σ defined by

$$\Sigma(k) = \theta(\chi(2k + 1) + 1 + \lceil \ln(2L(k + 1)) \rceil) + 1.$$

The next lemma is a slight variation of [23, Lemma 3], proved in [18, Lemma 2.8].

Lemma 3.4. *Let $L > 0$, $J \geq N \geq 2$, $\gamma \in (0, 1]$, (c_n) be a sequence bounded above by L, and $a_n = \frac{N}{\gamma(n+J)}$ for all $n \in \mathbb{N}$. Suppose that (s_n) is a sequence of nonnegative real numbers such that $s_0 \leq L$ and, for all $n \in \mathbb{N}$,*

$$s_{n+1} \leq (1 - \gamma a_{n+1})s_n + (a_n - a_{n+1})c_n.$$

Then, for all $n \in \mathbb{N}$,

$$s_n \leq \frac{JL}{\gamma(n+J)}.$$

4. VAM with errors for resolvents of m-accretive operators in Banach spaces

Let X be a normed space, $A : X \to 2^X$ an m-accretive operator such that zer $A \neq \emptyset$, and $f : X \to X$ an α-contraction for $\alpha \in [0, 1)$, that is, $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in X$.

We consider the iteration (x_n) defined as follows:

$$\text{VAMe} \quad x_0 = x \in X, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J^A_{\lambda_n} x_n + e_n, \tag{4.1}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $[0, 1]$, $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence in $(0, \infty)$, and $(e_n)_{n\in\mathbb{N}}$ is a sequence in X. Hence, (x_n) is obtained from the VAM iteration studied in [26] by adding error terms e_n.

For every $z \in$ zer A, let $(K_{z,n})_{n\in\mathbb{N}}$ be a sequence of real numbers defined as follows:

$$K_{z,0} = \max\left\{\|x - z\|, \frac{\|f(z) - z\|}{1 - \alpha}\right\}, \quad K_{z,n} = K_{z,0} + \sum_{i=0}^{n-1} \|e_i\| \text{ for all } n \geq 1. \tag{4.2}$$

Thus, $K_{z,n+1} = K_{z,n} + \|e_n\|$ for all $n \geq 0$.

Lemma 4.1. *For all $z \in$ zer A and $m, n \in \mathbb{N}$,*

(i) $\|x_n - z\|, \|f(x_n) - z\| \leq K_{z,n}$;

(ii) $\|x_{n+1} - x_n\| \leq 2K_{z,n+1}$;

(iii) $\|J^A_{\lambda_m} x_n - z\| \leq K_{z,n}$;

(iv) $\|J^A_{\lambda_m} x_n - x_n\|, \|J^A_{\lambda_m} x_n - f(x_n)\| \leq 2K_{z,n}$.

Proof. (i) We prove the two inequalities simultaneously by induction on n.

$n = 0$: $\|x_0 - z\| \leq K_{z,0}$ follows by (4.2). Furthermore, applying the fact that f is an α-contraction and (4.2), we get that

$$\|f(x_0) - z\| \leq \|f(x_0) - f(z)\| + \|f(z) - z\| \leq \alpha\|x_0 - z\| + (1 - \alpha)K_{z,0} \leq K_{z,0}.$$

$n \Rightarrow n+1$: we have that

$$\begin{aligned}\|x_{n+1}-z\| &= \left\|\alpha_n(f(x_n)-z)+(1-\alpha_n)(J^A_{\lambda_n}x_n - J^A_{\lambda_n}z)+e_n\right\| \quad \text{as } J^A_{\lambda_n}z=z\\ &\le \alpha_n\|f(x_n)-z\|+(1-\alpha_n)\|x_n-z\|+\|e_n\|\\ &\quad \text{as } J^A_{\lambda_n} \text{ is nonexpansive}\\ &\le K_{z,n}+\|e_n\| \quad \text{by the induction hypothesis}\\ &= K_{z,n+1}.\end{aligned}$$

Moreover, $\|f(x_{n+1})-z\| \le \alpha\|x_{n+1}-z\|+\|f(z)-z\| \le K_{z,n+1}$.

(ii) $\|x_{n+1}-x_n\| \le \|x_n-z\|+\|x_{n+1}-z\| \le K_{z,n}+K_{z,n+1} \le 2K_{z,n+1}$.

(iii) $\|J^A_{\lambda_m}x_n - z\| = \|J^A_{\lambda_m}x_n - J^A_{\lambda_m}z\| \le \|x_n-z\| \le K_{z,n}$.

(iv) $\|J^A_{\lambda_m}x_n - x_n\| \le \|J^A_{\lambda_m}x_n - z\|+\|x_n-z\| \le 2K_{z,n}$ and

$$\left\|J^A_{\lambda_m}x_n - f(x_n)\right\| \le \left\|J^A_{\lambda_m}x_n - z\right\|+\|f(x_n)-z\| \le 2K_{z,n}. \qquad \blacksquare$$

The following is the main inequality that will be used in the proof of one of our main results from Section 5.

Proposition 4.2. *For all $n \in \mathbb{N}$,*

$$\|x_{n+2}-x_{n+1}\| \le (1-(1-\alpha)\alpha_{n+1})\|x_{n+1}-x_n\|+M_{z,n}+\|e_{n+1}-e_n\|, \tag{4.3}$$
$$\|x_{n+2}-x_{n+1}\| \le (1-(1-\alpha)\alpha_{n+1})\|x_{n+1}-x_n\|+M^*_{z,n}+\|e_{n+1}-e_n\|, \tag{4.4}$$

where

$$\begin{aligned}M_{z,n} &= 2K_{z,n}\left(|\alpha_{n+1}-\alpha_n|+(1-\alpha_{n+1})\left|1-\frac{\lambda_{n+1}}{\lambda_n}\right|\right),\\ M^*_{z,n} &= 2K_{z,n}\left(|\alpha_{n+1}-\alpha_n|+(1-\alpha_{n+1})\left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|\right).\end{aligned}$$

Proof. We have that

$$\begin{aligned}x_{n+2}-x_{n+1} &= \left(\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})J^A_{\lambda_{n+1}}x_{n+1}\right)\\ &\quad - \left(\alpha_{n+1}f(x_n)+(1-\alpha_{n+1})J^A_{\lambda_n}x_n\right)\\ &\quad + \left(\alpha_{n+1}f(x_n)+(1-\alpha_{n+1})J^A_{\lambda_n}x_n\right)\\ &\quad - \left(\alpha_n f(x_n)+(1-\alpha_n)J^A_{\lambda_n}x_n\right)+e_{n+1}-e_n\\ &= \alpha_{n+1}(f(x_{n+1})-f(x_n))+(1-\alpha_{n+1})\left(J^A_{\lambda_{n+1}}x_{n+1}-J^A_{\lambda_n}x_n\right)\\ &\quad + (\alpha_{n+1}-\alpha_n)f(x_n)+(\alpha_n-\alpha_{n+1})J^A_{\lambda_n}x_n+e_{n+1}-e_n\\ &= \alpha_{n+1}(f(x_{n+1})-f(x_n))+(1-\alpha_{n+1})\left(J^A_{\lambda_{n+1}}x_{n+1}-J^A_{\lambda_n}x_n\right)\\ &\quad + (\alpha_{n+1}-\alpha_n)\left(f(x_n)-J^A_{\lambda_n}x_n\right)+e_{n+1}-e_n.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le \alpha_{n+1}\alpha\|x_{n+1}-x_n\| + (1-\alpha_{n+1})\left\|J^A_{\lambda_{n+1}}x_{n+1} - J^A_{\lambda_n}x_n\right\| \\ &\quad + |\alpha_{n+1}-\alpha_n|\left\|f(x_n) - J^A_{\lambda_n}x_n\right\| + \|e_{n+1}-e_n\| \\ &\le \alpha_{n+1}\alpha\|x_{n+1}-x_n\| + (1-\alpha_{n+1})\left\|J^A_{\lambda_{n+1}}x_{n+1} - J^A_{\lambda_n}x_n\right\| \\ &\quad + 2K_{z,n}|\alpha_{n+1}-\alpha_n| + \|e_{n+1}-e_n\| \quad \text{by Lemma 4.1 (iv).}\end{aligned}$$

As

$$\begin{aligned}\left\|J^A_{\lambda_{n+1}}x_{n+1} - J^A_{\lambda_n}x_n\right\| &\le \left\|J^A_{\lambda_{n+1}}x_{n+1} - J^A_{\lambda_{n+1}}x_n\right\| + \left\|J^A_{\lambda_{n+1}}x_n - J^A_{\lambda_n}x_n\right\| \\ &\le \|x_{n+1}-x_n\| + \left\|J^A_{\lambda_{n+1}}x_n - J^A_{\lambda_n}x_n\right\|,\end{aligned}$$

it follows that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| \le (\alpha_{n+1}\alpha + 1 - \alpha_{n+1})\|x_{n+1}-x_n\| + (1-\alpha_{n+1})\left\|J^A_{\lambda_{n+1}}x_n - J^A_{\lambda_n}x_n\right\| \\ + 2K_{z,n}|\alpha_{n+1}-\alpha_n| + \|e_{n+1}-e_n\|.\end{aligned}$$

By (2.2) and Lemma 4.1 (iv), we have that

$$\left\|J^A_{\lambda_{n+1}}x_n - J^A_{\lambda_n}x_n\right\| \le \left|1-\frac{\lambda_{n+1}}{\lambda_n}\right|\left\|J^A_{\lambda_n}x_n - x_n\right\| \le 2K_{z,n}\left|1-\frac{\lambda_{n+1}}{\lambda_n}\right|, \tag{4.5}$$

$$\left\|J^A_{\lambda_{n+1}}x_n - J^A_{\lambda_n}x_n\right\| \le \left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|\left\|J^A_{\lambda_{n+1}}x_n - x_n\right\| \le 2K_{z,n}\left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|. \tag{4.6}$$

Apply (4.5) and (4.6) to conclude that (4.3) and (4.4) hold. ∎

4.1. Quantitative hypotheses on the parameter sequences

We consider the following hypotheses on the parameter sequences (α_n), (λ_n), (e_n) from the definition (4.1) of the VAMe iteration (x_n):

$(H1\alpha_n)$ $\displaystyle\sum_{n=0}^{\infty}\alpha_n = \infty$ with divergence rate σ_1;

$(H2\alpha_n)$ $\displaystyle\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$ with Cauchy modulus σ_2;

$(H3\alpha_n)$ $\displaystyle\lim_{n\to\infty}\alpha_n = 0$ with rate of convergence σ_3;

$(H1\lambda_n)$ $\displaystyle\sum_{n=0}^{\infty}\left|1-\frac{\lambda_{n+1}}{\lambda_n}\right| < \infty$ with Cauchy modulus γ_1;

$(H1\lambda_n^*)$ $\displaystyle\sum_{n=0}^{\infty}\left|1-\frac{\lambda_n}{\lambda_{n+1}}\right| < \infty$ with Cauchy modulus γ_1^*;

$(H2\lambda_n)$ $\Lambda \in \mathbb{N}^*$ and $N_\Lambda \in \mathbb{N}$ are such that $\lambda_n \geq \frac{1}{\Lambda}$ for all $n \geq N_\Lambda$;

$(H3\lambda_n)$ $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ with Cauchy modulus γ_3;

$(H1e_n)$ $\sum_{n=0}^{\infty} \|e_n\| < \infty$ with Cauchy modulus θ_1;

$(H2e_n)$ $\lim_{n\to\infty} \|e_n\| = 0$ with rate of convergence θ_2;

$(H3e_n)$ $E \in \mathbb{N}^*$ is an upper bound on $\sum_{n=0}^{\infty} \|e_n\|$.

Lemma 4.3. (i) *Assume that* $(H3e_n)$ *holds. For every* $z \in \mathrm{zer}\, A$*, let* $K_z \in \mathbb{N}^*$ *be such that*

$$K_z \geq \max\left\{\|x - z\|, \frac{\|f(z) - z\|}{1-\alpha}\right\} + E.$$

Then, $K_{z,n} \leq K_z$ *for all* $n \in \mathbb{N}$*. Hence, Lemma 4.1 and inequalities (4.3), (4.4) hold with* K_z *instead of* $K_{z,n}$ *or* $K_{z,n+1}$*.*

(ii) *Suppose that* $(H1e_n)$ *holds. Then,* $(H2e_n)$ *is satisfied with* $\theta_2(k) = \theta_1(k) + 1$ *and* $(H3e_n)$ *is satisfied with* $E = \lceil \sum_{i=0}^{\theta_1(0)} \|e_i\| \rceil + 1$.

Proof. Let us denote, for all $m \in \mathbb{N}$, $\tilde{e}_m = \sum_{i=0}^{m} \|e_i\|$.

(i) It is obvious, by (4.2).

(ii) Let $k \in \mathbb{N}$ and $n \geq \theta_2(k)$. We get that $\|e_n\| = \tilde{e}_n - \tilde{e}_{n-1} \leq \frac{1}{k+1}$, as $n - 1 \geq \theta_1(k)$, so we can apply $(H1e_n)$.
Obviously, if $n < \theta_1(0)$, we have that $\tilde{e}_n \leq \tilde{e}_{\theta_1(0)} < E$. Let $n \geq \theta_1(0)$. By $(H1e_n)$, we get that $\tilde{e}_n - \tilde{e}_{\theta_1(0)} \leq 1$; hence, $\tilde{e}_n \leq E$. ∎

Lemma 4.4. *Assume* $(H2\lambda_n)$ *and* $(H3\lambda_n)$ *hold. Then,* $(H1\lambda_n)$ *and* $(H1\lambda_n^*)$ *hold with*

$$\gamma_1(k) = \gamma_1^*(k) = \max\{N_\Lambda, \gamma_3(\Lambda(k+1) - 1)\}. \tag{4.7}$$

Proof. Let us denote $\widetilde{\lambda}_n = \sum_{i=0}^{n} |\lambda_i - \lambda_{i+1}|$. We get that, for all $n \geq \gamma_1(k)$ and all $p \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{i=0}^{n+p} \left|1 - \frac{\lambda_{i+1}}{\lambda_i}\right| - \sum_{i=0}^{n} \left|1 - \frac{\lambda_{i+1}}{\lambda_i}\right| \\
&= \sum_{i=n+1}^{n+p} \frac{1}{\lambda_i} |\lambda_i - \lambda_{i+1}| \leq \sum_{i=n+1}^{n+p} \Lambda |\lambda_i - \lambda_{i+1}| \quad \text{by } (H2\lambda_n) \\
&= \Lambda(\widetilde{\lambda}_{n+p} - \widetilde{\lambda}_n) \leq \frac{1}{k+1} \quad \text{by } (H3\lambda_n).
\end{aligned}$$

The fact that $(H1\lambda_n^*)$ holds is obtained similarly. ∎

5. Rates of asymptotic regularity, $(J_{\lambda_n}^A)$-asymptotic regularity and, for all $m \in \mathbb{N}$, $J_{\lambda_m}^A$-asymptotic regularity

Throughout this section, X is a Banach space, $A : X \to 2^X$ is an m-accretive operator such that $\operatorname{zer} A \neq \emptyset$, $f : X \to X$ is an α-contraction for $\alpha \in [0, 1)$, $x \in X$, and (x_n) is the VAMe iteration starting with x, defined by (4.1).

The first main result of the paper gives effective rates of asymptotic regularity of (x_n).

Theorem 5.1. *Suppose that* $(H1\alpha_n)$, $(H2\alpha_n)$, $(H1\lambda_n)$, *and* $(H1e_n)$ *hold. Let* $z \in \operatorname{zer} A$, $K_z \in \mathbb{N}^*$, *be such that*

$$K_z \geq \max\left\{\|x - z\|, \frac{\|f(z) - z\|}{1 - \alpha}\right\} + \left\lceil \sum_{i=0}^{\theta_1(0)} \|e_i\| \right\rceil + 1, \tag{5.1}$$

and

$$\chi(k) = \max\{\sigma_2(6K_z(k + 1) - 1), \gamma_1(6K_z(k + 1) - 1), \theta_1(6k + 5)\}.$$

Then, (x_n) *is asymptotically regular with rate* $\Phi : \mathbb{N} \to \mathbb{N}$ *defined by*

$$\Phi(k) = \sigma_1\left(\left\lceil \frac{\chi(2k + 1) + 1 + \lceil \ln(4K_z(k + 1)) \rceil}{1 - \alpha} \right\rceil + 1\right).$$

Proof. We show that we can apply Proposition 3.3 with $s_n = \|x_{n+1} - x_n\|$, $L = 2K_z$,

$$a_n = (1 - \alpha)\alpha_{n+1}, \quad \text{and} \quad c_n = 2K_z\left(|\alpha_{n+1} - \alpha_n| + \left|1 - \frac{\lambda_{n+1}}{\lambda_n}\right|\right) + \|e_{n+1} - e_n\|.$$

Let us remark first that (3.1) holds, as a consequence of (4.3) and Lemma 4.3 (i). Furthermore, by Lemmas 4.1 (ii) and 4.3 (i), we have that L is an upper bound on (s_n).

For the rest of the proof, let $k \in \mathbb{N}$ be arbitrary. Define

$$\theta(k) = \max\left\{\sigma_1\left(\left\lceil \frac{k}{1 - \alpha} \right\rceil + 1\right) - 1, 0\right\}.$$

It follows that

$$\begin{aligned}
\sum_{n=0}^{\theta(k)} a_n &= (1 - \alpha)\left(\sum_{n=0}^{\theta(k)+1} \alpha_n - \alpha_0\right) \geq (1 - \alpha)\left(\sum_{n=0}^{\sigma_1(\lceil \frac{k}{1-\alpha} \rceil + 1)} \alpha_n - \alpha_0\right) \\
&\geq (1 - \alpha)\left(\left\lceil \frac{k}{1 - \alpha} \right\rceil + 1 - \alpha_0\right) \quad \text{by } (H1\alpha_n) \\
&\geq (1 - \alpha)\left\lceil \frac{k}{1 - \alpha} \right\rceil \quad \text{as } \alpha_0 \leq 1 \\
&\geq k.
\end{aligned}$$

Thus, θ is a rate of divergence of $\sum_{n=0}^{\infty} a_n$.

Denote, for all $m \in \mathbb{N}$,

$$\widetilde{\alpha}_m = \sum_{i=0}^{m} |\alpha_{i+1} - \alpha_i|, \quad \widetilde{\lambda}_m = \sum_{i=0}^{m} \left|1 - \frac{\lambda_{i+1}}{\lambda_i}\right|,$$
$$\tilde{e}_m = \sum_{i=0}^{m} \|e_i\|, \quad \text{and} \quad \tilde{c}_m = \sum_{i=0}^{m} c_i.$$

We get that, for all $n \geq \chi(k)$ and all $p \in \mathbb{N}^*$,

$$\begin{aligned}
\tilde{c}_{n+p} - \tilde{c}_n &= 2K_z((\widetilde{\alpha}_{n+p} - \widetilde{\alpha}_n) + (\widetilde{\lambda}_{n+p} - \widetilde{\lambda}_n)) + \sum_{i=n+1}^{n+p} \|e_{i+1} - e_i\| \\
&\leq \frac{4K_z}{6K_z(k+1)} + \sum_{i=n+1}^{n+p} \|e_{i+1} - e_i\| \quad \text{by } (H2\alpha_n) \text{ and } (H1\lambda_n) \\
&\leq \frac{2}{3(k+1)} + \sum_{i=n+1}^{n+p} (\|e_{i+1}\| + \|e_i\|) \\
&= \frac{2}{3(k+1)} + (\tilde{e}_{n+1+p} - \tilde{e}_{n+1}) + (\tilde{e}_{n+p} - \tilde{e}_n) \\
&\leq \frac{2}{3(k+1)} + \frac{2}{6(k+1)} \quad \text{as } n \geq \theta_1(6k+5), \text{ so we can apply } (H1e_n) \text{ twice} \\
&= \frac{1}{k+1}.
\end{aligned}$$

Thus, $\sum_{n=0}^{\infty} c_n$ converges with Cauchy modulus χ.

We can apply Proposition 3.3 to conclude that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ with rate of convergence

$$\begin{aligned}
\Sigma(k) = \theta(P) + 1 &= \max\left\{\sigma_1\left(\left\lceil \frac{P}{1-\alpha} \right\rceil + 1\right) - 1, 0\right\} + 1 \\
&= \max\left\{\sigma_1\left(\left\lceil \frac{P}{1-\alpha} \right\rceil + 1\right), 1\right\},
\end{aligned}$$

where $P = \chi(2k+1) + 1 + \lceil \ln(4K_z(k+1)) \rceil$. As $\lceil \frac{P}{1-\alpha} \rceil + 1 \geq P + 1 \geq 2 + \lceil \ln 4 \rceil = 4$, it follows, by Lemma 3.1, that $\sigma_1(\lceil \frac{P}{1-\alpha} \rceil + 1) \geq 2$; hence,

$$\Sigma(k) = \sigma_1\left(\left\lceil \frac{P}{1-\alpha} \right\rceil + 1\right) = \Phi(k). \qquad \blacksquare$$

Remark 5.2. Theorem 5.1 holds if we replace in the hypothesis $(H1\lambda_n)$ with $(H1\lambda_n^*)$ and in the rates γ_1 with γ_1^*. In the proof, we apply (4.4) instead of (4.3).

Remark 5.3. By Lemma 4.4, Theorem 5.1 also holds if we assume $(H2\lambda_n)$ and $(H3\lambda_n)$ instead of $(H1\lambda_n)$. Then, γ_1 is given by (4.7).

The second main result shows that, given a rate of asymptotic regularity of (x_n), one can compute, under some quantitative hypotheses on the parameter sequences, rates of $(J^A_{\lambda_n})$-asymptotic regularity and of $J^A_{\lambda_m}$-asymptotic regularity for every $m \in \mathbb{N}$.

Theorem 5.4. *Suppose that Φ is a rate of asymptotic regularity of (x_n), $(H2e_n)$ holds, $z \in \operatorname{zer} A$, and $K_z \in \mathbb{N}^*$ satisfies (5.1).*

(i) *Assume that $(H3\alpha_n)$ holds. Define $\Psi : \mathbb{N} \to \mathbb{N}$ by*

$$\Psi(k) = \max\{\sigma_3(6K_z(k+1)-1), \Phi(3k+2), \theta_2(3k+2)\}.$$

Then, Ψ is a rate of $(J^A_{\lambda_n})$-asymptotic regularity of (x_n).

(ii) *Assume that $(H3\alpha_n)$ and $(H2\lambda_n)$ hold. Define, for every $m \in \mathbb{N}$, $\Theta_m : \mathbb{N} \to \mathbb{N}$ by*

$$\Theta_m(k) = \max\{N_\Lambda, \Psi(\Lambda_m \Lambda(k+1)-1), \Psi(2k+1)\},$$

where $\Lambda_m \in \mathbb{N}^*$ is such that $\Lambda_m \geq \lambda_m$.
Then, for every $m \in \mathbb{N}$, Θ_m is a rate of $J^A_{\lambda_m}$-asymptotic regularity of (x_n).

Proof. (i) Remark first that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|J^A_{\lambda_n}x_n - x_{n+1}\| &= \|J^A_{\lambda_n}x_n - (\alpha_n f(x_n) + (1-\alpha_n)J^A_{\lambda_n}x_n + e_n)\| \\ &= \|\alpha_n(J^A_{\lambda_n}x_n - f(x_n)) - e_n\| \leq \alpha_n\|J^A_{\lambda_n}x_n - f(x_n)\| + \|e_n\| \\ &\leq 2\alpha_n K_z + \|e_n\| \quad \text{by Lemmas 4.1 (iv) and 4.3 (i).}\end{aligned}$$

It follows that, for all $n \geq \Psi(k)$,

$$\begin{aligned}\|J^A_{\lambda_n}x_n - x_n\| &\leq \|J^A_{\lambda_n}x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\alpha_n K_z + \|x_{n+1} - x_n\| + \|e_n\| \\ &\leq \frac{1}{3(k+1)} + \frac{1}{3(k+1)} + \frac{1}{3(k+1)} \\ &= \frac{1}{k+1},\end{aligned}$$

by $(H3\alpha_n)$, the fact that Φ is a rate of asymptotic regularity of (x_n), and $(H2e_n)$.

Thus, Ψ is a rate of $(J^A_{\lambda_n})$-asymptotic regularity of (x_n).

(ii) Let $m \in \mathbb{N}$. For all $n \in \mathbb{N}$, we have that

$$\begin{aligned}\|J^A_{\lambda_m}x_n - x_n\| &\leq \|J^A_{\lambda_m}x_n - J^A_{\lambda_n}x_n\| + \|J^A_{\lambda_n}x_n - x_n\| \\ &\leq \frac{|\lambda_n - \lambda_m|}{\lambda_n}\|x_n - J^A_{\lambda_n}x_n\| + \|J^A_{\lambda_n}x_n - x_n\| \quad \text{by (2.2)} \\ &= \left(\frac{|\lambda_n - \lambda_m|}{\lambda_n} + 1\right)\|J^A_{\lambda_n}x_n - x_n\|.\end{aligned}$$

Let $n \geq \Theta_m(k)$. We have two cases:

(a) $\lambda_m \geq \lambda_n$. Then, $\frac{|\lambda_n - \lambda_m|}{\lambda_n} + 1 = \frac{\lambda_m - \lambda_n}{\lambda_n} + 1 = \frac{\lambda_m}{\lambda_n}$, so

$$\|J^A_{\lambda_m} x_n - x_n\| \leq \frac{\lambda_m}{\lambda_n} \|J^A_{\lambda_n} x_n - x_n\| \overset{(H2\lambda_n)}{\leq} \Lambda_m \Lambda \|J^A_{\lambda_n} x_n - x_n\| \leq \frac{1}{k+1},$$

as $n \geq \Psi(\Lambda_m \Lambda(k+1) - 1)$.

(b) $\lambda_m < \lambda_n$. Then, $\frac{|\lambda_n - \lambda_m|}{\lambda_n} + 1 = \frac{\lambda_n - \lambda_m}{\lambda_n} + 1 = 2 - \frac{\lambda_m}{\lambda_n}$, so

$$\|J^A_{\lambda_m} x_n - x_n\| \leq \left(2 - \frac{\lambda_m}{\lambda_n}\right) \|J^A_{\lambda_n} x_n - x_n\| < 2\|J^A_{\lambda_n} x_n - x_n\| \leq \frac{1}{k+1},$$

as $n \geq \Psi(2k+1)$.

Thus, Θ_m is a rate of $J^A_{\lambda_m}$-asymptotic regularity of (x_n). ■

As it is the case with applications of proof mining, we obtain effective uniform rates that have a very weak dependency on the normed space X and the m-accretive operator A, only via K_z given by (5.1) for some zero z of A. The rates are computed for arbitrary parameter sequences (α_n), (λ_n), (e_n) satisfying the quantitative hypotheses stated in Theorems 5.1, 5.4 or Remarks 5.2, 5.3 and depend on the different moduli associated to these hypotheses. As we will see in Section 5.3, we get linear rates for concrete instances of such sequences.

Furthermore, if one forgets about the quantitative aspects, one gets, as an immediate consequence, qualitative asymptotic regularity results for the VAMe iteration (x_n).

Corollary 5.5. *Assume that* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$, *and one of the following holds:*

(a) $\sum_{n=0}^{\infty} |1 - \frac{\lambda_{n+1}}{\lambda_n}| < \infty$,

(b) $\sum_{n=0}^{\infty} |1 - \frac{\lambda_n}{\lambda_{n+1}}| < \infty$,

(c) $\inf_{n \in \mathbb{N}} \lambda_n > 0$ *and* $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.

Then, $\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0$.

Corollary 5.6. *Suppose that* $\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0$ *and* $\lim_{n \to \infty} \|e_n\| = 0$.

(i) *If* $\lim_{n \to \infty} \alpha_n = 0$, *then* $\lim_{n \to \infty} \|x_n - J^A_{\lambda_n} x_n\| = 0$.

(ii) *If* $\lim_{n \to \infty} \alpha_n = 0$ *and* $\inf_{n \in \mathbb{N}} \lambda_n > 0$ *hold, then* $\lim_{n \to \infty} \|x_n - J^A_{\lambda_m} x_n\| = 0$ *for every* $m \in \mathbb{N}$.

5.1. Rates for the VAM iteration

By letting $e_n = 0$ for all $n \in \mathbb{N}$, the VAMe iteration becomes the VAM iteration:

$$\text{VAM} \quad x_0 = x \in X, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J^A_{\lambda_n} x_n,$$

where A is an m-accretive operator.

Let $z \in \operatorname{zer} A$ and $K_z^* \in \mathbb{N}^*$ satisfy

$$K_z^* \geq \max\left\{\|x - z\|, \frac{\|f(z) - z\|}{1-\alpha}\right\}. \tag{5.2}$$

By a slight modification of the proofs of Theorems 5.1, 5.4, taking into account that $e_n = 0$ for all $n \in \mathbb{N}$ and that Lemma 4.3 (i) holds with $E = 0$, we obtain rates for the VAM iteration.

Proposition 5.7. *Assume that $(H1\alpha_n)$, $(H2\alpha_n)$, $(H1\lambda_n)$ hold and define*

$$\chi^*(k) = \max\{\sigma_2(4K_z^*(k+1)-1), \gamma_1(4K_z^*(k+1)-1)\},$$
$$\Phi^*(k) = \sigma_1\left(\left\lceil \frac{\chi^*(2k+1) + 1 + \lceil \ln(4K_z^*(k+1)) \rceil}{1-\alpha} \right\rceil + 1\right).$$

Then, Φ^* is a rate of asymptotic regularity of the VAM iteration (x_n).

Remarks 5.2 and 5.3 are true for the VAM iteration too.

Proposition 5.8. *Let Φ^* be a rate of asymptotic regularity of (x_n). Define*

$$\Psi^*(k) = \max\{\sigma_3(4K_z^*(k+1)-1), \Phi^*(2k+1)\} \quad \textit{if } (H3\alpha_n) \textit{ holds,}$$

and, for $m \in \mathbb{N}$ and $\Lambda_m \in \mathbb{N}^*$ such that $\Lambda_m \geq \lambda_m$,

$$\Theta_m^*(k) = \max\{N_\Lambda, \Psi^*(\Lambda_m \Lambda(k+1)-1), \Psi^*(2k+1)\}$$
$$\textit{if both } (H3\alpha_n) \textit{ and } (H2\lambda_n) \textit{ hold.}$$

Then, Ψ^* is a rate of $(J_{\lambda_n}^A)$-asymptotic regularity of (x_n) and, for every $m \in \mathbb{N}$, Θ_m^* is a rate of $J_{\lambda_m}^A$-asymptotic regularity of (x_n).

Corollaries 5.5 and 5.6 (with the hypotheses $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\lim_{n\to\infty} \|e_n\| = 0$ removed) hold also for the VAM iteration.

We remark that in [26] the VAM iteration (x_n) is studied in a more general setting, by considering an accretive operator A, an α-contraction $f : C \to C$, and $x \in C$, where $\emptyset \neq C \subseteq X$ is a nonempty closed convex subset of X satisfying $\overline{\operatorname{dom} A} \subseteq C \subseteq \operatorname{ran}(\mathrm{Id} + \gamma A)$ for all $\gamma > 0$. It is easy to see that the results from Section 4 specialized to $e_n = 0$ hold in this setting with basically the same proofs. Hence, Propositions 5.7 and 5.8 are true in this more general setting, too.

5.2. Rates for the HPPA iteration

Another particular case of the VAMe iteration is the (inexact) HPPA iteration:

$$\text{HPPA} \quad x_0 = x \in X, \quad x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\lambda_n}^A x_n + e_n,$$

obtained by letting $f(x) = u \in X$ in the definition (4.1) of VAMe. Obviously, the constant mapping $f(x) = u$ is an α-contraction with $\alpha = 0$.

Theorems 5.1 and 5.4 hold for the HPPA iteration with $K_z \in \mathbb{N}^*$ such that

$$K_z \geq \max\{\|x - z\|, \|u - z\|\} + \left\lceil \sum_{i=0}^{\theta_1(0)} \|e_i\| \right\rceil + 1.$$

Furthermore, Corollaries 5.5 and 5.6 are true for the HPPA (x_n), too.

By letting $e_n = 0$, we get that Propositions 5.7 and 5.8 hold with $K_z^* \in \mathbb{N}^*$ such that

$$K_z^* \geq \max\{\|x - z\|, \|u - z\|\}.$$

Methods of proof mining were applied in [17, 20] to the HPPA iteration associated to a maximal monotone operator A in a Hilbert space X to obtain quantitative results on its asymptotic behavior, including rates of $((J^A_{\lambda_n}),\ J^A_{\lambda_m}\ (m \in \mathbb{N}))$-asymptotic regularity.

In this paper, we compute such rates for the more general setting of m-accretive operators in Banach spaces.

5.3. Linear rates for concrete instances of the parameter sequences

[23, Lemma 3] or its slight variation, Lemma 3.4, were applied recently to obtain linear rates of asymptotic regularity for the Tikhonov–Mann and modified Halpern iterations [6], the alternating Halpern–Mann iteration [18], and different Halpern-type iterations [7]. In the sequel, we use Lemma 3.4 to compute linear rates for the VAMe iteration for two specific choices of the parameter sequences.

In the following, for all $n \in \mathbb{N}$,

$$\alpha_n = \frac{2}{(1-\alpha)(n+J)}, \quad \text{where } J = 2\left\lceil \frac{1}{1-\alpha} \right\rceil.$$

As (α_n) is decreasing, we have that

$$\alpha_n \leq \alpha_0 = \frac{2}{(1-\alpha)J} \leq 1.$$

Thus, α_n is a sequence in $[0, 1]$.

5.3.1. A first example. For all $n \in \mathbb{N}$, consider

$$\lambda_n = \lambda > 0 \text{ and } e_n = 0.$$

Then, (x_n) is the VAM iteration with a single mapping J^A_λ, which is nonexpansive. It follows that (x_n) is a particular case of the viscosity version of the Halpern iteration (where one considers an arbitrary nonexpansive mapping T instead of J_λ) introduced by Xu [25] and studied by Sabach and Shtern [23] under the name of sequential averaging method (SAM). As an application of [23, Lemma 3], Sabach and Shtern obtained linear rates of (T-)asymptotic regularity for SAM. Cheval and the second author [7] applied Lemma 3.4

to compute such linear rates in the more general setting of W-hyperbolic spaces; these rates hold in our setting, too.

Consider the following mappings, defined in [7, Section 3.2, (15), (16)], with notations adapted to this paper:

$$\Phi_0(k) = 4K_z^*\left\lceil \frac{1}{1-\alpha} \right\rceil^2 (k+1) - 2\left\lceil \frac{1}{1-\alpha} \right\rceil,$$
$$\Psi_0(k) = \left(4K_z^*\left\lceil \frac{1}{1-\alpha} \right\rceil^2 + 4K_z^*\left\lceil \frac{1}{1-\alpha} \right\rceil\right)(k+1) - 2\left\lceil \frac{1}{1-\alpha} \right\rceil,$$

where $z \in \operatorname{zer} A$ and $K_z^* \in \mathbb{N}^*$ satisfies (5.2).

Then, (x_n) is asymptotically regular with rate Φ_0 and J_λ^A-asymptotically regular with rate Ψ_0. As $\lambda_n = \lambda$ for all $n \in \mathbb{N}$, obviously $(J_{\lambda_n}^A)$-asymptotic regularity and $J_{\lambda_m}^A$-asymptotic regularity (for $m \in \mathbb{N}$) coincide with J_λ^A-asymptotic regularity of (x_n).

5.3.2. A second example. Let us take, for all $n \in \mathbb{N}$,

$$\lambda_n = \frac{n+J}{n+J-1} \quad \text{and} \quad e_n = \frac{1}{(n+J)^2}e^*, \quad \text{where } e^* \in X.$$

Since $\sum_{n=0}^{\infty} \frac{1}{(n+J)^2} < \frac{1}{J-1}$, it follows that $(H3e_n)$ holds with

$$E = \left\lceil \frac{\|e^*\|}{J-1} \right\rceil.$$

Let $z \in \operatorname{zer} A$ and $K_z \in \mathbb{N}^*$ satisfying

$$K_z \geq \max\left\{\|x-z\|, \frac{\|f(z)-z\|}{1-\alpha}\right\} + \left\lceil \frac{\|e^*\|}{J-1} \right\rceil.$$

Proposition 5.9. *For all $n \in \mathbb{N}$,*

$$\|x_{n+1} - x_n\| \leq \frac{3JK_z + \|e^*\|}{(1-\alpha)(n+J)}. \tag{5.3}$$

Thus,

$$\begin{aligned}\Phi_0(k) &= \left(3JK_z + \lceil\|e^*\|\rceil\right)\left\lceil \frac{1}{1-\alpha} \right\rceil (k+1) - J \\ &= 6K_z\left\lceil \frac{1}{1-\alpha} \right\rceil^2 (k+1) + \lceil\|e^*\|\rceil\left\lceil \frac{1}{1-\alpha} \right\rceil (k+1) - 2\left\lceil \frac{1}{1-\alpha} \right\rceil\end{aligned}$$

is a linear rate of asymptotic regularity of (x_n).

Proof. By Lemma 4.3 (i), we have that Lemma 4.1 and (4.3) hold with K_z defined as above instead of $K_{z,n}$ or $K_{z,n+1}$. Applying (4.3), we get that, for all $n \in \mathbb{N}$,

$$\|x_{n+2} - x_{n+1}\| \leq (1-(1-\alpha)\alpha_{n+1})\|x_{n+1} - x_n\| + P_z,$$

where

$$P_z = 2K_z\left(|\alpha_{n+1}-\alpha_n| + (1-\alpha_{n+1})\left|1-\frac{\lambda_{n+1}}{\lambda_n}\right|\right) + \|e_{n+1}-e_n\|$$
$$= |\alpha_{n+1}-\alpha_n|\left(2K_z\left(1+\frac{(1-\alpha_{n+1})}{|\alpha_{n+1}-\alpha_n|}\left|1-\frac{\lambda_{n+1}}{\lambda_n}\right|\right) + \frac{\|e_{n+1}-e_n\|}{|\alpha_{n+1}-\alpha_n|}\right).$$

As

$$|\alpha_{n+1}-\alpha_n| = \alpha_n - \alpha_{n+1} = \frac{2}{(1-\alpha)(n+J)(n+1+J)},$$
$$1-\alpha_{n+1} = \frac{(1-\alpha)(n+1+J)-2}{(1-\alpha)(n+1+J)},$$
$$\left|1-\frac{\lambda_{n+1}}{\lambda_n}\right| = 1-\frac{\lambda_{n+1}}{\lambda_n} = \frac{1}{(n+J)^2},$$

we have that

$$\frac{(1-\alpha_{n+1})}{|\alpha_{n+1}-\alpha_n|}\left|1-\frac{\lambda_{n+1}}{\lambda_n}\right| = \frac{\big((1-\alpha)(n+1+J)-2\big)(n+J)}{2}\cdot\frac{1}{(n+J)^2}$$
$$\leq \frac{n+J-1}{2(n+J)} < \frac{1}{2}.$$

Furthermore,

$$\frac{\|e_{n+1}-e_n\|}{|\alpha_{n+1}-\alpha_n|} = \frac{(2(n+J)+1)\|e^*\|(1-\alpha)}{2(n+J)(n+J+1)} \leq \frac{\|e^*\|}{n+J} \leq \frac{\|e^*\|}{J}.$$

It follows that, for all $n \in \mathbb{N}$,

$$\|x_{n+2}-x_{n+1}\| < (1-(1-\alpha)\alpha_{n+1})\|x_{n+1}-x_n\| + (\alpha_n-\alpha_{n+1})\left(3K_z + \frac{\|e^*\|}{J}\right).$$

One can easily see that Lemma 3.4 can be applied with

$$s_n = \|x_{n+1}-x_n\|, \quad L = 3K_z + \frac{\|e^*\|}{J}, \quad N = 2,$$
$$J = 2\left\lceil\frac{1}{1-\alpha}\right\rceil, \quad \gamma = 1-\alpha, \quad a_n = \alpha_n, \quad c_n = 3K_z + \frac{\|e^*\|}{J}$$

to conclude that (5.3) holds, and, as a consequence, Φ_0 is a rate of asymptotic regularity of (x_n). ∎

Proposition 5.10. *Define*

$$\Psi_0(k) = 18K_z\left\lceil\frac{1}{1-\alpha}\right\rceil^2(k+1) + 3\lceil\|e^*\|\rceil\left\lceil\frac{1}{1-\alpha}\right\rceil(k+1) - 2\left\lceil\frac{1}{1-\alpha}\right\rceil,$$
$$\Theta_0(k) = 36K_z\left\lceil\frac{1}{1-\alpha}\right\rceil^2(k+1) + 6\lceil\|e^*\|\rceil\left\lceil\frac{1}{1-\alpha}\right\rceil(k+1) - 2\left\lceil\frac{1}{1-\alpha}\right\rceil.$$

Then, Ψ_0 *is a linear rate of* $(J^A_{\lambda_n})$*-asymptotic regularity of* (x_n) *and* Θ_0 *is a linear rate of* $J^A_{\lambda_m}$*-asymptotic regularity of* (x_n) *for every* $m \in \mathbb{N}$*.*

Proof. We can apply Theorem 5.4, as $(H3\alpha_n)$ holds with $\sigma_3(k) = Jk$, $(H2e_n)$ holds with $\theta_2(k) = \max\{\lceil\sqrt{\|e^*\|(k+1)}\rceil - J, 0\}$, $(H2\lambda_n)$ holds with $\Lambda = 1$, $N_\Lambda = 0$, and $\Lambda_m = 2 \geq \lambda_m$ for all $m \in \mathbb{N}$.

Using also Proposition 5.9, it follows that (x_n) is $(J^A_{\lambda_n})$-asymptotically regular with rate

$$\Psi(k) = \max\{\sigma_3(6K_z(k+1)-1), \Phi_0(3k+2), \theta_2(3k+2)\}.$$

Since

$$\begin{aligned}\sigma_3(6K_z(k+1)-1) &= 6JK_z(k+1) - J,\\ \Phi_0(3k+2) &= 9JK_z\left\lceil\frac{1}{1-\alpha}\right\rceil(k+1) + 3\lceil\|e^*\|\rceil\left\lceil\frac{1}{1-\alpha}\right\rceil(k+1) - J,\\ \theta_2(3k+2) &= \max\left\{\left\lceil\sqrt{3\|e^*\|(k+1)}\right\rceil - J, 0\right\},\end{aligned}$$

we have that $\sigma_3(6K_z(k+1)-1), \theta_2(3k+2) < \Phi_0(3k+2)$; hence,

$$\Psi(k) = \Phi_0(3k+2) = \Psi_0(k).$$

Applying Theorem 5.4 (ii), we get that, for every $m \in \mathbb{N}$, (x_n) is $J^A_{\lambda_m}$-asymptotically regular with rate

$$\begin{aligned}\Theta_m(k) &= \max\{N_\Lambda, \Psi_0(\Lambda_m\Lambda(k+1)-1), \Psi_0(2k+1)\}\\ &= \Psi_0(2k+1) = \Theta_0(k). \qquad \blacksquare\end{aligned}$$

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