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A Smale–Barden manifold admitting K-contact but not Sasakian structure

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Abstract. We give the first example of a simply connected compact 5-manifold (Smale–Barden manifold) which admits a K-contact structure but does not admit any Sasakian structure, settling a long-standing question of Boyer and Galicki.

Keywords: Sasakian, K-contact, Smale–Barden manifold.

1. Introduction

In geometry, a central question is to determine when a given manifold admits a specific geometric structure. Complex geometry provides numerous examples of compact manifolds with rich topology, and there is a number of topological properties that are satisfied by Kähler manifolds [1, 15]. If we forget about the integrability of the complex structure, then we are dealing with symplectic manifolds. There has been enormous interest in the construction of (compact) symplectic manifolds that do not admit Kähler structures, and in determining its topological properties [30]. The fundamental group is one of the more direct invariants that constrain the topology of Kähler manifolds [15], whereas any finitely presented group can be the fundamental group of a compact symplectic manifold [18]. For this reason, the problem becomes more relevant if we ask for simply connected compact manifolds. On the other hand, the difficulties increase as we look for manifolds of the lowest possible dimension. For instance, the lowest dimension for a compact simply connected manifold admitting a symplectic but not a Kähler structure and having *non-formal* rational homotopy type is 8. Such an example was first provided by Fernández and the author in [16]. Also, a compact simply connected manifold admitting both a symplectic and a complex structure but not a Kähler structure can only happen in dimensions higher than 6. The first example of such instance in the lowest dimension 6 is given by Bazzoni, Fernández and the author in [4].

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A symplectic manifold always admits an almost-Kähler structure (which is a metric structure), so the topological question above can be rephrased as to finding manifolds which admit almost-Kähler but no Kähler structures. In odd dimensions, the analogues of Kähler and almost-Kähler manifolds are Sasakian and K-contact manifolds, respectively (and the analogue of symplectic manifold is contact manifold). These are metric structures which are endowed with a 1-dimensional foliation and a transversal structure which is Kähler or almost-Kähler, respectively (see Section 2.2 for precise definitions). Sasakian geometry has become an important and active subject since the treatise of Boyer and Galicki [7], and there is much interest on constructing K-contact manifolds which do not admit Sasakian structures. As mentioned in [7, Charpter 10], now there is a gap between contact and K-contact, and the problem of finding a manifold admitting a contact but not Sasakian structure is easily solved. However, finding manifolds which admit a K-contact but not a Sasakian structure is harder.

The parity of b_1 was used to produce the first examples of K-contact manifolds with no Sasakian structure [7, Example 7.4.16]. More refined tools are needed in the case of even Betti numbers. The cohomology algebra of a Sasakian manifold satisfies a hard Lefschetz property [12], and using it examples of K-contact non-Sasakian manifolds are produced in [10] in dimensions 5 and 7. These examples are nilmanifolds with even Betti numbers, so in particular they are not simply connected. The fundamental group can also be used to construct K-contact non-Sasakian manifolds [13]. Also it has been used to provide an example of a solvmanifold of dimension 5 which satisfies the hard Lefschetz property and which is K-contact and not Sasakian [11].

When one moves to the case of simply connected manifolds, K-contact non-Sasakian examples of any dimension ≥ 9 were constructed in [19] using the evenness of the third Betti number of a compact Sasakian manifold. Alternatively, using the hard Lefschetz property for Sasakian manifolds, one can find examples [22] of simply connected K-contact non-Sasakian manifolds of any dimension ≥ 9. In [6], the rational homotopy type is used to construct examples of simply connected K-contact non-Sasakian manifolds in dimensions ≥ 17. In dimension 7, there are examples in [26] of simply connected K-contact non-Sasakian manifolds. However, Massey products are not suitable for the analysis of lower-dimensional manifolds. The problem of the existence of simply connected K-contact non-Sasakian compact manifolds is still open in dimension 5, despite numerous attempts.

Open Problem ([7, p. 340, Open Problem 10.2.1]). *Do there exist Smale–Barden manifolds which carry K-contact but do not carry Sasakian structures?*

A simply connected compact 5-manifold is called a *Smale–Barden manifold*. These manifolds are classified [2, 29] by $H_2(M, \mathbb{Z})$ and the second Stiefel–Whitney class (see Section 2.1). This makes sensible to pose classification problems of manifolds admitting diverse geometric structures in the class of Smale–Barden manifolds.

A Sasakian manifold M always admits a *quasi-regular* Sasakian structure. This gives M the structure of a Seifert bundle over a cyclic Kähler orbifold X (see Section 2.3). In the case of a 5-manifold, X is a singular complex surface with cyclic quotient singularities. The Sasakian structure is *semi-regular* if the isotropy locus is only formed by codimension 2 submanifolds, that is, if X is a smooth complex surface and the isotropy consists of smooth complex curves (maybe intersecting). A similar statement holds for K-contact manifolds, where the base is now an almost-Kähler orbifold (that is, symplectic with a compatible almost complex structure, which always exists) with cyclic singularities, and the isotropy locus is formed by symplectic surfaces.

In [21], Kollár determines the topology of simply connected 5-manifolds which are Seifert bundles over semi-regular 4-orbifolds. The torsion in $H_2(M, \mathbb{Z})$ is determined by the genera and isotropy coefficients of the isotropy surfaces. He uses this to produce simply connected 5-manifolds which are Seifert bundles (that is, they admit a fixed point free circle action) but which do not admit a Sasakian structure. If the structure is semi-regular, the isotropy surfaces must satisfy the adjunction equality, so an example violating it will produce such example. In general, a Kähler orbifold can have isolated singularities which cause serious difficulties, since the classification of singular complex surfaces is far more complicated than that of smooth complex surfaces. Kollár uses the case $b_2(X) = 1$, where there is a bound on the number of singular points, and taking enough curves for the isotropy locus ensures that some of them satisfy the adjunction equality.

To produce K-contact Smale–Barden manifolds, one needs to construct symplectic 4-manifolds (or 4-orbifolds with cyclic quotient singularities) with symplectic surfaces of given genus inside. If the isotropy coefficients are not coprime, these surfaces are forced to be disjoint (and linearly independent in homology). Therefore, there is a bound on the number k of surfaces in the isotropy locus, $k \leq b_2(X)$. The genus of the isotropy surfaces, the isotropy coefficients, and whether they are disjoint, are translated to the homology group $H_2(M, \mathbb{Z})$ of the 5-manifold M. This is used in [25] to produce a homology Smale–Barden manifold (that is, a 5-manifold M with $H_1(M, \mathbb{Z}) = 0$ instead of simply connected) which admits a semi-regular K-contact but not a semi-regular Sasakian structure. For this, we construct a simply connected symplectic 4-manifold X with k *disjoint* symplectic surfaces of positive genus where $k = b_2(X)$, and linearly independent in homology. To prove that this is not semi-regular Sasakian, we have to check that there is no complex surface Y with $b_1(Y) = 0$, and k disjoint complex curves of positive genus where $k = b_2(Y) > 1$, and linearly independent in homology, at least for the case where the genera match our symplectic example. The existence of so many disjoint complex curves of positive genus and generating the rational homology is certainly a rare phenomenon, and we conjecture that it does never happen. Unfortunately, as the example in [9, Section 3] shows, this can happen for singular complex manifolds with cyclic singularities. For this reason, we do not know whether the example in [25] can admit a quasi-regular Sasakian structure.

Later, in [8] we extend the ideas of [25] to produce the first example of *simply connected* 5-manifold which admits a semi-regular K-contact structure but not a semi-regular Sasakian structure. Again, we have not been able to remove the semi-regularity assumption. The purpose of this paper is to completely settle the question in [7, p. 340, Open Problem 10.2.1].

Theorem 1.1. *There exists a Smale–Barden manifold M which admits a K-contact structure but does not admit a Sasakian structure.*

More precisely, a manifold M in Theorem 1.1 can be explicitly given as follows. There is some $N > 0$ large enough, and distinct primes p_{nm} with $p_{nm} > \max(3, n, m)$, $1 \leq n, m \leq N$, so that M is the Smale–Barden manifold characterized by the fact M is spin and its homology is

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus \bigoplus_{n,m=1}^{N} (\mathbb{Z}_{p_{nm}}^{18n^2+2} \oplus \mathbb{Z}_{p_{nm}^2}^{18m^2+2} \oplus \mathbb{Z}_{p_{nm}^3}^{20}).$$

Let us describe the philosophy behind the construction in Theorem 1.1, although the technical details, which will be carried out in the following sections, get quite involved. As we said before, for our Seifert bundles $M \to X$ we can keep track of the genera and the disjointness of the isotropy surfaces. With this, we try to push the construction to record also the value of $b_2^+(X)$. We note that when Y is a complex surface with $k = b_2(Y)$ complex curves spanning $H_2(Y, \mathbb{Q})$, then $b_2^+(Y) = 1$, whereas the same property does not necessarily hold for symplectic 4-manifolds. If X is symplectic and $b_2^+(X) > 1$, then when we have $k = b_2(X)$ disjoint symplectic surfaces, we can take positive multiples of those with positive self-intersection. This gives N families of $k = b_2(X)$ disjoint symplectic surfaces, which can be used as isotropy locus, for any $N \gg 0$ as large as we want.

The proof that the resulting 5-manifold M does not admit a Sasakian structure now requires to check that there is no singular complex surface Y with cyclic singularities with a large number of families, each consisting of $k = b_2(Y)$ disjoint complex curves. First we need to bound the number of singular points (universally, i.e., independently of Y) as it was done in [21] for the easy case $b_2(Y) = 1$. This serves to bound geometric quantities, like the Euler characteristic, K^2, or the self-intersection of negative curves. For orbifolds, the intersection and self-intersection numbers and K^2 can be rational (instead of just integers), so it is necessary to bound the denominators (independently of Y).

As the number of singular points is bounded, we have that most of the families of disjoint complex curves avoid the singular points. However, now the genera of the curves have increased (the arguments of [8, 25] deal with cases of low genus curves, so they are not helpful now). In the families of disjoint complex curves, it cannot happen that the curves are multiples of k fixed curves (incidentally, note that this was the way in which the symplectic example X is produced), because that would imply that $b_2^+(Y) > 1$, and this does not hold for an algebraic surface. This forces to have from the initial N families of k disjoint complex curves (these are orthogonal bases of $H_2(Y, \mathbb{Q})$), many of them whose elements are not proportional to each other (what we call proj-equivalent bases). The final step is to prove the impossibility of this situation, by writing K^2 with respect to each of the orthogonal basis, and use the bounds on the denominators of the rational numbers. We get a collection of diophantine equalities, and choosing N large enough, these become incompatible.

2. Basic notions

2.1. Smale–Barden manifolds

A 5-dimensional simply connected manifold is called a *Smale–Barden manifold*. These manifolds are classified by their second homology group over $\mathbb{Z}$ and the Barden invariant [2, 29]. In more detail, let M be a compact smooth simply connected 5-manifold, and write $H_2(M,\mathbb{Z})$ as a direct sum of cyclic groups of prime power order

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus \Big(\bigoplus_{p,i} \mathbb{Z}_{p^i}^{c(p^i)}\Big), \tag{2.1}$$

where $k = b_2(M)$. Equality (2.1) is actually an isomorphism of abelian groups. We can arrange so that the second Stiefel–Whitney class map

$$w_2\colon H_2(M,\mathbb{Z}) \to \mathbb{Z}_2$$

is zero on all but one summand (or zero on all if $w_2 = 0$). For that, take an element on which it is not zero, and complete to a generating system with elements in the kernel. If w_2 is non-zero on $\mathbb{Z}_{2^j}$, we set $i(M) = j$; if w_2 is non-zero on a summand $\mathbb{Z}$, we set $i(M) = \infty$; if $w_2 = 0$, then we set $i(M) = 0$. The number $i(M)$ is called the Barden invariant and determines w_2 up to isomorphism of abelian groups. A Smale–Barden manifold M is uniquely characterized by its homology (2.1) and $i(M)$.

We shall not use the following, but we include for completeness. The geometric description of Smale–Barden manifolds corresponding to these abelian groups is given as follows.

Theorem 2.1 ([7, Theorem 10.2.3]). *Any simply connected closed 5-manifold is diffeomorphic to one of the spaces*

$$M_{j;k_1,\dots,k_s;r} = X_j \# rM_\infty \# M_{k_1} \# \cdots \# M_{k_s},$$

where the manifolds X_{-1}, X_0, X_j, X_∞, M_j, M_∞ are characterized as follows:

$$1 < k_i < \infty, \quad k_1 \mid k_2 \mid \dots \mid k_s,$$

and

- $X_{-1} = \mathrm{SU}(3)/\mathrm{SO}(3)$, $H_2(X_{-1},\mathbb{Z}) = \mathbb{Z}_2$, $i(X_{-1}) = 1$,
- $X_0 = S^5$, $H_2(X_0,\mathbb{Z}) = 0$, $i(X_0) = 0$,
- X_j, $0 < j < \infty$, $H_2(X_j,\mathbb{Z}) = \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$, $i(X_j) = j$,
- $X_\infty = S^2 \tilde{\times} S^3$ *is the unique non-trivial S^3-bundle over S^2, it has* $H_2(X_\infty,\mathbb{Z}) = \mathbb{Z}$, $i(X_\infty) = \infty$,
- M_k, $1 < k < \infty$, $H_2(M_k,\mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_k$, $i(M_k) = 0$,
- $M_\infty = S^2 \times S^3$, $H_2(M_\infty,\mathbb{Z}) = \mathbb{Z}$, $i(M_\infty) = 0$.

2.2. Sasakian and K-contact manifolds

Let (M, η) be a co-oriented contact manifold with a contact form $\eta \in \Omega^1(M)$, i.e., $\eta \wedge (d\eta)^n > 0$ everywhere, with $\dim M = 2n + 1$. The manifold M is automatically oriented. We say that (M, η) is *K-contact* if there is an endomorphism Φ of TM such that

- $\Phi^2 = -\operatorname{Id} + \xi \otimes \eta$, where ξ is the Reeb vector field of η (that is, $i_\xi \eta = 1$, $i_\xi(d\eta) = 0$),
- the contact form η is compatible with Φ in the sense that $d\eta(\Phi X, \Phi Y) = d\eta(X, Y)$ for all vector fields X, Y,
- $d\eta(\Phi X, X) > 0$ for all non-zero $X \in \ker \eta$, and
- the Reeb field ξ is Killing with respect to the Riemannian metric defined by the formula $g(X, Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y)$.

In other words, the endomorphism Φ defines a complex structure on $\mathcal{D} = \ker \eta$ compatible with $d\eta$, hence Φ is orthogonal with respect to the metric $g|_{\mathcal{D}}$. By definition, the Reeb vector field ξ is orthogonal to $\mathcal{D}$, and it is a Killing vector field.

Let (M, η, ξ, Φ, g) be a K-contact manifold. Consider the contact cone as the Riemannian manifold $C(M) = (M \times \mathbb{R}_+, t^2 g + dt^2)$. One defines the almost complex structure I on $C(M)$ by

- $I(X) = \Phi(X)$ on $\ker \eta$,
- $I(\xi) = t\frac{\partial}{\partial t}$, $I(t\frac{\partial}{\partial t}) = -\xi$ for the Killing vector field ξ of η.

We say that (M, η, ξ, Φ, g) is *Sasakian* if I is integrable. Thus, by definition, any Sasakian manifold is K-contact.

The Sasakian structure can also be defined by the integrability of the almost contact metric structure. More precisely, an almost contact metric structure (η, ξ, Φ, g) is called *normal* if the Nijenhuis tensor N_Φ associated to the tensor field Φ, defined by

$$N_\Phi(X, Y) := \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y],$$

satisfies the equation

$$N_\Phi = -d\eta \otimes \xi.$$

Then a Sasakian structure is a normal contact metric structure.

A Sasakian (compact) manifold M has a 1-dimensional foliation defined by the Reeb vector field, which gives an isometric flow, and the transversal structure is Kähler. The Sasakian structure is called *quasi-regular* if the leaves of the Reeb flow are circles, in which case the leaf space X is a Kähler cyclic orbifold and the quotient map $\pi\colon M \to X$ has the structure of a Seifert bundle [7, Theorem 7.13]. Remarkably, a manifold M admitting a Sasakian structure also has a quasi-regular one [28]. So from the point of view of whether M admits a Sasakian structure, we can assume that it is a Seifert bundle over a Kähler cyclic orbifold. The Sasakian structure is *regular* if X is a Kähler manifold (no isotropy locus), and *semi-regular* if the isotropy locus has only codimension 2 strata (maybe intersecting), or equivalently if X has underlying space which is a topological manifold.

In the case of a K-contact manifold, the situation is analogous, with the difference that the transversal structure is almost-Kähler. We define regular, quasi-regular and semi-regular K-contact structures with the same conditions. Any K-contact manifold admits a quasi-regular K-contact structure [26], and hence a K-contact manifold is a Seifert circle bundle over a symplectic cyclic orbifold. In the case $\dim M = 5$, $\dim X = 4$, such orbifold has isotropy locus which is a collection of symplectic surfaces and points. From the point of view of whether a manifold admits a K-contact structure, we can always assume that the manifold is a Seifert bundle over a symplectic cyclic orbifold.

2.3. *Cyclic orbifolds*

Let X be a 4-dimensional (oriented) cyclic orbifold. For the notions about orbifolds, the reader can consult [6,7,23]. Let $x \in X$ be a point. A neighbourhood of x is an open subset $U = \widetilde{U}/\mathbb{Z}_m$, where $\widetilde{U} \subset \mathbb{C}^2$ and the action of $\mathbb{Z}_m = \langle \varepsilon \rangle$, $\varepsilon = e^{2\pi i/m}$, is given by

$$\varepsilon \cdot (z_1, z_2) = (\varepsilon^{j_2} z_1, \varepsilon^{j_1} z_2), \tag{2.2}$$

where j_1, j_2 are defined modulo m, and $\gcd(j_1, j_2, m) = 1$. We say that $m = m_x$ is the isotropy of x, and $\mathbf{j}_x = (m, j_1, j_2)$ are the local invariants for x.

We say that $D \subset X$ is an isotropy surface of multiplicity m if D is a closed 2-dimensional suborbifold, and the regular set $D^\circ \subset D$ is a connected smooth surface with $m_x = m$ for $x \in D^\circ$. The *local invariants* for D are those of a point in D°, that is, $\mathbf{j}_D = (m, j)$. Locally, $D = \{(z_1, 0)\}$ and the action is given by $\varepsilon = e^{2\pi i/m}$, $\varepsilon \cdot (z_1, z_2) = (z_1, \varepsilon^j z_2)$.

Now to describe action (2.2) at a point x, we set $m_1 = \gcd(j_1, m)$, $m_2 = \gcd(j_2, m)$. Note that $\gcd(m_1, m_2) = 1$, so we can write $m_1 m_2 d = m$ for some integer d. Put $j_1 = m_1 e_1$, $j_2 = m_2 e_2$. Then we have that [25, Proposition 2]

$$\mathbb{C}^2/\mathbb{Z}_m = ((\mathbb{C}/\mathbb{Z}_{m_2}) \times (\mathbb{C}/\mathbb{Z}_{m_1}))/\mathbb{Z}_d,$$

where $\mathbb{C}/\mathbb{Z}_{m_2} \times \mathbb{C}/\mathbb{Z}_{m_1}$ is homeomorphic to $\mathbb{C}^2$ via the map $(z_1, z_2) \mapsto (w_1, w_2) = (z_1^{m_2}, z_2^{m_1})$. The points of $D_1 = \{(z_1, 0)\}$ and $D_2 = \{(0, z_2)\}$ define two surfaces intersecting transversally, and with multiplicities m_1, m_2, respectively, and the action of $\mathbb{Z}_d$ on $\mathbb{C}^2$ is given by $\varepsilon \cdot (w_1, w_2) = (e^{2\pi i e_1/d} w_1, e^{2\pi i e_2/d} w_2)$, where $\gcd(e_1, d) = \gcd(e_2, d) = 1$. Thus the point x has as link a lens space $S^3/\mathbb{Z}_d$, and the images of D_1 and D_2 are the points with non-trivial isotropy, with multiplicities m_1, m_2, respectively.

We say that $x \in X$ is a singular point if $d > 1$ and smooth if $d = 1$, and we denote $d = d_x$. Let $P \subset X$ be the (finite) collection of singular points. We say that two surfaces $D_1, D_2 \subset X$ *intersect nicely* if at every intersection point $x \in D_1 \cap D_2$, there are adapted coordinates (z_1, z_2) at x such that $D_1 = \{(z_1, 0)\}$ and $D_2 = \{(0, z_2)\}$ in a model $\mathbb{C}^2/\mathbb{Z}_m$, as above. If the point $x \in X$ is smooth, then D_1, D_2 intersect transversally and positively. In this situation, the surfaces D_i are said to be *nice*.

A symplectic (cyclic) 4-orbifold (X, ω) is a 4-orbifold X with an orbifold 2-form $\omega \in \Omega^2_{\mathrm{orb}}(X)$ such that $d\omega = 0$ and $\omega^2 > 0$. At every point $x \in X$, there are orbifold

Darboux charts [24, Proposition 11], that is, an orbifold chart as above, where ω has the standard form on $\mathbb{C}^2$ (and hence $\mathbb{Z}_m$ acts symplectically). In this case, the isotropy surfaces D_i are symplectic surfaces (or more accurately, symplectic suborbifolds), and their intersections are nice, which in this case means that they intersect symplectically orthogonal and positively.

A Kähler (cyclic) 4-orbifold (X, J, ω) consists of a symplectic form ω and a compatible orbifold almost complex structure J, whose Nijenhuis tensor $N_J = 0$ vanishes. In this case, at every point $x \in X$ there are complex charts of the form $\mathbb{C}^2/\mathbb{Z}_m$ as above. The isotropy surfaces D_i are complex curves and the singular points are (cyclic) complex singularities.

A cyclic orbifold is recovered from the singular points $P \subset X$ and the isotropy surfaces as follows.

Proposition 2.2 ([23, Propositions 22 and 23]). *Let X be an oriented cyclic 4-orbifold whose isotropy locus is of dimension 0 (that is, the singular set P). Let D_i be embedded surfaces intersecting nicely, and take coefficients $m_i > 1$ such that $\gcd(m_i, m_j) = 1$ if D_i, D_j intersect. Then there is a cyclic orbifold structure on X that we denote by X', with isotropy surfaces D_i of multiplicities m_i, and singular points $x \in P$ of multiplicity $m_x = d_x \prod_{x \in D_i} m_i$.*

Moreover, if X is a Kähler (symplectic) cyclic orbifold and the surfaces D_i are complex (symplectic), then the resulting orbifold X' is a Kähler (symplectic) cyclic orbifold.

Once we have the orbifold X' given in Proposition 2.2, we need to assign local invariants. This is not automatic, but the following result is enough for our purposes.

Proposition 2.3 ([23, Proposition 25]). *Suppose that X is a cyclic 4-orbifold and the isotropy surfaces D_i which pass through points of P are disjoint. Take integers j_i with $\gcd(m_i, j_i) = 1$ for each D_i. Then there exist local invariants for all surfaces D_i and all points $x \in P$.*

A Seifert bundle over a cyclic 4-orbifold X, endowed with local invariants, is an oriented 5-manifold M equipped with a smooth S^1-action such that X is the space of orbits, and the projection $\pi\colon M \to X$ satisfies that for an orbifold chart $U = \widetilde{U}/\mathbb{Z}_m$ of X, we have that

$$\pi^{-1}(U) = (\widetilde{U} \times S^1)/\mathbb{Z}_m,$$

where the action of S^1 is given by

$$\varepsilon \cdot (z_1, z_2, u) = (\varepsilon^{j_2} z_1, \varepsilon^{j_1} z_2, \varepsilon u),$$

and the action on the base is (2.2).

For a Seifert bundle $\pi\colon M \to X$, there is a well-defined Chern class [25, Definition 13]

$$c_1(M) \in H^2(X, \mathbb{Q}).$$

If we set

$$\ell = \operatorname{lcm}(m_x \mid x \in X), \quad \mu = \operatorname{lcm}(m_i),$$

where m_i are the multiplicities of D_i, then $M/\mathbb{Z}_\ell$ is a line bundle over X, and $M/\mathbb{Z}_\mu$ is a line bundle over $X - P$. Therefore, $c_1(M/\mathbb{Z}_\ell) = \ell c_1(M) \in H^2(X,\mathbb{Z})$ and $c_1(M/\mathbb{Z}_\mu) \in H^2(X - P,\mathbb{Z})$ are integral classes.

Proposition 2.4 ([7, Theorem 7.1.3]). *Let (M,η,ξ,Φ,g) be a quasi-regular K-contact manifold. Then the space of leaves X has a natural structure of an almost-Kähler cyclic orbifold where the projection $M \to X$ is a Seifert bundle. Furthermore, if (M,η,ξ,Φ,g) is Sasakian, then X is a Kähler orbifold.*

Conversely, let (X,ω,J,g) be an almost-Kähler cyclic orbifold with $[\omega] \in H^2(X,\mathbb{Q})$, and let $\pi\colon M \to X$ be a Seifert bundle with $c_1(M) = [\omega]$. Then M admits a K-contact structure (M,η,ξ,Φ,g) such that $\pi^*(\omega) = d\eta$.

2.4. Topology of a Seifert bundle

Let X be an oriented cyclic 4-orbifold, $P \subset X$ the set of singular points, and $D_i \subset X$ the isotropy surfaces with coefficients $m_i > 1$. Suppose that there are local invariants $\mathbf{j}_{D_i} = (m_i, j_i)$ for each D_i and $\mathbf{j}_x$, for every $x \in P$. Let $0 < b_i < m_i$ be such that $j_i b_i \equiv 1 \pmod{m_i}$, and let B be a complex line bundle on X. Then there is a Seifert bundle $\pi\colon M \to X$ with the given local invariants and first Chern class

$$c_1(M) = c_1(B) + \sum_i \frac{b_i}{m_i}[D_i].$$

As we aim for simply connected 5-manifolds M, we need to characterize the first homology group by using the following result.

Proposition 2.5 ([23, Theorem 36]). *Suppose that $\pi\colon M \to X$ is a quasi-regular Seifert bundle with isotropy surfaces D_i with multiplicities m_i, and singular locus $P \subset X$. Let $\mu = \operatorname{lcm}(m_i)$. Then $H_1(M,\mathbb{Z}) = 0$ if and only if*

(1) $H_1(X,\mathbb{Z}) = 0$,

(2) $H^2(X,\mathbb{Z}) \to \bigoplus_i H^2(D_i,\mathbb{Z}_{m_i})$ *is surjective,*

(3) $c_1(M/\mathbb{Z}_\mu) \in H^2(X - P,\mathbb{Z})$ *is a primitive class.*

Moreover, $H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus (\bigoplus_i \mathbb{Z}_{m_i}^{2g_i})$, $g_i = g(D_i)$ the genus of D_i, $k + 1 = b_2(X)$.

To construct a K-contact manifold from a Seifert bundle, we use the following.

Lemma 2.6 ([23, Lemma 39]). *Let (X,ω) be a symplectic cyclic 4-orbifold with isotropy locus given by surfaces D_i and singular locus P. Assume given local invariants $\{\mathbf{j}_{D_i} = (m_i, j_i), \mathbf{j}_x, x \in P\}$ for X. Let b_i with $j_i b_i \equiv 1 \pmod{m_i}$, $\mu = \operatorname{lcm}(m_i)$. Then there is a Seifert bundle $\pi\colon M \to X$ such that*

(1) *It has Chern class $c_1(M) = [\hat{\omega}]$ for some orbifold symplectic form $\hat{\omega}$ on X.*

(2) *If $\sum \frac{b_i \mu}{m_i}[D_i] \in H^2(X - P,\mathbb{Z})$ is primitive and the second Betti number $b_2(X) \geq 3$, then we can further have that $c_1(M/\mathbb{Z}_\mu) \in H^2(X - P,\mathbb{Z})$ is primitive.*

Finally, in order to control the fundamental group, we introduce the following.

Definition 2.7. Let X be an oriented cyclic 4-orbifold with singular locus P and isotropy surfaces D_i of multiplicity m_i. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(X)$ is defined as

$$\pi_1^{\mathrm{orb}}(X) = \pi_1\big(X - ((\bigcup D_i) \cup P)\big)/\langle \gamma_i^{m_i} = 1\rangle,$$

where $\langle \gamma_i^{m_i} = 1\rangle$ denotes the relation $\gamma_i^{m_i} = 1$ on $\pi_1(X - ((\bigcup D_i) \cup P))$ for any small loop γ_i around the surface D_i.

We have the following exact sequence [7, Theorem 4.3.18]:

$$\cdots \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(X) \to 1.$$

If $H_1(M, \mathbb{Z}) = 0$ and $\pi_1^{\mathrm{orb}}(X)$ is abelian, then $\pi_1(M)$ must be trivial. This holds since if $H_1(M, \mathbb{Z}) = 0$, then $\pi_1(M)$ has no abelian quotients. As $\pi_1^{\mathrm{orb}}(X)$ is assumed abelian, we find that $\pi_1(M)$ is a quotient of $\mathbb{Z}$, hence again abelian. This implies that $\pi_1(M) = 1$. In this case, M is a Smale–Barden manifold.

2.5. *Symplectic constructions*

We review different constructions in symplectic geometry that we will use later. We start with the Gompf symplectic sum [18]. Let S_1 and S_2 be closed symplectic 4-manifolds, and $F_1 \subset S_1$, $F_2 \subset S_2$ symplectic surfaces of the same genus and with $F_1^2 = -F_2^2$. Fix a symplectomorphism $F_1 \cong F_2$. If ν_j is the normal bundle to F_j, then there is a reversing-orientation bundle isomorphism $\psi\colon \nu_1 \to \nu_2$. Identifying the normal bundles ν_j with tubular neighbourhoods $\nu(F_j)$ of F_j in S_i, one has a symplectomorphism $\varphi\colon \nu(F_1) - F_1 \to \nu(F_2) - F_2$ by composing ψ with the diffeomorphism $x \mapsto \frac{x}{\|x\|^2}$ that turns each punctured normal fiber inside out. The Gompf symplectic sum is the manifold obtained from $(S_1 - F_1) \sqcup (S_2 - F_2)$ by gluing with φ above. It is proved in [18] that this surgery yields a symplectic manifold, denoted $S = S_1 \#_{F_1=F_2} S_2$. The Euler characteristic of the Gompf symplectic sum is given by $\chi(S) = \chi(S_1) + \chi(S_2) - 2\chi(F)$, where $F = F_1 = F_2$.

In [25, Lemma 24], it is proved that if $D_1 \subset S_1$ and $D_2 \subset S_2$ are symplectic surfaces intersecting transversally and positively with F_1, F_2, respectively, such that $D_1 \cdot F_1 = D_2 \cdot F_2 = d$, then D_1, D_2 can be glued to a symplectic surface

$$D = D_1 \# D_2 \subset S_1 \#_{F_1=F_2} S_2$$

with self-intersection $D^2 = D_1^2 + D_2^2$ and genus $g(D) = g(D_1) + g(D_2) + d - 1$. This can be done with several surfaces simultaneously.

The following result is very useful to make intersections nice.

Lemma 2.8 ([25, Lemma 6]). *Let (X, ω) be a symplectic 4-manifold, and suppose that $D, D' \subset X$ are symplectic surfaces intersecting transversally and positively. Then we can isotop D (small in the C^0-sense, only around the points of $D \cap D'$), so that the image of D under the isotopy is symplectic, D and D' intersect nicely (symplectically orthogonal).*

To produce symplectic cyclic 4-orbifolds, we shall contract chains of symplectic surfaces.

Definition 2.9. Let X be a symplectic 4-manifold. A *chain of symplectic surfaces* $\mathcal{C} = C_1 \cup \cdots \cup C_l$ consists of $l \geq 1$ symplectic surfaces C_i, of genus $g = 0$ and self-intersection $C_i^2 = -b_i \leq -2$, such that $C_i \cap C_j = \emptyset$ for $|i - j| > 1$ and $C_i \cap C_{i+1}$ is a nice intersection, $i = 1, \dots, l - 1$.

Note that if we have a chain $\mathcal{C} = C_1 \cup \cdots \cup C_l$ as in Definition 2.9 where the intersections $C_i \cap C_{i+1}$ are only transverse and positive, then Lemma 2.8 allows to perturb the surfaces so that the chain satisfies that the intersections are nice.

Proposition 2.10. *Suppose that $\mathcal{C} = C_1 \cup \cdots \cup C_l$ is a chain of symplectic surfaces with $C_i^2 = -b_i \leq -2$, $i = 1, \dots, l$. Then there is a symplectic cyclic 4-orbifold $\overline{X}$ with a singular point p_0, and a map $\pi\colon X \to \overline{X}$ such that $\pi^{-1}(p_0) = \mathcal{C}$, and $\pi\colon X - \mathcal{C} \to \overline{X} - \{p_0\}$ is a symplectomorphism. Moreover, if we write the continuous fraction $[b_1, \dots, b_l] = \frac{d}{r}$, $\gcd(d, r) = 1$, then the orbifold point p_0 is of the form $\mathbb{C}^2/\mathbb{Z}_d$, where $\varepsilon \cdot (z_1, z_2) = (\varepsilon z_1, \varepsilon^r z_2)$, $\varepsilon = e^{2\pi i/d}$.*

Proof. Write the continuous fraction

$$\frac{d}{r} = [b_1, \dots, b_l] = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots}}$$

and consider the action of the cyclic group $\mathbb{Z}_d$ on $\mathbb{C}^2$ given by $(z_1, z_2) \mapsto (\varepsilon z_1, \varepsilon^r z_2)$, where $\varepsilon = e^{2\pi i/d}$, $0 < r < d$ and $\gcd(r, d) = 1$. By [9, Lemma 15], the complex resolution $\varpi\colon X' \to \mathbb{C}^2/\mathbb{Z}_d$ of $\mathbb{C}^2/\mathbb{Z}_d$ has an exceptional divisor formed by a chain of smooth rational curves of self-intersections $-b_1, -b_2, \dots, -b_l$. Let $\mathcal{C}' = C_1' \cup \cdots \cup C_l'$ denote the chain in the Kähler manifold X'.

In [23, Theorem 16], a symplectic neighbourhood theorem for chains of length $l = 2$ was proven, but it applies equally to chains of length $l \geq 2$. It says the following: suppose that (X, ω), (X', ω') denote the corresponding symplectic forms, and suppose that $\langle [\omega], [C_i] \rangle = \langle [\omega'], [C_i'] \rangle$ for all $i = 1, \dots, l$ (that is, the areas of the symplectic surfaces match). Assume also that $C_i^2 = C_i'^2$, so that the normal bundles $\nu_{C_i} \cong \nu_{C_i'}$ are isomorphic. Then, there are tubular neighbourhoods $\mathcal{C} \subset U \subset X$ and $\mathcal{C}' \subset U' \subset X'$ which are symplectomorphic via $\varphi\colon U \to U'$, with $\varphi(C_i) = C_i'$ for all i. Then we can take a small ball $B = B_\varepsilon(0) \subset \mathbb{C}^2/\mathbb{Z}_d$ such that $V' = \varpi^{-1}(B) \subset U'$, and let $V = \varphi^{-1}(V')$. Now we glue $X - \mathcal{C}$ to B to get a symplectic cyclic 4-orbifold $\overline{X}$, with a map $\pi\colon X \to \overline{X}$ as required.

To arrange the condition on the areas, write $[\omega] = \sum a_i [C_i]$ for $a_i \in \mathbb{R}$. Take a_{i_0} the maximum of the a_i. If $a_{i_0} \geq 0$, then

$$\langle [\omega], [C_{i_0}] \rangle = -a_{i_0} b_{i_0} + a_{i_0 - 1} + a_{i_0 + 1} \leq a_{i_0}(-b_{i_0} + 2) \leq 0,$$

which cannot occur since the symplectic area $\langle [\omega], [C] \rangle$ of a symplectic surface C is always positive. Hence $a_{i_0} < 0$ and therefore all $a_i < 0$. Next we compactify $\mathbb{C}^2/\mathbb{Z}_d$ to $\mathbb{C}P^2/\mathbb{Z}_d$, by adding the line at infinity which is away from the orbifold point. Consider the resolution $\widetilde{X}$ of $\mathbb{C}P^2/\mathbb{Z}_d$, and let H be the hyperplane class. As this is projective, it has a Kähler class of the form $T = H + \sum a_i'[C_i']$, with $a_i' < 0$. The Nakai–Moishezon ampleness criterion says that T is ample if $T^2 > 0$ and $T \cdot C > 0$ for every effective curve C. Hence for every non-exceptional curve C, $H \cdot C \geq \sum(-a_i')C_i' \cdot C$, and so $C_i' \cdot C \leq mH \cdot C$, for some $m > 0$. Now the class $T' = kH + \sum a_i[C_i'] \in H^2(\widetilde{X}, \mathbb{R})$ is a Kähler class for $k > 0$ large, since $T'^2 > 0$ and $T' \cdot C > 0$ for every non-exceptional curve C. For $C = C_i'$, $T' \cdot C_i' = \langle [\omega], [C_i] \rangle > 0$. Then there is a Kähler form Ω on $\widetilde{X}$ that restricts to a Kähler form ω' on V' such that $[\omega'] = \sum a_i[C_i']$, and so

$$\langle [\omega'], [C_i'] \rangle = \langle [\omega], [C_i] \rangle. \qquad \blacksquare$$

Another tool that we need is to transform Lagrangian submanifolds into symplectic ones.

Lemma 2.11 ([25, Lemma 27]). *Let (M, ω) be a 4-dimensional compact symplectic manifold. Assume that $[F_1], \dots, [F_k] \in H_2(M, \mathbb{Z})$ are linearly independent homology classes represented by Lagrangian surfaces $F_1, \dots, F_k$ which intersect transversally, not three of them intersect in a point, and the intersection pattern has no cycles. Then there is an arbitrarily C^∞-small perturbation ω' of the symplectic form ω such that all $F_1, \dots, F_k$ become symplectic.*

Note that if the Lagrangian F_i intersects transversally a symplectic surface S, after the perturbation we will have two symplectic surfaces intersecting transversally. With the given conditions, we can arrange the orientation of the homology classes $[F_i]$ suitably such the intersections will be positive, and using Lemma 2.8, we can make the intersection nice.

3. Construction of a K-contact Smale–Barden manifold

We are going to start by constructing a simply connected symplectic cyclic 4-orbifold with $b_2 = b_2^+ = 3$, and having three symplectic surfaces which are disjoint and span $H_2(X, \mathbb{Q})$.

We take the rational elliptic surface S with singular fibers $I_9 + 3A_1$ that appears in [5, p. 568]. To construct S, take the pencil of cubic curves in $\mathbb{C}P^2$ with equation $X^2Y + Y^2Z + Z^2X + tXYZ = 0$. We blow up twice at each of the three points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, which are the nodes of the singular curve of the pencil $XYZ = 0$. This produces a cycle of nine curves with another curve intersecting three of them (the image of the smooth curve of the pencil $X^2Y + Y^2Z + Z^2X = 0$), see Figure 1. We blow up the three intersections points to get the desired elliptic fibration. There is a cycle of nine rational (-2)-curves $C_1, \dots, C_9$, and three sections $\sigma_1, \sigma_2, \sigma_3$ with σ_j intersecting C_{3j+1}, $j = 1, 2, 3$. The sections σ_j are rational (-1)-curves. The three nodal curves are given by the values of $t = -3$, $3e^{\pi i/3}$, $-3e^{\pi i/3}$.

Next, let F be a smooth fiber of the elliptic fibration $S \to \mathbb{C}P^1$ obtained from the cubic pencil after the blow-ups, and take an isomorphism $H_1(F, \mathbb{Z}) \cong \mathbb{Z}^2$. The monodromy of the fibration, which appears listed in [17, Table 3, No. 63], is described by the

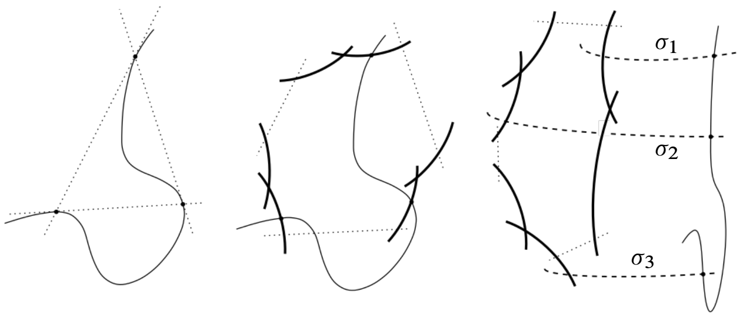

Fig. 1. Construction of the rational elliptic surface S with singular fibers $I_9 + 3A_1$.

equality $CX_{[1,-2]}X_{[2,-1]}A^9 = \mathrm{I}$ (the reverse order is due to the fact that we are understanding the matrices as composition of endomorphisms). The notation is

$$X_{[p,q]} = \begin{pmatrix} 1+pq & -p^2 \\ q^2 & 1-pq \end{pmatrix},$$

and $A = X_{[1,0]}$, $C = X_{[1,1]}$. So the monodromy representation is written as

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^9 = \mathrm{I}.$$

The vanishing cycle of $X_{[p,q]}$ is (p,q), with the choice of path to the critical point taken in [17]. Therefore, monodromies corresponding to going around the nodal curves are $C = X_{[1,1]}$, $X_{[1,-2]}$ and $X_{[2,-1]}$, whence the vanishing cycles are $(1,1)$, $(1,-2)$ and $(2,-1)$.

We take two copies S_1, S_2 of S as above, with two smooth fibers F_1, F_2. They have $F_1^2 = F_2^2 = 0$. We choose a symplectomorphism $\varphi\colon F_1 \to F_2$ such that the vanishing cycles match, that is, the identity in homology $\varphi_*\colon H_2(F_1,\mathbb{Z}) \to H_2(F_2,\mathbb{Z})$. Take the Gompf symplectic sum

$$X = S_1 \#_{F_1=F_2} S_2 = (S_1 - \nu(F_1)) \bigcup_{\nu(F_1)-F_1 \cong \nu(F_2)-F_2} (S_2 - \nu(F_2)).$$

As $b_2(S_1) = b_2(S_2) = 10$, then $\chi(S_1) = \chi(S_2) = 12$. So $\chi(X) = 24$, and hence $b_2(X) = 22$.

Let $C_1, \ldots, C_9$ be the I_9-cycle of S_1, with sections σ_1, σ_2, σ_3 as before. Analogously, let $C_1', \ldots, C_9'$ be the I_9-cycle of S_2, with sections σ_1', σ_2', σ_3' as before. By using [25, Lemma 24], we can glue the sections to produce symplectic surfaces E_1, E_2, E_3 of square (-2).

Lemma 3.1. *The* 4*-manifold X is simply connected.*

Proof. The elliptic surface S_1 is simply connected, hence $\pi_1(S_1 - \nu(F_1))$ is generated by a loop around F_1. But this is contracted by using one of the sections of the elliptic fibration. Hence $\pi_1(S_1 - \nu(F_1)) = 1$. Also $\pi_1(S_2 - \nu(F_2)) = 1$, so $\pi_1(X) = 1$. ∎

Fix a fiber $F \subset \partial\nu(F_1) = \partial\nu(F_2) \subset X$. Take the vanishing cycle $a = (1,1)$ in F, and the two vanishing thimbles D_1, D_2 in $S_1 - \nu(F_1)$, $S_2 - \nu(F_2)$, respectively. We glue them to form a Lagrangian (-2)-sphere D. Next take as dual curve in F the curve $b = (1,-2)$, intersecting a transversally and positively at three points. This follows since in $H_1(F_0, \mathbb{Z})$ we have $\langle a, b\rangle = \langle (1,1), (1,-2)\rangle = 3$, since the intersection form is anti-symmetric. Take the torus $T = b \times S^1 \subset F \times S^1 = \partial\nu(F_1) = \partial\nu(F_2)$. This produces a pair of surfaces D, T with

$$D^2 = -2, \quad D \cdot T = 3, \quad T^2 = 0,$$

where D is a Lagrangian sphere and T is a Lagrangian torus.

With the second vanishing cycle $b = (1,-2)$, we do the same thing using another fiber $F' \subset \partial\nu(F_1) = \partial\nu(F_2) \subset X$. We obtain a pair D', T' of Lagrangian (-2)-sphere and Lagrangian torus, disjoint from D, T. To arrange this, first move the S^1-factor in $\partial\nu(F_1) = F_1 \times S^1$, which maps to $S^1 \subset \mathbb{C}P^1$ around the point giving the fiber F_1, in outward direction to make it disjoint. This produces that T, T' are disjoint. Second, note that the vanishing thimbles D, D' map to paths from the point defining the fiber to the point of the nodal fiber, and these can be made disjoint; third, to avoid the possible intersections $D \cap T'$, $D' \cap T$, we use the dual curve to one of the vanishing cycle, which is equal to the other vanishing cycle, and move these loops in a parallel direction along F. Finally, we use Lemma 2.11 to change slightly the symplectic form, so that D, D' become symplectic (-2)-surfaces, and T, T' become symplectic tori.

Next, we see that there is a chain of 17 rational (-2)-curves

$$\mathcal{C} = C_8 \cup \cdots \cup C_2 \cup C_1 \cup E_1 \cup C_1' \cup C_2' \cup \cdots \cup C_8',$$

where E_1 is defined before Lemma 3.1 (see Figure 2). We contract D and D' to two points p and p' of multiplicity 2. Using Proposition 2.10, we contract the chain $\mathcal{C}$ to a point q of multiplicity 18. Note that $[2, \overset{(17)}{\ldots}, 2] = \frac{18}{17}$, so the point has a local model $(z_1, z_2) \mapsto (\varepsilon z_1, \varepsilon^{-1} z_2)$, with $\varepsilon = e^{2\pi i/18}$. Denote by $\bar{X}$ the resulting symplectic cyclic orbifold with singular set $P = \{p, p', q\}$. It has $b_2(\bar{X}) = 22 - 2 - 17 = 3$, and it is simply connected.

Proposition 3.2. *There is a collection of smooth symplectic surfaces T_n, $n \geq 1$, in a neighbourhood of $T \cup D$, of genus $g_n = 9n^2 + 1$, not intersecting D, and such that all the T_n intersect pairwise nicely.*

Proof. Let K be the canonical class associated to the symplectic form. Note that $K \cdot T = 0$, $K \cdot D = 0$, hence $K \cdot (aD + bT) = 0$, for any $a, b \geq 0$.

We start constructing a curve $T_1 \equiv 2T + 3D$ as follows. By the symplectic neighbourhood theorem (Proposition 2.10), we can assume that we have a holomorphic model consisting of complex curves D, T in a complex surface. Let q_1, q_2, q_3 be the points of $T \cap D$. We arrange two parallel copies of T, say T', T'', which intersect transversely D at six points $q_1', q_2', q_3', q_1'', q_2'', q_3''$, where $q_i', q_i'' \in D$ are close to q_i, for $i = 1, 2, 3$. Take the normal bundle to $D = \mathbb{C}P^1$, which is $\mathcal{O}_D(-2)$. Take three meromorphic sections σ_1, σ_2,

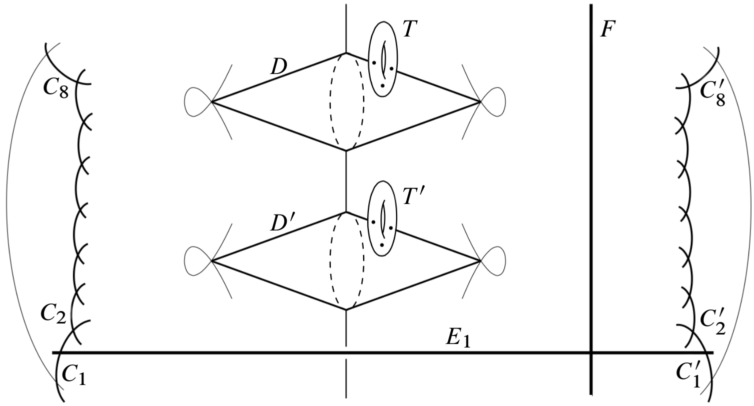

Fig. 2. The 4-manifold X and all the surfaces constructed along the text.

σ_3 of $\mathcal{O}_D(-2)$, where σ_i has poles at the points q_i', q_i''. At each of the six points we do as follows, we do it with q_1' for concreteness. Take an adapted chart (z, w) around q_1', where $T' = \{z = 0\}$, $D = \{w = 0\}$. Around q_1', we can assume that $\sigma_1 = \frac{1}{z}$. We glue the graph $\{(z, \frac{1}{z})\}$ with the graph $\{(\frac{1}{w}, w)\}$ in the normal bundle to T' (which is trivial). We use a cut off function to push this graph down to the graph $T' = \{(0, w)\}$, as in [8, Section 3.5]. The result is a symplectic surface. This has some self-intersections, which come from the intersections of the sections $\sigma_1, \sigma_2, \sigma_3$. These can be resolved symplectically to get a smooth symplectic surface of genus $g_1 = 10$, as in [25, Section 5.1] (basically changing the model $xy = 0$ by $xy = \epsilon$). Note that $2g_1 - 2 = T_1^2 = (2T + 3D)^2 = 12 \cdot 3 + 9 \cdot (-2) = 18$. The surface T_1 does not intersect D. Note that $T_1 \cdot D = (2T + 3D) \cdot D = 6 - 6 = 0$.

For given $n \geq 2$, take a collection of symplectic surfaces $\Sigma_1, \ldots, \Sigma_n$ as graphs in the normal bundle to T_1, and all intersecting transversally and positively. Using [25, Section 5.1], we can glue symplectically the Σ_i, $1 \leq i \leq n$, at the intersection points $\Sigma_i \cap \Sigma_j$ to obtain a symplectic surface $T_n \equiv \Sigma_1 + \cdots + \Sigma_n \equiv nT_1$. Then $T_n^2 = 18n^2$ and the genus $g_n = 9n^2 + 1$ satisfies $2g_n - 2 = 18n^2$ since $K \cdot T_n = 0$. Moreover, if we have different curves T_n, all can be taken to intersect transversally, and after perturbation as in Lemma 2.8, the intersections can be arranged to be nice. ∎

Proposition 3.3. *Let F be a fiber of the fibration that intersects the chain $\mathcal{C}$ transversally at a point of E_1. Consider the configuration of symplectic surfaces $\mathcal{C} \cup F$. Then there is a symplectic surface A of genus $g_A = 10$, in a neighbourhood of $\mathcal{C} \cup F$, not intersecting the chain.*

Proof. Take a cohomology class of the form

$$A \equiv 2F + a_0\sigma + a_1(C_1 + C_1') + a_2(C_2 + C_2') + \cdots + a_8(C_8 + C_8')$$

to arrange that it is disjoint from the curves E_1 and C_i, C_i', we need $0 = A \cdot E_1 = 2 - 2a_0 + 2a_1$, $0 = A \cdot C_1 = a_0 - 2a_1 + a_2$, $0 = A \cdot C_i = a_{i-1} - 2a_i + a_{i+1}$, $2 \leq i \leq 7$, and $0 = A \cdot C_8 = a_7 - 2a_8$, whose solution is $a_8 = 1, a_7 = 2, \ldots, a_1 = 8, a_0 = 9$. Note that

$$A^2 = 4a_0 + \sum 4a_{k-1}a_k - 2a_0^2 - \sum 4a_k^2 = 18,$$

hence $2g - 2 = K \cdot A + A^2 = 18$, so $g = 10$.

To construct the curve A, consider the push-down map $\pi \colon X \to \bar{X}$, which contracts $\mathcal{C}$ to the singularity q. The image $A = \pi(A)$ should be a smooth symplectic curve avoiding the singular point. We denote by the same letter since it does not pass through the singular point. We construct A directly in $\bar{X}$. As noted before, the singularity q is cyclic of order 18, and of type $\mathbb{C}^2/(\varepsilon, \varepsilon^{-1})$, $\varepsilon = e^{2\pi i/18}$. As explained in [27], there are 18 affine charts covering the 17 rational curves plus the coordinate axis $L_1 = \{y = 0\}$, $L_2 = \{x = 0\}$ (expressed in coordinates (x, y)). Each of these charts is centered at a point of intersection of two consecutive curves in the chain. They are given by the coordinates

$$(\xi_0, \eta_0) = \left(x^{18}, \frac{y}{x^{17}}\right), \quad (\xi_1, \eta_1) = \left(\frac{x^{17}}{y}, \frac{y^2}{x^{16}}\right), \quad (\xi_2, \eta_2) = \left(\frac{x^{16}}{y^2}, \frac{y^3}{x^{15}}\right), \quad \ldots,$$
$$(\xi_8, \eta_8) = \left(\frac{x^{10}}{y^8}, \frac{y^9}{x^9}\right), \quad (\xi_9, \eta_9) = \left(\frac{x^9}{y^9}, \frac{y^{10}}{x^8}\right), \quad \ldots, \quad (\xi_{16}, \eta_{16}) = \left(\frac{x^2}{y^{16}}, \frac{y^{17}}{x}\right),$$
$$(\xi_{17}, \eta_{17}) = \left(\frac{x}{y^{17}}, y^{18}\right).$$

The axis L_1 is $\eta_0 = 0$ (i.e., $y = 0$). The $(i + 1)$-th curve in the chain is defined by $\xi_i = 0$, $i = 0, \ldots, 16$. The second axis L_2 is $\xi_{17} = 0$ (i.e., $x = 0$). The curve $\bar{F} = \pi(F)$ passing through the mid-point of the 9-th curve is given by the equation $\eta_8 = 1$, that is, $\frac{y^9}{x^9} = 1$. This is equivalent to $y^9 - x^9 = 0$. The curve $2\bar{F}$ is thus $(y^9 - x^9)^2 = 0$, that we can perturb to a smooth curve as follows:

$$(y^9 - x^9)^2 = \epsilon xy + \zeta \tag{3.1}$$

with $0 < \zeta \ll \epsilon \ll 1$, in the chart $\mathbb{C}^2/\mathbb{Z}_{18}$. This avoids the singular point (the origin), and it is easily seen to be smooth. It is $\mathbb{Z}_{18}$-equivariant, so it descends to a smooth curve. We have to glue it to two copies of F, therefore we have to see that the boundary (of the intersection of (3.1) with a ball in the affine chart around the singular point) is a collection of two circles. In this way, we obtain the curve A sought for.

For proving that the boundary of (3.1) consists of two circles, note that we can see this for $\zeta = 0$, since the extra perturbation will merely slightly move the boundary, and so will not change it topologically. Note first that $y^9 - x^9 = 0$ is a collection of 9 lines, interchanged by $\mathbb{Z}_{18}$. Actually, the image is the quotient of $y - x = 0$ by $(x, y) \mapsto (-x, -y)$, which is the remaining $\mathbb{Z}_2$-action. Its boundary is the circle $\{(x, x) \mid x = e^{it}\}/(t \sim t + \pi)$. The equation $(y^9 - x^9)^2 = 0$ has as boundary again the same circle, but with multiplicity two. When we perturb $(y^9 - x^9)^2 = \epsilon xy$, the curve $(x, y = x)$ gets moved to $(x, y = x + a(x))$, and we can compute easily a Taylor expansion

$$a(x) = \pm \frac{1}{9}\sqrt{\epsilon}x^{-7} + \frac{1}{162}\epsilon x^{-15} + \cdots .$$

As we see, there are two solutions depending on the leading term. This means that there are two circles in the boundary (the other option would have been a double valued function $a(x)$). As there are only odd powers of x, the $\mathbb{Z}_2$-action $(x, y) \mapsto (-x, -y)$ goes down to $a \mapsto -a$, via $t \mapsto t + \pi$. That is, it acts on each circle, and not swapping the circles. So in the quotient, there are two circles remaining, as claimed. ■

Take the push-down map $\pi\colon X \to \bar{X}$, which contracts D, D' to singularities p, p' and the chain $\mathcal{C}$ to the singularity q. Consider the collection of symplectic surfaces T_n, $1 \leq n \leq N$, and T'_m, $1 \leq m \leq N$, and the symplectic surface $A = \pi(A)$. We shall fix a large $N > 0$ later on. None of the surfaces pass through singular points. We arrange all intersections to be nice, so that we can assign coefficients to all surfaces and make $\bar{X}$ into a cyclic orbifold X' by using Proposition 2.2. We can assign local invariants by using Proposition 2.3.

We take coefficients as follows. The genus of T_n and T'_n is $g_n = 9n^2 + 1$, $n \geq 1$. For each $1 \leq n, m \leq N$, take a prime p_{nm}. The collection of chosen primes should be different, $p_{nm} > 3$ and satisfy $p_{nm} > n, m$. We assign multiplicities as follows:

$$m_{T_n} = \prod_{m=1}^{N} p_{nm}, \quad m_{T'_m} = \prod_{n=1}^{N} p_{nm}^2, \quad m_A = \prod_{n,m=1}^{N} p_{nm}^3.$$

Note that for T_n, T_s, $n \neq s$, which are intersecting surfaces, we have that the primes p_{nk}, p_{sl} are different, hence $\gcd(m_{T_n}, m_{T_s}) = 1$. Analogously, $\gcd(m_{T'_m}, m_{T'_l}) = 1$ for $m \neq l$. Also note that $\gcd(m_{T_n}, m_A) \neq 1$ and $\gcd(m_{T'_m}, m_A) \neq 1$ for all $n, m \geq 1$. Also for any n, m, p_{nm} divides m_{T_n} and $m_{T'_m}$, hence $\gcd(m_{T_n}, m_{T'_m}) \neq 1$. This is in accordance with the fact that the involved surfaces are disjoint.

Theorem 3.4. *For any $N \geq 1$, there is a Seifert bundle $\pi\colon M \to X'$ which is K-contact and satisfies $H_1(M, \mathbb{Z}) = 0$ and*

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus \bigoplus_{n,m=1}^{N} (\mathbb{Z}_{p_{nm}}^{18n^2+2} \oplus \mathbb{Z}_{p_{nm}^2}^{18m^2+2} \oplus \mathbb{Z}_{p_{nm}^3}^{20}).$$

Moreover, M is spin.

Proof. We need to check the conditions of Proposition 2.5. First, clearly $H_1(\bar{X}, \mathbb{Z}) = 0$ because $\pi_1(\bar{X}) = 1$, by Lemma 3.1. Second, we have to see the surjectivity of the map $H^2(\bar{X}, \mathbb{Z}) \to \oplus_i H^2(D_i, \mathbb{Z}_{m_i})$. For this, we look at every prime. Let $p = p_{nm}$ and look at the map

$$\varpi\colon H^2(\bar{X}, \mathbb{Z}) \to H^2(T_n, \mathbb{Z}_p) \oplus H^2(T'_m, \mathbb{Z}_{p^2}) \oplus H^2(A, \mathbb{Z}_{p^3}).$$

Recall that $T_n \equiv nT_1$, $T'_m \equiv mT'_1$, $A \equiv 2\bar{F}$ in $H_2(\bar{X}, \mathbb{Z})$. The image $\varpi(T_1) = (nT_1 \cdot T_1, 0, 0) = (18n, 0, 0)$, $\varpi(T'_1) = (0, mT'_1 \cdot T'_1, 0) = (0, 18m, 0)$, $\varpi(A) = (0, 0, A^2) = (0, 0, 18)$, noting that $T_1, T'_1, A \in H^2(\bar{X}, \mathbb{Z}) = H_2(\bar{X} - P, \mathbb{Z})$. Now n, m are coprime with p (since $p > n, m$) and $p \geq 5$, so that $\gcd(p, 18) = 1$.

To proceed, we need to choose $c_1(M) \in H^2(\bar{X}, \mathbb{Q})$ so that it is a symplectic class, and also $c_1(M/\mathbb{Z}_\mu) \in H^2(\bar{X} - P, \mathbb{Z})$ is primitive. This follows from Lemma 2.6 if we can assure that

$$x = \mu\Big(\sum \frac{b_n}{m_{T_n}}[T_n] + \sum \frac{b'_m}{m_{T'_m}}[T'_m] + \frac{b}{m_A}[A]\Big) \in H^2(\bar{X} - P, \mathbb{Z})$$

is primitive, where b_n, b'_n, $n \geq 1$, and b are the corresponding b_i associated to the local invariants. Note that $\mu = m_A = \prod_{n,m} p_{nm}^3$. If we choose $b = 1$, then the coefficient of $[A]$ is 1. Cupping with $[A] \in H_2(\bar{X} - P, \mathbb{Z})$, we obtain $\langle x, [A]\rangle = [A]^2 = 18$. So the only possible divisors of x are 2 or 3. Now we note that $T_n \equiv nT_1$. Then the coefficient of T_1 in x is

$$\frac{b_1\mu}{m_{T_1}} + \sum_{n\geq 2} \frac{b_n \mu n}{m_{T_n}}.$$

As μ is not divisible by 6, if we choose $b_1 = 1$ and b_n divisible by 6 for $n \geq 2$, then this number is coprime with 6. Then x is not divisible by 2 or 3, as required.

By [18, (14)], the second Stiefel–Whitney class of M is

$$w_2(M) = \pi^* w_2(\bar{X} - P) + \sum (m_i - 1)\pi^{-1}(D_i).$$

As all m_i are odd, then $w_2(M) = \pi^* w_2(\bar{X} - P)$. Note that $K \cdot T = 0$, $K \cdot T' = 0$, $K \cdot A = 0$, hence $K = 0$, and so $w_2(\bar{X} - P) = 0$, hence $w_2(M) = 0$. So M is spin. ∎

Finally, we compute the fundamental group.

Theorem 3.5. *The orbifold fundamental group $\pi_1^{\mathrm{orb}}(X') = 1$, and $\pi_1(M) = 1$.*

Proof. Recall that X is simply connected by Lemma 3.1, and that we contract the surfaces D, D' and the chain $\mathcal{C}$. The singular points of the orbifold $\bar{X}$ are $P = \{p, p', q\}$. Then the fundamental group of

$$X^o = X - (D \cup D' \cup \mathcal{C}) = \bar{X} - P$$

is generated by loops around the singular points, that is, a, a' around p, p', respectively, and b around q. Note that $a^2 = 1$, $a'^2 = 1$, $b^{18} = 1$.

First fix a smooth fiber F_0. Recall that the vanishing cycles in F_0 are $(1, 1)$, $(1, -2)$, $(2, -1)$. Let $\alpha, \beta \in \pi_1(F_0)$ be the standard generators of the torus F_0. The third vanishing cycle contracts without touching any of the curves, because the vanishing thimbles can be taken to be disjoint. Hence $\alpha^2\beta^{-1} = 1$ in $\pi_1(X^o)$, so $\beta = \alpha^2$ and the group generated by α, β is generated by α, and it is abelian.

Next, take the surface A which lies in a neighbourhood of $F \cup \mathcal{C}$, and has genus 10, and self-intersection $A^2 = 18$. Let $\alpha_1, \beta_1, \ldots, \alpha_{10}, \beta_{10}$ be the loops generating $\pi_1(A)$, and let γ be a small loop around A, that is, a meridian. We order the loops so that α_1, β_1, α_2, β_2 are homotopic to α, β in the fiber F, and the other α_j, β_j are close to the singular point, so of the form b^k for some k (the value k depending on the loop). Then

$\gamma^{18} = \prod_{j=1}^{10} [\alpha_j, \beta_j] = 1$. Adding the relation γ^{m_A} and recalling that $\gcd(m_A, 18) = 1$ (since we have chosen all primes $p > 3$), we get $\gamma = 1$ in $\pi_1^{\rm orb}(X')$.

Now the fiber F_0 intersects the chain $\mathcal{C}$ in the central curve E_1. The loops around the curves $C_8, \ldots, C_1, E_1, C_1', \ldots, C_8'$ are given as, in this order, $b, \ldots, b^8, b^9, b^{10}, \ldots, b^{17}$. Then the loop around E_1 is b^9, which produces the relation $b^9 = [\alpha, \beta] = 1$. Now we use the fact that there are two extra sections E_2, E_3 of the fibration. These avoid T, D, T', D', E_2 intersects C_4, C_4', and E_3 intersects C_7, C_7'. They intersect A in two points. The loop around A is trivial $\gamma = 1 \in \pi_1^{\rm orb}(X')$. So we get relations $b^5 = b^{13}$ and $b^2 = b^{16}$. So $b^8 = 1$, that together with $b^9 = 1$ imply that $b = 1$.

Finally, take the surfaces T_n, $n \geq 1$, lying in a neighbourhood of $T \cup D$. Let c_j be a small loop around T_j, $j \geq 0$, and recall that a is a small loop around D, and $a^2 = 1$. All curves T_j intersect transversally, so $[c_j, c_k] = 1$, for all j, k. Let c_0 be a small loop around T, then $c_0 = a^3 = a$, since $T \cdot D = 3$. Move T slightly off to get a relation $c_1^9 c_2^{18} \cdots c_N^{9N} a = [\alpha\beta^{-2}, \gamma] = 1$ (the last loops are the generators of $\pi_1(T)$), using that $T \cdot T_k = 9k$. The group generated by $a, c_1, \ldots, c_N$ is abelian. We write the relations additively,

$$9(c_1 + 2c_2 + \cdots + Nc_N) + a = 0. \tag{3.2}$$

Put for brevity $m_j = m_{T_j}$. In $\pi_1^{\rm orb}(X')$, we have the extra relations $m_j c_j = 0$. Multiply (3.2) by $M_k = \prod_{j \neq k} m_j$ to get $9M_k k c_k = 0$. Now $m_k c_k = 0$, and $\gcd(m_k, M_k) = 1$ since $\gcd(m_j, m_k) = 1$ for $j \neq k$, and also $\gcd(k, m_k) = 1$ as we chose $\gcd(p_{km}, k) = 1$, and $m_k = \prod_m p_{km}$, and $\gcd(m_k, 3) = 1$ as all primes are $p > 3$. All together give $c_k = 0$ for $k \geq 1$. Then (3.2) gives $a = 0$ as well in $\pi_1^{\rm orb}(X')$.

Once that $\pi_1^{\rm orb}(X') = 1$, we get $\pi_1(M) = 0$ by the argument at the end of Section 2.4 using that $H_1(M, \mathbb{Z}) = 0$ from Theorem 3.4. ∎

Therefore, the K-contact manifold from Theorem 3.4 is a Smale–Barden manifold.

4. Bounding the number of singular points

Our last step is to prove that the manifold M from Theorem 3.4 does not admit a Sasakian structure. Suppose that M admits a Sasakian structure. Then there is Seifert bundle

$$\pi\colon M \to Y,$$

where Y satisfies the following conditions that we state explicitly.

Conditions 4.1. The orbifold Y is a Kähler cyclic orbifold with $b_2 = 3$, $b_1 = 0$. Associated to each prime p_{nm}, $1 \leq n, m \leq N$, there is a collection of three complex curves

$$D_1^{nm},\ D_2^{nm},\ D_3^{nm}, \tag{4.1}$$

which have genus $g(D_1^{nm}) = 9n^2 + 1$, $g(D_2^{nm}) = 9m^2 + 1$, $g(D_3^{nm}) = 10$. For each (n, m), the three curves (4.1) are disjoint, and span $H_2(Y, \mathbb{Q})$. Moreover, these curves are all nice, and intersect pairwise nicely.

This follows from the homology of M appearing in Theorem 3.4, and the relation to the homology of the base of a Seifert bundle given in Proposition 2.5. Curves (4.1) are the components of the isotropy locus. Conditions 4.1 imply that at most two different curves can go through a point of the singular set $P \subset X$. Some of the curves could be equal for different values of (n,m), e.g., $D_1^{nk} = D_1^{nl}$, $k \neq l$; or $D_2^{km} = D_2^{lm}$, $k \neq l$; or $D_1^{nk} = D_2^{ln}$; or $D_3^{nm} = D_3^{kl}$. Clearly, this can only happen if the genera of the involved curves are the same.

To prove that M cannot admit a Sasakian structure, we are going to get a contradiction if we assume the existence of a Kähler orbifold Y satisfying Conditions 4.1, for some $N \gg 0$ large enough. After preparatory work in Sections 4, 5 and 6, this will be proved in Theorem 7.1 in Section 7.

To start with, let Y be a Kähler cyclic orbifold satisfying Conditions 4.1. We do not assume $\pi_1^{\mathrm{orb}}(Y) = 0$. Our first task is to obtain a universal bound on the number of singular points $\#P$.

Let $\pi\colon \widetilde{Y} \to Y$ be the minimal resolution of singularities. For every cyclic singularity $p \in Y$, $\pi^{-1}(p) = E_p = C_1 \cup \cdots \cup C_l$ is a chain of rational curves with self-intersection $C_j^2 = -b_j \leq -2$. For any curve A in Y, we denote the proper transform as $\widetilde{A}$. Let A be a nice curve through p (for the sake of simplicity, assume that there is only one singular point). Then $\widetilde{A}$ intersects transversely just one of the extremal curves of the chain C_1, C_l. For concreteness, say it is C_1. We have that (see [3, p. 80])

$$\pi^* A = \widetilde{A} + \sum r_i C_i,$$

where $\frac{r_{k+1}}{r_k} = [b_{k+1}, \ldots, b_l]^{-1}$ for $0 \leq k \leq l-1$, where $r_0 = 1$. Note that $\frac{r_{k+1}}{r_k} < 1$, hence $0 < r_l < r_{l-1} < \cdots < r_1 < 1$. Next, let $\widetilde{K}$ be the canonical divisor of $\widetilde{Y}$, and K the canonical divisor of Y. In this case, $\widetilde{K}$ is not the proper transform of K. We have a formula (again assuming only one singular point)

$$\widetilde{K} = \pi^* K - \sum \lambda_i C_i, \tag{4.2}$$

where $\lambda_i \geq 0$. If there are more singular points, then we have to add the contribution over each $p \in P$.

Lemma 4.2. *Let A, B be two effective divisors in Y, and let $\widetilde{A}$, $\widetilde{B}$ be the proper transforms. Then $A \cdot B \geq \widetilde{A} \cdot \widetilde{B}$.*

Proof. It is enough to prove it for A, B two irreducible curves in Y. Then

$$A \cdot B = \pi^* A \cdot \pi^* B = \pi^* A \cdot \widetilde{B} = (\widetilde{A} + E) \cdot \widetilde{B} \geq \widetilde{A} \cdot \widetilde{B}.$$

The second equality follows since $\pi^* A \cdot C_i = A \cdot \pi_* C_i = 0$ for any exceptional divisor C_i. The third, because $\pi^* A = \widetilde{A} + E$, where $E = \sum r_i C_i$, where $r_i \in \mathbb{Q}$, $r_i \geq 0$, and C_i are exceptional divisors. The last equality is due to $C_i \cdot \widetilde{B} \geq 0$, being two distinct effective curves. ∎

Now let D_1, D_2, D_3 be three nice curves, which are disjoint, and span $H_2(Y,\mathbb{Q})$. We have the following.

Lemma 4.3. *The $\mathbb{Q}$-divisor $K + D_1 + D_2 + D_3$ is effective. Also $K + D_i$ is effective, $i = 1, 2, 3$.*

Proof. We have the exact sequence

$$0 \to \mathcal{O}(\widetilde{K}) \to \mathcal{O}(\widetilde{K} + \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3) \to \bigoplus \mathcal{O}_{\widetilde{D}_i}(K_{\widetilde{D}_i}) \to 0,$$

because the $\widetilde{D}_i$ are disjoint, and using the adjunction formula. As $g(\widetilde{D}_i) = g_i \geq 1$, we have $H^0(\mathcal{O}_{\widetilde{D}_i}(K_{\widetilde{D}_i})) = \mathbb{C}^{g_i} \neq 0$, by Riemann–Roch. As $b_1(\widetilde{Y}) = 0$, so $H^1(\widetilde{K}) = 0$. Also $H^0(\widetilde{K}) = 0$, since $h^{0,2}(\widetilde{Y}) = 0$, because $H^2(\widetilde{Y},\mathbb{C})$ is spanned by complex curves, so $h^{1,1}(\widetilde{Y}) = b_2(\widetilde{Y})$. Therefore, $h^0(\mathcal{O}(\widetilde{K} + \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3)) = g_1 + g_2 + g_3 > 0$, and hence there is some effective divisor $\Sigma' \equiv \widetilde{K} + \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3$. Pushing down, $\Sigma = \pi(\Sigma') \equiv K + D_1 + D_2 + D_3$ in Y.

The last assertion is proved in the same way. ■

Put $B = D_1 + D_2 + D_3$. We need to check that $K + B$ is log canonical, whose definition appears in [20, Definition 1.16]. This is checked at each singular point $p \in P$. Suppose that $p \in D_1 = D$ (the case that p is not in any divisor D_i is similar). Assume for simplicity that there are no more singular points, and write

$$\widetilde{K} = \pi^*(K + D) - \widetilde{D} + \sum_{i=1}^{l} a_i C_i,$$

where $\widetilde{D}$ is the proper transform of D, C_i are the exceptional divisors, ordered so that $\widetilde{D} \cdot C_1 = 1$. We have to check that $a_i \geq -1$. Letting $a_0 = -1$ and setting $a_{l+1} = 0$, we have the equalities $\widetilde{K} \cdot C_i + C_i^2 = -2$, which give $-a_i b_i + a_{i-1} + a_{i+1} - b_i = -2$. This is rewritten as

$$(a_{i-1} - a_i) + (a_{i+1} - a_i) = (a_i + 1)(b_i - 2). \tag{4.3}$$

Let i_0 be such that a_{i_0} is minimum. Then the left-hand side of (4.3) for $i = i_0$ is ≥ 0. Therefore, if $b_{i_0} > 2$, then $a_{i_0} + 1 \geq 0$, and so $a_i \geq a_{i_0} \geq -1$ for all i. If $b_{i_0} = 2$, then $a_{i-1} = a_i = a_{i+1}$ and we proceed recursively.

Recall the definition of the orbifold Euler–Poincaré characteristic of an orbifold with isolated singularities,

$$e_{\mathrm{orb}}(Y) = e(Y) - \sum_{p \in P} \left(1 - \frac{1}{d_p}\right),$$

where d_p denotes the multiplicity of the singular point $p \in P$.

Theorem 4.4. *Now let D_1, D_2, D_3 be three disjoint nice curves that span $H_2(Y,\mathbb{Q})$. Then $e_{\mathrm{orb}}(Y - (D_1 \cup D_2 \cup D_3)) \geq 0$.*

Proof. Let $B = D_1 + D_2 + D_3$. We already know that (Y, B) is log canonical and effective. If we have that $K + B$ is nef, then [20, Theorem 10.14] implies that

$$3c_2(\Omega^1_Y(\log B)) \geq c_1(\Omega^1_Y(\log B))^2 = (K + B)^2 \geq 0,$$

where the last inequality is due to the fact that $K + B$ is effective and nef. By [20, Theorem 10.8], we have that $c_2(\Omega^1_Y(\log B)) = e_{\mathrm{orb}}(Y - B)$ and the result follows.

It remains to see that $K + B$ is nef. Let $A \subset Y$ be an irreducible curve, and let us check that $(K + B) \cdot A \geq 0$. First assume that $A = D_i$. By Lemma 4.2, $K + D_i$ is effective. Moreover, $H^0(\widetilde{K} + \widetilde{D}_i) \to H^0(\mathcal{O}_{\widetilde{D}_i}(K_{\widetilde{D}_i}))$ is bijective, and as $K_{\widetilde{D}_i}$ is base-point free, $\widetilde{K} + \widetilde{D}_i$ can be represented by a divisor not containing $\widetilde{D}_i$. Hence there is $C \equiv K + D_i$ not containing D_i, and thus $(K + B) \cdot A = (K + D_i) \cdot D_i = C \cdot D_i \geq 0$.

So we can suppose now that $A \neq D_i$, $i = 1, 2, 3$. Let $\Sigma \equiv K + B$ be an effective $\mathbb{Q}$-divisor. Write $\Sigma = rA + T$, where T does not contain A. If $A^2 \geq 0$, then $(K + B) \cdot A = \Sigma \cdot A \geq 0$, and we are done. So we can assume that $A^2 < 0$. By Lemma 4.2, we also have $\widetilde{A}^2 < 0$.

Next suppose that $\widetilde{K} \cdot \widetilde{A} \geq 0$. Then as $K = \pi_* \widetilde{K}$ (although $\widetilde{K}$ is not the strict transform of K),

$$K \cdot A = \pi_* \widetilde{K} \cdot A = \widetilde{K} \cdot \pi^* A = \widetilde{K} \cdot (\widetilde{A} + E) \geq \widetilde{K} \cdot \widetilde{A} \geq 0,$$

where E is an effective $\mathbb{Q}$-divisor consisting of exceptional curves $E = \sum r_i C_i$, $r_i \geq 0$. For any C_i, it is $\widetilde{K} \cdot C_i \geq 0$, since they are rational curves with $C_i^2 \leq -2$. As $B \cdot A \geq 0$, then $(K + B) \cdot A \geq 0$, as required.

So we are left with the case of an irreducible curve $\widetilde{A}$ with $\widetilde{A}^2 < 0$, $\widetilde{K} \cdot \widetilde{A} < 0$. As $p_a(\widetilde{A}) = \widetilde{A}^2 + \widetilde{K} \cdot \widetilde{A} < 0$, then $\widetilde{A}$ must be a smooth rational curve, with $\widetilde{A}^2 = -1$ and $\widetilde{K} \cdot \widetilde{A} = -1$, that is, an exceptional divisor for a minimal model of $\widetilde{Y}$.

If $A \cdot D_1 = A \cdot D_2 = 0$, then $A \equiv \lambda D_3$, for some $\lambda > 0$, which is impossible since $A \cdot D_3 \geq 0$ and $D_3^2 < 0$. If $A \cdot D_2 = A \cdot D_3 = 0$, then $A \equiv \lambda D_1$ for some $\lambda > 0$, and hence $A^2 > 0$, contrary to our current assumption. Finally, if $A \cdot D_1 = 0$, then $A \equiv \lambda_2 D_2 + \lambda_3 D_3$ for some $\lambda_i \in \mathbb{Q}$. Since $A \cdot D_i \geq 0$, then $\lambda_i \leq 0$, $i = 2, 3$. Hence $A \leq 0$, which is a contradiction as A is effective.

So we can assume $A \cdot D_1 > 0$ and $A \cdot D_2 > 0$ (after swapping D_2, D_3 if necessary). Now $(\widetilde{K} + \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3) \cdot \widetilde{A} = -1 + \sum \widetilde{D}_i \cdot \widetilde{A}$. If $\widetilde{D}_i \cdot \widetilde{A} \geq 1$ for some i, then $(\widetilde{K} + \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3) \cdot \widetilde{A} \geq 0$. Recall that there is an effective $\Sigma' \equiv \widetilde{K} + \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3$ with $\pi(\Sigma') = \Sigma \equiv K + B$. So

$$(K + B) \cdot A = \Sigma' \cdot \pi^* A = \Sigma' \cdot (\widetilde{A} + E) \geq 0,$$

where we have $E = \sum r_i C_i$, $r_i \geq 0$, and $\Sigma' \cdot C_i \geq 0$ because $\widetilde{D}_j \cdot C_i \geq 0$ and $\widetilde{K} \cdot C_i \geq 0$.

Hence we can further assume that $\widetilde{D}_i \cdot \widetilde{A} = 0$ for all i. As $A \cdot D_1 > 0$, $A \cdot D_2 > 0$, this means that $\widetilde{A}$ intersects a chain of exceptional divisors E_p, for a singularity $p \in A \cap D_i$, for both cases $i = 1, 2$. By Lemma 4.2, we have

$$(K + D_1 + D_2 + D_3 + A) \cdot A \geq (\widetilde{K} + \widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3 + \widetilde{A}) \cdot \widetilde{A} + \sum_{p \in P} \ell_p = -2 + \sum_{p \in P} \ell_p,$$

where ℓ_p denotes a local contribution of the intersection at p. There is contribution to ℓ_p only if $p \in A \cap D_i$. This happens at least for two singular points, hence it is enough to see that $\ell_p \geq 1$ if $p \in A \cap D_i$. Once we have checked this, we have that $(K + D_1 + D_2 + D_3 + A) \cdot A \geq 0$. As we are assuming $A^2 < 0$, we have $(K + D_1 + D_2 + D_3) \cdot A \geq 0$, i.e., $(K + B) \cdot A \geq 0$, completing the proof.

Let us finally see that $\ell_p \geq 1$. The proper transform $\widetilde{A}$ intersects the chain $E_p = C_1 \cup \cdots \cup C_l$, but not $\widetilde{D}_i$. Let $\alpha_j = \widetilde{A} \cdot C_j \in \mathbb{Z}_{\geq 0}$. Take (a germ of) a curve A'_j that intersects transversally C_j and no other C_k. Then $\widetilde{A} \equiv \sum \alpha_j A'_j$ in a neighbourhood of E_p, and so $A \equiv \sum \alpha_j A_j$, where $A_j = \pi(A'_j)$. Then the contribution at p is

$$\begin{aligned}\ell_p &= ((K + D_1 + D_2 + D_3 + A) \cdot A)_p \\ &= \Big(\Big(K + D_1 + D_2 + D_3 + \sum \alpha_j A_j\Big) \cdot \sum \alpha_k A_k\Big)_p \\ &= \sum \alpha_j ((K + D_1 + D_2 + D_3 + A_j) \cdot A_j)_p \\ &\quad + \sum_{j \neq k} \alpha_j \alpha_k (A_j \cdot A_k)_p + \sum (\alpha_j^2 - \alpha_j)(A_j^2)_p.\end{aligned}$$

The local intersection number is defined in [14]. As $(A_j \cdot A_k)_p \geq 0$, we see that it is enough to prove the result for $A = A_j$. We assume this henceforth.

To compute $((K + D_i + A) \cdot A)_p$, note that A only intersects C_j. We contract $C_1 \cup \cdots \cup C_{j-1}$ and $C_{j+1} \cup \cdots \cup C_l$ and get an orbifold $\overline{Y}$, such that there are contractions $\widetilde{Y} \to \overline{Y} \to Y$. The map $\varpi\colon \overline{Y} \to Y$ has an exceptional divisor $\overline{E}$ with two orbifold points p_1, p_2 of multiplicities d_1, d_2 respectively (it is $d_1 = 1$ if $j = 1$, and $d_2 = 1$ if $j = l$). The proper transform of D_i is $\overline{D}_i$ with $p_1 = \overline{D}_i \cap \overline{E}$, which is a nice intersection. The proper transform of A, denoted by A again, intersects $\overline{E}$ transversally at a smooth point.

We have the following intersection numbers (see [14]). Let $\frac{a_1}{d_1} = [b_{j-1}, \ldots, b_1]^{-1}$, $\frac{a_2}{d_2} = [b_{j+1}, \ldots, b_l]^{-1}$ be the continuous fractions associated to the singularities (with $a_j = 0$ if $d_j = 1$), and let $\frac{a'_1}{d_1} = [b_1, \ldots, b_{j-1}]^{-1}$, $\frac{a'_2}{d_2} = [b_l, \ldots, b_{j+1}]^{-1}$ be the dual ones. Then, writing $b = b_j$,

$$(\overline{D}_i \cdot \overline{E})_{p_1} = \frac{1}{d_1}, \quad \overline{E}^2 = -b + \frac{a_1}{d_1} + \frac{a_2}{d_2}, \quad (\overline{D}_i^2)_{p_1} = \frac{a'_1}{d_1}.$$

Using the adjunction equality for a nice curve,

$$K_{\overline{Y}} \cdot C + C^2 = -e_{\mathrm{orb}}(C) = 2g(C) - 2 + \sum_{p \in C} \Big(1 - \frac{1}{d_p}\Big),$$

the corresponding local contributions for $C = \overline{E}$ and $\overline{D}_i$ give

$$K_{\overline{Y}} \cdot \overline{E} = b - \frac{a_1 + 1}{d_1} - \frac{a_2 + 1}{d_2}, \quad (K_{\overline{Y}} \cdot \overline{D}_i)_p = 1 - \frac{a'_1 + 1}{d_1}.$$

Recall that we aim to compute

$$\ell_p = ((K + D_i + A) \cdot A)_p = ((\bar{K} + \bar{D}_i + A) \cdot \varpi^* A)_p,$$

where the right-hand side accounts for the contribution to the intersection along the exceptional divisor. We write $\varpi^* A = A + x\bar{E}$, and compute $x \in \mathbb{Q}$ knowing that $\varpi^* A \cdot \bar{E} = 0$ and $A \cdot \bar{E} = 1$. Then

$$x = -\frac{1}{\bar{E}^2} = \frac{d_1 d_2}{b d_1 d_2 - a_1 d_2 - a_2 d_1},$$

and hence

$$\begin{aligned} \ell_p &= ((K + D_i + A) \cdot A)_p = ((\bar{K} + \bar{D}_i + A) \cdot \varpi^* A)_p \\ &= ((\bar{K} + \bar{D}_i + A) \cdot (A + x\bar{E}))_p \\ &= \left(b - \frac{a_1+1}{d_1} - \frac{a_2+1}{d_2} + \frac{1}{d_1} + 1\right) \frac{d_1 d_2}{b d_1 d_2 - a_1 d_2 - a_2 d_1} \\ &= 1 + \frac{d_1(d_2 - 1)}{b d_1 d_2 - a_1 d_2 - a_2 d_1} \geq 1, \end{aligned}$$

as required. ∎

By Theorem 4.4, the orbifold Euler–Poincaré characteristic is

$$e_{\mathrm{orb}}(Y - (D_1 \cup D_2 \cup D_3)) = 5 - \sum (2 - 2g_i) - \sum_{\substack{p \in Y \\ p \notin D_1 \cup D_2 \cup D_2}} \left(1 - \frac{1}{d_p}\right) \geq 0,$$

where g_1, g_2, g_3 are the genus of D_1, D_2, D_3, respectively. As $d_p \geq 2$, we deduce

$$\#\{p \in P, p \notin D_1 \cup D_2 \cup D_2\} \leq 2(2g_1 + 2g_2 + 2g_3 - 1). \tag{4.4}$$

Let us have three collections of curves (D_1, D_2, D_3), (D_1', D_2', D_3'), (D_1'', D_2'', D_3'') in the same situation, and suppose that all curves are distinct. Let $A = \{p \in P, p \in D_1 \cup D_2 \cup D_3\}$, $A' = \{p \in P, p \in D_1' \cup D_2' \cup D_3'\}$, $A'' = \{p \in P, p \in D_1'' \cup D_2'' \cup D_3''\}$. If $p \in Y$, then it can be at most in two curves (since they intersect nicely). Therefore, either $p \notin A$, $p \notin A'$ or $p \notin A''$. So $P \subset (P - A) \cup (P - A') \cup (P - A'')$. By equality (4.4) above,

$$\begin{aligned} \#P &\leq 2(2g_1 + 2g_2 + 2g_3 - 1) + 2(2g_1' + 2g_2' + 2g_3' - 1) \\ &\quad + 2(2g_1'' + 2g_2'' + 2g_3'' - 1), \end{aligned} \tag{4.5}$$

where g_i, g_i', g_i'' denote the genus of the respective curves.

Corollary 4.5. *Suppose that we have five bases with genera* $\{g_1, g_2, 10\}$, $\{g_1', g_2', 10\}$, $\{g_1'', g_2'', 10\}$, $\{g_1''', g_2''', 10\}$, $\{g_1'''', g_2'''', 10\}$, *and all* g_1, g_2, g_1', g_2', g_1'', g_2'', g_1''', g_2''', g_1'''', g_2'''' *and* 10 *are distinct numbers. Then there is some* τ_0 *(independent of* Y*) such that* $\#P \leq \tau_0$.

Proof. For checking this, we use Definition 5.1 from upcoming Section 5 (the results that we use for this proof are independent of Section 5). If among the five curves of genus 10, say D_3, D_3', D_3'', D_3''', D_3'''', there are only two distinct curves, then three of them coincide. Suppose that $D_3 = D_3' = D_3''$. Then Lemma 5.3 (below) implies that two of the bases (say $\mathcal{E}$, $\mathcal{E}'$) are proj-equivalent, and hence $D_2' = \lambda_2 D_2$ with $\lambda_2 > 0$. As $D_2 \neq D_2'$ because they have different genus, we get $D_2 \cdot D_2' \geq 0$. But then $\lambda_2 D_2^2 \geq 0$, which is a contradiction, as $D_2^2 < 0$.

Therefore, there are three of the bases with all curves distinct, and we take τ_0 as the right-hand side of formula (4.5). ■

The assumption of Corollary 4.5 is achieved as soon as we take $N \geq 11$ for (4.1).

5. Many collections of orthogonal bases of curves

Let Y be a Kähler cyclic orbifold with $b_1 = 0$ and $b_2 = 3$. Let P be the collection of singular points. Suppose that the ramification locus consists of a collection of nice curves $D_i^{(k)}$, $i = 1, 2, 3$, $1 \leq k \leq K$, such that

$$\mathcal{E}^{(k)} = (D_1^{(k)}, D_2^{(k)}, D_3^{(k)})$$

are orthogonal bases for $H_2(Y, \mathbb{Q})$, formed by curves which are disjoint. As Y is a Kähler orbifold, $h^{1,1}(Y) = b_2(Y) = 3$, because the homology is spanned by complex curves. The intersection form of $H^2(Y, \mathbb{R})$ is of signature $(1, 2)$. So we can order the curves so that $(D_1^{(k)})^2 = m_1^{(k)} > 0$, $(D_2^{(k)})^2 = -m_2^{(k)} < 0$, $(D_3^{(k)})^2 = -m_3^{(k)} < 0$. The genera are $g_1^{(k)} = g(D_1^{(k)})$, $g_2^{(k)} = g(D_2^{(k)})$, $g_3^{(k)} = g(D_3^{(k)}) \geq 1$.

For $k \neq l$, it may happen that $D_i^{(k)} = D_j^{(l)}$ in which case $g_i^{(k)} = g_j^{(l)}$, and also either $i = j = 1$ or $i, j \in \{2, 3\}$ (since the self-intersection coincides). On the other hand, if the curves are distinct, then it must be $D_i^{(k)} \cdot D_j^{(l)} \geq 0$.

Definition 5.1. Let $\mathcal{E} = (D_1, D_2, D_3)$, $\mathcal{E}' = (D_1', D_2', D_3')$ be two bases from the above list. We write $[\mathcal{E}] = [\mathcal{E}']$ if the elements are proportional, that is, up to reordering, $D_i' = \lambda_i D_i$ with $\lambda_i > 0$. We say that the bases are *proj-equivalent*.

Note that if $[\mathcal{E}] = [\mathcal{E}']$, then, by the discussion above, we have that $D_2 = D_2'$ and $D_3 = D_3'$.

Let K be the orbifold canonical class of Y. Let $\mathcal{E} = (D_1, D_2, D_3)$ be one of the basis provided above. Then we write $K = \sum a_i D_i$. We have the orbifold adjunction equality

$$K \cdot D + D^2 = -e_{\mathrm{orb}}(D)$$

for a smooth orbifold (nice) curve $D \subset Y$. As $b_2^+ = 1$, we have that $D_1^2 = m_1 > 0$, $D_i^2 = -m_i < 0$ for $i = 2, 3$, where $m_i \in \mathbb{Q}$. Let g_i be the genus of D_i. Let

$$\chi_i = -e_{\mathrm{orb}}(D_i) = 2g_i - 2 + \sum_{p \in D_i} \Big(1 - \frac{1}{d_p}\Big),$$

where d_p is the order of the singular point $p \in D_i$. Then $\chi_i \geq 2g_i - 2$. Using the adjunction formula, then $a_1 = \frac{\chi_1 - m_1}{m_1}$, $a_i = -\frac{\chi_i + m_i}{m_i}$ for $i = 2, 3$, so

$$K = \frac{\chi_1 - m_1}{m_1} D_1 - \frac{\chi_2 + m_2}{m_2} D_2 - \frac{\chi_3 + m_3}{m_3} D_3. \tag{5.1}$$

Note that $\chi_i + m_i > 0$ for $i = 2, 3$.

By Lemma 4.3, $K + D_2$ is effective. But

$$K + D_2 = \frac{\chi_1 - m_1}{m_1} D_1 - \frac{\chi_2}{m_2} D_2 - \frac{\chi_3 + m_3}{m_3} D_3.$$

If $m_1 \geq \chi_1$, then this is anti-effective, which is a contradiction. Hence we always have

$$0 < m_1 < \chi_1. \tag{5.2}$$

Lemma 5.2. *Let* $\mathcal{E} = (D_1, D_2, D_3)$, $\mathcal{E}' = (D'_1, D'_2, D'_3)$ *be two bases. If* $D'_1 = \lambda_1 D_1$, *then* $[\mathcal{E}] = [\mathcal{E}']$. *In particular,* $D_2 = D'_2$ *and* $D_3 = D'_3$.

Proof. We restrict to $V = \langle D_1 \rangle^\perp \subset H_2(Y, \mathbb{R})$, which is a vector space with a (negative) definite scalar product. If $D_2 = \lambda_2 D'_2$ (up to reordering), then it must be $D_3 = \lambda_3 D'_3$ and $[\mathcal{E}] = [\mathcal{E}']$. If D_2, D_3 are not proportional to D'_2, D'_3, then $D_i \cdot D'_j \geq 0$ for $i, j \in \{2, 3\}$. If we take coordinates on V so that $\{D_2, D_3\}$ is the standard basis, then $D'_l = \sum -a_{jl} D_j$ with $a_{jl} \geq 0$. This is impossible since the first is effective and the second anti-effective. ∎

Lemma 5.3. *Let* $\mathcal{E} = (D_1, D_2, D_3)$, $\mathcal{E}' = (D'_1, D'_2, D'_3)$, $\mathcal{E}'' = (D''_1, D''_2, D''_3)$ *be three bases. If* D_3, D'_3, D''_3 *are proportional, then two of the bases are proj-equivalent.*

Proof. Let $W = \langle D_1, D_2 \rangle$, which is a vector space of dimension 2 and signature $(1, 1)$. Take an orthonormal basis $\{e_1, e_2\}$ with $e_1 = \frac{D_1}{\sqrt{m_1}}$, $e_2 = \frac{D_2}{\sqrt{m_2}}$. If either $D'_1 = D_1$ or $D'_2 = D_2$, then $[\mathcal{E}'] = [\mathcal{E}]$. Otherwise $D'_1 \cdot D_1$, $D'_1 \cdot D_2$, $D'_2 \cdot D_1$, $D'_2 \cdot D_2 \geq 0$. In the above basis, $D'_1 = (a_1, -b_1)$, $D'_2 = (a_2, -b_2)$, with $a_j, b_j \geq 0$. As they are orthogonal, $a_1 a_2 - b_1 b_2 = 0$, hence $D'_2 = \mu(b_1, -a_1)$ with $\mu > 0$. In an analogous manner, $D''_1 = (c_1, -d_1)$, $D''_2 = \mu'(d_1, -c_1)$ with $c_1, d_1 \geq 0$, $\mu' > 0$. Then $D'_1 \cdot D''_1 = a_1 c_1 - b_1 d_1 \geq 0$ and $D'_2 \cdot D''_2 = \mu\mu'(b_1 d_1 - a_1 c_1) \geq 0$. So it must be $D''_1 \perp D'_1$, and hence D''_1 is proportional to D'_2. ∎

Definition 5.4. We call a curve D_i *good* if it does not pass through any singular point. We call a basis $\mathcal{E} = (D_1, D_2, D_3)$ good if the three curves D_1, D_2, D_3 are good. In this case, $m_1 = D_1^2$, $m_2 = -D_2^2$ and $m_3 = -D_3^2$ are positive integers. Also $\chi_i = 2g_i - 2 \in \mathbb{Z}$, and their homology classes lie in $H_2(X - P, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$.

Fix some $N_0 > 0$ to be determined later. Now we focus on bases of curves with genera $\{g_n = 9n^2 + 1, g_a = 9a^2 + 1, 10\}$ for $1 < a \leq N_0$, and $N_0 < n \leq N$. For each $a \in [2, N_0]$, we take a prime number $n_a \in [N_0 + 1, N]$. In particular, $10 < g_a < g_{n_a}$. We require the numbers $n_a \neq n_{a'}$ for $a \neq a'$. We say that a number a is *bad* if the basis (D_1, D_2, D_3) of genera $\{g_{n_a}, g_a, 10\}$ is not good.

Proposition 5.5. *There are at most* $4\tau_0$ *bad numbers* a*, where* τ_0 *is given in Corollary 4.5.*

Proof. By Corollary 4.5, the number of orbifold points is $\#P \leq \tau_0$. At an orbifold point, there are at most two (nice) curves through it. Let $\mathcal{E}_{n_a a}$ be the basis associated to the genera $\{g_{n_a}, g_a, 10\}$. For any bad a, $\mathcal{E}_{n_a a}$ contains a curve through a point of P. Let us see that a point $p \in P$ can be at most in four bases $\mathcal{E}_{n_a a}$. Therefore, the number of bad numbers a is $\leq 4\tau_0$.

To check the assertion, fix a point $p \in P$ and suppose that a_1, a_2, a_3, a_4, a_5 are bad with curves through p. Let $\mathcal{E}_{n_{a_i} a_i} = (D_1^{(i)}, D_2^{(i)}, D_3^{(i)})$, $1 \leq i \leq 5$. Note that $g_{a_i} \neq g_{a_j}$ and $g_{n_{a_i}} \neq g_{n_{a_j}}$ for $i \neq j$ since $n_{a_i} \neq n_{a_j}$, and $g_{a_i} \neq g_{n_{a_k}}$ for i, k since $a_i \leq N_0 < n_{a_k}$. By Lemma 5.3, there must be three different curves among $D_3^{(i)}$. Reordering, we can suppose this for $i = 1, 2, 3$. Then all curves in $\mathcal{E}_{n_{a_i} a_i}$, $i = 1, 2, 3$ are different. But it cannot be more than two curves through p, a contradiction. ∎

Taking $N_0 = 4\tau_0 + 1$ guarantees the existence of some a which is not bad. Let now

$$N_1 = \max(n_a \mid a \in [2, N_0]). \tag{5.3}$$

This is a universal quantity, i.e., independent of Y.

6. Universal geometric bounds

Now we want to get universal bounds on some geometric quantities associated to an orbifold Y satisfying Conditions 4.1. As before, let $\pi\colon \widetilde{Y} \to Y$ be the minimal resolution of singularities, and let $\widetilde{K}$ and K be the canonical divisors of $\widetilde{Y}$ and Y, respectively. In this section, we use N_1 from (5.3).

Lemma 6.1. *There is a universal* τ_1 *such that* $\widetilde{K}^2 \leq K^2 \leq \tau_1$.

Proof. The first equality follows from (4.2). Next, by Proposition 5.5, there is some $a \leq N_0$ which is not bad. This means that the curves in the basis $\mathcal{E}_{n_a a} = (D_1, D_2, D_3)$, with genera $\{g_{n_a}, g_a, 10\}$, do not pass through singular points. As $n_a \leq N_1$, we have $g_{n_a} = 9n_a^2 + 1 \leq 9N_1^2 + 1$. By using (5.1), we have

$$\begin{aligned} K^2 &= \frac{(2g_{n_a} - 2 - m_1)^2}{m_1} - \frac{(2g_a - 2 + m_2)^2}{m_2} - \frac{(18 + m_3)^2}{m_3} \\ &\leq \frac{(2g_{n_a} - 2 - m_1)^2}{m_1}, \end{aligned}$$

where we have assumed that the curve D_1 is the positive one (the other cases can be done similarly). As $0 < m_1 < 2g_{n_a} - 2$ by (5.2), we can compute the maximum value of the expression above to be $(2g_{n_a} - 3)^2 \leq (18N_1^2 - 1)^2$. So K^2 is universally bounded. ∎

Lemma 6.2. *There is a universal* τ_2 *such that* $K^2 \geq \widetilde{K}^2 \geq -\tau_2$.

Proof. We already know that $K^2 \geq \widetilde{K}^2$. Now we take $a \leq N_0$ which is not bad, and let $\{g_{n_a}, g_a, 10\}$ be the genera of a good basis of curves (D_1, D_2, D_3). As they do not pass through singular points, we denote the proper transforms under the resolution map $\pi \colon \widetilde{Y} \to Y$ by the same letters D_1, D_2, D_3. Recall that we denote by C_j the exceptional divisors.

To bound $\widetilde{K}^2$, we note that $H^0(\widetilde{K} + D_1) \cong H^0(D_1, K_{D_1}) = \mathbb{C}^{g_{n_a}}$. We assume that D_1 is the positive curve, the other cases are similar. Write the linear system $|\widetilde{K} + D_1| = Z + |F|$, where Z is the base-point locus and F is a free divisor. Then $Z \cdot D_1 = 0$ since $H^0(D_1, K_{D_1})$ is base-point free. Write the divisor $Z = T + \sum a_i D_i + \sum b_j C_j$ for $a_i \geq 0$, $b_j \geq 0$, and $T \geq 0$ not containing D_i and C_j. As $Z \cdot D_1 = 0$, we have $a_1 = 0$ and $T \cdot D_1 = 0$. In the rational equivalence class, we have $T \equiv \sum \alpha_i D_i + \sum \beta_j C_j$. Again, $\alpha_1 = 0$ and $\alpha_i \leq 0$ because $T \cdot D_i \geq 0$, $i = 2, 3$. Also $T \cdot C_j \geq 0$ for all j, implies that $\beta_j \leq 0$ for all j. This implies that T is anti-effective and effective, hence $T = 0$. Thus $Z = \sum a_i D_i + \sum b_j C_j$, hence $\widetilde{K} \cdot Z \geq 0$ because $\widetilde{K} \cdot D_i \geq 0$ and $\widetilde{K} \cdot C_j \geq 0$. Next

$$(\widetilde{K} + D_1 - Z)^2 = F^2 \geq 0.$$

So $(\widetilde{K} + D_1)^2 - 2(\widetilde{K} + D_1) \cdot Z + Z^2 \geq 0$, and hence $(\widetilde{K} + D_1)^2 \geq 0$. This reads $\widetilde{K}^2 + 2(2g_{n_a} - 2 - m_1) + m_1 \geq 0$, whence $\widetilde{K}^2 \geq 4 - 4g_{n_a} + m_1 \geq 5 - 4g_{n_a} \geq 1 - 36N_1^2$, using that $g_{n_a} = 9n_a^2 + 1 \leq 9N_1^2 + 1$. ■

Lemma 6.3. *There is a universal τ_3 such that $e(\widetilde{Y}) \leq \tau_3$.*

Proof. As $h^{1,1} = b_2$, we have that the geometric genus is $p_g = h^{2,0} = 0$. Also $b_1 = 0$ implies that the irregularity is $q = 0$. Therefore, the holomorphic Euler characteristic is $\chi(\mathcal{O}_{\widetilde{Y}}) = 1 - q + p_g = 1$. By the Noether formula, $\widetilde{K}^2 + e(\widetilde{Y}) = 12\chi(\mathcal{O}_{\widetilde{Y}}) = 12$, hence $e(\widetilde{Y}) = 12 - \widetilde{K}^2 \leq 12 + \tau_2 = \tau_3$. ■

Proposition 6.4. *There is a universal τ_4 such that if C is a nice curve with $C^2 = -m < 0$ and genus $g = g(C) \geq 1$, then $m \leq 2g + \tau_4$.*

Proof. We apply [20, Theorem 10.14] to the smooth variety $\widetilde{Y}$. First we check that $\widetilde{K} + \widetilde{C}$ is effective, which follows as in Lemma 4.3. If we have that $\widetilde{K} + \widetilde{C}$ is nef, then [20, Theorem 10.14] says that

$$(\widetilde{K} + \widetilde{C})^2 \leq 3e(\widetilde{Y} - \widetilde{C}) = 3e(\widetilde{Y}) + 6g - 6. \tag{6.1}$$

To check that $\widetilde{K} + \widetilde{C}$ is nef, let A be an effective curve. If $A = \widetilde{C}$, then $(\widetilde{K} + \widetilde{C}) \cdot \widetilde{C} = 2g - 2 \geq 0$. So suppose $A \neq \widetilde{C}$. If $\widetilde{K} \cdot A \geq 0$, then $(\widetilde{K} + \widetilde{C}) \cdot A \geq 0$. Also if $A^2 \geq 0$, then write for an effective $\Sigma \equiv \widetilde{K} + \widetilde{C}$, $\Sigma = rA + T$, $r \geq 0$, T not containing A, and thus $(\widetilde{K} + \widetilde{C}) \cdot A = rA^2 + T \cdot A \geq 0$.

So we are left with $\widetilde{K} \cdot A < 0$ and $A^2 < 0$. Then A is a (-1)-curve. If $A \cdot \widetilde{C} \geq 1$, then we are again done. So also $A \cap \widetilde{C} = \emptyset$. Blow down A and let $\widetilde{Y} \to \overline{Y}$ be the blow-down map. We can assume inductively that in $\overline{Y}$ we have $(\overline{K} + \widetilde{C})^2 \leq 3e(\overline{Y}) + 6g - 6$. So $(\widetilde{K} + \widetilde{C})^2 - 1 \leq 3e(\widetilde{Y}) - 3 + 6g - 6$, and (6.1) follows.

Now from (6.1), $\widetilde{K}^2 + 2\widetilde{K} \cdot \widetilde{C} + \widetilde{C}^2 \leq 3e(\widetilde{Y}) + 6g - 6$, which reads $12 - e(\widetilde{Y}) + 4g - 4 - \widetilde{C}^2 \leq 3e(\widetilde{Y}) + 6g - 6$. Therefore,

$$-\widetilde{C}^2 \leq 4e(\widetilde{Y}) + 2g - 14 \leq 4\tau_3 + 2g - 14 = 2g + \tau_4$$

with $\tau_4 = 4\tau_3 - 14$. Finally, Lemma 4.2 says that for $C = \pi(\widetilde{C})$, then $C^2 \geq \widetilde{C}^2$, so $-C^2 \leq -\widetilde{C}^2 \leq 2g + \tau_4$. ∎

7. Proof of the non-Sasakian property

Our final purpose is to complete the main result (Theorem 1.1).

Theorem 7.1. *There is some N large enough such that the K-contact manifold M from Theorem 3.4 does not admit a Sasakian structure.*

If M admits a Sasakian structure, then it also admits a quasi-regular Sasakian structure. Therefore, there is a Seifert bundle $\pi: M \to Y$, where Y is a Kähler cyclic orbifold. From the homology of M given by Theorem 3.4, we have that $b_1(Y) = 0$, $b_2(Y) = 3$ and the ramification locus is given by a collection of curves

$$\mathcal{E}^{nm} = (D_1^{nm}, D_2^{nm}, D_3^{nm}),$$

which satisfy that $D_1^{nm}, D_2^{nm}, D_3^{nm}$ are disjoint and span $H_2(Y, \mathbb{Q})$ for each n, m. They can coincide or intersect for different values of (n, m). The genera of $D_1^{nm}, D_2^{nm}, D_3^{nm}$ are $\{9n^2 + 1, 9m^2 + 1, 10\}$.

We start with the collection of bases $\mathcal{E}_n = (D_1^n, D_2^n, D_3^n)$ associated to $m = 2$, $n \in [3, N]$. The genera of the curves are $\{g_n = 9n^2 + 1, 37, 10\}$ with $g_n > 37$.

Recall the bound $\#P \leq \tau_0$ from Corollary 4.5. Then there are at most $2\tau_0$ curves among D_1^n, D_2^n, D_3^n passing through points of P. All the curves D_1^n are distinct, but there can be repetitions among D_2^n, D_3^n. By (5.2), if D_1^n is the positive curve, then we have $m_1 < \chi_1 = 2g_n - 2 = 18n^2$.

Proposition 7.2. *There are some (universal) $n_0 > 0$ and positive integers R and $N > n_0$ such that there exist two prime numbers $n, n' \in [n_0 + 1, N]$ with*

$$R\Big(\frac{n^4}{m_1} - \frac{n'^4}{m_1'}\Big) \in \mathbb{Z},$$

where $m_1 = (D_1^n)^2$, $m_1' = (D_1^{n'})^2 \in \mathbb{Z}$, and $0 < m_1 < 18n^2$, $0 < m_1' < 18n'^2$.

We can select n, n' from a previously given infinite collection of primes $\mathcal{P} \subset \mathbb{Z}_{>0}$. Only N depends on $\mathcal{P}$, otherwise it is universal.

Proof. Divide the set $[3, N] \cap \mathcal{P}$ into classes $\mathcal{A}_1, \dots, \mathcal{A}_l$ according to proj-equivalence of the basis $\mathcal{E}_n$, that is, $[\mathcal{E}_n] = [\mathcal{E}_m]$ if and only if $n, m \in \mathcal{A}_i$ for some i. It may happen that $D_2^n = D_2^m$ if $n \in \mathcal{A}_i$, $m \in \mathcal{A}_j$, $i \neq j$, but it cannot be for three different classes, by Lemma 5.3. If this happens for $\mathcal{A}_i$, $\mathcal{A}_j$, we retain $\mathcal{A}_i$ and discard $\mathcal{A}_j$, so that

$\#\mathcal{A}_i \geq \#\mathcal{A}_j$. Let $\mathcal{A}_{i_1}, \ldots, \mathcal{A}_{i_t}$ be the retained classes, and note that $\#(\bigcup \mathcal{A}_{i_k}) \geq \frac{\tau}{2}$, where $\tau = \#([3, N] \cap \mathcal{P})$. Repeat the same process with the curves D_3^n. The remaining classes $\mathcal{A}_{j_1}, \ldots, \mathcal{A}_{j_s}$ have cardinality $\#(\bigcup \mathcal{A}_{i_k}) \geq \frac{\tau}{4}$. Then two bases in $\bigcup \mathcal{A}_{i_k}$ are either proj-equivalent, or their curves are all distinct.

If a class $\mathcal{A}_{j_k}$ contains two primes $n, n' > n_0$ (n_0 will be chosen later), then let $\mathcal{E}_n = (D_1, D_2, D_3)$, $\mathcal{E}_{n'} = (D_1', D_2', D_3')$ be the proj-equivalent bases. As $D_2 = D_2'$, $D_3 = D_3'$, then $D_1 \cdot D_1' \geq 0$, so D_1, D_1' are the positive curves. We compute

$$\begin{aligned} K^2 &= \frac{(\chi_1 - m_1)^2}{m_1} - \frac{(\chi_2 + m_2)^2}{m_2} - \frac{(\chi_3 + m_3)^2}{m_3} \\ &= \frac{(\chi_1' - m_1')^2}{m_1'} - \frac{(\chi_2 + m_2)^2}{m_2} - \frac{(\chi_3 + m_3)^2}{m_3}, \end{aligned}$$

with the usual meaning for χ_j, m_j and χ_j', m_j'. Then

$$\frac{(\chi_1 - m_1)^2}{m_1} - \frac{(\chi_1' - m_1')^2}{m_1'} = 0.$$

Now suppose that one $\mathcal{A}_{j_k}$ contains $2\tau_0 + 2$ primes $n > n_0$. At most $2\tau_0$ of the curves D_1^n are not good. So there are two primes $n, n' > n_0$ associated to good curves, and hence using that $2g_n - 2 = 18n^2$, we have

$$\frac{(18n^2 - m_1)^2}{m_1} - \frac{(18n'^2 - m_1')^2}{m_1'} = 0$$

with m_1, m_1' integers. This gives the result (actually with $R = 18^2$).

Now suppose that all classes $\mathcal{A}_{j_k}$ contain at most $2\tau_0 + 1$ primes $n > n_0$. Take $N > 0$ so that in $[n_0 + 1, N] \cap (\bigcup \mathcal{A}_{i_k})$ there are more than $(2\tau_0 + 1)2\tau_0 + 2$ primes. This can be arranged if

$$\frac{\tau}{4} - (n_0 - 2) \geq (2\tau_0 + 1)2\tau_0 + 2. \tag{7.1}$$

Choose N large enough so that τ is large enough for (7.1) to hold. Now remove all classes $\mathcal{A}_{j_k}$ that contain a curve which is not good. There are at most $2\tau_0$ of them. Therefore, there must be two primes n, n' still left after this. In that case, the bases (D_1, D_2, D_3), (D_1', D_2', D_3') are both good, and in different classes.

Let us see first that D_1, D_1' are the positive curves. Suppose, for instance, that D_2 is the positive curve. Then

$$K^2 = \frac{(72 + m_2)^2}{m_2} - \frac{(2g_n - 2 + m_1)^2}{m_1} - \frac{(18 + m_3)^2}{m_3}.$$

The first and last terms are bounded by (5.2) and Proposition 6.4. So

$$K^2 \leq \tau_5 - \frac{(2g_n - 2 + m_1)^2}{m_1}$$

for some universal τ_5. This implies the bound $K^2 \leq \tau_5 - (8g_n - 8)$. By Lemma 6.2, $-\tau_2 \leq \tau_5 - 8g_n + 8$ and so $g_n = 9n^2 + 1 \leq 1 + \frac{1}{8}(\tau_5 + \tau_2)$. This means that there is

$n_0 = [\frac{1}{72}(\tau_5 + \tau_2)] + 1$ such that for $n \geq n_0$, $D_1 = D_1^n$ is the positive curve. This n_0 is universal.

Now take $n, n' \geq n_0 + 1$. Then D_1, D_1' are positive curves, we have

$$K^2 = \frac{(2g_n - 2 - m_1)^2}{m_1} - \frac{(72 + m_2)^2}{m_2} - \frac{(18 + m_3)^2}{m_3}$$
$$= \frac{(2g_{n'} - 2 - m_1')^2}{m_1'} - \frac{(72 + m_2')^2}{m_2'} - \frac{(18 + m_3')^2}{m_3'}.$$

Recalling that $2g_n - 2 = 18n^2$, we have

$$\frac{18^2 n^4}{m_1} - \frac{72^2}{m_2} - \frac{18^2}{m_3} - \frac{18^2 n'^4}{m_1'} + \frac{72^2}{m_2'} + \frac{18^2}{m_3'} \in \mathbb{Z},$$

where $0 < m_1 < 18n^2$, $0 < m_1' < 18n'^2$. By Proposition 6.4, $m_2 \leq \tau_4 + 74$ and $m_3 \leq \tau_4 + 20$. Then take $R = 18^2 \cdot \mathrm{lcm}(2, 3, 4, \ldots, \tau_4 + 74)$, and we get the statement.

The number N has to be chosen large enough so that τ satisfies inequality (7.1). It depends on $\mathcal{P}$ clearly. ■

Now take $n, n' > n_0$ prime numbers satisfying the condition in Proposition 7.2. Take $d = \gcd(m_1, m_1')$ and write $m_1 = da$, $m_1' = da'$ with $\gcd(a, a') = 1$. Then $\frac{n^4 R}{a} - \frac{n'^4 R}{a'}$ is an integer, from where $a \mid n^4 R$ and $a' \mid n'^4 R$. Given that $a < 18n^2$ and $a' < 18n'^2$, there is a finite set of possibilities for a, a'. Let $D = \{d_1, \ldots, d_t\}$ be the divisors of R. Then $a \in \{d_i, d_i n, d_i n^2\}$, and $a' \in \{d_i, d_i n', d_i n'^2\}$. Therefore,

$$\frac{m_1}{m_1'} = \frac{d_i}{d_j} \frac{n^\beta}{n'^\gamma} \tag{7.2}$$

with $\beta, \gamma = 0, 1, 2$, $d_i, d_j \in D$.

Next K^2 is bounded by Lemma 6.1, hence

$$\frac{(18n^2 - m_1)^2}{m_1} \leq \tau_6$$

for some universal τ_6, using also Proposition 6.4 to bound m_2, m_3. Then m_1 lies in the interval

$$m_1 \in \left[18n^2 + \frac{\tau_6}{2} - \sqrt{18n^2\tau_6 + \frac{\tau_6^2}{4}}, 18n^2\right).$$

In particular,

$$18n^2 - \sqrt{18\tau_6}n \leq m_1 < 18n^2, \tag{7.3}$$

and analogously for m_1'. Now

$$\frac{m_1}{m_1'} \in \left(\frac{18n^2 - \sqrt{18\tau_6}n}{18n'^2}, \frac{18n^2}{18n'^2 - \sqrt{18\tau_6}n'}\right). \tag{7.4}$$

Consider the set $\mathcal{R} = \{s = \frac{d_i}{d_j} \mid d_i \in D\}$. Let $\epsilon = \min(|1-s| \mid s \in \mathcal{R}, s \neq 1) > 0$. This is a universal number. Enlarging n_0, we have that for primes $n, n' \geq n_0 + 1$, quotient (7.4) is within ϵ of $\frac{n^2}{n'^2}$, i.e., in the interval

$$\Big((1-\epsilon)\frac{n^2}{n'^2}, (1+\epsilon)\frac{n^2}{n'^2}\Big). \tag{7.5}$$

This n_0 is again universal (depends on R and τ_6).

We choose our collection of primes $\mathcal{P} = \{n_1, n_2, \ldots\}$ in Proposition 7.2 in increasing order as follows. First choose $n_0 \geq R(1+\epsilon)$, so that $n_i > n_0 \geq R(1+\epsilon)$. Next take $n_{i+1} > (1-\epsilon)^{-1} R n_i^2$ for $i \geq 1$.

Now given $n = n_i$, $n' = n_j$, $i > j$, then all numbers (7.2) are away from (7.5). This is proved as follows: first all quotients $s = \frac{d_i}{d_j} \in [\frac{1}{R}, R]$. Next,

$$(1-\epsilon)\frac{n^2}{n'^2} \geq nR,$$

which is bigger than any of the expressions $s, sn, s\frac{1}{n'}, s\frac{n}{n'}, s\frac{1}{n'^2}, s\frac{n}{n'^2}$. Also $(1+\epsilon)\frac{n^2}{n'^2} \leq \frac{1}{R}\frac{n^2}{n'}$, which is smaller than any of the expressions $s\frac{n^2}{n'}, sn^2$. Hence it must be

$$\frac{m_1}{m_1'} = \frac{n^2}{n'^2}$$

since $s = \frac{d_i}{d_j} \notin (1-\epsilon, 1+\epsilon)$ unless $s = 1$. Therefore, $m_1 = d_i n^2$, $m_1' = d_i n'^2$, for some $d_i \in D$. By (7.3), this is impossible.

This contradiction shows that for such N in Proposition 7.2, Theorem 7.1 holds.

Remark 7.3. All the numbers $\tau_0, \tau_1, \ldots, \tau_6, n_0, R, N_0, N_1$ and N that have appeared along the proof can be determined. So N in Theorem 7.1 can be found explicitly.

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References

[1] Amorós, J., Burger, M., Corlette, K., Kotschick, D., Toledo, D.: Fundamental groups of compact Kähler manifolds. Math. Surveys Monogr. 44, American Mathematical Society, Providence, RI (1996) Zbl 0849.32006 MR 1379330

[2] Barden, D.: Simply connected five-manifolds. Ann. of Math. (2) **82**, 365–385 (1965) Zbl 0136.20602 MR 184241

[3] Barth, W., Peters, C., Van de Ven, A.: Compact complex surfaces. Ergeb. Math. Grenzgeb. (3) 4, Springer, Berlin (1984) Zbl 0718.14023 MR 749574

[4] Bazzoni, G., Fernández, M., Muñoz, V.: A 6-dimensional simply connected complex and symplectic manifold with no Kähler metric. J. Symplectic Geom. **16**, 1001–1020 (2018) Zbl 1414.53071 MR 3917726

[5] Beauville, A.: Les familles stables de courbes elliptiques sur $\mathbf{P}^1$ admettant quatre fibres singulières. C. R. Acad. Sci. Paris Sér. I Math. **294**, 657–660 (1982) Zbl 0504.14016 MR 664643

[6] Biswas, I., Fernández, M., Muñoz, V., Tralle, A.: On formality of Sasakian manifolds. J. Topol. **9**, 161–180 (2016) Zbl 1339.55014 MR 3465845

[7] Boyer, C. P., Galicki, K.: Sasakian geometry. Oxford Math. Monogr., Oxford University Press, Oxford (2008) Zbl 1155.53002 MR 2382957

[8] Cañas, A., Muñoz, V., Rojo, J., Viruel, A.: A K-contact simply connected 5-manifold with no semi-regular Sasakian structure. Publ. Mat. **65**, 615–651 (2021) Zbl 1479.53055 MR 4278760

[9] Cañas, A., Muñoz, V., Schütt, M., Tralle, A.: Quasi-regular Sasakian and K-contact structures on Smale–Barden manifolds. Rev. Mat. Iberoam. **38**, 1029–1050 (2022) Zbl 1497.53092 MR 4413764

[10] Cappelletti-Montano, B., De Nicola, A., Marrero, J. C., Yudin, I.: Examples of compact K-contact manifolds with no Sasakian metric. Int. J. Geom. Methods Mod. Phys. **11**, article no. 1460028 (2014) Zbl 1304.53043 MR 3270291

[11] Cappelletti-Montano, B., De Nicola, A., Marrero, J. C., Yudin, I.: A non-Sasakian Lefschetz K-contact manifold of Tievsky type. Proc. Amer. Math. Soc. **144**, 5341–5350 (2016) Zbl 1351.53059 MR 3556276

[12] Cappelletti-Montano, B., De Nicola, A., Yudin, I.: Hard Lefschetz theorem for Sasakian manifolds. J. Differential Geom. **101**, 47–66 (2015) Zbl 1322.58002 MR 3356069

[13] Chen, X.: On the fundamental groups of compact Sasakian manifolds. Math. Res. Lett. **20**, 27–39 (2013) Zbl 1285.53035 MR 3126719

[14] Cogolludo-Agustín, J. I., Martín-Morales, J., Ortigas-Galindo, J.: Local invariants on quotient singularities and a genus formula for weighted plane curves. Int. Math. Res. Not. IMRN **2014**, 3559–3581 (2014) Zbl 1304.14034 MR 3229762

[15] Deligne, P., Griffiths, P., Morgan, J., Sullivan, D.: Real homotopy theory of Kähler manifolds. Invent. Math. **29**, 245–274 (1975) Zbl 0312.55011 MR 382702

[16] Fernández, M., Muñoz, V.: An 8-dimensional nonformal, simply connected, symplectic manifold. Ann. of Math. (2) **167**, 1045–1054 (2008) Zbl 1173.57012 MR 2415392

[17] Fukae, M.: Monodromies of rational elliptic surfaces and extremal elliptic $K3$ surfaces. arXiv:math/0205062 (2002)

[18] Gompf, R. E.: A new construction of symplectic manifolds. Ann. of Math. (2) **142**, 527–595 (1995) Zbl 0849.53027 MR 1356781

[19] Hajduk, B., Tralle, A.: On simply connected K-contact non-Sasakian manifolds. J. Fixed Point Theory Appl. **16**, 229–241 (2014) Zbl 1320.53097 MR 3346752

[20] Kollár, J.: Flips and abundance for algebraic threefolds. A summer seminar at the University of Utah (Salt Lake City, 1991). Astérisque **211**, 1–258 (1992) Zbl 0782.00075

[21] Kollár, J.: Circle actions on simply connected 5-manifolds. Topology **45**, 643–671 (2006) Zbl 1094.57020 MR 2218760

[22] Lin, Y.: Lefschetz contact manifolds and odd dimensional symplectic geometry. arXiv:1311.1431 (2016)

[23] Muñoz, V.: Gompf connected sum for orbifolds and K-contact Smale–Barden manifolds. Forum Math. **34**, 197–223 (2022) Zbl 1492.57018 MR 4360012

[24] Muñoz, V., Rojo, J. A.: Symplectic resolution of orbifolds with homogeneous isotropy. Geom. Dedicata **204**, 339–363 (2020) Zbl 1510.53090 MR 4056707

[25] Muñoz, V., Rojo, J. A., Tralle, A.: Homology Smale–Barden manifolds with K-contact but not Sasakian structures. Int. Math. Res. Not. IMRN **2020**, 7397–7432 (2020) Zbl 1461.57011 MR 4176828

[26] Muñoz, V., Tralle, A.: Simply connected K-contact and Sasakian manifolds of dimension 7. Math. Z. **281**, 457–470 (2015) Zbl 1432.53066 MR 3384880

[27] Reid, M.: Surface cyclic quotient singularities and Hirzebruch–Jung resolutions. http://homepages.warwick.ac.uk/~masda/surf/more/cyclic.pdf, visited on 15 July 2024 (2012)

[28] Rukimbira, P.: Chern–Hamilton's conjecture and K-contactness. Houston J. Math. **21**, 709–718 (1995) Zbl 0865.53023 MR 1368340

[29] Smale, S.: On the structure of 5-manifolds. Ann. of Math. (2) **75**, 38–46 (1962) Zbl 0101.16103 MR 141133

[30] Tralle, A., Oprea, J.: Symplectic manifolds with no Kähler structure. Lecture Notes in Math. 1661, Springer, Berlin (1997) Zbl 0891.53001 MR 1465676