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New Challenges in the Interplay between Finance and Insurance

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Abstract. The aim of this workshop was to convene experts for fostering the discussion and the development of innovative approaches in insurance and financial mathematics. New challenges like price instability, huge insurance claims and climate change are affecting the markets, while at the same time the possibility of using large volumes of data and continuously increasing computer power as well as recently developed mathematical methods offer new opportunities for modelling and risk assessment. Here we present an overview of these recent developments by providing the abstracts of the talks that were given during the week, together with a brief summary of the covered topics.

Mathematics Subject Classification (2020): 60, 62, 90, 91, 93.

Introduction by the Organizers

The last years have been a challenging period for financial and insurance markets. While stock markets experienced unexpected large price jumps, insurance and reinsurance companies suffered huge claims, but at the same time had the opportunity to use large volumes of data for their modelling, and the continuously increasing level of computer power gives rise to new approaches to make use of them. The impact of climate change poses a further challenge to both fields, and the consideration of sustainable investment policies and strategies becomes increasingly important.

This workshop brought together leading experts in all these fields to foster the discussion and the development of new and innovative approaches. In the following, we will provide the abstracts of the talks that were given during the week, and start with a brief summary of the covered topics.

Ralf Korn started with formulating stochastic control problems in the context of sustainable finance, and Peter Tankov gave an account of mean-field approaches for the decarbonization of financial markets. Emanuela Rosazza Gianin and Silvana Pesenti presented new results on consistency and robustness of dynamic risk measures, and on the application side for insurers. Filip Lindskog presented multi-period approaches for the valuation of liabilities, and Michael Schmutz gave an update of the current view on risk measures from the regulatory perspective of Switzerland. Concerning challenges in life insurance, Peter Hieber talked about an approach to give policyholders more control in participating life contracts, Griselda Deelstra showed some new insights when combining financial and mortality risks, and Damir Filipovic presented a new flexible non-parametric data-driven approach to model long-term interest rates, which is an important challenge for life insurers facing long-tailed risks. Stéphane Loisel gave an account on how classical actuarial techniques may be used for the analysis of insurance risks prone to climate change, which was nicely complemented with a presentation of Valérie Chavez-Demoulin on techniques in the statistics of extremes when dealing with non-stationary situations like the one due to climate change. Johanna Ziegel and Pierre-Olivier Goffard then presented some recent advances on certain aspects of statistical methodology. There were several interesting contributions on model uncertainty in the context of optimal investment, with talks by Frank Riedel, Nicole Bäuerle, Mogens Steffensen and Katharina Oberpriller. Multivariate portfolio choice via quantiles was discussed by Carole Bernard. Christa Cuchiero showed how to use polynomial processes to model the capital distribution curves of financial markets, and, also along the lines of stochastic portfolio theory in the spirit of R. Fernholz, Josef Teichmann talked about ergodic robust maximization of asymptotic growth with stochastic factor processes. Extending classical mathematical finance concepts in other directions, Thilo Meyer-Brandis introduced cooperation in arbitrage theory, Irene Klein dealt with large financial markets and Cosimo Munari considered the case of frictions. Finally, Eckhard Platen gave an update of his alternative benchmark approach to financial modelling. On a conceptual side, Berenice Anne Neumann talked about Markovian randomized equilibria in general Dynkin games, Gudmund Pammer presented new results on stretched Brownian motion, Brandon Garcia Flores presented a new approach to use techniques from optimal transport for the identification of optimal reinsurance treaties, and Sigrid Källblad showed how to use optimal transport for adapted distance between the laws of SDEs. Furthermore, Monique Jeanblanc shaded new light on shrinked semimartingales, Anna Aksamit studied multi-action options under information delay, while Claudia Ceci and Alessandra Cretarola presented results on reinsurance using backward SDEs an dynamic contagion models. David Criens presented results on controlled mean field SPDEs, and Caroline Hillairet gave an account of recent advances in the study of Hawkes processes, which are relevant for instance in the insurance of cyber risk.

The week was very stimulating, with many scientific and social interactions of participants and seeds of new ideas and approaches, many of which will be pursued in the time to come.

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Workshop: New Challenges in the Interplay between Finance and Insurance

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Abstracts

Optimal Portfolios with Sustainable Assets – Aspects for Life Insurers

Ralf Korn
(joint work with A. Nurkanovic)

The talk has been based on [4]. With the task to transform our society to a more environmental-friendly and fair one, the interest in investing in sustainable assets has increased. Even more, potential customers have to be asked about their interest in sustainable investment before they enter a pension contract. Hence, the provider has to be prepared to offer suitable investment opportunities.

For various reasons, life insurers have already decided to invest in sustainable assets as part of their actuarial reserve fund. We therefore provide a new framework for optimal portfolio decisions of a life insurer and suggest new modeling approaches for the evolution of the demand for sustainable assets, for the hedging of the risk of sustainability rating changes and for the evolution of asset prices depending on their sustainability rating. While solving various portfolio problems under sustainability constraints explicitly and suggesting further research topics, we take a particular look at the role of the actuarial reserve fund and the annual declaration of its return.

We thus consider a portfolio optimization problem with asset price dynamics $B(t), S_i(t), i = 1, ..., d, t \in [0, T]$ (where $B(t)$ denotes the evolution of the money market account, $S(t)$ is the vector of stock price processes) and square integrable, progressively measurable portfolio processes $\pi(t), t \in [0, T]$. As new ingredients, our framework for sustainable investment contains

- the dynamics $D(t)$ of the cumulative demand of the customers for sustainable investments expressed in percent of their invested sum,
- the dynamics of sustainability ratings $R_i(t)$ of the different assets,
- and their possible influence on the dynamics of the asset prices.

The portfolio problem with a sustainability constraint has the form

$$\max_{\pi(.)\in A(x)} E\left(U\left(X^{\pi}(T)\right)\right) \tag{1}$$

$$\text{such that } R(t) \geq D(t)\ \forall t \in [0, T] \tag{2}$$

For the special choice of $U(x) = \ln(x)$ we can solve this problem in an explicit way and demonstrate various affects of the presence of the sustainability constraint. In particular, we highlight the special situation of a life insurer that is able to use its actuarial reserve fund as an asset

- with a sustainability rating and a constant rate of return for a full year,
- that can be rebuilt with respect to its sustainability rating over a one-year time span,
- and that can possibly be used as the basis for an insurance product against the threat of a sustainability rating downgrade.

As in current models the sustainability constraint leads to an optimal solution that is worse than an unconstrained optimal solution, a natural task is to provide a framework such that it will also be optimal to (mainly) include sustainable assets in the portfolio. Political decisions such as a special taxation on fossile resources based products or the promotion of sustainable production methods can lead to a different potential of future dividends of the corresponding companies and thus motivates the suggestion of new stock price models with a rating- or a demand-dependent drift that itself can depend on the sustainability rating. A possible form can be

$$dS(t) = S(t)\left[(b + \lambda(\hat{D} - D(t)))dt + \sigma dW(t)\right], \tag{3}$$

$$dD(t) = \delta\left(\hat{D} - D(t)\right)dt + \sigma\sqrt{D(t)(1 - D(t))}dW_D(t) \tag{4}$$

with the two Brownian motions $W(t)$ and $W_D(t)$ possibly being correlated. I.e. we are using a Jacobi process (see [2] or [1] for its properties) for modeling the demand fluctuations over time. Considering a simple portfolio problem with a money market account and just this one stock, the optimal portfolio process can be shown to be given as

$$\pi(t) = \frac{1}{1-\gamma}\frac{b + \lambda(\hat{D} - D(t)) - r}{\sigma^2} \tag{5}$$

for the case of $U(x) = x^\gamma/\gamma$ for $\gamma < 1, \gamma \neq 0$ if the two Brownian motions are independent. In the dependent case, we will obtain a further term that depends on $D(t)$. A proof for this and the explicit form of the optimal portfolio in this case is current work and will be presented soon.

Further model and conceptual challenges in the area of optimal investment with sustainable assets for life insurers can be found in [4].

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Decarbonization of financial markets: a mean-field game approach

Peter Tankov

(joint work with P. Lavigne)

Decarbonization of industry is an essential ingredient for a successful environmental transition, and the financial sector has a key role to play in meeting the financing needs of green companies and directing the funds away from brown, carbon intensive projects. The amount of assets invested in climate-aware funds increased more than two-fold in each year between 2018 and 2021, reaching USD 408 billion at the end of 2021, and several authors aimed to quantify the impact of these additional funding flows on the emission reductions in the real economy. Such impact can be achieved only if green-minded investors target a sufficiently large proportion of companies [1], and the environmental performance of each company depends on factors which are not directly controlled by investors, such as the general economic situation, financial health of the company, and future climate policies. The decarbonization of a financial market is therefore the result of interaction of a large number of companies, operating in an uncertain environment, and should be modeled as a dynamic stochastic game with a large number of players.

Here we develop a dynamic model for the decarbonization of a large financial market, arising from an equilibrium dynamics involving companies and investors, and built using the analytical framework of mean-field games. Mean-field games, introduced in [3] and [4] provide a rigorous way to pass to the limit of a continuum of agents in stochastic dynamic games with a large number of identical agents and symmetric interactions. In the limit, the representative agent interacts with the average density of the other agents (the mean field) rather than with each individual agent. This limiting argument simplifies the problem, leading to explicit solutions or efficient numerical methods for computing the equilibrium dynamics.

The key ingredient of our framework is the notion of mean-field financial market, which describes a large financial market with a continuum of small firms, where the performance of each firm is driven by idiosyncratic noise and a finite number of market-wide risk factors (common noise). We assume that the investors in this market are 'large' meaning that in every investor's portfolio the idiosyncratic risk of small firms is completely diversified, and the portfolio value depends only on market-wide risk factors. Consequently, and consistently with the classical finance theories, only market-wide risk factors are priced, and the stochastic discount factor depends only on the common noise and the 'mean-field'.

We then consider a mean-field market where shares of a continuum of carbon-emitting firms are traded. Each firm determines its dynamic stochastic emission schedule based on its own information and on the market-wide risk factors and market-wide decarbonization dynamics, rather than on the individual decisions of each other small firm, which it cannot observe. To fix its emission level, each firm optimizes a criterion depending on its financial and environmental performance. The financial performance is measured by the market value of the firm's shares and therefore depends on the stochastic discount factor, introducing an interaction between the firms. The environmental performance is measured by carbon emissions, which are penalized in the optimization functional of the firm. The strength of this emission penalty is stochastic, reflecting the uncertainty of climate transition risk. This "stochastic carbon penalty" is a key feature of our model, allowing us to analyze the impact of climate policy uncertainty on market decarbonization and asset prices in a diffusion setting. We show that higher uncertainty about future climate policies and transition risks creates incentive for all companies to emit more carbon and leads to higher share prices and higher spreads between share prices of carbon efficient and carbon intensive companies, confirming the findings of [2] in a more realistic setting with stochastic emission schedules.

The second key ingredient of our model is the interaction between two large investors (or two classes of investors), with different views about the future: while the regular investor uses the real-world measure, the green-minded investor uses an alternative measure, which may, for example, overweight the probability of some environmental policies, making the costs of climate transition more material. In the presence of such green-minded investors, all companies will reduce their emissions and pay lower dividends, leading to lower share prices. However, carbon intensive companies are affected much stronger than climate-friendly carbon efficient companies. This pressure on share prices, in turn, spurs the polluting companies to decrease their emissions.

We summarize the interaction channels and the structure of the game of the present article in figure 1 below. The interaction goes as follows:

- On the one hand, given a stochastic discount factor ξ, the firms choose optimal emissions ψ, driving their economic values V;
- On the other hand, the investors $i \in \{r, g\}$ optimize their wealth W_T^i depending on their greenness;
- All the players (the firms and the investors) are coupled through the terminal market clearing condition: the wealth of the investors equals the economic value of the firms.

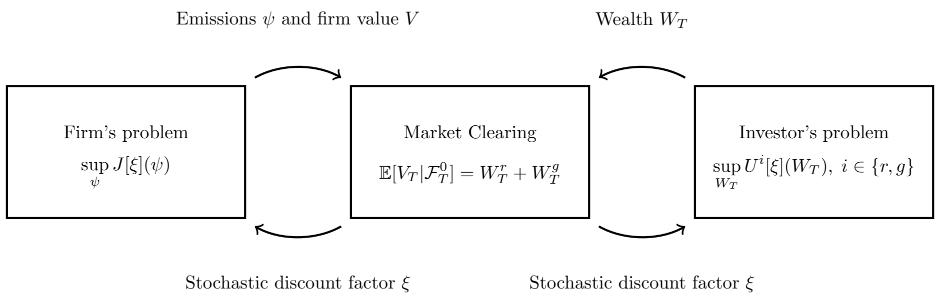

Figure 1. Structure of the game.

We rigorously prove the existence and uniqueness of the mean-field game Nash equilibrium for the contunuum of firms interacting through market prices of their shares, providing a robust solution to the stochastic "decarbonization game" in a competitive environment. The equilibrium is materialized by the equilibrium stochastic discount factor, which can be used to compute share prices and emission strategies for each firm. We then develop a convergent numerical algorithm to compute the equilibrium and use it to study the impact of climate transition risk and green investors on the market decarbonization dynamics and share prices.

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Fully-dynamic risk measures: horizon risk, time-consistency, and relations with BSDEs and BSVIEs

Emanuela Rosazza Gianin

(joint work with Giulia Di Nunno)

In a dynamic framework, we identify a new concept associated with the risk of assessing the financial exposure by a measure that is not adequate to the actual time horizon of the position. This will be called *horizon risk*. We clarify that *dynamic risk measures* are subject to horizon risk, so we propose to use the *fully-dynamic* version. To quantify horizon risk, we introduce *h-longevity* as an indicator. We investigate these notions together with other properties of risk measures as normalization, restriction property, and different formulations of time-consistency. We also consider these concepts for fully-dynamic risk measures generated by backward stochastic differential equations (BSDEs), backward stochastic Volterra integral equations (BSVIEs), and families of these. In particular, both for BSDEs and for BSVIEs, we show that h-longevity, restriction and the different formulations of time-consistency can be obtained under suitable conditions on the driver of the BSDE/BSVIE. Within this study, we provide new results for BSVIEs such as a converse comparison theorem and the dual representation of the associated risk measures.

Finally, inspired by the recent literature on cash-subadditive risk measures, we analyze - in full generality and in the framework of (families of) BSDEs - the case where cash-additivity of fully-dynamic risk measures is dropped. An example based on the generalized entropic risk measure (and the corresponding BSDE) will be also provided.

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Uncertainty Propagation and Dynamic Robust Risk Measures

Silvana M. Pesenti
(joint work with Marlon R. Moresco, Mélina Mailhot)

As uncertainty prevents perfect information from being attained, decision makers are confronted with the consequences of their risk assessments made under partial information. Incorporating model misspecification and Knightian uncertainty into dynamic decision making, thus robustifying one's decisions, has been studied in various fields, including economics [10, 15], mathematical finance [4, 12], and risk management [1]. Many circumstances require sequential decisions, where risk assessments are made over a finite time horizon and are based on the flow of information. Importantly, these decisions need to be time-consistent and account for the propagation of uncertainty. As uncertainty may change over time, we consider the dynamic risk of the entire processes rather than the total losses amount at terminal time. While the theory of time-consistent dynamic risk measures is growing [13, 6, 2, 9, 7, 3, 8], the time evolution of uncertainty is little explored.

In this work, we propose an axiomatic framework for quantifying uncertainty of discrete-time stochastic processes. Specifically, we introduce *dynamic uncertainty sets* consisting of a family of time-t uncertainty sets. Each time-t uncertainty set is a set of $\mathcal{F}_t$-measurable random variables summarising the uncertainty of the entire stochastic process at time t. The dynamic uncertainty sets may vary with each stochastic process, as the uncertainty of two processes may differ, even if they share the same law. That is, a time-t uncertainty set is a map $X_{t:T} \mapsto u_t(X_{t:T}) \subset L_t^\infty$ for any bounded discrete process X. This general framework includes, to the authors knowledge, all uncertainty sets encountered in the literature, from moment constraints, f-divergences, semi-norms, and the popular (adapted) Wasserstein distance.

Equipped with a dynamic risk measure represented by a family of one-step risk measures $\{\rho_t\}_{t\in\mathcal{T}}$ and a dynamic uncertainty set $\{u_t\}_{t\in\mathcal{T}}$, we define *dynamic robust risk measures* as sequences of conditional robust risk measures by taking the supremum of all risks in the uncertainty set. Mathematically, a time-t robust risk measure takes the form

$$R_{t:T}(X_{t+1:T}) = \operatorname{ess\,sup}\{\rho_t(Y) : Y \in u_{t+1}(X_{t+1:T})\},$$

for all discrete bounded process $X_{t+1:T}$ from time $t+1$ to T. In this procedure, the first step is to summarise the uncertainty and information of the process $X_{t+1:T}$ into a set of $(t+1)$-measurable random variables, the uncertainty set. The second step is to evaluate the risk of each of the candidate random variables and choose the largest.

This work proceeds by studying conditions on the dynamic uncertainty set that lead to well-known properties of dynamic robust risk measures such as convexity and coherence. To guarantee that the conditions are not overly strong, we seek not only sufficient conditions but also necessary ones. However, two different uncertainty sets can induce the same dynamic robust risk measure, and in fact, for each uncertainty set that satisfies a sufficient condition for a property of interest on the robust risk measure, one can find another uncertainty set that also satisfy it. Therefore, we introduce the *dynamic consolidated uncertainty set* $\{U_t\}_{t\in\mathcal{T}}$, which is the union of all uncertainty sets that agree on the dynamic robust risk measurement. We show that this consolidated uncertainty set also induces the same robust risk measure and can be written as

$$U_{t+1}(X_{t+1:T}) = \left\{Y \in L^\infty_{t+1} : \rho_t(Y) \leq R^{\mathfrak{u}}_{t:T}(X_{t+1:T})\right\}.$$

Theorem 1 in the pre-print [11] connects the properties in the consolidated uncertainty set with the axioms of the dynamic robust risk measure.

Crucial to the dynamical framework are notions of time-consistencies, of which many have been introduced and studied in the literature. The most common is strong time-consistency, leading to a dynamic programming principle [6, 14, 5]. While the majority of works assume normalisation of the dynamic risk measures, in a robust setting, uncertainty does generally not lead to normalisation. Indeed, an important subject of debate is whether the value of zero – or more generally an $\mathcal{F}_{t-1}$-measurable random variable – contains uncertainty – at time t. We find that uncertainty sets induced by the f-divergence are normalised, while those generated by the Wasserstein distance or norms are not. Consequently, we introduce the new concept of non-normalised time-consistency to account for non-normalised uncertainty sets. We also work with weaker notions of time-consistency, such as rejection and weak time-consistency. We discuss time-consistency of the uncertainty sets, and show, in Theorem 2, that they are equivalent to the notions of time consistency in the robust risk measure. Figure 1 and Proposition 5 in the pre-print [11] summarise the relationship between the most common notions of time-consistencies.

One of the manuscript's key theorem generalises results from the seminal works of [6, 14]. Specifically, we show that a dynamic robust risk measure is strong or non-normalised time-consistent if and only if it admits a recursive representation of one-step robust risk measures. Furthermore, these one-step robust risk measures are characterised by dynamic uncertainty sets which possess the property of static. Static uncertainty sets arise in one-period settings and do not contain future information. Thus, we show that it is enough to consider the simpler subclass of static uncertainty sets when working with time-consistent dynamic robust risk measures. That is:

Theorem 4 (Recursive Relation)**.** *A (normalised) dynamic robust risk measure R is non-normalised (strong) time-consistent if and only if there exists a static (and normalised) uncertainty set $\mathfrak{u}^{\varsigma} := \{u_t^{\varsigma}\}_{t\in\mathcal{T}}$ such that*

$$R_{t,T}\left(X_{t+1:T}\right) = R_t^{\mathfrak{u}^{\varsigma}}\left(Y_{t+1} + R_{t+1}^{\mathfrak{u}^{\varsigma}}\Big(Y_{t+2} + R_{t+2}^{\mathfrak{u}^{\varsigma}}\big(Y_{t+3} + \ldots + R_{T-1}^{\mathfrak{u}^{\varsigma}}(Y_T)\ldots\big)\Big)\right),$$

where $Y_t := X_t - R_t^{\mathfrak{u}^{\varsigma}}(0)$ for all $t \in \mathcal{T}$.

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Valuation of liability cash flows subject to capital requirements

Filip Lindskog
(joint work with N. Engler, H. Engsner, K. Lindensjö and J. Thøgersen)

I present two closely related approaches to valuation of liability cash flows motivated by current regulatory frameworks for the insurance industry.

In the first part I study market-consistent valuation of liability cash flows motivated by current regulatory frameworks for the insurance industry. The value assigned to an insurance liability is the consequence of (1) considering a hypothetical transfer of an insurance company's liabilities, and financial assets intended to hedge these liabilities, to an empty corporate entity, and (2) considering the circumstances under which a capital provider would want to achieve and maintain ownership of this corporate entity given limited liability for the owner and that capital requirements have to be met at any time for continued ownership. I focus on the consequences of the capital provider assessing the value of continued ownership in terms of a least favorable expectation of future dividends, meaning that I consider expectations with respect to probability measures in a set of equivalent martingale measures. I present natural conditions on the set of probability measures that imply that the value of a liability cash flow is given in terms of a solution to a backward recursion. This part of my talk is based on joint work with H. Engsner, K. Lindensjö and J. Thøgersen in [2] and [3].

The approach presented in the first part is attractive because it provides a general framework for market-consistent valuation of liability cash flows, taking repeated capital requirements and limit liability into account. However, it typically gives rise to computational challenges when accurate numerical estimates are required. The second part considers a specialized setting, yet sufficiently general for a wide range of applications, aiming for computational tractability.

This approach is motivated by computational challenges arising in multi-period valuation in insurance. Aggregate insurance liability cashflows typically correspond to stochastic payments several years into the future. However, insurance regulation requires that capital requirements are computed for a one-year horizon, by considering cashflows during the year and end-of-year liability values. This implies that liability values must be computed recursively, backwards in time, starting from the year of the most distant liability payments. Solving such backward recursions with paper and pen is rarely possible, and numerical solutions give rise to major computational challenges. The aim of the presented approach is to provide explicit and easily computable expressions for multi-period valuations that appear as limit objects for a sequence of multi-period models that converge in terms of conditional weak convergence. Such convergence appears naturally if one considers large insurance portfolios such that the liability cashflows, appropriately centered and scaled, converge weakly as the size of the portfolio tends to infinity. This part of my talk is based on joint work with N. Engler in [1].

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The risk margin and market risks

Michael Schmutz

(joint work with Christoph Möhr, Laurent Dudok de Wit)

The risk-based solvency frameworks Solvency II in Europe and the Swiss Solvency Test (SST) assess the capitalisation of insurance companies based on a risk evaluation over a one-year interval. The one-year horizon signifies that insurance companies, even in the case of long-term multi-year insurance contracts, basically only have to maintain capital at the regulatory required level of protection for one year. For the further settlement of the contracts, the risk margin (market value margin in the SST) should allow in later years to finance the capital necessary for the regulatory required level of protection or to raise this capital if required.

The risk margin thus plays a fundamental role in these frameworks. In practical implementations, it is often calculated via the sum of the multiplication of a cost of capital rate with the suitably discounted future expected capital requirements. The cost of capital rate represents the premium above the risk-free interest that an investor would demand from the insurance company for covering the corresponding risks. In a recent article [1], the risk margin and, in particular, the cost of capital rate are discussed in the context of an economic triangle of policyholders, shareholders, and regulator. The article uses well-established valuation procedures for illiquid balance sheet items and assumes that the insurance claims are nonhedgeable and independent of the financial market. In view of the in reality often present and sometimes substantial dependencies of insurance claims on financial market risks, we examine here somehow “the opposite”. Namely, the dependency of the cost of capital rate on risks in traded financial assets. We focus here only on these risks and ignore further components, such as a potentially considerable illiquidity premium. Using substantial simplifications, we subsequently analytically discuss the fundamental influence of market risks on the cost of capital rate. Our aproach combines the practitioner’s perspective with insights from Platen’s benchmark approach to quantitative finance, cf. e.g. [5].

More concretely, we analyse the cost of capital from an investor’s perspective. Let T denote the time at which all contracts have been settled and assume for simplicity that this date just falls at the end of a year. The capital realized at T is denoted by $\tilde{C}_T$ (i.e. value of assets − value of liabilities). The investor can exercise its limited liability put option if $\tilde{C}_T < 0$. Thus, the investor may price

$\tilde{C}_T^+$ at time $t = T - 1$ using "risk-neutral valuation", i.e.

$$C_t = \mathbb{E}_{\mathbb{Q}}\left(\frac{B_t \tilde{C}_T^+}{B_T}\Big|\mathcal{F}_t\right),$$

where $\mathbb{Q}$ is a suitable "risk-neutral" valuation measure, $(B_t)_{t\geq 0}$ represents the risk-free cash account, and $a^+ = \max(0, a)$ for $a \in \mathbb{R}$. Clearly, to prevent solvency problems, $C_t \geq SCR_t$ should hold for the regulatory required solvency capital SCR_t.

For simplicity, we consider from now on a Continuous Financial Market (CFM), cf. [5], with additional assumptions. The risky tradeables $((S_t^j)_{t\geq 0})_{j=1}^d$ therefore satisfy

$$dS_t^j = S_t^j a_t^j dt + S_t^j \sum_{k=1}^{d} b_t^{j,k} dW_t^k ,$$

with $((W_t^1, \ldots, W_t^d)_t)$ a d-dimensional standard Brownian Motion, $((a_t^j)_t)$ a suitable "drift", and $((b_t^{j,k})_t)$ a suitable volatility with respect to the k-th source of market risk. For simplicity, let $((b_t^{j,k})_t)_{j,k=1}^d$ be invertible for each t, with inverse matrix $((\bar{b}_t^{j,k})_t)$. For more detail, see e.g. [5, Chapter 10].

The *numéraire*-, or growth-optimal, portfolio $((S_t^*)_t)$, whose existence is assumed here, results from an "admissible trading strategy" in the chosen CFM and satisfies a number of interesting properties through which it can be defined differently but, under appropriate assumptions, equivalently. In particular, for any value process $((S_t^\delta)_t)$ of an "admissible trading strategy" with the same initial value as $((S_t^*)_t)$, the process $((\hat{S}_t^\delta)_t) = ((S_t^\delta / S_t^*)_t)$ is a *supermartingale*, i.e. $\hat{S}_s^\delta \geq \mathbb{E}_{\mathbf{P}}(\hat{S}_t^\delta | \mathcal{F}_s)$ for all $s \leq t$. For general background, see e.g. [5] or [4] for a kind of fundamental theorem that links the existence of this portfolio with an Absence of Arbitrage concept in an equivalent way. However, note that, according to the assumptions made on the existence of $\mathbb{Q}$, we are working within classical option pricing theory as it is e.g. also often used (in an extended form) for life insurance contracts. It turns out, see e.g. [5], that the numéraire portfolio S^* in our CFM can be represented by the following SDE:

$$dS_t^* = S_t^*(r_t + |\theta_t|^2)dt + S_t^*|\theta_t|dW_t ,$$

i.e. $S_t^* = \exp(\int_0^t (r_s + |\theta_s|^2)ds + \int_0^t |\theta_s| dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds)$, for $S_0^* = 1$, where (r_t) stands for the short-rate of the risk-free cash account, and (W_t), given by $dW_t = \frac{1}{|\theta_t|}\sum_{k=1}^d \theta_t^k dW_t^k$, is itself a (real-world, one-dimensional) standard Brownian motion by Lévy's characterization theorem. Here, $|\theta_t|$ stands for the "Total Market Price of Risk" given by $|\theta_t| = \sqrt{\sum_{k=1}^d (\theta_t^k)^2}$ and $\theta_t^k = \sum_{j=1}^d (a_t^j - r_t)\bar{b}_t^{k,j}$.

We assume in the above CFM that the density process (Z_t) is given by

$$Z_t = \frac{d\mathbb{Q}}{d\mathbf{P}}_{|\mathcal{F}_t} = \frac{B_t}{S_t^*},$$

and that it is a true **P**-martingale. The investor's view of the value C_t can then be reformulated via the generalized Bayes rule to give

$$\mathbb{E}_{\mathbf{P}}\left(\frac{S_t^*\tilde{C}_T^+}{S_T^*}\Big|\mathcal{F}_t\right) = \mathbb{E}_{\mathbf{P}}\left(\frac{\tilde{C}_T^+}{\exp\left(\int_t^T (r_s + |\theta_s|^2)ds + \int_t^T |\theta_s| dW_s - \frac{1}{2}\int_t^T |\theta_s|^2 ds\right)}\Big|\mathcal{F}_t\right).$$

Thus, $|\theta_t|$ should have a link to the cost of capital rate. The cost of capital rate is typically taken as a constant rate of return above risk-free interest. For a Risk Return analysis on Multiple-Factor Beta Models, we refer e.g. to [3]. For a representation of the cost of capital rate based on the quotient of a conditional real-world and a risk-neutral expectation, we refer to [2]. The advantage of the following approach is that it leads to very explicit expressions with clear dependencies on parameters of the underlying financial market.

Unfortunately, $\tilde{C}_T^+$ is often too complicated for an analytical approach to the cost of capital rate. To gain insights into basic mechanisms, we assume, again very simplistically, that $\tilde{C}_T^+$ can be approximated by an Itô-process of the following form

$$d\tilde{C}_t^+ = \mu_t \tilde{C}_t^+ dt + \sigma_t \tilde{C}_t^+ d\tilde{W}_t\,,$$

for suitable drift and volatility processes μ and σ, where $(\tilde{W}_t)$ also stands for a real-world standard Brownian motion and where for the covariation $[W, \tilde{W}]_t = \int_0^t \rho_s ds$ shall hold for a suitable process ρ. Itô -Calculus yields

$$d\left(\frac{\tilde{C}_t^+}{S_t^*}\right) = \frac{\tilde{C}_t^+}{S_t^*}(\mu_t - r_t - \sigma_t|\theta_t|\rho_t)dt + \frac{\tilde{C}_t^+}{S_t^*}\sigma_t d\tilde{W}_t - \frac{\tilde{C}_t^+}{S_t^*}|\theta_t| dW_t\,.$$

We use this to approximate $\frac{\tilde{C}_T^+}{S_T^*}$ *very roughly* from $t = T-1$ to T:

$$\frac{\tilde{C}_T^+}{S_T^*} \approx \frac{\tilde{C}_t^+}{S_t^*}(1 + (\mu_t - (r_t + \sigma_t|\theta_t|\rho_t))\Delta t) + \frac{\tilde{C}_t^+}{S_t^*}\sigma_t\sqrt{\Delta t}\tilde{Z} - \frac{\tilde{C}_t^+}{S_t^*}|\theta_t|\sqrt{\Delta t}Z\,,$$

where $\Delta t = 1$, $\tilde{Z}$ and Z are standard normally distributed random variables independent of $\mathcal{F}_t$, and all other terms are $\mathcal{F}_t$-measurable. With $1 + x \approx \exp(x)$ one approximatively obtains

$$C_t = \mathbb{E}_{\mathbf{P}}\left(\frac{S_t^*\tilde{C}_T^+}{S_T^*}\Big|\mathcal{F}_t\right) \approx \frac{\mathbb{E}_{\mathbf{P}}\left(\tilde{C}_T^+|\mathcal{F}_t\right)}{1 + r_t + \sigma_t|\theta_t|\rho_t}\,.$$

This provides a concrete link to classical Discounted Cash-Flow valuation methods from corporate finance, making $\sigma_t|\theta_t|\rho_t$ a concrete candidate for the cost of capital rate under the imposed assumptions. (This would then have to be supplemented by additional components such as an illiquidity premium.) The observation suggests, among other things, that market risks on a specific balance sheet can have a substantial impact on the cost of capital rate, with the covariation playing a major role, along with the volatility σ_t of the capital and, of course, the total market price of risk $|\theta_t|$. The concrete form of this representation paves the way for relating the cost of capital rate to concrete models for financial markets.

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Insurer's management discretion: Self-hedging participating life insurance

Peter Hieber

(joint work with Karim Barigou)

The performance of participating life insurance contracts depends on an underlying investment portfolio. For the policyholder, risks are limited as the insurance provider assures a minimum return. If the underlying portfolio performs well, the policyholder participates in its return.

The majority of scientific articles on participating life insurance assumes an exogenously given investment strategy for the underlying asset portfolio. This, however, strongly simplifies reality as the insurance provider has full control over the investment strategy of the underlying investment portfolio. He may adapt the portfolio's risk over time, for example contingent on the value of liabilities or asset-liability ratios. In this talk, we depart from the assumption of exogenously given investment strategies and consider more general endogenous investment strategies that adapt dynamically to market developments. The talk has three parts:

(1) Existing literature: We review approaches in the literature on endogenous strategies that are mostly based on the assumption of a complete financial market where all financial risks can be fully hedged. Examples include [2], [4], [3]. [3] transform the non-standard valuation problem into a fixed-point problem using the martingale method, which requires the evaluation of conditional expectations of highly path-dependent payoffs. They then use the Least-square Monte-Carlo (LSMC) approach to approximate such conditional expectations. [2] considers perfect hedging of a participating contract and derived a numerical method for the valuation. However, in both cases ([2], [3]), the focus is on the valuation problem and the determination of the optimal underlying hedging strategy remains an open research question.

(2) Solution in an incomplete market setting: As participating contracts invest for long time horizons, a more realistic assumption is that financial risks cannot be fully hedged. We discuss an objective function that minimizes the hedging risk and determine the corresponding optimal investment strategy. The financial model we consider is a Vasicek-Black-Scholes

model where interest rates are modelled stochastically by a Vasicek model. We consider a multi-period participating contract with an annual guarantee, a product that is very common in central Europe (Belgium, Germany, Switzerland). The implementation follows the neural network approach introduced in [1]. For special cases, we obtain closed form solutions for the optimal investment strategies that serve as a benachmark for our numerical results (see also [5]).

(3) Comparison to the complete market case: As a last step, we link our results to the complete market case and the results existing in the literature ([2], [4]). We specifically point at the resulting optimal hedging strategies. We stress the importance of endogenous investment strategies and their effect on the risk management of participating life insurance contracts. More specifically, we compare the solvency risks and contract values of participating life insurance contracts if investment strategies are (A) exogenously given and (B) chosen endogenously.

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Hybrid life insurance valuation based on a new standard deviation premium principle in a stochastic interest rate framework

Griselda Deelstra
(joint work with Oussama Belhouari, Pierre Devolder)

In this talk, we focus on the pricing of a class of hybrid life insurance products, which are dependent on both mortality and financial risks, and this in a stochastic interest rate framework.

Assuming a complete, arbitrage-free financial market, the valuation of future (purely) financial cash-flows can be based upon risk-neutral expectations and is related to the existence of hedging strategies. In insurance, the calculation of premiums is based on best estimate values and safety loadings, assuming that the law of large numbers can be applied by pooling independent contracts. Of course, in finance, markets appear in practice very often to be incomplete, whereas insurance risks are not always perfectly diversifiable (for instance by the presence of longevity risks or catastrophic risks).

Moreover, as hybrid life insurance contracts depend on both financial and insurance risks, defining a fair valuation of hybrid contracts requires a hybrid valuation principle combining the notions of financial and actuarial valuation. Different principles have been proposed in the literature in order to price these hybrid products (see, e.g., [5], [6], [1], [3], [2] and many others). In order to be consistent with the financial market, the concept of market-consistency is used in the literature, see e.g. [4] or [6]; whilst to be consistent with the actuarial market, the concept of actuarial-consistency has been introduced, see e.g. [2].

Focusing on the pricing of hybrid products in the presence of stochastic interest rates, we first conduct a profound study of the axioms that a valuation operator should verify in the presence of stochastic interest rates (see e.g. [1]) and we study both the market-consistency and actuarial-consistency properties. In particular, we present a generalized standard deviation premium principle in a stochastic interest rate framework, and discuss its integration in different valuation operators suggested in the literature, namely by [5], [6] and [3]. We illustrate our methods with a classical application in life insurance, namely a pure endowment with profit.

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Shrinking the term structure

Damir Filipović

(joint work with Markus Pelger and Ye Ye)

We develop a conditional factor model for the term structure of Treasury bonds, which unifies non-parametric curve estimation with cross-sectional asset pricing. Our factors are investable portfolios and estimated with cross-sectional ridge regressions. They correspond to the optimal non-parametric basis functions that span the discount curve and are based on economic first principles. Cash flows are covariances, which fully explain the factor exposure of coupon bonds. Empirically, we show that four factors explain the discount bond excess return curve and term structure premium, which depends on the market complexity measured by the time-varying importance of higher order factors. The fourth term structure factor capturing complex shapes of the term structure premium is a hedge for bad economic times and pays off during recessions.

Concretely, we denote by $d_t(x)$ the price at date t of a discount bond with time to maturity x [years]. The excess return over $t-1$ to t of this discount bond is then given by

$$r_t(x) = \frac{d_t(x)}{d_{t-1}(x+\Delta_t)} - \frac{1}{d_{t-1}(\Delta_t)},$$

where Δ_t denotes the time [in years] between business days $t-1$ and t. The goal of this project is to estimate and study the empirical properties of the unobserved discount bond excess return curve $r_t : [0,\infty) \to \mathbb{R}$. What is observed at any t are M_t coupon bond securities with prices $P_{t,i}$, cash flows $C_{t,ij}$ at cash flow dates $0 < x_1 < \cdots < x_N$, and their excess returns $R^{\text{bond}}_{t,i} = \frac{P_{t,i}+C_{t-1,ii+1}}{P_{t-1,i}} - \frac{1}{d_{t-1}(\Delta_t)}$. By the absence of arbitrage, we know that a coupon bond is a portfolio of discount bonds. Formally, we obtain

$$R^{\text{bond}}_t = \underbrace{Z_{t-1} r_t(\boldsymbol{x})}_{\text{fundamental returns}} + \underbrace{\epsilon_t}_{\text{return errors}}$$

where we define the normalized discounted cash flows $Z_{t-1,ij} := \frac{C_{t-1,ij+1} d_{t-1}(x_j+\Delta_t)}{P_{t-1,i}}$, and we denote by $f(\boldsymbol{x}) := (f(x_1), \ldots, f(x_n))^\top$ the array of function values queried at $\boldsymbol{x} = (x_1, \ldots, x_N)^\top$, for any function f.

We estimate r_t by solving the following regularized optimization problem

$$\min_{r_t \in \mathcal{H}_\alpha} \Bigg\{ \underbrace{\frac{1}{M_t} \left\| R^{\text{bond}}_t - Z_{t-1} r_t(\boldsymbol{x}) \right\|_2^2}_{\text{return error}} + \underbrace{\lambda \left\| r_t \right\|^2_{\mathcal{H}_\alpha},}_{\text{smoothness}} \Bigg\}. \tag{1}$$

We choose the regularization penalty by awarding smoothness of r_t. Smoothness of the return curve is motivated by economic principles, it puts limits to excessive returns of investments such as the butterfly trade $r_t(x-\Delta) - 2r_t(x) + r_t(x+\Delta) \approx r_t''(x)\Delta^2$. Our hypothesis space $\mathcal{H}_\alpha$ therefore consists of twice weakly differentiable functions satisfying the natural boundary conditions $r_t(0) = 0$ and $\lim_{x\to\infty} r_t'(x) = 0$, and finite weighted Sobolev type norm

$$\|r_t\|^2_{\mathcal{H}_\alpha} := \int_0^\infty r_t''(x)^2 \mathrm{e}^{\alpha x}\, dx.$$

We prove that $\mathcal{H}_\alpha$ is a reproducing kernel Hilbert space with kernel k given in closed form. Problem (1) is a kernel ridge regression with unique solution $\hat{r}_t$ in $\mathcal{H}_\alpha$, which is spanned by the N kernel basis functions $k(x_1, \cdot), \ldots, k(x_N, \cdot)$. We orthonormalize the basis functions as follows. We show that the kernel matrix $\boldsymbol{K}_{ij} := k(x_i, x_j)$ is invertible, and thus admits spectral decomposition $\boldsymbol{K} = VSV^\top$, with eigenvectors $V = [v_1|\cdots|v_N]$, and strictly positive eigenvalues $s_1 \geq \cdots \geq$

$s_N > 0$. We obtain the orthonormal system of functions in $\mathcal{H}_\alpha$ given by $\boldsymbol{u} = (u_1, \dots, u_N)^\top := S^{-1/2} V^\top k(\cdot, \boldsymbol{x})$.

After this transformation we obtain the following result.

Theorem 1 (Conditional Factor Model Representation)**.** *The unique solution* $\hat{r}_t$ *to* (1) *can be represented as factor model*

$$\hat{r}_t(\cdot) = \boldsymbol{u}(\cdot)^\top \hat{F}_t, \tag{2}$$

where the factors $\hat{F}_t$ *are unique solution to the cross-sectional ridge regression*

$$\min_{F_t \in \mathbb{R}^N} \left\{ \frac{1}{M_t} \left\| R_t^{\text{bond}} - \beta_{t-1}^{\text{bond}} F_t \right\|_2^2 + \lambda \left\| F_t \right\|_2^2 \right\},$$

where the conditional loadings $\beta_{t-1}^{\text{bond}}$ *are given in terms of the normalized discounted cash flows (bond characteristics)* Z_{t-1} *by*

$$\beta_{t-1}^{\text{bond}} := Z_{t-1} V S^{1/2}.$$

The factors $\hat{F}_t$ *are given in closed form by*

$$\hat{F}_t = \omega_{t-1} R_t^{\text{bond}},$$

which are the excess returns of traded bond portfolios with portfolio weights

$$\omega_{t-1} := \left({\beta_{t-1}^{\text{bond}}}^\top \beta_{t-1}^{\text{bond}} + \lambda M_t I_N \right)^{-1} {\beta_{t-1}^{\text{bond}}}^\top.$$

In summary, this is a flexible non-parametric data-driven approach, the smoothness penalty $\lambda > 0$ and maturity weight $\alpha > 0$ are selected empirically by cross-validation. We perform an extensive empirical analysis on a large sample of daily U.S. Treasury bond returns ranging from June 1961 to December 2020. In particular, we shrink the term structure and study low-dimensional approximations of the N-factor model (2), and empirically show that the first n factors describe the data accurately well, for $n = 4$. The paper is available at SSRN: `https://ssrn.com/abstract=4182649`, which contains an extensive list of references.

Climate change, insurance mathematics and optimal prevention

Stèphane Loisel

(joint work with H. Albrecher, C. Constantinescu, R. Gauchon, D. Kortschak, P. Ribereau, J.L. Rullière, J. Trufin)

In this blackboard talk, we start by describing the various impacts of climate change on the insurance industry. We present some theoretical results that demonstrate that quantitative risk management of uncertain and potentially worsening risks is completely different in presence of climate change. We also show the impact of the level of access to information of insurance risk managers on their ability to keep the insurance business safe enough. In presence of full uncertainty, without any possibility to adjust premium, we use our previous results obtained by Albrecher and Constantinescu to show that increasing capital requirements is not enough to make the ruin probability decrease to zero, and that there is a positive probability to be ruined anyway. Besides, the asymptotic rate of decay towards this positive probability level with respect to the initial capital is much slower than usually as well. In the opposite case, assuming that one is “magically” able to adjust instantaneously income premium rate to the worsened risk level, in the regular variation case, we use results obtained with Kortschak and Ribereau to study the effect of claim size distribution worsening. We consider two approaches: either the shape or the scale parameter changes over time. Comparing the two approaches, we note that when risks initially have infinite variance, a change in the scale parameter may have more impact than a change in the shape parameter. We also note that the company may cease its business due to climate change for several reasonns, including ruin, insolvency or mass lapse due to the rise of insurance premium to an unacceptable level. We then present recent works and works in progress to propose a risk management partial solution to this problem. We believe that one key ingredient is risk prevention. We briefly present some results of our recent works with Gauchon, Rullière and Trufin and explain the differences between our optimal prevention problem and classical optimal reinsurance problems. We highlight some results and explain in particular that the optimal prevention level does not depend on the initial surplus level in presence of one single kind of claims, while it depends on the initial surplus when there are two kinds of claims and when prevention only has some effect on one of them. We mention some work in process with Minier and Mamode Khan about prevention with INAR and BINAR processes. Following discussions during this Oberwolfach workshop, some concrete future collaborations have been started with Hansjoerg Albrecher on risk models in presence of climate change, as well as with Michael Schmutz on insurance regulation of long-term risk and short-term bias. In Oberwolfach discussions, we also planted the seed for other future collaborations, notably with Valérie Chavez-Demoulin on climate change risk for hailstorm risk management and with Caroline Hillairet on prevention and thinning.

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Extreme value theory in a changing world

Valérie Chavez-Demoulin
(joint work with Linda Mhalla)

The past few decades have seen extreme climate events affecting all regions of the world with catastrophic impacts on human society. Extreme value theory is the field of statistics dedicated to the study of events with low occurrence frequencies and large amplitudes. Such events are necessarily rare in relation to the bulk of a population, which makes them hard to model and difficult to predict. Classical methods of extreme value theory are based on the assumption that the data are independent and identically distributed (iid) or at least stationary and, in this case, the classical approaches rely on theoretical foundations that are well established and understood. In practice the iid or stationarity assumptions are generally violated, the nature of the series being non-stationary or depending on covariates. In this talk I have reviewed extreme value theory in the univariate and multivariate settings and under non-stationarity, attempting, in this case, to capture different sorts of dependence when estimating risk measures. Part of the work I presented contributes to the development of flexible frameworks for taking into account the effect of covariates on the (tail) dependence structure between two variables. In the context of multivariate extremes, we develop in [1] flexible, semi-parametric method for the estimation of non-stationary multivariate Pickands dependence functions. Related works in multivariate extremes, allowing extremal dependence structures that may vary with covariates are [2] and [3]. A new field of interest and very much linked to the understanding of effect of covariates is causality. The study of causality for extremes is in its infancy. Examples of related work are [4], who defined recursive max-linear models on directed acyclic graphs, [5], who define a causal tail coefficient capturing asymmetries in the extremal dependence between two random variables, [6], who use multivariate generalized Pareto distributions to study probabilities of necessary and sufficient causation as defined in the counterfactual theory of Pearl, and [7], who construct a causal inference method for tail quantities relying on Kolmogorov complexity of extreme conditional quantiles. [8] review the related basic probability schemes, inference techniques, and statistical hypotheses for extreme event attribution. In preparation, we are currently writing a Chapter about causality of extremes in a book entitled "Handbook on Statistics of Extremes".

Part of my presentation was related to a book entitled "Risk Revealed: Cautionary Tales, Understanding and Communication" I co-authored with Paul Embrechts and Marius Hofert, which will appear in 2024 in Cambridge University Press.

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Isotonic distributional regression

Johanna Ziegel
(joint work with Sebastian Arnold, Tilmann Gneiting, Alexander Henzi, Gian-Reto Kleger, Eva-Maria Walz)

Isotonic distributional regression (IDR) is a nonparametric distributional regression approach under a monotonicity constraint [9]. It has found application as a generic method for uncertainty quantification [12], in statistical postprocessing of weather forecasts [9, 11], and it is an integral part of distributional single index models [7, 2]. In this abstract, the construction and main properties of IDR are reviewed and it is explained how IDR can be generalized from empirical distributions to arbitrary distributions yielding isotonic conditional laws.

Assume that the covariate X takes values in a partially ordered space $(\mathcal{X}, \leq)$, and the outcome Y is real-valued. The main assumption of IDR is that when the covariate X increases, we expect an increase of the outcome Y. Mathematically, we assume that $x \leq x'$ for $x, x' \in \mathcal{X}$ implies $\mathcal{L}(Y \mid X = x) \preceq_{\mathrm{st}} \mathcal{L}(Y \mid X = x')$, where $\mathcal{L}(Y \mid X = x)$ denotes the conditional distribution of Y given $X = x$ and $\preceq_{\mathrm{st}}$ denotes the usual stochastic order.

For given data pairs $(x_i, y_i)_{i=1}^n$ with $(x_i, y_i) \in \mathcal{X} \times \mathbb{R}$, the IDR estimator is defined as the vector $\hat{\mathbf{F}} = (\hat{F}_i)_{i=1}^n = (\hat{F}_{Y|X=x_i})_{i=1}^n$ of cumulative distribution functions (cdfs) that satisfies

$$\hat{\mathbf{F}} = \arg \min_{(F_1,\ldots,F_n)} \sum_{\ell=1}^n \mathrm{CRPS}(F_\ell, y_\ell), \tag{1}$$

where the minimum is taken over all vectors of cdfs $(F_1, \ldots, F_n)$ that satisfy $F_i \preceq_{\mathrm{st}} F_j$ whenever $x_i \leq x_j$. Here, the continuous ranked probability score (CRPS) is defined as

$$\mathrm{CRPS}(F, y) = \int_{\mathbb{R}} \left(F(z) - \mathbb{1}\{y \leq z\}\right)^2 \,\mathrm{d}z$$

for a cdf F and $y \in \mathbb{R}$.

The optimization problem at (1) has a unique solution that can be stated explicitly as a min-max formula. It turns out that for each $y \in \mathbb{R}$, $\hat{F}_1(y), \ldots, \hat{F}_n(y)$ is the antitonic least-squares regression of the binary outcomes $\mathbb{1}\{y_1 \le y\}, \dots, \mathbb{1}\{y_n \le y\}$ [3]. Furthermore, the IDR solution is universal in the sense that the same solution arises when replacing the CRPS in (1) by any quantile- or threshold-weighted CRPS [6]. IDR can be efficiently computed using the pool adjacent violators (PAV) algorithm for each threshold $y \in \{y_1, \dots, y_n\}$ if there is a total order on the covariate space $\mathcal{X}$. For partial orders, the solution can be obtained as a quadratic programming problem for each $y \in \{y_1, \dots, y_n\}$. There is an R package and a Python implementation available [8]. The IDR solution is defined at observed covariate values only but predictions at new covariate values can be readily obtained by suitable interpolation techniques.

Statistical consistency results for IDR can be found in [5] for ordinal covariates, in [10] for real-valued covariates, and in [9] for vector-valued covariates. Furthermore, in [7, 2], the authors show that even if the partial is estimated from the data, consistency still holds.

Suppose that a vector $(X, Y) \in \mathcal{X} \times \mathbb{R}$ has distribution $(1/n) \sum_{i=1}^n \delta_{(x_i, y_i)}$. Then, IDR provides an approximation to the joint distribution of (X, Y) such that all conditional distributions of Y given X are ordered with respect to the stochastic order. It is a natural question to ask if such an isotonic approximation can be constructed starting with any distribution for (X, Y), where we assume that (X, Y) are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The answer is positiv as shown in [1], where the solution is termed the isotonic conditional law of Y given X. The isotonic conditional law of Y given X is constructed as the conditional law of Y given the σ-lattice generated by X.

More precisely, a σ-lattice $\mathcal{C} \subseteq \mathcal{F}$ is a system of sets that contains $\emptyset, \Omega$ and is closed under countable unions and countable intersections. A random variable Z is $\mathcal{C}$-measurable if $\{Z > a\} \in \mathcal{C}$ for all $a \in \mathbb{R}$. The conditional expectation $\mathbb{E}(Z \mid \mathcal{C})$ with respect to the σ-lattice $\mathcal{C}$ can be defined as the L^2-projection of Z onto the closed convex cone of $\mathcal{C}$-measurable random variables [4]. The conditional law $\mathcal{L}(Y \mid \mathcal{C})$ of Z with respect to $\mathcal{C}$ is then a Markov kernel from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\omega \mapsto \mathcal{L}(Z \mid \mathcal{C})(\omega, (a, \infty))$ is a version of $\mathbb{E}(\mathbb{1}\{Z > a\} \mid \mathcal{C})$ for any $a \in \mathbb{R}$. Furthermore, let $\mathcal{U}$ be the collection of all upper sets in $(\mathcal{X}, \le)$. It is a σ-lattice, and a function $f : \mathcal{X} \to \mathbb{R}$ is increasing if and only f is $\mathcal{U}$-measurable, that is, $\{f > a\} \in \mathcal{U}$ for all $a \in \mathbb{R}$. Finally, for an ordered metric space $(\mathcal{X}, d, \le)$, the σ-lattice generated by X is defined as

$$\mathcal{A}(X) = \{X^{-1}(B) \mid B \in \mathcal{B}(\mathcal{X}) \cap \mathcal{U}\}.$$

IDR is the isotonic conditional law of Y given X if the joint distribution of (X, Y) has finite support. Isotonic conditional laws can also be identified as CRPS minimizers in a suitable sense [1].

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Approximate Bayesian Computation for Insurance and Finance

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(joint work with Patrick Laub)

Approximate Bayesian Computation (ABC) is a statistical learning technique to calibrate and select models by comparing observed data to simulated data. This technique bypasses the use of the likelihood and requires only the ability to generate synthetic data from the models of interest. We apply ABC to fit and compare insurance loss models using aggregated data. The talk is based on the work Goffard and Laub [3].

Over a fixed time period, an insurance company experiences a random number of claims called the *claim frequency*, and each claim requires the payment of a randomly sized compensation called the *claim severity*. The two could be associated in an equivalent way with a policyholder, a group of policyholders or even an entire nonlife insurance portfolio. The claim frequency is a counting random variable while the claim sizes are non-negative continuous random variables. Let us say that the claim frequency and the claim severity distributions are specified by the parameters θ_{freq} and θ_{sev} respectively, with $\theta = (\theta_{\text{freq}}; \theta_{\text{sev}})$. For each time $s = 1, \dots, t$ the number of claims n_s and the claim sizes $u_s := (u_{s,1}, u_{s,2}, \dots, u_{s,n_s})$ are distributed as

$$n_s \sim p_N(n|\theta_{\text{freq}}) \quad \text{and} \quad (u_s|n_s) \sim f_U(u|n, \theta_{\text{sev}}).$$

Fitting these distributions is key for claim management purposes. For instance, it allows one to estimate the expected cost of claims and set the premium rate accordingly. The mixed nature of claim data, with a discrete and a continuous component, has lead to two different claim modelling strategies. The first strategy is to handle the claim frequency and the claim severity separately, see for instance [1]. The second approach gathers the two constituents in a compound model for which data in aggregated form suffices. We take the later approach as we assume that the claim count and amounts $\{(n_1, u_1), \ldots, (n_t, u_t)\}$ are unobservable. Instead, we only have access to some real-valued *summaries* of the claim data at each time, denoted by

$$x_s = \Psi(n_s, u_s), \quad s = 1, \ldots, t. \tag{1}$$

Standard actuarial practice uses the aggregated claim sizes, defined as $\Psi(n, u) = \sum_{i=1}^{n} u_i$, and assumes that the claim frequency is Poisson distributed while the severities are governed by a gamma distribution, we refer to the works of [4].

A Bayesian approach to estimating θ would be to treat θ as a random variable and find (or approximate) the *posterior distribution* $\pi(\theta|x)$. Bayes' theorem tells us that

$$\pi(\theta|x) \propto p(x|\theta)\, \pi(\theta), \tag{2}$$

where $p(x|\theta)$ is the *likelihood* and $\pi(\theta)$ is the *prior distribution*. The prior represents our beliefs about θ before seeing any of the observations and is informed by our domain-specific expertise. The posterior distribution is a very valuable piece of information that gathers our knowledge over the parameters. A point estimate $\hat{\theta}$ may be derived by taking the mean or mode of the posterior. For an overview on Bayesian statistics, we refer to the book of [2].

The posterior distribution (2) rarely admits a closed-form expression, so it is approximated by an empirical distribution of samples from $\pi(\theta|x)$. Posterior samples are typically obtained using Markov Chain Monte Carlo (MCMC), yet a requirement for MCMC sampling is the ability to evaluate (at least up to a constant) the likelihood function $p(x|\theta)$. When considering the definition of x in (1), we can see that there is little hope of finding an expression for the likelihood function even in simple cases (e.g. when the claim sizes are **i.i.d.**). If the claim sizes are not **i.i.d.** or if the number of claims influences their amount, then the chance that a tractable likelihood for x exists is extremely low. Even when a simple expression for the likelihood exists, it can be prohibitively difficult to compute (such as in a big data regime), and so a likelihood-free approach can be beneficial.

We advertise here a likelihood-free estimation method known as *approximate Bayesian computation* (ABC). This technique has attracted a lot of attention recently due to its wide range of applicability and its intuitive underlying principle. One resorts to ABC when the model at hand is too complicated to write the likelihood function but still simple enough to generate artificial data. Given some observations x, the basic principle consists in iterating the following steps:

(1) generate a potential parameter from the prior distribution $\tilde{\theta} \sim \pi(\theta)$;
(2) simulate 'fake data' $\tilde{x}$ from the likelihood $(\tilde{x}|\tilde{\theta}) \sim p(x|\theta)$;
(3) if $\mathcal{D}(x, \tilde{x}) \leq \epsilon$, where $\epsilon > 0$ is small, then store $\tilde{\theta}$,

where $\mathcal{D}(\cdot,\cdot)$ denotes a distance measure and ϵ is an acceptance threshold. The algorithm provides us with a sample of θ's whose distribution is close to the posterior distribution $\pi(\theta|x)$.

The basic ABC algorithm outlined above is, arguably, the simplest method of all types of statistical inference in terms of conceptual difficulty. At the same time, this simple method is perhaps the most difficult form of inference in terms of computational cost. We must use a modified form of this basic regime to minimize (though not eliminate) the gigantic computational costs of ABC. ABC is a somewhat young field (like machine learning), and the methodology of ABC and the other likelihood-free algorithms are currently the subject of intense research. As such, there are many variations of ABC which are under investigation, and there is no ironclad consensus on which variation of the ABC algorithm is the best. For a comprehensive overview on ABC, we refer to the monograph of [7]; in finance and insurance, ABC has been considered in the context of operational risk management by [5] and for reserving purposes by [6].

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Sharing Model Uncertainty

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(joint work with Chiaki Hara, Sujoy Mukerji, Jean-Marc Tallon)

Uncertainty, as opposed to risk, is a major concern in today's societies. Be it financial markets during the 2007-2009 crisis, policy makers when a new virus emerged, or farmers hit by climate change - in all situations, decision makers faced and face uncertainties that cannot be easily quantified probabilistically. It is therefore of crucial importance to understand whether and how economic institutions can deal with and possibly hedge against this uncertainty. In this paper, we study this question in the framework of identifiable environments in which (Knightian) uncertainty is resolved ex post, at least partially, when sufficient amounts of data have been collected, and agents exhibit smooth ambiguity-averse preferences ([17]), a setting that that has recently been axiomatized by [11]. Identifiability is a necessary condition for statistical learning to occur. We thus put ourselves in a framework where such perfect learning is possible, in principle. In experiments, we could think of an Ellsberg experiment in which the composition of urns is revealed after the experiment. In statistics, ergodic environments suffice for identifiability. In real life, perfect identification is not always achievable, of course. However, in the case of financial markets, e.g., the past volatility of a stock price is very well known ex post. A virus, to take another example, is understood much better aa couple of years after its first appearance. Even in climate change, it might be possible to state after a sufficiently long time that average temperature, e.g., has risen by at least one or two degrees. Our study thus sheds also light on the issue of learning under ambiguity, a notoriously difficult task so far.

In identifiable environments, agents can make their consumption plans contingent on *models*, thus allowing to make ex post insurance payments that depend on a certain probabilistic model being true. The farmer, to take up an example from above, can thus write an insurance contract on a temperature change of a certain amount being true after thirty years or so. This possibility allows to study uncertainty sharing in much more detail and to obtain more results than in general models in which uncertainty is not identifiable.

We are thus able to study models with aggregate uncertainty, in contrast to much of the literature on risk and uncertainty sharing that has focused on the simpler case of no aggregate uncertainty so far. We are able to identify the environments in which a representative agent of smooth ambiguity type exists. In such settings, we can compute quite explicitly the efficient uncertainty sharing rules and study how consumption shares vary with different uncertainty scenarios, depending on the respective individuals' risk and ambiguity aversion relative to society's risk and ambiguity aversion.

We investigate consequences of ambiguous model uncertainty on efficient allocations in an exchange economy, and departing from the literature, allow for ambiguous aggregate risk and heterogeneously ambiguity averse consumers. A model – a statistical view of the world, comprising of parameters and distinctive mechanisms– implies a specific probabilistic forecast about the states of the world. Furthermore, the parameters and mechanisms driving a model may be estimated and identified on the basis of objective data. However, at the point of decision-making, the data relevant to identifying the model is still missing. Hence, consumers are unsure what would be the appropriate probability measure to apply to evaluate consumption contingent on a state space Ω and keep in consideration a set $\mathcal{P}$ of alternative probabilistic laws. Importantly, because models are identified, the usual assumption that consumption plans are contingent on events in the state space now means that they can be made effectively contingent on models too.

We study the case where consumers in the economy are heterogeneously ambiguity averse with *smooth ambiguity* preferences [17]. Our primary focus lies in those economies that admit a *representative* consumer who is also of the smooth ambiguity type. This setting offers valuable and precise insights into efficient sharing rules and the characteristics of the representative consumer. Another advantage of the setting is that the insights obtained, initially assuming that $\mathcal{P}$ is *point-identified*, robustly extend to the case where models are only *set-identified*. When aggregate risk is unambiguous we show, quite generally, that ambiguity aversion makes no difference to the qualitative nature of efficient allocations: they are comonotone just as under expected utility. An economy with a smooth ambiguity averse representative consumer is characterized by consumers who exhibit linear risk tolerance with the same marginal risk tolerance. When aggregate risk is ambiguous, efficient sharing rules systematically deviate from the linearity that would arise under expected utility. The deviations –which make the slope and intercept of the linear rule model-contingent– arise to allow the more ambiguity averse consumers to have smoother expected utility across models.

Macro-finance models that study effects of ambiguity aversion consider single consumer economies with ambiguous aggregate risk. We show if we introduce heterogeneous ambiguity aversion the nature of the representative consumer can be very different from what is widely assumed in the literature. For instance, even if individual consumers have *constant* relative ambiguity aversion, the representative consumer is shown to have *decreasing* relative ambiguity aversion. Such a representative consumer makes for more compelling asset-pricing predictions than one based on homogeneous ambiguity aversion.

Related literature. Efficient risk-sharing in expected-utility economies was first studied by [4], further refined for the HARA class of utility functions by [25], [5] and [14] among others. Under ambiguity, [8] extended the comonotonicity result obtained under expected utility to Choquet expected utility with common capacity. [3], [22] and [12] further studied the case in which aggregate endowment is non-risky and preferences are more general than Choquet-expected-utility preferences (including, for the two latter references, the smooth ambiguity model). [23] and [9] characterized properties of efficient risk-sharing when the aggregate endowment is risky but not ambiguous. [2] extends some of these results to cases where agents have possibly heterogeneous, non-convex ambiguity sensitive preferences. [24] proves that, under HARA with common risk tolerance, a two-fund theorem holds for maxmin-expected-utility economies (and hence efficient allocations are comonotonic). To the best of our knowledge, no paper has studied risk-sharing with *ambiguous* aggregate endowments and *heterogeneous* ambiguity aversion.

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Optimal investment in ambiguous financial markets with learning

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(joint work with Antje Mahayni)

We investigate the effects of model ambiguity preferences on optimal investment decisions in a multi asset Black Scholes market. Since the seminal paper by [5], we know that decision makers may have a non-neutral attitude towards model ambiguity. As a result, preferences are decomposed into risk preferences (based on known probabilities) and preferences concerning the degree of uncertainty about the (unknown) model parameters and are evaluated separately. This is in particular relevant for portfolio optimization problems. [4] suggests that model ambiguity is at least as prominent as risk in making investment decisions.

There are different ways to incorporate model ambiguity in decision making. In our setting, model ambiguity refers to the drift uncertainty in the dynamics of asset prices and we apply the smooth ambiguity approach of [7] to deal with it. The risk in asset prices itself is evaluated by a utility function applied to the terminal wealth. Thus, the expected utility is itself a random variable (determined by the prior distribution of the drift parameters) which is evaluated by a second utility function (ambiguity function) capturing the model ambiguity. As in [1] we assume that both the risk aversion and ambiguity aversion of the investor are described by (CRRA) power functions. While [1] consider pre-commitment strategies, we take into account for the possibility that the investor is able to gradually learn about the drift by observing the asset price movements. Using duality results we are able to solve the problem analytically. To the best of our knowledge this has not yet been achieved before in our setting. Further, based on our theoretical results, we are able to shed light on the impact and consequences of ambiguity preferences.

The underlying financial market consists of d stocks and one riskless bond (for simplicity assumed to be identical to 1), defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with finite time horizon $T > 0$. The price process $S = (S_1(t), \dots, S_d(t))_{t\in[0,T]}$ of the d stocks will for $i = 1, \dots, d$ be given by

$$dS_i(t) = S_i(t)\Big[\mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t)\Big] = S_i(t)\Big[\sum_{j=1}^{d} \sigma_{ij} dY_j(t)\Big], \tag{1}$$

where $W = (W_1(t), \dots, W_d(t))^\top_{t\in[0,T]}$ is a d-dimensional Brownian motion, $\mu_i \in \mathbb{R}, \sigma_{ij} \in \mathbb{R}_+, i, j = 1, \dots, d$ and $\sigma = (\sigma_{ij})$ is regular. We further set

$$Y(t) := W(t) + \Theta t, \quad \Theta^\top := \sigma^{-1}\mu, \quad \mu := (\mu_1, \dots, \mu_d),$$

where Θ denotes the market price per unit of risk. We further assume that μ is not known and thus a random variable. This implies that the market price of risk Θ is also not known to the investor. However, she has a prior knowledge about Θ in form of a prior distribution $\mathbb{P}$ on $\mathbb{R}^d$.

Due to the self-financing condition, trading strategies $\pi = (\pi_1, \dots, \pi_d)$ are d-dimensional stochastic processes, where $\pi_k(t)$ describes the amount invested in the k-th stock at time $t \in [0, T]$. Strategies π should be $\mathcal{F}^Y$-progressively measurable (which is the filtration generated by Y or equivalently S). This means that the agent is able to *learn* the right market price of risk. The associated wealth process denoted by $(X_t^\pi)_{t\in[0,T]}$ is given by

$$dX_t^\pi = \sum_{k=1}^{d} \pi_k(t) \frac{\mathrm{d}S_k(t)}{S_k(t)} = \pi(t)\sigma dY(t) \tag{2}$$

with initial capital $x_0 \in \mathbb{R}$. In what follows let $u(x) = \frac{1}{\alpha}x^\alpha, \alpha < 1, \alpha \neq 0$.

The investor aims to maximize her expected utility of terminal wealth. First we assume that the investor is ambiguity-neutral w.r.t. the unknown parameter and consider

$$V(x_0) = \sup_{\pi} \int \mathbb{E}_{\vartheta}[u(X_T^{\pi})]\mathbb{P}(d\vartheta) \tag{3}$$

where the supremum is taken over all $\mathcal{F}^Y$-adapted strategies π for which the stochastic integral and the expectations are defined and $X_T^{\pi} \geq 0$. We denote this set by $\mathcal{A}$. $\mathbb{E}_{\vartheta}$ is the conditional expectation, given $\Theta = \vartheta$. This problem is the well-known Bayesian adaptive portfolio problem. We summarize its solution in the following theorem ([6, 8]) (where $\|\cdot\|$ is the usual Euclidean norm):

Theorem 1. *The maximal expected utility attained in* (3) *is given by*

$$V(x_0) = \frac{x_0^{\alpha}}{\alpha}\left(\int_{\mathbb{R}^d}\left(\int \exp\left(z\cdot\vartheta - \frac{1}{2}\|\vartheta\|^2 T\right)\mathbb{P}(d\vartheta)\right)^{\gamma}\varphi_T(z)dz\right)^{1/\gamma}, \quad x_0 > 0 \tag{4}$$

where $\gamma = 1/(1-\alpha)$, φ_T is the density of the d-dimensional normal distribution $\mathcal{N}(0, TI)$ (I being the identity matrix). The optimal fractions invested in the stocks are also given by an explicit formula.

Now we are interested in an investor who takes model ambiguity into account, i.e. instead of (3) we consider for $v(x) = \frac{1}{\lambda}\, x^{\lambda}, \lambda < 1, \lambda \neq 0$ the problem ([1])

$$\sup_{\pi\in\mathcal{A}} v^{-1}\int v\circ u^{-1}\mathbb{E}_{\vartheta}[u(X_T^{\pi})]\mathbb{P}(d\vartheta) = \sup_{\pi\in\mathcal{A}}\left(\int \left(\mathbb{E}_{\vartheta}[(X_T^{\pi})^{\alpha}]\right)^{\lambda/\alpha}\mathbb{P}(d\vartheta)\right)^{1/\lambda} \tag{5}$$

This means that model ambiguity, represented by an uncertain market price of risk, is evaluated with a second utility function v which is here of the same form but with possibly different parameter. In case $\alpha > 0$ problem (5) is equivalent to

$$\sup_{\pi\in\mathcal{A}}\left(\mathbb{E}\left[\left(\mathbb{E}_{\Theta}[(X_T^{\pi})^{\alpha}]\right)^{\lambda/\alpha}\right]\right)^{\alpha/\lambda}, \tag{6}$$

Here we restrict to the case that $\lambda > \alpha > 0$ and define $\mathbf{p} := \lambda/\alpha > 1$. The economic interpretation is that the agent is ambiguity-loving (the ambiguity-averse case is similar). By using the $L^{\mathbf{p}}$ norm $\|\cdot\|_{\mathbf{p}}$ we can write problem (6) as

$$\sup_{\pi\in\mathcal{A}} \|\mathbb{E}_{\Theta}[(X_T^{\pi})^{\alpha}]\|_{\mathbf{p}} \tag{7}$$

where the norm is w.r.t. Θ. It is well-known that the $L^{\mathbf{p}}$ norm has the following dual representation for a r.v. $X \geq 0$, where $1/\mathbf{p} + 1/\mathbf{q} = 1$ (see e.g. [9]):

Lemma 1. *If $\mathbf{p} := \lambda/\alpha > 1$ we obtain for non-negative $X \in L^{\mathbf{p}}$*

$$\|X\|_{\mathbf{p}} = \sup\left\{\int X d\mathbb{Q} : \left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|_{\mathbf{q}} \leq 1\right\}. \tag{8}$$

where on the right-hand side of (8) *the supremum is taken over all measures $\mathbb{Q}$ (not necessarily probability measures) which are absolutely continuous w.r.t. $\mathbb{P}$ and satisfy the constraint. Moreover, an optimal measure $\mathbb{Q}^*$ exists.*

In what follows define the set of measures $\mathfrak{Q}$ as the set of measures which satisfy the constraints in (8). This gives immediately rise to the following solution algorithm for our problem:

Theorem 2. *In the model of this subsection we have*

$$\sup_{\pi\in\mathcal{A}}\sup_{\mathbb{Q}\in\mathfrak{Q}}\int\mathbb{E}_\vartheta[(X_T^\pi)^\alpha]\mathbb{Q}(d\vartheta)=\sup_{\mathbb{Q}\in\mathfrak{Q}}\sup_{\pi\in\mathcal{A}}\int\mathbb{E}_\vartheta[(X_T^\pi)^\alpha]\mathbb{Q}(d\vartheta)=\int\mathbb{E}_\vartheta[(X_T^{\pi^*})^\alpha]\mathbb{Q}^*(d\vartheta).$$

After normalizing $\mathbb{Q}$, the inner optimization problem is exactly the Bayesian portfolio problem with distribution $\tilde{\mathbb{Q}} := \mathbb{Q}/\mathbb{Q}(\mathbb{R})$ for the unknown parameter. So solving (6) *boils down to solving the classical Bayesian portfolio problem first with value given in Theorem 1 and then in a second step finding the optimal prior distribution implied by $\mathbb{Q}^*$ which is obtained from the outer optimization problem. The optimal strategy π^* is then the one in Theorem 1 with $\mathbb{P}$ replaced by $\mathbb{Q}^*$.*

An approach like this may be generalized to situations where uncertainty and ambiguity are measured by other means (see e.g. [3]). The extended abstract is based on [2].

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Investment under Uncertain Preferences

Mogens Steffensen

(joint work with S. Desmettre and J. Søe)

We consider classes of dynamic decision problems where an investor maximizes utility but faces random preferences. We consider three versions of the problem.

In one version, the investor optimizes expected utility where the expectation is taken with respect to both financial and preference uncertainty. That is based on Steffensen and Søe (2023). We formalize a consumption–investment–insurance problem with the distinction of a state-dependent relative risk aversion. The state dependence refers to the state of the finite state Markov chain that also formalizes insurable risks such as health and lifetime uncertainty. We derive and analyze the implicit solution to the problem and compare it with special cases in the literature.

Two other versions are based on certainty equivalents. We tackle the time-consistency issues arising from that formulation by applying the equilibrium theory approach.

In one version, the investor learns nothing about his preferences as time passes. That is based on Desmettre and Steffensen (2023). We provide the proper definitions and prove a rigorous verification theorem. We complete the calculations for the cases of power and exponential utility. For power utility, we illustrate in a numerical example, that the equilibrium stock proportion is independent of wealth, but decreasing in time, which we also supplement by a theoretical discussion. For exponential utility, the usual constant absolute risk aversion is replaced by its expectation.

The main results of Desmettre and Steffensen (2023) are gathered in the following verification theorem and corollary. Definitions and proofs can be found in Desmettre and Steffensen (2023). We model the parameter of a utility function γ as a real-valued random variable. Examples are the constant (known) relative and absolute risk aversions that are replaced by random variables. We form an optimization problem based on the idea to *maximize the certainty equivalent of terminal wealth w.r.t. a random risk aversion* in an equilibrium sense, i.e. we want to maximize the reward functional

$$J^{\pi}(t,x) := \int (u^{\gamma})^{-1}\left(\mathbb{E}_{t,x}[u^{\gamma}(X^{\pi}(T))]\right)\, d\Gamma(\gamma)\,, \tag{1}$$

where Γ is the Cumulative Distribution Function (CDF) of γ, and we integrate over the support of the corresponding CDF. Moreover, we assume that the dependence of the utility function $u^{\cdot}$ on $\gamma \sim \Gamma$ is such that the integral in (1) is always well-defined. Note that now we decorate the utility function by subscript γ to highlight its dependence on risk aversion.

We now first formalize the equilibrium problem and then characterize its solution in a verification theorem. We introduce

$$y^{\pi,\gamma}(t,x) := \mathbb{E}_{t,x}\left[u^{\gamma}\left(X^{\pi}(T)\right)\right]\,, \tag{2}$$

such that the objective of the investor is to maximize the reward functional

$$J^{\pi}(t,x) := \int (u^{\gamma})^{-1}\left(y^{\pi,\gamma}(t,x)\right) d\Gamma(\gamma) \tag{3}$$

in a given sense.

Theorem 1 (Verification Theorem)**.** *Assume that there exist functions $U \in C^{1,2}$, $Y^{\gamma} \in C^{1,2}$ for all γ, such that*

$$\begin{aligned} U_t(t,x) = \inf_{\pi}\Big\{ &-(r+\pi(\alpha-r))\,xU_x(t,x) - 0.5\pi^2x^2\sigma^2U_{xx}(t,x) + H_t(t,x) \\ &+ (r+\pi(\alpha-r))xH_x(t,x) + 0.5\pi^2x^2\sigma^2H_{xx}(t,x) \\ &- \int \iota^{\gamma}(Y^{\gamma}(t,x))(Y^{\gamma}_t(t,x) \\ &+ (r+\pi(\alpha-r)x)Y^{\gamma}_x(t,x) + 0.5\sigma^2\pi^2x^2Y^{\gamma}_{xx}(t,x))\, d\Gamma(\gamma)\Big\}\,, \end{aligned} \tag{4}$$

and

$$Y_t^\gamma(t,x) = -(r+\hat{\pi}(\alpha-r))xY_x^\gamma(t,x) - 0.5\sigma^2\hat{\pi}^2x^2Y_{xx}^\gamma(t,x)\,, \tag{5}$$

where $H(t,x) = \int (u^\gamma)^{-1}(Y^\gamma(t,x))d\Gamma(\gamma) \in C^{1,2}$ *and*

$$\begin{aligned}\hat{\pi} = \arg\inf_{\pi}\Big\{ &-(r+\pi(\alpha-r))\,xU_x(t,x) - 0.5\pi^2x^2\sigma^2U_{xx}(t,x) + H_t(t,x)\\ &+(r+\pi(\alpha-r))xH_x(t,x) + 0.5\pi^2x^2\sigma^2H_{xx}(t,x)\\ &-\int \iota^\gamma(Y^\gamma(t,x))(Y_t^\gamma(t,x)\\ &+(r+\pi(\alpha-r)x)Y_x^\gamma(t,x) + 0.5\sigma^2\pi^2x^2Y_{xx}^\gamma(t,x))\,d\Gamma(\gamma)\Big\}\,,\end{aligned} \tag{6}$$

with boundary conditions

$$U(T,x) = x, \text{ and } Y^\gamma(T,x) = u^\gamma(x) \text{ for all } \gamma\,. \tag{7}$$

Furthermore assume that U, H, *and* Y^γ *for all* γ, *belong to the space* $L^2(X^{\hat{\pi}})$. *Then* $\hat{\pi}$ *is an equilibrium control, and we have that*

$$V(t,x) = U(t,x)\,, \tag{8}$$

$$y^{\hat{\pi},\gamma}(t,x) = Y^\gamma(t,x) \text{ for all } \gamma\,. \tag{9}$$

For the special form of H given by

$$H(t,x) = \int (u^\gamma)^{-1}(Y^\gamma(t,x))\,d\Gamma(\gamma)$$

we obtain as an immediate consequence:

Corollary 1. *From the pseudo HJB* (4) *we obtain by using*

$$\begin{aligned}H(t,x) &= \int (u^\gamma)^{-1}(Y^\gamma(t,x))\,d\Gamma(\gamma)\,,\\ H_t(t,x) &= \int \iota^\gamma(Y^\gamma(t,x))Y_t^\gamma(t,x)\,d\Gamma(\gamma)\,,\\ H_x(t,x) &= \int \iota^\gamma(Y^\gamma(t,x))Y_x^\gamma(t,x)\,d\Gamma(\gamma)\,,\\ H_{xx}(t,x) &= \int \iota^\gamma(Y^\gamma(t,x))Y_{xx}^\gamma(t,x)\,d\Gamma(\gamma) + \int (\iota^\gamma)'(Y^\gamma(t,x))(Y_x^\gamma(t,x))^2\,d\Gamma(\gamma)\,,\end{aligned}$$

the following form:

$$\begin{aligned}U_t(t,x) = \inf_{\pi}\Big\{ &-(r+\pi(\alpha-r))xU_x(t,x) - 0.5\sigma^2\pi^2x^2U_{xx}(t,x)\\ &+0.5\sigma^2\pi^2x^2\int (\iota^\gamma)'(Y^\gamma(t,x))(Y_x^\gamma(t,x))^2\,d\Gamma(\gamma)\Big\}\,.\end{aligned} \tag{10}$$

In this formulation, the non-linearity arising within the time-inconsistent control problem is clearly visible, cf. [1, Section 16.2].

Risk aversion is an observed stochastic process in another version (work in progress, new results). That version can, e.g., be motivated by preferences that directly depend on the state of health. We introduce the notion of preferences concerning preference risk and find a case where the investor invests as if the (conditional) expected risk aversion were realized.

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Reduced-form framework under model uncertainty

Katharina Oberpriller

(joint work with Francesca Biagini, Andrea Mazzon)

The talk is based on [3],[4] and [5]. In this talk we introduce a reduced-form framework for multiple ordered default times under model uncertainty and study some applications in insurance and finance. To this purpose we define a sublinear conditional operator with respect to a family of probability measures possibly mutually singular to each other in presence of multiple ordered default times. In this way we extend the classical literature on credit risk in presence of multiple defaults, see for example [11], [12], [13] and [17] to the case of a setting where many different probability models can be taken into account.
Over the last years, several different approaches have been developed in order to establish such robust settings which are independent of the underlying probability distribution, see among others [1], [7], [8], [9], [10], [15], [16], [19], [20], [22], [23], [24] and [25]. However, the above results hold only on the canonical space endowed with the natural filtration. In credit risk and insurance modeling it is fundamental to model multiple random events occurring as a surprise, such as defaults in a network of financial institutions or the loss occurrences of a portfolio of policy holders. This requires to consider filtrations with a dependence structure. Such a problem is mentioned in [2] and solved for an initial enlarged filtration. In [6] they define a sublinear conditional operator with respect to a filtration which is progressively enlarged by one random time.

In this paper we extend the approach in [6] and define a sublinear conditional operator with respect to a filtration progressively enlarged by multiple ordered stopping times. Such an extension is connected to several additional technical challenges with respect to the construction in [6].

First, we cannot consider default times in all generality, but we need to focus on a family of ordered stopping times. In particular, we work in the setting of the top-down model for increasing default times introduced in [11], in order to model the loss of CDOs, as a generalization to the well known Cox model in [18]. More specifically, we start with a reference filtration $\mathbb{F}$ and define a family of ordered stopping times $\tau_1, ..., \tau_N$, in a similar way as done in [11]. We then progressively enlarge $\mathbb{F}$ with the filtrations $\mathbb{H}^n$ generated by $(\mathbf{1}_{\{\tau_n \leq t\}})_{t\geq 0}$, $n = 1, ..., N$, and define $\mathbb{G}^{(n)} := \mathbb{F} \vee \mathbb{H}^1 \vee ... \vee \mathbb{H}^n$, $n = 1, ..., N$. In our case, we construct $\tau_1 < ... < \tau_N$ in such a way that $\tilde{\tau}_n := \tau_n - \tau_{n-1}$ is independent of $\mathcal{H}_t^{n-1}$ for any $n = 2, ..., N, t \geq 0$ conditionally on $\mathcal{F}_\infty$. In particular, the intensities of the stopping times are driven by $\mathbb{F}$-adapted stochastic processes which may be used to model dependence structures driven by common risk factors and also contagion effects. We first address the problem of computing $\mathbb{G}^{(N)}$-conditional expectations of a given random variable under one given prior in terms of a sum of $\mathbb{F}$-conditional expectations depending on how many defaults have happened before time t. This is also a new contribution to the literature on ordered multiple default times in the classical case, i.e., in presence of only one probability measure. For an analogous result following the density approach for modeling successive default times, we refer to [12]. The main technical issue in our setting is to compute conditional expectations when a strictly positive number of defaults, but not *all* the N defaults, have happened. Already under a fixed prior the results for multiple ordered default times are not a trivial extension of the ones in a single default setting.

We then use this representation to define the sublinear conditional operator $\tilde{\mathcal{E}}^N$ under model uncertainty with respect to the progressively enlarged filtration $\mathbb{G}^{(N)}$. As in [6], our definition makes use of the sublinear conditional operator introduced by Nutz and van Handel in [21] with respect to $\mathbb{F}$. To this purpose we assume that $\mathbb{F}$ is given by the canonical filtration. In particular, we show that our construction is consistent with the ones in [21] in presence of no default and in [6] for $N = 1$, respectively. The main technical challenge is to prove a weak dynamic programming principle for the operator as done in [6] for the single default setting, as it requires to exchange the order of integration between the operator and expectations under a given prior. We then use the conditional sublinear operator to evaluate credit portfolio derivatives under model uncertainty. In particular, we focus on the valuation of the so called i-th to default contingent claims $\mathrm{CCT}^{(i)}$, for $i = 1, ..., N$. Moreover, we discuss if the valuation of such financial or insurance products with the sublinear conditional operator corresponds to a sensitive pricing rule. As done in [6] for the single default case, we can establish a relation between the sublinear conditional operator and the superhedging problem in a multiple default setting for a generic payment streams under given conditions. Furthermore, we show that the sublinear conditional operator can be used to price a contingent claim such that the extended market allows no arbitrage of the first kind under model uncertainty as in [7]. This result requires assumptions about the trading strategies which are, however, not restrictive in an insurance setting. By modeling the intensity processes as an affine process under uncertainty, introduced for example in [14] and [3], the valuation of several relevant payoffs can be numerically computed by solving non-linear PDEs.

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Multivariate Portfolio Choice via Quantiles

Carole Bernard
(joint work with Andrea Perchiazzo, Steven Vanduffel)

The talk was organized as follows. First, I recalled the quantile approach of [8] for an agent maximizing a one-dimensional objective function that is law-invariant and non-decreasing. The quantile approach builds on the concept of cost-efficiency originally proposed by [5, 6] and further discussed in [1]. Then I related the multivariate portfolio choice (see (1) below) to a risk sharing problem (see (3) hereafter) as studied e.g., by [3] in the context of a multivariate expected utility setting. We then show how the quantile approach used for univariate optimal portfolio choice can be also useful to solve the multivariate portfolio choice as in (1) below. To do so, we use the concept of multivariate cost-efficiency ([2]). Finally, two examples are fully solved: the optimization of a sum of expected CRRA utility functions and the infconvolution of the Range Value-at-Risk (RVaR). For this latter example, we make use of the explicit results of [7] and show that the portfolio problem that minimizes the sum of d RVaRs can be rewritten as a portfolio that maximizes a one-dimensional objective function, i.e., a distorted expectation. Furthermore, this problem has been explicitly solved in [4] and [9].

Specifically, we assume a frictionless and arbitrage-free financial market living on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a non-empty sample space, $\mathcal{F}$ is the σ-algebra generated by Ω and $\mathbb{P}$ denotes the probability measure on Ω. We consider a fixed investment horizon $T > 0$ without intermediate consumption in which a final payoff X_T received at time T has an initial price given as $\mathbb{E}[\xi_T X_T]$ where ξ_T is the pricing kernel, agreed by all agents, with positive density on $\mathbb{R}_+ \setminus \{0\}$. Let $V(\cdot)$ be a multivariate law-invariant objective function. We consider the problem

$$\sup_{(X_1,X_2,\ldots,X_d)\in\mathcal{A}} V(X_1, X_2, \ldots, X_d), \tag{1}$$

where $\mathcal{A} = \left\{(X_1, X_2, \ldots, X_d) \in \mathcal{K} \text{ s.t. } \mathbb{E}\left[\xi_T \sum_{i=1}^d X_i\right] = w_0\right\}$, $\mathcal{K}$ is the set of random d-vector and $w_0 > 0$ denotes the total budget that must be allocated in d dimensions. The goal is to optimize a multivariate law-invariant objective function $V(\cdot)$ over a set of admissible $(X_1, \ldots, X_d) \in \mathcal{A}$ such that the total budget w_0 is allocated.

We assume that the objective function $V(\cdot)$ is law-invariant (that is, if two vectors $(X_1,\ldots,X_d)$ and $(Y_1,\ldots,Y_d)$ are equal in distribution, then $V(X_1,\ldots,X_d) = V(Y_1,\ldots,Y_d)$). Furthermore, we assume that $V(\cdot)$ is strictly increasing in at least one of the dimensions. Without loss of generality, we can thus assume that for any constant $a \in \mathbb{R}_+ \setminus \{0\}$, $V(X_1+a, X_2,\ldots,X_d) > V(X_1,X_2,\ldots,X_d)$. Finally, we assume that the general portfolio problem (see (1)) is well-posed in that there exists an optimal multivariate portfolio $(X_1^\star,\ldots,X_d^\star)$ leading to a maximum finite value for $V(X_1,\ldots,X_d)$.

To solve the general multivariate portfolio problem in (1), we first solve a multivariate risk sharing problem in the absence of a financial market that we then use to provide the solution to (1).

Let S be a random variable. Define the risk sharing of S as the following set of random vectors associated to S

$$A_d(S) := \left\{(X_1,X_2,\ldots,X_d) \in \mathcal{K} : \sum_{i=1}^{d} X_i = S\right\}. \tag{2}$$

The optimal multivariate risk sharing associated to the total risk S solves

$$\sup_{(X_1,X_2,\ldots,X_d)\in A_{d(S)}} V(X_1,\ldots,X_d). \tag{3}$$

Denote by

$$(Y_1(S),\ldots,Y_d(S))$$

a solution to (3). In the context of the additive multivariate utility function, i.e., where $V(X_1,\ldots,X_d)$ is of the form $V(X_1,\ldots,X_d) = \sum_{i=1}^{d} U_i(X_i)$ in which U_i for $i=1,\ldots,d$ are univariate exponential utility functions or univariate CRRA (Constant Relative Risk Aversion) utility functions, the multivariate risk sharing problem (3) can easily be solved explicitly. In the case of an objective function based on quantile risk measures (e.g., RVaR), a solution for the multivariate risk sharing problem is found in [7].

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Polynomial interacting particle systems and non-linear SPDEs for capital distribution curves

Christa Cuchiero
(joint work with Florian Huber)

The stability of the *capital distribution curves* over time, as shown in Figure 1, can be seen as a universal phenomenon in finance. By this we here mean a robust

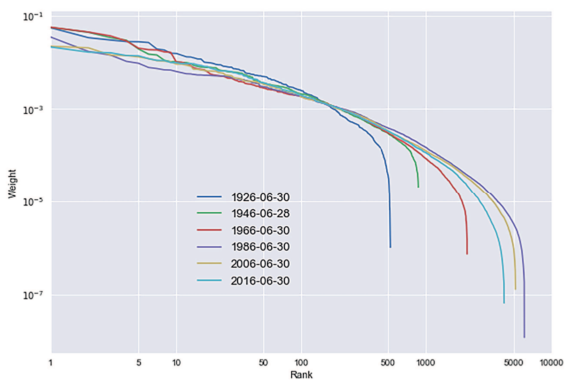

Figure 1. Capital distribution curves: 1926 - 2016, source [4]

empirical feature that holds universally across different markets, asset classes and in particular over time. Each of the above curves depicts the relative market capitalization in ranked order of the major US markets' stocks on a log-log scale from 1926 to 2016. The *relative market capitalization* or *market weight* is defined as the percentage of the market capitalization of a fixed company, i.e., the number of outstanding shares times the current price of one share, with respect to the capitalization of the whole market. The striking feature of these curves is their remarkably stable shape over the last century. Although the market weights of each company fluctuate stochastically the shape of the capital distribution curves differs (in first order) over the years only by the number of stocks present in the considered market. This fundamental observation was the starting point for R. Fernholz to develop *stochastic portfolio theory* about 20 years ago, see [1].

On the mathematical side of financial modeling we also encounter universal structures, such as the interplay of potentially infinitely many factors as well as mean field interactions and limits. Universal model classes that are able to capture these phenomena and appear throughout in mathematical finance, but also in other fields like population genetics and physics, are (infinite dimensional) affine and polynomial processes.

One goal of this work is to combine mathematical with financial universality and to model the capital distribution curves via polynomial processes, which have empirically proved to provide a very good fit to these curves.

More precisely, we extend volatility stabilized market models, a particular class of polynomial models introduced by Fernholz et al [2], by allowing for a common noise term such that the models remains polynomial. Indeed, we consider the following model for the N individual market capitalizations

$$dS_i(t) = \beta \sum_{j=1}^{N} S_j(t)dt + \sqrt{\alpha}\sqrt{S_i(t)\sum_{j=1}^{N} S_j(t)}dW_t^i + \sqrt{(N-\alpha)}S_i(t)dW_t^0,$$

where $\alpha \geq 0$, $\beta \geq \frac{\alpha}{2}$ and W^i for $i \in \{1, \ldots, N\}$ are the idiosyncratic Brownian motions and W^0 the common one. The introduction of this common noise term permits to overcome the absence of correlation between the individual stocks in the original model of [2].

Inspired by M. Shkolnikov [5] who studied large volatility stabilized markets, we then analyze the limit as $N \to \infty$. To do so we rescale time, i.e. let time go slower as we add particles, and consider $X(t) := S(t/N)$

$$dX_i(t) = \frac{\beta}{N}\sum_{j=1}^{N} X_j(t)dt + \sqrt{\frac{\alpha}{N}}\sqrt{X_i(t)}\sqrt{\sum_{j=1}^{N} X_j(t)}dW_t^i + \sqrt{1-\frac{\alpha}{N}}X_i(t)dW_t^0.$$

Taking formal limits and denoting the typical particle in the limit by Y then yields

$$dY(t) = \beta\mathbb{E}[Y(t)|\sigma(W^0)]dt + \sqrt{\alpha Y(t)\mathbb{E}[Y(t)|\sigma(W^0)]}dW_t + Y(t)dW_t^0. \tag{1}$$

for some Brownian motion W independent of W^0, and where $\sigma(W^0)$ denotes the sigma-algebra generated by W^0. To make this rigorous we consider, as usual for McKean-Vlasov equations, the particles' empirical probability measure on path space, i.e.

$$\rho^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i}$$

and its "mean-field limit" $(\rho_t^N)_{t\in[0,T]} \to (\rho_t)_{t\in[0,T]}$ in $C([0,T]; M_1(\mathbb{R}_+))$, where $T > 0$ denotes some finite time and $M_1(\mathbb{R}_+)$ probability measures over $\mathbb{R}_+$ with finite first moment, i.e.

$$M_1(\mathbb{R}_+) = \{\mu \in M(\mathbb{R}_+) | \int_{\mathbb{R}_+} x\mu(dx) =: \langle x, \mu\rangle < \infty\},$$

equipped with the Wasserstein-1 distance. Then, we show that the limit ρ, which is the unique solution of a degenerate, non-linear SPDE, corresponds to the conditional law of the typical particle Y, i.e. $\rho = \mathcal{L}(Y(\cdot)|\sigma(W^0))$. and $\langle \rho_t, \mathrm{id}_x\rangle = \mathbb{E}[Y(t)|\sigma(W^0)]$. Indeed, our two main results read as follows:

Theorem 1. *Under minor conditions on the initial values of the particle system, each convergent subsequence of* $(\rho_\cdot^N)_{N\in\mathbb{N}}$ *converges a.s. in* $C([0,T],M_1(\mathbb{R}_+))$ *to the unique probabilistically strong, analytically weak,* $M_1(\mathbb{R}_+)$*-valued solution* ρ *of the non-linear SPDE*

$$d\rho_t = (\frac{\alpha}{2}\langle\rho_t,\mathrm{id}_x\rangle\partial_x^2(x\rho_t) + \frac{1}{2}\partial_x^2(x^2\rho_t) - \beta\langle\rho_t,\mathrm{id}_x\rangle\partial_x\rho_t)dt - \partial_x(x\rho_t)dW_t^0. \tag{2}$$

Theorem 2. *Consider* (1) *with* $0 < Y(0) \in L^2(\Omega)$*, independent of* W^0*, and let* ρ *be the unique solution of* (2) *with* $\rho_0 = \mathcal{L}(Y(0))$*.*

- *Then, any solution to* (1) *satisfies* $\rho = \mathcal{L}(Y(\cdot)|\sigma(W^0))$ *as well as*

$$\mathbb{E}[Y(t)|\sigma(W^0)] = \langle\rho_t,\mathrm{id}_x\rangle = \langle\rho_0,\mathrm{id}_x\rangle\exp((\beta-\frac{1}{2})t+W_t^0) =: S(t).$$

- *The two-dimensional process* $(Y,\mathbb{E}[Y(t)|\sigma(W^0)]) =: (Y,S)$ *is a polynomial diffusion on* $\mathbb{R}^2_{++}$ *which is unique in law. Its dynamics are given by*

$$\begin{aligned} dY(t) &= \beta S(t)dt + \sqrt{\alpha}\sqrt{Y(t)S(t)}dW_t + Y(t)dW_t^0 \\ dS(t) &= \beta S(t)dt + S(t)dW_t^0, \quad S_0 = \langle\rho_0,\mathrm{id}_x\rangle. \end{aligned}$$

One of the mathematical subtleties of these results lies in the uniqueness proof which involves fine estimates with respect to weighted Sobolev norms. This uniqueness result then also allows us to conclude uniqueness in law of the polynomial process $(Y,\mathbb{E}\left[Y\mid\sigma(W^0)\right])$ which was open so far.

Let us remark that behind the intriguing polynomial property of $(Y,\mathbb{E}[Y(t)|\sigma(W^0)])$ is a generic structure. Indeed, consider (for simplicity) a one-dimensional conditional McKean-Vlasov SDE of the form

$$\begin{aligned} dZ_t &= b(Z_t,\mathbb{E}[Z_t^1|\sigma(W^0)],\dots,\mathbb{E}[Z_t^k|\sigma(W^0)])dt \\ &\quad+ \sqrt{c(Z_t,\mathbb{E}[Z_t^1|\sigma(W^0)],\dots,\mathbb{E}[Z_t^k|\sigma(W^0)])}dW_t \\ &\quad+ c^0(Z_t,\mathbb{E}[Z_t^1|\sigma(W^0)],\dots,\mathbb{E}[Z_t^k|\sigma(W^0)])dW_t^0, \quad 0\le t\le T. \end{aligned}$$

Then, if c is quadratic in the first variable and b and c^0 are affine in the first variable, the conditional moments become a k-dimensional autonomous standard Itô-SDE driven by W^0. Provided that a (pathwise) unique solution exists for this SDE, its components then correspond to the conditional moments $\mathbb{E}[Z^i|\sigma(W^0)]$ for $i = 1,\dots,k$. From the theory of time-inhomogeneous polynomial processes (see [3]), one should then be able to deduce existence and uniqueness for a large class of conditional McKean-Vlasov SDEs beyond the standard conditions of Lipschitz continuity and uniform ellipticity. Proving this conjecture rigorously is subject of ongoing work.

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Ergodic robust maximization of asymptotic growth with stochastic factor processes

Josef Teichmann
(joint work with David Itkin, Martin Larsson, Benedikt Koch)

We consider a robust asymptotic growth problem under model uncertainty in the presence of stochastic factors. We fix two inputs representing the instantaneous covariance for the asset price process X, which depends on an additional stochastic factor process Y, as well as the invariant density of X together with Y. The stochastic factor process Y has continuous trajectories, but is not even required to be a semimartingale. Our setup allows for drift uncertainty in X and model uncertainty for the local dynamics of Y. There are several interpretation of Y: it could model stochastic covariance as it often happens in Finance, but it could also be a numerical model for uncertainty of the instantanous covariance function for X.

This work builds upon a recent paper of Kardaras & Robertson (AAP 2022), where the authors consider an analogous problem, however, without the additional stochastic factor process. Under suitable, quite weak assumptions we are able to characterize the robust optimal trading strategy and the robust optimal growth rate. The optimal strategy is shown to be functionally generated and, remarkably, does not depend on the factor process Y. We also construct a worst case model for the functionally generated strategy thereby fully solving the min-max problem.

Our result provides a comprehensive answer to a question proposed by Fernholz in 2002. We also show that the optimal strategy remains optimal even in the more restricted case where Y is a semimartingale and the joint covariation structure of X and Y is prescribed.

Our results are obtained using a combination of techniques from partial differential equations, calculus of variations, and generalized Dirichlet forms.

Collective Arbitrage and the Value of Cooperation

Thilo Meyer-Brandis
(joint work with Francesca Biagini, Alessandro Doldi, Jean-Pierre Fouque, Marco Frittelli)

The theory developed in this paper aims at expanding the classical Arbitrage Pricing Theory to a setting where N agents are investing in stochastic security markets and are allowed to cooperate through suitable exchanges. More precisely, we suppose that each agent is allowed to invest in a subset of the available assets

$(X^1, \dots, X^J)$, for a given $J \in \mathbb{N}$, and in a common riskless asset. Note that we do not exclude that such subset coincides with the full set $(X^1, \dots, X^J)$. The novel notions of Collective Arbitrage and Collective Super-replication, are based on the possibility that the N agents may additionally enter in a zero-sum risk exchange mechanism, where no money is injected or taken out of the overall system. Cooperation and the multi-dimensional aspect are the key features of Collective Arbitrage and Collective Super-replication. In this setting agents not only may invest in their respective market but may additionally cooperate to improve their positions by taking advantage of the risk exchanges. In the case of one single agent, the theory reduces to the classical one. There is an extensive literature in recent years on variations around the concept of one-agent No Arbitrage or No Free Lunch and we refer to the books Delbaen Schaermayer (2006) [5] and Föllmer and Schied (2014) [6], and references therein, for a detailed overview of the topic. Departing from this stream of literature, the main aim of this paper is to understand the consequences of the cooperation between several agents in relation to Arbitrage and Super-replication.

Before moving into the details of our new setup, we briefly summarize the classical one-agent situation. Let a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, with $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$, $\mathcal{T} = \{1, \dots, T\}$ be given, and denote by $X = (X^1, \dots, X^J)$ the J adapted stochastic processes representing the prices of J securities. The set of admissible trading strategies is denoted by $\mathcal{H}$ and let K be the set of time-T stochastic integral of $H \in \mathcal{H}$ with respect to X. The set K represents all the possible terminal time-T payoffs available in the market from admissible trading strategies and having zero initial cost.

An arbitrage opportunity is an admissible trading strategy $H \in \mathcal{H}$, having zero initial cost and producing a non negative final payoff $k \in K$, being strictly positive with positive probability. Equivalently, we have no arbitrage in this setting if the only non negative element in K is P-a.s. equal to 0, or more formally $K \cap L^0_+(\Omega, \mathcal{F}, P) = \{0\}$.

In this paper, we generalize the setting to multiple agents that might cooperate with each other. This leads to the new concepts of Collective Arbitrage and Collective Super-replication which we shortly describe in the following.

Collective Arbitrage. Since each agent $i = 1, \dots, N$ is allowed to invest only in a subset of the available assets $(X^1, \dots, X^J)$, we define, similarly to the notion of the set K, the market K_i of agent i, that is the space of all the possible time-T payoffs that agent i can obtain by using admissible trading strategies in his/her allowed investments and having zero initial cost.

Inspired by [4] we consider the set of all zero-sum risk exchanges

$$\mathcal{Y}_0 = \left\{ Y \in (L^0(\Omega, \mathcal{F}, P))^N \mid \sum_{i=1}^{N} Y^i = 0 \ P\text{-a.s.} \right\},$$

and the set $\mathcal{Y}$ of possible/allowed exchanges

$$\mathcal{Y} \subseteq \mathcal{Y}_0 \text{ such that } 0 \in \mathcal{Y}.$$

We stress that even if the overall sum is P-a.s. equal to 0, each components Y^i of $Y \in \mathcal{Y}$ is in general a random variable. If Y^i is positive on some event, agent i is receiving, on that event, from the collection of the other agents some (positive) amount of cash. So $Y \in \mathcal{Y}$ represents the amount that the agents may exchange among themselves with the requirement that the overall amount distributed is equal to zero.
A *Collective Arbitrage* is a vector $(k^1, \dots, k^N)$, where $k^i \in K_i$ for each i, and a vector $Y = (Y^1, \dots, Y^N) \in \mathcal{Y}$ such that

$$k^i + Y^i \geq 0, \quad P\text{-a.s. for all } i \in \{1, \dots, N\},$$

and

$$P(k^j + Y^j > 0) > 0 \text{ for at least one } j \in \{1, \dots, N\}.$$

One may immediately notice that if $N = 1$, then $Y \in \mathcal{Y}$ must be equal to 0 and thus a Collective Arbitrage reduces to a Classical Arbitrage.
However, for $N \geq 2$, in a Collective Arbitrage, agents are entangled by the vector of exchanges $Y \in \mathcal{Y}$: this additional possible cooperation may create a Collective Arbitrage even if there is No Arbitrage for each single agent.

We study the implications of the assumption of No Collective Arbitrage with respect to the set $\mathcal{Y}$, which we denote in short by $\mathbf{NCA}(\mathcal{Y})$. We also write $\mathbf{NA}_i$ for the No Arbitrage condition (in the classical sense) for agent i in market K_i and **NA** for the No Arbitrage condition (in the classical sense) in the global market K.

It is easy to verify that under very reasonable conditions the following implications hold

$$\mathbf{NA} \Rightarrow \mathbf{NCA}(\mathcal{Y}) \Rightarrow \mathbf{NA}_i \ \forall i \in \{1, \dots, N\},$$

but none of the reverse implication holds true in general. We show that the strongest condition **NA** is equivalent to $\mathbf{NCA}(\mathcal{Y})$ for the "largest" choice $\mathcal{Y} = \mathcal{Y}_0$, while the weakest condition, $\mathbf{NA}_i \ \forall i$, is equivalent to $\mathbf{NCA}(\mathcal{Y})$ for the "smallest" choice $\mathcal{Y} = \mathcal{Y}_0 \cap (L^0(\Omega, \mathcal{F}_0, P))^N$. The latter space actually consists of zero-sum deterministic vectors, when $\mathcal{F}_0$ is the trivial sigma algebra. However, for general sets $\mathcal{Y}$ the notions of $\mathbf{NCA}(\mathcal{Y})$ give rise to new concepts.

We analyse the conditions under which a new type of Fundamental Theorem of Asset Pricing holds, that we label Collective FTAP (CFTAP). Differently from the classical version, the CFTAP depends of course on the properties of the set of exchanges $\mathcal{Y}$, and so we provide several versions of such a theorem. On the technical side, in the classical case the **NA** condition implies that the set $(K - L^0_+(\Omega, \mathcal{F}, P))$ is closed in probability. This property is paramount to prove the FTAP and the dual representation of the super-replication price. Analogously, in our collective setting we need to show the closedness in probability of the analogue set denoted by $K^{\mathcal{Y}}$. We show such closure under some specific assumptions on the set $\mathcal{Y}$ and under the assumption of $\mathbf{NCA}(\mathcal{Y})$.

The key novel feature in the CFTAP is that equivalent martingale measures have to be replaced by vectors $(Q^1, \dots, Q^N)$ of equivalent martingale measures, one for each agent and theirs corresponding market, fulfilling in addition the polarity property

$$\sum_{i=1}^{N} E_{Q^i}[Y^i] \leq 0\,, \quad \forall Y \in \mathcal{Y}. \tag{1}$$

We stress that the findings of this paper take particularly tractable, yet informative and meaningful forms in a finite probability space setup. Indeed, the fact that the agents are allowed to cooperate and the assumption of $\mathbf{NCA}(\mathcal{Y})$ has several consequences also in the pricing of contingent claims. This is particularly evident in the super-replication of N contingent claims.

Collective Super-replication. We consider the problem of N agents each super-replicating a contingent claim g^i, $i = 1, \ldots, N$, which is a $\mathcal{F}$-measurable random variable. We set $g = (g^1, \ldots, g^N)$. As an immediate extension of the classical super-replication price, we first introduce the overall super-replication price

$$\rho_+^N(g) := \inf \left\{ \sum_{i=1}^{N} m^i \mid \exists k_i \in K_i, m \in \mathbb{R}^N \text{ s.t. } m^i + k^i \geq g^i \ \forall i \right\}.$$

If we use $\rho_{i,+}(g^i)$ for the classical super-replication of the single claim g^i, we may easily recognize that

$$\rho_+^N(g) = \sum_{i=1}^{N} \rho_{i,+}(g^i). \tag{2}$$

In the spirit of Systemic Risk Measures with random allocations in [2], we introduce the Collective super-replication of the N claims $g = (g^1, \ldots, g^N)$ as

$$\rho_+^{\mathcal{Y}}(g) := \inf \left\{ \sum_{i=1}^{N} m^i \mid \exists k_i \in K_i, m \in \mathbb{R}^N, Y \in \mathcal{Y} \text{ s.t. } m^i + k^i + Y^i \geq g^i \ \forall i \right\},$$

and show that under $\mathbf{NCA}(\mathcal{Y})$ the definition is well posed. The functional $\rho_+^{\mathcal{Y}}(g)$ and $\rho_+^N(g)$ both represent the minimal total amount needed to super-replicate simultaneously all claims $(g^1, ..., g^N)$. For the Collective super-replication price $\rho_+^{\mathcal{Y}}(g)$ we allow an additional exchange among the agents, as described by $\mathcal{Y}$. As $0 \in \mathcal{Y}$, we clearly have $\rho_+^{\mathcal{Y}} \leq \rho_+^N$. Thus Collective super-replication is less expensive than classical super-replication: cooperation helps to reduce the cost of super-replication and $(\rho_+^N(g) - \rho_+^{\mathcal{Y}}(g)) \geq 0$ is the *value of cooperation* with respect to g.

Under the $\mathbf{NCA}(\mathcal{Y})$ assumption and using the closure of the set $K^{\mathcal{Y}}$, we prove the following version of the pricing-hedging duality

$$\rho_+^{\mathcal{Y}}(g) = \sup_{Q \in \mathcal{M}^{\mathcal{Y}}} \sum_{i=1}^{N} E_{Q^i}[g^i], \tag{3}$$

where $\mathcal{M}^{\mathcal{Y}}$ is the set of vectors of martingale measures satisfying the polarity condition (1). When problem (3) admits an optimum $\widehat{Q} = (\widehat{Q}^1, \ldots, \widehat{Q}^N)$, which

clearly will depend on $\mathcal{Y}$, we derive the following formula
(4)
$$\rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^{N} \inf \left\{ m \in \mathbb{R} \mid \exists k^i \in K_i, Y^i \text{ with } E_{\widehat{Q}^i}[Y^i] = 0 \text{ s.t. } m + k^i + Y^i \geq g^i \right\}.$$

Note that in (4), $(Y^1, \ldots, Y^N)$ is not required to belong to $\mathcal{Y}$, but every Y^i must have zero cost under each component of the endogenously determined pricing vector $\widehat{Q}$. This is a strong fairness property associated to the value $\rho_+^{\mathcal{Y}}(g)$. Indeed, each term in the summation on the RHS of (4) is the *individual* super-replication price of the claim g^i under the assumption that the agent i is "pricing"using the pricing functional assigned by $\widehat{Q}^i$, so that both k^i and Y^i have zero value under $\widehat{Q}^i$. Thus the interpretation of $\rho_+^{\mathcal{Y}}(g)$ is twofold:

(i) $\rho_+^{\mathcal{Y}}(g)$ is the super-replication of the N claims $(g^1, \ldots, g^N)$ when agents are allowed to exchange scenario dependent amounts under the condition that the overall exchanges $\sum_{i=1}^{N} Y^i$ is equal to 0;

(ii) $\rho_+^{\mathcal{Y}}(g)$ is the sum of the individual super-replication price of each claim g^i under the assumption that each agent is using the pricing measure $\widehat{Q}^i$.

This fairness aspect is discussed in the spirit of [3] and [1].

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Some thoughts on large financial markets under model uncertainty (discrete time)

Irene Klein

(joint work with Christa Cuchiero, Thorsten Schmidt)

All the ideas in the talk are based on joint work in progress with Christa Cuchiero and Thorsten Schmidt. Theorems 2 and 3 below currently are in the state of well-founded conjectures. The proofs still have to be made precise with all details.

We present some ideas for large financial markets in discrete time under model uncertainty. We consider a classical model of a large financial market (LFM) on a sequence of probability spaces as in Kabanov and Kramkov (1994) [3]. For each $n \geq 1$, the "small" market n in the sequence is defined as follows. Let $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t=0,1,\ldots,T_n})$ be a filtered measure space defined as in Bouchard and Nutz (2015) [1]. As there, let $\mathcal{P}^n$ be a convex set of probability measures on $(\Omega^n, \mathcal{F}^n)$. The risky assets are $d(n)$ Borel-measurable stocks $S_t^n = (S_t^{n,1}, ..., S_t^{n,d(n)}) : \Omega_t^n \to \mathbb{R}^{d(n)}$, where, for each $t = 0, 1, \ldots, T_n$ the set Ω_t^n is defined as in [1], i.e., the t-fold Cartesian product of a Polish space Ω_1^n and Ω_0^n is a singleton. Let $\mathcal{H}^n$ be the set of all predictable $\mathbb{R}^{d(n)}$-valued processes on $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t=0,1,\ldots,T_n})$. Then, a portfolio in market n with strategy $H^n \in \mathcal{H}^n$ is given by

$$X_t^n := (H^n \cdot S^n)_t = \sum_{k=1}^{d(n)} \sum_{u=1}^{t} H_u^{n,k}(S_u^{n,k} - S_{u-1}^{n,k}), \quad t = 1, \ldots T_n,$$

where $X_0^n = 0$. Now we give the definition of a LFM under model uncertainty.

Definition 1. *A large financial market under model uncertainty is a sequence of small markets n as given above with $d(n)$ risky stocks in discrete time and time horizons $T_n < \infty$.*

As usual in the theory of large financial markets, we will assume that each small market n satisfies no arbitrage (NA) where we use the robust NA condition of [1]:

Definition 2. *The market n satisfies the condition NA($\mathcal{P}^n$) if for all $H^n \in \mathcal{H}^n$*

$$(H^n \cdot S^n)_{T_n} \geq 0 \quad \mathcal{P}^n\text{-q.s.} \quad \textit{implies} \quad (H^n \cdot S^n)_{T_n} = 0 \quad \mathcal{P}^n\text{-q.s.}$$

In the above definition q.s. stands for *quasi surely*. A property is said to hold $\mathcal{P}^n$-q.s. if it holds outside a polar set A' for $\mathcal{P}^n$, that is, a set A' such that $A' \subset A$ for some $A \in \mathcal{F}^n$ with $P^n(A) = 0$ for all $P^n \in \mathcal{P}^n$.

Let us recall the connection to martingale measures from [1]. On market n we define the following set $\mathcal{Q}^n$ of probability measures. (Note that, as $\mathcal{P}^n$ is a convex set by assumption, also $\mathcal{Q}^n$ is convex).

Definition 3.

$$\mathcal{Q}^n = \{Q^n \lll \mathcal{P}^n : Q^n \text{ is a martingale measure for } S^{n,k}, k = 1, \ldots, d(n)\},$$

where $Q^n \lll \mathcal{P}^n$ means that for $Q^n \in \mathcal{Q}^n$ there exists $P^n \in \mathcal{P}^n$ such that $Q^n \ll P^n$.

As a consequence of the NA assumption on each small market n the following existence of martingale measures hold:

Theorem 1 (FTAP (Bouchard, Nutz 2015))**.** *The following are equivalent:*

(1) *NA($\mathcal{P}^n$) holds.*
(2) *For all $P^n \in \mathcal{P}^n$ there exists $Q^n \in \mathcal{Q}^n$ such that $P^n \ll Q^n$.*
(3) *$\mathcal{P}^n$ and $\mathcal{Q}^n$ have the same polar sets.*

We suggest now to define a notion of asymptotic arbitrage with model uncertainty on the large financial market. We will adapt here the concept of asymptotic arbitrage of first kind (AA1) as of [3]. Observe that this kind of asymptotic arbitrage is, if all Ω^n coincide, equivalent to the concept unbounded profit with bounded risk defined in Karatzas and Kardaras (2007) [5]. Note that this is a particularly important arbitrage property due to its connection to the growth optimal portfolio of Eckhard Platen.

Definition 4. *We say that the robust large financial market has an asymptotic arbitrage of first kind (AA1($\mathcal{P}^n$)) if the following holds: there exists a subsequence of markets n_k and a sequence of portfolios $X^k = (H^{n_k} \cdot S^{n_k})$ and a sequence of positive real numbers $\varepsilon_k \to 0$ such that*

(1) *for all $k \geq 1$ and all $t = 0, 1, \ldots, T(n_k)$, $X^k_t \geq -\varepsilon_k$ $\mathcal{P}^{n_k}$-q.s.*
(2) *there exists a sequence $(P^k)_{k\geq 1}$ with $P^k \in \mathcal{P}^{n_k}$ such that*

$$P^k(X^k_{T(n_k)} \geq \alpha) \geq \alpha,$$

for some $\alpha > 0$ and all $k \geq 1$.

We say that no asymptotic arbitrage of first kind *(NAA1($\mathcal{P}^n$)) is satisfied if the above does not exists.*

We can now suggest the following fundamental theorem of asset pricing under model uncertainty for large financial markets in discrete time. Observe that it looks very similar to Theorem 1 but now on the large financial market.

Theorem 2 (A (conjectured) FTAP under model uncertainty)**.** *NAA1($\mathcal{P}^n$) $\Leftrightarrow$ for each sequence $(P^n)_{n\geq 1}$ with $P^n \in \mathcal{P}^n$, for all n, there exists a sequence $(Q^n)_{n\geq 1}$ with $Q^n \in \mathcal{Q}^n$, for all n, such that $(P^n) \triangleleft (Q^n)$.*

Note that $(P^n) \triangleleft (Q^n)$ basically is the generalization of absolute continuity of measures to a sequences of measures and means that for each sequence $A^n \in \mathcal{F}^n$ with $Q^n(A^n) \to 0$ for $n \to \infty$ we have that $P^n(A^n) \to 0$ for $n \to \infty$.

Some ideas for the proof of Theorem 2: work in progress

Similarly as in [4] the idea is to find a generalized quantitative version of the Halmos-Savage Theorem. Here we suggest a version for convex sets of probability measures, see Theorem 3 below. With the help of this result it is quite standard to get Theorem 2 by using the superreplication of [1] which fits perfectly to the current setting. On the way we use the following characterization of NAA1 under model uncertainty which we can prove with all details.

Lemma 1. *NAA1($\mathcal{P}^n$) $\Leftrightarrow$ $\forall\varepsilon > 0$ $\exists\delta > 0$ such that $\forall n \geq 1$ and $\forall A^n \in \mathcal{F}^n$ such that $\exists P^n \in \mathcal{P}^n$ with $P^n(A^n) \geq \varepsilon$ there $\exists Q^n \in \mathcal{Q}^n$ with $Q^n(A^n) \geq \delta$.*

So, if our conjectured Theorem 3 below can be proved in the given form, the proof of Theorem 2 is done. Let us now formulate the conjectured Theorem 3, i.e., the quantitative Halmos–Savage–type result for convex sets of probability measures we are aiming at.

Theorem 3 (Conjecture: Quantitative Halmos-Savage Theorem for convex sets of probability measures)**.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be a convex sets of probability measures on $(\Omega, \mathcal{F})$ such that $\mathcal{Q} \lll \mathcal{P}$. For fixed $\varepsilon > 0$ and $\delta > 0$ the following statement is true: Assume that for each $A \in \mathcal{F}$ such that there exists $P \in \mathcal{P}$ with $P(A) \geq \varepsilon$*

there exists $Q \in \mathcal{Q}$ such that $Q(A) \geq \delta$. Then for each $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}$ such that for each $A \in \mathcal{F}$ with $P(A) \geq 2\varepsilon$ we have that $Q(A) \geq \frac{\varepsilon\delta}{2}$.

Note that $\mathcal{Q} \lll \mathcal{P}$ in the statement of the theorem means that for every $Q \in \mathcal{Q}$ there exists $P \in \mathcal{P}$ such that $Q \ll P$.

Ideas for the Proof of Theorem 3. As a technical tool for the proof we will define a **convex** set $D^{\varepsilon,P}$: fix $P \in \mathcal{P}$ and $\varepsilon > 0$. Define

$$D^{\varepsilon,P} = \{h \in \bigcap_{P' \in \mathcal{P}} L^\infty(P') : 0 \leq h \leq 1 \quad \mathcal{P}-\text{q.s. and } E_P[h] \geq 2\varepsilon\}.$$

The assumption of Theorem 3 will lead to the following inequality:

$$\inf_{h \in D^{\varepsilon,P}} \sup_{Q \in \mathcal{Q}} E_Q[h] \geq \varepsilon\delta.$$

Now by finding appropriate dual locally convex topological vector spaces (E, E') and using a general Banach-Alaoglu-Bourbaki Theorem we think to be able to show that the convex set $D^{\varepsilon,P} \subset E'$ is $\sigma(E', E)$-compact. Then we aim at applying a Minmax Theorem as in Sion (1958) [6] to the given bilinear functional with a continuity property with respect to the chosen topology to get that:

$$\sup_{Q \in \mathcal{Q}} \inf_{h \in D^{\varepsilon,P}} E_Q[h] \geq \varepsilon\delta.$$

With this the statement of Theorem 3 follows. □

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Fundamental theorem of asset pricing with acceptable risk in markets with frictions

Cosimo Munari

We revisited the problem of market-consistent valuation of insurance liabilities from a financial economics perspective. The challenge is to define a range of prices at which an insurance company that has access to an outstanding financial market and is subject to a regulatory capital adequacy regime should be prepared to buy/sell a contract outside of the financial market. Our proposal was to call a price *market consistent with acceptable risk* (MCP) if there exists no portfolio of traded assets that can be bought/sold at a lower/higher price in the market and that super/sub-replicates the contract's payoff at an acceptable level of risk as prescribed by the regulatory solvency test. In the spirit of classical arbitrage pricing theory, the main goal was to provide a characterization of MCPs by way of special stochastic discount factors, called (strictly) consistent price deflators, that have to be chosen to respect market frictions as well as to be consistent with the regulator's solvency test. The presentation unfolded as follows:

- Formalization of the financial market and the capital adequacy test.
- Definition of MCPs.
- Primal characterization of MCPs based on super/sub-replication prices.
- Definition of (scalable) good deals as generalizations of arbitrage opportunities.
- Definition of (strictly) consistent price deflators as generalizations of stochastic discount factors.
- Extension of the fundamental theorem of asset pricing: The market is free of scalable good deals if and only if there exists a strictly consistent price deflator.
- Dual characterization of MCPs based on strictly consistent price deflators.
- Examples of price deflators that are strictly consistent with respect to Expected Shortfall and expectiles.

A number of future challenges was mentioned at the end, including at least:

- Extension to multi-period models.
- Extension to settings without a dominating probability.
- Characterization of optimal hedging portfolios/strategies with acceptable risk.
- Comparison with market-consistent valuation rules used in practice (best estimate of insurance liabilities plus risk margin).

We believe that the last point is especially pressing to bridge the gap between theory and practice and should ideally contribute to the ongoing discussion on the broad topic "valuation" in insurance regulation.

This work is related to the literature on good deal pricing. The goal there is to restrict the interval of arbitrage-free prices by discarding some "extreme" stochastic discount factors and the main problem is that of identifying, by way of an inverted fundamental theorem of asset pricing, the corresponding pricing bounds, the so-called good deal bounds. We refer, e.g., to:

- Arai, T., & Fukasawa, M. (2014). Convex risk measures for good deal bounds. *Math Financ*, 24(3), 464-484.
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Apart from the general motivation, the key difference with our results is that the bulk of this literature focuses on frictionless markets and the only versions of the fundamental theorem of asset pricing involve, in our language, only consistent, instead of strictly consistent, price deflators. In particular, these versions cannot be used to characterize MCPs in dual terms.

Benchmark-Neutral Pricing for Entropy-Maximizing Dynamics

Eckhard Platen

The paper applies the benchmark approach to the modeling, pricing, and hedging of long-term contingent claims involving the growth optimal portfolio (GOP) of a large stock market. It employs the entropy-maximizing dynamics of the GOP of the stocks for modeling. Instead of risk-neutral or real-world pricing, the paper proposes the method of benchmark-neutral pricing, where it uses the GOP of the stocks as numéraire and the respective new benchmark-neutral pricing measure for taking conditional expectations. Under the entropy-maximizing dynamics of the GOP for stocks, the benchmark-neutral pricing measure turns out to be an equivalent probability measure. The risk-neutral pricing measure does not represent a probability measure. Consequently, benchmark-neutral pricing provides the minimal possible prices and hedges, whereas risk-neutral pricing becomes more expensive than necessary. The implementation of benchmark-neutral pricing and hedging is demonstrated. It is shown that the minimal possible prices, which benchmark-neutral pricing provides, can be significantly lower for long-term contingent claims than the respective risk-neutral ones.

The paper makes the following three key assumptions:
A1: The GOP exists.
A2: The normalized GOP forms a stationary scalar diffusion and its volatility is a function of its value.
A3: The market maximizes the relative entropy of the stationary density of the normalized GOP.

The first assumption is about the existence of the GOP and represents an intuitive and easily verifiable *no-arbitrage condition* because [4] have shown that the existence of the GOP is equivalent to their *No Unbounded Profit with Bounded*

Risk (NUPBR) condition. This no-arbitrage condition is weaker than the NFLVR condition of [1].

The maximization of the relative entropy is known to be equivalent to the minimization of the information rate; see [5]. Consequently, the resulting entropy-maximizing market dynamics does not leave any room for exploitable information and charaterizes the undisturbed market dynamics.

Conservation laws simplify in many areas the undisturbed dynamics of complex dynamical systems. According to [6], the maximization of a Lagrangian in the presence of Lie-group symmetries leads to the identification of conservation laws for the resulting model dynamics. The entropy-maximizing stationary dynamics of the normalized GOP turn out to have Lie-group symmetries and emerge as those of a time-transformed square root process, with conserved dimension four, and conserved logarithmic mean zero.

The modeling is performed on a filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$, satisfying the usual conditions. We consider $d+1$, $d \in \{1,2,...\}$ adapted, nonnegative assets, denoted by $S_t^0, S_t^1, ..., S_t^d$, which we call the d *primary security accounts*, where all dividends or interests are reinvested. We interpret the d primary security accounts $S_t^1, ..., S_t^d$ as stocks, which are here denominated in units of the savings account $S_t^0 = 1$. Furthermore, we assume for the investment universe given by the d stocks that a continuous *growth optimal portfolio* (GOP) S_t^*, the *stock GOP*, exists. Every primary security account $\tilde{S}_t^j = \frac{S_t^j}{S_t^*}$, $j \in \{1,...,d\}$, when denominated in the stock GOP, forms a right-continuous, integrable $(P, \underline{\mathcal{F}})$-local martingale. The stochastic differential equation (SDE) for the continuous stock GOP S_t^* is assumed to be of the form

$$\frac{dS_t^*}{S_t^*} = \lambda_t^* dt + \theta_t(\theta_t dt + dW_t)$$

for $t \in [0,\infty)$ with $S_0^* > 0$. We extend the above market formed by the d stocks by adding the savings account S_t^0 as an additional primary security account. In line with Theorem 7.1 in [3], the GOP S_t^{**} of the extended market satisfies the SDE

$$\frac{dS_t^{**}}{S_t^{**}} = \frac{\lambda_t^* + (\theta_t)^2}{\theta_t}\left(\frac{\lambda_t^* + (\theta_t)^2}{\theta_t} dt + dW_t\right)$$

for $t \in [0,\infty)$ and $S_0^{**} = 1$. For a replicable contingent claim $H_T \geq 0$ with maturity T the *real world pricing formula*

$$H_t = S_t^{**} \mathbf{E}^P \left(\frac{H_T}{S_T^{**}} | \mathcal{F}_t\right)$$

describes its unique fair price H_t at time $t \in [0,T]$, see [2]. Other pricing rules are possible but do never provide lower prices. The numéraire for real-world pricing is the GOP S_t^{**} of the extended market, which is, in reality, a highly leveraged portfolio and difficult to construct. Therefore, a change of numéraire is suggested that uses the strictly positive stock GOP S_t^* as numéraire. The Radon-Nikodym derivative

$$\Lambda_{S^*}(t) = \frac{dQ_{S^*}}{dP}|_{\mathcal{F}_t} = \frac{\frac{S_t^*}{S_t^{**}}}{\frac{S_0^*}{S_0^{**}}}$$

characterizes the respective *benchmark-neutral pricing measure* Q_{S^*}. For the entropy-maximizing dynamics, the Radon-Nikodym derivative $\Lambda_{S^*}(t)$ is shown to be a true $(P, \underline{\mathcal{F}})$-martingale and Q_{S^*} to be an equivalent probability measure. We call the new pricing method *benchmark-neutral pricing*, which uses the stock GOP S_t^* as numéraire and the benchmark-neutral pricing measure Q_{S^*} as pricing measure. One obtains directly the *benchmark-neutral pricing formula*

$$H_t = S_t^* \mathbf{E}^{Q_{S^*}}(\frac{H_T}{S_T^*}|\mathcal{F}_t)$$

for $t \in [0, T]$. The process $W^0 = \{W_t^0, t \in [0, \infty)\}$, satisfying the SDE

$$dW_t^0 = \sigma_{S^*}(t)dt + dW_t$$

for $t \in [0, \infty)$ with $W_0^0 = 0$, is under Q_{S^*} a Brownian motion. This result is of practical importance because it allows one to use the stock GOP as numéraire for pricing and hedging. Under benchmark-neutral pricing there is no need to estimate λ_t^* because this drift parameter becomes absorbed in the measure transformation.

Hedging under the benchmark-neutral pricing measure can be performed analogously as shown in [2] under the real world probability measure P, and can also be extended for non-replicable contingent claims.

When using a total return stock index as proxy for the stock GOP, it has been shown for zero-coupon bonds that long-term hedging over many decades can be accurately performed with very small hedge errors. These findings give access to new production methods for life insurance, pension, climate, and other long-term contracts that use the stock index as numéraire.

Since the risk-neutral pricing measure turns out to be not an equivalent probability measure under the entropy-maximizing dynamics, formally applied risk-neutral prices and hedges can become considerably more expensive than the minimal possible ones, which can be obtained via benchmark-neutral pricing and hedging.

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Markovian randomized equilibria for general Markovian Dynkin games in discrete time

Berenice Anne Neumann
(joint work with Sören Christensen, Kristoffer Lindensjö)

In discrete time Dynkin games each player $i \in \{1,2\}$ chooses a stopping time τ_i in order to maximize her expected reward

$$\mathbb{E}\left[F^i_{\tau_i}\mathbb{I}_{\{\tau_i<\tau_j\}} + G^i_{\tau_j}\mathbb{I}_{\{\tau_j<\tau_i\}} + H^i_{\tau_i}\mathbb{I}_{\{\tau_i=\tau_j\}}\right],$$

where $j = 3-i$ and F^i, G^i, H^i are integrable discrete time processes (with a suitable interpretation of H^i_n for $n=\infty$). In the case that $F^1 \leq H^1 \leq G^1$ and $F^2 \leq H^2 \leq G^2$ these games are well-understood. Under suitable integrability assumptions existence and characterization of Nash equilibria have been established [2, 3, 4]. However, the situation becomes more involved if we drop the assumption $F^1 \leq H^1 \leq G^1$ and $F^2 \leq H^2 \leq G^2$. First of all it is now necessary to consider mixed strategies [5]. Moreover, also using this class of strategies there are simple examples without a Nash equilibrium [6]. In general, only the existence of ϵ-equilibria can be established [7, 8].

In this talk we restricted our attention to discrete time Markovian Dynkin games. In this setting $(X_n)_{n\in\mathbb{N}}$ is a homogeneous Markov process with state space E and the reward of player i reads

$$\mathbb{E}_x\left[\alpha^{\tau_i} f_i(X_{\tau_i})\mathbb{I}_{\{\tau_i<\tau_j\}} + \alpha^{\tau_j} g_i(X_{\tau_j})\mathbb{I}_{\{\tau_j<\tau_i\}} + \alpha^{\tau_i} h_i(X_{\tau_i})\mathbb{I}_{\{\tau_i=\tau_j<\infty\}}\right],$$

where $j = 3-i$, α is the discount factor satisfying $0<\alpha<1$ and $f_i, g_i, h_i : E \to \mathbb{R}, i = 1,2,$ are measurable functions satisfying an integrability assumption. In the talk we motivated that Markovian randomized stopping times are a natural class of randomized stopping times for these games. These Markovian randomized stopping times are stopping times, where at each time step n the player stops with a certain probability that only depends on the current state X_n of the underlying Markov process. Relying on this type of strategies we provide an explicit characterization and verification result of Wald-Bellman type. This result then allows us to construct equilibria in certain classes of zero-sum and symmetric games and to obtain necessary and sufficient conditions for the non-existence of pure strategy equilibria in zero-sum games. Moreover, we establish the existence of an equilibrium in Markovian randomized stopping times for general games whenever the state space of the underlying Markov chain is countable.

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Stretched Brownian motion: Analysis of a fixed-point scheme

Gudmund Pammer
(joint work with Beatrice Acciaio, Antonio Marini)

A central challenge in the theory of mathematical finance is the pricing of financial derivatives. In the classical theory this question is closely tied to the notion of martingale measures: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis and $S = (S_t)_{t\geq 0}$ be the $(\mathcal{F}_t)_{t\geq 0}$-adapted asset-price process. Under the no-arbitrage assumption, that is, we exclude the possibility of making profit without risk, the task of pricing a financial derivative Φ boils down to finding an equivalent martingale measure $\mathbb{Q}$. An equivalent martingale measure is simply a measure equivalent to $\mathbb{P}$ under which S is a martingale.

However, the true dynamics of the market, including the stochastic basis and the asset-price process, are unknown. Rather than directly specifying a model, we can extract information on the pricing measure $\mathbb{Q}$ from market data. The cornerstone of this approach is the famous observation by Breeden–Litzenberger [3], which culminates in the fitting problem (FP) in mathematical finance: The task is to find a stochastic basis supporting a martingale $S = (S_t)_{t\geq 0}$ that adheres to prescribed marginal constraints $S_t \sim \mu_t$ for $t \in I$ derived from market observations. Here $(\mu_t)_{t\in I}$ are one-dimensional marginals that are derived from market observations at a given time index set $I \subseteq \mathbb{R}_+$. Building on the Bass solution to the Skorokhod embedding problem and optimal transport, Backhoff, Beiglbock, Huesmann, and Kallblad [1] propose a solution to (FP) for the two-marginal problem, i.e., with constraints on two specific time-points $I = \{0, 1\}$. The stretched Brownian motion M^* is the unique-in-law optimizer of

$$\sup\{\mathbb{E}[M_1 \cdot B_1] : M \text{ solves (FP)}\},$$

where B is some Brownian motion. Notably rich in structure, this process is an Ito diffusion and a continuous, strong Markov martingale that emulates the behaviour of Brownian motion locally.

Following a similar approach, Conze and Henry-Labordere [2] recently introduced a novel alternative to the local volatility model. This model, rooted in an extension of the Bass construction, is efficiently computable through a fixed-point scheme. The goal is to find a fixed point of the map

$$\mathcal{A}\colon \text{CDF} \to \text{CDF}\colon F \mapsto F_{\mu_0} \circ \left(\gamma_1 * F_{\mu_1}^{-1}\left(\gamma_1 * F\right)\right),$$

where F is a cumulative distribution function (CDF), F_μ denotes the CDF of μ and γ_1 a normal distribution with variance 1. When α is a distribution whose CDF $\hat{F}$ is a fixed-point, then the process $\hat{M} = (\hat{M})_{t\geq 0}$, determined by

$$\hat{M}_t := \mathbb{E}[T(B_1)|B_t] = (\gamma_{1-t} * F_{\mu_1}^{-1}(\gamma_1 * \hat{F}))(B_t),$$

solves (FP).

In this work, we explore the intricate relationship between the fixed-point scheme and the stretched Brownian motion, revealing that in law $\hat{M} = M^*$. Furthermore, we give a precise criterion for the existence of a fixed-point and demonstrate its convergence. This study unveils that solving the fixed-point equation provides a highly efficient alternative to computing stretched Brownian motion. In particular, when μ_0 is concentrated on finitely many points, the fixed-point scheme exhibits linear convergence.

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On random reinsurance contracts and optimal transport

Brandon Garcia Flores

(joint work with Beatrice Acciaio, Hansjörg Albrecher)

Building upon the concept of random reinsurance treaties from [3] and [4], we establish a general framework for the study of optimal reinsurance problems. Traditionally, an optimal reinsurance problem consists in minimizing a risk measure $\mathcal{P}$ defined on a set of functions. The minimization is subject to the solution being in a set of constraints $\mathcal{S}$, which usually relates to demands set by either the cedent or the reinsurer. In this generality, one can hardly show the existence of any contract and is therefore restricted to deal with specific instances of the problem. The introduction of random reinsurance treaties is then reminiscent to the Monge-Kantorovich formulation of optimal transport (OT) which is used as a way of convexifying the problem, thus ensuring the existence of optimal solutions.

A random reinsurance treaty η is a probability measure in $\mathbb{R}^n \times \mathbb{R}^n$ supported in the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid 0 \leq y_i \leq x_i, i = 1, \ldots, n\}$ and such that $\pi_{1\#}\eta = \mu$, where μ is the distribution of the original claims. Here, $\pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the projection in the first coordinate and $\pi_{1\#}$ denotes the push-forward map induced by π_1. Denoting by X the original portfolio of claims, contracts of this kind can be simply seen as the joint distribution of X and the final risk exposure of the reinsurer, which now is not necessarily determined by X in a functional way. By means of standard OT methods, one can easily prove the following:

Theorem 1. *Let $\mathcal{M}$ denote the space of random reinsurance treaties endowed with the weak topology induced by bounded continuous functions. If $\mathcal{P} : \mathcal{M} \to \overline{\mathbb{R}}$ is lower semi-continuous and $\mathcal{S}$ is closed, then an optimal reinsurance contract η^* exists.*

While existence is guaranteed under relatively mild assumptions, one is then faced with the identification of optimal contracts. The rest of our work addresses this matter by using the idea of (local) linearization, a concept widely used in the are of optimization.

Assuming that the set of constraints is given as

$$\mathcal{S} = \{\eta \in \mathcal{M} \mid \mathcal{G}(\eta) \leq 0\}$$

for a lower semi-continuous function $\mathcal{G} = (g_1, \ldots, g_m) : \mathcal{M} \to \mathbb{R}^m$, one of the main results of our work is the following:

Theorem 2. *Let η^* be an optimal reinsurance contract and assume there exist continuous functions $p_{\eta^*} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ and $g_{\eta^*} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^m$ such that*

$$\lim_{t \to 0^+} \frac{\mathcal{P}((1-t)\eta^* + t\vartheta) - \mathcal{P}(\eta^*)}{t} = \int_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} p_{\eta^*}(x,y)(\vartheta - \eta^*)(dx, dy)$$

and

$$\lim_{t \to 0^+} \frac{\mathcal{G}((1-t)\eta^* + t\vartheta) - \mathcal{G}(\eta^*)}{t} = \int_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} g_{\eta^*}(x,y)(\vartheta - \eta^*)(dx, dy)$$

for every $\vartheta \in \mathcal{M}$. Moreover, assume that the partial minimization function,

$$m(x) = \inf_{y \in [0,x]} r p_{\eta^*}(x,y) + \lambda \cdot g_{\eta^*}(x,y)$$

is measurable for every $r \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}^m_+$. Then, there exist $r^* \in \mathbb{R}_+$ and $\lambda^* \in \mathbb{R}^m_+$ such that $\lambda^* \cdot \mathcal{G}(\eta^*) = 0$ and

$$\eta^* \left(\{(x,y) \in \mathcal{A}_R \mid y \in \operatorname{argmin}_{t \in [0,x]} r^* p_{\eta^*}(x,t) + \lambda^* \cdot g_{\eta^*}(x,t)\} \right) = 1.$$

If $\mathcal{G}$ is constant or there exists $\vartheta \in \mathcal{M}$ such that

$$\mathcal{G}(\eta^*) + \int_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} g_{\eta^*}(x,y)(\vartheta - \eta^*)(dx, dy) < 0,$$

then r^* can be taken to be equal to 1.

This theorem thus identifies the support of optimal reinsurance contracts relative to the functions p_{η^*} and g_{η^*}, and λ^*, all of which depend on η^*. However, in several common applications, p_{η^*} and g_{η^*} depend on the optimal contract through a (finite) set of parameters. Together with λ^*, one can then treat this set of parameters as variables and optimize over them, thus reducing the problem to a finite dimensional optimization problem, for which several techniques can be used.

One example that prominently falls into this category is when the risk measure is given by

$$\mathcal{P}(\eta) = f\left(\int_{\mathbb{R}_+^n} p_1(x,y)\,\eta(dx,dy),\ldots,\int_{\mathbb{R}_+^n} p_\ell(x,y)\,\eta(dx,dy)\right)$$

subject to the constraints $\mathcal{G} = (g_1,\ldots,g_m)$ given by

$$g_i(\eta) = h_i\left(\int_{\mathbb{R}_+^n} q_{i,1}(x,y)\,\eta(dx,dy),\ldots,\int_{\mathbb{R}_+^n} q_{i,\ell_i}(x,y)\,\eta(dx,dy)\right)$$

where all the p_i's and $q_{i,j}$'s are continuous functions and f and the h_i's are differentiable. This type of risk measure includes, but is not limited to the cases where one would like to minimize the expectation, variance, skewness, coefficient of variation, etc. of the total retained amount subject on constraints depending on similar measures. Several of the optimal reinsurance problems that fall under this umbrella are treated in [1] and [5]. Adapting for non-continuities and differentiability, the techniques can be slightly generalized to deal with distortion risk measures, such as those dealt with in [2], which shows the generality of our approach.

Throughout the previous discussion, it was imperative that the set $\mathcal{S}$ was described by a finite set of inequalities. The final portion of our study then relaxes the requirement for $\mathcal{S}$ to be finitely representable by inequalities. Still inspired by the idea of local linearization, we make the following assumptions:

(1) If $\eta^* \in \mathcal{S}$ is an optimal reinsurance contract, then for every $\eta \in \mathcal{S}$ and $0 \le t \le 1$, we have

$$\mathcal{P}(\eta^*) \le \mathcal{P}((1-t)\eta^* + t\eta).$$

(2) For every $\eta \in \mathcal{S}$, $d\mathcal{P}(\eta;\cdot)$ exists for every direction in $\mathcal{S}-\eta$ and is given as an integral operator, i.e., there exists a measurable function $p_\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for every $\vartheta \in \mathcal{S}$,

$$d\mathcal{P}(\eta;\vartheta-\eta) = \int p_\eta(x,y)(\vartheta-\eta)(dx,dy)$$

These two assumptions jointly imply that

$$\int p_{\eta^*}(x,y)\,\eta^*(dx,dy) = \min_{\eta\in\mathcal{S}} \int p_{\eta^*}(x,y)\,\eta(dx,dy).$$

Letting q_{η^*} denote the function on $\mathbb{R}^n \times \mathbb{R}^n$ such that $q_{\eta^*}(x,y) = \infty$ on the complement $\mathcal{A}_R$ and otherwise being equal to p_{η^*}, the previous equation can be stated as

$$\int q_{\eta^*}(x,y)\,\eta^*(dx,dy) = \min_{\nu\in\pi_2(\mathcal{S})} \mathcal{C}(\mu,\nu), \tag{1}$$

where

$$\mathcal{C}(\mu,\nu) = \min_{\eta\in\Pi(\mu,\nu)\cap\mathcal{S}} \int q_{\eta^*}(x,y)\,\eta(dx,dy), \tag{2}$$

and $\Pi(\mu, \nu)$ is the set of couplings between μ and ν. Equations (1) and (2) mean that the optimal contract satisfies a double minimization property, where the inner minimum is a constrained optimal transport problem. We conclude our work by showing how, by taking a point of view inspired by this OT approach, we are enabled to use tools from the area to provide novel solutions to old and new optimal reinsurance problems.

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Adapted Wasserstein distance between the laws of SDEs

Sigrid Källblad

(joint work with Julio Backhoff-Veraguas, Ben Robinson)

In applications where filtrations and the flow of information play a key role, the concepts of weak convergence and Wasserstein distances have proven to be insufficient for specifying convergence and distances between stochastic processes. For instance, neither usual stochastic optimisation problems (such as optimal stopping or utility maximisation) nor Doob–Meyer decompositions behave continuously with respect to these topologies. Over the last decades, several approaches have been proposed to overcome these shortcomings; we focus here on one such notion, namely the so-called adapted Wasserstein distance.

We refer to [1, 2, 3, 6] for more on the motivation and history of adapted distances and the closely related concepts of causal and bi-causal couplings.

Specifically, in this talk we study the adapted Wasserstein distance between the laws of solutions of one-dimensional Markovian SDEs when the space of continuous functions is equipped with the L^p-metric. We address this problem by embedding it into a class of bi-causal optimal transport problems featuring a specific type of cost function. Imposing fairly general conditions on the (Markovian) coefficients of the SDEs, we will discuss methods and results which can be summarised as follows:

(i) characterisation of the coupling attaining the infimum for a class of bi-causal optimal transport problems including the adapted Wasserstein distance;

(ii) a time-discretisation method allowing derivation of most continuous-time statements from their more elementary discrete-time counterparts;

(iii) a stability result for optimisers to some bi-causal optimal transport problems;

(iv) a result stating that the topology induced by the adapted Wasserstein distance coincides with several topologies (including the weak topology) when restricting to SDEs whose coefficients belong to an equicontinuous family;

(v) examples illustrating what to expect for path-dependent SDEs and in higher dimensions.

At a conceptual level, we connect two hitherto unrelated objects: the *synchronous coupling* of SDEs, which is the coupling arising when letting a single Wiener process drive two SDEs; and the *Knothe–Rosenblatt* rearrangement, which is a celebrated discrete-time adapted coupling that preserves the lexicographical order. In particular, we provide an optimality property for the Knothe–Rosenblatt rearrangement which extends earlier results of [4, 7]. We then make use of this result to argue that in a certain sense, the synchronous coupling is the continuous-time counterpart of the Knothe–Rosenblatt rearrangement.

Concerning the contributions (i) and (iv) above, similar statements have been made in the pioneering work of Bion-Nadal and Talay [5] for the problem of optimally controlling the correlation between SDEs with smooth coefficients. We here show that the bi-causal optimal transport problem, for general cost functions and between laws of possibly path-dependent SDEs, admits such a control reformulation. A posteriori, it is thus clear that (i) and (iv) were established for the adapted Wasserstein distance and smooth coefficients already in [5]. Our results in this direction can be understood as using probabilistic methods to generalise their findings to more general cost functions and SDEs.

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Shrinkage of semimartingales

Monique Jeanblanc
(joint work with Tomasz R. Bielecki, Jacek Jakubowski, Pavel V. Gapeev and Mariusz Niewkeglowski)

In this talk we study projections of semi-martingales on various filtrations, under specific assumptions. More precisely, $\mathbb{F}$ and $\mathbb{G}$ being two filtrations with $\mathbb{F} \subset \mathbb{G}$, and $Y^{\mathbb{G}}$ being a $\mathbb{G}$-semimartingale, we define the optional projection of $Y^{\mathbb{G}}$ as $Y_t = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t], \forall t \geq 0$ which is an $\mathbb{F}$-semimartingale under some conditions (see [7]) and we find some relationships between the decomposition of $Y^{\mathbb{G}}$ and Y.

1. A simple case

Let $\vartheta^{\mathbb{G}}$ be a $\mathbb{G}$-adapted bounded process. It is well known that

$$\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds|\mathcal{F}_t] = M_t + \int_0^t \vartheta_s ds$$

where M is an $\mathbb{F}$-martingale and $\vartheta_s = \mathbb{E}[\vartheta_s^{\mathbb{G}}|\mathcal{F}_s]$. (See, e.g., [5, lemma 8.3])

The goal is to identify M in terms of the $\vartheta^{\mathbb{G}}$ and one specific martingale which satisfy predictable representation property (PRP) on $\mathbb{F}$.

Assume for example that $\mathbb{F}$ is a Brownian filtration generated by W. In that case PRP holds, i.e., for any $\mathbb{F}$-martingale M there exists an $\mathbb{F}$-predictable process ψ such that $M_t = M_0 + \int_0^t \psi_s dW_s$.

For any $\mathbb{F}$-adapted bounded process φ one has, using tower property in the first equality

$$\begin{aligned}
\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \int_0^t \varphi_s dW_s] &= \mathbb{E}[\,\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds\,|\mathcal{F}_t] \int_0^t \varphi_s dW_s] \\
&= \mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] + \mathbb{E}[M_t \int_0^t \varphi_s dW_s]
\end{aligned}$$

hence

$$\begin{aligned}
&\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \int_0^t \varphi_s dW_s] - \mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] \\
= \; &\mathbb{E}[M_t \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \psi_s \varphi_s ds]
\end{aligned}$$

To proceed, we need to apply integration by parts to the product of $\mathbb{G}$-semimartingales $\int_0^\cdot \vartheta_s^{\mathbb{G}} ds$ and $\int_0^\cdot \varphi_s dW_s$ (if $\int_0^\cdot \varphi_s dW_s$ is a $\mathbb{G}$-semimartingale!) which leads to

$$\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) dW_s]$$

We now assume that there exists a $\mathbb{G}$-adapted process $\alpha^{\mathbb{G}}$ such that W is a $\mathbb{G}$-semimartingale with decomposition

$$W_t = W_t^{\mathbb{G}} + \int_0^t \alpha_s^{\mathbb{G}} ds$$

where $W^{\mathbb{G}}$ is a $\mathbb{G}$-Brownian motion, then

$$\mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) dW_s] = \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) \alpha_s^{\mathbb{G}} ds]$$

(See some conditions in [1, Ch 4 and 5]).

Using tower property in the second equality

$$\begin{aligned} \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \int_0^t \varphi_s dW_s] &= \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) \alpha_s^{\mathbb{G}} ds] \\ &= \mathbb{E}[\int_0^t \vartheta_s \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) \alpha_s^{\mathbb{G}} ds] \end{aligned}$$

we get (one has to check carefully that all local martingales that appear are true martingales) noting that

$$\mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \vartheta_s \left(\int_0^s \varphi_u dW_u\right) ds]$$

$$\mathbb{E}[\int_0^t \psi_s \varphi_s ds] = \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) \alpha_s^{\mathbb{G}} ds]$$

and this being true for any φ, this yields

$$\psi_s = \mathbb{E}[\alpha_s \int_0^s \vartheta_u^{\mathbb{G}} du | \mathcal{F}_s] \,.$$

Remarks: If $\vartheta^{\mathbb{G}}$ is $\mathbb{F}$- adapted $M = 0$ and $\vartheta^{\mathbb{G}} = \vartheta$. This can be recover from

$$\psi_s = \mathbb{E}[\alpha_s^{\mathbb{G}} \int_0^s \vartheta_u^{\mathbb{G}} du | \mathcal{F}_s] = \int_0^s \vartheta_u du \, \mathbb{E}[\alpha_s^{\mathbb{G}} | \mathcal{F}_s] = 0$$

since $\mathbb{E}[\alpha_s^{\mathbb{G}} | \mathcal{F}_s] = 0$.

This can be easily extended to the case where $\mathbb{F}$ has a process (may be multi-dimensional or having jumps) which enjoy PRP for example if $\mathbb{F}$ is generated by a pair $(W, \widetilde{\mu})$ where W is a Brownian motion independent of a compensated marked point process $\widetilde{\mu}$.

2. Martingales

Let $\mathbb{F}$ be a filtration, M an $\mathbb{F}$-martingale (possibly multidimensional, or with jumps) enjoying PRP.

Let $\mathbb{G}$ be a filtration larger than $\mathbb{F}$ which enjoy PRP with respect to $M^{\mathbb{G}}$ where $M^{\mathbb{G}}$ is a (possibly multidimensional or with jumps) $\mathbb{G}$-martingale such that any $\mathbb{G}$-martingale $Y^{\mathbb{G}}$ has a decomposition as

$$Y_t^{\mathbb{G}} = \int_0^t \psi_s^{\mathbb{G}} dM_s^{\mathbb{G}}$$

We note with a superscript $\mathbb{G}$ processes that are $\mathbb{G}$-adapted as $Y^{\mathbb{G}}$.

Our goal is to find the decomposition of the $\mathbb{F}$-martingale $Y_t = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = \int_0^t \psi_s dM_s$.

The r.v. Y_t is characterized by

$$\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dM_s] = \mathbb{E}[Y_t \int_0^t \varphi_s dM_s]$$

for any $\varphi \in \mathbb{F}$.

In the one hand, using tower property

$$\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dM_s] = \mathbb{E}[Y_t \int_0^t \varphi_s dM_s] = \mathbb{E}[\int_0^t \psi_s \varphi_s d\langle M\rangle_s]\,.$$

To compute using integration by parts $\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dM_s]$, we need to assume that M is a $\mathbb{G}$-semimartingale with decomposition

$$M_t = \int_0^t \beta_s^{\mathbb{G}} dM_s^{\mathbb{G}} + \int_0^t \alpha_s^{\mathbb{G}} d\langle M^{\mathbb{G}}\rangle_s\,.$$

This yields

$$\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dM_s] = \mathbb{E}[\int_0^t Y_s^{\mathbb{G}} \varphi_s dM_s] + \mathbb{E}[\int_0^t \Big(\int_0^s \varphi_u dM_u\Big) dY_s^{\mathbb{G}}] + \mathbb{E}[\langle Y^{\mathbb{G}}, \int_0^\cdot \varphi_s dM_s\rangle_t]$$

where in the first integral in the righthand side M is a $\mathbb{G}$-semimartingale as well as in the bracket and the second term is null. We compute the two remaining parts using that the local martingales are true martingales, this can proved by means of Burkolder Davis Gundy.

$$\mathbb{E}[\int_0^t Y_s^{\mathbb{G}} \varphi_s dM_s] = \mathbb{E}[\int_0^t Y_s^{\mathbb{G}} \varphi_s \alpha_s^{\mathbb{G}} d\langle M^{\mathbb{G}}\rangle_s]$$

and

$$\mathbb{E}[\langle Y^{\mathbb{G}}\,, \int_0^\cdot \varphi_s dM_s\rangle_t] = \mathbb{E}[\int_0^t \varphi_s \psi_s^{\mathbb{G}} \beta_s^{\mathbb{G}} d\langle M^{\mathbb{G}}\rangle_s]$$

$$\mathbb{E}[\int_0^t \psi_s \varphi_s d\langle M\rangle_s] = \mathbb{E}[\int_0^t \varphi_s \big(Y_s^{\mathbb{G}} \alpha_s^{\mathbb{G}} + \psi_s^{\mathbb{G}} \beta_s^{\mathbb{G}}\big) d\langle M^{\mathbb{G}}\rangle_s]$$

hence

$$\psi_s = \frac{\mathbb{E}[\big(Y_s^{\mathbb{G}} \alpha_s^{\mathbb{G}} + \psi_s^{\mathbb{G}} \beta_s^{\mathbb{G}}\big) d\langle M^{\mathbb{G}}\rangle_s|\mathcal{F}_s]}{d\langle M\rangle_s}$$

and, since $d\langle M\rangle = (\beta^{\mathbb{G}})^2 d\langle M^{\mathbb{G}}\rangle$

$$\psi_s = \mathbb{E}[\frac{Y_s^{\mathbb{G}}\alpha_s^{\mathbb{G}} + \psi_s^{\mathbb{G}}\beta_s^{\mathbb{G}}}{(\beta^{\mathbb{G}})_s^2}|\mathcal{F}_s]$$

See [3, 4] for details.

3. Semimartingales

It is well known, from [7], that if X is a $\mathbb{G}$-semimartingale and is $\mathbb{F}$-adapted where $\mathbb{F} \subset \mathbb{G}$, then X is an $\mathbb{F}$-semimartingale.

Note that if the $\mathbb{G}$-special semimartingale decomposes as $X = M + A$ and is $\mathbb{F}$-adapted, it may happen that M and A are not $\mathbb{F}$-adapted (see [7] or [2]). However, in our case X can be decomposed in both filtrations as (ℓ being a truncation function)

$$X_t = X_0 + X_t^{c,\mathbb{G}} + \int_0^t \int_E \ell(x)(\mu(dt,dx) - \nu^{\mathbb{G}}(dt,dx)) + B_t^{\mathbb{G}}(\ell) = M_t^{\mathbb{G}} + B_t^{\mathbb{G}}(\ell)$$

$$X_t = X_0 + X_t^{c,\mathbb{F}} + \int_0^t \int_E \ell(x)(\mu(dt,dx) - \nu^{\mathbb{F}}(dt,dx)) + B_t^{\mathbb{F}}(\ell) = M_t^{\mathbb{F}} + B_t^{\mathbb{F}}(\ell)$$

where B is a predictable process with finite variation. The process B is the first characteristic, the second characteristic is $\langle X\rangle$, the third characteristic is ν.

There exists a $\mathbb{G}$-predictable, locally integrable increasing process, say $A^{\mathbb{G}}$, predictable processes $b^{\mathbb{G}}$, $c^{\mathbb{G}}$ and a transition kernel K such that

$$B^{\mathbb{G}} = b^{\mathbb{G}}\cdot A^{\mathbb{G}}, \qquad C^{\mathbb{G}} = c^{\mathbb{G}}\cdot A^{\mathbb{G}}, \qquad \nu^{\mathbb{G}}(dt,dx) = K_t^{\mathbb{G}}(dx)dA_t^{\mathbb{G}}.$$

We assume that

$$A_t^{\mathbb{G}} = \int_0^t a_u^{\mathbb{G}} du,$$

where $a^{\mathbb{G}}$ is a $\mathbb{G}$ progressively measurable process. Then it can be shown (see [6]) that the $\mathbb{F}$- characteristic triple of X is given as

$$dB^{\mathbb{F}} = \int_0^{\cdot} {}^{o,\mathbb{F}}(b_s^{\mathbb{G}}a^{\mathbb{G}})_s ds, \quad C^{\mathbb{F}} = C^{\mathbb{G}}, \quad \nu^{\mathbb{F}}(dt,dx) = \left(K_t^{\mathbb{G}}(dx)a_t^{\mathbb{G}}dt\right)^{p,\mathbb{F}}$$

where ${}^{o,\mathbb{F}}Z$ is the $\mathbb{F}$-projection of Z and $U^{p,\mathbb{F}}$ is the dual predictable projection of U (see, e,g, [1]).

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Robust duality for multi-action options with information delay

Anna Aksamit
(joint work with Ivan Guo, Shidan Liu, Zhou Zhou)

We establish pricing–hedging duality under model uncertainty for multi-action options. Multi-action options form a class of contracts whose pay-off depends on the actions taken by a buyer of such contract. As an example we may consider American options, baskets of American options with constraints on execution times, or swing options.

We thus generalize the duality obtained in [2] to the case of exotic options that allow the buyer to choose some action from an action space, countable or uncountable, at each time step in the setup of [3]. Our ideas, however, go beyond that model and can be applied in various frameworks – including dominated setup.

We solve above problem by introducing an enlarged canonical space in order to reformulate the superhedging problem for such exotic options as a problem for European options. Then in a discrete time market with the presence of finitely many statically traded liquid options, we prove the pricing-hedging duality for such exotic options as well as the European pricing-hedging duality in the enlarged space. For the sake of simplicity we focus on the case without statically traded options in what follows.

Consider the discrete-time model introduced in [3]. Fix a time horizon $N \in \mathbb{N}$, and let $\mathbb{T} := \{0, 1, \dots, N\}$ be the time periods in this model. Let $\Omega_0 = \{\omega_0\}$ be a singleton and Ω_1 be a Polish space. For each $k \in \{1, \dots, N\}$, define $\Omega_k := \Omega_0 \times \Omega_1^k$ as the k-fold Cartesian product. For each k, define $\mathcal{G}_k := \mathcal{B}(\Omega_k)$ and let $\mathcal{F}_k$ be its universal completion. In particular, we see that $\mathcal{G}_0$ is trivial and denote $\Omega := \Omega_N$, $\mathcal{F} := \mathcal{F}_N$ and $\mathbb{F} = (\mathcal{F}_k)_k$.

Consider a market with $d \in \mathbb{N}$ financial assets that can be traded dynamically without transaction costs. We model the dynamically traded assets by an $\mathbb{R}^d$-valued process $S = (S_t)_{t\in\mathbb{T}}$ such that S_t is $\mathcal{G}_t$-measurable for $t \in \mathbb{T}$. For an $\mathbb{F}$-predictable, $\mathbb{R}^d$-valued process H, the terminal wealth of the hedging portfolio is given by $(H \circ S)_N = \sum_{j,k} H_k^j (S_k^j - S_{k-1}^j)$.

Model uncertainty is expressed via the family of possible models $\mathcal{P}$ which is constructed in the following manner. For a given $k \in \{0, \dots, N-1\}$ and $\omega \in \Omega_k$, we have a non-empty convex set $\mathcal{P}_{k,k+1}(\omega) \subseteq \mathfrak{P}(\Omega_1)$ of probability measures, representing the set of all possible models for the $(k+1)$-th period, given the state ω at time k. We assume that for each $k \in \{0, \dots, N\}$, $\text{graph}(\mathcal{P}_{k,k+1}) \subseteq \Omega_k \times \mathcal{P}(\Omega_1)$ is analytic. We can then introduce the set $\mathcal{P} \subseteq \mathfrak{P}(\Omega)$ of possible models for the multi-period market up to time N by

$$\mathcal{P} := \{\mathbb{P}_{0,1} \otimes \mathbb{P}_{1,2} \otimes \cdots \otimes \mathbb{P}_{N-1,N} : \mathbb{P}_{k,k+1}(\cdot) \in \mathcal{P}_{k,k+1}(\cdot)\}\,.$$

Let $\mathcal{A}$ be the space of actions at each time and introduce $\mathcal{C} := \mathcal{A}^{N+1}$ to be the collection of all possible plans, equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{C})$ and a canonical filtration $(\mathcal{F}_k^c)_{0\leq k\leq N}$. In such set-up we are interested in the action dependent pay-off function $\Phi : \Omega \times \mathcal{C} \to \overline{\mathbb{R}}$, and its superhedging price given by

$$\pi(\Phi) := \inf\left\{x : \exists H \in \mathcal{H},\ \text{s.t.},\ x + (H(\cdot,c)\circ S)_N \geq \Phi(\cdot,c)\ \mathcal{P}\text{-q.s.},\forall c \in \mathcal{C}\right\}$$

and define the set of dynamic trading strategies

$$\mathcal{H} := \left\{H : \Omega \times \mathcal{C} \times \mathbb{T} \to \mathbb{R}^d \mid H(\cdot,\cdot,k+1) =: H_{k+1}(\cdot,\cdot) \text{ is } \mathcal{F}_k \otimes \mathcal{F}_k^c\text{-measurable}\right\}.$$

Our main theorem states the duality result where dual representation of this superhedging price is established:

Theorem 1. *Suppose that the no arbitrage condition NA($\mathcal{P}$) holds, and let $\Phi : \Omega \times \mathcal{C} \to \overline{\mathbb{R}}$ be upper semianalytic. Then, one has*

$$\pi(\Phi) = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\chi\in\mathcal{D}} \mathbb{E}^{\mathbb{Q}}\left[\Phi_\chi\right].$$

In the above theorem set $\mathcal{D}$ consists of all feasible action plans $\chi : \Omega \times \mathbb{T} \to \mathcal{A}$ such that $\chi(\cdot,k)$ is $\mathcal{F}_k$-measurable for each k. Set $\mathcal{D}$ generalizes the set of stopping times to a multi-action set-up. Set $\mathcal{M}$ denotes the set of martingale measures for a process S on Ω, and is given by

$$\mathcal{M} = \left\{\mathbb{Q} \in \mathfrak{P}(\Omega) : \mathbb{Q} \lll \mathcal{P} \text{ and } \mathbb{E}^{\mathbb{Q}}[\Delta S_k \mid \mathcal{F}_{k-1}] = 0, \forall k = 1,\ldots,N\right\}.$$

To prove Theorem 1, we apply the idea of space enlargement motivated by [2], which enables to view multi-action option as an European option on the space $\Omega \times \mathcal{C}$. Crucial argument is re-establishing dynamic programming principle based on Jankov-von Neumann analytic selection theorem. Since our framework allows for uncountable action space this argument becomes significantly more involved.

We complement our duality result with the study of the superhedging price of a multi-action option in the case of information delay. More precisely we cover the case where the seller of the option does not possess perfect information about the actions taken by the buyer, and is able to observe them with a delay. This framework takes into account this different type of uncertainty. The resulting duality for the superhedging price with information delay $\pi^{del}(\Phi)$ takes the following form:

$$\pi^{del}(\Phi) = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\chi\in\mathcal{D}^{ant}} \mathbb{E}^{\mathbb{Q}}\left[\Phi_\chi\right],$$

where, instead of previously appearing set of adapted feasible action plans $\mathcal{D}$, we have the set of the *anticipating* feasible action plans $\mathcal{D}^{ant}$. The dual side can be interpreted as the price which may be achieved by the buyer able to look into the future. *Looking into the future* feature is present here as information delay puts more constraints on the superhedging side.

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Optimal reinsurance via BSDEs in a partially observable model with jump clusters

Claudia Ceci

(joint work with Matteo Brachetta, Giorgia Callegaro, Carlo Sgarra)

Optimal reinsurance problems have attracted special attention during the past few years and they have been investigated in many different model settings. Insurance companies can hardly deal with all the different sources of risk in the real world, so they hedge against at least part of them, by re-insuring with other institutions. A reinsurance agreement allows the primary insurer to transfer part of the risk to another company and it is well known that this is an effective tool in risk management. Moreover, the subscription of such contracts is required by some financial regulators, see e.g. the Directive Solvency II in the European Union. Large part of the existing literature focuses mainly on classical reinsurance contracts such as the proportional and the excess-of-loss, which were extensively investigated under a variety of optimization criteria, e.g. ruin probability minimization, dividend optimization and expected utility maximization. Here we are interested in the latter approach (see Irgens and Paulsen [12], Mania and Santacroce [15], Brachetta and Ceci [3] and references therein). Some of the classical papers devoted to the subject assume a diffusive dynamics for the surplus process, while the more recent literature considers surplus processes including jumps.

The pioneering risk model with jumps in non-life insurance is the classical Cramér-Lundberg model, where the claims arrival process is a Poisson process with constant intensity. This assumption implies that the instantaneous probability that an accident occurs is always constant, which is in a way too restrictive in the real world, as already motivated by Grandell [10]. In recent years, many authors made a great effort to go beyond the classical model formulation. For example, Cox processes were employed to introduce a stochastic intensity for the claims arrival process, see e.g. Albrecher and Asmussen [1], Bjork and Grandell [2], Embrechts et al. [9]. Moreover, other authors introduced Hawkes processes in order to capture the self-exciting property of the insurance risk model in presence of catastrophic events. Hawkes processes were introduced by Hawkes [11] to describe geological phenomena with clustering features like earthquakes. Hawkes processes with general kernels are not Markov processes: they can eventually include long-range dependence, while Hawkes processes with exponential kernel exhibit the appealing property that the couple process-intensity is Markovian.

Dassios and Zhao [7] proposed a model which combines the two approaches by introducing a Cox process with shot noise intensity and a Hawkes process with exponential kernel for describing the claim arrival dynamics. Recently Cao, Landriault and Li [5] investigated the optimal reinsurance-investment problem in the model setting proposed by Dassios and Zhao [7] with a reward function of mean-variance type.

A different line of research related to the optimal-reinsurance investment problem focuses on the possibility that the insurer does not have access to all the information when choosing the reinsurance strategy. As a matter of fact, only the claims arrival and the corresponding disbursements are observable. In this case we need to solve a stochastic optimization problem under partial information. Liang and Bayraktar [14] were the first to introduce a partial information framework in optimal reinsurance problems. They consider the optimal reinsurance and investment problem in an unobservable Markov-modulated compound Poisson risk model, where the intensity and jump size distribution are not known, but have to be inferred from the observations of claim arrivals. Ceci, Colaneri and Cretarola [6] derive risk-minimizing investment strategies when information available to investors is restricted and they provide optimal hedging strategies for unit-linked life insurance contracts. Jang, Kim and Lee [13] present a systematic comparison between optimal reinsurance strategies in complete and partial information framework and quantify the information value in a diffusion setting.

More recently, Brachetta and Ceci [4] investigate the optimal reinsurance problem under the criterion of maximizing the expected exponential utility of terminal wealth when the insurance company has restricted information on the loss process in a model with claim arrival intensity and claim sizes distribution affected by an unobservable environmental stochastic factor.

In the present paper we investigate the optimal reinsurance strategy for a risk model with jump clustering properties in a partial information setting. The risk model is similar to that proposed by Dassios and Zhao [7] and it includes two different jump processes driving the claims arrivals: one process with constant intensity describing the exogenous jumps and another with stochastic intensity representing the endogenous jumps, that exhibits self-exciting features. The externally-excited component represents catastrophic events, which generate claims clustering increasing the claim arrival intensity. The endogenous part allows us to capture the clustering effect due to self-exciting features. That is, when an accident occurs, it increases the likelihood of such events. The insurance company has only partial information at disposal, more precisely the insurer can only observe the cumulative claims process. The externally-excited component of the intensity is not observable and the insurer needs to estimate the stochastic intensity by solving a filtering problem. Our approach is substantially different from that of Cao et Al. [5] in several respects: firstly, we work in a partial information setting; secondly, the intensity of the self-excited claims arrival exhibits a slight more general dependence on the claims severity; finally, we maximize an exponential utility function instead of following a mean-variance criterion. In a partially observable framework, our goal is to characterize the value process and the optimal strategy. The optimal stochastic control problem in our case turns out to be infinite dimensional and the characterization of the optimal strategy cannot be performed by solving a Hamilton-Jacobi-Bellman equation, but via a BSDE approach.

A difficulty naturally arises when dealing with Hawkes processes: the intensity of the jumps is not bounded a priori, although a non-explosive condition holds. Hence we are not able to exploit some relevant bounds, which are usually required to prove a verification theorem and results on existence and uniqueness of the solution for the related BSDE. Nevertheless, we are going to show that the optimal stochastic control problem has a solution, which admits a characterization in terms of a unique solution to a suitable BSDE.

Our paper aims to contribute in different directions to the literature on optimal reinsurance problems: first, we provide a rigorous and formal construction of the dynamic contagion model. Second, we study the filtering problem associated to our model, providing a characterization of the filter process in terms of the Kushner-Stratonovich equation and the Zakai equation as well. To the best of our knowledge, this problem has not been addressed insofar in the existing literature. We refer to Dassios and Jang [8] for a similar problem without the self-exciting component. Third, we solve the optimal reinsurance problem under the expected utility criterion.

We remark that our study differs from Brachetta and Ceci [4] in many key aspects. The risk model is substantially different, requires a strong effort to be rigorously constructed and the study of a new filtering problem. What is more, a crucial assumption in Brachetta and Ceci [4] is the boundedness of the claims arrival intensity, which is not satisfied in our case, thus leading to additional technicalities in most of the proofs. This is what happens, for example, when one needs to prove existence and uniqueness of the solution of the BSDE. Moreover, we perform the optimization over a class of admissible contracts, instead of maximizing over the retention level. This feature allows us to cover a larger class of problems. Finally, we do not require the existence of an optimal control for the derivation of the BSDE, hence the general presentation turns out to be different.

The paper is organized as follows. In Section 1 we are going to introduce the risk model and to specify what information is available to the insurer. A rigorous mathematical construction is provided, based on a measure change approach, necessary to develop the following analysis in full details. In Section 2 the filtering problem is investigated in order to reduce the optimal stochastic control problem to a complete information setting. The stochastic differential equation satisfied by the filter is obtained, by exploiting both the Kushner-Stratonovich and the Zakai approaches. In Section 3 the optimal stochastic control problem is formulated, while in Section 4 a characterization of the value process associated with the optimal stochastic control problem is illustrated. Due to the infinite dimension of the filter, the approach based on the Hamilton-Jacobi-Bellman equation cannot be exploited, so the value process is characterized as the unique solution of a BSDE. In Section 5 the optimal reinsurance strategy is investigated under general assumptions and some relevant cases are discussed.

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Utility maximization for reinsurance policies in a dynamic contagion claim model

Alessandra Cretarola
(joint work with Claudia Ceci)

Optimal reinsurance and optimal investment problems for various risk models have gained a lot of interest in the actuarial literature in recent years. Thanks to the development of effective strategies, insurers can reduce potential claim risk (insurance risk) and optimize capital investments. Indeed, acquiring reinsurance serves as a safeguard for insurers against unfavorable claim experiences, while investing also enables insurers to diversify risks and potentially achieve higher returns on the cash flows within their insurance portfolio. Within the extensive body of literature devoted to risk theory, a classical task is to deal with optimal risk control and optimal asset allocation for an insurance company. Mainly in the case of classical reinsurance contracts such as the proportional and the excess-of-loss, different decision criteria have been adopted in the study of these problems e.g. ruin probability minimization, dividend optimization and expected utility maximization. Here, we focus on the latter approach (see Irgens and Paulsen [9], Mania and Santacroce [10], Brachetta and Ceci [4] and references therein). Earlier seminal papers on the topic adopt a diffusive dynamics for the surplus process, whereas more recent literature explores surplus processes that incorporate jumps.

The first risk model specification incorporating jumps in non-life insurance is represented by the classical Cramér-Lundberg model, in which the claims arrival process follows a Poisson process with a constant intensity. Since it is an assumption which is seriously violated in a large number of insurance contexts (e.g., climate risks), many researchers have suggested to employ a stochastic intensity for the claim arrival dynamics. For instance, clustering features due to exogenous (externally-excited) factors, such as earthquakes, flood, and hurricanes, might be captured using a Cox process, see e.g. Albrecher and Asmussen [1], Bjork and Grandell [2], Embrechts et al. [7]. Moreover, clustering effects due to endogenous (self-excited) factors, such as aggressive driving habits and poor health conditions, can be effectively described by a Hawkes process, see e.g. Hawkes [8]. Dassios and Zhao [6] introduced a dynamic contagion model by generalizing both the Cox process with shot noise intensity and the Hawkes process.

In recent years, Cao, Landriault and Li [5] analyzed the optimal reinsurance-investment problem for the compound dynamic contagion process introduced by Dassios and Zhao [6] via the time-consistent mean–variance criterion. Brachetta et al. [3] very recently investigated the optimal reinsurance strategy for a risk model with jump clustering features similar to that proposed by Dassios and Zhao [6] under partial information.

In this work, we study the optimal reinsurance problem via expected utility maximization in the risk model with jump clustering properties introduced in Brachetta et al. [3] under full information for general reinsurance contracts. Note that, the problem considered in Brachetta et al. [3] is the same but analyzed in a partial information setting. The study of the problem in the case of complete information is not addressed in the literature, and furthermore, it could allow for comparative analyses in a more tractable context than that of partial information. We discuss two different methodologies: the classical stochastic control approach based on the Hamilton-Jacobi-Bellman equation and a backward stochastic differential equation approach. It is important to stress that proving the existence of a classical solution to the Hamilton-Jacobi-Bellman equation corresponding to the optimal stochastic control problem under investigation is challenging due to its inherent complexity. This difficulty stems from the equation's nature as a partial integro-differential equation, compounded by an optimization component embedded within the associated integro-differential operator. This motivated the application of an alternative approach based on backward stochastic differential equations. It is worth noting that the resulting backward stochastic differential equation, whose unique solution characterizes the value process, differs from that studied in Brachetta et al. [3], due to the presence of an additional jump component.

The paper, still in progress, is organized as follows. Firstly, we introduce the mathematical framework including the dynamic contagion process. Then, we formally introduce the problem under investigation, which involves the controlled surplus process and the objective function. Afterwards, we discuss the Hamilton-Jacobi-Bellman approach in order to solve the resulting optimal stochastic control problem and represent the value process as the unique solution of a suitable backward stochastic differential equation. We also characterize the optimal strategy for a general reinsurance premium and provide more explicit results in some relevant cases. Currently, we are performing a comparison analysis, which should underline the risk due to the self-exciting component.

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Set-Valued Propagation of Chaos for Controlled Mean Field SPDEs

David Criens

The area of controlled McKean–Vlasov dynamics, also known as mean field control, has rapidly developed in the past years More recently, there is also increasing interest in infinite dimensional systems, see, e.g., [1, 6] for equations appearing in financial mathematics. We also refer to the recent paper [2], where the authors investigate controlled mean field stochastic PDEs (SPDEs) for which they establish well-posedness of the state equation, the dynamic programming principle and a Bellman equation.

Mean field dynamics are typically motivated by particle approximations (related to propagation of chaos). It is an important task to make the heuristic motivation rigorous. For finite dimensional frameworks, a suitable limit theory was developed in the seminal paper [8].

In this talk, we discuss recent results established in the paper [3] for an infinite dimensional variational SPDE framework as initiated by Pardoux [9] and Krylov–Rozovskii [7]. To reduce the technical level of the talk, we consider a specific interacting systems of controlled porous media equations of the form

$$dY_t^k = \Big[\Delta(|Y_t^k|^{q-2}Y_t^k) + \frac{1}{n}\sum_{i=1}^{n}(Y_t^k - Y_t^i) + \int c(f)\,\mathfrak{m}^k(t,df)\Big]dt + \sigma dW_t^k,$$
$$Y_0^k = x,$$

with $q \geq 2$ and $k = 1, \ldots, n$. Here, $\mathfrak{m}^1, \mathfrak{m}^2, \ldots, \mathfrak{m}^n$ denote kernel that model the control variables, and $W^1, \ldots, W^n$ are independent cylindrical Brownian motions. This corresponds to a relaxed control framework in the spirit of [4, 5].

Let $\mathcal{R}^n(x)$ be the set of joint empirical distributions of such particles together with their controls (latter are captured via $\mathfrak{m}^k(t,df)dt$ in a suitable space of Radon measures). The associated set of mean field limits is denoted by $\mathcal{R}^0(x)$. It consists of probability measures supported on the set of laws of $(Y, \mathfrak{m}(t,df)dt)$, where Y solves a controlled McKean–Vlasov equation of the form

$$dY_t = \Big[\Delta(|Y_t|^{q-2}Y_t) + (Y_t - E[Y_t]) + \int c(f)\,\mathfrak{m}(t,df)\Big]dt + \sigma dW_t \quad Y_0 = x.$$

For this setting, we discuss two types of results. Conceptually, the first one is probabilistic and deals with the convergence of the controlled particle systems, while the second one sheds light on the mean field limits from a stochastic optimal control perspective.

The probabilistic result states that the sets $\mathcal{R}^n(x)$ and $\mathcal{R}^0(x)$ are nonempty and compact (in a suitable Wasserstein space) and that

$$\mathcal{R}^n(x) \to \mathcal{R}^0(x)$$

in the Hausdorff metric topology. This result is considered as *set-valued propagation of chaos*. Indeed, when the sets $\mathcal{R}^n(x^n)$ and $\mathcal{R}^0(x^0)$ are singletons, we recover a classical formulation of the propagation of chaos property. To the best of our knowledge, the concept and formulation of set-valued propagation of chaos has not appeared in the literature before.

The optimal control result states that the value functions associated with $\mathcal{R}^n(x)$ and $\mathcal{R}^0(x)$ converge to each other (uniformly on compacts in their initial values x), i.e.,

$$\Big(x \mapsto \sup_{P\in\mathcal{R}^n(x)} E^P\big[\,\psi\,\big]\Big) \to \Big(x \mapsto \sup_{P\in\mathcal{R}^0(x)} E^P\big[\,\psi\,\big]\Big)$$

compactly, for any continuous input function ψ on the suitable Wasserstein space that is of certain growth. As a consequence, one also obtains limit theorems in the spirit of the seminal work [8]. Namely, it follows that all accumulation points of sequences of n-state nearly optimal controls maximize the mean field value function, and that any optimal mean field control can be approximated by a sequence of n-state nearly optimal controls.

The talk is concluded with the open problem to relax some weak monotonicity conditions from [3]. This problem appears to be challenging due to the non-local structure of McKean–Vlasov equations.

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Hawkes processes, Malliavin calculus and application to financial and actuarial derivatives

Caroline Hillairet

(joint work with Anthony Réveillac, Mathieu Rosenbaum)

In this talk, we are interested in the evaluation of financial or actuarial derivatives whose payoff depends on a cumulative loss (L_t)

$$L_t := \sum_{i=1}^{N_t} X_i, \quad t \in [0,T]$$

where $N := (N_t)_{t\in[0,T]}$ is a counting process (jumping at time $(\tau_i)_{i\in\mathbb{N}^*}$) that represents the claims arrival (frequency component) and the $(X_i)_{i\in\mathbb{N}^*}$ (iid random variables) are the claims sizes (severity component).

In the classical Cramer-Lundberg model, N is assumed to be a Poisson process, meaning that inter-arrivals $(\tau_i - \tau_{i-1})$ are assumed to be iid (with exponential distribution). Nevertheless, self-exciting and contagion effects have been highlighted such as for example in credit risk and in cyber risk, in favor of modeling the claims arrivals by a Hawkes process, that is adapted to model aftershocks of claims. A (linear) Hawkes process H is characterized by its stochastic intensity $\lambda(t)$ fully specified by the process H itself, namely

$$\lambda(t) := \lambda_0(t) + \int_{(0,t)} \Phi(t-s)dH_s = \lambda_0(t) + \sum_{\tau_n<t} \Phi(t-\tau_n) \quad t \in [0,T],$$

where Φ is the (deterministic) excitation kernel and λ_0 is the (deterministic) baseline intensity (hereafter taken as a constant μ). The main contribution is to derive an explicit closed form pricing formula for contracts with underlying a cumulative loss indexed by a Hawkes process.

From the probabilistic point of view, we consider a payoff of the form $K_T h(L_T)$ where (K_t) and (L_t) are two loss processes indexed by the same Hawkes. This quantity is at the core for determining the premium of a large class of insurance derivatives or risk management instruments : reinsurance contracts (such as Stop-Loss contracts), or credit derivatives (such as tranches of Collaterized Debt Obligations), or computation of the expected shortfall of contingent claims. It can be expressed as $\int_{(0,T]} Z_t dH_t F$ where Z is a predictable process and $F := h(L_T)$ is a functional of the Hawkes process. In the case where the counting process is a Poisson process (or a Cox process), Malliavin calculus enables one to transform this quantity. More precisely, if $H = N$ is an homogeneous Poisson process with intensity $\mu > 0$ (in other words the self-exciting kernel Φ is put to 0), the Malliavin integration by parts formula (Mecke formula, see [7]) allows us to derive that

$$\mathbb{E}\left[\int_{(0,T]} Z_t dN_t F\right] = \mu \int_0^T \mathbb{E}\left[Z_t F \circ \epsilon_t^+\right] dt, \tag{1}$$

where the notation $F \circ \epsilon_t^+$ denotes the functional on the Poisson space where a deterministic jump is added to the paths of N at time t. This expression turns out to be particularly interesting from an actuarial point of view since adding a jump at some time t corresponds to realising a stress test by adding artificially a claim at time t. Naturally, in case of a Poisson process, the additional jump at some time t only impacts the payoff of the contract by adding a new claim in the contract but it does not impact the dynamic of the counting process N.

We provide a generalization of Equation (1) in case the counting process is a Hawkes process H. The main ingredient consists in using a representation of a Hawkes process known as the "Poisson imbedding" (related to the "Thinning Algorithm", see [5]) in terms of a Poisson measure N on $[0,T] \times \mathbb{R}_+$ to which the Malliavin integration by parts formula can be applied.

$$\begin{cases} H_t = \int_{(0,t]} \int_{\mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_s\}} N(ds, d\theta), \\ \lambda_t = \mu + \int_{(0,t)} \Phi(t-u) dH_u. \end{cases} \tag{2}$$

As the adjunction of a jump at a given time impacts the dynamic of the Hawkes process, we refer to the obtained expression more to an "expansion" rather than an "integration by parts formula" for the Hawkes process, as it involves what we name "shifted Hawkes processes" $H^{v_n,\ldots,v_1}$ for which jumps at deterministic times $0 < v_n < \cdots < v_1$ are added to the process accordingly to the self-exciting kernel Φ. To illustrate this, a one shift Hawkes process at time v in $(0,T)$ can be expressed as follows

$$\begin{cases} H_t^v = \mathbf{1}_{[0,v)}(t)H_t + \mathbf{1}_{[v,T]}(t)\left(H_{v-}^v + 1 + \int_{(v,t]}\int_{\mathbb{R}_+}\mathbf{1}_{\{\theta\le\lambda_s^v\}}N(ds,d\theta)\right) \\ \lambda_t^v = \mathbf{1}_{(0,v]}(t)\lambda_t + \mathbf{1}_{(v,T]}(t)\left(\mu^{v,1}(t) + \int_{(v,t)}\Phi(t-u)dH_u^v\right), \\ \mu^{v,1}(t) := \mu + \int_{(0,v]}\Phi(t-u)dH_u^v = \mu + \int_{(0,v)}\Phi(t-u)dH_u + \Phi(t-v). \end{cases}$$

The main result is the following **expansion formula** (see [2]): Assuming Z a bounded $\mathbb{F}$-predictable process, F a bounded $\mathcal{F}_T$-measurable random variable and $\|\Phi\|_1 < 1$. Then

$$\mathbb{E}\left[F\int_{[0,T]} Z_t dH_t\right] = \mu\int_0^T \mathbb{E}\left[Z_v F^v\right]dv$$
$$+\mu\sum_{n=2}^{+\infty}\int_0^T\int_0^{v_1}\cdots\int_0^{v_{n-1}}\prod_{i=2}^n \Phi(v_{i-1}-v_i)\mathbb{E}\left[Z_{v_1}^{v_n,\dots,v_2}F^{v_n,\dots,v_1}\right]dv_n\cdots dv_1.$$

The first term $\mu\int_0^T \mathbb{E}\left[Z_v F^v\right]dv$ corresponds to the formula for a Poisson process (setting the self-exciting kernel Φ at zero). The sum in the second term can be interpreted as a correcting term due to the self-exciting property of the counting process H. The shifted processes $H^{v_n,\dots,v_1}$ appearing in the form of the premium are of the same complexity than the original Hawkes process H. However, they exhibit deterministic jumps at some times $v_1,\dots,v_n$ which are weighted by correlation factors of the form $\Phi(v_i - v_{i-1})$. We benefit from this formulation to derive a lower and an upper bound respectively for the quantity $\mathbb{E}[K_T h(L_T)]$: by controlling the different types of jumps of the shifted Hawkes process, one can perform bounds that are more accurate than those available so far.

As an extension (still assuming $\|\Phi\|_1 < 1$), we indicate how this methodology combining Poisson imbedding and Malliavin calculus, can be used to provide new results on Hawkes processes such as

- **Explicit "Pseudo-Chaotic" expansion** (see [3])

$$H_T = \sum_{k=1}^{+\infty}\int_{\mathbb{X}^k}\frac{1}{k!}c_k(x_1,\dots,x_k)N(dx_1)\cdots N(dx_k),$$

$$\begin{cases} c_1(x_1) = \mathbf{1}_{\{\theta_1\le\mu\}}, \\ c_k(x_1,\dots,x_k) = D^{k-1}_{(x_1,x_2,\dots,x_{k-1})}\mathbf{1}_{\{\theta_k\le\lambda_{t_k}\}} \end{cases}$$

  where $\mathbb{X} := [0,T]\times\mathbb{R}_+$; $x := (t,\theta)$; $dx = d\theta dt$ and D is the Malliavin derivative $(D_x F) := F\circ\epsilon_x^+ - F$.

- **Explicit correlation of a general Hawkes process** (see [4]). For $s \le t$

$$Cov(H_s,H_t) = \mu\int_0^s\left(1+\int_0^v\Psi(w)dw\right)\left(1+\int_v^s\Psi(y-v)dy\right)\left(1+\int_v^t\Psi(y-v)dy\right)dv,$$

where $\Psi := \sum_{n=1}^{+\infty} \Phi_n$ and Φ_n are the iterated convolution of the excitation kernel $\Phi_1 := \Phi, \quad \Phi_n(t) := \int_0^t \Phi(t-s)\Phi_{n-1}(s)ds, \quad t \in \mathbb{R}_+,\ n \in \mathbb{N}^*$.

- **Quantitative TCL** (see [1]). "Berry Esseen" bounds Central Limit Theorems for the compound Hawkes process $L_T := \sum_{i=1}^{H_T} X_i$ (with X_i iid and independent of H) using Malliavin-Stein method (as in Nourdin Pecatti [6])

$$d_W\left(\frac{L_T - m\int_0^T \lambda_s ds}{\sqrt{T}}, G\right) \leq \frac{C_{\Phi,\nu}}{\sqrt{T}}, \quad \forall T > 0, \quad G \sim \mathcal{N}(0,\sigma^2),$$

with $m = \mathbb{E}(X)$ and $\sigma^2 = \frac{\mu\mathbb{E}(X^2)}{1-||\Phi||_1}$.

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