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Mini-Workshop: Combinatorial and Algebraic Structures in Rough Analysis and Related Fields

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Abstract. Recent years have seen an explosion of algebraic methods to study singular stochastic and rough dynamics. These include developments in geometric rough path theory based on the algebra of words, the introduction of decorated trees in regularity structures, and the recent approach to singular stochastic partial differential equations based on multi-indices. These developments have furthermore led to important links with numerical analysis, machine learning, stochastic quantisation, and the study of symmetries of physical systems. The aim of this mini-workshop was to bring together experts working on these fields using algebraic structures that appear in rough dynamics. The goal was to facilitate the exchange of ideas and encourage further connections to be established.

Mathematics Subject Classification (2020): 60L10, 60L20, 60L30, 60L70, 16T05, 16S10, 18M60, 18G45, 18M60, 65M12.

Introduction by the Organizers

Organizational details

The mini-workshop *Combinatorial and Algebraic Structures in Rough Analysis and Related Fields*, organised by Yvain Bruned (Université de Lorraine), Carlo Bellingeri (TU Berlin), Ilya Chevyrev (University of Edinburgh) and Rosa Preiß (University of Postdam) was attended by 16 participants currently based in France, Germany, Norway, Poland and the UK. The program consisted of 16 talks (45 minutes each), each being followed by a discussant's presentation (15 minutes each), leaving sufficient time for additional questions from the audience.

Due to some participants becoming ill at short notice in connection with the Covid-19 pandemic, this event took place in a hybrid format having 4 participants attending online. In accordance with Oberwolfach's tradition, the schedule was not known in advance by the participants. The days' schedules were sent each evening to the group. Further informal discussions took place in between and after the talks. The Zoom session was managed with the precious help of Carlo Bellingeri, and Usama Nadeem took care of the the report.

Motivation

The main purpose of the mini-workshop was to gather together early career researchers working in the development of new algebraic structures to study non-linear singular random dynamics arising from rough analysis and connected areas. In particular, we wanted to encourage collaborative work and the sharing of recent contributions among different research groups, including groups working on SPDEs with regularity structures and multi-indices, signatures, numerical analysis, data science, and operad theory. Combinatorial and algebraic structures arise naturally in non-linear dynamics when we want to describe in a compact way higher order expansions of differential equations.

Consider for instance a autonomous system

$$x'(t) = f(x(t)), \quad x(0) = x_0 \,.$$

Then, applying iteratively the Taylor formula, we can write x as the asymptotic series indexed by trees. The same trees can be used also to describe higher order Runge–Kutta methods to solve the system numerically [3]. More generally, by introducing appropriate algebraic structures, an important example of which is the Butcher–Connes–Kreimer Hopf algebra [5], it is possible to derive a consistent theory of numeric integration for ordinary differential equations [10] and to renormalise Feynman diagrams in quantum field theory. Moving beyond numerical analysis, formal expansions can be used analytically to establish well-posedness of singular stochastic dynamics. We refer principally to rough differential equations (RDEs)

$$dY_t = g(Y_t)dW_t, \quad Y(0) = Y_0$$

and singular stochastic partial differential equations (SPDEs) of the form

$$(\partial_t - L)u = F(u, \nabla u)\xi, \,, \quad u(0) = v \,.$$

Both systems are characterised by the presence of noise terms, which are associated respectively to a highly oscillating driving noise W and random distribution ξ, making the equation singular. The resolution of these systems is performed in a series of papers [8, 2, 4, 1], taking their roots in Lyons theory of rough paths [11, 6, 7], and that have culminated in the formation of the recent field of rough analysis, see [9]. In this context, combinatorial and algebraic structures are adopted to construct truncated Taylor-type expansions of the solutions of the previous equations.

We should indeed mention the following offshoots which have been widely discussed by the participants:

- Multi-indices, which are a different way to encode the expansions for solutions of singular SPDEs. The idea is to index the expansion according to the elementary differentials (coefficients arising from the nonlinearities) instead of the iterated integrals. Talks on the subject were given by Bruned, Linares and Tempelmayr.
- Regularity Structures via decorated trees where the characterisation of symmetries in a general combinatorial context remains a challenge. For example, there is no unification between the chain rule in the geometric and quasilinear KPZ equations and gauge-covariance in Yang-Mills. Talks were given on this topic by Chevyrev and Nadeem.
- Numerical Analysis for dispersive PDEs where a resonance analysis allows us to get low regularity schemes for a large class of equations. The combinatorial structure used is very similar to decorated trees developed for singular SPDEs. Talks were given by Alama Bronsard and Schratz.
- Rough paths where its geometry is much better understood via the isomorphisms between words and trees. Talks were given on this topic by Bellingeri, Ferrucci, Rahm and Tapia.
- The potential of these combinatorial structures could be seen via other fields of application brought by the participants such as Algebraic geometry for rough paths (Preiß), Algebraic operads (Tamaroff), Perturbative Quantum Field Theory (Klose), Stochastic analysis in Frobenius manifold (Combe) and 2D signature via multiparemeter iterated integrals (Diehl).

Acknowledgement: The workshop organizers would like to thank MFO for the nice environment provided for this event.

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Timetable of the Mini-workshop

	Monday	Tuesday	Wednesday	Thursday	Friday
9h30	Bellingeri	Combe	Schratz	Tamaroff	Nadeem
11h00	Klose	Chevyrev	Diehl	Preiß	Bruned
15h30	Tempelmayr	Rahm		Tapia	
17h00	Linares	Alama Bonsard		Ferrucci	

Speakers and discussants

Speaker	Title	Discussant
Bellingeri	Algebraic structures in the rough change of variable formula	Alama Bronsard
Klose	Perturbation theory for the Φ^4_3 measure, revisited with Hopf algebras	Tempelmayr
Tempelmayr	Recentering for rough paths and regularity structures via multi-indices	Tamaroff
Linares	Algebraic renormalization of rough paths and regularity structures based on multi-indices	Bellingeri
Combe	Semimartingales with values in a (pre-)Frobenius manifolds	Ferrucci
Chevyrev	Symmetries in stochastic Yang–Mills equations	Linares
Rahm	Planarly Branched Rough Paths Are Geometric	Combe
Alama Bronsard	Numerical approximations to rough solutions of dispersive equations	Diehl
Schratz	Resonances as a computational tool	Tapia
Diehl	Multiparameter iterated integrals	Schratz
Tamaroff	From bialgebras to algebraic operads	Nadeem
Preiß	An algebraic geometry of (rough) paths	Rahm
Tapia	Branched Itô formula	Klose
Ferrucci	Natural Itô-Stratonovich isomorphism	Bruned
Nadeem	Solution theory for quasilinear generalised KPZ Equation	Chevyrev
Bruned	Novikov algebras and multi-indices in regularity structures	Preiß

Mini-Workshop: Combinatorial and Algebraic Structures in Rough Analysis and Related Fields

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Abstracts

Algebraic structures in the rough change of variable formula

Carlo Bellingeri

Given a smooth function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ and a continuous bounded variation path $x\colon [0,T] \to \mathbb{R}^d$, $x = (x^1, \ldots, x^d)$ the fundamental theorem of calculus tells us the well-known identity

$$\varphi(x_t) - \varphi(x_s) = \sum_{i=1}^{d} \int_s^t \frac{\partial \varphi(x_r)}{\partial x_i} dx^i_r \,.$$

This formula is a cornerstone of standard calculus. In particular, when x does not satisfy this property, the integral in might not be well defined because x is not a.e. differentiable and Lebesgue integration theory is not useful any more. Surprisingly, thanks to the theory of rough paths [5] it is still possible to write a similar change of variable formula. However, in this case, the formula is not unique, depending on the underlying *algebraic theory* defining the integrals. The goal of this talk is to fully explore the possible identities known in the theory and prepare the discussion for the talks of Emilio Ferrucci and Nikolas Tapia on [2].

A first possibility is represented by introducing the *Young integral*, see [11], defined as the limit of the Riemann-type sum

$$\int_s^t f(x_r) d\hat{x}^i_r : \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} f(x_u)(x^i_v - x^i_u)\,,$$

where π is a generic partition of $[s,t]$ with size $|\mathcal{P}|$. This sum converges if and only if x is γ-Hölder with $\gamma \in (1/2, 1)$ and one has the formula

$$\varphi(x_t) - \varphi(x_s) = \sum_{i=1}^{d} \int_s^t \frac{\partial \varphi(x_r)}{\partial x_i} d\hat{x}^i_r \,.$$

More generally, using the standard theory of *geometric rough paths*, see [10], the starting point is not anymore a path but an *extended path* $X\colon [0,T]^2 \to \mathcal{G}(\mathbb{R}^d)$ with values in the character group of the shuffle Hopf algebra $(T(\mathbb{R}^d), ⧢, \Delta_c)$. Using the additional components of X we can indeed define for any $\gamma \in (0,1)$ the *geometric rough integral*

$$\int_s^t f(X_r) dX^i_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \sum_{|w|<N-1} \frac{\partial f(X_u)}{\partial x_{i_1} \ldots \partial x_{i_w}} \langle wi, X_{u,v} \rangle\,,$$

where we sum over a set of words with a length smaller than a finite N depending on γ. It follows from elementary considerations of Taylor's formula and the shuffle product that in this case one has the identity

$$\varphi(x_t) - \varphi(x_s) = \sum_{i=1}^{d} \int_s^t \frac{\partial \varphi(x_r)}{\partial x_i} dX^i_r \,.$$

Similar computations were also provided in [1] where the shuffle product is deformed into a quasi-shuffle product [8]. The main feature of this approach starts with some *apriori* relations among the components of X and then one derives the formula using standard combinatorial relations.

In case we not want to assume any apriori relations we need to start from a *branched rough paths* [6, 7] i.e. our starting path will take value in the character group of the Butcher–Connes–Kreimer Hopf algebra $(\mathcal{H}(\mathbb{R}^d), ., \Delta_{ck})$ [3] where the product is free. A first general theory to express these identities was given in the last chapter of David Kelly's PhD Thesis [9, Chap. 5]. This condition allows us to obtain an extremely general formula but at the same time, this notion requires to satisfy some additional properties and it is not unique, which makes this definition more arduous for applications. Some parts of [2] are dedicated to providing a new formula in this context. Interestingly we will have a final identity of the form

$$\varphi(x_t) - \varphi(x_s) = \sum_{i=1}^{d} \int_s^t \frac{\partial \varphi(x_r)}{\partial x_i} dX_r^i + \text{“higher order integrals”}\,.$$

where in the remaining integral we will integrate not just with respect to the component of X but the value of X on its primitive elements, whose properties were deeply analysed in [4].

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Perturbation theory for the Φ^4_3 measure, revisited with Hopf algebras

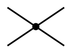
Tom Klose
(joint work with Nils Berglund)

The Φ^4_3 model, defined on the 3-dimensional torus $\mathbb{T}^3$, is probably one of the simplest non-trivial models in Euclidean quantum field theory. At cut-off scale $N \in \mathbb{N}$, it can be written as

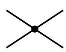
$$\mu^N_{\Phi^4_3}(\mathrm{d}\phi) = \frac{1}{Z_N(\varepsilon)} \exp\Big(-\int_\Lambda \Big(\frac{\|\nabla\phi(x)\|^2}{2} + \frac{1-\varepsilon^2 C_N^{(2)}}{2}{:}\,\phi(x)^2{:} + \frac{\varepsilon}{4}{:}\,\phi(x)^4{:} + \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}\Big)\mathrm{d}x\Big)\mathrm{d}\phi$$

where $C_N^{(k)}$, $k = 1, \ldots, 4$ are suitable explicit renormalisation constants and ${:}\cdot{:}$ is the Wick product w.r.t. the covariance $C_N^{(1)}$. The purpose of this talk is to revisit perturbation theory for the renormalised log partition function

(1)
$$-\log\frac{Z_N(\varepsilon)}{Z_N(0)} = \gamma - \log \mathbb{E}^{\mu_N}\left[e^{-\alpha X - \beta Y}\right] = \gamma - \sum_{n=2}^{\infty}\frac{\kappa_n}{n!},$$

$\kappa_n = \mathbb{E}^{\mu_N}_c \Big[\Big(\alpha$  $+\,\beta$ $\Big)^n\Big]$

associated with this measure, where μ_N is the Gaussian measure with covariance $(-\Delta+1)^{-1}$, regularised at scale N,

(2) $X \equiv$  $\equiv \int_{\mathbb{T}^3} {:}\,\phi(x)^4{:}\mathrm{d}x, \quad Y \equiv$ $\equiv \int_{\mathbb{T}^3} {:}\,\phi(x)^2{:}\mathrm{d}x,$

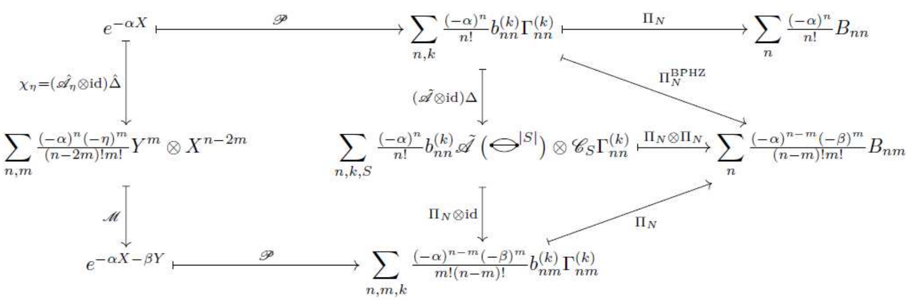
and the parameters α, β, and γ are defined as $\alpha := \varepsilon/4$, $\beta := \frac{\varepsilon^2}{2}C_N^{(2)}$, and $\gamma := \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}$. The last equality in (1) is an expansion in terms of *cumulants* κ_n and it is well-known (see, e.g. [10]) that they can be expressed in terms of *connected* Feynman diagrams $\Gamma^{(k)}_{nm}$ with m vertices of valency 4, $n-m$ vertices of valency 2, and $n+m$ edges. These diagrams come with a degree $\deg(\Gamma^{(k)}_{nm}) = 2n-m-3$ for all k and are associated with a real number via a canonical valuation map Π_N. Even though all diagrams Γ with valuation $\Pi_N(\Gamma) \leq 0$ are divergent as $N \to \infty$, it is unfortunately *not* the case that all the diagrams with positive valuation converge; this is known as the problem of (nested) *subdivergences*. It has been overcome in the celebrated work by Bogoliubov, Parasiuk, Hepp, and Zimmermann [3, 9, 11], who constructed a renormalised *BPHZ valuation* map Π_N^{BPHZ} for which $\deg(\Gamma) > 0$ implies that $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ is uniformly bounded in N; see also the recent work by Hairer [8] for a self-contained formulation. The main result of our work [2, Thm. 3.5] is the following theorem:

Theorem 1. *The following equality holds in the sense of formal power series:*

(3)
$$\gamma - \sum_{n=2}^{\infty}\frac{\kappa_n}{n!} = \sum_{p=4}^{\infty}\frac{1}{p!}(-\alpha)^p \sum_k b^{(k)}_{pp}\Pi_N^{\mathrm{BPHZ}}\left(\Gamma^{(k)}_{pp}\right)$$

where the $b_{pp}^{(k)}$'s denote combinatorial factors. Since $\deg(\Gamma_{pp}^{(k)}) = p - 3 \geq 1$ *for $p \geq 4$, this implies that all terms in the perturbative cumulant expansion* (1) *are bounded uniformly in the cut-off parameter N.*

This result is already known but our proof is new. The theorem follows from the commutativity of the diagram below which we establish in [2, Sec. 3.6].



In this diagram, the RHS is well-known. Since we work with connected diagrams, note that the only possible divergent sub-diagram in our setting is the "bubble" $\ominus$.

Furthermore, the co-product Δ describes the extraction-contraction procedure due to Connes and Kreimer [5, 6], $\tilde{\mathscr{A}}$ is the twisted antipode w.r.t. the BPHZ valuation, and $\mathscr{C}_S\Gamma$ is the graph Γ with all bubbles with labels in S contracted to a vertex, see [2, Sec. 3.3 and 3.4] and the references therein for details.

Inspired by the work of Ebrahimi-Fard et al. [7], the main novelty of our approach is represented on the LHS of the diagram: We consider the polynomial Hopf algebra H spanned by the monomials X and Y given in (2), equipped with the classical co-product $\hat{\Delta}$; on top of that, we build a map $\hat{\mathscr{A}}_\eta$ that resembles the twisted antipode $\tilde{\mathscr{A}}$ such that the map χ_η then resembles the BPHZ renormalisation procedure of Feynman diagrams on the RHS. The map $\mathscr{M}$ describes the multiplication $\mathscr{M} : H \otimes H \to H$ and the connection between the LHS and the RHS is given by the map $\mathscr{P}$, which formalises the pairings in Wick's formula and projects onto connected diagrams, see [2, Sec. 3.5] for details.
Interaction with the other participants. The discussant, Markus Tempelmayr, did a wonderful job and raised several interesting questions. The first question concerns the generality of our approach, in particular with regards to

- the *full subcritical regime* of the $\Phi^4_{4-\delta}$ model that was recently investigated in [4] or even
- the case of the *critical* Φ^4_4 model, the triviality of which was established by Aizenman and Duminil-Copin [1].

A potential answer to this question is linked to another question raised by the discussant, namely: Can we characterise the algebraic structure on the LHS of the commutative diagram above? In our work, we have left that problem open but we believe that one should be able to recast (a modification of) our construction in the language of the above-mentioned work by Ebrahimi-Fard et al. [7]. While this question remains open for now, the workshop has provided the author with the opportunity to initiate a discussion with Nikolas Tapia, another participant and co-author of the article [7], which could potentially lead to a follow-up project.

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Recentering for rough paths and regularity structures via multi-indices

Markus Tempelmayr
(joint work with Pablo Linares and Felix Otto)

Following [3, Section 6], we review the Hopf algebra structure underlying recentering in multi-index based regularity structures introduced in [4]. To simplify this exposition, we consider instead of a PDE the rough differential equation

$$\frac{d}{dt}u(t) = a(u(t))\xi(t), \quad u(t=0) = 0. \tag{1}$$

Here, we think of $a : \mathbb{R} \to \mathbb{R}$ as being a smooth nonlinearity, and of ξ as a random Schwartz distribution, say $\xi \in C^{\alpha-1}$ for $\alpha \in (0, 1)$. For such ξ, the product $a(u)\xi$ is a product of a function with a distribution which falls short of the Young regime.

The basic idea to develop a solution theory in [4] is to parameterize the model, which captures the local solution behaviour, by partial derivatives w.r.t. the nonlinearity a. We thus make the ansatz

$$u(t) - u(s) = \sum_{\beta} \Pi_{s\beta}(t) \prod_{k=0}^{\infty} \Big(\frac{1}{k!} \frac{d^k a}{du^k} \big(u(s)\big) \Big)^{\beta(k)}$$

for a base point $s \in \mathbb{R}$, where $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ is a multi-index. With help of the coordinates z_k on the space of nonlinearities given by $\mathsf{z}_k[a] := \frac{1}{k!} \frac{d^k a}{du^k}(0)$, the above ansatz takes the more compact form of

$$u(t) - u(s) = \sum_{\beta} \Pi_{s\beta}(t)\, \mathsf{z}^\beta [a(\cdot + u(s))],$$

where the monomials z^β are given by $\mathsf{z}^\beta := \prod_{k=0}^{\infty} \mathsf{z}_k^{\beta(k)}$. This power series does in general not converge. We thus "algebraize" our ansatz by not evaluating the coordinates at a nonlinearity a, and consider instead formal power series in the abstract variables $\{\mathsf{z}_k\}_{k=0}^{\infty}$,

$$\Pi_s(t) := \sum_{\beta} \Pi_{s\beta}(t)\, \mathsf{z}^\beta \in \mathbb{R}[[\mathsf{z}_k]].$$

Plugging this ansatz into (1) and comparing coefficients yields

$$\frac{d}{dt}\Pi_{s\beta}(t) = \sum_{k=0}^{\infty} \sum_{e_k + \beta_1 + \cdots + \beta_k = \beta} \Pi_{s\beta_1}(t) \cdots \Pi_{s\beta_k}(t) \xi(t), \quad \Pi_s(t = s) = 0,$$

where we denote by e_k the multi-index mapping l to δ_k^l. Some examples are

$$\tfrac{d}{dt}\Pi_{s\, e_0} = \xi, \quad \tfrac{d}{dt}\Pi_{s\, e_0+e_1} = \Pi_{s\, e_0}\xi, \quad \tfrac{d}{dt}\Pi_{s\, 2e_0+e_2} = \Pi_{s\, e_0}^2 \xi,$$
$$\tfrac{d}{dt}\Pi_{s\, 2e_0+e_1+e_2} = \Pi_{s\, 2e_0+e_2}\xi + 2\Pi_{s\, e_0}\Pi_{s\, e_1}\xi.$$

Comparison to rough paths. We compare this construction to branched rough paths [1]. For rooted trees $\tau_1, \dots, \tau_k$ and a tree $\tau = \overset{\tau_1 \cdots \tau_k}{\vee}$ the rough path $\mathbb{X}(\tau)$ is recursively defined by

$$\frac{d}{dt}\mathbb{X}_{s,t}(\tau) = \mathbb{X}_{s,t}(\tau_1) \cdots \mathbb{X}_{s,t}(\tau_k)\xi, \quad \mathbb{X}_{s,t=s}(\tau) = 0.$$

Some examples are

$$\tfrac{d}{dt}\mathbb{X}_s(\bullet) = \xi, \quad \tfrac{d}{dt}\mathbb{X}_s([\bullet]) = \mathbb{X}_s(\bullet)\xi, \quad \tfrac{d}{dt}\mathbb{X}_s([\bullet\bullet]) = \mathbb{X}_s(\bullet)\mathbb{X}_s(\bullet)\xi,$$
$$\tfrac{d}{dt}\mathbb{X}_s([[\bullet\bullet]]) = \mathbb{X}_s([\bullet\bullet])\xi, \quad \tfrac{d}{dt}\mathbb{X}_s([\bullet[\bullet]]) = \mathbb{X}_s(\bullet)\mathbb{X}_s([\bullet])\xi.$$

As these examples correctly suggest, every model component $\Pi_{s\beta}$ can be expressed as a linear combination of rough paths $\mathbb{X}_s(\tau)$.

Proposition 1. For every β we have

$$\Pi_{s\beta}(t) = \sum_{\tau \in \mathcal{T}_\beta} \tfrac{\sigma(\beta)}{\sigma(\tau)}\, \mathbb{X}_{s,t}(\tau),$$

where $\mathcal{T}_\beta$ is the set of all trees having $\beta(k)$ nodes with k children for all $k \in \mathbb{N}_0$, $\sigma(\beta) := \prod_{k=0}^{\infty}(k!)^{\beta(k)}$ is a symmetry factor of a multi-index, and $\sigma(\tau)$ is the symmetry factor of the tree τ.

This induces a dictionary ϕ from (linear combinations of) multi-indices T to (linear combinations of) trees $\mathcal{T}$, given by $\phi(\beta) = \sum_{\tau\in\mathcal{T}_\beta} \frac{\sigma(\beta)}{\sigma(\tau)}\,\tau$, such that

$$\Pi = \mathbb{X} \circ \phi.$$

Recentering. We turn to recentering, and aim to relate Π_s to $\Pi_{\bar{s}}$. Observe that for given $u_{\bar{s}} \in \mathbb{R}$ and $u[a(\cdot + u_{\bar{s}})]$ the solution to (1) with a replaced by $a(\cdot + u_{\bar{s}})$,

$$\bar{u} := u\big[a(\cdot + u_{\bar{s}})\big] + u_{\bar{s}}$$

satisfies (1) with initial condition $\bar{u}(0) = u_{\bar{s}}$. Allowing the shift to depend on a, we might hope to choose $u_{\bar{s}}[a]$ such that $\bar{u}(\bar{s}) = 0$. We thus informally identified the transformation $\Gamma^*_{\bar{s}0}$ that recenters solutions from 0 to $\bar{s}$ by

$$(\Gamma^*_{\bar{s}0}u)[a] = u\big[a(\cdot + u_{\bar{s}}[a])\big] + u_{\bar{s}}[a].$$

The goal is to translate this to the level of Π, where the above can not directly be applied as Π is not a well-defined functional of a (only a formal power series!). Instead, we consider the infinitesimal generator D of the u-shift of a defined by

$$(Du)[a] := \tfrac{d}{dv}\big|_{v=0} u[a(\cdot + v)],$$

which as a derivation is well defined on formal power series $\mathbb{R}[[\mathsf{z}_k]]$. Analogously, the generator of the a-dependent u-shift is given by $\mathsf{z}^\beta D \in \mathrm{Der}(\mathbb{R}[[\mathsf{z}_k]])$. The linear span $\mathsf{L} := \mathrm{span}\{\mathsf{z}^\beta D \,|\, \beta \text{ multi-index}\}$ is then a pre-Lie algebra when equipped with

$$\mathsf{z}^\beta D \triangleright \mathsf{z}^\gamma D := (\mathsf{z}^\beta D.\mathsf{z}^\gamma)D,$$

the dot meaning the application of the derivation $\mathsf{z}^\beta D$ to the power series z^γ. Note that its universal enveloping algebra $U(\mathsf{L})$ is naturally a Hopf algebra. We define the space of "forests" of multi-indices T^+ via the pairing

$${}_{U(\mathsf{L})}\langle \mathsf{z}^{\beta_1}\cdots\mathsf{z}^{\beta_k} D\cdots D, \gamma_1\cdots\gamma_l\rangle_{\mathsf{T}^+} = \delta^l_k \delta^{\gamma_1}_{\beta_1}\cdots\delta^{\gamma_k}_{\beta_k},$$

where $\mathsf{z}^{\beta_1}\cdots\mathsf{z}^{\beta_k} D\cdots D$ can be given a sense in $U(\mathsf{L})$ by help of the pre-Lie product $\triangleright$, which crucially does not depend on any order as the multiplication of monomials z^β commutes. This turns T^+ into a Hopf algebra. From its character group, we can analogous to the theory of regularity structures [2] with help of a comodule build a (structure) group G containing endomorphisms $\Gamma_{s\bar{s}}$, such that their duals $\Gamma^*_{s\bar{s}} \in \mathrm{End}(\mathbb{R}[[\mathsf{z}_k]])$ satisfy

$$\Gamma^*_{s\bar{s}}\Pi_{\bar{s}} = \Pi_s.$$

By working on the "dual side", we thus obtained a geometric interpretation of the Hopf algebra at play in regularity structures for recentering.

Comparison to rough paths again. On the tree-side we consider the Grossman-Larson pre-Lie algebra $\mathcal{L} := \mathrm{span}\{\tau \,|\, \tau \text{ tree}\}$ equipped with

$$\tau_1 \rightsquigarrow \tau_2 := \sum_\tau n(\tau_1, \tau_2, \tau)\tau,$$

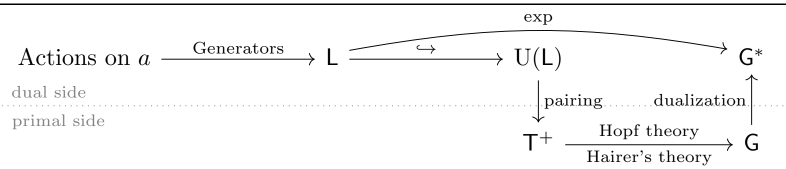
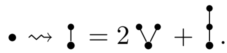


FIGURE 1. Algebraic construction of the group G^*.

where $n(\tau_1, \tau_2, \tau)$ is the number of single cuts performed on τ with branch τ_1 and trunk τ_2, e.g.



The dictionary ϕ can by the pairings ${}_{\mathsf{T}}\langle \beta, \mathsf{z}^\gamma D\rangle_{\mathsf{L}} = \delta_\beta^\gamma$ and ${}_{\mathcal{T}}\langle \tau_1, \tau_2\rangle_{\mathcal{L}} = \delta_{\tau_1}^{\tau_2}$ be transposed to obtain $\phi^\dagger : \mathcal{L} \to \mathsf{L}$, given by $\phi^\dagger(\tau) = \frac{\sigma(\beta)}{\sigma(\tau)}\, \mathsf{z}^\beta D$ provided $\tau \in \mathcal{T}_\beta$.

Proposition 2. $\phi^\dagger$ is a pre-Lie morphism, i.e.

$$\phi^\dagger(\tau_1 \rightsquigarrow \tau_2) = \phi^\dagger(\tau_1) \triangleright \phi^\dagger(\tau_2).$$

By the universality property it lifts to a Hopf algebra morphism $\phi^\dagger : U(\mathcal{L}) \to U(\mathsf{L})$. Yet another pairing on forests of trees between the Grossman-Larson Hopf algebra $U(\mathcal{L})$ and the Connes–Kreimer Hopf algebra $\mathcal{H}$ defined by

$${}_{U(\mathcal{L})}\langle \tau_1 \cdots \tau_k, \sigma_1 \cdots \sigma_l\rangle_{\mathcal{H}} = \delta_k^l \delta_{\tau_1}^{\sigma_1} \cdots \delta_{\tau_k}^{\sigma_k},$$

allows to transpose $\phi^\dagger$ once more and yields a Hopf algebra morphism $\phi : \mathsf{T}^+ \to \mathcal{H}$.

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Algebraic renormalization of rough paths and regularity structures based on multi-indices

PABLO LINARES

The theories of rough paths [13], [8, 9] and regularity structures [10] provide local well-posedness results for RDEs (respectively SPDEs): The fundamental analytic objects they are based on take the form of local expansions with respect to nonlinear functionals of the driving noise, collected in what is called *rough path* (respectively *model*). In their usual approaches, these local expansions and their natural transformations (re-expansions, multiplication, renormalization) are encoded in Hopf algebras of trees, leading to a systematic treatment of semi-linear subcritical singular SPDEs [2, 4, 6]. More recently, and in the context of quasi-linear SPDEs, [14] obtained a priori bounds in a regularity structures set-ups based on multi-indices instead of trees: A deeper algebraic understanding of this new bookkeeping, and particularly of the recentering operation based on multi-indices, was later given in [12].

The purpose of this talk, based on [5, 11], is to provide a complete description of the algebraic structures emerging from multi-indices for a general class of semi-linear equations of the form

$$\mathcal{L}u = \sum_{\mathfrak{l}\in\mathfrak{L}} a_{\mathfrak{l}}(\mathbf{u})\xi^{\mathfrak{l}}, \;\; u:\mathbb{R}^d\to\mathbb{R}, \;\; \mathbf{u} = \big(\tfrac{1}{\mathbf{m}!}\partial^{\mathbf{m}}u\big)_{\mathbf{m}\in\mathbb{N}_0^d}.$$

More precisely, we give a systematic construction of a regularity structure based on multi-indices, including a careful study of the recentering transformations leading to the so-called structure group; we introduce finite counterterms in the original equation and describe a recursive procedure to construct an algebraically renormalized smooth model; and we provide, in the rough path case, the construction of a renormalization group based on multi-indices.

The basis of the constructions consists of variables $\{\mathsf{z}_{(\mathfrak{l},k)}\}$, $\mathfrak{l}\in\mathfrak{L}$, $k\in M(\mathbb{N}_0^d)$, which are placeholders for the derivatives of the nonlinearities $a_{\mathfrak{l}}$; and $\{\mathsf{z}_{\mathbf{n}}\}$, $\mathbf{n}\in\mathbb{N}_0^d$, which represent Taylor coefficients of a local parameterization of the manifold of solutions. Multi-indices arise when considering monomials in these variables z^β. Next to this, we have infinitesimal generators of shifts in the space of solutions (denoted $D^{(\mathbf{n})}$, $\mathbf{n}\in\mathbb{N}_0^d$), which act like pre-Lie products; and of shifts in space-time (denoted $\boldsymbol{\partial}_i$, $i=1,...,d$), which are commutative linear maps. Appealing to the construction of Guin and Oudom [7], combined with suitable grading properties, we derive a Lie algebraic PDE in mild form describing a smooth model, namely

$$\Pi_x = K*\rho\big(\exp(\boldsymbol{\Pi}_x)\big)\sum_{\mathfrak{l}\in\mathfrak{L}}\mathsf{z}_{(\mathfrak{l},0)}\xi^{\mathfrak{l}} + \sum_{\mathbf{n}\in\mathbb{N}_0^d}\mathsf{z}_{\mathbf{n}}(\cdot-x)^{\mathbf{n}},$$

where exp is a symmetric exponential and ρ some action onto the algebra of $\{\mathsf{z}_{(\mathfrak{l},k)}\}$ $\cup\,\{\mathsf{z}_{\mathbf{n}}\}$. Similar techniques give rise to the structure group as a group of symmetric exponentials.

In the description of an algebraically renormalized model, the introduction of a counterterm is reflected in a shift of the form $\mathsf{z}_{(0,0)}\mapsto\mathsf{z}_{(0,0)}+c$, where we think of $\xi^0=1$ and c is a polynomial of $\{\mathsf{z}_{(\mathfrak{l},k)}\}\cup\{\mathsf{z}_{\mathbf{n}}\}$; this, in particular, connects with translation of rough paths [3] and preparation maps [1]. It is also possible to characterize the infinitesimal generators of local counterterms to seek a non-recursive construction of renormalized models; these generators create another pre-Lie algebra, which in the simpler rough path case allows to write translation maps as symmetric exponentials (the general SPDE case would require an enlargement of the structure via extended decorations, cf. [4]).

As reflected in [5, 11, 12], multi-indices encode linear combinations of trees, and have been proven more efficient e. g. when bookkeeping renormalization constants. Understanding if there are any analytic or stochastic interpretations in this grouping of trees is an open problem which was brought up in the posterior discussion.

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Symmetries in stochastic Yang–Mills equations

Ilya Chevyrev

(joint work with Ajay Chandra, Martin Hairer, and Hao Shen)

Recent works [2, 3, 5] have made sense of the stochastic quantisation equations of Yang–Mills (YM) on the torus $\mathbb{T}^d$, $d = 2, 3$, that read (in the DeTurck gauge)

$$\partial_t A^\varepsilon = -\,\mathrm{d}^*_{A^\varepsilon} F_{A^\varepsilon} - \mathrm{d}_{A^\varepsilon}\,\mathrm{d}^* A^\varepsilon + C^\varepsilon A + \xi^\varepsilon = \Delta A^\varepsilon + A^\varepsilon \partial A^\varepsilon + (A^\varepsilon)^3 + C^\varepsilon A^\varepsilon + \xi^\varepsilon\ .$$

Here $A^\varepsilon \colon \mathbb{R}_+ \times \mathbb{T}^d \to \mathfrak{g}^d$ is a 1-form and $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, ξ^ε is an adapted mollification at scale $\varepsilon > 0$ of a $\mathfrak{g}^d$-valued white noise ξ on $\mathbb{R} \times \mathbb{T}^d$, and $\{C^\varepsilon\}_{\varepsilon>0} \subset L(\mathfrak{g}, \mathfrak{g})$ are renormalisation counterterms. In the final expression we write the heuristic form of the non-linearities in the equation.

For $d = 2$ or $d = 3$, there exist choices for C^ε such that, as $\varepsilon \to 0$, the solutions A^ε converge (modulo blow-up) to a space-time distribution A that we call a solution to the stochastic YM equations (SYM) with mass $\{C^\varepsilon\}_{\varepsilon>0}$.

In this report, we describe the argument in [3] based on small-noise limits that shows there is distinguished choice for C^ε such that the solution A is *gauge-covariant* in the following way: if $A(t)$ and $\bar{A}(t)$ are solutions of SYM with mass $\{C^\varepsilon\}_{\varepsilon>0}$ and gauge equivalent initial conditions $A(0) \sim \bar{A}(0)$, then $[A(t)]$ is equal in law to $[\bar{A}(t)]$ (modulo blow-up). Here $[A] = \{B : B \sim A\}$ is the gauge orbit of A where $\sim$ denotes gauge equivalence which, roughly speaking, means that there exists $g\colon \mathbb{T}^d \to G$ such that $A^g := \mathrm{Ad}_g A - (\mathrm{d}g)g^{-1} = B$. Since solutions to SYM are distributions, gauge equivalence needs to be interpreted appropriately.

This result is shown for $d = 2$ and $d = 3$ in [2] and [3] respectively (see also [4] for a survey). It in particular implies that the projected process $[A(t)]$ is Markov. In the case $d = 2$, one can furthermore choose $C^\varepsilon \equiv C$ independent of ε, which is 'atypical' for a singular stochastic PDE. We also mention that in [5] it is shown, for $d = 2$, that the Markov process $[A(t)]$ has a unique invariant measure which is the YM measure on $\mathbb{T}^2$ associated with trivial principal G-bundle and that the operator $C^\varepsilon \equiv C \in L(\mathfrak{g}, \mathfrak{g})$ with the above property is unique.

The first step in both [2, 3] in the proof of the gauge-covariance proprety is the following result that follows from the general theory of regularity structures.

Proposition 1. *There exist operators $C^\varepsilon_{\mathrm{BPHZ}}, \tilde C^\varepsilon, \tilde C^{0,\varepsilon} \in L(\mathfrak{g}, \mathfrak{g})$ such that, for any fixed $\mathring{C}_1, \mathring{C}_2 \in L(\mathfrak{g}, \mathfrak{g})$, the solutions to*

$$\text{(1)} \quad \partial_t B = \Delta B + B\partial B + B^3 + \mathrm{Ad}_g \xi^\varepsilon + (C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_1)B + (\tilde C^\varepsilon + \mathring{C}_2)(\mathrm{d}g)g^{-1}\ ,$$

$$\text{(2)} \quad \partial_t \bar A = \Delta \bar A + \bar A\partial \bar A + \bar A^3 + (\mathrm{Ad}_{\bar g} \xi)^\varepsilon + (C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_1)\bar A + (\tilde C^{0,\varepsilon} + \mathring{C}_2)(\mathrm{d}\bar g)\bar g^{-1}\ ,$$

converge to the same limit in probability as $\varepsilon \downarrow 0$, where g and $\bar g$ solve

$$\partial_t g = \Delta g - (\partial_j g)g^{-1}(\partial_j g) + [Z_j, (\partial_j g)\bar g^{-1}]g \ \textit{ with initial condition } g(0)$$

with Z taken as B and $\bar A$ respectively.

The relevance of this result is that, if we choose $C^\varepsilon = C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_1$, then $B := A^g$ solves (1) provided that $\tilde C^\varepsilon + \mathring{C}_2 = C^\varepsilon$. On the other hand, provided that $\tilde C^{0,\varepsilon} + \mathring{C}_2 = 0$, then by Itô isometry, since $(\mathrm{Ad}_{\bar g}\xi)^\varepsilon$ is equal in law to ξ^ε (which is where we use that $(\mathrm{Ad}_{\bar g}\xi)^\varepsilon$ and thus $\bar g$ are adapted), $\bar A$ is equal in law to A. The gauge-covariance property would thus follow once we show that the limits

$$\text{(3)} \qquad \lim_{\varepsilon\downarrow 0} \tilde C^\varepsilon - C^\varepsilon_{\mathrm{BPHZ}} \ \text{ and } \ \lim_{\varepsilon\downarrow 0} \tilde C^{0,\varepsilon} \ \text{ exist.}$$

This is because, if these limits exist, then we can choose $\mathring{C}_2 = -\lim_{\varepsilon\downarrow 0} \tilde C^{0,\varepsilon}$ and $\mathring{C}_1 = \lim_{\varepsilon\downarrow 0}\{\tilde C^\varepsilon - C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_2\}$ to satisfy the the above conditions with $C^\varepsilon = C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_1$. However, since renormalisation constants generically diverge, it is not clear a priori that (3) holds.

In [2] for $d = 2$, the claim (3) is shown by direct calculation since the number of diverging diagrams involved is rather small (three to be precise).

For $d = 3$, the argument in [3] is different and inspired by the work [1] on manifold-valued stochastic heat equations. We demonstrate this method by showing that $\limsup_{\varepsilon\downarrow 0} |\tilde C^{0,\varepsilon}| < \infty$ without knowing the precise form of $\tilde C^{0,\varepsilon}$.

Arguing by contradiction, suppose $\limsup_{\varepsilon\downarrow 0} |\tilde C^{0,\varepsilon}| = \infty$ and let $\tilde C^{0,\varepsilon}_\sigma$ denote the renormalisation constant arising from a rescaled noise $\sigma\xi$. It is not difficult to see that there exist $\sigma_\varepsilon \downarrow 0$ such that $\tilde C^{0,\varepsilon}_{\sigma_\varepsilon} \to \hat C \neq 0$ as $\varepsilon \downarrow 0$ along a subsequence. Take now bare masses $\mathring{C}_1 = 0$, $\mathring{C}_2 = -\hat C$ in the equation for $\bar A$. Then, by continuity in the noise, $\bar{A}$ converges to the solution of $\partial_t \bar{A} = \Delta \bar{A} + \bar{A}\partial\bar{A} + \bar{A}^3 - \hat{C}\,\mathrm{d}\bar{g}\bar{g}^{-1}$. On the other hand, $\bar{A}$ is equal in law to the solution of

$$\partial_t \tilde{A} = \Delta \tilde{A} + \tilde{A}\partial\tilde{A} + \tilde{A}^3 + C^{\varepsilon}_{\mathrm{BPHZ},\sigma_\varepsilon}\tilde{A} + \sigma_\varepsilon \tilde{\xi}^\varepsilon + (\tilde{C}^{0;\varepsilon}_{\sigma_\varepsilon} - \hat{C})\,\mathrm{d}\tilde{g}\tilde{g}^{-1}\ .$$

Treating $\tilde{C}^{0;\varepsilon}_{\sigma_\varepsilon} - \hat{C}$ as a bare mass that converges to 0, by joint continuity in noise and bare mass, $\tilde{A}$ converges to the deterministic YM heat flow $\partial_t \tilde{A} = \Delta \tilde{A} + \tilde{A}\partial\tilde{A} + \tilde{A}^3$. The limits of $\bar{A}$ and $\tilde{A}$ are not equal since $\hat{C} \neq 0$, which yields a contradiction.

A similar but slightly different argument based on gauge-covariance of the deterministic YM heat flow shows that $\limsup_{\varepsilon\downarrow 0} |\tilde{C}^\varepsilon - C^\varepsilon_{\mathrm{BPHZ}}| < \infty$. With further work, one can show that the limits in (3) actually exist, completing the proof of gauge-covariance. This argument raises the natural question of whether there is an algebraic framework to describe and unify the symmetries appearing in [1, 2, 3].

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Semimartingales with values in a (pre-)Frobenius manifolds

Noémie C. Combe

In the sixties, Pierre Cartier proposed a generalisation of probability theory on richer structures such as manifolds. In this paper we follow this idea. We show that there exists a class of symmetric spaces of Cartan–Hadamard type for which Itô's integrals of 1-forms along semimartingales with value in such a manifold have no divergences. In particular, one can omit the approach relying on perturbative expansion of the functional integral appearing as a sum labelled by Feynman graphs. This is explained by the fact that the manifolds investigated below are Hessian manifolds satisfying the properties of a pre-Frobenius potential manifold and they contain a submanifold which is a Frobenius manifold.

For this class of manifolds, covariant derivatives form a pre-Lie algebra. The fibres of the Frobenius manifold's tangent bundle have the structure of a Frobenius algebra. The fact that one can omit perturbative expansions here relates—among others—to the phenomenon that F-manifold algebras are the corresponding semi-classical limits of pre-Lie formal deformations of commutative associative algebras. Moreover, by [6], the class of Frobenius algebras is a class closed under deformations. Finally, applying the geometric flavoured argument (the "no-go theorem") of [4] ends the discussion.

1. A new approach

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered complete probability space. For X to be a martingale with values in a vector space V with some given connection ∇, it is necessary and sufficient that for any 1-form ω, $Y_t = \nabla \int_{X_0^t} \omega$ is a real local martingale.

We consider the problem of defining semimartingales with values in a Riemannian manifold $(\mathcal{M}, g)$. Let $(\mathcal{M}, g)$ be a Riemannian manifold of dimension n and consider its corresponding frame bundle, with frame $(H_i)_{i=1}^n$. An $\mathcal{M}$-valued semimartingale X is defined throughout the set $Z = (Z^i)_{i=1}^n$ of real semimartingales such that the Stratonovitch formula is

$$Z_t^i = \int_{X_0^t} \omega^i, \tag{1}$$

where $\omega^i \in T^*\mathcal{M}$ is a 1-form in the cotangent bundle $T^*\mathcal{M}$ (see [5] for a precise definition of (1)).

A classical problem in stochastic differential equations is to understand how to reconstruct X from a real semimartingale Z. That is, given a real valued semimartingale Z one looks for the $\mathcal{M}$-valued semimartingale X with given X_0 and satisfying

$$d^2 X_t = h_{X_t}(d^2 Z_t),$$

where h is defined by putting $dx^i = h_a^i \omega^a$.

The Stratonovitch like formula leads to the Itô formula. This step leads to highlighting relations to connections on a manifold:

$$X_t^i - X_0^i = \int_{X_0^t} dx^i = \int_{X_0^t} h_a^i \omega^a = \int_0^1 h_a^i(X_b) \cdot dZ_s^a.$$

Indeed, this implies that $dX_t^i = h_a^i(X_s) dZ_s^a + \frac{1}{2} d\langle h_a^i(X_s), Z_s^a\rangle$. But since

$$d\langle h_a^i(X_s), Z_s^a\rangle = D_j h_a^i(X_s) d\langle X_s^a, Z_s^a\rangle\,, \quad d\langle X^j, Z^a\rangle = h_b^j(X_s) d\langle Z^b, Z^a\rangle_s\,,$$

so $dX_t^i = h_a^i(X_s) dZ_s^a + \frac{1}{2}(D_j h_a^i(X_s) h_b^j(X_s) d\langle Z_s^b, Z_s^a\rangle)$. Symbolically this amounts to writing

$$dX_t = dZ_t^a \cdot H_a(X).$$

2. Results on $(\mathcal{M}, g)$-valued semimartingales

Theorem 1. *Consider one of the following symmetric Riemannian manifolds* $GL_n(\mathbb{R})/SO_n$, $GL_n(\mathbb{C})/SU_n$, $GL_n(\mathbb{H})/Sp_n$ $GL_3(\mathbb{O})/F_4$ *and* $O(1, n-1)/O(n-1) \oplus \mathbb{R}$ *(or a linear combination of those), where* $n \geq 2$ *is a positive integer. If* $\mathcal{M}$ *is one of those manifolds then it possess all required conditions for* $(\mathcal{M}, g)$*-valued semimartingales to be well defined.*

The proof of this relies on the approaches of L. Schwartz, M. Emery, P A. Meyer. We illustrate this on the following fact. Let $X^1, \cdots, X^n$ be continuous semimartingales and $f \in C^2(\mathbb{R}^n)$. Then $Y = f(X)$ is a semimartingale and $dY = \sum_i D_i f(X) dX^i + \frac{1}{2} \sum_{i,j} D_{ij} f(X) d\langle X^i, X^j\rangle$. The rightmost part of the equation is ruled by the connection of the manifold. Connections for the listed above manifolds are torsion-free and the covariant derivatives form a pre-Lie algebra. By [1] those manifolds are potential pre-Frobenius manifolds. This implies that there exists everywhere locally a potential function (given by the Koszul–Vinberg characteristic function) such that the Hessian is non-degenerate.

Furthermore, following [1] any of the spaces listed above obey to a decomposition into two submanifolds:

(1) a flat torus being a totally geodesic submanifold of $(\mathcal{M}, g)$. It carries the structure of a Frobenius manifold; all geodesics lie in that subspace.
(2) A homogeneous Hadamard space, having strictly negative sectional curvature.

This implies the following.

Proposition 1. *Let $(\mathcal{M}, g)$ be a Riemannian manifold. Suppose that $(\mathcal{M}, g)$ is one of the following $GL_n(\mathbb{R})/SO_n$, $GL_n(\mathbb{C})/SU_n$, $GL_n(\mathbb{H})/Sp_n$ $GL_3(\mathbb{O})/F_4$ and $O(1, n-1)/O(n-1) \oplus \mathbb{R}$ (or a linear combination of those), where $n \geq 2$ is a positive integer. Consider $\overline{\mathcal{M}}$ the Frobenius manifold (a flat torus) in $(\mathcal{M}, g)$. Each point of $\overline{\mathcal{M}}$ has an open neighborhood $U \subset \mathcal{M}$ such that for every U-valued martingale X with $X_1 \in \overline{\mathcal{M}}$ a.s the whole process $(X_t)_{0 \leq t \leq 1}$ lives in the Frobenius manifold $\overline{\mathcal{M}}$.*

The proof is based on works of M. Emery [3] and of the theorem in [1].

Proposition 2. *The Frobenius manifold in $(\mathcal{M}, g)$ (where $(\mathcal{M}, g)$ is defined as above) is the locus in which exist pure fluctuations / local martingales.*

Remark 1. *The above Riemannian manifolds parametrise the space of Wishart probability distributions. Wishart laws being exponential we can proceed to a direct application of our statements above and of the main theorem of* [2]*. In the latter, the existence of a Frobenius manifold in a space of probability distributions of exponential type is shown.*

So, as a corollary we have:

Corollary 1. *Let $(\mathcal{M}^W, g^W)$ be a manifold of Wishart distributions (finite dimensional). Then, there exists $\overline{\mathcal{M}}^W$ a Frobenius manifold of $(\mathcal{M}^W, g^W)$ such that for each point of $\overline{\mathcal{M}}$ one has an open neighborhood $U \subset \mathcal{M}$ and for every U-valued martingale X with $X_1 \in \overline{\mathcal{M}}^W$ a.s the whole process $(X_t)_{0 \leq t \leq 1}$ lives in the Frobenius submanifold $\overline{\mathcal{M}}^W$.*

Conclusion. We have explored some aspects of the question raised by P. Cartier. Further developments concerning a discussion and classification of manifolds satisfying good properties for semimartingales is expected.

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Planarly Branched Rough Paths Are Geometric

Ludwig Rahm
(joint work with Kurusch Ebrahimi-Fard)

In 2002 Foissy published the important work [3], where he characterized all finite-dimensional comodules and all endomorphisms of the Butcher–Connes–Kreimer Hopf algebra $\mathcal{H}_{BCK} = (\mathcal{F}, \odot, \Delta_{BCK})$ of non-planar rooted trees. He furthermore constructed a recursively defined projection map onto the primitive elements of the Hopf algebra, and showed a Hopf algebra ismomorphism to the shuffle Hopf algebra generated by the primitives. Almost all of his proofs were based on the so-called natural growth operation

$$\top : \mathcal{H}_{BCK} \otimes \mathcal{H}_{BCK} \to \mathcal{H}_{BCK},$$

and its relation to the reduced coproduct:

$$\hat{\Delta}_{BCK}(x \top y) = x \otimes y + x^{(1)} \otimes x^{(2)} \top y, \tag{1}$$

where y is a primitive element and we use Sweedler's notation for the reduced coproduct

$$\hat{\Delta}_{BCK}(x) = x^{(1)} \otimes x^{(2)}.$$

The Hopf algebra isomorphism constructed by Foissy was later used by Boedihardjo and Chevyrev to interpret branched rough paths as being geometric rough paths [1]. This allowed the authors to consider important results on the well-studied theory of geometric rough paths, and obtain the same results for branched rough paths.

Rough path theory is a very successful theory for solving rough differential equations. A rough path is a two-parameter path taking values in the character group of a Hopf algebra. A branched rough path takes values in the BCK Hopf algebra, and a geometric rough path takes values in a shuffle Hopf algebra. Both of these rough path theories are used for rough differential equations on Euclidean spaces. In [2], the authors constructed so-called planarly branched rough paths to solve rough differential equations on homogeneous spaces. These rough paths are valued in the Munthe-Kaas-Wright Hopf algebra $\mathcal{H}_{MKW} = (\mathcal{OF}, \sqcup, \Delta_{MKW})$.

In this talk we note that the MKW Hopf algebra can be endowed with a natural growth operation, meaning a map that satisfies equation (1) for the reduced MKW coproduct $\hat{\Delta}_{MKW}$. This lets us apply the results Foissy obtained for $\mathcal{H}_{BCK}$, to $\mathcal{H}_{MKW}$. In particular, we obtain a Hopf algebra isomorphism between $\mathcal{H}_{MKW}$ and a shuffle Hopf algebra. We also obtain a way to find the primitive elements via a recursively defined projection map. Following the approach of Boedihardjo and Chevyrev, we can then interpret planarly branched rough paths as being geometric rough paths by using the Hopf algebra isomorphism. Results for geometric rough paths can then be transfered to results for planarly branched rough paths. As an example of this, we obtain the result that two planarly branched rough paths have the same signature if and only if they are tree-like equivalent.

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Numerical approximations to rough solutions of dispersive equations

Yvonne Alama Bronsard

An introduction was given on *resonance-based schemes*, a class of schemes which allows for the approximation at low-regularity to the following class of nonlinear dispersive equation:

$$
\begin{aligned}
&i\partial_t u(t,x) + \mathcal{L}\left(\nabla\right) u(t,x) = p\left(u(t,x), \overline{u}(t,x)\right) \\
&u(0,x) = u_0(x), \quad x \in \mathbb{T}^d,
\end{aligned} \tag{1}
$$

with $\mathcal{L}$ real operator, p polynomial nonlinearity. The idea behind their construction was illustrated on the prototypical Nonlinear Schrödinger equation (NLS):

$$
i\partial_t u(t,x) = -\Delta u(t,x) + |u(t,x)|^2 u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d, \ + \text{I.C},
$$

where one goes to Fourier variables in space to carefully extract dominant and lower-order contributions appearing from the interaction of the linear evolution and the nonlinearity. This idea was worked out by [6] in the first order case. A generalization was made by [7] which allows for first and second order low-regularity approximation to a class of nonlinear evolution equation set on more general domains. These new schemes, together with their optimal local error, allow for convergence under lower regularity assumptions than required by classical methods, such as exponential integrator or splitting methods.

Higher order extensions were then presented, following new techniques based on decorated trees series inspired by singular SPDEs via regularity structures. The work of [4] was first presented, where the authors derive resonance-based schemes up to arbitrary order for solving the class of equations (1).

We then presented the work [2] which considers the case of a randomized initial condition of the form:

$$u(0,x) = v^{\eta}(x) = \sum_{k\in\mathbb{Z}^d} v_k \eta_k(\omega) e^{ikx}, \tag{2}$$

with $(\eta_k)_{k\in\mathbb{Z}}$ i.i.d standard complex Gaussians. By letting u be solution of (1) starting from the randomized initial data (2), we obtained higher order approximations to the second moment $\mathbb{E}(|u_k(t,v^{\eta})|^2)$, together with a formal local error bound. This second order moment is a central quantity of interest for the derivation of the Wave Kinetic equation. This equation is widely used in oceanography for the forecasting of waves in the ocean.

A limitation of the former resonance-based approaches was, since the algorithm for extracting dominant parts depended on Fourier computations, the method is restricted to spatial domains which are periodic. In the work [3] we consider systematizing the higher order derivation of low-regularity schemes for the following class of nonlinear evolution equations set on more general domains:

$$\partial_t u - \mathcal{L}u = \sum_{\mathfrak{l}} f_{\mathfrak{l}}(u,\overline{u}) V_{\mathfrak{l}}, \qquad (t,x)\in\mathbb{R}\times\Omega,\ \Omega\subseteq\mathbb{R}^d.$$

This work was inspired by the work [1] which dealt with the first and second order low-regularity approximation to the Gross-Pitaevski equation. In the work [3] we extended it to higher orders and for a more general class of nonlinear evolution equations, using the commutators and filtering functions introduced in [7].

We finished by presenting a symmetric low-regularity schemes for the NLS equation, which exactly conserves time-reversibility of the underlying equation. We explained on the one hand the construction of this symmetric scheme, which inherits much better structure preserving properties on the discrete level than previous low-regularity schemes. On the other hand we presented rigorous low-regularity error analysis results.

Higher order construction of a class of symmetric schemes using the previously introduced tree formalism was briefly discussed through our recent joint work [4].

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Resonances as a computational tool

Katharina Schratz

A large toolbox of numerical schemes for dispersive equations has been established, based on different discretization techniques such as discretizing the variation-of-constants formula (e.g., exponential integrators) or splitting the full equation into a series of simpler subproblems (e.g., splitting methods). In many situations these classical schemes allow a precise and efficient approximation. This, however, drastically changes whenever non-smooth phenomena enter the scene such as for problems at low regularity and high oscillations. Classical schemes fail to capture the oscillatory nature of the solution, and this may lead to severe instabilities and loss of convergence. In this talk I present a new class of resonance based schemes. The key idea in the construction of the new schemes is to tackle and deeply embed the underlying nonlinear structure of resonances into the numerical discretization.

Let me explain the key idea behind resonances as a computational tool on the nonlinear PDE[1]

$$\partial_t u(t,x) + i\mathcal{L}\left(\nabla, \varepsilon^{-1}\right) u(t,x) = f\big(u(t,x)\big), \quad u(0,x) = u_0(x) \tag{1}$$

which covers a large class of important models, e.g., Schrödinger ($\mathcal{L} = -\Delta$), KdV ($\mathcal{L} = -i\partial_x^3$) and half-wave ($\mathcal{L} = \sqrt{-\Delta}$) equations, wave maps, Zakharov, Kadomtsev–Petviashvili, and many more systems.

The symmetric differential operator $\mathcal{L}\left(\nabla, \varepsilon^{-1}\right)$ thereby triggers oscillations (in space and/or in time) and, unlike for parabolic problems, no smoothing can be expected. At low regularity, e.g., for rough solutions and in highly oscillatory regimes $\varepsilon \to 0$, it is therefore crucial to capture these oscillations numerically. Most classical schemes were originally developed for linear problems and fail to resolve the nonlinear frequency interactions in system (1).

The key idea to overcome this is to understand, control and deeply embed the nonlinear resonance structure (driven by the nonlinear frequency interaction of the operator $\mathcal{L}$ and nonlinearity f in (1)) into the numerical discretisation. In order to achieve this we have to first understand the behaviour of the nonlinear PDE (1). Duhamel's formula (suppressing the x-dependence) reads

[1] We include the parameter ε^{-1} to also cover relativistic regimes, e.g., relativistic Klein–Gordon with $\mathcal{L} = \varepsilon^{-1}\sqrt{\varepsilon^{-2} - \Delta}$

$$u(t) = e^{-it\mathcal{L}}u(0) + \int_0^t \mathrm{e}^{-i(t-\xi)\mathcal{L}} f(u(\xi))\mathrm{d}\xi \tag{2}$$

with the next iteration (i.e., using that $u(\xi) = \mathrm{e}^{-i\xi\mathcal{L}}u(0) + \int_0^\xi \dots \mathrm{d}\xi_1$) given by

$$u(t) = e^{-it\mathcal{L}}u(0) + \int_0^t \mathrm{e}^{-i(t-\xi)\mathcal{L}} f\left(\mathrm{e}^{-i\xi\mathcal{L}}u(0)\right)\mathrm{d}\xi + \int_0^t \int_0^\xi \dots \mathrm{d}\xi\mathrm{d}\xi_1. \tag{3}$$

At first order we can neglect the higher order terms (i.e., the double integral) and observe that the underlying structure of the solution is driven by the nonlinear frequency interaction of $\mathcal{L}$ and f with central oscillations of the form

$$\mathrm{e}^{i\xi\mathcal{L}} f\left(\mathrm{e}^{-i\xi\mathcal{L}}u(0)\right). \tag{4}$$

Classical numerical methods are based on linear frequency approximations, (e.g., splitting schemes, Gautschi-type, exponential and Lawson methods with possible filter functions, and in general neglect the nonlinear interactions in (4). For instance, in case of splitting or an exponential approach the underlying frequency approximations read

$$\text{(splitting)} \qquad \mathrm{e}^{i\xi\mathcal{L}} f\left(\mathrm{e}^{-i\xi\mathcal{L}}u(0)\right) \approx f\left(u(0)\right) \tag{5}$$

While such linearised frequency approximations are computational very handy (as on the right-hand side of (5) no oscillations anylonger appear), they dramatically destroy the underlying structure of the PDE (1). This is due to the fact that nonlinear frequency interactions play an essential role (especially on bounded domains, where no dispersion can be expected) and can heavily impact the solution: Note that while the influence of $i\mathcal{L}$ can be small, the influence of the interaction of $+i\mathcal{L}$ with $-i\mathcal{L}$ can be huge, and *vice versa*. The central idea lies in a new nonlinear approach: Instead of linearising the frequency interactions in the central oscillations (4) (as done in (5)) the key idea is to filter out the dominant parts of the oscillations and solve them exactly while only approximating the lower order terms in spirit of

$$\mathrm{e}^{i\xi\mathcal{L}} f\left(\mathrm{e}^{-i\xi\mathcal{L}}u(0)\right) \approx \left[\mathrm{e}^{i\xi\mathcal{L}_{\mathrm{dom}}} f_{\mathrm{dom}}(u(0))\right] f_{\mathrm{noc}}(u(0)) + \text{lower order terms.} \tag{6}$$

Here, $\mathcal{L}_{\mathrm{dom}}$ denotes a suitable dominant part of the high frequency interactions and f_{noc} the corresponding non-oscillatory part. A first attempt of so-called *resonance-based schemes* (Schratz et al. [4]), based on the approximation (6), was profoundly inspired by major breakthroughs in the theoretical analysis of dispersive equations at low regularity (Bourgain [3], Tao [8]) and rough path theory (Gubinelli [6]) and provides a powerful tool which in many situations allows for approximations in a much more general setting (i.e., for rougher data) than classical schemes (e.g., Splitting with $\mathcal{L}_{\mathrm{dominant}} = 0$ cf. (5)), see also the recent important works [2, 5, 7, 9, 10] and references therein.

The severe shortcoming of the approach (6), however, lies in the fact that the corresponding resonance-based schemes are *not* structure preserving as they do not take the underlying geometric structure of PDEs into account. Lack of structure preservation is also observed drastically in numerical experiments and, as for classical schemes, breaks down the earlier and earlier the rougher the solutions becomes.

This is an open question, and up to now only symmetric schemes could be found [1].

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Multiparameter iterated integrals

Joscha Diehl

Iterated sums and integral have in the last decade found great success in data science applications. Whereas the original domain of their application is to data indexed by one parameter, i.e. time series, there are recent investigations of multiparameter generalizations [2, 8, 4, 3].

The success of iterated sums/integrals is partly explained by the fact that their calculation is possible in linear time, owing to a dynamic programming principle. It finds its algebraic counterpart in *Chen's formula*, which establishes a connection between the concatenation of words and the concatenation of time series.

For multi-parameter objects, the situation is more complicated. There is no canonical way to concatenate two objects, and, apart from special cases, none of the algebraic structures in the mentioned papers is compatible with the different concepts of concatenation. This has the consequence that the calculation of the multiparameter sums from [2] or the multiparameter integrals from [3] is, in general, not possible in linear time (lower complexity bounds for special cases are established in [2]).

Let us illustrate the problem with a simple example. Consider the following integral of a two-parameter function $Z : [0,1]^2 \to \mathbb{R}$:

$$\int_{\substack{0\le r_1^1<r_1^2\le t_1\\ 0\le r_2^1<r_2^2\le t_2}} Z_{r_1^1,r_2^1} Z_{r_1^2,r_2^2} dr$$

Now, we try to split this integral in the horizontal direction at some point $u_1 < t_1$:

$$\ldots = \int_{\substack{0\le r_1^1<r_1^2<u_1\le t_1\\ 0\le r_2^1<r_2^2\le t_2}} Z_{r_1^1,r_2^1} Z_{r_1^2,r_2^2} dr + \int_{\substack{u_1<r_1^1<r_1^2\le t_1\\ 0\le r_2^1<r_2^2\le t_2}} Z_{r_1^1,r_2^1} Z_{r_1^2,r_2^2} dr$$
$$+ \int_{\substack{0\le r_1^1<u_1<r_1^2\le t_1\\ 0\le r_2^1<r_2^2\le t_2}} Z_{r_1^1,r_2^1} Z_{r_1^2,r_2^2} dr.$$

Note that the last term presents an issue, since the integral cannot be split into a product of two integrals, as it would in the one-parameter case.

The problem of (naive) non-multiplicativity of multi-parameter integrals is well-known in category theory and it has been addressed with techniques from higher categories, see for example [1, 7] for entry points into the literature.

In the work in progress presented, which is joint with Ilya Chevyrev, Kurusch Ebrahimi-Fard, and Nikolas Tapia, we build on the work of [5] to realize an analog to the classical iterated-integrals signature that *does* satisfy a Chen-like identity and allows for a linear-time calculation. An important ingredient is the notion of crossed modules of Lie algebras, in particular the free crossed module of Lie algebras over the free Lie algebra over $\mathbb{R}^n$. Here n is the dimension of the ambient space of the data.

The techniques are closely related to the recent work [6], but the two approaches differ in at least two aspects:

(1) We work with the free crossed module of Lie algebras, whereas [6] works with a specific crossed module. Our current expectation is that the object obtained by us is universal in the sense that any “surface development” in another crossed module can be arbitrarily well approximated by terms in our object.
(2) We consequently do calculations in the Lie *algebra*, in what can be considered a Magnus-like expansion.

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From bialgebras to algebraic operads

Pedro Tamaroff

Algebraic homotopy theory explains that, in order to find new robust and more general definitions of algebras (algebras up to homotopy), one can find (possibly non-minimal) models for operads. These also allows us understand the homotopy theory of their (co)algebras and, in particular, define their deformation theory. This has been done in many situations: for associative algebras [1], commutative algebras, Lie algebras [2], Gerstenhaber and Poisson algebras [3], and Batalin–Vilkovisky algebras [4, 5, 6], among others. In the context of the current workshop, it is useful to remark that pre-Lie algebras fall within the scope of Koszul duality theory, through which most of the previous examples are handled, and other interesting recent generalizations due to P. Laubie [7] involving Greg trees decorated by coalgebras, have been proved to also fall within the scope of Koszul duality.

The present talk explained how to take an operadic point of view of the well known fact that, for any bialgebra H, the category of left H-modules admits an internal tensor product —defined through the so called diagonal action of H— coming from its coproduct. This means that it makes sense to consider associative algebras in the category ${}_H\mathrm{mod}$ of left H-modules, which we show are controlled by an algebraic operad. There is a functor $H \longmapsto \mathsf{Ass}_H$ that assigns to each weight graded bialgebra H a weight graded operad Ass_H so that an associative algebra in ${}_H\mathrm{mod}$ is the same as an Ass_H-algebra. The idea of producing such functors from certain "amenable" categories to study operads and related structures, or even producing endofunctors on operads themselves, has already appeared several times in the literature, see [18, 11, 19] and [13]*Chapter 4, for example.

Unraveling the definitions, we see that the way the associative product x_1x_2 of an Ass_H-algebra and an operation T_h coming from $h \in H$ behave with respect to each other is dictated by the coproduct of H: using Sweedler notation, we require that the following compatibility relation holds (á la Boardmann–Vogt [8]) $T_h(x_1x_2) = T_{h_{(1)}}(x_1)T_{h_{(2)}}(x_2)$. This relation is not quadratic, so the operad Ass_H falls outside the scope of the theory of Koszul duality, in strong contrast to the examples we mentioned above. To counter this, we use the methods of V. Dotsenko [19] (word operads) and pertubation theoretic methods in the spirit of B. Vallette and S. Merkulov [9] to show how to obtain a minimal model of Ass_H from an associated quadratic operad $q\mathsf{Ass}_H$ which, in case H is Koszul, is itself a Koszul operad. Moreover, we showed that this functor behaves well with respect to Gröbner bases: one can directly compute one of Ass_H in case H admits a Gröbner basis. In particular, we showed that $q\mathsf{Ass}_H$ is *strongly Koszul* —that it admits a quadratic Gröbner basis— in case H is itself strongly sKoszul.

The takeaway is that we can explicitly describe the differential of the minimal model of Ass_H provided we can do this for the Koszul model of H and its coproduct. This problem, pertaining to the domain of algebras and coalgebras, is usually a simpler problem to tackle, so our result gives a useful bridge to solve from a much familiar problem a seemingly more complicated one. The theory of Koszul duality for usual associative algebras, on the other hand, has existed for almost five decades since its inception in [10], and now extensive literature and methods exist to deal with them and with many of their variants; see for example [12, 14, 15, 16, 17].

As a by product, the talk introduced key concepts in the study of algebraic operads and their theory, which lead to and allowed for a detailed discussion of the results of P. Laubie [7] regarding families of pre-Lie algebras with a common Lie bracket by participant U. Nadeem. In particular, Koszulness of the operads constructed by Laubie, which follow from the existence of Gröbner bases for them, were discussed, and compared to the results presented in the talk.

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An algebraic geometry of (rough) paths

Rosa Preiss

In previous work, see e.g. [1], the complex projective Zariski closure of the finite dimensional semialgebraic set that is $\sigma^{(k)}(\mathcal{X}_\ell)$, where $\mathcal{X}_\ell$ is piecewise linear paths/polynomial paths/log-linear rough paths of order ℓ.

Our new approach, however, is to introduce a Zariski topology and algebraic geometry on the infinite dimensional path space itself, see the preprint [2].

The algebraic and combinatorial structure we are working with is

$$(T(\mathbb{R}^d), \sqcup\!\sqcup, \Delta_\bullet, \mathcal{A}, \succ, \prec),$$

where $(T(\mathbb{R}^d), \sqcup\!\sqcup, \Delta_\bullet, \mathcal{A})$ is the well known shuffle-deconcatenation Hopf algebra. Let the right $\succ$ and left $\prec$ halfshuffles be recursively defined by

$$w \succ \mathtt{i} := w\mathtt{i}, \qquad \mathtt{i} \prec w := \mathtt{i}w$$

$$w \succ v\mathtt{i} := (w \succ v + v \succ w)\mathtt{i}, \qquad \mathtt{i}v \prec w := \mathtt{i}(w \prec v + v \prec w)$$

Then $x \sqcup\!\sqcup y = x \succ y + y \succ x = x \prec y + y \prec x$ and $\mathcal{A}(x \succ y) = \mathcal{A}y \prec \mathcal{A}x$, $\mathcal{A}(x \prec y) = \mathcal{A}y \succ \mathcal{A}x$. Let $\langle W \rangle_\succ$ denote the $\succ$-ideal generated by W.

In classical algebraic geometry, affine varieties in $\mathbb{R}^d$ are sets of the form $V(P) = \{x \in \mathbb{R}^d | p(x) = 0\, \forall p \in P\}$, where P is a set of polynomials $p : \mathbb{R}^d \to \mathbb{R}$.

Similarly, we now consider varieties in the space $C^{2^- \text{-var}}(\mathbb{R}^d)$ of continuous paths in $\mathbb{R}^d$ with finite p-variation for some $p < 2$. We call an *affine path variety* any subset of the form

$$\mathcal{V}(W) := \{X \in C^{2^- \text{-var}}(\mathbb{R}^d) | \langle \sigma(X), x \rangle = 0\, \forall x \in W\}, \quad W \subseteq T(\mathbb{R}^d)$$

They form the closed sets of what we introduce as the *path Zariski topology*. Path varieties are in 1-to-1 correspondance to the 2^--var 'radical' shuffle ideals

$$\mathcal{I}(U) := \{x \in T(\mathbb{R}^d) | \langle \sigma(X), x \rangle = 0\, \forall X \in U\}, \quad U \subseteq C^{2^- \text{-var}}(\mathbb{R}^d).$$

$\mathcal{V} \circ \mathcal{I}$ is the closure operator, and $\mathcal{I} \circ \mathcal{V}$ is the 2^--var radical operator. Our first main result is the following.

Theorem 1. *Whenever a set of paths U contains history, i.e. all left subpaths of reduced paths, $\mathcal{I}(U)$ is a $\succ$-ideal. Whenever I is a $\succ$-ideal, $\mathcal{V}(I)$ contains history.*

The next corollary is of key importance.

Corollary 1. *Let $p : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map with $p(0) = 0$. Then $\mathcal{V}(\langle \varphi(p_i), i \rangle_\succ)$ is the variety $P_{\in M}$ of all paths X such that $\check{X} - X_0$ lies in the point variety M defined by the vanishing of all p_i.*

This allows us to define rough paths on point varieties! Indeed, it makes sense to demand that geometric rough paths living on an affine point variety should be those which are limits of smooth paths living on that point variety. This is a strictly stronger property than just the underlying path living on that point variety!

Our second main result concerns another way of using the time ordered aspect of paths, through concatenation.

Theorem 2. *If $M \subseteq C^{2^- \text{-}var}(\mathbb{R}^d)$ is a set of paths closed under concatenation, then the variety $\bar{M}$ is closed under concatenation, time reversal and taking admissible roots, and $\mathcal{I}(M)$ is a Hopf ideal.*

Corollary 2. *The set of lattice paths $\mathfrak{L}$ is Zariski dense in $C^{2^- \text{-}var}(\mathbb{R}^d)$.*

To summerize, if an affine path variety V contains history then $\mathcal{I}(V)$ is a half-shuffle ideal, and thus its coordinate ring $\mathbb{R}[V] := T(\mathbb{R}^d)/\mathcal{I}(V)$ is a Zinbiel algebra again.

If an affine path variety V is stable under concatenation, then $\mathcal{I}(V)$ is a Hopf ideal, and this means $\mathbb{R}[V] := T(\mathbb{R}^d)/\mathcal{I}(V)$ is a Hopf algebra.

An important remark, however, is that to understand the geometrical structure of V, we need the algebraic structure of the coordinate ring $\mathbb{R}[V]$ **plus** the 2^--var radical operator on the power set of $\mathbb{R}[V]$. At least until we can find a purely algebraic characterization of $\mathcal{I} \circ \mathcal{V}$, and can answer whether the radical operator can be derived from the ring structure of $\mathbb{R}[V]$ alone, or not.

In the discussion led by Ludwig Rahm we answered that while understanding $\mathcal{I} \circ \mathcal{V}$ for $C^{2^- \text{-var}}(\mathbb{R}^d)$ is a very hard problem, the solution to which would in particular solve the important open problem about how to characterize the image of the signature, understanding the radical operator for piecewise linear paths and polynomial paths should be feasible much earlier. Furthermore, as also brought up in a question by Ludwig Rahm, generalizations of our approach to maps from subsets of $\mathbb{R}^n$ to $\mathbb{R}^d$, instead of just time dependend paths, will become relevant. Finally, as asked by Ilya Chevyrev, a generalization of the notion of variety to vanishings $\langle \sigma(X), x \rangle = 0$ for infinite series $x \in T((\mathbb{R}^d))$ which can be paired with the signature is another opportunity for future work.

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Branched Itô formula

Nikolas Tapia
(joint work with Carlo Bellingeri, Emilio Ferrucci)

Branched rough paths were introduced by Gubinelli [8], as an extension of Lyons' original approach [9], in order to encode iterated integrals of processes that do not satisfy an integration-by-parts rule. These are defined as families of characters over the Connes–Kreimer Hopf algebra $\mathcal{H}_{\mathrm{CK}}$ [3] of non-planar decorated rooted trees satisfying certain regularity and compatibility conditions. By leveraging Foissy's decomposition of this Hopf algebra in terms of its primitive elements [6] and iterations of the *natural growth* operator [2], we show that nonetheless an integration by parts rule is still satisfied. Primitive elements can be interpreted as higher-order variations of the process, analogous to the stochastic bracket present in classical Itô calculus, in the sense that they describe the correction terms in said formula. The algebraic structure precisely describing this new integration-by-parts identities is that of a $\mathbf{B}_\infty$-algebra [7].

Let $\mathcal{P}$ denote the space of primitive elements, $\pi\colon \mathcal{H}_{\mathrm{CK}} \to \mathcal{P}$ be Foissy's projection and $\mathcal{Q} := \operatorname{im}(\pi)^\perp$. Denote by $\mathcal{F}_+$ the set of non-empty forests. We define rough differential equations with drifts, as solutions to RDEs of the form

$$\mathrm{d}y = \sum_{f \in \mathcal{F}_+} F_{\pi^*(f)}(Y)\, \mathrm{d}X^{\pi(f)},$$

where $F \in \mathcal{L}(\mathcal{Q}, C^\infty(\mathbb{R}^n, \mathbb{R}^n))$ is a given collection of vector fields, and show they satisfy the following change of variable formula: there exists a family of differential operators $\mathbf{F}\colon \mathcal{L}(\mathcal{Q}, \mathrm{Diff}(\mathbb{R}^n))$ such that for any smooth observable $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ we have

$$\varphi(y_t) = \varphi(y_0) + \sum_{f \in \mathcal{F}_+} \int_0^t \mathbf{F}_{\pi^*(f)} \varphi(y_u)\, \mathrm{d}\mathbf{X}_u^{\pi(f)}, \tag{1}$$

where the integrals are defined in the rough sense.

In the case of quadratic drift, i.e., an Itô SDE, (1) coincides with the classical Itô formula. The definition of $\mathbf{F}_q$ fully relies on the pre-Lie structure of vector fields on Euclidean space, and the proof relies on an extended form of Davie expansion including corrections induced by the drifts.

We also show that quasi-geometric rough paths correspond to a particular quotient of branched rough paths, and therefore an analog of formula (1) holds in that case. This is connected to already-known formulas [1, 5].

Question 1. Any coalgebra equipped with a family of 1-cocycles indexed by its primitive elements is cofree. What other kinds of rough paths can be show to satisfy integration-by-parts, and therefore an Itô formula? This question has been partially answered by K. Ebrahimi-Fard and L. Rahm.

Question 2. In regularity structures Hopf algebras also play an important role. Is it possible to obtain a similar decomposition for positive and/or negative renormalization? If so, what would be the interpretation of primitive elements in that context?

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Natural Itô-Stratonovich isomorphism

Emilio Ferrucci

(joint work with Carlo Bellingeri, Nikolas Tapia)

Following the theory introduced by Nikolas Tapia in a previous talk, we consider the Connes–Kreimer Hopf algebra $\mathcal{H}_{\mathrm{CK}}$ as a commutative $\mathbf{B}_\infty$-algebra over its primitive elements $\mathcal{P}$. After introducing the Eulerian idempotent of a Hopf algebra and some of its properties, we use it, together with Foissy's idempotent $\pi\colon \mathcal{H}_{\mathrm{CK}} \to \mathcal{P}$, to define an explicit Hopf isomorphism from the shuffle algebra over $\mathcal{P}$ to $\mathcal{H}_{\mathrm{CK}}$. This isomorphism can be used to transform branched rough paths to geometric ones of inhomogeneous regularity over a larger space. Compared to the work of [1, 2], who considered this problem previuosly, our isomorphism has the distinguishing property of being a natural transformation when $\mathcal{H}_{\mathrm{CK}}$ and the shuffle algebra are viewed as a covariant functor in the decorating vector spaces. The motivation for naturality comes from, among other things, the requirement that our theory continue to work when shifted to the setting of smooth manifolds. We compare our isomorphism with Hoffman's exponential [3], which can be obtained from it, but which contains strictly fewer terms: those not present come from interactions between forests that may contain edges. Our work has since appeared on arXiv [4].

During the Q&A, the question of uniqueness (subject to naturality) came up. During the discussion portion, Yvain Bruned brought up the question of how the theory might develop along similar lines when $\mathcal{H}_{\mathrm{CK}}$ is replaced with more recently-introduced Hopf algebras that appear in the context of regularity structures. When cofreeness no longer holds, it might not be reasonable to find an isomorphism, and a natural epimorphism may be the next best thing.

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Solution theory for quasilinear generalised KPZ Equation

Muhammad Usama Nadeem
(joint work with Yvain Bruned and Mate Gerencser)

In this work, we provide a (local-in-time) solution theory for the so called quasi-linear generalised KPZ equation defined on the 1-d torus, that takes the following form:

$$\partial_t u - a(u)\partial_x^2 u = \underbrace{f(u)(\partial_x u)^2 + k(u)\partial_x u + h(u)}_{F_1} + \underbrace{g(u)}_{F_\Xi}\xi. \tag{1}$$

Here f, k, h, and g are assumed to be smooth (although one may be able to survive with functions regular enough), a is smooth with the additional constraint of being bounded by some $c \in \mathbb{R}_+$, and ξ is a random spacetime distribution - the quintessential example being that of the spacetime white noise. This equation falls under the umbrella of singular Stochastic Partial Differential Equation (SPDEs) of the parabolic type and as such the regularising effect of the dynamics fall short of facilitating the pathwise understanding of certain products in the equation. In the equation above and the sort of random fields we are after, $g(u)(\partial_x u)^2$ for example, does not make sense.

In the semilinear case (i.e. when $a(u) \equiv 1$) the advent of the theories such as regularity structures [7] have provided a definite solution to this problem. A major component of this theory is the notion of (negative) renormalisation, which amounts to subtracting infinite constants (dubbed renormalisation constants) from the equation to cure the divergence caused by the ill-defined products in the equation. From a generalisation of Hairer's work [2] we can quote the renormalised equation for the semilinear gKPZ:

$$\partial_t u_\varepsilon - \partial_x^2 u_\varepsilon = f(u_\varepsilon)(\partial_x u_\varepsilon)^2 + k(u_\varepsilon)\partial_x u_\varepsilon + h(u_\varepsilon) + \sum_\tau \frac{\Upsilon_F[\tau]}{S(\tau)} C_\varepsilon(\tau) + g(u_\varepsilon)\xi_\varepsilon, \tag{2}$$

where ξ_ε is a mollified version of the noise, τ denotes rooted trees that encode multiple stochastic integral, $S(\tau)$ is a symmetry factor, $C_\varepsilon(\tau)$ are the renormalisation constants, and $\Upsilon_F[\tau]$ are elementary differentials that are defined by taking derivatives of F_1 and F_Ξ. This has also inspired investigation into potential adaptation of these techniques to quasilinear problems. We refer the reader to [5, 8, 9, 10, 11] for the existing state of this work. The present work is an extension of the work [5], wherein the authors employ an innovative extension of the Hairer's original work. What the authors realise in [5], is that the local-in-time solution of (1) solves the following the system of equations:

$$
\begin{aligned}
&u = I(a(u), \hat{F}), \qquad v_\alpha = I_\alpha(a(u), \hat{F}), \\
&\hat{F} = \left[q(f - a') + a(a')^2 v_{(2,0)} + a a'' v_{(1,0)}\right] (\partial_x u)^2 \\
&\quad + 2(aa')(u)(\partial_x u) v_{(1,1)} + a'(u)(\partial_x u) v_{(0,1)} + \hat{g}(u)\xi.
\end{aligned} \tag{3}
$$

The benefit of this reformulation is that the existing results of the semilinear SPDEs can be applied with only minor changes. Unfortunately, due to the restrictions of the methodology (1) remains out of reach of this work. We ameliorate this condition by introducing some new abstract derivatives ∂_{v_α} for $\alpha \in \mathbb{N}^2$, wherewith we define new elementary differential equations: $\Upsilon_{\hat{F}}$ and Υ_{V_α}. This allows us to prove the following theorem:

Theorem 1. *The renormalised version of* (1) *is given by:*

$$
\begin{aligned}
\partial_t u_\varepsilon - a(u_\varepsilon)\partial_x^2 u_\varepsilon = f(u_\varepsilon)(\partial_x u_\varepsilon)^2 + k(u_\varepsilon)\partial_x u_\varepsilon + h(u_\varepsilon) + g(u_\varepsilon)\xi_\varepsilon \\
+ \sum C_\varepsilon^c(\tau) \frac{\Upsilon_{\hat{F}}[\tau](u_\varepsilon)}{S(\tau)}.
\end{aligned} \tag{4}
$$

Also, the local solutions u_ε on $\mathbb{T}$ endowed with an initial condition $u_\varepsilon(0, \cdot) = \varphi \in \mathcal{C}^{2\delta}(\mathbb{T})$, converge in probability in $\mathcal{C}_\star^\delta$ to a nontrivial limit u.

The defect that [5] suffers from, and by extension our work, is that there is no systematic way of proving that the renormalisation constants C_ε^c do not depend on the non-local terms v_α that were introduced into the equation when transforming the equation into the non-divergence form. The way this is dealt with by them, is to prove that the constants satisfy some integration by parts [4, Lemma 2.4], and then use this result to check for each τ that the non-local terms cancel out [4, Section 3.4]. The problem with this approach is that it very easily becomes unwieldy, due to the amount of calculations involved. Our solution to this problem is to recognise that the integration-by-parts formula is just a specific case of the chain rule that was derived in [3]. To leverage the results of that paper we need to specify a set of covariant derivatives that are capable of generating the space of τ, and at the same time are independent of the non-local terms. By positing these, we suspect the following result is immediate from the arguments in [3]:

Conjecture 1. *The renormalised equation of* (1) *is given by:*

$$
\begin{aligned}
&\partial_t u_\varepsilon - a(u_\varepsilon)\partial_x^2 u_\varepsilon = \\
&f(u_\varepsilon)(\partial_x u_\varepsilon)^2 + k(u_\varepsilon)\partial_x u_\varepsilon + h(u_\varepsilon) + g(u_\varepsilon)\xi_\varepsilon + \sum_\tau C_\varepsilon^{a(u_\varepsilon)}(\tau)\frac{\Upsilon_F[\tau](u_\varepsilon)}{S(\tau)}
\end{aligned}
$$

where the $C_\varepsilon^{a(u)}(\tau)$ satisfy certain chain rule identities. Moreover u_ε converges in the same sense as before.

Some possible open problems in this programme include that of global-in-time solutions, identification of the Butcher series for these problems á la [1], and finally one could look at the same problem in some other manifold.

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Novikov algebras and multi-indices in regularity structures

Yvain Bruned

(joint work with Vladimir Dotsenko)

We are looking at the class of subcritical semi-linear stochastic partial differential equations (SPDEs) of the form

$$(\partial_t - \mathcal{L})\, u = \sum_{\mathfrak{l} \in \mathfrak{L}^-} a^{\mathfrak{l}}(\mathbf{u}) \xi_{\mathfrak{l}}. \tag{1}$$

where $\mathfrak{L}^-$ is a finite set, $\mathcal{L}$ is a differential operator, $\xi_{\mathfrak{l}}$ are space-time noises and $a^{\mathfrak{l}}(\mathbf{u})$ are non-linearities depending on the solution u and its derivatives. This class of equations have been successfully treated via the theory of Regularity Structures [9, 3, 1, 6]. The resolution is based on new Taylor expansions whose monomials are recentered iterated integrals that can be described in a systematic way via decorated trees in [3]. More recently, another index set has been proposed in [13, 11] for quasi-linear SPDEs. It has been extended in [4] for covering equations of the form (1). The simplest possible instance of multi-indices corresponds to considering a set of abstract variables $(z_k)_{k \in \mathbb{N}}$, where the variable z_k encodes nodes of the tree that have k children. Multi-indices β over $\mathbb{N}$ can be represented as monomials

$$z^\beta := \prod_{k\in\mathbb{N}} z_k^{\beta(k)}.$$

The pre-Lie product on the vector space of such monomials is defined as

$$z^\beta \triangleright z^{\beta'} = z^\beta D(z^{\beta'}), \quad D = \sum_{k\in\mathbb{N}} (k+1) z_{k+1} \partial_{z_k}.$$

The action of this operator corresponds to adding one child to one of the nodes of our tree in all possible ways. We focus on multi-indices satisfying the so called "populated" condition [11]:

$$\sum_{k\in\mathbb{N}} (1-k)\beta(k) = 1.$$

It was conjectured by Dominique Manchon that populated multi-indices form the free Novikov algebra. A Novikov algebra is a vector space equipped with a bilinear product $x, y \mapsto x \triangleright y$, satisfying the identities

$$(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z),$$
$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright y.$$

This type of algebras was considered in [8, 5, 12]. It turns out that the corresponding theorem does exist in the literature; it goes back to [7].

Theorem 1. [7, 10, 2] *The Novikov algebra of populated multi-indices is isomorphic to the free algebra on one generator.*

One first extends this theorem to general multi-indices defined using formal variables of the form $z_{(\mathfrak{l},w)}$ with $\mathfrak{l}$ belongs to $\mathfrak{L}^-$ and w is a commutative monomial in the alphabet $A = \mathbb{N}^{d+1}$. One can define a collection of derivations $D^{(\mathbf{n})}$ indexed by A. These very general multi-indices have been proposed in [4]. One needs a new structure for these multi-indices called multi-Novikov algebra which is a vector space equipped with bilinear products $x, y \mapsto x \triangleright_a y$ indexed by a set A

$$(x \triangleright_a y) \triangleright_b z - x \triangleright_a (y \triangleright_b z) = (y \triangleright_a x) \triangleright_b z - y \triangleright_a (x \triangleright_b z),$$
$$(x \triangleright_a y) \triangleright_b z - x \triangleright_a (y \triangleright_b z) = (x \triangleright_b y) \triangleright_a z - x \triangleright_b (y \triangleright_a z),$$
$$(x \triangleright_a y) \triangleright_b z = (x \triangleright_b z) \triangleright_a y,$$

for all $a, b \in A$. This is analogue to the generalisation from pre-Lie algebras to multi-pre-Lie algebras in [1]. One gets a new version of Theorem 1.

Theorem 2. [2] *The multi-Novikov algebra of populated general multi-indices is isomorphic to free algebra generated by the set $\mathfrak{L}^-$.*

For capturing the complexity of the multi-indices for singular SPDEs, one has to introduce other derivations ∂_i, $0 \le i \le d$, that satisfy, together with the derivations $D^{(\mathbf{n})}$, the following relations:

$$D^{(\mathbf{n})} D^{(\mathbf{m})} = D^{(\mathbf{m})} D^{(\mathbf{n})}, \quad \partial_i \partial_j = \partial_j \partial_i$$
$$D^{(\mathbf{n})} \partial_i = \partial_i D^{(\mathbf{n})} + n_i D^{(\mathbf{n}-e_i)},$$

where e_i is the standard basis vector of $\mathbb{N}^{d+1}$. There is a corresponding generalisation of multi-indices which we shall call SPDE multi-indices.

Theorem 3. [2] *The multi-Novikov algebra of populated SPDE multi-indices is isomorphic to free algebra generated by the set* $\mathbb{N}^{d+1} \times \mathfrak{L}^-$.

After free multi-pre-Lie, one has a new free structure useful for expanding solutions of singular SPDEs. They are several applications/open problems to such a result:

- One can try to find other combinatorial sets and their free structures that will be different from multi-indices and decorated trees.
- One can get a more operadic perspective as it was initiated in [14] that recovers as an example the multi-Novikov algebra.
- One can study symmetries in the contex of multi-indices like the chain rule or Itô isometry by defining maps from the free Novikov structure.

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