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Mini-Workshop: Homological aspects of TDLC-groups

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Abstract. This mini-workshop aimed at bringing together experts and early career researchers on finiteness conditions for discrete groups, and experts on varying aspects of locally compact groups to find a common framework to develop a systematic theory of homological finiteness conditions for totally disconnected locally compact groups. Whereas the homological theory of finiteness conditions of discrete groups is well developed and the structure theory of totally disconnected locally compact groups has seen some important breakthroughs in the last decade, the homological theory for (non-compact) totally disconnected locally compact groups is an emerging research area. Specific topics include finiteness conditions for locally compact groups, Mackey functors and Bredon cohomology for topological groups, connections to condensed mathematics, connections to ℓ^2-invariants and Σ-invariants.

Mathematics Subject Classification (2020): 18Gxx, 20Exx, 20Fxx, 20Jxx, 22Dxx, 57Txx.

Introduction by the Organizers

The class of locally compact (= LC) groups plays a central role among topological groups. With the solution of Hilbert's 5th problem, the understanding of the structure of connected LC-groups has significantly increased. Since every LC-group is an extension of a connected LC-group by a totally disconnected LC-group, the contemporary structure problem focuses on totally disconnected LC-groups (= TDLC-groups). In the last decades there has been a significant progress in the study of the structure theory of TDLC-groups, see e.g. [2, 4, 14, 10, 12, 13]. However, the study of homological finiteness conditions has, so far, been rather disjointed and piece-meal. Stefan Witzel gave a 3-lectures survey on finiteness properties for LC-groups, discussing the notions introduced by Castellano–Corob Cook [6] and Abels–Tiemeyer [1] and stressing further on the missing pieces of the theory. Ged Corob Cook, during his talk, suggested a new strategy to investigate finiteness properties for TDLC-groups based on new model structures on k-spaces and simplicial k-spaces [8]. Homological finiteness conditions are very well understood for discrete groups [3], and there is a rich theory for cohomology (both discrete and profinite) for profinite groups. A link, however, to the theory of TDLC-groups, so far, is only very superficial. In recent years, partly due to the theory developed by Castellano–Weigel [7], the study of cohomological finiteness conditions for TDLC-groups has had a little bit of a resurgence [6, 5, 11]. Thomas Weigel's lectures introduced the state of the art of the rational discrete cohomology for TDLC-groups [7], and highlighted the connection with zeta functions for groups. Stable categories for the rational discrete modules of a TDLC-group were considered by Rudradip Biswas, whereas Sofiya Yatsyna introduced Gedrich and Gruenberg invariants for TDLC-groups. Bianca Marchionna presented her recent work concerning double coset zeta functions of TDLC-groups acting on trees, and Laura Bonn discussed the relation between the finiteness properties of a discrete group and those of its Schlichting completion. Peter Kropholler offered three lectures related to finiteness properties of discrete groups that culminated in a lecture on condensed mathematics and profinite groups. He highlighted the advantages of condensed mathematics over other theories: homological algebra in condensed mathematics takes place in abelian categories, which provides an apparently easier approach, though there is still much work to do in that area. Ian Leary presented several embeddings theorems for discrete groups whose TDLC analogue is still unknown, and Lewis Molyneux discussed finiteness properties of groups generalizing Richard Thompson's group F. Dawid Kielak [9] and Yuri Santos Rego offered different perspectives on profinite rigidity.

This mini-workshop was attended by the 16 invited participants, who all travelled to Oberwolfach. Amongst the 20h lectures, there were four mini series of 3h each given by four experienced mathematicians, eight 1h talks, mostly given by early career researchers, and one very lively problem session.

The staff of the Mathematisches Forschungsinstitut Oberwolfach have excelled, providing all the support that we could have wished, and all in a very courteous manner. We are very grateful for the additional funding for 2 young PhD students through Oberwolfach-Leibniz-Fellowships. We strongly believe that such opportunities enable Ph.D. and junior researchers to get integrated within the research community at an early stage in their academic career, and broaden their networking activities. In conclusion, the meeting was a success.

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Abstracts

Geometric and Cohomological Finiteness Conditions in Group Theory

Peter H. Kropholler

The background to Leary's results on groups of type FP_2. In 1937, Bernhard Neumann proved that there are uncountably many 2-generator groups. Subsequently, in 1949, Higman, Neumann, and Neumann [3] introduced what are now known as HNN extensions and used them to prove that every countable group can be embedded in a 2-generator group. A parallel can be drawn with Liouville's discovery of a transcendental number in 1844. In essence Liouville showed that the number

$$\sum_{n=1}^{\infty} 10^{-n!}$$

is far better approximated by rational numbers than is possible for any irrational root of a polynomial with integer coefficients, and therefore this number must be transcendental. Liouville's result was trumped by Cantor's proof that there are $2^{\aleph_0}$ transcendental numbers but only $\aleph_0$ algebraic numbers. Of course, Cantor's extraordinary insight took many year to develop because Set Theory was in its infancy and the theory of infinite cardinals needed to be developed. In particular, clear proofs (avoiding the axiom of choice) of the Schroeder–Bernstein Theorem (which is needed to show that the order relation on cardinal numbers satisfies the law of trichotomy) did not emerge until the turn of the century and until that matter was settled, Cantor's diagonal argument showing that the set of real numbers is uncountable remained one piece of a jig-saw.

Liouville's number admits many variations and it is easy to generate $2^{\aleph_0}$ numbers with the essential property concerning rational approximation. In 1844, Liouville and others would have been aware that there was now a whole family of transcendental numbers but would not have been able to formulate this in terms of countability or uncountability.

By 1937, Bernhard Neumann's construction of $2^{\aleph_0}$ finitely generated groups was a milestone similar in nature to Liouville's discovery of $2^{\aleph_0}$ transcendental numbers and 1949 paper [3] cements this discovery with the more remarkable embedding theorem which can be compared with Cantor's discover that almost all real numbers are transcendental.

Since there are only countably many finitely presented groups up to isomorphism the question arises: which finitely generated groups can be embedded into finitely presented groups. Higman answered this in 1961 by exhibiting a remarkable connection with logic, [2]: a finitely generated group admits a finitely presented overgroup if and only if it is recursively presentable.

By this point, cohomological finiteness conditions emerged in the work of Serre. A group G if of type FP_n if there is a projective resolution $P_* \to \mathbb{Z}$ of the trivial $\mathbb{Z}G$-module with P_j finitely generated for $j < n+1$. It was easy to see that type

FP_1 is equivalent to finite generation, and that finitely presented groups are of type FP_2. Therefore the natural question arose:

Do there exist groups of type FP_2 that are not finitely presented? The question was answered in 1997 by the celebrated work of Bestvina and Noel Brady, [1]. They developed a combinatorial form of Morse theory suited to cube complexes and were able to exhibit whole families of examples. However, unlike the case of Louiville's numbers where we can see now that Liouville has a continuum of examples of similar kinds of number, the Bestvina–Brady examples were only countable in number. This fact seemed to pass unnoticed but around 2014 at a gathering in the Oxford Mathematical Institute, Charles Miller III raised the question. By 2018, Ian Leary had the answer both in the spirit of Bernhard Neumann's theorem [5]:

There are uncountably many groups of type FP. And in the spirit of the Higman–Neumann–Neumann theorem [4]:

Every countable group can be embedded in a group of type FP_2. So arguably, Leary's results build on Bestvina and Brady's work in the same way that Cantor's diagonal argument transcends Liouville's examples of 1844.

This raises a number of questions. The obvious one, and only one I will mention in this short abstract is:

For which n is it possible to embed every countable group into a group of type FP_n? It is natural to suspect that the answer is $n = \infty$ but there is not method known at present even to address the case $n = 3$.

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Discrete cohomology, the Hattori-Stallings rank, the Euler characteristic and formal Dirichlet series for t.d.l.c. groups

Thomas Weigel
(joint work with Ilaria Castellano, Gianmarco Chinello, Bianca Marchionna and George Willis)

In recent years totally disconnected locally compact (= t.d.l.c.) groups have raised much attention. As reflected by D. van Dantzig's theorem (cf. [6]), which states that every t.d.l.c. group contains a compact open subgroup, the structure theory of t.d.l.c. groups is significantly different from the theory of Lie groups. T.d.l.c. groups arise in many different areas in Mathematics.

Example 1. (A) If X is an affine group scheme defined over $\mathbb{Z}$ and $\mathbb{F}$ is a t.d.l.c. field, then $X(\mathbb{E})$ carries naturally the structure of a t.d.l.c. group.
(B) If $\mathscr{T}$ is a locally finite tree then $\mathrm{Aut}(\mathscr{T})$ carries naturally the structure of a t.d.l.c. group.
(C) Let Λ be a connected graph and let $\mathscr{G}$ be a graph of groups based on Λ such that $\mathscr{G}_x$ are profinite groups for all $x \in V(\Lambda) \cup E(\Lambda)$, and that $\alpha_e \colon \mathscr{G}_e \to \mathscr{G}_{t(e)}$ are open immersions for all edges $e \in E(\Lambda)$. Then $\pi_1(\mathscr{G}, \Lambda, x_0)$ carries naturally the structure of a t.d.l.c. group.
(D) If $\mathbb{F}$ is a field of characteristic 0, and $\mathbb{E}/\mathbb{F}$ is a field extension of finite transcendence degree then $\mathrm{Aut}_{\mathbb{F}}(\mathbb{E}) = \{\, \alpha \in \mathrm{Aut}(\mathbb{E}) \mid \alpha|_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}} \,\}$ carries naturally the structure of a t.d.l.c. group (cf. [1, §6.3]).
(E) For every crystallographic Coxeter group (W, S) there exists a simply-connected root group datum $\mathscr{D}$ and a Tits functor $X_{\mathscr{D}}$. Evaluating this functor on a finite field $\mathbb{F}$ and taking the completion with respect to its action on the positive part of the twin building $\Delta_{\pm}$ one obtains the topological Kac-Moody group $\hat{X}_{\mathscr{D}}(F)$.

1. Discrete cohomology. Let G be a t.d.l.c. group, let $R \in \{\mathbb{Z}, \mathbb{Q}\}$ and let M be a left $R[G]$-module. Then

$$\mathrm{d}M = \{\, m \in M \mid \mathrm{stab}_G(m) \text{ open in } G \,\} \tag{1}$$

is an $R[G]$-submodule of M, the largest discrete left $R[G]$-submodule of M. One calls the left $R[G]$-module *discrete*, if $M = \mathrm{d}M$. The full subcategory ${}_{R[G]}\mathbf{dis}$ of ${}_{R[G]}\mathbf{mod}$, the objects of which are discrete left $R[G]$-modules, is an abelian category with enough injectives and thus allows to define cohomology with coefficients in ${}_{R[G]}\mathbf{dis}$ by $\mathrm{d}H^\bullet(G, \_) = \mathscr{R}^\bullet(\_^G)$. For $R = \mathbb{Z}$ these cohomology groups are quite difficult to compute. Nevertheless, an interesting question in this context which has not yet obtained the attention it deserves, is the following:

Question 1. *Let $G = \mathrm{Aut}_{\mathbb{F}}(\mathbb{E})$, where $\mathbb{F}$ is a field of characteristic 0 and let $\mathbb{E}/\mathbb{F}$ be a field extension of finite transcendence degree over $\mathbb{F}$. What is $\mathrm{d}H^1(G, \mathbb{E}^\times)$?*

In [3] the authors addressed many problems concerning the category ${}_{\mathbb{Q}[G]}\mathbf{dis}$. However, several questions remained unanswered. E.g.:

Question 2. *Let G be a t.d.l.c. group. For which closed subgroups H of G is $\mathrm{res}^G_H(\_)$ mapping injectives to injectives. Of particular interest would be the case when H is co-compact in G, or when H is discrete in G.*

For $R = \mathbb{Q}$ calculations of $\mathrm{d}H^\bullet(G, \_)$ become easier due to the following fact

Fact 1. *The category ${}_{\mathbb{Q}[G]}\mathbf{dis}$ is an abelian category with enough projectives. Indeed for any compact, open subgroup $\mathcal{O}$ of G the left $\mathbb{Q}[G]$-permutation module $\mathbb{Q}[G/\mathcal{O}]$ is a projective object in ${}_{\mathbb{Q}[G]}\mathbf{dis}$.*

If M and N are two rational discrete left G-modules it is straightforward to verify that $M \otimes_{\mathbb{Q}} N$ is again a rational discrete left G-module. Neverthelesss the following question remained unanswered.

Question 3. *Let P be a projective object in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ and let M be a rational discrete left G-module. Is $P \otimes_{\mathbb{Q}} M$ necessarily a projective object in ${}_{\mathbb{Q}[G]}\mathbf{dis}$?*

The existence of enough projectives in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ allows one to define discrete rational homology $\mathrm{d}H_\bullet(G, \_)$ with coefficients in ${}_{\mathbb{Q}[G]}\mathbf{dis}$, and yields also a natural notion of being rationally of type FP_∞. The setup allows one to define the rational discrete cohomological dimension of a t.d.l.c. group G by

$$\mathrm{cd}_{\mathbb{Q}}(G) = \sup\{\, n \in \mathbb{N} \mid \mathrm{d}H^{n+1}(G, \_) = 0 \,\} \in \mathbb{N} \cup \{\infty\}. \tag{2}$$

E.g., for a discrete group G this number coincides with the cohomological $\mathbb{Q}$-dimension of G. It is a direct consequence of Bass-Serre theory that the t.d.l.c. groups $\pi = \pi_1(\mathscr{G}, \Lambda, x_0)$ described in Example (C) satisfy $\mathrm{cd}_{\mathbb{Q}}(\pi) \leq 1$. Let G be a t.d.l.c. group and let μ be a fixed left-invariant Haar measure on G. One says that G is *c/o-bounded*, if there exists a positive real number c such that for every compact open subgroup $\mathcal{O}$ of G one has $\mu(\mathcal{O}) \leq c$. The following theorem can be seen as a second[1] t.d.l.c. version of the Stallings-Swan theorem (cf. [4]).

Theorem 2 (I. Castellano, B. Marchionna, T.W.)**.** *Let G be a unimodular c/o-bounded, compactly generated t.d.l.c. group satisfying $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$. Then there exists a graph of groups $\mathscr{G}$ like in Example (C) based on a finite connected graph Λ such that $G \simeq \pi_1(\mathscr{G}, \Lambda, x_0)$.*

Question 4. *Does Theorem 2 remain true without the hypothesis of c/o boundedness, and/or unimodularity?*

Although there is no group algebra one may associated to the abelian category ${}_{\mathbb{Q}[G]}\mathbf{dis}$, there is a canonical rational discrete bimodule

$$\mathrm{Bi}(G) = \varinjlim_{\mathcal{O} \subseteq_{c/o} G} \mathbb{Q}[G/\mathcal{O}] \tag{3}$$

which plays a similar role as the integral group algebra for discrete groups. E.g., the t.d.l.c. group G is said to be a *rational duality group of dimension $d \in \mathbb{N}$*, if

(a) G is rationally of type FP_∞,
(b) $\mathrm{cd}_{\mathbb{Q}}(G) = d$,

[1] A first version has been obtained by I. Castellano in [2].

(c) $\mathrm{d}H^k(G, \mathrm{Bi}(G)) = 0$ for $k \neq d$.

From work of Michael Davis one concludes that the t.d.l.c. group $G = \hat{X}(\mathbb{F})$ of Example 1(E) is a rational duality groups of dimension $d \geq 1$ if, and only if, (W, S) is a $\mathbb{Q}$-duality group of dimension d (cf. [3]). Although many features of Coxeter groups have been studied in detail, the author could not find a satisfactory[2] answer to the following question.

Question 5. *What crystallographic Coxeter groups are $\mathbb{Q}$-duality groups of dimension $d \geq 1$?*

2. The Hattori-Stallings rank and the Euler-Poincaré characteristic. In case that G is unimodular, the Hom-$\otimes$ identity in combination with the evaluation morphism $\phi_{\mathcal{O}}\colon P \otimes_G \mathrm{Hom}_G(P, \mathrm{Bi}(G)) \to \mathbb{Q} \cdot \mu$ assigns do the identity of every finitely generated projective object P in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ a rational multiple $\mathbf{hs}(P) \in \mathbb{Q} \cdot \mu$ of a normalized Haar measure μ, and thus can be considered as a generalized Hattori-Stallings rank (cf. [5]), e.g., $\mathbf{hs}(\mathbb{Q}[G/\mathcal{O}]) = \mu_{\mathcal{O}}$, where $\mu_{\mathcal{O}}$ is the Haar measure on G which restriction to $\mathcal{O}$ is a probability measure. The following theorem may be considered as a t.d.l.c. version of a theorem of I. Kaplansky (cf. [5]).

Theorem 3 (I. Castellano, G. Chinello, T.W.)**.** *Let G be a t.d.l.c. group, let $\mathcal{O}$ be a compact open subgroup of G, and let $P \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$ be projective. Then $\mathbf{hs}(P) \in \mathbb{Q}_0^+ \cdot \mu_{\mathcal{O}}$. In particular, $\mathbf{hs}(P) = 0$ if, and only if, $P = 0$.*

Let G be a t.d.l.c. group which is

(i) unimodular,
(ii) rationally of type FP_∞,
(iii) of finite rational cohomological dimension.

For such a group let $(P_\bullet, \partial_\bullet)$ be a finite and finitely generated projective resolution of the trivial left $\mathbb{Q}[G]$-module in the category ${}_{\mathbb{Q}[G]}\mathbf{dis}$. Then one defines the Euler-Poincaré characteristic χ_G of G by

$$\chi_G = \sum_{k \in \mathbb{N}_0} (-1)^k \cdot \mathbf{hs}(P_k). \tag{4}$$

Example 2. (A) If $G = \pi_1(\mathscr{G}, \Lambda, x_0)$ is the fundamental group of a profinite graph of groups based on the finite graph Λ (cf. Ex.1(C)), one has

$$\chi_G = \sum_{v \in V(\Lambda)} \mu_{\mathscr{G}_v} - \sum_{\mathbf{e} \in \mathbb{E}^g(\Lambda)} \mu_{\mathscr{G}_{\mathbf{e}}} \tag{5}$$

(B) If $G = \hat{X}_{\mathscr{D}}(\mathbb{F})$ for some finite field of cardinality q (cf. Ex.1(E)) one obtains

$$\chi_G = \frac{1}{p_{W,S}(q)} \cdot \mu_{\mathrm{Iw}}, \tag{6}$$

where Iw is the stabilizer of a chamber in the building Δ_+.

[2] Obviously, affine and crystallographic hyperbolic Coxeter groups share this property, but one may speculate whether the class of examples is much larger or not.

3. Formal Dirichlet series associated to t.d.l.c. groups. Let G be a t.d.l.c. group, let μ be a left-invariant Haar measure of G, let $\mathcal{O} \subseteq_{c/o} G$ and let $\mathscr{R} \subset G$ be a set of coset representatives for $\mathcal{O}\backslash G/\mathcal{O}$. One says that G has *bounded coset growth with respect to* $\mathcal{O}$, if for all $n \in \mathbb{N}$ one has

$$a_n = |\{ r \in \mathscr{R} \mid \mu(\mathcal{O}r\mathcal{O}) = n \cdot \mu(\mathcal{O}) \}| < \infty. \tag{7}$$

If G has bounded coset growth with respect to some compact open subgroup, then it has bounded coset growth with respect to all compact open subgroups. For such a t.d.l.c. group G one defines the formal Dirichlet series

$$\zeta_{G,\mathcal{O}}(s) = \sum_{n\in\mathbb{N}} a_n \cdot n^{-s}. \tag{8}$$

In many cases one verifies that $\zeta_{G,\mathcal{O}}$ defines a meromorphic function $\hat{\zeta}_{G,\mathcal{O}} \colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. In this case one calls $(G, \mathcal{O})$ a *meromorphic pair*. A question we have investigated for many examples is the following:

Question 6. *For which merohoric pairs* $(G, \mathcal{O})$ *satisfying (i)-(iii) is it true that*

$$\chi_G = \tfrac{1}{\zeta_{G,\mathcal{O}}(-1)} \cdot \mu_{\mathcal{O}} \tag{9}$$

Question 6 has an affirmative answer for $(\hat{X}_{\mathscr{D}}(\mathbb{F}), \mathrm{Iw})$ for every crystallographic Coxeter group (W, S) (cf. Ex. 1(E)). The same is true if $G = X(\mathbb{E})$ for a Chevalley group scheme X, a t.d.l.c. field $\mathbb{E}$ and $\mathcal{O} \subset G$ a parahoric subgroup of G (cf. [5]). However, recently B. Marchionna found examples of merohoric pairs which do not satisfy (4) and also many merohoric pairs satisfying (4) (for t.d.l.c. groups G without Bruhat decomposition).

The abscissa of convergency $a = \mathbf{abs}(\zeta_{G,\mathcal{O}})$ as well as the order of the pole $\mathbf{ord}(\zeta_{G,\mathcal{O}})$ at a, do not depend on the choice of compact open subgroup, and thus are invariants of G.

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Cohomological finiteness conditions and stable categories on rational discrete modules over TDLC groups

Rudradip Biswas

1. Objective. We want to associate to the abelian category of rational discrete modules over a TDLC group a "stable category" with good behaviour (i.e. satisfying the equivalences shown in **Theorem C**).

2. A useful homological result. For any TDLC group G, it is known (Proposition 3.2 of [3]) that the rational discrete modules form an abelian category with enough projectives and enough injectives - we will be denoting this category by $\mathfrak{A}_G$. Denote the supremum over all $\mathfrak{A}_G$-objects with finite projective (resp. injective) dimension by $\operatorname{Fin}\operatorname{ProjDim}(\mathfrak{A}_G)$ (resp. $\operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G)$).

Theorem A. (partly new, partly inspired by [5], and partly covered by Thm. VII .2.2 of [2]) The following are equivalent for any TDLC group G.

a) Every object in $\mathfrak{A}_G$ admits a complete projective resolution, i.e. every object has a projective resolution that eventually agrees with a totally acyclic complex of projectives.

b) $\operatorname{silp}(\mathfrak{A}_G)$ (defined as the supremum over the injective dimension of $\mathfrak{A}_G$-projectives) and $\operatorname{spli}(\mathfrak{A}_G)$ (defined as the supremum over the projective dimension of $\mathfrak{A}_G$-injectives) are finite.

c) Every object in $\mathfrak{A}_G$ admits a complete injective resolution, i.e. every object has an injective resolution that eventually agrees with a totally acyclic complex of injectives, and $\operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G) < \infty$.

d) Complete cohomology computed with projective resolutions agrees with complete cohomology computed with injective resolutions (in the style of Nucinkis [5]).

Proposition B. When any of the equivalent statements of **Theorem A** are satisfied (an easy example is when $G = \mathrm{SL}_n(\mathbb{Q}_p)$ as it has finite virtual cohomological dimension), we have

$\operatorname{silp}(\mathfrak{A}_G) = \operatorname{spli}(\mathfrak{A}_G) = \operatorname{Fin}\operatorname{ProjDim}(\mathfrak{A}_G) = \operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G) < \infty$

We can add some more invariants here like the finitistic and the global Gorenstein dimensions. The main use of **Theorem A** is that it gives very neat conditions on when every object has complete resolutions which is useful in constructing a well-behaved stable category as we describe below.

3. Candidates for stable categories and equivalences.

Theorem C. (new, in the spirit of [1, 4]) Let G be a TDLC group such that $\operatorname{silp}(\mathfrak{A}_G)$ and $\operatorname{spli}(\mathfrak{A}_G)$ are finite. Then, the following triangulated categories are equivalent (and are therefore equivalently adequate candidates for our stable category):

(i) $(\mathfrak{A}_G, \widehat{\operatorname{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ (here, the objects are all modules in $\mathfrak{A}_G$ and the Hom-sets are given by the zero-th complete cohomology groups computed with projective resolutions)

(ii) $(\mathfrak{A}_G, \widetilde{\mathrm{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ (here, the objects are all modules in $\mathfrak{A}_G$ and the Hom-sets are given by the zero-th complete cohomology groups computed with injective resolutions)

(iii) $\mathcal{D}^b(\mathfrak{A}_G)/\,\mathrm{K}^b(\mathrm{Proj}\text{-}\mathfrak{A}_G)$ (the Verdier quotient of the derived bounded category on $\mathfrak{A}_G$ and the homotopy category of bounded complexes of $\mathfrak{A}_G$-projectives)

(iv) $\mathcal{D}^b(\mathfrak{A}_G)/\,\mathrm{K}^b(\mathrm{Inj}\text{-}\mathfrak{A}_G)$ (the Verdier quotient of the derived bounded category on $\mathfrak{A}_G$ and the homotopy category of bounded complexes of $\mathfrak{A}_G$-injectives)

(v) The homotopy category of totally acyclic complexes of $\mathfrak{A}_G$-projectives.

(vi) The homotopy category of totally acyclic complexes of $\mathfrak{A}_G$-injectives.

(vii) $\underline{\mathrm{GProj}}(\mathfrak{A}_G)$ (Gorenstein projectives of $\mathfrak{A}_G$, i.e. $\mathfrak{A}_G$-objects arising as cycles in totally acyclic complexes of $\mathfrak{A}_G$-projectives, form a Frobenius category with the class of projective-injectives given by the $\mathfrak{A}_G$-projectives; $\underline{\mathrm{GProj}}(\mathfrak{A}_G)$ denotes its stable category where we keep all Gorenstein projectives as objects and kill all morphisms that factor through an $\mathfrak{A}_G$-projective). To get to (iii) from (i), consider a module as a complex concentrated in degree 0; for (iii) to (v), take complete projective resolutions (possible due to Thm A); for (v) to (vii), take the zero-th syzygy functor (see Def. 3.7 of [4]); and for (vii) to (i), use the inclusion functor. Composing these, get $(i) \cong (iii) \cong (v) \cong (vii)$. Repeat the analogous treatment with "module as a deg 0 concentrated complex", "taking complete injective resolutions" (again, possible due to Thm A), and the zero-th cosyzygy functor, to get $(ii) \cong (iv) \cong (vi)$. $(i) \cong (ii)$ by Thm A, and $(iii) \cong (iv)$ as $\mathrm{silp}(\mathfrak{A}_G), \mathrm{spli}(\mathfrak{A}_G) < \infty$.

Note that all the categories in **Theorem C** except two, namely $(\mathfrak{A}_G, \widehat{\mathrm{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ and $(\mathfrak{A}_G, \widetilde{\mathrm{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$, are clearly triangulated categories. Without the assumption that $\mathrm{silp}(\mathfrak{A}_G)$ and $\mathrm{spli}(\mathfrak{A}_G)$ are finite, these two categories need not even be triangulated. Since we know very little about $\mathfrak{A}_G$-injectives, the presence of (vi) above is noteworthy.

4. Finishing remarks. We are insisting on these equivalences because they are useful for making progress on stratification questions in the spirit of, for example, Benson-Iyengar-Krause or Barthel-Heard-Sanders. This is why even if we can replace $\mathfrak{A}_G$ with a more refined abelian category associated to G using the full force of *Condensed Maths*, our use of and dependence on complete resolutions (both projective and injective) to establish "good behaviour" of our stable category will remain.

Ending open question. Can we achieve **Theorem C** with a finiteness assumption on just one of $\mathrm{silp}(\mathfrak{A}_G)$ and $\mathrm{spli}(\mathfrak{A}_G)$?

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Finiteness properties and locally compact groups

Stefan Witzel

Topological finiteness properties of discrete groups are studied for two principal reasons. The classical one is that they give sufficient conditions for group cohomology to be effectively computable using a CW-model for a classifying space. The modern one is that certain finiteness properties are coarse invariants and therefore allow to distinguish the large-scale geometry of groups. Finiteness properties for locally compact groups have been proposed. What is missing is the universal object whose finiteness they are supposed to describe.

1. Discrete groups

Topological finiteness properties are about the existence of classifying spaces with good finiteness properties so we start with these. If G is a discrete group, a $K(G,1)$ is a homotopy type CG of a pointed CW-complex such that $\pi_1(CG) = G$ and the map $[(Z,z_0),CG] \to \operatorname{Hom}(\pi_1(Z,z_0),G)$ is bijective. Another perspective is via principal bundles: a principal G-bundle $p\colon EG \to BG$ is universal if every numerable principal G-bundle arises as a pullback $f^*(p)$ along a map $f\colon Z \to Y$.

For a discrete group G, a principal G-bundle $X \to Y$ is the same as a normal covering with group of deck transformations G which by covering space theory corresponds to a homomorphism $\pi_1(Y) \to G$. In particular, the universal bundle $EG \to BG$ can be recovered from BG and $K(G,1)$ and BG are the same thing. This is specific to discrete groups and for non-discrete groups it may be more reasonable to generalize EG.

A discrete group is *of type* F_n if it admits a CW-model for EG whose n-skeleton is compact modulo G, equivalently if acts freely and cocompactly on an $(n-1)$-connected CW complex. It is of type F_∞ if it is of type F_n for all n.
A basic fact about these finiteness properties is that if $1 \to N \to G \to Q \to 1$ is an extension in which N is of type F_n then G of type F_{n+1} implies Q of type F_{n+1} and Q of type F_n implies G of type F_n.

The main tool in determining finiteness properties is Brown's criterion [3, Theorems 2.2, 3.2]:

Theorem. *Fix G and n. Let X be an $(n-1)$-connected G-CW complex. Let $(X_i)_i$ be a filtration by G-CW subcomplexes. Assume that G_σ is of type $F_{n-\dim\sigma}$ for all σ. Then G is of type F_n if and only if $(X_i)_i$ is essentially $(n-1)$-connected.*

The filtration is essentially n-connected if the directed system $(\pi_k(X_i))_i$ is essentially trivial for every $k \le n$ which in turn means that for every i there exists a

j such hat $\pi_k(X_i \to X_j) = 0$. Brown's criterion can be decomposed into two parts: the stabilizer part says that a CW-complex on which G acts can be replaced by one on which G acts freely without affecting cocompactness on the n-skeleton provided the stabilizers have the right finiteness properties; and the filtration part says that if G acts freely on an $(n-1)$-connected CW complex, its finiteness properties are detected by any cocompact filtration.

2. Universal spaces for locally compact groups

Moving on to locally compact groups, we would like to define finiteness properties F_n that capture the finiteness of some universal free G-space EG. It turns out that is relatively easy to agree on what the properties F_n will be but not so clear what the universal space EG is that they describe finiteness of, a basic problem being that non-discrete groups do not act freely and continuously on CW-complexes (in contrast there is no problem to define a classifying space for proper actions of a tdlc group, for instance). One may hope for EG to be an object in a model category for topological spaces with continuous G-actions that has a notion of dimension and then F_n would be universality with respect to n-dimensional objects. Milnor's EG, the infinite join of G with itself appropriately topologized, is bound to be model for EG in the sense to be established, but studying finiteness properties becomes interesting only once one can vary the model within its (equivariant) homotopy class. Corob Cook [4] has results in this direction with the additional ambition of recovering G from BG as a topological group, but their universal properties are unclear and connected groups will likely interfere with the homotopy theory.

Finiteness properties of locally compact groups should generalize the notions of being compactly generated (F_1) and of being finitely presented (F_2) and these special cases are instructive: G is compactly generated if there is a compact subset C that generated G as an abstract group. This can be reformulated to say that G acts cocompactly on a topological graph with vertex set G and edge set $G \times C$ whose *underlying discrete graph* is connected. A natural extension to G being of type F_n would be to ask for the existence of a simplicial space Δ (consisting of topological spaces $\Delta[k]$ of k-simplices and continuous face and degeneracy maps) on which G acts freely such that the action on $\Delta[k], k \leq n$ is cocompact and that the geometric realization of the underlying simplicial set $|F\Delta|$ is $(n-1)$-connected (where F is the forgetful functor from topological spaces to topological sets). Or alternatively to act freely on a topological CW-complex (whose n-cells are indexed by a topological space rather than a set) cocompactly on k-cells, $k \leq n$, and that the underlying CW-complex (with discrete set of cells) be $(n-1)$-connected. But for now these are just ad-hoc notions without justification by some form of universality.

There are a few more lessons to be learned from looking at compact generation. First, the models for EG we are looking for will not be G-CW complexes, which are built out of equivariant cells $G/H \times D^k$: while the 0-skeleton of the topological graph is a single equivariant cell $G/\{1\} \times D^0$, already the edges are parametrized by $G \times C$, so a single G/H-factor is not sufficient to capture the amount of non-discreteness. Second, a connected group G is not compactly generated simply because $G \to G\backslash G$ is a bundle with connected total space but rather because it is generated by any identity neighborhood and this neighborhood can be taken to be compact. This leads to a warning when moving beyond the locally compact setting: if $G\backslash X$ is compact it may not be true that there is a compact $C \subseteq X$ with $G.C = X$.

3. Finiteness properties for locally compact groups

The reason that the correct notion of finiteness properties for locally compact groups is uncontroversial is that certain simple assumptions determine them uniquely. For instance:

Observation. *Suppose $(T_n)_{n\in\mathbb{N}}$ are properties of groups (and T_∞ means T_n for all n) such that the following hold:*

(1) If $1 \to N \to G \to Q \to 1$ is an extension with N of type T_∞ then G is T_n if and only if Q is T_n.
(2) If G acts properly and transitively on a contractible manifold then it is of type T_∞.

Then G is T_n if and only if $G/G^{(0)}$ is T_n. If in addition

(3) Brown's criterion holds (at least for proper actions)

then the properties T_n are uniquely determined.

Proof. If G is compact then it is T_∞ by (2). If G is connected Lie then it admits a maximal compact subgroup C [7, Theorem 14.1.3] and $C\backslash G$ is contractible [7, Theorem 14.3.11] so G is T_∞ by (2). If G is connected then by the Gleason–Yamabe theorem it is pro-Lie. Since the Lie groups involved have bounded dimension, the inverse system eventually consists of coverings. Since the fundamental group of a Lie group is finitely generated abelian [7, Theorem 12.4.14], only finitely many of these can be infinitely-sheeted. It follows that G is (pro-finite)-by-Lie and hence of type T_∞ by (1). If G is a general locally compact group with connected component $G^{(0)}$ it follows by another application of (1) that G is T_n if and only if $G/G^{(0)}$ is, reducing to the tdlc case. Finally if G is tdlc then by van Dantzig's theorem there is a compact open subgroup C. Then G acts properly on the free simplicial set over G/C, which is contractible, so it is of type T_n if and only if some/any cocompact filtration is essentially $(n-1)$-connected. □

The existing notions of finiteness properties for locally compact groups stipulate some assumption of this form to get a definition. The compactness properties C_n by Abels–Tiemeyer [2] stipulate that the filtration part of Brown's criterion should hold for the filtration of the free simplicial set over G filtered by G-orbits of free simplicial sets over compact subsets of G. The finiteness properties F_n by Castellano–Corob-Cook [5], which are only defined for tdlc groups, stipulate that compact groups should be F_∞ and that the stabilizer part of Brown's criterion should hold for proper actions. Applying [2, Theorem 3.2.2] and [5, Theorems 4.7,4.10] to the free simplicial set over G/C, C a compact open subgroup, one finds that both notions coincide on tdlc groups.

4. Examples

Examples of locally compact groups with interesting finiteness properties exist in the literature although they are usually formulated for discrete groups because of the unclear meaning of finiteness properties of locally compact groups. This is true specifically of the solvable groups discussed in [12, 10]. Note that Brown's criterion together with the fact that arithmetic groups are F_∞ allows to reduce the determination of finiteness properties of S-arithmetic groups to that of finiteness properties of algebraic groups over local fields, see [11, Theorem 3.1]. For instance finiteness properties of $A(\mathbb{Z}[1/p])$ are equivalent to compactness properties of $A(\mathbb{Q}_p)$. Beyond this equivalence, however, the proof that $A(\mathbb{Z}[1/p])$ is of type F_{n-1} but not of type F_n by filtering a Bruhat–Tits building applies in verbatim to prove that $A(K)$ is of type F_{n-1} but not of type F_n even if K is local field of positive characteristic, using the version [5, Theorems 4.7,4.10] of Brown's criterion.

5. Coarse geometry

The second motivation for studying finiteness properties mentioned in the introduction is coarse geometry. In this context the situation is much clearer, even beyond locally compact groups. A metric space X is *coarsely n-connected* if the Vietoris–Rips filtration $\mathrm{VR}_r(X)$ is essentially n-connected. Alonso [1] observed that this is a coarse invariant and by Brown's criterion being of type F_n coincides with being coarsely $(n-1)$-connected for countable groups. A locally compact σ-compact group G carries an adapted pseudo-metric that is unique up to coarse equivalence. If G is compactly generated then the metric can be taken to be coarsely geodesic and is then unique up to quasi-isometry (see [6, Milestones 4.A.8 and 4.B.13]. This generalizes statements for discrete groups that are countable and finitely generated, respectively. Thus from a geometric perspective the natural notion for a locally compact (σ-compact) group to be of type F_n is to be coarsely $(n-1)$-connected. If one is willing to work with coarse structures that need not be metrizable (see [8]) one can go further: Rosendal [9] defines a coarse structure on every topological group that is unique up to coarse equivalence.

An analogous form of coarse and large-scale geometry also exists for approximate groups leading to similar questions. In particular, it would be interesting to know how coarse connectivity relates to finiteness of cohomology.

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Profinite rigidity, homology, and Coxeter groups

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(joint work with Petra Schwer)

Dropping 'locally' from TDLC, a (topological, Hausdorff) group G is called *profinite* if it is totally disconnected (TD) and compact (C). Many infinite groups have TDC counterparts. Given $G \in \mathcal{FGRF}$ = class of finitely generated residually finite (discrete) groups, its *profinite completion* $\widehat{G}$ is the topological closure

$$\widehat{G} := \overline{\iota(G)} \leq \prod_{N \trianglelefteq G \,\&\, [G:N]<\infty} G/N, \text{ where } \iota(g) = (gN)_N \text{ is the diagonal embedding}$$

and each finite quotient G/N is given the discrete topology (thus $\widehat{G}$ is TDC).

1. Profinite rigidity and homological aspects

To what extent does the collection of (isomorphism classes of) finite quotients of an infinite group G determine its algebraic structure? This problem, whose origin is traced back to questions of Grothendieck and others in the 1970s, is a common point of interest for geometric group theory and the theory of TDLC groups. An ambitious first version is whether finite quotients determine isomorphism types.

Definition 1. Given a subclass $\mathcal{C} \subseteq \mathcal{FGRF}$ we say $G \in \mathcal{FGRF}$ is *profinitely rigid relative to* $\mathcal{C}$ if ($\widehat{G} \cong \widehat{H} \implies G \cong H$) holds for all $H \in \mathcal{C}$. If $\mathcal{C} = \mathcal{FGRF}$ we call G *absolutely* profinitely rigid. If (up to isomorphism) there are only finitely many $H \in \mathcal{C}$ with $\widehat{H} \cong \widehat{G}$ but $H \ncong G$, we call G *almost* profinitely rigid (rel. $\mathcal{C}$).

Below is a widely nonexhaustive list around the current state of knowledge. We tacitly assume our discrete groups to lie in $\mathcal{FGRF}$ unless explicitly said otherwise.

(1) (Folklore) $\mathbb{Z}$ and the infinite dihedral group are absolutely profinitely rigid.
(2) (Baumslag) $\exists B_1, B_2 \in \mathcal{FGRF}$ metacyclic with $\widehat{B_1} \cong \widehat{B_2}$ but $B_1 \not\cong B_2$.
(3) (Pickel) Nilpotent groups are almost absolutely profinitely rigid.
(4) (Remeslennikov; open problem) Are free groups absolutely profinitely rigid?
(5) (Wilton) Free groups are profinitely rigid relative to limit groups.
(6) (Liu) Fundamental groups of hyperbolic 3-manifolds of finite volume are almost profinitely rigid relative to 3-manifold groups.

Adapting the question, what kinds of features are witnessed by finite quotients?

Definition 2. A group-theoretic property (P) and a group-theoretic invariant $\eta(-)$ are said to be *profinite relative to* $\mathcal{C} \subseteq \mathcal{FGRF}$ in case the implications

- $(G$ has property (P), G, $H \in \mathcal{C}$, and $\widehat{G} \cong \widehat{H}) \implies H$ has property (P),
- $(G, H \in \mathcal{C}$, and $\widehat{G} \cong \widehat{H}) \implies \eta(G) = \eta(H)$

hold, respectively. If $\mathcal{C} = \mathcal{FGRF}$ we call the property (P) (resp. $\eta(-)$) *profinite*.

For instance, the first integral homology group $H_1(-, \mathbb{Z})$ is a profinite invariant. In fortunate cases, further homological information is detected by completions, or homological tools aid in computing completions, motivating the following (broad) program: Given a class of groups $\mathcal{C} \subseteq \mathcal{FGRF}$, ...

(1) ...find (co)homological invariants relative to $\mathcal{C}$,
(2) ...use (co)homological methods to check whether $\widehat{G} \cong \widehat{H}$ for $G, H \in \mathcal{C}$.

Here we mention some important contributions in this direction.

Theorem 3 (Platonov–Tavgen)**.** *Consider $G_1, G_2 \in \mathcal{FGRF}$ and suppose there exists a finitely presented group $Q \notin \mathcal{FGRF}$ for which $\widehat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$ and such that there are epimorphisms $\pi_1 : G_1 \to Q$ and $\pi_2 : G_2 \to Q$. Then the fiber product associated to π_1 and π_2 has the same profinite completion as $G_1 \times G_2$.*

Theorem 4 (Lück; Bridson–Conder–Reid)**.** *The rational Euler characteristic is a profinite invariant relative to $\mathcal{C}$ = lattices in $\mathbb{P}\mathrm{SL}_2(\mathbb{R})$.*

Theorem 5 (Kammeyer–Kionke–Raimbault–Sauer)**.** *Let $\mathcal{C}$ be the class of arithmetic groups with the congruence subgroup property. Then the rational Euler characteristic* is not *a profinite invariant relative to $\mathcal{C}$, though its sign* is so.

Theorem 6 (Jaikin-Zapirain; Hughes–Kielak)**.** *Let $\mathcal{C}$ be the class of finitely presented subgroup-separable groups. Then the property "the BNS invariant $\Sigma^1(-; R)$ contains antipodal points (for any commutative unital ring R)" is profinite rel. $\mathcal{C}$.*

2. Coxeter groups

Profinite topics have been actively studied for groups of strong geometric flavor. Surprisingly, not much is known for the (arguably) core examples of such groups: a *Coxeter group* W (of rank n) is a group admitting a *Coxeter presentation*

$$W \cong \langle s \in S \mid (st)^{m_{s,t}} \text{ for every pair } s, t \in S \text{ with } m_{s,t} < \infty \rangle,$$

i.e., $|S| = n$ and the orders $m_{s,t}$ satisfy $m_{s,s} = 1$ and $m_{s,t} = m_{t,s} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.

Example 7. $\mathrm{Sym}(n)$ is a prototypical finite Coxeter group, and $D_\infty \cong C_2 * C_2$ is the 'smallest' infinite Coxeter group. The following Coxeter group of rank 5,

$$W \cong \langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, e^2, (ab)^3, (bc)^5, (ad)^2, (bd)^2, (be)^2, (ce)^2 \rangle,$$

is also isomorphic to the (hyperbolic) Coxeter triangle group $\Delta(6, 10, \infty)$.

To our knowledge, profinite literature around the class $\mathcal{G}$:= Coxeter groups of finite rank $\subset \mathcal{FGRF}$ is at most a decade old. We record:

(Kropholler–Wilkes [4]; Corson–Hughes–Möller–Varghese [3]) Right-angled Coxeter groups are profinitely rigid relative to $\mathcal{G}$.
(Bridson–McReynolds–Spitler–Reid [2]) 14 hyperbolic Coxeter triangle groups are known to be absolutely profinitely rigid.
(Möller–Varghese [5]) Relative to $\mathcal{G}$, irreducible and affine imply rigidity.

Let us highlight two results addressing our motivating program for Coxeter groups.

Theorem 8 (Corson–Hughes–Möller–Varghese [3])**.** *The right-angled Coxeter group* $(C_2 * C_2 * C_2 * C_2) \times (C_2 * C_2 * C_2 * C_2)$ *is not absolutely profinitely rigid.*

Proof sketch. Apply Theorem 3 of Platonov–Tavgen taking $G_1 = G_2 = C_2 * C_2 * C_2 * C_2$ and using R. Thompson's simple group V as the quotient Q, and check that the corresponding fiber product is not isomorphic to $G_1 \times G_2$. □

Theorem 9 (Santos Rego–Schwer [6])**.** *Write* $\mathcal{G}_{\leq 3}$ *for the class of Coxeter groups that admit some Coxeter presentation of rank three or less. Then every* $W \in \mathcal{G}_{\leq 3}$ *is profinitely rigid relative to* $\mathcal{G}_{\leq 3}$*. Moreover, Coxeter triangle groups satisfy*

$$\widehat{\Delta(p,q,r)} \cong \widehat{\Delta(p',q',r')} \iff \{p,q,r\} = \{p',q',r'\}.$$

Proof strategy. Rule out spherical groups and clear the affine case by looking at $H_1(-, \mathbb{Z})$ and comparing completions of virtually abelian groups. In the hyperbolic case the presence of von Dyck subgroups implies that Theorem 4 still applies. Compare Euler characteristics and invoke profinite techniques of [1] to finish. □

Theorem 9 applies to some groups of higher rank, see Example 7. Besides using homological tools, Theorem 9 shows that the geometry of the given groups (e.g., covolume, having cusps, being hyperbolic) is encoded by finite quotients. We pose:

Problem 10. *Which (co)homological properties and invariants are profinite relative to the class* $\mathcal{G}$ *of Coxeter groups of finite rank?*

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Double-coset zeta functions for groups acting on trees

Bianca Marchionna

Double cosets play a prominent role in the study of totally disconnected locally compact (= t.d.l.c.) groups, e.g. in Representation Theory or Geometric Group Theory. From now on, we focus on double cosets of a t.d.l.c. group G with respect to a compact open subgroup $K \leq G$. In this case, each double coset KgK has

$$\mu_K(KgK) = |K : K \cap gKg^{-1}| \in \mathbb{Z}_{\geq 1}, \tag{1}$$

where $\mu_K(\_)$ is the left Haar measure on G such that $\mu_K(K) = 1$. For a given pair (G, K), consider the following (formal) Dirichlet series:

$$\zeta_{G,K}(s) := \sum_{KgK \in K\backslash G/K} \mu_K(KgK)^{-s}, \tag{2}$$

which is called the *double-coset zeta function associated to* (G, K) (cf. [2]).

The key for studying $\zeta_{G,K}(s)$ is to find a favourable enumeration of the K-double cosets and an explicit formula for each $\mu_K(KgK)$. Consider for instance $G = \mathrm{SL}_2(\mathbb{Q}_p)$ and $K = \mathrm{SL}_2(\mathbb{Z}_p)$, where $\mathbb{Q}_p$ is the p-adic field and $\mathbb{Z}_p$ is its ring of integers. By the Cartan decomposition of G with respect to K one has

$$G = \bigsqcup_{d \in \mathbb{Z}_{\geq 0}} K \cdot \underbrace{\mathrm{diag}(p^{-d}, p^d)}_{=:g_d} \cdot K. \tag{3}$$

Via a direct computation, $\mu_K(Kg_dK) = |K : K \cap g_d K g_d^{-1}|$ is 1 if $d = 0$, and equals $(p+1)p^{2d-1}$ otherwise. Therefore,

$$\zeta_{G,K}(s) = \sum_{d=0}^{+\infty} \mu_K(Kg_dK)^{-s} = 1 + \frac{(p+1)^{-s}p^{-s}}{1 - p^{-2s}}. \tag{4}$$

There is another (geometric) way to obtain the same result, which goes beyond matrix computations and opens to further generalizations. It is based on the fact (cf. [6, Ch. II, § 1]) that $G = \mathrm{SL}_2(\mathbb{Q}_p)$ acts on a $(p+1)$-regular (simplicial) tree T_{p+1} and $K = \mathrm{SL}_2(\mathbb{Z}_p)$ is the stabilizer of a vertex, say v_0. Remarkably, the G-action on T_{p+1} is *locally ∞-transitive*, i.e., every vertex-stabilizer G_v acts transitively on the set of geodesics $\{[v, w] \subset T_{p+1} : \mathrm{length}([v, w]) = k\}$, for every $k \geq 0$. Hence there is a 1-to-1 correspondence mapping Kg_dK, $d \geq 0$, to the orbit $K \cdot [v_0, g_d \cdot v_0] = \{[v_0, w] \subset T_{p+1} : \mathrm{length}([v_0, w]) = 2d\}$. Moreover, by (1) one has

$$\mu_K(Kg_dK) = |K \cdot [v_0, g_d \cdot v_0]| = |\{[v_0, w] \subset T_{p+1} : \mathrm{length}([v_0, w]) = 2d\}.$$

By the regularity of T_{p+1}, one recovers the formula of $\mu_K(Kg_dK)$ claimed before.

This second argument easily extends to every *t.d.l.c. group G acting locally ∞-transitively on a locally finite tree T with compact open vertex-stabilizers*, taking K as a vertex-stabilizer (or, up to minor changes, an edge-stabilizer).

More generally, we can assume that G acts *weakly locally ∞-transitively* on T, i.e., every vertex-stabilizer G_v acts transitively on the set of geodesics from v in T having the same image via the quotient map $\pi : T \longrightarrow \Gamma := G\backslash T$. If the G-action on T is edge-transitive, this condition coincides with locally ∞-transitivity. In general, however, it comprises many other (non-edge transitive) examples, like the groups of automorphisms of locally finite trees preserving a vertex-coloring (cf. [7, § 5]) or universal groups associated to certain local action diagrams (cf. [5]).

In this more general setting, let $K = G_v$ be a compact open vertex-stabilizer. Then, each KgK corresponds to a loop at $\pi(v)$ in Γ, i.e., $\pi([v, g\cdot v])$, and $\mu_K(KgK)$ is the number of geodesics from v in T lifting the loop $\pi([v, g \cdot v])$ via π. A similar argument can be found in [1, § 3]. Hence, for computing $\zeta_{G,K}(s)$, we need only two tools: the quotient graph $\Gamma = G\backslash T$ regarded as a *Serre-graph*[1], and a weight $\omega(e) \in \mathbb{Z}_{\geq 1}$ on each $e \in \mathrm{Edg}(\Gamma)$ giving the number of edges in the Serre-graph associated to T lifting e and with a common origin. With a similar argument as in [3], one deduces what follows.

Theorem A. *Let G be a t.d.l.c. group acting weakly locally ∞-transitively on a locally finite tree T with compact open vertex-stabilizers, and let $K = G_v$ be a vertex-stabilizer. Let $\Gamma = G\backslash T$ be finite, and $\omega(e) \geq 3$ for every $e \in \mathrm{Edg}(\Gamma)$. Then $\zeta_{G,K}(s)$ converges at some $s \in \mathbb{C}$ and it can be meromorphically continued to $\mathbb{C}$ as*

$$\zeta_{G,K}(s) = \frac{\det(I - W(s) + U_{\pi(v)}(s))}{\det(I - W(s))}.$$

Here, $W(s)$ and $U_{\pi(v)}(s)$ are $|\mathrm{Edg}(\Gamma)|$-dimensional matrices whose entries are entire functions in $s \in \mathbb{C}$ depending only on Γ and $\omega(\_)$.

Following [2], we can often recover the *Euler characteristic*[2] χ_G of the group G from the meromorphic continuation of $\zeta_{G,K}(s)$ as follows.

Theorem B. *Let (G, K) as in Theorem A. If Γ is a tree, then $\chi_G = \zeta_{G,K}(-1)^{-1}\mu_K$.*

Unlike the examples studied in [2], if Γ is not a tree, the conclusion of Theorem B is no longer true in general (cf. [4]). At the current stage, however, there is not a complete characterization of all pairs (G, K) for which $\chi_G = \zeta_{G,K}(-1)^{-1}\mu_K$ holds yet.

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Profinite rigidity of fibring

Dawid Kielak
(joint work with Sam Hughes)

The starting point for this series of three lectures is the theorem of Jaikin-Zapirain [5].

Theorem 1. *Let M and N be compact connected orientable aspherical three-manifolds. If their fundamental groups are profinitely isomorphic, then M fibres over the circle if and only if N does.*

Here, being *profinitely isomorphic* means precisely that the profinite completions of the groups are isomorphic. When the groups are finitely generated, as is the case for the groups above, this amounts to saying that for every finite group, it is a quotient of one of the groups if and only if it is a quotient of the other.

For irreducible three-manifolds, fibring over the circle can be understood algebraically: it was shown by Stallings [7] that it is equivalent to *algebraic fibring*, that is, admitting an epimorphism to $\mathbb{Z}$ with finitely generated kernel (this fact has two more modern proofs [3, 1]). Hence, it becomes natural to ask under what circumstances does being algebraically fibred pass between groups with the same profinite completions.

Theorem 2 (Hughes–K. [4])**.** *Let G and H be two finitely presented LERF groups, and suppose that the profinite completions $\widehat{G}$ and $\widehat{H}$ are isomorphic. If G is algebraically fibred, then so is H.*

Recall that a group G is *LERF (locally extended residually finite*) if and only if for every finitely generated subgroup $A \leqslant G$ and an element $b \in G \smallsetminus A$, there exists a quotient map $\rho\colon G \to Q$ with finite image such that $\rho(b) \notin \rho(A)$.

The above is actually an instant of a more general result. To state it, we need to introduce the concept of BNS-invariants. Given a ring R, the *nth BNS invariant of G over R*, denoted $\Sigma^n(G;R)$, is a subset of the set of non-zero homomorphisms $G \to \mathbb{R}$ consisting of maps $\phi\colon G \to \mathbb{R}$ for which

$$\mathrm{H}_i(G; \widehat{RG}^{\phi}) = 0$$

for all $i \leqslant n$. Here $\widehat{RG}^{\phi}$ is the *Novikov ring* associated to ϕ, defined as the ring of functions $G \to R$ whose support intersects $\phi^{-1}((-\infty, \kappa))$ in a finite set for every $\kappa \in \mathbb{R}$.

The BNS-invariants are related to our previous discussion, since for a character $\phi\colon G \to \mathbb{Z}$, lying in $\Sigma^n(G; R) \cap -\Sigma^n(G; R)$ is equivalent to having kernel of type $\mathtt{FP}_n(R)$.

A character $G \to \mathbb{Z}$ lying in $\Sigma^n(G; R) \cup -\Sigma^n(G; R)$ will be called *n-semi-fibred.*

In [4] we introduced the following definition.

Definition 3. Let R be an integral domain. A group G lies in $\mathrm{TAP}_n(R)$ if and only if n-semi-fibred characters are precisely the characters for which all twisted Alexander polynomials over R in dimensions i for all $i \leqslant n$ do not vanish.

The twisting considered above is that by an epimorphism from G to a finite group. For every group, if we are given an n-semi-fibred character, then its twisted Alexander polynomials over R in dimensions i for all $i \leqslant n$ do not vanish; the interesting part of the definition is the reverse implication.

It was first observed by Friedl–Vidussi [2] that twisted Alexander polynomials are important in recognising fibred 3-manifolds. In the language we just introduced, the main theorem of [2] states that fundamental groups of connected compact orientable three-manifolds with empty or toroidal boundary lie in $\mathrm{TAP}_1(R)$ for every Noetherian UFD R.

We now have more examples of such groups.

Theorem 4 (Hughes–K. [4])**.** *Let R be a commutative ring.*

- *If G is a LERF group of type $\mathtt{FP}_2$ over any commutative ring, then G lies in $\mathrm{TAP}_1(R)$.*
- *The class $\mathrm{TAP}_1(R)$ is closed under finite products.*
- *Products of limits groups lie in $\mathrm{TAP}_\infty(R)$.*

Once a group is shown to lie in $\mathrm{TAP}_1(\mathbb{F})$ over a finite field $\mathbb{F}$ we can use it to study profinite rigidity of BNS-invariants, thanks to the following result, heavily inspired by ideas of Jaikin-Zapirain [5] and Liu [6].

Theorem 5 (Hughes–K. [4])**.** *Let G and H be groups of type $\mathtt{FP}_n(\mathbb{Z})$ that are n-good in the sense of Serre, and that have isomorphic profinite completions. If G lies in $\mathrm{TAP}_n(R)$ and $\Sigma^n(G; R) = \emptyset$ then $\Sigma^n(H; R) = \emptyset$ as well.*

The class $\mathrm{TAP}_1(R)$ remains quite mysterious.

Conjecture 6. *Do all {finitely generated free}-by-cyclic groups lie in $\mathrm{TAP}_\infty(R)$?*

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Classifying Spaces

Ged Corob Cook

An important tool for studying abstract groups G is the use of group actions on spaces. When we care about understanding these actions up to homotopy, we typically work in the categories like CW-complexes or G-CW-complexes. There is a CW-complex BG which is a classifying space for G: it classifies principal G-bundles, it is an Eilenberg-MacLane space for G, and so on. As noted by Stefan Witzel at this workshop, we would like to phrase questions about homological finiteness conditions for totally disconnected, locally compact groups (type FP_n, cohomological dimension) in terms of properties of such a classifying space. When group theorists think of doing this for topological groups, they tend to take the approach that, at any rate, there can be no Eilenberg-MacLane space for a topological group, since the homotopy group(oid)s of such a space would be abstract groups. The purpose of this talk is to argue that most, if not all, of the work done by classifying spaces for abstract groups can be recovered for topological groups.

This generalisation is done by category theory in [1]. First, we replace the category of topological spaces with a convenient category of spaces, that is, a cartesian closed category. In my existing work, this is the category $\mathcal{U}$ of compactly generated, weakly Hausdorff spaces (k-spaces); in future iterations, it will probably be condensed sets. The first benefit of this comes when we want to define topological homotopy groups of topological spaces X: putting the compact-open topology on the set of (pointed) continuous maps $S^n \to X$, and the quotient topology from this on $\pi_n(X)$, does not give a topological group in general. The problem is precisely the failure of topological spaces to be cartesian closed. But using the version of the compact-open topology internal to $\mathcal{U}$ (that is, the k-ification of the usual compact-open topology) instead, $\pi_n(X)$ becomes an internal group object of $\mathcal{U}$: a k-group.

We can put a model structure on $\mathcal{U}$, the CH-structure, such that the fibrant-cofibrant objects (that is, our analogue of CW-complexes) are retracts of KW-complexes, spaces built by 'attaching spaces of n-cells' in dimension n, instead of attaching a discrete sets of n-cells. Formally, a KW-complex X is a colimit (in k-spaces) of a sequence

$$X^0 \to X^1 \to^2 \to \cdots$$

in which $X^0 = T_0$ is a disjoint union of compact Hausdorff spaces, and each maps $X^n \to X^{n+1}$ is given by a pushout of a diagram

$$X^n \leftarrow S^n \times T_{n+1} \to B^{n+1} \times T_{n+1},$$

where S^n is the n-sphere, B^{n+1} the $n+1$-ball, and T_{n+1} is a disjoint union of compact Hausdorff spaces.

This model structure has fine enough weak equivalences that weak equivalences induce isomorphisms of all homotopy k-groups. On the other hand, maps that induce such isomorphisms are not weak equivalences in the CH-structure in general. For this reason, we also need to track a weaker structure, called the regular structure, in which the weak equivalences are those which induce isomorphisms of homotopy group objects in the Barr-exact completion of $\mathfrak{U}$. Both the CH-structure and the regular structure have analogous versions in $s\mathfrak{U}$, the category of simplicial objects in $\mathfrak{U}$, which we will also need.

Finally, we can use this category-theoretic work to start constructing classifying spaces for totally disconnected, locally compact groups, in [2]. For an abstract group G, the classifying space can be constructed as the geometric realisation of a simplicial set S with S_n given by the n-fold product G^n, and we copy this construction for k-groups, building the classifying space BG from a simplicial k-space S with $S_n = G^n$. By van Dantzig's theorem, S_n is a disjoint union of compact Hausdorff spaces when G is totally disconnected, locally compact, so BG is a KW-complex. (For k-groups more generally, one takes a cofibrant replacement of S before the geometric realisation.) The main technical result is the following:

Theorem 1. *Suppose C is an open cover of $X \in \mathfrak{U}$, closed under intersections; we think of C as a poset ordered by inclusion. Then $\underline{\mathrm{Sing}}(X)$ is weakly equivalent (in the regular structure on $s\mathfrak{U}$) to the homotopy colimit (in the CH-structure on $s\mathfrak{U}$) of $\{\underline{\mathrm{Sing}}(U)\}_{U \in C}$.*

From this, we show the main result.

Theorem 2. *If a k-group G is totally path-disconnected, BG is an Eilenberg–Mac Lane space $K(G,1)$ for G.*

This applies, in particular, to totally disconnected, locally compact groups. I conjecture that if one replaces $\mathfrak{U}$ with the category of condensed sets, the weak equivalences in the CH-structure and the regular structure will be the same; this should strengthen the equivalent of Theorem 1 and allow further results to be proved in this direction.

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From discrete to t.d.l.c.

Laura Bonn

For discrete groups there are a lot of results about finiteness conditions, see [3]. The question is now, how we can transform these results into the world of totally disconnected locally compact groups. As a first step we generalize the concept of normal subgroups.

Definition. (also see [1]) Let Γ be a group and $\Lambda \leq \Gamma$. Then Λ is called a **commensurated** subgroup of Γ, if $\Lambda \cap g\Lambda g^{-1}$ has finite index in Λ and in $g\Lambda g^{-1}$.

In the following we want Γ to be a discrete group and $\Lambda \subseteq \Gamma$ to be a commensurated subgroup. Then Γ acts on Γ/Λ by left multiplication, such that we can define a map $\alpha\colon \Gamma \to \mathrm{Sym}(\Gamma/\Lambda)$. We can equipped $\mathrm{Sym}(\Gamma/\Lambda)$ with the topology of pointwise convergence and then we define the following group.

Definition. The group $\Gamma//\Lambda := \overline{\alpha(\Gamma)}$ is called the **Schlichting completion**.

Sometimes this construction is called the profinite completion of Γ relative to Λ.

Remark. (also see [2])

- If Λ is a normal subgroup, then $\Gamma//\Lambda = \Gamma/\Lambda$.
- $\Gamma//\Lambda$ is a totally disconnected locally compact group.
- If Λ is a not normal subgroup, then $\Gamma \hookrightarrow \Gamma//\Lambda$ is a dense embedding.
- $\overline{\alpha(\Lambda)}$ is a compact open subgroup of $\Gamma//\Lambda$.

For the Schlichting completion the following lemma holds.

Lemma 1. [1, Lemma 6.3 and 6.4] $\Gamma//\Lambda = \overline{\alpha(\Lambda)}\alpha(\Gamma)$ *and* $\overline{\alpha(\Lambda)} \cap \alpha(\Gamma) = \alpha(\Lambda)$.

We want to transform some finiteness conditions of the discrete group along the Schlichting completion to the tdlc case.

Theorem. [1, Theorem 6.1] *If Γ is finitely presented and $\Lambda \subseteq \Gamma$ is a finitely generated commensurated subgroup then $\Gamma//\Lambda$ is compactly presented.*

Sketch of proof: For the full proof see [1, Theorem 6.1].
The compact generation set is $\overline{\alpha(\Lambda)} \cup S$, where S is a finite generation set of $\alpha(\Gamma)$. Give four types of relations, G inherits all relations of $\alpha(\Gamma)$, the intersection of $\overline{\alpha(\Lambda)}$ and S gives the second type, the two last types comes from lemma 1. □

In the setting of discrete groups the following result about finiteness properties is known [4, Section 6], for a short exact sequence $0 \to N \to G \to H \to 0$, with N is of type F_{n-1} and G of type F_n, then H is of type F_n and if N and H of type F_n, then G is of type F_n, too.
Here we have seen, if Γ is of type F_2 and the commensurated subgroup Λ is of type F_1, then $\Gamma//\Lambda$ is of type F_2.
I am currently working on generalizing Le Boudec's theorem to higher finiteness conditions.

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Some cohomological invariants for tdlc-groups

Sofiya Yatsyna

When looking at the homological aspects of totally disconnected locally compact (tdlc) groups, one may come across rational discrete cohomology theory introduced by Castellano–Weigel in [1]. Specifically, given a commutative ring R with identity and tdlc group G, let $R[G]$ denote the R-group algebra, and ${}_{R[G]}\mathbf{mod}$ – the abelian category of left $R[G]$-modules. A left $R[G]$-module D is *discrete* if and only if for each $d \in D$, the stabilizer $\mathrm{stab}_G(d)$ is an open subgroup of G. In [1], the authors establish ${}_{R[G]}\mathbf{dis}$, the full subcategory of ${}_{R[G]}\mathbf{mod}$, whose objects are discrete left $R[G]$-modules; whereby in the case of $R = \mathbb{Q}$, ${}_{\mathbb{Q}[G]}\mathbf{dis}$ is an abelian category with enough injectives, and rational discrete cohomology theory for tdlc groups can be defined.

The question naturally arises: *Can this cohomology theory be used to find analogous tdlc versions of known results?* One such interesting result is given by Gedrich–Gruenberg in [3] looking at two homological finiteness conditions on a ring R: the supremum of projective lengths (dimensions) of injective R-modules ($\mathrm{spli}(R)$) and the supremum of injective lengths of projective R-modules ($\mathrm{silp}(R)$). By way of these invariants, one can show the following:

Theorem (Gedrich–Gruenberg [3])**.** *Let R be a commutative noetherian ring of finite R-injective dimension t. If Λ is a R-projective Hopf R-algebra, then*

$$\mathrm{silp}\,\Lambda \leq \mathrm{spli}\,\Lambda + t.$$

Results of Cornick–Kropholler in [2] expand on the relationship of the above Gedrich–Gruenberg invariants with the finitistic dimension when R is the group algebra of a hierarchically decomposable group. Using rational discrete cohomology, it would be interesting to develop the theory of tdlc analogues. Furthermore, it turns out that ${}_{\mathbb{Z}[G]}\mathbf{dis}$ does not have enough projections (unlike its ${}_{\mathbb{Q}[G]}\mathbf{dis}$ counterpart) – whether the above cohomological invariants could be defined for ${}_{\mathbb{Z}[G]}\mathbf{dis}$ is also interesting and worth exploring.

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Finiteness Properties of Algebraic Bieri-Strebel Groups

Lewis Molyneux

(joint work with Brita Nucinkis, Yuri Santos Rego)

Given a subinterval of the real numbers I, a subgroup of positive real numbers under multiplication P, and a $\mathbb{Z}[P]$-submodule of the real numbers A, the Bieri-Strebel group $G(I, A, P)$ is the group of piecewise linear, orientation preserving homeomorphisms of I, with slopes in P and breakpoints in A. Bieri and Strebel constructed these groups[1] as a generalisation of Thompson's group F. We consider groups of the form $G([0,1], \mathbb{Z}[\beta], \langle\beta\rangle)$, where β is the positive real root of the polynomial $(\sum_{i=1}^{n} a_i x^i) - 1, a_i \in \mathbb{N}$. We call these Algebraic Bieri-Strebel groups.

A frequently useful property of Thompson's group F is the ability to express each element of the group as an ordered pair of rooted binary trees. [2] This representation of elements has proven useful in proofs of properties of Thompson's group, particularly finiteness properties such as F_n (and in particular F_∞) and the BNSR-invariant. For instance, Thompson's group can be shown to be F_∞ via its action on a space of pairs of rooted binary forests and rooted binary trees, as summarised by Zaremsky [3]. In addition, while the initial calculation of the BNSR invariant of Thompson's group was performed by Bieri, Geoghegan and Kochloukova [4], Zaremsky and Witzel were able to recalculate the invariant using Morse theory and an adaptation of the previous forest-tree space [5].

For polynomials of the form $a_2x^2 + a_1x - 1$, Winstone was able to show that tree-pair representations for all elements of the associated group are only possible when $a_2 \leq a_1$ [6]. Initially, Cleary was able to demonstrate the F_∞ property for groups with associated polynomial of the form $x^2 + nx - 1, n > 0$ [8]. This proof has since been generalised using tree-pair representations, demonstrating the F_∞ property for all quadratic Bieri-Strebel groups with complete tree-pair representations.

Molyneux, Nucinkis and Santos Rego [7] were able to apply Bieri, Geoghegan and Kochloukova's method in order to calculate the BNSR-invariant of F_τ, but the BNSR-invariant for Algebraic Bieri-Strebel groups in general remains an open problem. For those with complete tree-pair representations, A complex similar to that constructed by Stein and Farley is producible, and the Morse Theory used by Zaremsky and Witzel should be applicable.

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Embedding theorems for discrete groups

Ian J. Leary

This talk gave a survey of three embedding theorems and discussed some related open questions. The theorems, together with their dates of publication, are stated below.

Theorem 1 (Higman–Neumann–Neumann [4], 1949). *Every countable group embeds in a 2-generator group.*

Theorem 2 (Higman [3], 1961). *A finitely generated group embeds in a finitely presented group if and only if it is recursively presented.*

Theorem 3 ([6], 2018). *Every countable group embeds in a group of type FP_2.*

Remark 4. *The questions of whether every finitely presented group embeds in a group of type F_3 and whether every group of type FP_2 embeds in a group of type FP_3 remain open. Arguably the question for FP_2 and FP_3 should be easier because computability seems not to arise.*

Remark 5. *Can one state and either prove or find a counterexample to a version of Theorem 1 for tdlc groups? Ilaria Castellano points out that free products, which are used in the proof of Theorem 1, are not available in this context* [9].

Although the statement of Theorem 3 is similar to that of Theorem 1, the proof is modelled closely on Valiev's proof of the Higman embedding theorem (Theorem 2) [11, 7]. The only recent ingredient needed in the proof is the existence of a family of groups of type FP indexed by subsets of $\mathbb{Z}$ [5].

Even the most streamlined versions of the proof of the Higman embedding theorem are difficult. Very roughly there are three steps: reduction to subgroups of the free group $\mathbb{F}_2$ using Theorem 1; reduction to subsets of $\mathbb{N}$; encoding suitable subsets of $\mathbb{N}$ inside finitely presented groups.

A key definition is that of a *benign* subgroup $H \leq G$ of a finitely generated group G. This is a subgroup H such that the HNN-extension

$$\langle G, t \ : \ tht^{-1} = h \ \ h \in H \rangle$$

can be embedded in a finitely presented group.

Higman's rope trick states that if H is a benign normal subgroup of finitely generated G then G/H embeds in a finitely presented group. With the rope trick and the HNN-embedding theorem, one is reduced to showing that every recursively generated subgroup of the free group $\mathbb{F}_2$ is benign.

Words in two generators and their inverses are encoded as subsets of $\mathbb{N}$ via a Gödel numbering, with the digits $1, 2, 3, 4$ standing for the generators and their inverses, concatenation being unchanged, and 0 standing for the empty word. This gives a coding process that replaces *subsets* of $\mathbb{F}_2 = \langle a, b\rangle$ by *subgroups* of $\mathbb{F}_3 = \langle c, d, e\rangle$ of the form $\langle c^n de^n \ : \ n \in S\rangle$ for some $S \subseteq \mathbb{N}$ in an algorithmic way.

After this step, one is reduced to showing that any recursively enumerable subset of $\mathbb{N}$ can be 'encoded' within a finitely presented group. For this the proof in Lyndon and Schupp [7] uses a technique that was not available to Higman in 1961: Matiyasevich's theorem (building on work of Davis, Putnam and Robinson) that recursively enumerable subsets of $\mathbb{Z}$ are Diophantine, i.e., of the form

$$\{x \in \mathbb{Z} \ : \ \exists y_1, \dots, y_n \in \mathbb{Z} \ f(x, y_1, \dots, y_n) = 0\}$$

for some n and for some integer polynomial f [8, 2, 10].

In the 1990's Bestvina and Brady constructed the first groups of type FP that are not finitely presented [1], resolving a well-known problem that had been open for at least 30 years. Around 20 years later, I discovered a way to generalize the Bestvina–Brady construction to produce an uncountable family of groups of type FP; in particular for any $S \subseteq \mathbb{Z}$ with $0 \in S$ I could construct a group $J = J(S)$ of type FP and elements $j_1, \dots j_4 \in J$ so that $j_1^n j_2^n j_3^n j_4^n = 1$ iff $n \in S$ [5]. This showed that the class of subgroups of groups of type FP_2 is larger than the class of subgroups of finitely presented groups. Note that the map $S \mapsto J(S)$ encodes *any* subset of $\mathbb{Z}$ that contains 0 inside the presentation of a group of type FP.

This also suggested the possibility of proving the new Theorem 3, by modifying Valiev's proof of Theorem 2. Define a subgroup H of a finitely generated group G to be *homologically benign* if the HNN-extension $\langle G, t\rangle$ as defined earlier can be embedded in a group of type FP_2. Next check that there is a homological version of the Higman rope trick: if $H \leq G$ is homologically benign, then G/H embeds in a group of type FP_2. Just as in the proof of the Higman embedding theorem, this plus the HNN-embedding theorem gives a reduction: to prove Theorem 3 it suffices to show that every normal subgroup of the free group $\mathbb{F}_2$ is homologically benign. The encoding of subsets of $\mathbb{F}_2$ via subgroups of $\mathbb{F}_3$ and then subsets of $\mathbb{N}$ can be used essentially unchanged. Since we already know how to encode arbitrary subsets of $\mathbb{N} \subseteq \mathbb{Z}$ inside presentations of groups of type FP_2, this gives Theorem 3.

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Open problems on TDLC-groups

The mini-workshop also featured a discussion session on the open problems and further questions related to totally disconnected locally compact groups. In addition to problems already stated in the extended abstracts presented in this report, this section compiles further problems that remain unresolved and are ordered by those who have expressed interest in them.

1. Dawid Kielak

Question 1. *Let G be a group of type $\mathrm{FP}_n(\mathrm{F})$, for $\mathrm{F} \in \{\mathbb{Q}, \mathbb{F}_p : p$ prime$\}$. In general, G is not of type $\mathrm{FP}_n(\mathbb{Z})$ $(n \geq 1)$, but there exist examples where it holds (e.g. right-angled Artin group (RAAG)).*

(i) In which classes of groups is it true?
(ii) Is it true if there exists an exact sequence $G \to H \to \mathbb{Z}$, where H of type F?

2. Rudradip Biswas

Theorem 1. [1, Theorem 19.1] *Let $\Gamma \in \mathbf{LH}\mathfrak{F}$ (locally in Kropholler's Hierarchy with all finite groups as the base class), and suppose Γ is of type $\mathrm{FP}_\infty(\mathbb{Z})$. Then Γ has only finitely many conjugacy classes of finite elementary abelian p-groups.*

Question 2. *Is it possible to formulate a TDLC version of this theorem?*

Remark 2. *It follows from the discussion that it may be useful to consider groups of type $\mathrm{FP}_\infty(\mathbb{Q})$ or F_∞.*

3. Ilaria Castellano

Question 3. *Let G, H be compactly generated TDLC groups, and suppose G is quasi-isometric to H, with G and H both having finite cohomological dimension over $\mathbb{Q}$. Does $\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}_{\mathbb{Q}}(H)$?*

Question 4. *Is there a suitable TDLC analogue of right-angled Artin groups (RAAGs)?*

4. Ian Leary & Ilaria Castellano

Question 5. *Suppose G is a σ-compact TDLC group. Does G embed into a compactly generated TDLC group?*

5. Roman Sauer

Question 6. *Does a version of the Atiyah conjecture hold for TDLC groups? Or do there exist examples of TDLC groups G with irrational L^2-Betti numbers, i.e. $b_n^{(2)}(G, \mu) \notin \mathbb{Q}$, where μ is the Haar measure normalised to be 1 on a compact-open subgroup of G.*

Remark 3. *For a locally compact group G and lattice $\Gamma < G$, then $b_n^{(2)}(\Gamma) = \operatorname{covol}(\Gamma) b_n^{(2)}(G)$. There exist non-compactly generated examples with irrational co-volume.*

Question 7. *Are there examples of compactly generated TDLC groups with lattices of irrational covolume?*

6. Yuri Santos Rego

Question 8. *Under which conditions are Coxeter groups virtually residually finite rationally solvable (RFRS)?*

Remark 4. *Kielak in* [2] *shows that for a finitely generated virtually RFRS group G, virtual fibering is equivalent to the vanishing of the first L^2-Betti number, i.e.* $b_1^{(2)}(G) = 0$.

Question 9. *Do there exist Coxeter groups W_1, W_2 with isomorphic profinite completions $\widehat{W_1} \cong \widehat{W_2}$ but different rational Euler characteristic $\chi(W_1) \neq \chi(W_2)$?*

7. Thomas Weigel

Question 10. *Let N_q be the Neretin group acting on the q-regular rooted tree.*
- *(i) Is $H^k(N_q, \mathrm{Bi}(N_q)) = 0$ for all k?*
- *(ii) Is $H_c^k(\underline{E}_{\mathcal{O}} N_q, \mathbb{Q}) = 0$ for all k?*

Question 11. *Are there "non-good" Coxeter groups?*

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