Mathematisches Forschungsinstitut Oberwolfach

Report No. 50/2023

DOI: 10.4171/OWR/2023/50

# Mini-Workshop: Standard Subspaces in Quantum Field Theory and Representation Theory

Organized by
Maria Stella Adamo, Tokyo
Gandalf Lechner, Erlangen
Roberto Longo, Rome
Karl-Hermann Neeb, Erlangen

29 October – 3 November 2023

Abstract. Real standard subspaces of complex Hilbert spaces are long known to provide the right language for Tomita-Takesaki modular theory of von Neumann algebras. In recent years they have also become an object of prominent interest in mathematical quantum field theory (QFT) and unitary representation theory of Lie groups. This workshop brought together mathematicians and physicists working with standard subspaces, particularly in QFT (construction of QFT models, characterization of entropy, information-theoretic aspects), nets of standard subspaces on causal homogeneous spaces and aspects of reflection positivity and euclidean models related to standard subspaces and modular theory.

*Mathematics Subject Classification (2020):* 22Exx, 46Lxx, 47Bxx.

## Introduction by the Organizers

Standard subspaces originate from the theory of von Neumann algebras, where they encode the modular data of a von Neumann algebra w.r.t. a cyclic and separating (standard) vector. They can however also be defined independently of von Neumann algebras in the simple setting of a complex Hilbert space; a standard subspace is then a closed real linear subspace which contains no complex line and has dense complex linear span. In recent years it has become increasingly clear that this point of view allows for a rich and still unfolding theory that is of interest in its own right and has fascinating applications in various fields.

The mini-workshop *Standard Subspaces in Quantum Field Theory and Representation Theory* was a meeting designed to bring together researchers working with standard subspaces from different perspectives, with an emphasis on people in quantum field theory and representation theory of Lie groups.

In quantum field theory (QFT), standard subspaces serve as a means to encode localization regions in a spacetime manifold and are thus a basic aspect of any model QFT. Typical questions involve the modular data of standard subspaces belonging to particular localization regions (for massive QFT, the modular group for a double cone is still unknown), the usage of standard subspaces in the formulation of examples of interacting QFTs, the role played by standard subspaces to define an intrinsic notion of entropy, or the interplay of standard subspaces with KMS-condition and reflection positivity, which appears in reconstruction theorems for Euclidean field theories.

In the presence of a spacetime symmetry Lie group, one considers nets of standard subspaces transforming under a unitary representation of this group, which immediately explains the close link to Lie group representations. Here typical questions concern the interplay between the geometric configurations, such as wedge regions for modular flows on causal homogeneous spaces and the types of unitary group representations that can host corresponding nets of standard subspaces. Another important aspect is to detect natural finite-dimensional spaces of distribution vectors for the representations, specified by a suitable KMS condition, that are invariant under large subgroups and from which well-behaved nets of real subspaces can be constructed by a smearing process.

The mini-workshop format has turned out to be the perfect choice for discussing these questions in an efficient and productive manner. The areas of expertise of the 17 participants were close enough to allow for easy discussions, and at the same time far enough apart for learning new results, points of view and ideas from each other. For the younger participants the event also offered the highly appreciated opportunity to get to know more colleagues, discuss and present their projects, and grow their scientific networks.

Thanks to the mini-workshop format, we could also successfully implement some informal discussion sessions in addition to more typical seminar talks. In these sessions, participants presented ideas, observations and questions in an unfinished format which led to long and intense discussions between many people.

An example of such a discussion was a session on inclusions of standard subspaces. Here the main question is how to decide whether an inclusion $K \subset H$ is irreducible, and the discussion related this to questions in von Neumann algebras (split property, modular nuclearity), entropy (the boundedness of the cutting projection decides about the existence of irreducible extensions), symmetric inner functions and the distribution of their zeros (closely connected to $\dim(K' \cap H)$ in particular examples), and more.

Another group discussion was centered around positive energy representations of gauge groups. As these groups are infinite-dimensional, the highly developed finite-dimensional structure theory does not apply to these groups, but their positive energy representations appear naturally in physical models, such as Conformal Field Theory (CFT), where positive energy representations of loops groups are crucial in the construction for models, such as the $U(1)$-current and its derivatives. The extension of the elaborate geometric side of standard subspaces for finite-dimensional groups to important classes of infinite-dimensional ones is an important problem for future research.

Some further group discussions concerned the, by far not fully understood, aspect of reflection positivity and the existence of euclidean models. In this context it is not clear how the modular objects, such as modular operator and conjugation, corresponding to standard subspaces, should be represented on the euclidean side. Natural candidates involve unitary representations of the non-connected group $O_2(\mathbb{R})$, satisfying suitable positivity conditions.

The following abstracts provide an excellent picture of the current state of the art and the diverse research directions concerning various aspects of standard subspaces and their applications. Topics that appeared in several presentations were aspects of entropy (Longo, Cadamuro), deformations of second quantization processes (Lechner, Correa da Silva), reflection positivity and euclidean models (Adamo, Tanimoto), and connections between nets of standard subspaces and unitary representations (Morinelli, Ólafsson, Beltiţă, Neeb).

*Acknowledgement:* The organizers thank the director Prof. Dr. Gerhard Huisken, and the Oberwolfach staff for offering an outstanding environment for this workshop and support in all phases of the planning.

The MFO and the workshop organizers would also like to thank the Simons Foundation for supporting Gestur Ólafsson in the "Simons Visiting Professors" program at the MFO.

## Mini-Workshop: Standard Subspaces in Quantum Field Theory and Representation Theory

## Table of Contents

# Abstracts

## Standard subspaces in Representation Theory

Vincenzo Morinelli
(joint work with K.-H. Neeb, G. Ólafsson)

A model in Algebraic Quantum Field Theory (AQFT) is specified by a map associating to any open region of the spacetime, its von Neumann algebra of local observables acting on a fixed complex Hilbert space $\mathcal{H}$ (the state space), satisfying fundamental quantum and relativistic assumptions as Isotony, Locality, Poincaré covariance, positivity of the energy, cyclicity of the vacuum vector for local algebras [Haa96]. One can take as an example an AQFT on Minkowski spacetime. Here, the Rindler wedge and its Poincaré transforms are fundamental localization regions called wedges. They are determined by the one-parameter group of boost symmetries (properly parametrized) that fix them as a subset of the Minkowski spacetime. The algebraic canonical construction of the free field provided by Brunetti–Guido–Longo (BGL) builds on the the wedge-boost identification, the Bisognano-Wichmann (BW) property and the PCT Theorem, cf. [BGL02]. In particular, given a particle, namely an irreducible representation of the proper Poincaré group $U$ that is unitary on the connected component of $\mathbf{1}$ and antiunitary on the connected component of $-\mathbf{1}$, it is possibile to canonically determine the states in the Hilbert space $\mathcal{H}_U$ supporting $U$ localized in any wedge, having as a fundamental input the unitary representation of the one-parameter group of boosts associated to the wedge and the antiunitary operator implementing the wedge reflection. For instance, consider the wedge region $W_R = \{x \in \mathbb{R}^{1+d} : |x_0| < x_1\}$, the real standard subspace[1] $\mathsf{H}(W_R) \subset \mathcal{H}_U$ of states localized in a wedge region $W_R$ is uniquely determined as follows: let

$$\Delta^{it}_{\mathsf{H}(W_R)} := U(\Lambda_{W_R}(-2\pi t)), \qquad \text{(BW) property}$$

$$J_{\mathsf{H}(W_R)} := U(r_{W_R}), \qquad \text{(PCT) Theorem}$$

where we have that $\Lambda_{W_R}(t)x = (\cosh(t)x_0 + \sinh(t)x_1, \sinh(t)x_0 + \cosh(t)x_1, \mathbf{x})$, $j_R x = (-x_0, -x_1, \mathbf{x})$, for $x \in \mathbb{R}^{1+d}$, $\mathbf{x} \in \mathbb{R}^{d-1}$, then $\mathsf{H}(W_R) = \ker(1 - J_{\mathsf{H}(W_R)}\Delta^{\frac{1}{2}}_{\mathsf{H}(W_R)})$. Note that $S_{\mathsf{H}(\mathsf{W_R})} = J_{\mathsf{H}(\mathsf{W_R})}\Delta^{\frac{1}{2}}_{\mathsf{H}(\mathsf{W_R})}$ is the *Tomita operator* of the standard subspace $\mathsf{H}(W_R)$ (for the Tomita theory of standard subspaces we refer to [Lon08]). Then for every open region $O = \bigcap_{W \supset O} W$ with $W$ wedge region, one can define the set of states localized in $O$ by intersection $\mathsf{H}(O) = \bigcap_{W \supset O} \mathsf{H}(W)$. The free field net of von Neumann algebras is then constructed via second quantization, see [BGL02, LRT78].

In this presentation, we will provide an overview on the analysis developed in the last years together with K.-H. Neeb and G. Ólafsson where we generalize this one-particle picture from a geometrical perspective. The core of this analysis relies on the understanding of a deep connection between the geometry of standard

[1] a real closed subspace $\mathsf{H} \subset \mathcal{H}$ is standard if $\overline{\mathsf{H} + i\mathsf{H}} = \mathcal{H}$ and $\mathsf{H} \cap i\mathsf{H} = \{0\}$

subspaces, given by the Tomita modular operator and modular conjugation, and the geometry of specific elements in the Lie algebra of a Lie group $G$ called Euler elements and their representation theory. This approach provides feedbacks for representation theory and for the algebraic approach to Quantum Field Theory without restrictions to second quantization models.

Let $G$ be a connected Lie group and let $h$ be an Euler element in its Lie algebra $\mathfrak{g}$, namely $\mathrm{ad}\, h$ is diagonalizable and $\mathrm{Spec}(\mathrm{ad}\, h) = \{-1, 0, 1\}$, then $\tau_h = \exp(i\pi\, h)$ generates an involution on $\mathfrak{g}$. Simple Lie algebras containing Euler elements are classified, cf. [MN21, Kan00]. Assume that $\tau_h$ integrates to an involution on $G$ and let $G_{\tau_h}$ be the semidirect product group generated by $G$ and the involution $\tau_h$. A $G$-equivariant set of wedges is defined by

$$\mathcal{G}_E = \{W = (x, \tau_x) \in \mathfrak{g} \times \tau_h G : x \text{ is and Euler element}\}$$

where the $G$-action is defined by $g.W = (\mathrm{Ad}\, g(x), g\tau_x g^{-1})$. Once a cone $C$ in the Lie algebra $\mathfrak{g}$ is given, then wedge inclusions can be defined. Furthermore, the causal complement of a abstract wedge is given by $W' = (-x, \tau_x)$. In particular we have defined on a abstract level a local poset of abstract wedge regions, for the general picture see [MN21].

Let $W = (x_W, \sigma_{x_W}) \in \mathcal{G}_E$, given an (anti-)unitary representation of $G_{\tau_h}$ on a Hilbert space $\mathcal{H}$, then $U(\exp(-2\pi t\, x_W))$ and $U(\sigma_{x_W})$ identify the standard subspace $\mathsf{H}(W) \subset \mathcal{H}$ by the Tomita theory. Due to the general theory of standard subspaces we can define a generalized framework for one particle nets of standard subspaces that strictly extends the set of one-particle models from AQFT that can be constructed through the BGL-construction, cf. [MN21].

Causal homogeneous spaces $M = G/H$ play the role of the spacetime in AQFT models. On these spaces concrete wedge regions, as positive subsets for the Euler element flow, can be defined, see [NÓ22, NÓ23, MNÓ23a, MNÓ23b]. If $G$ is centerfree and the wedge subset is connected in $M$, then there is a correspondence between abstract wedges (Euler couples) and wedge subsets of a causal homogeneous space [MN23]. Given a unitary representation $U$ of $G$, one can associate real subspaces on general open regions by using the language of distribution vectors [FNÓ23, NÓ23, NØÓ21, NÓ21]. Bosonic second quantization associates to a one-particle net an isotonous, G-covariant net of von Neumann algebras acting on the Fock space [MN21].

We are in the position of defining an axiomatic framework for nets of von Neumann algebra on abstract wedges as well as on open regions of a causal symmetric spaces. One can deduce properties of wedge symmetries and wedge von Neumann algebras for this generalized AQFT from local properties of the net.

The following results are contained in [MN23]. Firstly, given a one-parameter subgroup $\lambda(t)$ of a connected Lie group $G$, a unitary representation $U$ of $G$ with discrete kernel on an Hilbert space $\mathcal{H}$ and a standard subspace inclusion $\mathsf{K} \subset \mathsf{H} \subset \mathcal{H}$ such that $\Delta_{\mathsf{H}}^{it} = U(\lambda(-2\pi t))$ (BW property) and $U(g)\mathsf{K} \subset \mathsf{H}$ when $g$ is in an open neighbourhood of the identity (regularity property), then $\lambda(t)$ is generated by an Euler element. So, *Euler elements appear naturally in this framework as a consequence of the (BW) and the regularity properties.*

Let $G$, $U$ and $\lambda$ as before. Assume that $\lambda(t)$ is generated by an anti-elliptic element of $\mathfrak{g}$ and consider a von Neumann algebra inclusion $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ with a common unique $G$-fixed cyclic and separating vector in $\mathcal{H}$ (vacuum vector). If the Bisognano-Wichmann property holds, namely $\Delta^{it}_{\mathcal{A},\Omega} = U(\lambda(-2\pi t))$ where $\Delta^{it}_{\mathcal{A},\Omega}$ is the modular group of $\mathcal{A}$ with respect to $\Omega$, and an analogue regularity property holds for the von Neumann algebra inclusion $\mathcal{N} \subset \mathcal{M}$ with respect to the adjoint $G$-action, then the algebra $\mathcal{M}$ is a type $\mathrm{III}_1$ factor with respect to Connes' classification.

## References


[BGL02] Brunetti, R., Guido, D., and R. Longo, *Modular localization and Wigner particles*, Rev. Math. Phys. **14** (2002), 759–785

[FNÓ23] Frahm, J., K.-H. Neeb, and G. Ólafsson, *Nets of standard subspaces on non-compactly causal symmetric spaces*, arxiv:2303.10065

[Haa96] Haag, R., "Local Quantum Physics. Fields, Particles, Algebras," Second edition, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1996

[Kan00] Kaneyuki, S.: Graded Lie algebras and pseudo-hermitian symmetric space. In: Faraut, J., et al. (eds.) Analysis and Geometry on Complex Homogeneous Domains. Progress in Math., vol. 185. Birkhäuser, Boston (2000)

[LRT78] Leyland, P., J. Roberts, and D. Testard, *Duality for quantum free fields*, Unpublished manuscript, Marseille(1978)

[Lon08] Longo, R., *Real Hilbert subspaces, modular theory, SL(2, R) and CFT* in "Von Neumann Algebras in Sibiu", 33-91, Theta Ser. Adv. Math. **10**, Theta, Bucharest

[MN21] Morinelli, V., and K.-H. Neeb, *Covariant homogeneous nets of standard subspaces*, Comm. Math. Phys. **386** (2021), 305–358; arXiv:math-ph.2010.07128

[MNÓ23a] Morinelli, V., K.-H. Neeb, and G. Ólafsson, *From Euler elements and 3-gradings to non-compactly causal symmetric spaces*, Journal of Lie Theory **23:1** (2023), 377–432; arXiv:2207.1403

[MNÓ23b] Morinelli, V., K.-H. Neeb, and G. Ólafsson, *Modular geodesics and wedge domains in non-compactly causal symmetric spaces*, to appear in Annals of Global Analysis and Geometry

[MN23] Morinelli, V., and K.-H. Neeb, *From local nets to Euler elements*, in preparation

[NÓ21] Neeb, K.-H., and G. Ólafsson, *Nets of standard subspaces on Lie groups*, Advances in Math. **384** (2021), 107715, arXiv:2006.09832

[NÓ22] Neeb, K.-H., and G. Ólafsson, *Wedge domains in non-compactly causal symmetric spaces*, Geometriae Dedicata **217:2** (2023), Paper No. 30; arXiv:2205.07685

[NÓ23] Neeb, K.-H., and G. Ólafsson, *Wedge domains in compactly causal symmetric spaces*, Int. Math. Res. Notices **2023:12** (2023), 10209–10312; arXiv:math-RT:2107.13288

[NØÓ21] Neeb, K.-H., Ørsted, B., and Ólafsson, G. *Standard subspaces of Hilbert spaces of holomorphic functions on tube domains*, Communications in Math. Phys. **386** (2021), 1437–1487; arXiv:2007.14797

## Signal communications and modular theory

Roberto Longo

We propose a conceptual frame to interpret the prolate differential operator

$$W = \frac{d}{dx}(1 - x^2)\frac{d}{dx} - x^2 \,,$$

which appears in Communication Theory, as an entropy operator; indeed, we write its expectation values as a sum of terms, each subject to an entropy reading by an embedding suggested by Quantum Field Theory.

This adds meaning to the classical work by Slepian et al. on the problem of simultaneously concentrating a function and its Fourier transform, in particular to the "lucky accident" that the truncated Fourier transform $\mathbf{F}_B$

$$\mathbf{F}_B = E_B \mathbf{F} E_B$$

commutes with the prolate operator; here, $\mathbf{F}$ is the unitary Fourier transform on $L^2(\mathbb{R})$ and $E_B$ the orthogonal projection onto $L^2(B)$, with $B = (-1, 1)$ the unit ball.

The key is the notion of entropy $S(\Phi \| H)$ of a vector $\Phi$ of a complex Hilbert $\mathfrak{H}$ space with respect to a real linear subspace $H$, recently introduced by the author, and extended with collaborators, by means of the Tomita-Takesaki modular theory of von Neumann algebras; if $H$ is a factorial standard subspace of $\mathfrak{H}$ with modular operator $\Delta_H$, we have

$$S(\Phi \| H) = \Re(\Phi, iP_H i \log \Delta_H \Phi) = \Re(\Phi, \mathbf{E}_H \Phi) \,.$$

Here, $P_H$ is the cutting projection

$$P_H : \Phi + \Phi' \mapsto \Phi \,, \quad \Phi \in H, \ \Phi' \in H' \,,$$

with $H'$ the symplectic complement of $H$. $P_H$ can be explicitly expressed in terms of the modular data.

$\mathbf{E}_H = i \log \Delta_H$ is a real-linear, selfadjoint, positive operator, that we regard as an entropy operator inasmuch as its expectation values are entropy quantities.

We consider a generalization of the prolate operator to the higher dimensional case and show that it admits a natural extension commuting with the truncated Fourier transform; this partly generalizes the one-dimensional result by Connes to the effect that there exists a natural selfadjoint extension to the full line commuting with the truncated Fourier transform.

We consider the entropy operator $\mathbf{E}_H$ when $\mathfrak{H}$ is the one-particle Hilbert space of a free, massless, scalar Boson, $H$ is the local subspace associated with the unit ball $B$ in $\mathbb{R}^d$, and $\Phi \in \mathfrak{H}$ is a wave packet. Then $S(\Phi \| H)$ is the information contained by $\Phi$ in $B$.

In this case, on Cauchy data in $L^2(B) \oplus L^2(B)$, $\mathbf{E}_H$ has (up to constants) two components: $-L$, with $L = \nabla(1 - r^2)\nabla$ the Legendre operator, and $M$, the multiplication operator by $(1 - r^2)$.

We infer that the prolate operator is an entropy operator, thus a natural a priori candidate to commute with $\mathbf{F}_B$.

## References

[Lon19] R. Longo, *Entropy of coherent excitations*, Lett. Math. Phys. **109** (2019), 2587—2600.

[CLR20] F. Ciolli, R. Longo, G. Ruzzi, *The information in a wave*, Commun. Math. Phys. **379**, (2020) 979–1000.

[CLRR22] F. Ciolli, R. Longo, G. Ruzzi, A. Ranallo, *Relative entropy and curved spacetimes*, J. Geom. Phys. **172**, (2022), 104416.

[Lon23] R. Longo, *Signal communications and modular theory*, Commun. Math. Phys. **403**, (2023) 473–494.


## The massive modular Hamiltonian for a double cone

Daniela Cadamuro

Since it has been set up in the 1970's due to works by Tomita that became public with lectures by Takesaki [Tak70], as well as by Araki [Ara76], Tomita-Takesaki modular theory has been one of the most important developments in the theory of operator algebra, as well as in quantum theory. However, in relevant examples from quantum (field) theory, obtaining an "explicit" form of the modular generator $\log \Delta$ has been the strenous work of many researchers along the time. At least in the following situations, a model-independent answer is known:

- If $\mathcal{M}$ is the algebra of all observables and $\Omega$ represents a thermal equilibrium state (KMS condition), then $\log \Delta$ is the generator of time translations (up to a factor) [HHW67].
- If $\mathcal{M} = \mathcal{A}(\mathcal{W})$ is the algebra associated with a spacelike wedge region $\mathcal{W}$ in quantum field theory, and $\Omega$ is the Minkowski vacuum, then $\log \Delta$ is the generator of boosts along the wedge [BW75].

But what about the algebra of a double cone, $\mathcal{M} = \mathcal{A}(\mathcal{O})$, in a quantum field theory? To answer this question we consider the example of a real scalar free field $\phi$ of mass $m > 0$. We consider the Fock vacuum as our cyclic and separating vector. The local algebras, as well as the modular operator [EO73], are determined by second quantization, so that we only need to consider the modular operator at one-particle level, which is defined as follows.

On the (complex) one-particle Hilbert space $\mathcal{H}_1$ of the theory, we consider a (closed real) local subspace $\mathcal{L}_1(\mathcal{O}) \subset \mathcal{H}_1$, which is "standard" and "factorial" ($\overline{\mathcal{L}_1 + i\mathcal{L}_1} = \mathcal{H}_1$, $\mathcal{L}_1 \cap \mathcal{L}_1' = \mathcal{L}_1 \cap i\mathcal{L}_1 = \{0\}$), where "prime" denotes the symplectic complement. We define the one-particle Tomita operator on $\mathcal{H}_1$ as

$$T_1 : f + ig \mapsto f - ig, \quad f, g \in \mathcal{L}_1(\mathcal{O}), \tag{1}$$

the polar decomposition of its closure is $T_1 = J_1 \Delta_1^{1/2}$. (We shall drop the index "1" from now on.)

We can rewrite the one-particle modular generator as follows. Let $P$ be the real-linear projector onto $\mathcal{L} \subset \mathcal{H}$ with kernel $\mathcal{L}' \subset \mathcal{H}$. Then, on a certain domain, we write

$$P = (1 + T)(1 - \Delta)^{-1}. \tag{2}$$

A computation then shows that

$$\log \Delta = -2\,\mathrm{arcoth}(P - iPi - 1). \tag{3}$$

This determines $\log \Delta$ from $P$, and hence from $\mathcal{L}$ [FG89]. We now write this formula in a different manner, by writing $\mathcal{H}$ in time-0 formalism in configuration space. Here, $\mathcal{H}$ is parametrized by time-0 initial data of field and field momentum $f = (f_+, f_-)$. The scalar product and the complex structure, with $A = -\nabla^2 + m^2$, are given by

$$\mathrm{Re}\langle f, g\rangle_{\mathcal{H}} = \left\langle f, \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} g \right\rangle_2, \quad i_A = \begin{pmatrix} 0 & A^{-1/2} \\ -A^{1/2} & 0 \end{pmatrix}. \tag{4}$$

The local subspaces are defined as follows. Let $\mathcal{B}$ be the base of $\mathcal{O}$ at time 0, then we define

$$\mathcal{L} = \overline{\mathcal{C}_0^\infty(\mathcal{B}) \oplus \mathcal{C}_0^\infty(\mathcal{B})}, \quad P = \chi \oplus \chi, \tag{5}$$

where $\chi$ multiplies with the characteristic function of $\mathcal{B}$. Inserting this in formula (3), we have

$$\log \Delta = i_A \begin{pmatrix} 0 & M_- \\ -M_+ & 0 \end{pmatrix}, \tag{6}$$

where

$$M_\pm = 2A^{\pm\frac{1}{4}}\,\mathrm{arcoth}(B)A^{\pm\frac{1}{4}}, \quad B = \overline{A^{1/4}\chi A^{-1/4}} + \overline{A^{-1/4}\chi A^{1/4}} - 1. \tag{7}$$

Hence, $\Delta$ is determined from $\chi$ and $A$. However, "explicitly" finding the spectral decomposition of $B$ as a selfadjoint operator on $L^2(\mathbb{R}^s)$ is very difficult. There are however known examples:

- If $\mathcal{O}$ is the wedge in $x_1$-direction, then $M_-$ multiplies with $2\pi x_1$, indepedent of $m$.
- If $\mathcal{O}$ is a double cone of radius $r$ and $m = 0$, then $M_-$ multiplies with $\pi(r^2 - \|\boldsymbol{x}\|^2)$ [HL82].

Now the questions we would like to answer in the case of double cones and $m > 0$ are the following: Is $M_-$ mass indepedent? Is $M_-$ a multiplication operator? Since answering these questions anaytically is very difficult, we do it numerically, namely we evaluate $B$ and $M_-$ numerically to check this hypothesis.

Using numerical approximation means approximating $A^s$ and $\chi$ with finite-dimensional matrices. For that, we need to choose an orthonormal basis and finite dimensional in one summand of $\mathcal{H}$, and we need to approximate $A^{\pm 1/4}$ and $\chi$ with a matrix in this basis. Then, we can apply numerical eigendecomposition in order to evaluate the arcoth, and therefore approximate the operator $B$. We do this with no rigorous estimates on the approximation. Explicitly, $A, \chi$ acts on $L^2_{\mathbb{R}}(\mathbb{R})$ by $A = -\partial_x^2 + m^2$, and $\chi$ is determined by the region considered: $\chi(x) = \Theta(x)$ for a wedge, or $\chi(x) = \Theta(1 + x)\Theta(1 - x)$ for the standard double cone.

As our basis functions, we choose suitable piecewise linear functions [BCM23], and the discretization is first done for $A^{-1/4}$ which is bounded and has a known convolution kernel; we then obtain $A^{1/4}$ by numerical matrix inversion. We can then approximate (the integral kernel of) $M_-$ using the formula (7); this is done by

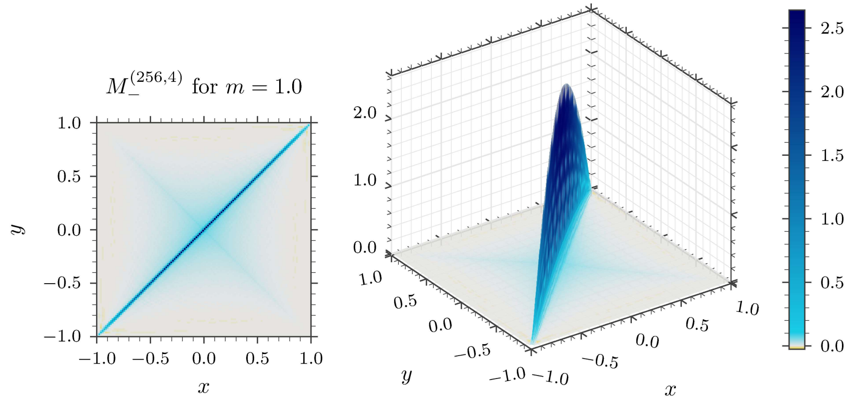


FIGURE 1

functional calculus of matrices, and the computation turns out to require extended floating point precision of 400–600 decimal digits. We expect convergence against the undiscretized result in the weak sense, i.e., if $M^{(N,b)}$ denotes the integral kernel at a number $N$ of basis elements covering the interval $[-b, b]$,

$$\iint g(x)M_-^{(N,b)}(x,y)h(y)dx\,dy \xrightarrow[N,b\to\infty]{} \iint g(x)M_-(x,y)h(y)dx\,dy.$$

We choose $g = h$ to be a Gaussian located near a point $\mu$, then we vary this point $\mu$.

The results in the wedge case turn out to be compatible with known results. In the case of a double cone, we find that the discretized kernel $M_-$ is concentrated predominantly on the diagonal, see Figure 1. There appear to be some contributions along the antidiagonal, but it is unclear whether this is due to numerical errors or whether there is really a subdominant non-diagonal contribution. An explicit expression for the curves displayed in Figure 1 and Figure 2 is not known. The smeared version of the discretized kernel $M_-$, see Figure 2, shows that the kernel is mass-dependent. In particular, the black parabola corresponds to the case $m = 0$ and therefore to the quadratic result of Hislop-Longo, while the two straight black lines (piecewise linear) for large mass correspond to the result of a left and a right wedge. Indeed, large masses correspond to small correlation lengths, and hence a heuristic explanation for the approximate “double wedge” structure may be that at one end of the interval, the contribution from the other end of the interval is very small, so that the modular operator for the interval approximately behaves like the one for a half-line.

A similar analysis can be done for a double cone in the 3+1-dimensional field using its spherical symmetry. It turns out in this case that the modular operator also depends on angular momentum [BCM23].

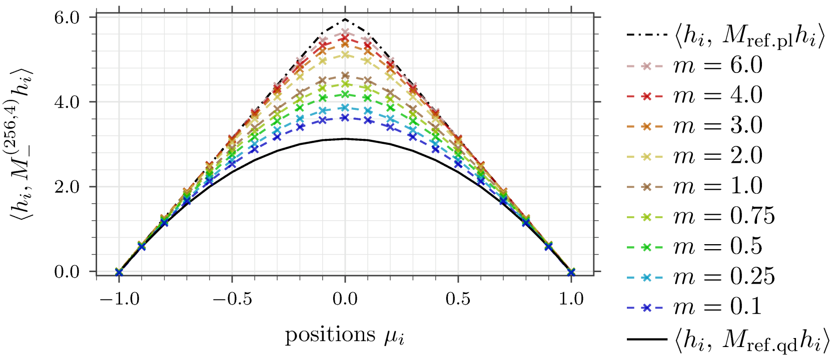


Figure 2

## References


[Ara76] H. Araki, *Relative entropy of states of von Neumann algebras*, Pub. RIMS **11(3)**, 809–833 (1976).

[BW75] J. J. Bisognano and E. H. Wichmann, *On the duality condition for a Hermitian scalar field*, J. Math. Phys. **16(4)**, 985–1007 (1975).

[BCM23] H. Bostelmann and D. Cadamuro and C. Minz, *On the Mass Dependence of the Modular Operator for a Double Cone*, Ann. H. Poincaré **24(9)**, 3031–3054 (2023).

[EO73] J.-P. Eckmann and K. Osterwalder, *An application of Tomita's theory of modular Hilbert algebras: Duality for free Bose fields*, J. Funct. Anal. **13(1)**, 1–12 (1973).

[FG89] F. Figliolini and D. Guido, *The Tomita operator for the free scalar field*, Annales de l'Institut Henri Poincaré Physique théorique **51(4)**, 419–435 (1989).

[HHW67] R. Haag and N. M. Hugenholtz and M. Winnink, *On the equilibrium states in quantum statistical mechanics*, Commun. Math. Phys. **5(3)**, 215–236 (1967).

[HL82] P. D. Hislop and R. Longo, *Modular structure of the local algebras associated with the free massless scalar field theory*, Commun. Math. Phys. **84(1)**, 71–85 (1982).

[Tak70] M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Lecture Notes in Mathematics **1**(1970).


## Modular Generators Implementing Conformal Flows in 1 + 1 de Sitter Space


Christian Jäkel

(joint work with Urs Achim Wiedemann)


We construct *modular Hamiltonians*, which satisfy the Virasoro algebra relations. They give rise to one-parameter groups of unitary operators in the Fock representation for the free massless field, which implement the geometric flows associated to the conformal Killing vector fields on the 1+1-dimensional de Sitter space **dS**. Previous results by Longo and Kawahigashi [KL05] and Longo, Martinetti, and Rehren [LMR10] on chiral quantum fields suggest that the modular Hamiltonians on de Sitter space should be given by *Connes' spatial derivatives* for pairs of *product states*, build up from *rescaled vacuum states*. We show that this is indeed the case, and that we can provide explicit expressions for Connes' spatial derivatives in terms of *higher ladder operators* on Fock space.

To be more specific, we first compute the *conformal Killing vector fields* on the two-dimensional de Sitter space $\mathbf{dS} \doteq \big\{x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = -r^2\big\}$. We show that these vector fields can be analytically continued to conformal Killing vector fields defined on $\mathbb{S}^2 \setminus \{(ir,0,0),(-ir,0,0)\}$, where $\mathbb{S}^2 \doteq \big\{(ix_0,x_1,x_2) \in (i\mathbb{R}) \times \mathbb{R}^2 \mid x_0^2 + x_1^2 + x_2^2 = r^2\big\} \subset \mathbf{dS}_{\mathbb{C}}$ is the Euclidean sphere embedded in the complexified de Sitter space. Only six vector fields allow analytic continuations to all of $\mathbb{S}^2$, as this is the maximum number of conformal Killing vector fields allowed on $\mathbb{S}^2$. The other (analytically continued) vectors fields diverge at the poles $(\pm i,0,0)$ of $\mathbb{S}^2$. We solve the flows equations for the conformal Killing vector fields and the analytically continued conformal Killing vector fields. Inspecting Killing vector fields for higher $k > 2$, we find that also there, analytic continuation of the flow to $t \to i\pi/k$ yields a discrete space-time transformation that amounts to mirroring any point $a \in \mathbf{dS}$ at the source or sink of the corresponding vector field that lies the closest to $a$.

Next, we provide [BJM23] a new realisation of the representations $D_1^\pm$ first studied by Bargmann in his classification of the unitary irreducible representations of $SO_0(1,2)$. While Bargmann used functions supported on the forward light cone in $\mathbb{R}^{1+2}$, our representation space consists of functions supported on the Cauchy surface $\mathcal{C} = \{x \in \mathbf{dS} \mid x_0 = 0\}$, sometimes called the *time-zero circle*. The scalar product[1]

$$(1) \qquad \langle h_1, h_2\rangle = -\tfrac{1}{2}\int_{\mathcal{C}} \mathrm{d}\psi\, \overline{h_1(\psi)} \int_{\mathcal{C}} \mathrm{d}\psi'\, \ln\big(2 - 2\cos(\psi-\psi')\big) h_2(\psi)\ .$$

and the generators of the rotations and the two Lorentz boosts, denoted by $k_0, l_1$ and $l_2$, have a particular simple form in our formulation: the corresponding unitary groups are

$$D_1^\pm(R_0(\alpha)) = \mathrm{e}^{i\alpha k_0^\pm}\ , \qquad k_0^\pm = -i\tfrac{\mathrm{d}}{\mathrm{d}\psi}\ .$$

and

$$D_1^\pm(\Lambda_1(t)) = \mathrm{e}^{it\nu r\cos_\psi}\ , \quad (\cos_\psi h)(\psi) \doteq \cos\psi \cdot h(\psi)\ , \quad h \in \mathfrak{h}^\pm\ .$$

The unitary group of the second boost is $D_1^\pm(\Lambda_2(t)) = \mathrm{e}^{it\nu r\sin_\psi}$.

In the sequel, we associate to any (generalized) function $h$ in the one-particle Hilbert space $\mathfrak{h}$, a distribution $\widehat{h}$ on the de Sitter space, with support on the Cauchy surface $\mathcal{C}$. With the help of the fundamental solution $E$, we construct solutions $\mathbf{\Phi}_{\widehat{h}} \doteq E * \widehat{h}$ of the wave equation with *Cauchy data*

$$\phi_{\widehat{h}} = -(\nu r)^{-1}\Im h \quad \text{and} \quad \pi_{\widehat{h}} = \Re h\ , \qquad h \in \mathfrak{h}\ .$$

We verify that the generators $k_i$ and $l_i$, $i = 0,1,2$, implement the geometric flows $\Phi_t^{\mathfrak{X}_i}$ and $\Phi_t^{\mathfrak{Y}_i}$ associated to the (conformal) Killing vector fields $\mathfrak{X}_i$ and $\mathfrak{Y}_i$, $i =$

[1] The logarithmic singularity of the kernel is integrable and hence ultra-violet divergencies are mild, while infra-red divergencies are absent due to the fact that $\mathcal{C}$ is compact.

0, 1, 2. For instance,

$$\Re\, \mathrm{e}^{itl_i} h = \big(\tfrac{\partial}{\partial\eta} E * (\widehat{h} \circ \Phi_t^{\mathfrak{X}_i})\big)(0, \,.\,)\ ,$$
$$-(\nu r)^{-1} \Im\, \mathrm{e}^{itl_i} h = -\big(E * (\widehat{h} \circ \Phi_t^{\mathfrak{X}_i})\big)(0, \,.\,)\ , \qquad i = 1, 2\ .$$

This geometric picture allows us to look out for Cauchy data supported in an interval $I \subset \mathcal{C}$. The lack of a zero mode requires care when defining the standard subspaces $\mathfrak{h}_I$ associated to an interval $I$ on the Cauchy surface: the Cauchy data $(f', g) \in \mathfrak{h}_I$ gives rise to a solution of the wave equation which has support in the space-time points that can be connected with light rays to points in the closed interval $\overline{I}$, *once an overall constant solution is subtracted*. As expected, $\mathfrak{h}_I = \mathfrak{h}_I^+ \oplus \mathfrak{h}_I^-$ can be decomposed into right-movers and left-movers without destroying the localisation properties, as left and right movers are uncorrelated. The map $I \mapsto \mathfrak{h}_I$ has a number of desirable properties: isotony, preservation of intersections, additivity, locality, Haag duality, anti-locality and the one-particle Reeh-Schlieder property all hold. Rescaling is a delicate operation on the one-particle Hilbert space $\mathfrak{h}$ due to the fact that the Cauchy surface is compact. The *higher conformal flows* defined maps on the set of solutions of the wave function, and therefore also on the Cauchy data. The positive and negative frequencies remain separated.

The next step is to build up the Tomita operator associated to a wedge $W \subset \mathbf{dS}$ from the boost leaving the wedge invariant, and the reflection at the edge of the wedge. Since both building blocks arise from group theory, the Tomita operator associated to a wedge is given intrinsically by the representation theory of the space-time symmetry group. We show that modular localisation, invented by to Brunetti, Guido and Longo [BGL02], assigns $\mathbb{R}$-linear subspaces $\mathfrak{h}(\mathcal{O})$ of $\mathfrak{h}$ to causally complete space time regions $\mathcal{O} \subset \mathbf{dS}$, in agreement with our physical expectations, as it yields the localisation of Cauchy data for solutions of the wave equation because

$$\mathfrak{h}(\mathcal{O}_I) := \bigcap_{W \subset \mathcal{O}_I} \mathfrak{h}(W) = \mathfrak{h}_I$$

for all double cones $\mathcal{O}_I$ with base $I \subset \mathcal{C}$.

The generators $l_0, l_1, l_2$ and $k_1, k_2$ are modular Hamiltonians, *i.e.*, they are the generators of modular groups associated to $\mathbb{R}$-linear subspaces $\mathfrak{h}(X)$ of $\mathfrak{h}$. The localisation region $X \subset \mathbf{dS}$ turns out to be the region where the associated Killing vector field (select one among $\mathfrak{Y}_0, \mathfrak{X}_1, \mathfrak{X}_2$ and $\mathfrak{Y}_1, \mathfrak{Y}_2$) is time-like and future directed. As the vector field $\mathfrak{X}_0$ is nowhere time-like, the angular momentum operator $k_0$ is not a modular Hamiltonian. Beside the five modular Hamiltonians already mentioned, there exits many more such Hamiltonians which arise by applying $SO_0(2,2)$ transformations to the space-time regions already considered. A particular interesting case arises by shrinking a wedge to a double cone.

In the sequel, we introduce Fock space and we define smeared-out field operators

$$\varphi(h) \doteq \tfrac{1}{\sqrt{2}}\big(a^*(h) + a(h)\big)\ , \qquad h \in \mathfrak{h}\ ,$$

satisfying the canonical commutation relations $\big[\varphi(f), \varphi(g)\big] = i\Im\langle f, g\rangle_{\mathfrak{h}}$, $f, g \in \mathfrak{h}$. The *canonical momenta* are defined by setting $\pi(g) \doteq \varphi(i\nu r g)$, $\nu r g \in \mathfrak{h}$, $g$ real valued. Next, we provide unitary operators on Fock space which implement the higher conformal flows. The generators

$$X_1^{(n)} \doteq \tfrac{1}{2}\big(K_n^+ + K_{-n}^+ + K_n^- + K_{-n}^-\big) ,$$
$$Y_1^{(n)} \doteq \tfrac{1}{2}\big(K_n^+ + K_{-n}^+ - K_n^- - K_{-n}^-\big) ,$$

can be expressed in terms of *higher ladder operators* $K_n^+$, $K_{-n}^+$, $K_n^-$ and $K_{-n}^-$ satisfying the *Virasoro algebra* relations

$$\big[K_n^{\pm}, K_m^{\pm}\big] = (n-m)K_{n+m}^{\pm} + \tfrac{1}{12}\, n(n^2-1)\delta_{n,-m}\, 1 ,$$

while $[K_n^{\pm}, K_m^{\mp}] = 0$ for all $n, m \in \mathbb{Z}$. The generators satisfy

$$i\big[X_1^{(n)}\, , \nabla\varphi(f) + \pi(g)\big] = \pi\big(\cos_{n\psi} \tfrac{\mathrm{d}}{\mathrm{d}\psi} f\big) + \nabla\varphi\big(\cos_{n\psi} \tfrac{\mathrm{d}}{\mathrm{d}\psi} g\big) ,$$
$$i\big[Y_1^{(n)}\, , \nabla\varphi(f) + \pi(g)\big] = \nabla\varphi\big(\cos_{n\psi} \tfrac{\mathrm{d}}{\mathrm{d}\psi} f\big) + \pi\big(\cos_{n\psi} \tfrac{\mathrm{d}}{\mathrm{d}\psi} g\big) .$$

Similar formulas hold for $X_2^{(n)}$ and $Y_2^{(n)}$, $n \in \mathbb{N}$. In the the sequel we show that these operators are the generators of modular groups for product states consisting of rescaled vacuum state.

## References

[BGL02] Romeo Brunetti, Daniele Guido, and R Longo. Modular localization and Wigner particles. *Reviews in Mathematical Physics*, 14(07n08):759 – 785, 00 2002.

[BJM23] João C. A. Barata, Christian Jäkel, and Jens Mund. The $\mathcal{P}(\varphi)_2$ Model on de Sitter Space. *Mem. Amer. Math. Soc.*, 281, 2023.

[LMR10] Roberto Longo, Pierre Martinetti, and Karl-Henning Rehren. Geometric modular action for disjoint intervals and boundary conformal field theory. *Reviews in Mathematical Physics*, 22(03):331–354, 2010.

[KL05] Yasuyuki Kawahigashi and Roberto Longo. Noncommutative spectral invariants and black hole entropy. *Communications in Mathematical Physics*, 257(1):193–225, 2005.

# Standard Subspaces and Distribution Vectors

Gestur Ólafsson
(joint work with J. Frahm, V. Morinelli, K.-H. Neeb and I. Sitiraju)

Construction of fields of standard subspaces using antiunitary representations and the geometry of causal symmetric spaces has recently been studied in a series of articles including [BN23, FNÓ23, MN21, MNÓ23a, MNÓ23b, NØÓ21, NÓ23a, NÓ23b, ÓS23]. We give here a short overview of the main ideas.

In the following $G$ will always denote a connected semisimple Lie group with finite center, $\tau : G \to G$ a nontrivial involution and $(G^\tau)_e \subseteq H \subset G^\tau$, where $G^\tau = \{x \in G \mid \tau(x) = x\}$. We denote by $G_{\tau_h}$ the semidirect product of $G$ and $\{1, \tau_h\}$. The derivative of $\tau$ induces a Lie algebra homomorphism $\tau_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ which leads to the $\pm1$-eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{h}$ is the Lie algebra of $H$ and $\mathfrak{q}$ is isomorphic as a $H$-space to the tangent space of $M$ at $x_0 = eH$. We assume that the symmetric space $M = G/H$ is noncompactly causal, i.e., there exists an $H$-invariant closed pointed and generating cone $C \subset \mathfrak{q}$ such that the elements of $C^o$ are hyperbolic.

We always choose a Cartan involution that commutes with $\tau$ and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition. Let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Denote by $\Delta$ the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$ and let $\Omega_{\mathfrak{a}} = \{x \in \mathfrak{a} \mid (\forall \alpha \in \Delta)\, |\alpha(x)| < \pi/2\}$. We then let $A_{i\Omega} = \exp i\Omega$ and define $\Xi_{G/K} = \Xi = GA_{i\Omega_{\mathfrak{a}}}x_0 \subset G_{\mathbb{C}}/K_{\mathbb{C}}$, the crown of the Riemannian symmetric space $G/K$. We also define $\Xi_G = GA_{i\Omega}K_{\mathbb{C}} = q_{G/K}^{-1}(\Xi_G)$, the crown domain in $G_{\mathbb{C}}$. Here $q_{G/K} : G \to G/K$ denotes the canonical projection $x \mapsto xK$. $\Xi$ is open in $G_{\mathbb{C}}/K_{\mathbb{C}}$ and $\Xi_G$ is open in $G_{\mathbb{C}}$. Both sets are by definition $G$-invariant. Denote by $\partial_d \Xi$ the distinguished boundary of $\Xi$ (see [GK02]).

**Theorem 1** (Gindikin-Krötz, [GK02]). *Assume that $G \subset G_{\mathbb{C}}$, $G_{\mathbb{C}}$ simply connected. Let $M = Gz_0K_{\mathbb{C}}$ be an open $G$-orbit in $\partial_d\Xi$. If $M$ is a symmetric space then $M$ is ncc and $z_0^{-1}H_{\mathbb{C}}z_0 = K_{\mathbb{C}}$, $H = G^{z_0}$. Furthermore, up to covering, every ncc space $G/H$ can be realized in this way.*

**Example 2** (The de Sitter space). For $v = (v_0, \mathbf{v}), w = (w_0, \mathbf{w}) \in \mathbb{R}^{1+d}$ let $[v, w] = v_0w_0 - \langle \mathbf{v}, \mathbf{w}\rangle$. Let $G = \mathrm{SO}_{1,d}(\mathbb{R}) \supset H = \mathrm{SO}_{1,d-1}(\mathbb{R})$. We have $G(ie_0) = G/K = \mathbb{H}^d = \{iv \in i\mathbb{R}^{d+1} \mid [v, v] = 1\}$ and $Ge_1 = G/H = \mathrm{dS}^d$. Let $h : (x_0, x_1, \tilde{x}) \mapsto (x_1, x_0, 0, \cdots, 0)$. Then $h$ is an Euler element and $\mathfrak{a} = \mathbb{R}h$. Furthermore, $\Xi_{\mathbb{H}^d} = (\mathbb{R}^{d+1} + iV_+) \cap \mathrm{dS}^d$ where $V_+$ is the forward light cone. We have $\exp(-ith)(ie_0) = i(\cos t)e_0 + (\sin t)e_1 \to e_1$ as $t \to \pi/2$ and $\mathrm{dS}^d$ is the only open $G$-orbit in the distinguished boundary of $\Xi$.

Let $(U, \mathcal{H})$ be an anti-unitary representation of $G_{\tau_h}$. Recall that a real subspace $\mathtt{V} \subset \mathcal{H}$ is *standard* if $\mathtt{V}$ is closed, $\mathtt{V} \cap i\mathtt{V} = \{0\}$ and $\mathtt{V} + i\mathtt{V} \subset \mathcal{H}$ is dense. Denote by $\mathcal{H}^\infty$ the space of smooth vectors and by $\mathcal{H}^{-\infty}$ the conjugate linear dual of $\mathcal{H}$. For a finite dimensional $H$-invariant subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ and $\mathcal{O} \subseteq M = G/H$ open define

$$\mathtt{H}_{\mathtt{E}}(\mathcal{O}) := \overline{\operatorname{span}_{\mathbb{R}}\{U^{-\infty}(\phi)\mathtt{E} \mid \phi \in C_c^\infty(q_M^{-1}(\mathcal{O}), \mathbb{R})\}} \subset \mathcal{H}. \tag{1}$$

$\mathtt{H}_{\mathtt{E}}$ defines a net of real subspaces which is clearly isotone and covariant, but locality, Reeh-Schlieder and Bisonanao-Wichmann do not hold in general.

Let $h$ be an Euler element such that $z_0 = \exp(\frac{\pi i}{2}h)$, $z_0$ as in Theorem 1. Denote by $\alpha_t(m) = \exp(th)m$, $m \in M$, the modular flow and $X_h^M(m) = d/dt|_{t=0}\,\alpha_t(m)$. The positivity domain is $W_M^+(h) = \{m \in M \mid X_h^M(m) \in V_+(m)\}$ (see [MNÓ23a, NÓ23a, NÓ23b]).

Let $(U, \mathcal{H})$ be a irreducible antiunitary representation of $G_{\tau_h}$. Denote by $\mathcal{H}^{-\infty,8[\mathfrak{h}]}$ the space of $\mathfrak{h}$-finite distribution vectors and by $\mathcal{H}^{[K]}$ the space of $K$-finite vectors. If $G \subset G_{\mathbb{C}}$ is linear then for $\xi \in \mathcal{H}^{[K]}$ the orbit map $g \mapsto U_g\xi$ extends to a holomorphic function on $\Xi_G$ ([KS04]). Define

$$\beta^\pm(\xi) = \lim_{t \to \pm\pi/2} U(\exp -ith)\xi \in \mathcal{H}^{[\mathfrak{h}]}$$

whenever the limit exists.

**Theorem 3** (FNÓ23). *Assume that $G$ is linear or locally isomorphic to $\mathrm{SO}_{1,2}(\mathbb{R})$. Assume that $M = G/H$ is ncc. Let $(U, \mathcal{H})$ be an irreducible antunitary representation of $G_{\tau_h}$, $\mathcal{E} \subset \mathcal{H}^{[K]}$ finite dimensional subspace invariant under $K$ and $J = U_{\tau_h}$. Let $\mathtt{E} = \mathcal{E}^J$. Then the following holds:*
*1) $\beta^\pm : \mathcal{H}^{[K]} \to \mathcal{H}^{-\infty,[\mathfrak{h}]}$ exists and both maps are injective.*
*2) Let $\mathtt{E}_H = \beta^+(\mathtt{E}) \subseteq \mathcal{H}^{-\infty}$. Then the net $\mathtt{H}^M_{\mathtt{E}_H}$ defined in (1) on $M$ is isotone, covariant and has the Reeh-Schlieder and Bisognano-Wichmann property, where $W = W_M^+(h)_{eH}$ is the connected component of the positivity domain of $h$ on $M$, containing the base point.*

The holomorphic extension of the orbit map $g \mapsto U_g\xi$ leads to a sesquiholomorphic $G$-invariant positive definite kernel on $\Xi \times \Xi$, $\Phi_\xi^U(z, w) = \Phi_w(z) = \langle U_w\xi, U_z\xi\rangle$ (the inner product is linear in the second factor). The GNS construction then leads to a realization of $(U, \mathcal{H})$ in spaces of holomorphic functions on $\Xi_G$. If $\xi \in \mathcal{H}^K$ then $\Phi^U_\xi$ lives on $\Xi_{G/K}$ and $\phi^U_\xi = \Phi_{eK}$ is a spherical function if $\|\xi\| = 1$. Keeping one of the variable, say $z$, in $\Xi$ we have $y \to \Phi(z, y)$ extends to an analytic function on $M$ and hence a well defined distribution $M$. The question is then again if $\lim_{z\to eH} \Phi(z, \cdot)$ exists in $\mathcal{D}'(G/H)$. For this we discuss as an example the case of $\mathrm{dS}^d$. Let $U = U_\lambda$ be a unitary spherical representation of $\mathrm{SO}_{1,d}(\mathbb{R})_e$ with spectral parameter $\lambda$. In this case we have, with $\rho = (n-1)/2$:

$$\Psi(z, w) = {}_2F_1\left(\rho + \lambda, \rho - \lambda; \frac{n}{2}; \frac{1 - [z, \bar{w}]}{2}\right), \quad \lambda \in i\mathbb{R}_+ \cup (0, \rho].$$

The hypergeometric function ${}_2F_1(z)$ is holomorphic on $\mathbb{C} \setminus [1, \infty)$. For $x \in \mathrm{dS}^d$ let $\Gamma(x) = \{z \in \mathrm{dS}^d \mid [z - x, z - x] > 0\} = \Gamma^+(x) \cup \Gamma^-(x)$ where the $\pm$ indicate $\pm(z - x)_0 > 0$. We have

**Lemma 4.** 1) $\{[z, \bar{w}] \mid z, w \in \Xi\} = \mathbb{C} \setminus (-\infty, -1]$.
2) $[\mathrm{dS}^d, \Xi] \cap \mathbb{R} = (-1, 1)$.
3) *If $x \in \mathrm{dS}^d$ then $[x, y] \leq -1$ if and only if $x + y \in \Gamma(x)$.*

(1) Shows that $\Psi(z, w)$ is well defined on the crown. (2) shows that $\Psi(z, y)$ is well defined for $z \in \Xi$ and $y \in \mathrm{dS}^d$. Finally, it was shown in [FNÓ23, ÓS23] that the distributional limit exists and defines a $H$-invariant distribution on $\mathrm{dS}^d$.

There is also the *conjugate* crown, the crown of $G(-ie_0)$, so we have two natural crowns $\Xi^\pm$ and two kernels $\Psi^\pm$ and hence two distributions $\eta^\pm_\lambda$. The explicit formulas above show that $\eta^+ - \eta^-$ is supported on the closed forward and backward light cone (see [ÓS23, Cor. 6.9])

Similar calculations work for all $K$-types for principal series representations of $\widetilde{\mathrm{S}L}_2(\mathbb{R})$ expressed in terms of hypergeometric functions.

## References


[BN23] Beltiţă, D., and K.-H. Neeb, *Holomorphic extension of one-parameter operator groups*, to appear in Pure and Applied Funct. Anal.; arXiv:2304.09597

[FNÓ23] Frahm, J., K-H. Neeb, G. Ólafsson: Nets of standard subspaces on non-compactly causal symmetric spaces, to appear in Progress in Math., arXiv:2303.10065

[GK02] S. Gindikin and B. Krötz, *Complex crowns of Riemannian symmetric spaces and non-compactly causal symmetric spaces* Trans. Amer. Math. Soc. 354 (2002), 3299–3327.

[MN21] Morinelli, V., K.-H. Neeb, *Covariant homogeneous nets of standard subspaces*, Comm. Math. Phys. **386** (2021), 305–358

[MNÓ23a] Morinelli, V., K.-H. Neeb, and G. Ólafsson, *From Euler elements and 3-gradings to non-compactly causal symmetric spaces* J. Lie theory **33** (2023), 377–432

[MNÓ23b] Morinelli, V., K.-H. Neeb, and G. Ólafsson, *Modular geodesics and wedge domains in general non-compactly causal symmetric spaces.* To appear in: Annals of Global Analysis and Geometry

[NØÓ21] Neeb, K.-H., B. Ørsted, and G. Ólafsson, *Standard subspaces of Hilbert spaces of holomorphic functions on tube domains*, Comm. Math. Phys. **386** (2021), 1437–1487

[NÓ21] Neeb, K.-H., and G. Ólafsson, *Nets of standard subspaces on Lie groups*, Adv. Math. **384** (2021), 69 pp

[NÓ23a] Neeb, K.-H., and G. Ólafsson, *Wedge domains in non-compactly causal symmetric spaces*, Geometriae Dedicata **217:2** (2023), Paper No. 30

[NÓ23b] Neeb, K.-H., and G. Ólafsson, *Wedge domains in compactly causal symmetric spaces*, Int. Math. Res. Notices **2023:12** (2023), 10209–10312

[ÓS23] G. Ólafsson and I. Sitiraju: Analytic Wavefront Sets of Spherical Distributions on de Sitter Space. arXiv2309.10685

[KS04] Krötz, B and R. J. Stanton, *Holomorphic extensions of representations. I. Automorphic functions*, Annals of Mathematics, **159** (2004), 641–724

## Arrow of time and quantum physics

Detlev Buchholz
(joint work with Klaus Fredenhagen)

The arrow of time is a subject of ongoing debate ever since this term was coined by Eddington almost a hundred years ago. In brief, this topic can be described as follows: the time parameter that enters into the fundamental equations of physics can be reversed, which in principle seems to allow physical systems to move backwards in time. On the other hand, there is overwhelming evidence that this does not happen. The standard resolution of this apparent clash between theory and reality is based on the argument that such time reversed processes are exceedingly unlikely (Second Law). Therefore, they were and will never be observed.

In this contribution a complementary view is presented [BF23]. It is based on the hypothesis that time translations form a semi-group acting on all systems, there is no inverse and hence no return to the past. Information about the past is encoded in material bodies which accompany us, such as books or other devices and media, not least our brains. We can extract from them information about past events, observations, experiments, data taken, and theories developped on their basis. The informations obtained in this way can be described in common language, including mathematics. In order to check their truth value, one has to repeat past experiments. But this can only happen in the future. Thus the past may be regarded as factual, whereas the future is indeterminate.

While this hypothesis seems consistent with reality, it raises some questions concerning the *theoretical* treatment of time. These questions and the proposed answers are outlined in this abstract.

**1**. Is the hypothesis of an intrinsic arrow of time compatible with the successful theoretical treatment of time as a group? In order to answer this question one considers a unital C*-algebra $\mathfrak{A}(\mathcal{V}_o)$, describing local observables in a given future-directed lightcone $\mathcal{V}_o$ in Minkowski space $\mathcal{M}$. On this algebra acts the abelian semi-group of time translations $V_+ := \{\tau = t(1, \boldsymbol{v}) : t \geq 0\,,\ |\boldsymbol{v}| < 1\}$ by endomorphisms $\alpha$, *viz.* $\alpha_\tau(\mathfrak{A}(\mathcal{V}_o)) = \mathfrak{A}(\mathcal{V}_o + \tau)$. With this input one can identify vacuum states in the region $\mathcal{V}_o$. The following result can then be established.

**Proposition 1.** *In the GNS-representation induced by a vacuum state*

(i) *there exists a continuous unitary representation $U_0$ of the semi-group $V_+$ whose adjoint action implements the time translations on the observables.*

(ii) *there exists an extension of $U_0$ to a continuous unitary representation $U$ of the group of spacetime translations $\mathbb{R}^d$ on $\mathcal{M}$. It satisfies the relativistic spectrum condition. Its adjoint action on the represented observables in $\mathcal{V}_o$ defines a net of observables in all of Minkowski space $\mathcal{M}$.*

This proposition shows that the hypothesis of a fundamental arrow of time is compatible with the familiar theoretical assumption according to which the group of spacetime translations $\mathbb{R}^d$ acts on the observables. In this way a theoretical picture of the past is obtained that is consistent with the theory for future observations.

**2**. Are there uncertainties in the theoretical description of the past and how do they manifest themselves? It turns out that the unitary representation $U_0$ of the semigroup $V_+$ to a representation $U$ of the group $\mathbb{R}^d$ is in general not unique. The pertinent information is encoded in the largest projection $Z$ in the weak closure of the algebra of observables in $\mathcal{V}_o$ that annihilates the vacuum state. It commutes with all translations $U$. Whenever $Z < (1 - P_0)$, where $P_0$ is the projection onto the vacuum state, the extension $U$ is not unique. As a consequence, the theoretical description of the past is ambiguous. This feature becomes manifest in the structure of the energy-momentum spectrum.

**Proposition 2.** *If the past is ambiguous, i.e. the extensions $U$ of the time translation $U_0$ are not unique, their spectrum consists of the closed cone $\overline{V}_+$ in energy-momentum space. So there exist excitations of arbitrarily small mass. Conversely, the existence of massless excitations implies that the extensions $U$ are not unique.*

Since there are massless excitations in reality, the photons, complete information about the past (the wave function of the universe) is a theoretical fiction.

**3**. How big is the loss of information on the properties of states over time that arises from these ambiguities? An answer is given by noticing that the uncertainties concerning the past are due to states $\Phi$ in the kernel of the projection $Z$, involving massless excitations. The corresponding information in a lightcone $\mathcal{V}_o + \tau$ can be quantified by a convenient measure of information $I_\tau(\Phi)$, which is related to the concept of relative entropy introduced by H. Araki. It was invented by R. Longo, who used the theory of standard subspaces for its definition. Making use of this notion, the following result obtains.

**Proposition 3.** *Let $\Phi$ be a state in the kernel of $Z$. Then*

(i) $I_\tau(\Phi) \in [0, \infty]$ *and there is a dense set of states $\Phi$ for which this information is finite.*

(ii) $I_\tau(\Phi) \leq I_\sigma(\Phi)$ *if* $\tau - \sigma \in V_+$.

*The information contained in the stationary vacuum state is equal to* 0.

Thus the information contained in the states decreases in the course of time.

**4**. Does the arrow of time enforce the quantum features of operations that are described by common language (classical terms), extracted from past information? The preceding results suggest that a statistical description of future experiments is unavoidable in view of the lacking information about the past. It turns out that the arrow of time adds to it the non-commutative features of quantum physics.

We outline here the simple case of a classical, non-interacting, smooth field $x \mapsto \phi(x)$ in Minkowski space $\mathcal{M}$ with relativistic Lagrangian density

$$x \mapsto L(x)[\phi] = (1/2)(\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi(x)^2) \,.$$

One considers operations affecting the field which are induced by perturbations of the Lagrangian. In the present simple case these are given by

$$L(x)[\phi] \to L(x)[\phi] + c(x) + f(x)\phi(x) \,,$$

where $c, f$ are real test functions with compact support. Their spacetime integrals describe functionals $\phi \mapsto F[\phi]$ on the field. The support of the functionals $F$ in $\mathcal{M}$ is identified with the support of $f$. Constant functionals have empty support.

The effect of these pertubations on $\phi$, encoded in the functionals $F$, is described by symbols $S_L(F)$. They define a dynamical group $\mathcal{G}_L$. It is the free group generated by these symbols, modulo the following defining relations:

(1a) $S_L(F)S_L(G) = S_L(F+G)$ if $\operatorname{supp} F$ lies above and $\operatorname{supp} G$ lies beneath some Cauchy surface. Here the arrow of time enters, the swapped product is not fixed and depends on the dynamics. Only if the supports of $F$ and $G$ are spacelike separated, the product is commutative according to this definition.

(1b) $S_L(c) = e^{ic} 1 \,, \quad c \in \mathbb{R}$.

(2) $S_L(F) = S_L(F^{\phi_0} + \delta L(\phi_0))$, where $F^{\phi_0}[\phi] := F[\phi + \phi_0]$ for given external smooth field $\phi_0$ with compact support; $\delta L(\phi_0)$ is the corresponding variation of the action, determined by $L$. In this relation the Lagrangian and hence the dynamics enters. If $F = 0$, the underlying field is unaffected.

Proceeding from the group $\mathcal{G}_L$ to the enveloping dynamical C*-algebra $\mathcal{A}_L$, one arrives at the following result.

**Proposition 4.** *Let $L$ be the non-interacting Lagrangian, given above. The algebra $\mathcal{A}_L$ coincides with the Weyl algebra of a local quantum field, satisfying the Klein-Gordon equation. It is generated by the exponentials of the field, integrated with real test functions $f$.*

For Lagrangians $L$ describing interacting fields, the corresponding algebras $\mathcal{A}_L$ comply with all Haag-Kastler axioms of local quantum physics. These results obtain without imposing any quantization rules. It is the arrow of time which, together with the dynamics, leads to the quantization of the classical input. In view of these results, it seems worthwhile to take a fresh look at the foundations of quantum physics, based on this new paradigm.

## References

[BF23] D. Buchholz and F. Fredenhagen, Arrow of Time and Quantum Physics, Foundations of Physics (2023) 53:85. e-print arXiv:2305.11709

## Nets of standard subspaces on homogeneous spaces

Karl-Hermann Neeb
(joint work with Vincenzo Morinelli)

We discuss some recent results concerning nets of real subspaces indexed by open subsets $\mathcal{O}$ of a homogeneous space $M = G/H$ of a Lie group $G$. We assume that $M$ carries a $G$-invariant causal structure, i.e., a field of pointed generating closed convex cones $C_m \subseteq T_m(M)$ that is invariant under the $G$-action. Typical examples are time-oriented Lorentzian manifolds on which $G$ acts by time-orientation preserving symmetries or conformal maps. Natural properties of such nets are closely related to those of nets of von Neumann algebras.

For a unitary representation $(U, \mathcal{H})$ of a connected a Lie group $G$ and a homogeneous space $M = G/H$, we consider families $(\mathsf{H}(\mathcal{O}))_{\mathcal{O}\subseteq M}$ of closed real subspaces of $\mathcal{H}$, indexed by open subsets $\mathcal{O} \subseteq M$ with the following properties:

- (Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$
- (Cov) **Covariance:** $U(g)\mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.
- (RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.
- (BW) **Bisognano–Wichmann property:** There exists an open subset $W \subseteq M$ (called a *wedge region*) and $h \in \mathfrak{g}$, such that $\exp(\mathbb{R}h)W \subseteq W$ and $\mathsf{H}(W)$ is standard with modular group $\Delta_{\mathsf{H}(W)}^{-it/2\pi} = U(\exp th)$, $t \in \mathbb{R}$.

Given a unitary representation $(U, \mathcal{H})$, we would like to understand, and possibly classify, all nets with these properties. To this end, the first problem is to understand which Lie algebra elements $h$ and which regions $W \subseteq M$ occur in (BW). The following theorem shows that we may restrict our attention to the case where $h$ is an *Euler element*, i.e., $\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h)$ for $\mathfrak{g}_\lambda(h) = \ker(\lambda\mathbf{1} - \mathrm{ad}h)$.

**Theorem 1.** (Euler Element Theorem, [MN23]) *Let $G$ be a connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$. Let $(U, \mathcal{H})$ be a unitary representation of $G$ with discrete kernel. Suppose that $\mathtt{V}$ is a standard subspace and $N \subseteq G$ an identity neighborhood such that*

- (a) $U(\exp(th)) = \Delta_{\mathtt{V}}^{-it/2\pi}$ *for $t \in \mathbb{R}$, i.e., $\Delta_{\mathtt{V}} = e^{2\pi i\, \partial U(h)}$, and*
- (b) $\mathtt{V}_N := \bigcap_{g\in N} U(g)\mathtt{V}$ *is cyclic, i.e., $\mathtt{V}_N + i\mathtt{V}_N$ is dense in $\mathcal{H}$.*

*Then $h$ is an Euler element and the conjugation $J_{\mathtt{V}}$ satisfies*

$$J_{\mathtt{V}} U(\exp x) J_{\mathtt{V}} = U(\exp \tau_h(x)) \quad \textit{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}, x \in \mathfrak{g}. \tag{1}$$

If (Iso), (Cov), (RS) and (BW) are satisfied, then the preceding theorem applies with $\mathtt{V} = \mathsf{H}(W)$. For any relatively compact open subset $\mathcal{O} \subseteq W$ we find a symmetric $e$-neighborhood with $N.\mathcal{O} \subseteq W$, and then $\mathsf{H}(\mathcal{O}) \subseteq \mathtt{V}_N$ by (Iso), (Cov) and (BW). Hence (RS) implies that $\mathtt{V}_N$ is cyclic. Accordingly, we may assume in the following that $h$ is an Euler element, and by (1), that $U$ extends to a representation of the extended group $G_{\tau_h} = G \rtimes \{e, \tau_h\}$ with $U(\tau_h) = J_{\mathsf{H}(W)}$.

So we may now start with an (anti-)unitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ with discrete kernel, and consider the standard subspace $\mathtt{V} := \mathtt{V}(h, U)$, specified by

$$J_{\mathtt{V}} = U(\tau_h) \quad \text{and} \quad \Delta_{\mathtt{V}}^{-it/2\pi} = U(\exp th), t \in \mathbb{R}.$$

For any homogeneous space $M = G/H$ and an open $\exp(\mathbb{R}h)$-invariant subset $W \subseteq M$, we may then consider the nets

$$\mathsf{H}^{\max}(\mathcal{O}) := \bigcap_{g \in G, \mathcal{O} \subseteq gW} U(g)\mathtt{V} \quad \text{and} \quad \mathsf{H}^{\min}(\mathcal{O}) := \overline{\sum_{g \in G, gW \subseteq \mathcal{O}} U(g)\mathtt{V}}. \tag{2}$$

Both nets are easily seen to be isotone and covariant. It is also rather easy to verify that they satisfy (BW) in the sense that $\mathsf{H}^{\max}(W) = \mathsf{H}^{\min}(W) = \mathtt{V}$ if and only if we have the following inclusion of subsemigroups of $G$:

$$S_W := \{g \in G \colon gW \subseteq W\} \subseteq S_{\mathtt{V}} := \{g \in G \colon g\mathtt{V} \subseteq \mathtt{V}\}. \tag{3}$$

We refer to [MN23] for details. An interesting consequence is that the existence of a net $\mathsf{H}$ on open subsets of $M$, satisfying (Iso), (Cov) and (BW), implies (3) and that

$$\mathsf{H}^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}^{\max}(\mathcal{O})$$

for all open subsets of $\mathcal{O} \subseteq M$.

As this point, the next step consists in a better understanding of condition (3). Here $W$ and the semigroup $S_W$ are the most intricate points, but the semigroup $S_{\mathtt{V}}$ has a rather explicit description ([Ne22, Thms. 2.16, 3.4]):

$$S_{\mathtt{V}} = G_{\mathtt{V}} \exp(C_+ + C_-) = \exp(C_+) G_{\mathtt{V}} \exp(C_-), \tag{4}$$

where $G_{\mathtt{V}} = \{g \in G \colon \operatorname{Ad}(g)h = h, \tau_h(g)g^{-1} \in \ker(U)\}$, and

$$C_\pm := \pm C_U \cap \mathfrak{g}_{\pm 1}(h), \quad C_U := \{x \in \mathfrak{g} : -i\partial U(x) \geq 0\}.$$

In this sense $S_{\mathtt{V}}$ can be obtained from $h$, $\ker U$, and the positive cone $C_U$ of the representation $U$.

Natural choices of wedge regions $W \subseteq M$ are the connected components of the *positivity domain*

$$W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^\circ\}$$

of the so-called *modular vector field* $X_h^M(m) := \frac{d}{dt}\big|_{t=0} \exp(th).m$. These "wedge regions" have been studied for compactly and non-compactly causal symmetric spaces in [NÓ23b] and [NÓ23a, MNÓ23], respectively. In many situations $W_M^+(h)$ is connected, and then one has good information on $S_W$ (cf. [MN23, Prop. 2.9]):

$$\mathbf{L}(S_W) := \{x \in \mathfrak{g}\colon \exp(\mathbb{R}_+ x) \subseteq S_W\} = \mathfrak{g}_0(h) + (C_W \cap \mathfrak{g}_1(h)) - (C_W \cap \mathfrak{g}_{-1}(h)),$$

where $C_W := \{y \in \mathfrak{g}\colon (\forall m \in W)\ X_y^M(m) \in C_m\}$ contains the invariant cone

$$C_M := \{y \in \mathfrak{g}\colon (\forall m \in M)\ X_y^M(m) \in C_m\}.$$

In all examples for which we have explicit information on these cones, we have $C_M \cap \mathfrak{g}_{\pm 1}(h) = C_W \cap \mathfrak{g}_{\pm 1}(h)$. Note that $C_M$ can be considered as the "positive cone" of the $G$-action on the causal manifold $M$, so that the semigroups $S_{\mathsf{V}}$ and $S_W$ are closely related to the cones $C_U$ and $C_M$ in a similar fashion.

**Example 2.** For Minkowski space $M = \mathbb{R}^{1,d}$, $G = \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})^\uparrow$, the Poincaré group, the Lie algebra $\mathfrak{g}$ contains only one adjoint orbit of Euler elements, represented by the boost generator $h.x = (x_1, x_0, 0, \ldots, 0)$. Then

$$W = \{x \in \mathbb{R}^{1,d}\colon x_1 > |x_0|\}$$

is the Rindler wedge, $S_W = \overline{W} \rtimes (\mathrm{SO}_{1,1}(\mathbb{R})^\uparrow \times \mathrm{SO}_{d-1}(\mathbb{R}))$, and for an antiunirary representation of $G_{\tau_h} = \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})$, the compatibility condition $S_W \subseteq S_{\mathsf{V}}$ is equivalent to the positive energy condition that

$$V_+ = \{x \in \mathbb{R}^{1,d}\colon x_0 > \mathbf{x}^2, x_0 > 0\} \subseteq C_U.$$

If the semigroup $S_W$ is a connected group, then the compatibility condition $S_W \subseteq S_{\mathsf{V}}$ imposes no essential restriction on the representation, such as positive spectrum conditions. In this context, a central result of [MN23], based on the irreducible case that is dealt with in [FNÓ23], is:

**Theorem 3.** *For every connected reductive linear Lie group $G$ and any Euler element $h \in \mathfrak{g}$, there exists a causal symmetric space $M = G/H$ such that for* **all** *conneced components $W \subseteq W_M^+(h)$ and* **all** *(anti-)unitary representations $(U, \mathcal{H})$ of $G_{\tau_h}$, the net $\mathsf{H}^{\max}$ satisfies* (Iso), (Cov), (BW) *and* (RS).

## References


[FNÓ23] J. Frahm, K.-H. Neeb, and G. Ólafsson, *Nets of standard subspaces on non-compactly causal symmetric spaces*, in "Toshiyuki Kobayashi Festschrift", Progress in Mathematics, Springer-Nature, to appear; arxiv:2303.10065

[MN21] V. Morinelli, and K.-H. Neeb, *Covariant homogeneous nets of standard subspaces*, Comm. Math. Phys. **386** (2021), 305–358; arXiv:math-ph.2010.07128

[MN23] V. Morinelli, and K.-H. Neeb, *From local nets to Euler elements*, in preparation

[MNÓ23] V. Morinelli, K.-H. Neeb, and G. Ólafsson, *Modular geodesics and wedge domains in general non-compactly causal symmetric spaces*, Annals of Global Analysis and Geometry, to appear; arXiv:2307.00798

[Ne22] Neeb, K.-H., *Semigroups in 3-graded Lie groups and endomorphisms of standard subspaces*, Kyoto Math. Journal **62:3** (2022), 577–613; arXiv:OA:1912.13367

[NÓ21] Neeb, K.-H., and G. Ólafsson, *Nets of standard subspaces on Lie groups*, Advances in Math. **384** (2021), 107715, arXiv:2006.09832

[NÓ23a] Neeb, K.-H., and G. Ólafsson, *Wedge domains in non-compactly causal symmetric spaces*, Geometriae Dedicata **217:2** (2023), Paper No. 30; arXiv:2205.07685

[NÓ23b] Neeb, K.-H., and G. Ólafsson, *Wedge domains in compactly causal symmetric spaces*, Int. Math. Res. Notices **2023:12** (2023), 10209–10312; arXiv:math-RT:2107.13288

[NÓ23c] Neeb, K.-H., and G. Ólafsson, *Algebraic Quantum Field Theory and Causal Symmetric Spaces*, Eds Kielanowski, P., et.al., "Geometric Methods in Physics XXXIX. WGMP 2022", Trends in Mathematics, 2023; Birkhäuser/Springer, 207–231

## Quasi-free isomorphisms of second quantization von Neumann algebras and modular theory

Gerardo Morsella

(joint work with Roberto Conti)

### 1. Motivations

We present the work [CM], whose main motivation is to try to understand a classical result by Eckmann and Fröhlich on the local quasi-equivalence of vacua of different masses of the Klein-Gordon field [EF74] (proven with methods form constructive QFT) in terms of modular theory. The main tool employed towards this end is the quasi-equivalence criterion of Araki and Yamagami [AY82].

### 2. Abstract result

Our setting is the following. Let $H$ be a complex Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and let $e^H = \bigoplus_{n=0}^{+\infty} H^{\otimes s n}$ be the associated symmetric Fock space, in which the coherent vectors $e^x := \bigoplus_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} x^{\otimes n}$, $x \in H$, form a total set. We also consider on $e^H$ the Weyl unitaries $W(x), x \in H$, defined by their action on $\Omega := e^0$ (vacuum vector) and by the canonical commutation relations (CCR):

$$W(x)\Omega := e^{-\frac{1}{4}\|x\|^2} e^{ix/\sqrt{2}}, \qquad x \in H,$$

$$W(x)W(y) = e^{-\frac{i}{2}\Im\langle x,y\rangle} W(x+y), \qquad x, y \in H.$$

For any standard subspace $K$ of $H$ (i.e., a real subspace such that $\overline{K + iK} = H$ and $K \cap iK = \{0\}$), the von Neumann algebra

$$A(K) = \{W(h) \mid h \in K\}''$$

on $e^H$, is called the second quantization algebra of $K$. Moreover, $K$ defines a closed, densely defined conjugate linear operator

$$s : K + iK \to K + iK, \quad s(h + ik) = h - ik, \qquad h, k \in K.$$

and if $s = j\delta^{1/2}$ is the polar decomposition, $j$ and $\delta$ are the modular conjugation and the modular operator of $K$. Their second quantizations $J = \Gamma(j)$, $\Delta = \Gamma(\delta)$ are respectively the modular conjugation and the modular operator of $A(K)$ with respect to $\Omega$ [EO73, Lon].

A *Bogolubov transformation* between standard subspaces $K_1, K_2 \subset H$ is a real linear bijection $Q : K_1 \to K_2$ preserving the symplectic form, i.e., $\Im\langle Qh, Qk\rangle = \Im\langle h, k\rangle$, $h, k \in K_1$. Given such a map, the C*-algebras generated by the Weyl operators $W(k)$ and $W(Qk)$, $k \in K_1$, are isomorphic, and it is then natural to ask under which condition this isomorphism extends to an isomorphism between the respective von Neumann algebras $\phi : A(K_1) \to A(K_2)$. If this is the case, $\phi$ is called the *quasi-free isomorphism* induced by $Q$.

The problem of the existence of the quasi-free isomorphism is equivalent to the problem of the quasi-equivalence of the states $\omega$ and $\omega_Q$ on the C*-algebra generated by $W(k)$, $k \in K_1$, defined by

$$\omega(W(k)) = e^{-\frac{1}{4}\|k\|^2}, \quad \omega_Q(W(k)) = e^{-\frac{1}{4}\|Qk\|^2}, \qquad k \in K_1.$$

The relevance of the modular structures of $K_1$, $K_2$ for this problem can be understood from the fact that they relate the symplectic structures and the real Hilbert space ones of $K_1$, $K_2$. Indeed, if

$$R_j := i\frac{\delta_j - 1}{\delta_j + 1}, \qquad j = 1, 2,$$

is the *polariser* of $K_j$, there holds $\Im\langle h, k\rangle = \Re\langle h, R_j k\rangle$, $h, k \in K_j$.

Applying then the very general quasi-equivalence criterion of [AY82], one obtains the following result, in which $Q^\dagger : K_2 \to K_1$ is the adjoint of $Q$ w.r.t. the real scalar products of $K_1$, $K_2$, and $Q^\dagger Q$ is extended to $K_1 + iK_1$ by complex linearity.

**Theorem 1.** *The Bogolubov transformation $Q : K_1 \to K_2$ induces a quasi-free isomorphism if and only if:*

(i) *$Q$ is bounded (w.r.t. the norm of $H$);*
(ii) *$(1 + iR_1)^{1/2} - (Q^\dagger Q + iR_1)^{1/2}$ is Hilbert-Schmidt on $K_1 + iK_1$, endowed with the graph scalar product of $s_1$.*

A sufficient condition for (ii) is that $1 - Q^\dagger Q$ is of trace class on $K_1$, while a necessary condition is that $1 - Q^\dagger Q$ is Hilbert-Schmidt on $K_1$. Moreover, (ii) is also equivalent to the fact that the operators

$$(*) \qquad 1 - Q^\dagger Q, \quad \frac{1}{\sqrt{1+\delta_1}} - Q^{-1}\frac{1}{\sqrt{1+\delta_2}}Q$$

are both Hilbert-Schmidt on $K_1 + iK_1$. In the case in which $\delta_1$, $\delta_2$ are bounded, using powerful results of [BS67], it is possible to show that the fact that $1 - Q^\dagger Q$ is Hilbert-Schmidt on $K_1$ is equivalent to (ii).

## 3. Applications to QFT

The one particle space of the Klein-Gordon field in $d$ spatial dimensions is $H = L^2(\mathbb{R}^d)$. On it, the operator $\omega_m := (-\Delta + m^2)^{1/2}$ is defined by functional calculus. For $d = 1$ and $I \subset \mathbb{R}$ an open interval, the space

$$K_m(I) := \left\{\omega_m^{-1/2} f + i\omega_m^{1/2} g \,:\, f, g \in C_c^\infty(I, \mathbb{R}),\, \textstyle\int_I f = 0 = \int_I g\right\}^{-}, \qquad m \geq 0,$$

is a standard subspace of $H$ for $m > 0$, and of

$$H_0 := \left\{\omega_0^{-1/2} f + i\omega_0^{1/2} g \,:\, f, g \in C_c^\infty(\mathbb{R}, \mathbb{R}),\, \textstyle\int_{\mathbb{R}} f = 0 = \int_{\mathbb{R}} g\right\}^{-}$$

for $m = 0$. The restriction to zero average functions is needed to avoid the infrared divergence of the scalar field in $d = 1$. The map

$$Q : K_m(I) \to K_0(I), \quad \omega_m^{-1/2} f + i\omega_m^{1/2} g \mapsto \omega_0^{-1/2} f + i\omega_0^{1/2} g,$$

is a Bogolubov transformation, and it can be shown that $1 - Q^\dagger Q$ is of trace class [CM20], so, by the above results, it induces a quasi-free isomorphism of the respective second quantization von Neumann algebras. Equivalently, the restriction of the massive and massless vacua to the nets generated by the derivative of the time zero field and momentum are locally quasi-equivalent.

For $d = 2, 3$ and $B \subset \mathbb{R}^d$ the unit ball, the real subspace

$$K_m(B) := \mathcal{L}_-(B) + i\mathcal{L}_+(B), \qquad \mathcal{L}_\pm(B) := \overline{\omega_m^{\pm 1/2} C_c^\infty(B, \mathbb{R})}, \qquad m \geq 0.$$

is a standard subspace of $H$ for all $m \geq 0$, and, given $m > 0$,

$$Q : K_m(B) \to K_0(B), \quad \omega_m^{-1/2} f + i\omega_m^{1/2} g \mapsto \omega_0^{-1/2} f + i\omega_0^{1/2} g,$$

is a Bogolubov transformation, for which one can compute

$$Q^\dagger Q = \left( E_- \frac{\omega_m}{\omega_0} E_- + iE_+ \frac{\omega_0}{\omega_m} E_+ i \right) \Big|_{K_m(B)},$$

with $E_\pm : H \to \mathcal{L}_\pm(B)$ the real orthogonal projections. Contrary to the $d = 1$ case, now $1 - Q^\dagger Q$ is most likely not of trace class. However, the above formula can be used to estimate the integral kernel of $E_{K_m(B)}(1 - Q^\dagger Q)E_{K_m(B)}$ (with $E_{K_m(B)}$ the real orthogonal projection onto $K_m(B)$), and obtain the following partial result towards the existence of the quasi-free isomorphism induced by $Q$.

**Theorem 2.** *The operator $1 - Q^\dagger Q$ is Hilbert-Schmidt on $K_m(B) + iK_m(B)$ (and then on $K_m(B)$), and its Hilbert-Schmidt norm vanishes for $m \to 0^+$.*

Unfortunately, proving the Hilbert-Schmidt property of the second operator in $(*)$ seems to require a much more detailed knowledge of the massive modular operator than is presently available.

As a byproduct of the above result, if $\delta_{m,B}$ is the modular operator of $K_m(B)$, then the Hilbert-Schmidt norm (on $K_m(B) + iK_m(B)$) of $(\lambda - \delta_{m,B})^{-1} - Q^{-1}(\lambda - \delta_{0,B})^{-1}Q$, $\lambda \in \mathbb{C} \setminus [0, +\infty)$, vanishes for $m \to 0^+$, i.e., the resolvents of the local modular operators depend continuously on the mass.

## References

[CM] R. Conti, G. Morsella, *Quasi-free isomorphisms of second quantization von Neumann algebras and modular theory*, arXiv:2305.07606.

[EF74] J. P. Eckmann, J. Fröhlich, *Unitary equivalence of local algebras in the quasifree representation*, Ann. Inst. H. Poincaré Sect. A (N.S.) **20** (1974), 201-209.

[AY82] H. Araki, S. Yamagami, *On the quasi-equivalence of quasi-free states of the canonical commutation relations*, Publ. RIMS **18** (1982), 283-338.

[EO73] J. P. Eckmann, K. Osterwalder, *An application of Tomita's theory of modular Hilbert algebras: duality for free Bose fields*, J. Func. Anal. **13** (1973) 1-12.

[Lon] R. Longo, *Lectures on conformal nets. Part 2*, unpublished.

[BS67] M. Sh. Birman, M. Solomiak, *Double Stieltjes operator integrals*, Topics in Math. Phys. Vol. 1, 25-54. Consultants Bureau, New York, 1967.

[CM20] R. Conti, G. Morsella, *Asymptotic morphisms and superselection theory in the scaling limit II: analysis of some models*, Commun. Math. Phys. **376** (2020), 1767-1801.

## Twisted Araki-Woods Algebras: structure and inclusions

Ricardo Correa da Silva
(joint work with Gandalf Lechner)

Fock spaces and second quantization are central concepts in algebraic quantum theory and exist in various forms. From the physics perspective, the better known examples are the Boltzmann-Fock space $\mathcal{F}_0(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, the Bose-Fock space $\mathcal{F}_F(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{H}^{\otimes n}$, and the Fermi-Fock space $\mathcal{F}_{-F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{A}_n \mathcal{H}^{\otimes n}$, where $\mathcal{S}_n$ and $\mathcal{A}_n$ are the symmetrization and anti-symmetrization operators, which are used in the description of interaction-free Bosonic and Fermionic models [BR97]. The use of symmetrization and anti-symmetrization maps on the $n$-particle components capture the fact that bosons satisfy CCR and fermions CAR, respectively, and more general commutation relations such as the $q$-deformed commutation relations require the introduction of twisted Fock spaces [BS91], whose construction holds in much more generality than only $q$-deformed commutation relations and are relevant in studying representations of Wick algebras [JSW95]. Analogous spaces, called $S$-symmetric Fock spaces, are also relevant in integrable models in quantum field theory when a prescribed two-particle scattering matrix $S$ is given [Lec23], [AL17].

**Twisted Fock Spaces and Twisted Araki-Woods Algebras.** Following [BS91] and [JSW95], given a separable Hilbert $\mathcal{H}$ and an operator $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ with $\|T\| \leq 1$ we define, for $n \in \mathbb{N}$, the operators $T_j, R_{T,n}, P_{T,n} \in \mathcal{B}(\mathcal{H}^{\otimes n})$, $1 \leq j \leq n-1$, by $T_j = 1^{\otimes(j-1)} \otimes T \otimes 1^{\otimes(n-j-1)}$, $R_{T,n} := 1 + T_1 + T_1T_2 + \ldots + T_1 \cdots T_{n-1}$, $P_{T,1} := 1$, $P_{T,n+1} := (1 \otimes P_{T,n})R_{T,n+1}$. In case $P_{T,n}$ is positive for all $n \in \mathbb{N}$ we say that $T$ is a twist and denote the set of all twists $\mathcal{T}_{\geq}$.

In case $T \in \mathcal{T}_{\geq}$, we define $\mathcal{H}_{T,n}$ as the closure of the quotient $\mathcal{H}^{\otimes n}/\ker(P_{T,n})$ with respect to the inner product $\langle [\psi_n], [\phi_n] \rangle_{T,n} := \langle \psi_n, P_{T,n}\phi_n \rangle$, where the square-brackets denote equivalence classes and $\psi_n, \phi_n \in \mathcal{H}^{\otimes n}$. Finally, the twisted Fock space is $\mathcal{F}_T(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_{T,n}$ provided with the natural inner product $\langle \cdot, \cdot \rangle_T$. It is worth mentioning that, as the afore-used notation suggests, the case $T = 0$, $T = F$, and $T = -F$, where $F$ is the tensor flip, correspond respectively to the Boltzmann-, Bose-, and Fermi-Fock spaces, but there are many more operators that are twists. In fact, it is known that if $T = T^*$ and $T$ satisfies one of the following conditions (*i*) $\|T\| \leq \frac{1}{2}$; (*ii*) $T \geq 0$; (*iii*) $\|T\| \leq 1$ and the Yang-Baxter equation holds, *i.e.* $T_1T_2T_1 = T_2T_1T_2$, then $T \in \mathcal{T}_{\geq}$.

The recursive formula defining $P_{T,n}$ makes, for each $\xi \in \mathcal{H}$, the twisted left creation operator $a^{\star}_{L,T}(\xi) : \mathcal{H}^{\otimes n}/\ker P_{T,n} \to \mathcal{H}^{\otimes(n+1)}/\ker P_{T,n+1}$ given by the formula $a^{\star}_{L,T}(\xi)[\Psi_n] := [\xi \otimes \Psi_n]$ a well-defined operator which naturally extends to densely define operator on $\mathcal{F}_T(\mathcal{H})$ denoted by the same symbol. Its adjoint with respect to $\langle \cdot, \cdot \rangle_T$ can be calculated and turns out to be the twisted left annihilation operator $a_{L,T}(\xi)[\Psi_n] = [a_L(\xi) R_{T,n} \Psi_n]$, where $a_L(\xi)$ is the usual (untwisted) annihilation operator.

As usual, we can define the essentially self-adjoint field operators $\phi_{L,T}(\xi) = a^\star_{L,T}(\xi) + a_{L,T}(\xi)$ and define, following [CdSL23], for a standard subspace $H \subset \mathcal{H}$, the left twisted Araki-Woods algebras

**Definition 1.** Given a closed real subspace $H \subset \mathcal{H}$ and a twist $T \in \mathcal{T}_\geq$, we define the (left) $T$-twisted Araki-Woods von Neumann algebra

$$\mathcal{L}_T(H) := \{\exp(i\phi_{L,T}(h)) : h \in H\}'' \subset \mathcal{B}(\mathcal{F}_T(\mathcal{H})).$$

It is easy to prove that $H$ being cyclic in $\mathcal{H}$ implies the Fock vacuum $\Omega$ to be cyclic for $\mathcal{L}_T(H)$. The natural question to be asked is under which conditions $\Omega$ is separating for $\mathcal{L}_T(H)$ and what is the modular data of this pair.

**Twisted Araki-Woods Algebras and Standard Vectors.** Under the assumption that $\Omega$ is separating, we have two modular data to consider: The one originating from $H$, denoted by $J_H$, and $\Delta_H$ (see [Lon08]); and the one originating from the pair $(\mathcal{L}_T(H), \Omega)$, denoted by $J$ and $\Delta$. In order to have $\Delta_{|_{\mathcal{H} \cap \mathrm{Dom}(\Delta)}} = \Delta_H$, we introduce the concept of compatibility:

**Definition 2.** Let $H \subset \mathcal{H}$ be a standard subspace. The twists *compatible with H* are the elements of

$$\mathcal{T}_\geq(H) := \{T \in \mathcal{T}_\geq : [\Delta_H^{it} \otimes \Delta_H^{it}, T] = 0 \text{ for all } t \in \mathbb{R}\}.$$

Under the assumption of compatibility and $\Omega$ being separating for $\mathcal{L}_T(H)$, one can explore the KMS condition to prove two conditions about the twist:

(i) $T$ is braided, *i.e.* $T$ satisfies the Yang-Baxter equation $T_1 T_2 T_1 = T_2 T_1 T_2$;
(ii) $T$ is *crossing-symmetric*, *i.e.* for all $\psi_i \in \mathcal{H}$, $1 \leq i \leq 4$

$$T(t) := \langle \psi_1 \otimes \psi_2, (\Delta_H^{it} \otimes 1) T (1 \otimes \Delta_H^{-it}) \psi_3 \otimes \psi_4 \rangle$$

must have a continuous and bounded extension to the strip in the complex plane with $0 \leq \mathrm{Im}(t) \leq \frac{1}{2}$ and analytic in its interior satisfying the boundary condition

$$T\left(t + \frac{i}{2}\right) := \langle \psi_2 \otimes J_H \psi_4, (1 \otimes \Delta_H^{-it}) T (\Delta_H^{it} \otimes 1) J_H \psi_1 \otimes \psi_3 \rangle.$$

On the other hand, in case $T$ satisfies the Yang-Baxter equation and crossing-symmetry, the analogous construction for right twisted operators is possible and it is easy to see that the twisted right Araki-Woods algebra satisfies $\mathcal{R}_T(H') \subset \mathcal{L}_T(H)'$ where $H'$ is the symplectic complement of the standard subspace $H$. These results can be collect in the following theorem which is one of the main results on [CdSL23]:

**Theorem 3.** *Let $H \subset \mathcal{H}$ be a standard subspace and $T \in \mathcal{T}_\geq(H)$ a compatible twist. The following are equivalent:*

*(1) $\Omega$ is separating for $\mathcal{L}_T(H)$;*
*(2) $T$ is braided and crossing symmetric w.r.t. $H$.*

**Inclusions of Twisted Araki-Woods Algebras.** From the quantum field theory perspective, one is interested in a net of von Neumann algebras indexed by the open regions of a manifold. Among several other physically motivated conditions, we mention: $(i)$ *isotony*, meaning that if two space-time regions $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$; and *causality*, meaning that if $\mathcal{O}_1$ is space-like separated from $\mathcal{O}_2$, then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$. In the standard subspace language, it justifies considering the relative commutant of the inclusion $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$, where $K \subset H$ are standard subspaces, namely, $\mathcal{C}_T(K, H) := \mathcal{L}_T(H) \cap \mathcal{L}_T(K)'$.

Two situations are studied in [CdSL23] and [CdSL], one showing that the relative commutant can be very big (a type III von Neumann algebra) and the other showing that the relative commutant may consists only of multiples of the identity.

**Theorem 4.** *Let $K \subset H$ be an inclusion of standard subspaces and let $T \in \mathcal{T}_{\geq}(H)$ be a braided crossing-symmetric twist w.r.t to $H$ with norm $\|T\| < 1$.*

*(1) If $L^2$-nuclearity holds on the standard subspace level,* i.e. $\|\Delta_H^{1/4}\Delta_K^{-1/4}\|_1 < 1$, *where $\|\cdot\|_1$ is the trace norm on $\mathcal{H}$, and $T$ is also compatible with $K$, then $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$ satisfies $L^2$-nuclearity and is quasi-split. If, in addition, $\mathcal{L}_T(H)$ is of type III, also the relative commutant $\mathcal{C}_T(K, H)$ is of type III.*

*(2) If $\Delta_H^{\frac{1}{4}} E_K$ is not compact, where $E_K$ is the real orthogonal projection onto $K$, then $\mathcal{L}_T(K)' \cap \mathcal{L}_T(H) = \mathbb{C}1$.*

The assumptions on item (1) on the above theorem are, in general, too strong and item (2) shows that physical models with $\|T\| < 1$ are usually non-local. Understanding what happens in the situation when $\Delta_H^{\frac{1}{4}} E_K$ is compact, but $L^2$-nuclearity doesn't hold, and when $\|T\| = 1$ are still under investigation.

## References


[AL17] S. Alazzawi and G. Lechner. "Inverse Scattering and Locality in Integrable Quantum Field Theories". *Communications in Mathematical Physics* 354 (2017), pp. 913–956.

[BR97] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics II.* Springer, 1997.

[BS91] M. Bożejko and R. Speicher. "An Example of a Generalized Brownian Motion". *Communications in Mathematical Physics* 137.3 (1991), pp. 519–531.

[CdSL23] R. Correa Da Silva and G. Lechner. "Modular Structure and Inclusions of Twisted Araki-Woods Algebras". *Communications in Mathematical Physics* 402.3 (2023), pp. 2339–2386.

[CdSL] R. Correa da Silva and G. Lechner. Inclusions of Standard Subspaces. *to appear*

[JSW95] P. E. T. J orgensen, L. M. Schmitt, and R. F. Werner. "Positive Representations of General Commutation Relations Allowing Wick Ordering". *Journal of Functional Analysis* 134.1 (1995), pp. 33–99.

[Lec23] G. Lechner. "Polarization-Free Quantum Fields and Interaction". *Letters in Mathematical Physics* 64 (2003), pp. 137–154.

[Lon08] R. Longo. "Real Hilbert Subspaces, Modular Theory, SL(2, R) and CFT". *Von Neumann Algebras in Sibiu.* Theta Ser. Adv. Math. 10. Bucharest: Theta, 2008.

## Holomorphic extension in a locally convex setting and standard subspaces

Daniel Beltiţă
(joint work with Karl-Hermann Neeb)

In the framework of one-parameter operator groups on locally convex spaces, we discussed holomorphic extensions with respect to the parameter, from the real line to suitable horizontal strips in the complex plane. In the special case of one-parameter unitary groups $e^{itH} = (e^H)^{\mathrm{i}t}$ on Hilbert spaces, we recover the complex powers $e^{\mathrm{i}zH} = (e^H)^{\mathrm{i}z}$ of the positive operator defined as the exponential of the infinitesimal generator. This Hilbert space setting is however too special for the applications to certain constructions of nets of standard subspaces in the framework of Lie group representations, as they appear in Algebraic Quantum Field Theory in connection with the Kubo–Martin–Schwinger (KMS) boundary conditions. The constructions of this type are our main motivation. They require one-parameter operator groups on spaces of distribution vectors of unitary representations of Lie groups as presented below in some more detail.

**A general KMS boundary condition**. We assume the following setting:

- $\mathcal{S}_\pi := \mathbb{R} + \mathrm{i}(0,\pi) \subset \mathbb{R} + \mathrm{i}[0,\pi] =: \overline{\mathcal{S}}_\pi \subset \mathbb{C}$
- $\mathcal{Y}$ a complex Hausdorff locally convex space.
- for every subset $\Gamma \subseteq \mathbb{C}$ we denote by $\mathcal{O}_\partial(\Gamma, \mathcal{Y})$ the set of all continuous functions $f\colon \Gamma \to \mathcal{Y}$ that are weakly holomorphic on the interior of $\Gamma$
- $(U_t)_{t\in\mathbb{R}}$ is a 1-parameter subgroup of $\mathrm{GL}(\mathcal{Y})$
- $J\colon \mathcal{Y} \to \mathcal{Y}$ is an anti-linear continuous map
- the following compatibility condition is satisfied: $(\forall t \in \mathbb{R}) \quad JU_t = U_t J$

Then $v \in \mathcal{Y}$ is said to satisfy the *KMS condition* ($v \in \mathcal{Y}_{\mathrm{KMS}}$) if there exists a function $f \in \mathcal{O}_\partial(\overline{\mathcal{S}}_\pi, \mathcal{Y})$, satisfying the boundary condition

$$(\forall t \in \mathbb{R}) \quad f(t) = U_t v, \;\; f(t + \mathrm{i}\pi) = JU_t v \quad (= Jf(t)).$$

**A construction of standard subspaces in a representation theoretic setting**. We now assume the following:

- $G$ is a finite-dimensional real Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp_G\colon \mathfrak{g} \to G$.
- $U\colon G \to \mathcal{U}(\mathcal{H}),\ g \mapsto U_g$ is a unitary representation of $G$ with continuous orbit maps $U^\xi(g) = U_g\xi$.
- $\mathcal{H}^\infty := \{\xi \in \mathcal{H} : U^\xi \in \mathcal{C}^\infty(G, \mathcal{H})\}$ is endowed with its unique Fréchet topology for which the inclusion map $\mathcal{H}^\infty \hookrightarrow \mathcal{H}$ is continuous.
- $\mathrm{d}U\colon \mathfrak{g} \to \mathcal{L}(\mathcal{H}^\infty)$, $\mathrm{d}U(x)v := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} U_{\exp_G(tx)}v$;
- The space $\mathcal{H}^{-\infty}$ of continuous antilinear functionals on $\mathcal{H}^\infty$ is endowed with its weak-$*$-topology and we write

  $$\langle \cdot, \cdot \rangle\colon \mathcal{H}^\infty \times \mathcal{H}^{-\infty} \to \mathbb{C}$$

  for the antiduality pairing that coincides on $\mathcal{H}^\infty \times \mathcal{H}$ with the scalar product of $\mathcal{H}$ (antilinear in the first variable).

- $U^{\pm\infty}: G \to \mathrm{GL}(\mathcal{H}^{\pm\infty})$ are the representations naturally associated to the unitary representation $U: G \to \mathcal{U}(\mathcal{H})$.
- We also define for $h \in \mathfrak{g}$ and $t \in \mathbb{R}$

$$U_{h,t} := U(\exp_G(th)),\ U_{h,t}^{\pm\infty} := U^{\pm\infty}(\exp_G(th)).$$

- $J: \mathcal{H} \to \mathcal{H}$ is a conjugate-linear surjective isometry satisfying $JU_{h,t} = U_{h,t}J$ for every $t \in \mathbb{R}$, and moreover $J\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$, hence we also have its corresponding operators $J^{\pm\infty}: \mathcal{H}^{\pm\infty} \to \mathcal{H}^{\pm\infty}$.

We then obtain a standard subspace of $\mathcal{H}$ defined by

$$\mathtt{V} := \{v \in \mathcal{D}(\Delta^{1/2}) : \Delta^{1/2}v = Jv\} \quad \text{for} \quad \Delta := e^{2\pi i \mathrm{d}U(h)}$$

(cf. [NÓ17, §3.1]). Moreover, [NØÓ21, Prop. 2.1] implies $\mathtt{V} = \mathcal{H}_{\mathrm{KMS}}$.
**The position of the standard subspace within the space of distribution vectors**. Our main results can now be stated as follows:

- $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ is the (weak-$*$-)closure of $\mathtt{V}$ in $\mathcal{H}^{-\infty}$ ([BN23, Thms. 6.2 and 6.5]);
- $\mathcal{H}^{-\infty}_{\mathrm{KMS}} \cap \mathcal{H} = \mathtt{V}$ ([BN23, Thm. 6.4]).
- $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ is the annihilator of $J\mathtt{V} \cap \mathcal{H}^\infty$ with respect to the imaginary part of the pairing ([BN23, Cor. 6.8]).

Here we define the space $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ via the KMS boundary condition with respect to the 1-parameter group $(U^{-\infty}_{h,t})_{t\in\mathbb{R}}$ in $\mathrm{GL}(\mathcal{H}^{-\infty})$ and the continuous antilinear map $J^{-\infty}: \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}$.

### References


[BN23] D. Beltiţă, K.-H. Neeb, *Holomorphic extension of one-parameter operator groups.* Pure and Applied Functional Analysis. (to appear). Preprint arXiv:2304.09597v2

[NÓ17] K.-H. Neeb and G. Ólafsson, *Antiunitary representations and modular theory*, in *50th Sophus Lie Seminar*, eds. K. Grabowska, J. Grabowski, A. Fialowski, and K.-H. Neeb, Banach Center Publications **113** (2017), 291–362. Preprint arXiv:1704.01336.

[NØÓ21] K.-H. Neeb, B. Ørsted, and G. Ólafsson, *Standard subspaces of Hilbert spaces of holomorphic functions on tube domains*, Comm. Math. Phys. **386** (2021), 1437–1487. Preprint arXiv:2007.14797.


## Inclusions of Standard Subspaces


Gandalf Lechner

(joint work with Ricardo Correa da Silva)


Standard subspaces naturally appear in the context of von Neumann algebras, where any von Neumann algebra in standard form gives rise to a standard subspace encoding its modular data, and in quantum field theory, where standard subspaces encode localization regions. From this perspective, standard subspaces appear as auxiliary objects. There is however growing evidence that standard subspaces are interesting objects in their own right – for example, they lead to an independent notion of entropy [CLR19], can naturally be constructed on the basis of suitable Lie group representations [MN21], and lie at the basis of the recently introduced twisted Araki-Woods algebras [CdSL23].

In these and other applications, one is typically not interested in a single standard subspace (the set of all standard subspaces $H$ of a complex Hilbert space $\mathcal{H}$ can easily be classified, see [Lon08, Cor. 2.1.5]), but rather in families of standard subspaces and their intersection, inclusion and covariance properties. The topic of this talk was therefore to initiate an abstract discussion of inclusions

$$K \subset H \subset \mathcal{H}$$

of standard subspaces, without reference to von Neumann algebras or group representations. This can be seen as an analogue of the study of inclusions of von Neumann algebras, or more specifically subfactors.

We review some known results about inclusions of standard subspaces and then reported on joint work in progress with R. Correa da Silva [CdSL].

**Inclusions and irreducible inclusions.** Given a standard subspace $K$, can we embed it properly into a larger standard subspace $H$, or can we properly embed a smaller standard subspace into $K$? This question is answered in the following lemma:

**Lemma 1.** [FG00] *Let $K \subset \mathcal{H}$ be a standard subspace. Then the following are equivalent:*

*(1) There exists a standard subspace $H \subset \mathcal{H}$ such that $K \subsetneq H$.*
*(2) There exists a standard subspace $H \subset \mathcal{H}$ such that $H \subsetneq K$.*
*(3) The modular operator $\Delta_K$ is unbounded.*

Guided by the comparison with subfactor theory, we are particularly interested in understanding *irreducible* inclusions, which by definition are inclusions $K \subset H$ with $K' \cap H = \{0\}$. Here $K'$ denotes the symplectic complement of $K$. Clearly, this requires in particular $K' \cap K = \{0\}$, i.e. $K$ must be a factorial subspace (a factor, for short). Recall that a factor has a well-defined cutting projection $P_K : K + K' \to K$, $k + k' \mapsto k$ [CLR19].

The basic result in this regard is a reformulated version of a proposition from [FG00].

**Proposition 2.** *Let $K \subset \mathcal{H}$ be a standard subspace. Then the following are equivalent:*

*(1) There exists a standard subspace $H \subset \mathcal{H}$ such that $K \subsetneq H$ is irreducible.*
*(2) There exists a standard subspace $H \subset \mathcal{H}$ such that $H \subsetneq K$ is irreducible.*
*(3) The modular operator $\Delta_K$ is unbounded, $K$ is a factor, and the cutting projection $P_K$ of $K$ is unbounded.*

This proposition states that irreducible inclusions of standard subspaces exist in abundance. A central question is then how to detect whether a given inclusion is irreducible, or how to detect whether the relative symplectic complement $K' \cap H$ is cyclic (hence standard).

**Detecting irreducibility.** Let $K, H$ be a pair of standard subspaces. Then [BGL02, Prop. 4.1]

$$K' \cap H + i(K' \cap H) = \{v \in \mathrm{dom}(S_K^* S_H) \,:\, S_K^* S_H v = v\}.$$

This characterization is however often difficult to use as it leads to intricate domain questions. The same holds true for other characterizations that we derived for $K' \cap H$ in terms of polarizers and projections [CdSL].

Comparing with the von Neumann algebraic situation, two notions that are helpful tools in the understanding of relative commutants are the split property [DL84] and modular nuclearity [BDL90]. We give standard subspace formulations for both of them and investigate their consequences in [CdSL]. Here we focus on the nuclearity aspects.

**Definition 3.** An inclusion $K \subset H$ of standard subspaces is said to satisfy modular nuclearity if the real linear operator $\Delta_H^{1/4} E_K$, where $E_K : \mathcal{H} \to K$ is the real orthogonal projection onto $K$, is trace class.

Making use of [LRT78, BDL90, LS16], we then prove:

**Theorem 4.** [CdSL] *Let $K \subset H$ be an inclusion of factor standard subspaces satisfying modular nuclearity. Then* $\dim(K' \cap H) = \infty$.

**A class of examples.** As a concrete class of examples, we consider the irreducible one-dimensional standard pair, namely the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, \frac{dp}{p})$ and the standard subspace $H \subset \mathcal{H}$ given by the data (see [LL14, Sect. 4] for this and other equivalent formulations)

$$(\Delta_H^{it}\psi)(p) = \psi(e^{-2\pi t}p), \qquad (J_H\psi)(p) = \overline{\psi(p)}.$$

The one-parameter group of unitaries $(U(x)\psi)(p) = e^{ipx}\psi(p)$ acts half-sidedly by endomorphisms of $H$, namely $U(x)H \subset H$, $x \geq 0$. It is known that the semigroup of all unitaries $V \in \mathcal{U}(\mathcal{H})$ that commute with $U(x)$, $x \in \mathbb{R}$, and satisfy $VH \subset H$, are precisely the unitaries of the form $V = \varphi(P)$, where $P$ is the generator of $U$ and $\varphi$ an inner function on the upper half plane satisfying the symmetry condition $\varphi(-p) = \overline{\varphi(p)}$, $p > 0$ [LW10, Thm. 2.3].

We are therefore presented with the family of concrete inclusions $\varphi(P)H \subset H$. In the talk it was explained that the modular nuclearity condition fails except for quite specifically chosen inner functions $\varphi$. Nonetheless it is possible to understand and sometimes explicitly compute the relative symplectic complement $\varphi(P)H' \cap H$, which can be $\{0\}$, finite-dimensional, infinite-dimensional, or cyclic depending on $\varphi$. In particular, there are interesting relations relating the number of zeros of the inner function $\varphi$ and the dimension of $\varphi(P)H' \cap H$.

The structures found in this class of examples are currently being investigated alongside more general methods for analyzing relative symplectic complements of standard subspaces [CdSL].

## References


[BDL90] D. Buchholz, C. D'Antoni, and R. Longo. Nuclear maps and modular structures. I. General properties. *J. Funct. Anal.*, 88:233–250, 1990.

[BGL02] R. Brunetti, D. Guido, and R. Longo. Modular localization and Wigner particles. *Rev. Math. Phys.*, 14:759–786, 2002.

[CdSL23] R. Correa da Silva and G. Lechner. Modular Structure and Inclusions of Twisted Araki-Woods Algebras. *Commun. Math. Phys.*, 402:2339–2386, 2023.

[CdSL] R. Correa da Silva and G. Lechner. Inclusions of Standard Subspaces. *to appear*

[CLR19] F. Ciolli, R. Longo, and G. Ruzzi. The information in a wave. *Commun. Math. Phys.*, 379:979–1000, 2020.

[DL84] S. Doplicher and R. Longo. Standard and split inclusions of von Neumann algebras. *Invent. Math.*, 75:493–536, 1984.

[FG00] F. Figliolini and D. Guido. Inclusions of second quantization algebras. In H. Holden J. Jost S. Paycha M. Röckner S. Scarlatti F. Gesztesy, editor, *Stochastic Processes, Physics and Geometry: New Interplays. II: A Volume in Honor of Sergio Albeverio*, number 29, page in, Providence, Rhode Island, November 2000. CMS Conference Proceedings.

[LL14] G. Lechner and R. Longo. Localization in Nets of Standard Spaces. *Commun. Math. Phys.*, 336(1):27–61, 2015.

[Lon08] R. Longo. *Real Hilbert subspaces, modular theory, $SL(2,\mathbb{R})$ and CFT.* Theta Ser. Adv. Math. 10. Theta, 2008.

[LRT78] P. Leylands, J. E. Roberts, and D. Testard. Duality for Quantum Free Fields. *Preprint*, 1978.

[LS16] G. Lechner and Ko Sanders. Modular Nuclearity: A Generally Covariant Perspective. *Axioms*, 5(1), 2016.

[LW10] R. Longo and E. Witten. An Algebraic Construction of Boundary Quantum Field Theory. *Commun. Math. Phys.*, 303(1):213–232, 2011.

[MN21] V. Morinelli and K.-H. Neeb. Covariant homogeneous nets of standard subspaces. *Commun. Math. Phys.*, 386:305–358, 2021.


## Unitarity and reflection positivity in two-dimensional conformal field theory


Yoh Tanimoto

(joint work with Maria Stella Adamo, Yuto Moriwaki)


Two-dimensional conformal field theories (2d CFTs) have been studied extensively in various setting, from algebraic to analytic. One of the algebraic settings is Vertex Operator Algebras (VOAs), which formalize chiral components of a 2d CFT in terms of formal series in $z$. A VOA that corresponds to a quantum field theory must satisfy a condition called unitarity [DL14].

It is known [CKLW18] that one can construct Wightman fields on $S^1$ from quasi-primary fields in a unitary VOA satisfying so-called polynomial energy bounds. As $S^1$ can be seen as the one-point compactification of $\mathbb{R}$, one can see these Wightman field as Wightman fields on one of the lightrays in $\mathbb{R}^{1+1}$. On the other hand, there are Osterwalder-Schrader axioms (OS axioms) [OS73, OS75] that can accomodate many interacting QFTs, mostly the massive ones. From the Schwinger functions satisfying the OS axioms, one can reconstruct Wightman fields. As the

VOA formalism considers the Euclidean setting, it should be possible to construct Schwinger functions satisfying the OS axioms, at least for a nice class of 2d CFTs.

The OS axioms in the two-dimensional Euclidean space concern the Schwinger functions $\{S_n(z_1, \cdots, z_n)\}$, where $S_n$ is a distribution on a subset of $\mathbb{R}^{2n}$ excluding the coinciding points, $z_j \neq z_k$ for $j \neq k$. Among the OS axioms, we consider only reflection positivity, which assures the Hilbert space structure in the resulting Wightman field theory. Up to a conformal transformation [FFK89], this means that, for a finite sequence of test functions $\{f_n(z_1, \cdots, z_n)\}$ supported in the region $|z_1| < |z_2| < \cdots < |z_n|$, one should have

$$\begin{aligned} 0 \le \sum_{j,k} \int & \overline{f_j(\bar z_j^{-1}, \cdots, \bar z_1^{-1})} f_j(z_{j+1}, \cdots, z_{j+k}) S_{j+k}(z_1, \cdots, z_{j+k}) \\ & |J(z_1) \cdots J(z_n)| dx_1 dy_1 \cdots dx_{j+k} dy_{j+k}, \end{aligned} \tag{1}$$

where $z_j = x_j + iy_j \in \mathbb{R}^2$ and $J(z)dxdy$ is a measure on $\mathbb{R}^2$ invariant under the reflection $z \mapsto \bar z^{-1}$ when including the scaling factor coming from the conformal transformations for fields (we define $S_n(z_1, \cdots, z_n) = \langle \Omega, \phi(z_1) \cdots \phi(z_n)\Omega\rangle$, see below).

Let $V, Y(\cdot, z)$ be a unitary VOA and $v \in V$ be a quasi-primary vector. For a given $v$, we denote $\underline{\phi}(z) = Y(v, z) = \sum_n \phi_n z^{-n-d}$, where $d \in \mathbb{N}$ is the conformal dimension of $v$ and $\phi_n \in \mathrm{End}(V)$. Unitarity means that $V$ is equipped with a positive-definite inner product $\langle \cdot, \cdot \rangle$ and $V$ is generated by quasi-primary fields satisfying $(\phi_n)^* = \phi_{-n}$ [CKLW18]. We put $\phi(z) = \sum_n \phi_n z^{-n}$ and define

$$\begin{aligned} S_n(z_1, \cdots, z_n) &= \langle \Omega, \phi(z_1) \cdots \phi(z_n)\Omega\rangle \\ &= \sum_{k_1, \cdots, k_n} \langle \Omega, \phi_{k_1} \cdots \phi_{k_n}\Omega\rangle z_1^{-k_1} \cdots z_n^{-k_n}, \end{aligned} \tag{2}$$

where $\Omega \in V$ is the vacuum. This is, at this point, a formal series in $z_1, \cdots, z_n$.

We assume polynomial energy bounds for $V$ [CKLW18]. This means that $\|\phi_n \Psi\| \le C(|n| + 1)^s \|(L_0 + \mathrm{id})^p \Psi\|$ for some $C, s, p > 0$. Then one can show that the series (2) converges for $|z_1| < \cdots < |z_n|$. Using again polynomial energy bounds, it is also possible that $S_n$ defines a distribution as required in the OS axioms.

If we consider $z \in S^1$, we have the relation $(\phi(z)^*) = \phi(z)$. By analytically continuing this equation, we have (weakly) $\phi(z)^* = \phi(\bar z^{-1})$. As for reflection positivity, by the positive definiteness of the scalar product, we have $\langle \Psi, \Psi\rangle \ge 0$, where

$$\Psi = \sum_j \int f_j(z_1, \cdots, z_j) \varphi(a_1, z_1) \cdots \varphi(a_j, z_j)\Omega \, |J(z_1) \cdots J(z_j)| dx_1 dy_1 \cdots dx_j dy_j$$

is a vector in the completion $\overline{V}$ of $V$. One can show that $\langle \Psi, \Psi\rangle$ is equal to the right-hand side of (1), therefore, reflection positivity holds under unitarity and polynomial energy bounds. Other OS axioms can be checked as well [Mor22], and also the linear growth condition [OS75] from polynomial energy bounds. Altogether, quasi-primary fields in unitary VOA can generate Schwinger functions, from which one can construct Wightman fields as is done in Constructive QFT [GJ87].

We hope to extend this to full vertex operator algebras [Mor20], as we already have Wightman fields for a class of full CFT [AGT]. We hope to find Hilbert space structure for the Euclidean fields. This Euclidean construction could be useful when one tries to perturb CFTs to obtain massive models [JT].

## References


[AGT] Maria Stella Adamo, Luca Giorgetti, and Yoh Tanimoto. Wightman fields for two-dimensional conformal field theories with pointed representation category. *Commun. Math. Phys.* (2023). https://doi.org/10.1007/s00220-023-04835-1.

[CKLW18] Sebastiano Carpi, Yasuyuki Kawahigashi, Roberto Longo, and Mihály Weiner. From vertex operator algebras to conformal nets and back. *Mem. Amer. Math. Soc.*, 254(1213):vi+85, 2018. https://arxiv.org/abs/1503.01260.

[DL14] Chongying Dong and Xingjun Lin. Unitary vertex operator algebras. *J. Algebra*, 397:252–277, 2014.

[FFK89] G. Felder, J. Fröhlich, and G. Keller. On the structure of unitary conformal field theory. I. Existence of conformal blocks. *Comm. Math. Phys.*, 124(3):417–463, 1989. http://projecteuclid.org/euclid.cmp/1104179209.

[GJ87] James Glimm and Arthur Jaffe. *Quantum physics*. Springer-Verlag, New York, second edition, 1987. https://books.google.com/books?id=VSjjBwAAQBAJ.

[JT] Christian Jäkel and Yoh Tanimoto. Towards integrable perturbation of 2d CFT on de Sitter space. *Lett. Math. Phys.*, 113, 89 (2023). https://doi.org/10.1007/s11005-023-01709-4.

[Mor20] Yuto Moriwaki. Full vertex algebra and bootstrap – consistency of four point functions in 2d cft. 2020. https://arxiv.org/abs/2006.15859.

[Mor22] Yuto Moriwaki. Vertex operator algebra and colored parenthesized braid operad. 2022. https://arxiv.org/abs/2209.10443.

[OS73] Konrad Osterwalder and Robert Schrader. Axioms for Euclidean Green's functions. *Comm. Math. Phys.*, 31:83–112, 1973.

[OS75] Konrad Osterwalder and Robert Schrader. Axioms for Euclidean Green's functions. II. *Comm. Math. Phys.*, 42:281–305, 1975. With an appendix by Stephen Summers.


## Reflection Positivity for finite dimensional Lie groups


Maria Stella Adamo

(joint work with Karl-Hermann Neeb, Jonas Schober)


Reflection positivity appears as one of Osterwalder–Schrader axioms, used to study a large class of quantum field theories (QFTs) [OS73, OS75]. Such axioms are used in Constructive QFT, for example, to construct interacting or massive QFTs, see, e.g., [GJ87]. Osterwalder–Schrader axioms are given for a *Euclidean* field theory, providing tools to reconstruct Wightman fields for a Minkowskian (Lorentzian) quantum field theory by analytic continuation of Euclidean Schwinger functions.

As a consequence of a similar duality between the Euclidean motion Lie group and the Poincaré Lie group on the Minkowski space, see [LM75], one can investigate instead reflection positivity for unitary Lie group representations $\mathcal{U}$ on Hilbert spaces $\mathcal{H}$ equipped with a $\theta$-positive closed subspace $\mathcal{H}_+ \subseteq \mathcal{H}$, where $\theta$ is an involution on $\mathcal{H}$.

The represented Lie groups G are paired with a subsemigroup S and an involution $\tau$ such that $\tau(\mathrm{S})^{-1} = \mathrm{S}$. Note that S and $\tau$ play the role of $\mathcal{H}_+$ and $\theta$ respectively in the group theoretic context. A unitary representation $\mathcal{U} : \mathrm{G} \to \mathrm{U}(\mathcal{H})$ is said to be reflection positive if $\mathcal{U}$ is a representation of $(\mathrm{G}, \mathrm{S})$ on $(\mathcal{H}, \mathcal{H}_+)$, i.e., $\mathcal{U}(\mathrm{S})\mathcal{H}_+ \subseteq \mathcal{H}_+$, and $\mathcal{U}$ and $\theta$ verify a compatibility condition between $\theta$ and $\tau$ of the form $\theta\mathcal{U}(g)\theta = \mathcal{U}(\tau(g))$ for all $g \in \mathrm{G}$, for further reading, see [NO18]. When the other conditions are satisfied, the compatibility condition is usually difficult to establish.

From a quadruple $(\mathcal{U}, \mathcal{H}, \mathcal{H}_+, \theta)$ as before, in a canonical way, one obtains a new $*$-representation $\widehat{\mathcal{U}}$ of $(\mathrm{S}, \sharp)$ on $\widehat{\mathcal{H}}$, where $s^\sharp := \tau(s)^{-1}$ is the involution induced by $\tau$ in $S$. $\widehat{\mathcal{H}}$ indicates the completion of the quotient of $\mathcal{H}_+$ by the null vectors with respect to the norm induced by $\theta$. This construction of $\widehat{\mathcal{U}}$ from $\mathcal{U}$ involves the so-called Osterwalder-Schrader transform and so we regard $(\mathcal{U}, \mathcal{H}, \mathcal{H}_+, \theta)$ as a Euclidean realization of $(\widehat{\mathcal{U}}, \widehat{\mathcal{H}})$.

One of the simplest example of $(\mathrm{G}, \mathrm{S}, \tau)$, yet rich in information, is given by the real line $\mathbb{R}$ with its subsemigroup of the positive half line $\mathbb{R}_+$ and $\tau = -\mathrm{id}_{\mathbb{R}}$. Analogously, one can consider the triple $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_{\mathbb{Z}})$. In [ANS22] we consider only regular representations $\mathcal{U}$, namely those for which $\cap_{g\in\mathrm{G}}\mathcal{U}(g)\mathcal{H}_+ = \{0\}$ and $\overline{\cup_{g\in\mathrm{G}}\mathcal{U}(g)\mathcal{H}_+} = \mathcal{H}$. For these representations $\mathcal{U}$, every $\mathcal{U}(\mathrm{G})$-invariant subspace in $\mathcal{H}_+$ is trivial and the only $\mathcal{U}(\mathrm{G})$-invariant subspace that contains $\mathcal{H}_+$ is $\mathcal{H}$ itself.

For the real line $\mathbb{R}$, a regular representation $\mathcal{U}$ is a 1-parameter group, that by the spectral form of the Lax–Phillips Representation Theorem, is realized by multiplication on $\mathrm{L}^2(\mathbb{R}, \mathcal{M})$, and its positive subspace corresponds to the Hardy space $\mathrm{H}^2(\mathbb{C}_+, \mathcal{M})$ of the upper half-plane $\mathbb{C}_+$ with values in some higher dimensional multiplicity space $\mathcal{M}$ [LP81, NO18]. Furthermore, in [ANS22] we assume that the multiplicity space is one-dimensional. However, the Lax–Phillips Theorem doesn't recover the involution, and thus doesn't give information on the compatibility condition between $\theta$ and $\tau$. Thus, we investigate the issue of classifying the involution $\theta$ which verify the compatibility condition and thus produce reflection positive representations.

Under these assumptions, $\theta$ is of the form $\theta = \varphi R$, where $\varphi \in L^\infty(\mathbb{R})$ takes values on the unit circle and, for $x \in \mathbb{R}$, $Rf(x) := f(-x)$ is a reflection on the real line. By using a similar characterization of Hankel operators given for the unit disk $\mathbb{D}$, we show that $\theta$ defines a Hankel operator by $\mathrm{H}_\theta := P_+\theta P_+^*$. Therefore, we obtain a 1-1 correspondence between positive Hankel operators and unitary reflection positive representations, see [ANS22, Example 1.7 (a)], cf. [Nik02, Nik19, Par88, RR94].

Hankel operators can be characterized through Carleson measures. By Nehari's Theorem [ANS22, Nik02], such a measure has a symbol $h \in \mathrm{L}^\infty(\mathbb{R})$ with values in $\mathbb{S}^1$, and thus a kernel, that allows us to define a new weighted space $\mathrm{L}^2(\mathbb{R}, \nu) \cong \mathrm{L}^2(\mathbb{R})$ through a *-isometric isomorphism that preserves the Hardy space (the positive part) and produces a reflection positive representation $(\mathcal{U}, \mathrm{L}^2(\mathbb{R}, \nu), \mathrm{H}^2(\mathbb{C}_+, \nu), \theta_h)$ [ANS22, Theorem 4.5]. Recently, in [Sch23], if the positive Hankel operator on $\mathrm{H}^2(\mathbb{C}_+)$ is contractive, then there exists a involution $\theta_h$ such that $(\mathrm{L}^2(\mathbb{R}), \mathrm{H}^2(\mathbb{C}_+),$

$\theta_h$) is a reflection positive Hilbert space, without modification of the measure. Accordingly, by using the Wold decomposition as a normal form for regular unitary operators $\mathcal{U}$ on $\mathbb{Z}$, we obtain similar results for the triple $(\mathrm{L}^2(\mathbb{S}^1), \mathrm{H}^2(\mathbb{D}), \mathcal{U})$, where $\mathcal{U}$ acts as a multiplication operator by $z$.

For reflection positive representations on $\mathbb{Z}$ and $\mathbb{R}$ respectively, the positive part of the Hilbert space is realized as a Hardy space $\mathrm{H}^2$ on $\mathbb{D}$ and on $\mathbb{C}_+$ respectively. Such domains are biholomorphically equivalent to the $\beta$-strip $\mathbb{S}_\beta$ of all $z \in \mathbb{C}$ such that $\operatorname{Im} z \in (0, \beta)$. Nonetheless, the $\beta$-strip exhibits different geometrical features compared to $\mathbb{D}$ and $\mathbb{C}_+$, e.g., the biholomorphism to the upper half-plane $\mathbb{C}_+$ is given by the exponential map, which is not a Möbius transformation, and the boundary of $\mathbb{S}_\beta$ has two connected components, whereas the boundaries of $\mathbb{D}$ and $\mathbb{C}_+$ are both connected. Such a domain naturally appears when one studies reflection positivity for the circle group $(\mathbb{T}_\beta, \mathbb{T}_{\beta,+}, \tau_\beta)$ for $\beta > 0$, where $\mathbb{T}_\beta := \mathbb{R}/\beta\mathbb{Z}$, $\mathbb{T}_{\beta,+}$ is the half-circle and $\tau_\beta(z) := -\mathrm{id}_{\mathbb{T}_\beta}$. Compared to the previous cases of $\mathbb{R}$ and $\mathbb{Z}$, $\mathbb{T}_{\beta,+}$ is *not* a semigroup.

We will start our investigation of reflection positivity for the circle group $\mathbb{T}_\beta$ by looking at reflection positive functions $\varphi$, since they provide a way to produce reflection positive representations through GNS-like construction by using the positive definite kernels induced by $\varphi$ [NO15, NO18]. In the special case of reflection positive functions on the real line $\mathbb{R}$ which verify the $\beta$-KMS condition, they constitute a source of standard subspaces. Indeed, in [NO19] such functions on $\mathbb{R}$ are shown to be of the form $\varphi_{\mathbb{R}}(t) := \langle v, \Delta^{-it/\beta} v\rangle$ for $t \in \mathbb{R}$, where $v$ belongs to a standard subspace $\mathrm{V} \subseteq \mathcal{H}$ and $(\Delta, J)$ denotes its modular pair.

For the circle group $\mathbb{T}_\beta$, general reflection positive functions $\varphi_{\mathbb{T}_\beta}$ admit an integral representation with respect to a finite Borel measure $\mu$ on $\mathbb{R}_+$ [KL81, NO15]. This allows to extend $\varphi_{\mathbb{T}_\beta}$ to a continuous function on $\overline{\mathbb{S}_\beta}$, which is holomorphic in $\mathbb{S}_\beta$ and to obtain by restriction to the lower boundary of $\mathbb{S}_\beta$ a reflection positive function on $\mathbb{R}$ which verify the $\beta$-KMS condition [NO15, NO19]. Using the integral representation of $\varphi_{\mathbb{T}_\beta}$, such a restriction is given by the Fourier transform of a finite positive Borel measure $\nu$ on $\mathbb{R}$ that verifies $\beta$-reflection, i.e., $d\nu(-p) = e^{-\beta p} d\nu(p)$.

On the other hand, in [ANS], finite positive Borel measures $\nu$ on $\mathbb{R}$ for which $\beta$-reflection holds are in 1-1 correspondence with finite positive Borel measures $\mu$ on $\mathbb{R}_+$. To define $\varphi_{\mathbb{T}_\beta}$, we consider the restriction on $[0, \beta]i$ of the Fourier transform of the finite positive Borel measure $\nu$ on $\mathbb{R}$, which satisfies the $\beta$-reflection condition. Therefore, we can directly show that reflection positive functions $\varphi_{\mathbb{T}_\beta}$ are in 1-1 correspondence with reflection positive functions $\varphi_{\mathbb{R}}$ which verify the $\beta$-KMS-condition.

## References


[ANS] Adamo M. S., Neeb K.-H., Schober J., *Reflection positivity and Hardy spaces on disc, half plane and the strip*. In preparation.

[ANS22] Adamo M. S., Neeb K.-H., Schober J., *Reflection positivity and Hankel operators – the multiplicity free case*. J. Funct. Anal., **283**, 109493, 2022.

[GJ87] Glimm J., Jaffe A., *Quantum physics*, Springer-Verlag, New York, second edition, 1987.

[KL81] Klein A., Landau L., *Preiodic gaussian osterwalder–schrader positive process and the two-sided markov property on the circle*, Pacific J. Math., **94(2)**, 341-367, 1981.
[LP81] Lax P. D., Phillps R. S., *The translation representation theorem*, Integral Equ. Oper. Theory, **4**, 416-421, 1981.
[LM75] Luscher M., Mack G., *Global conformal invariance in quantum field theory*, Comm. Math. Phys., **41**, 203-234, 1975.
[NO15] Neeb K.-H., Ólafsson G., *Reflection positivity for the circle group*, In Proceedings of the 30th International Colloquium on Group Theoretical Methods, number 597 in Journal of Physics: Conference Series, page 012004, 2015.
[NO18] Neeb K.-H., Ólafsson G., *Reflection Positivity. A Representation Theoretic Perspective*, Number 32 in Springer Briefs in Mathematical Physics, Springer Cham, 2018.
[NO19] Neeb K.-H., Ólafsson G., *KMS conditions, standard real subspaces and reflection positivity on the circle group*, Pacific J. Math., **299:1**, 117-169, 2019.
[Nik02] Nikolski N., *Operators, Functions and Systems: An Easy Reading. Volume I: Hardy, Hankel, and Toeplitz*, volume 92 of Math. Surveys and Monographs, Amer. Math. Soc., 2002.
[Nik19] Nikolski N., *Hardy Spaces*, Cambridge University Press, 2019.
[OS73] Osterwalder K., Schrader R., *Axioms for Euclidean Green's functions*, Comm. Math. Phys., **31**, 83-112, 1973.
[OS75] Osterwalder K., Schrader R., *Axioms for Euclidean Green's functions 2*, Comm. Math. Phys., **42**, 281-305, 1975.
[Par88] Partington J., *An Introduction to Hankel Operators*, volume 13 London Math. Soc. Student Texts, 1988.
[RR94] Rosenblum M., Rosnyak J., *Topics on Hardy Classes and Univalent Functions*, Birkhäuser, 1994.
[Sch23] Schober J., *Regular one-parameter groups, reflection positivity and their application to Hankel operators and standard subspaces*, PhD Thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, 2023.

## Maximal Quantum $f-$Divergences in von Neumann Algebras

Alessio Ranallo
(joint work with Stefan Hollands)

Inspired by recent advances in the study of the capacity of quantum channels between finite-dimensional factors by means of *Geometric Rényi Divergences* [FF21], we study the notion of Maximal Quantum $f-$Divergences in the setting of von Neumann algebras and Algebraic Quantum Field Theory.

Divergences are used to distinguish between couples of probability measures (and quantum states). Araki's notion of relative entropy is an example of divergence. In Quantum Information Theory an account of *quasi-entropies* is given in [OP93] and a more systematic account of various types of Quantum $f-$Divergences can be found in [Hia19].

Consider two probability measures $p = \{p_i\}_i, q = \{q_i\}_i \in \mathcal{P}(X)$ on a finite set $X$, such that $p \ll q$. The relative entropy between these two reads

$$S(p\|q) = -\sum_i p_i \log\left(\frac{q_i}{p_i}\right) = -\sum_i p_i \log\left((R_p^q)_i\right) ,$$

where $R_p^q$ denotes the *Radon-Nikodym* derivative of $q$ w.r.t. $p$. Relative entropy can be generalized to the quantum setting in a number of ways. The Araki's notion of relative entropy comes from the Umegaki's one, where the Radon-Nikodym derivative is replaced by the *relative modular operator*. Indeed, let $\mathcal{M}$ be a von Neumann algebra in standard form acting on the Hilbert space $\mathfrak{H}$. Let $\psi, \varphi \in \mathcal{M}_{*,+}$ be two normal, bounded, and positive functionals with (standard) vector representatives $\Psi, \Phi \in \mathfrak{H}$. For the sake of simplicity, suppose that both $\psi$ and $\varphi$ are faithful, then

$$S(\varphi\|\psi) = -\langle \Phi \, , \log\left(\Delta_{\psi,\varphi}\right) \Phi \rangle$$

is the formula for Araki's relative entropy. It is then easy to show that in finite dimensions Araki's notion reduces to the Umegaki's original definition

$$S(\varphi\|\psi) = -\mathrm{Tr}\left(\rho_\varphi\left(\log\left(\rho_\psi\right) - \log\left(\rho_\varphi\right)\right)\right) ,$$

where $\rho_\varphi$, resp. $\rho_\psi$, denotes the matrix representing $\varphi$, resp. $\psi$. However, in finite dimensions, the choice of the operator $\rho_\psi^{1/2}\rho_\varphi^{-1}\rho_\psi^{1/2}$ induce another "entropy-like" quantity

$$S_{\mathrm{BS}}(\varphi\|\psi) := -\mathrm{Tr}\left(\rho_\varphi \log\left(\rho_\psi^{1/2}\rho_\varphi^{-1}\rho_\psi^{1/2}\right)\right) .$$

Here, BS is for Belavkin-Staszewski [BS82], where this entropy was introduced. Note that $S_{\mathrm{BS}}(\varphi\|\psi) = S(\varphi\|\psi)$ whenever $\rho_\varphi$ and $\rho_\psi$ commutes. Araki's relative entropy is an example of *standard divergence*, while the Belavkin-Staszewski notion is an example of *maximal divergence*, see [Hia19] for a more systematic treatise.

We prove that a Kosaki-type formula holds for the Belavkin-Staszewski divergence.

$$S_{\mathrm{BS}}(\varphi\|\psi) = \sup\sup\left\{\varphi(1)\log n - \int_{1/n}^{\infty}\left[\varphi(x_t x_t^*) + \frac{1}{t}\psi(y_t y_t^*)\right]\frac{dt}{t}\right\},$$

where the first sup is taken over $n \in \mathbb{N}$, while the second is over finite range step functions $x_{(\cdot)} : (\frac{1}{n}, \infty) \to \mathcal{M}$ such that $x_t = 1$ for sufficiently small $t$, such that $x_t = 0$ for sufficiently large $t$, and where $y_t := 1 - x_t$.

Given two normal linear maps $\alpha\, , \beta : \mathcal{N} \to \mathcal{M}$ that are completely positive and unital, normal *channels* for short, we are able to provide a notion of Belavkin-Staszewski divergence of $\alpha$ w.r.t. to $\beta$. The definition generalizes the one for matrix algebras introduced (for generalized divergences of which $S_{\mathrm{BS}}$ is an instance of) in [LKDW18], where one takes the sup over all states induced from states on the enlarged system obtained from coupling our initial system, e.g. $\mathcal{M}$ above, with an ancillary system $\mathcal{A}$ (arbitrary), and then precomposing with the dilated channels $\alpha \otimes \mathrm{id}_{\mathcal{A}}$, $\beta \otimes \mathrm{id}_{\mathcal{A}}$:

$$S_{\mathrm{BS}}(\alpha\|\beta) = \sup_{\substack{\mathcal{A} \\ \psi \in (\mathcal{M}\otimes\mathcal{A})_{*,+,1}}} S_{\mathrm{BS}}(\psi \circ (\alpha \otimes \mathrm{id}_{\mathcal{A}})\|\psi \circ (\beta \otimes \mathrm{id}_{\mathcal{A}})) .$$

The motivation behind this definition comes from the fact that refined information about the action of channels can be obtained through entanglement. In the case of von Neumann algebras of general type, we provide a generalization of this definition based on the notion of bimodules between two von Neumann algebras.

After presenting some results on channel divergences, we discuss briefly some open questions of relevance to the workshop.

## References


[HR23] Stefan Hollands and Alessio Ranallo Channel Divergences and Complexity in Algebraic QFT. *Communications in Mathematical Physics*, 404:927–962, 2023.

[Ara76] Huzihiro Araki. Relative entropy of states of von neumann algebras. *Publications of the Research Institute for Mathematical Sciences*, 11(3):809–833, 1976.

[BS82] Viacheslav P Belavkin and P Staszewski. $c^*$-algebraic generalization of relative entropy and entropy. In *Annales de l'IHP Physique théorique*, volume 37, pages 51–58, 1982.

[FF21] Kun Fang and Hamza Fawzi. Geometric rényi divergence and its applications in quantum channel capacities. *Communications in Mathematical Physics*, pages 1–63, 2021.

[FFRS20] Kun Fang, Omar Fawzi, Renato Renner, and David Sutter. Chain rule for the quantum relative entropy. *Physical review letters*, 124(10):100501, 2020.

[Hia19] Fumio Hiai. Quantum $f$-divergences in von Neumann algebras. II. Maximal $f$-divergences. *J. Math. Phys.*, 60(1):012203, 30, 2019.

[LKDW18] Felix Leditzky, Eneet Kaur, Nilanjana Datta and Mark M. Wilde. Approaches for approximate additivity of the Holevo information of quantum channels. *Phys. Rev. A*, 97(1):012332, 2018.

[OP93] Masanori Ohya and Dénes Petz. *Quantum Entropy and Its Use*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 1993.

[Ren60] Alfréd Rényi. On measures of entropy and information. *The 4th Berkeley Symposium on Mathematics, Statistic and Probability*, pages 547–561, 1960.


## Localization of positive energy representations for gauge groups on conformally compactified Minkowski space


Bas Janssens

(joint work with Karl-Hermann Neeb)


For a gauge theory associated to a principal $K$-bundle $P \to M$, the relevant group $\mathcal{G}$ of gauge transformations depends rather sensitively on the boundary conditions at infinity. It contains the group $\mathrm{Gau}_c(P) = \Gamma_c(M, \mathrm{Ad}(P))$ of compactly supported vertical automorphisms[1] of $P$ (the 'local' gauge transformations), but it is usually larger. For instance, if $P = M \times K$ is the trivial bundle, one would expect $\mathcal{G}$ to contain the group $K$ of constant gauge transformations, which are certainly not compactly supported.

On the other hand, if one requires that $\mathcal{G}$ preserves boundary conditions for the (classical) fields at infinity, then the relevant group $\mathcal{G}$ of gauge transformations may be significantly smaller than the group $\mathrm{Gau}(P) = \Gamma(M, \mathrm{Ad}(P))$ of *all* vertical automorphisms. Any input on the following question would be most welcome:

[1] My convention here is that $\mathrm{Ad}(P) = (P \times K)/K$ is the bundle of Lie groups over $M$ associated to $P$ by the conjugation, and $\mathrm{ad}(P) = (P \times \mathfrak{k})/K$ is the bundle of Lie algebras associated to $P$ by the adjoint action.

### Question

*What, in the gauge theory and geometric setting of your choice, would be examples of relevant groups* $\mathrm{Gau}_c(P) \subset \mathcal{G} \subset \mathrm{Gau}(P)$ *of gauge transformations?*

Different gauge theories and space-time geometries will probably lead to different answers. For instance, if the boundary conditions are in terms of a fall-off rate $1/r^k$ for the curvature $F \in \Omega^2(M, \mathfrak{k})$ of the principal connection, then the requirement on the infinitesimal gauge transformations $\xi \in \Gamma(\mathrm{ad}(P))$ will be that the fall-off rate for $\delta F = [\xi, F]$ does not exceed $1/r^k$ for any field $F$ with this property. If $K$ is abelian, then this condition is vacuous. If $K$ is semisimple however, then the above condition is fulfilled only if $\xi \sim 1$ is bounded. (See [Ash83] for a more refined version of this type of argument, taking into account the 'peeling-off' behaviour of $F$.)

Let me say a few words about the background of the above question, and sketch the implications of one possible answer. I will be rather brief because the details have appeared elsewhere [JN23].

Together with Karl-Hermann Neeb, we have proven a localization theorem for certain projective unitary representations of the compactly supported gauge group $\mathrm{Gau}_c(P)$. This is in an equivariant setting, with a Lie group $H$ of 'space–time symmetries' whose action on $M$ lifts to an action on $P$ by bundle automorphisms. In the Lie algebra $\mathfrak{h}$ of $H$ we specify a distinguished cone $\mathcal{C}$ of 'timelike generators'. If, for example, $M = \mathbb{R}^d$ is Minkowski space and $H = \mathbb{R}^d \rtimes \mathrm{SO}(d-1,1)$ is the Poincaré group, then it is natural to choose $\mathcal{C} = \{p \in \mathbb{R}^d\,;\, \eta(p,p) \leq 0\}$ to be the forward light cone.

We are interested in *positive energy representations*; projective unitary representations of $\mathrm{Gau}_c(P)$ that extend to the semidirect product of $\mathrm{Gau}_c(P)$ with the group $H$ of space-time symmetries in such a way that every timelike generator $p \in \mathcal{C}$ gives rise to a Hamilton operator $H(p)$ with spectrum bounded from below.

This positive energy condition is surprisingly restrictive. If the structure group $K$ of the principal fibre bundle $P \to M$ is compact, semisimple and 1-connected, one can prove the following result.

**Theorem 1** (Localization theorem)**.** *Suppose that the action of* $\mathcal{C}$ *on* $M$ *has no fixed points. Then for every positive energy representation* $(\overline{\rho}, \mathcal{H})$ *of the identity component* $\Gamma_c(M, \mathrm{Ad}(P))_0$*, there exists a 1-dimensional,* $H$*-equivariantly embedded submanifold* $S \subseteq M$ *and a positive energy representation* $\overline{\rho}_S$ *of* $\Gamma_c(S, \mathrm{Ad}(P))$ *such that the following diagram commutes,*

$$\begin{array}{ccc} \Gamma_c(M, \mathrm{Ad}(P))_0 & \xrightarrow{\ \overline{\rho}\ } & \mathrm{PU}(\mathcal{H}) \\ {\scriptstyle r_S}\downarrow & \nearrow {\scriptstyle \overline{\rho}_S} & \\ \Gamma_c(S, \mathrm{Ad}(P)), & & \end{array}$$

*where the vertical arrow denotes restriction to* $S$*.*

Loosely speaking: if there are no fixed points for the action of the space-time symmetry group $H$, then positive energy representations come from 1-dimensional $H$-orbits.

One way to fix boundary conditions on Minkowski space is to require the gauge fields (weighted by an appropriate conformal factor) to extend smoothly to the *conformal compactification* $M = S^1 \times S^{d-1}$. In this setting, the relevant gauge group is $\mathcal{G} := \Gamma(S^1 \times S^{d-1}, \mathrm{Ad}(P))$. This is larger than the group $\Gamma_c(\mathbb{R}^d, \mathrm{Ad}(P))$ of 'local' gauge transformation (because compactly supported gauge transformations extend trivially to infinity), but it is strictly smaller than the group $\Gamma(\mathbb{R}^d, \mathrm{Ad}(P))$ of *all* vertical automorphisms (which does not require any limiting behaviour at infinity). This is one example of a choice of boundary conditions for which

$$\Gamma_c(\mathbb{R}^d, \mathrm{Ad}(P)) \subseteq \mathcal{G} \subseteq \Gamma(\mathbb{R}^d, \mathrm{Ad}(P)).$$

If we take $H$ to be the connected Poincaré group, then our theorem does not immediately apply. The reason for this is that although the action of $H$ on null infinity is fixed point free, spacelike infinity $\iota_0$ (which is identified with timelike infinity $\iota_\pm$ in the compactification) is a fixed point.

Let us start by taking $M$ to be the noncompact manifold $M = S^1 \times S^1/\{\iota_0\}$, the conformal compactification of $\mathbb{R}^2$ with spatial infinity removed. The action of the Poincaré group then has three orbits: Minkowski space $\mathbb{R}^2$, and the two one-dimensional components $\mathcal{I}_{L/R} \simeq \mathbb{R}$ of null infinity (corresponding to left and right moving modes). So the only 1-dimensional Poincaré-invariant orbits are $\mathcal{I}_{L/R}$! In this setting, the localization theorem implies that every positive energy representation of $\Gamma_c(S^1 \times S^1/\{\iota_0\}, P)$ is determined entirely by two positive energy representations of the pointed loop group $\Gamma_c(S^1/\{\iota_0\}, \mathrm{Ad}(P))$, one for $\mathcal{I}_L \simeq S^1/\{\iota_0\}$ and one for $\mathcal{I}_R \simeq S^1/\{\iota_0\}$.

Although we cannot directly apply the above form of the localization theorem to the full compactification $S^1 \times S^1$, a more refined analysis reveals that for every positive energy representation $(\overline{\rho}, \mathcal{H})$ of $\Gamma(S^1 \times S^1, \mathrm{Ad}(P))$, the projective unitary operator $\overline{\rho}(g) \in \mathrm{PU}(\mathcal{H})$ associated to a gauge transformation $g \in \Gamma(S^1 \times S^1, \mathrm{Ad}(P))$ can only depend on the *values* of $g$ at null infinity $\mathcal{I}_{L/R}$, and on the *2-jets* at spatial infinity $\iota_0$.

For Minkowski space $\mathbb{R}^d$ with $d > 2$, the conformal compactification $S^1 \times S^{d-1}$ has one orbit of dimension $d$ (the open dense subset $\mathbb{R}^d$), one orbit of dimension $d-1$ (null infinity), and a single fixed point (spacelike infinity $\iota_0$, which is again identified with past and future timelike infinity $\iota_\pm$ in the compactification). If we again apply the localization theorem to the noncompact manifold $S^1 \times S^{d-1}/\{\iota_0\}$, we now find that every positive energy representation of $\Gamma(S^1 \times S^{d-1}, \mathrm{Ad}(P))$ is *trivial* on $\Gamma_c(S^1 \times S^{d-1}/\{\iota_0\}, \mathrm{Ad}(P))$. In other words: for every positive energy representation $(\overline{\rho}, \mathcal{H})$ of $\Gamma(S^1 \times S^{d-1}, \mathrm{Ad}(P))$, the projective unitary transformation $\overline{\rho}(g) \in \mathrm{PU}(\mathcal{H})$ assigned to a gauge transformation $g$ depends only on the *germ* of $g$ around spacelike infinity $\iota_0$.

In fact, a more refined analysis shows that $\overline{\rho}(g)$ depends only on the *1-jet* of $g$ at $\iota_0$. Taking into account the Poincaré group as well, this reduces the relevant symmetry group from $(\mathrm{SO}_0(d-1,1) \rtimes \mathbb{R}^d) \ltimes \Gamma(S^1 \times S^{d-1}, \mathrm{Ad}(P))$ to the *finite* *dimensional* Lie group $(\mathrm{SO}_0(d-1,1) \rtimes \mathbb{R}^d) \ltimes J^1_{\iota_0}\mathrm{Ad}(P)$. Now the group of 1-jets $J^1_{\iota_0}\mathrm{Ad}(P)$ is isomorphic to $K \ltimes \mathfrak{k} \otimes \mathbb{R}^d$, where the first term captures the values of the gauge transformation at $\iota_0$ and the (abelian) second term captures the first derivatives of the gauge transformation at $\iota_0$. Putting it all together, we end up with a semidirect product

$$(\mathrm{SO}_0(d-1,1) \times K) \ltimes (\mathbb{R}^d \oplus \mathfrak{k} \otimes \mathbb{R}^d)$$

of a semisimple Lie group $G := \mathrm{SO}_0(d-1,1) \times K$ with the abelian Lie group $V := \mathbb{R}^d \oplus \mathfrak{k} \otimes \mathbb{R}^d$ (considered as a vector space with addition). The projective unitary representations of this group can be found using Mackey's imprimitivity theorem; they are given by a $G$-orbit in $V$, together with a projective unitary representation of the little group $L_\nu \subseteq G$, the stabiliser of a point $\nu$ in the orbit.

For $d = 4$, $K = \mathrm{SU}(3)$ and $\nu = p \oplus X \otimes p$ with $\eta(p,p) = 0$, one obtains representations that are induced from the little group $L_\nu = E(2) \times \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$, where $E(2) \subseteq \mathrm{SO}(3,1)$ is the group of two-dimensional euclidean motions. It is tempting to speculate that these representations might be connected to symmetry breaking phases.

## References

[Ash83] A. Ashtekar *On the Boundary Conditions for Gravitational and Gauge Fields at Spatial Infinity*, Springer Lecture notes in Physics **202** (1983), 95–109.

[JN23] B. Janssens and K.-H. Neeb, *Positive Energy Representations of Gauge Groups I: Localization*, `arxiv:2108.03501`, to appear in Memoirs of the European Mathematical Society.

# On separable states

Ko Sanders

## 1. Introduction

Entanglement is the phenomenon in quantum physics where measurements in spacelike separated regions give rise to correlations that cannot be explained by classical physics. Although this defining feature of quantum physics is rather counter-intuitive, it is not at all rare. Physical systems naturally entangle themselves with their environment at no cost to the experimenter. On the contrary, to prevent this decoherence is difficult and expensive in terms of effort and energy.

The omnipresence of entanglement is reflected in the structure of relativistic quantum field theory (QFT). The Reeh-Schlieder Theorem [RS61] states that the vacuum vector $\Omega_0 \in \mathcal{H}$ of any Wightman QFT in Minkowski space is cyclic for all local algebras of observables. This entails that the vacuum is entangled between any two spacelike separated open regions $A$ and $B$ of spacetime, cf. Corollary 1 in Section 5.1 of [HS18]. This property is shared by many other states, cf. [San09, Wit18]. Indeed, $\mathcal{H}$ contains a dense $G_\delta$ of vectors with the Reeh-Schlieder property [DM71].

Even though entanglement is the rule in QFT, rather than the exception, there are at least two good reasons to have a closer look at separable states, i.e. states which are not entangled between $A$ and $B$. Firstly, most of our physical concepts are classical and hence arise in a context where all states are separable. Secondly, to quantify the amount of entanglement between $A$ and $B$ in a given state $\omega$, one uses an entanglement measure, which compares $\omega$ to the nearest separable state. Here, the word "nearest" can be made mathematically precise in various ways, leading to a range of entanglement measures, cf. [HS18] and references cited therein.

E.g., the entanglement entropy in vacuum typically falls off when the separation between $A$ and $B$ increases. This suggests an explanation as to why the physical world looks so classical on large scales: a smaller entanglement entropy should make it harder to exploit any entanglement present in the system and make it visible. Unfortunately, I am not aware of any results that quantify the word "harder" in terms of the energy (density) needed in relativistic QFT.

It is known that there exist normal separable states under quite general circumstances, namely when a QFT satisfies the split property, cf. [BDF87] (see also [Buc74]). However, normality is a rather weak condition on quantum states and one might like to ascertain further physical properties, e.g. that separable states can share the symmetry of a system and/or have a finite energy (density), etc. Furthermore, it would be interesting to know how much energy needs to be expended to create and/or maintain a separable state. In this talk, based on [San23], I present a result that gives partial answers to these questions in a toy model system. The proof of this result required novel methods involving test functions of positive type, which I will also discuss.

## 2. An existence theorem for separable states

To formulate the main result of [San23], let us fix an inertial coordinate frame in Minkowski space and write $x = (x_0, \mathbf{x})$. $\omega_2(x, x')$ denotes the two-point distribution of a state.

**Theorem.** Consider a free scalar QFT of mass $m > 0$ in 4-dimensional Minkoswki space. Given any $R > 0$ there exists a quasi-free, Hadamard, stationary, homogeneous, isotropic state $\omega$, s.t.

(i) $\omega_2(x, x') = 0$ if $(x, x') \in S = \{\|\mathbf{x} - \mathbf{x}'\| > R + |x_0 - x'_0|\}$,

(ii) $\omega(T^{\mathrm{ren}}_{00}(x)) \leq 10^{31} m^4 \frac{e^{-\frac{1}{4}mR}}{(mR)^8}$.

Item (i) ensures that $\omega$ is a product state between $A$ and $B$, as soon as these regions are separated by a distance $\geq R$. By a standard spacetime deformation argument one can also establish the existence of separable states for massless fields and in curved spacetimes with topology $\mathbb{R}^4$.

To find $\omega$, we will compare $\omega_2$ to the vacuum two-point distribution $\omega^0_2$, i.e.

$$\omega_2(x - x') = \omega^0_2(x - x') + w(x - x')\,,$$

where we exploited the translation invariance. $w$ must satisfy $(\Box + m^2)w = 0$ with initial data $w_0(\mathbf{x}) = w|_{x_0=0}(\mathbf{x})$ and $w_1(\mathbf{x}) = \partial_0 w|_{x_0=0} \equiv 0$ such that

(1) $w_0$ is real-valued, smooth and rotation invariant.
(2) $w_0$ is of positive type, i.e. $\widehat{w_0} \geq 0$.
(3) $w_0(\mathbf{x}) = -\omega_2^0(0, \mathbf{x})$ if $\|\mathbf{x}\| > R$.
(4) $\omega(T_{00}^{\text{ren}}(x)) = (-\Delta + m^2)w(0) \leq 10^{31} m^4 \frac{e^{-\frac{1}{4}mR}}{(mR)^8}$.

The strategy to find $w$ (and hence $\omega_2$) is to modify the initial data of $\omega_2^0$, taking

$$w_0(\mathbf{x}) = -\chi_\infty(\|\mathbf{x}\|)\omega_2^0(0, \mathbf{x}) + f(\mathbf{x})\,.$$

Here $\chi_\infty$ is a smooth, rotation invariant function that vanishes near $\mathbf{x} = 0$ and equals 1 when $\|\mathbf{x}\| \geq R$, removing all unwanted correlations. $f \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ is supported in the ball of radius $R$ and is needed to achieve the positive type of $w_0$, i.e. $\hat{f} \geq \mathcal{F}\left[\chi_\infty \omega_2^0|_{x_0=0}\right]$ (the Fourier transform), which leads to the study of test functions $f$ of positive type and *lower* bounds on $\hat{f}$. To my knowledge such bounds had not been considered before, except asymptotically for $|k| \to \infty$ [FF15].

## 3. Test functions of positive type

A standard construction of test functions starts with the characteristic function $\chi$ of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and takes repeated convolutions (cf. [Hor90]). Given $a = \sum_{n=1}^\infty a_n < \infty$ with $a_1 \geq a_2 \geq \cdots > 0$, one obtains a test function $f \in C_0^\infty([-a, a], \mathbb{R})$ by taking the limit

$$f := \chi\left(\frac{\cdot}{a_1}\right) * \chi\left(\frac{\cdot}{a_2}\right) * \cdots\,.$$

This construction leads to good control on $f$, e.g. on $\|\partial_x^n f\|_\infty$ for all $n \geq 0$ and on upper bounds on $|\hat{f}|$. Because $\widehat{\chi * \chi} = \hat{\chi}^2 \geq 0$ we can even get $\hat{f} \geq 0$. However, we have no good control over lower bounds on $\hat{f}$. Indeed, $\hat{f}$ will have zeroes. To remedy this, one can modify the construction and replace $\chi$ by $\eta = \frac{3}{2}(\chi * \chi)^2$ with

$$\frac{1}{1 + \frac{7}{40}k^2} \leq \hat{\eta}(k) \leq \frac{1}{1 + \frac{1}{20}k^2}\,.$$

For $f \in C_0^\infty(\mathbb{R}, \mathbb{R})$, $\hat{f}$ falls off faster than any polynomial and $|\hat{f}(k)| \leq e^{-|k|}$ iff $f \equiv 0$. More precisely, using the construction with $\chi$ Ingham [Ing34] showed:

**Theorem.** Given $l > 0$ and $\epsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ decreasing, there exists $f \in C_0^\infty([-l, l], \mathbb{R})$ such that $|\hat{f}(k)| \leq e^{-k\epsilon(|k|)}$ iff

$$\int_1^\infty \frac{\epsilon(k)}{k}\, dk < \infty\,. \tag{1}$$

In analogy, [San23] proves a lower bound using the construction with $\eta$:

**Theorem.** Given $l > 0$, $\epsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ decreasing and $\gamma \in (0, 1)$ such that (1) holds and $\lim_{k\to\infty} k^\gamma \epsilon(k) = \infty$, there exists a non-negative, even $g \in C_0^\infty([-l, l], \mathbb{R})$ such that $\int g(x)dx = 1$ and $|\hat{g}(k)| \geq e^{-k\epsilon(|k|)}$.

Analogous results hold in higher dimensions. Examples include test functions of arbitrarily small support that dominate Gevrey type functions. The constructions involved provide enough detailed control over test functions of positive type to prove the main theorem in Section 2, but the estimate on the energy density is not sharp. It would be interesting to see if the methods introduced here can be developed further to yield sharper results.

## References


[Buc74] D. Buchholz, *Product States for Local Algebras*, Commun. Math. Phys. **36**, 287–304 (1974).

[BDF87] D. Buchholz, C. D'Antoni and K. Fredenhagen, *The universal structure of local algebras*, Commun. Math. Phys. **111**, 123–135 (1987).

[DM71] J. Dixmier and O. Maréchal, *Vecteurs totalisateurs d'une algèbre de von Neumann*, Commun. Math. Phys. **22**, 44–50 (1971).

[FF15] C.J. Fewster and L. Ford, *Probability distributions for quantum stress tensors measured in a finite time interval*, Phys. Rev. D **92**, 105008 (2015).

[HS18] S. Hollands and K. Sanders, *Entanglement measures and their properties in quantum field theory*, Springer Briefs in Mathematical Physics Vol. 34, Springer Nature (2018).

[Hor90] L. Hörmander, *The analysis of linear partial differential operators*, Vol.I, Springer, Berlin Heidelberg (1990).

[Ing34] A.E. Ingham, *A note on Fourier transforms*, J. London Math. Soc. **S1-9**, 29 (1934).

[San23] K. Sanders, *On separable states in relativistic quantum field theory*, arXiv:2304.03120

[RS61] H. Reeh and S. Schlieder, *Bemerkungen zur Unitäräquivalenz von Lorentzinvarianten Feldern*, Nuovo Cimento **22**, 1051–1068 (1961).

[San09] K. Sanders, *On the Reeh-Schlieder Property in Curved Spacetime*, Commun. Math. Phys. **288**, 271–285 (2009).

[Wit18] E. Witten, *APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory*, Rev. Mod. Phys. **90**, 045003 (2018).


*Reporter: Ricardo Correa da Silva*

# Participants

**Dr. Maria Stella Adamo**
Department of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba
Tokyo 153-8914
JAPAN

**Dr. Daniel Beltita**
Institute of Mathematics
"Simion Stoilow" of the
Romanian Academy
21 Calea Grivitei Street
P.O. Box 1–764
014 700 Bucharest
ROMANIA

**Prof. Dr. Detlev Buchholz**
Mathematisches Institut
Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen
GERMANY

**Dr. Daniela Cadamuro**
Institut für Theoretische Physik
Universität Leipzig
Brüderstraße 16
04103 Leipzig
GERMANY

**Dr. Ricardo Correa da Silva**
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstr. 11
91058 Erlangen
GERMANY

**Dr. Christian D. Jaekel**
Department of Applied Mathematics
Institute of Mathematics and Statistics
University of São Paulo
Rua do Matão, 1010 - Vila Universitaria
São Paulo, SP 05508-090
BRAZIL

**Dr. Bas Janssens**
Delft Institute of Applied Mathematics
Delft University of Technology
Mekelweg 4
2628 CD Delft
NETHERLANDS

**Prof. Dr. Gandalf Lechner**
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstr. 11
91058 Erlangen
GERMANY

**Prof. Dr. Roberto Longo**
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1
00133 Roma
ITALY

**Dr. Vincenzo Morinelli**
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1
00133 Roma
ITALY

**Prof. Dr. Gerardo Morsella**
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1
I-00133 Roma
ITALY

**Prof. Dr. Karl-Hermann Neeb**
Department Mathematik
FAU Erlangen-Nürnberg
Cauerstraße 11
91058 Erlangen
GERMANY

**Prof. Dr. Gestur Olafsson**
Department of Mathematics
Lockett Hall 322
Louisiana State University
70803 Baton Rouge, LA 70803-4918
UNITED STATES

**Mr. Alessio Ranallo**
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1
00133 Roma
ITALY

**Prof. Dr. Jacobus Ambrosius Sanders**
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstr. 11
91058 Erlangen
GERMANY

**Dr. Yoh Tanimoto**
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1
00133 Roma
ITALY