Elemente der Mathematik 79 (2024), 89–97
DOI 10.4171/EM/525

On squares circumscribing quadrangles

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1 Introduction

It is well known ([1, p. 8], [2, p. 422]) that there are six squares whose sides pass through four given points in general position. In our discussion, we distinguish between the set of four points in general position, often called *the four-point* $ABCD$ [3, p. 246], and the quadrangles defined by this, the arguments being valid for all kinds of generic quadrangles, convex, non-convex, and self-intersecting ones.

Regarding the construction of such a square, this can be based on the following standard remark [2, p. 422]. From B, draw the orthogonal line to AC and take B^* such that BB^* is equal to AC. It is easily seen that the point B^* is on the side of a square that solves the problem. Once two points on a side, such as B^* and D, are known, the whole square is determined by drawing lines through $\{A, C\}$ orthogonal to DB^*, etc. (see Figure 1). Point B^{**} which is symmetric to B^* relative to B gives another solution.

The two squares shown in Figure 1 are in some sense *adjoint* in that their opposite sides pass through the same couple of points out of the four $\{A, B, C, D\}$. In the rest of this section, we discuss another aspect of this separation of the six squares in couples of adjoints,

Eine beliebte Aufgabe im Geometrieunterricht verlangt, ein Quadrat zu konstruieren, dessen Seiten durch vier gegebene Punkte in allgemeiner Lage gehen. Es ist bekannt, dass es sechs solcher Quadrate gibt. Der Autor des vorliegenden Artikels stellt nun zwei Strukturen vor, die der Konfiguration dieser sechs Quadrate zugrunde liegen. Die erste Struktur besteht aus sechs Hilfsquadraten, die sich von den sechs Lösungen unterscheiden und deren Seiten die zweite Struktur tragen. Diese besteht aus sechs Parallelogrammen, von denen je zwei kongruent sind. Es wird bewiesen, dass jedes Paar kongruenter Parallelogramme durch seine Diagonalen ein Paar adjungierter Quadrate definiert, welche die Lösungen des ursprünglichen Problems darstellen.

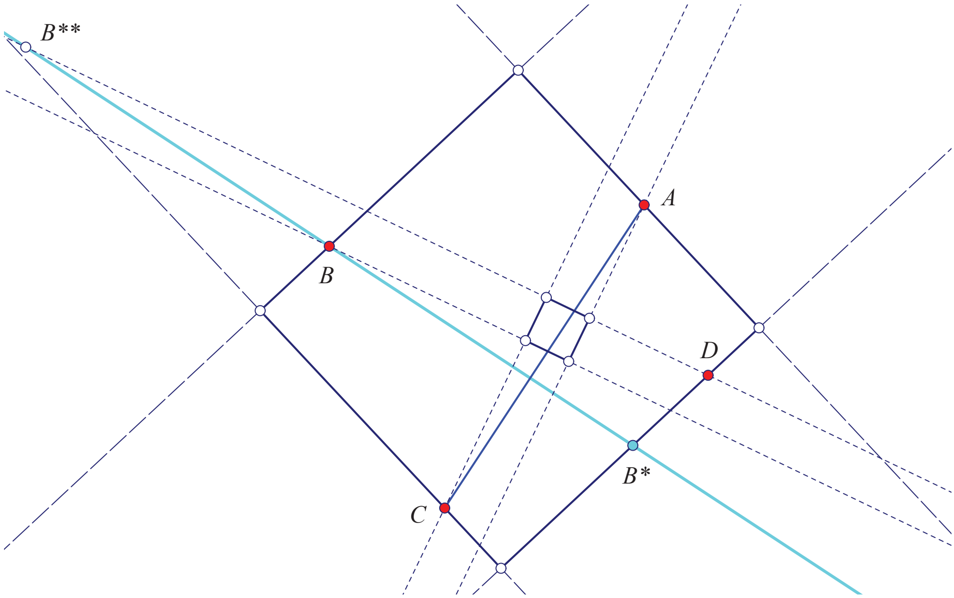

Figure 1. Two squares with sides passing through four given points

which correspond also to couples of auxiliary squares and to couples of parallelograms forming the two structures which make the core subject of this article.

The first structure consists of six auxiliary squares, other than the six solutions, whose sides carry the second structure consisting of six parallelograms congruent by two. It is proved that the two congruent parallelograms define through their diagonals a pair of *adjoint* squares representing the solutions of the original problem whose opposite sides pass through the same couples of points.

Returning to the procedure producing Figure 1 and doing the permutation denoted by $(A)(B)(C, D)$, which fixes $\{A, B\}$ and interchanges $\{C, D\}$, analogously the permutation $(B)(D)(A, C)$, and repeating the preceding constructions for each set of permuted vertices, we get six solutions displayed in Figure 2, well known to represent all the solutions of the problem for points $\{A, B, C, D\}$ in general position.

Looking at this figure, one can ask which structures underlie these six squares and how their mutual positions and the position of each relative to the given four-point $ABCD$ are related. This question is at the core of this article, and the whole discussion is based on the following simple fact.

A quadruple of points $\{A, B, C, D\}$ in general position defines three pairs of segments with no common ends,

$$(AB, CD), \quad (BC, AD), \quad (AC, BD).$$

If we choose two of these pairs to represent opposite sides, then the remaining pair represents the diagonals of the respective quadrangle. It is then easy to see that each square out of these six, *leaves out* one of these three pairs.

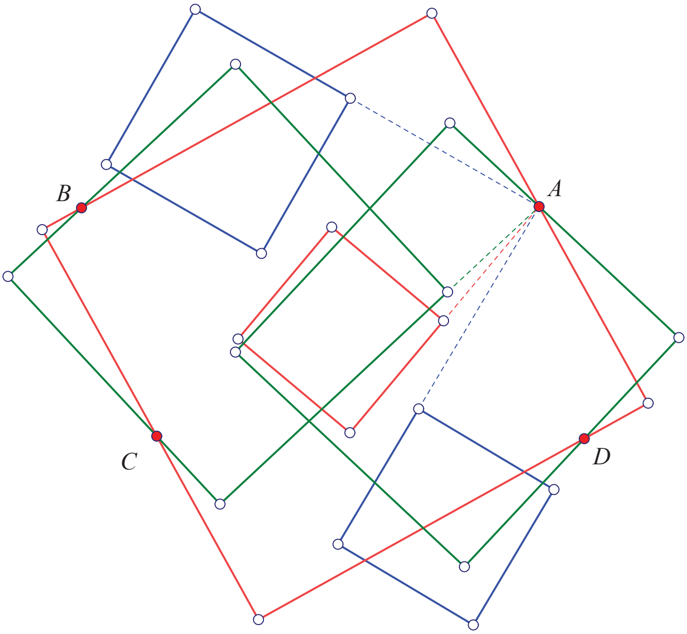

Figure 2. *Pairs of adjoint squares* through four given points $\{A, B, C, D\}$

This is meant in the following sense: in a definite square, like the biggest one in the figure say, its right angles (or their supplements or vertical angles in general) are viewing the four segments of two pairs representing the sides of a quadrangle, whereas the remaining pair of segments, representing the diagonals of that quadrangle, is not viewed in this manner. In the example of the biggest square of the figure, the segments viewed by the right angles of the square are those of the pairs $\{(AB, CD), (BC, AD)\}$. The segments of the pair (AC, BD), representing the diagonals of the quadrangle, have instead, each, their endpoints on two opposite sides of the square.

Thus, the set of six squares splits in subsets of two *adjoint* squares, each subset *leaving out* the same pair of segments. In the example of the biggest square of the figure, its adjoint is the smallest square, whose two opposite angles view the segments (AB, CD), the verticals of the two other opposite right angles viewing (BC, AD). In the following, we will study the set of two *adjoint* squares and see how the three sets of adjoints are related. First, however, we examine an alternative way to construct the squares and a couple of auxiliary lemmata.

2 Half-circles on the sides

The next lemma is a variant of a proposition attributed by F. G.-M. [2, p. 519] to Collignon. It brings into the play the circles having the sides of a quadrangle as diameters and with them an alternative way to construct the circumscribed squares.

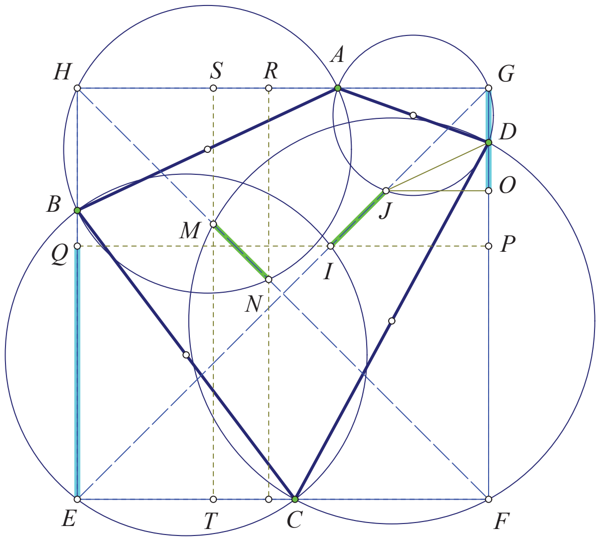

Figure 3. Equal segments

Lemma 1. *Let $EFGH$ be a square circumscribing the quadrangle $ABCD$. The diagonals $\{EG, FH\}$ of the square intersect the circles with diameters opposite sides of the quadrangle at points defining equal segments $\{IJ, MN\}$ (see Figure 3).*

Proof. Project J at O on the side FG of the square. Because J is the middle of the arc (AJD) of the circle on diameter AD and $JA = JD$, we have

$$GO = GJ\cos(\pi/4) = AD\sin(\widehat{GDJ})\cos(\pi/4) = AD\frac{JO\cos(\pi/4)}{JD} = \frac{AD \cdot JG}{2 \cdot JD}.$$

Using in the last expression Ptolemy's theorem,

$$AD \cdot GJ = GA \cdot JD + AJ \cdot GD = JD(GA + GD) \implies GO = \frac{1}{2}(GA + GD).$$

The equalities

$$EQ = \frac{1}{2}(EB + EC), \quad HR = \frac{1}{2}(HA + HB), \quad FT = \frac{1}{2}(FD + FC)$$

are proved analogously. Denoting the length of the side of the square by d, these equalities imply

$$\begin{aligned} HR + FT = d + SR &= \frac{1}{2}(HB + HA) + \frac{1}{2}(FD + FC), \\ GO + EQ = d - OP &= \frac{1}{2}(GA + GD) + \frac{1}{2}(EB + EC) \\ \implies 2d + (SR - OP) &= \frac{1}{2}(4d) \implies SR = OP \implies MN = IJ. \end{aligned}$$

∎

The lemma shows an alternative method to the one mentioned in the introduction, to construct circumscribing squares about a given quadrangle: draw the circles with diameters two opposite sides of the quadrangle and join the middles of the half-circle arcs, as e.g. does the segment IJ seen in the figure. The line defined by such a segment intersects a second time the two circles and defines a diagonal of the square, as e.g. the diagonal EG of the figure. Having the diagonal, the construction of the corresponding square is easily accomplished.

The next simple lemma and the definition related to it prepare the way to select in a systematic way the appropriate middles of half-circle arcs and produce by this method the six squares.

Lemma 2. *A quadruple of points $\{A, B, C, D\}$ in general position defines three pairs of segments with no common ends: $\{(AB, CD), (BC, AD), (AC, BD)\}$. Select two of them, for example $\{(BC, AD), (AC, BD)\}$, and consider the four circles with diameters these segments. Then the four other than the segment-endpoint intersections of circles corresponding to segments of different pairs lie by two on the lines of the remaining pair of segments (see Figure 4), $C', D' \in \varepsilon = AB$, $A', B' \in \varepsilon' = CD$, and $A'B'/C'D' = AB/CD$.*

Hint: Proof by the figure, in which the circle with diameter XY is denoted by (XY). As stated, we consider only intersections $\{A', B', C', D'\}$ of circles with diameters not pertaining to the same pair of segments. Thus, the intersections $(AC) \cap (BD)$ are not considered since $\{(AC, BD)\}$ belong to the same pair.

Definition 1. We call points $\{A', B', C', D'\}$ *two-pair cuts* of the two pairs of segments $\{(BC, AD), (AC, BD)\}$. We call also the eight middles of the half-circle arcs defined by these two pairs of segments *two-pair middles*.

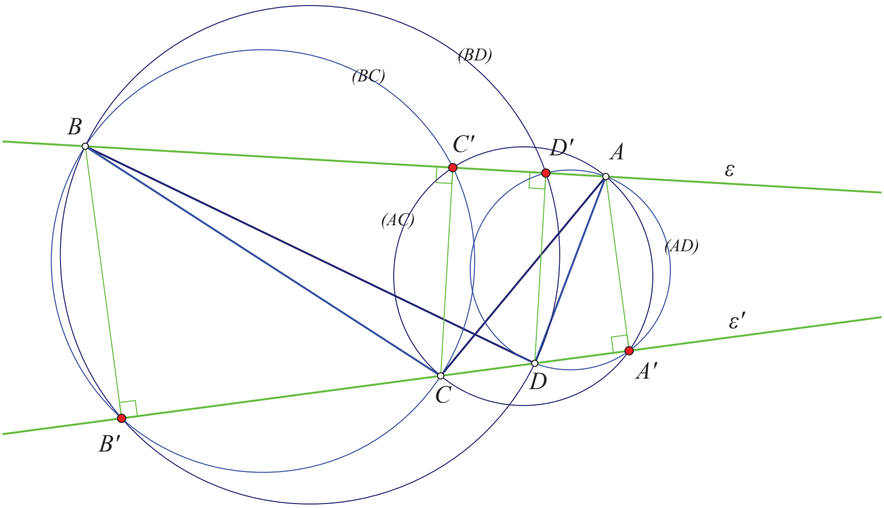

Figure 4. Collinear intersections of circles

The next section discusses the structure underlying such a set of *two-pair middles*.

3 The structure of two-pair middles

Theorem 1. *With the notation of Lemma 2 and Definition 1, we consider the* two-pair cuts $\{A', B', C', D'\}$ *of the pairs of segments in general position* $\{(BC, AD), (AC, BD)\}$ *and the corresponding* two-pair middles $\{K, L, M, N, P, Q, S, T\}$ *(see Figure 5). There results a configuration with the following properties:*

(1) *At each* two-pair cut, *out of the four* $\{A', B', C', D'\}$, *two lines passing, each, through two* two-pair middles *intersect orthogonally.*

(2) *The eight lines of* (1), *taken by four, define two squares, each square having two diagonal vertices coinciding with two two-pair cuts.*

(3) *The side lines of the two squares of* (2), *taken by four, define two equal parallelograms, each having vertices four two-pair middles and each resulting from the other by a right-angle rotation.*

(4) *The side ratio of the two squares is equal to the ratio of the segments of the remaining third pair of segments.*

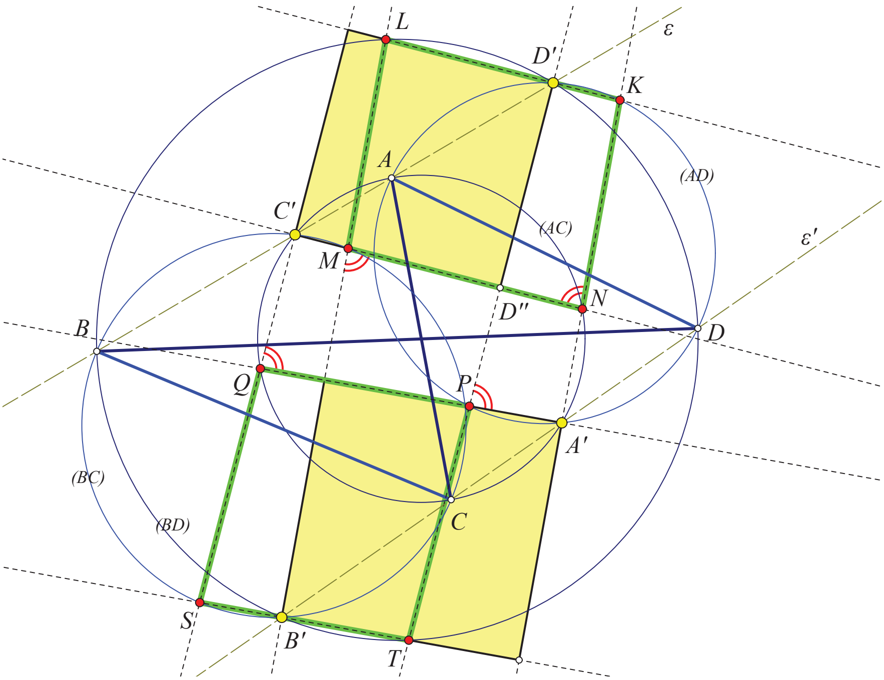

Figure 5. Configuration created by the *two-pair middles* $\{K, L, M, N, P, Q, S, T\}$

Proof. We consider the lines $\{\varepsilon = AB,\ \varepsilon' = BC\}$ of Lemma 2 and the angles they form with lines joining *two-pair middles*.

(1) Consider one such *two-pair cut*, $C' \in (BC) \cap (AC)$ say. Since $\{S, M\}$ are two-pair middles of the same circle (BC), we have

$$\widehat{SC'M} = \pi/2 \quad \text{and} \quad \widehat{BC'S} = \pi/4.$$

By the same argument and considering now C' as a point of circle (AC), we have

$$\widehat{QC'N} = \pi/2 \quad \text{and} \quad \widehat{NC'A} = \pi/4.$$

This shows that $M \in C'N$ and $Q \in C'S$ and proves the claim for the two-pair cut C'. The proof for the other two-pair cuts $\{A', B', D'\}$ is completely analogous.

(2) follows from (1) and the fact that the lines intersecting at $\{C', D' \in \varepsilon\}$ are inclined with respect to ε by an angle of measure $\pi/4$.

(3) The parallelogram property follows directly from (2). The equality of the angles of the parallelograms follows also directly from the fact that their sides are correspondingly orthogonal. For the same reason, their sides are equally inclined to corresponding sides of the square. This implies also the equality of corresponding sides of the parallelograms. For example, side NK is equal to QP since

$$NK = D'D''/\sin(\phi) \quad \text{and} \quad QP = C'D''/\sin(\phi) \quad \text{with } \phi = \widehat{C'NK} = \widehat{A'QC'}.$$

The rotation claim follows immediately from the preceding arguments. In fact, rotating $QSPT$ about Q by a right angle, we obtain two equal parallelograms with parallel corresponding equal sides. Then, translating by QM, we identify them. Since the composition of a rotation and a translation is a rotation by the same angle, we have the proof of the property.

(4) The side ratio of the two squares is equal to the ratio of their diagonals $A'B'/C'D'$, and this, by Lemma 2, is equal to AB/CD. ∎

Notice that the right-angle rotational equivalence of the two created parallelograms $\{KLMN, PQST\}$ implies the equality and orthogonality of their corresponding diagonals and gives an alternative proof of Lemma 1 without the use of any computation.

The next section discusses the creation of the two *adjoint* squares related to these two pairs of equal and orthogonal diagonals.

4 Adjoint squares

Theorem 2. *With the notation and conventions adopted so far, each pair of equal and orthogonal diagonals of the parallelograms of Theorem* 1 (3) *defines a square whose sides pass through the four points* $\{A, B, C, D\}$ *(see Figure* 6*).*

Proof. We discuss the proof for the pair $\{LN, PS\}$ of orthogonal diagonals of the two parallelograms. The proof for the other pair of diagonals $\{KM, QT\}$ is completely analogous.

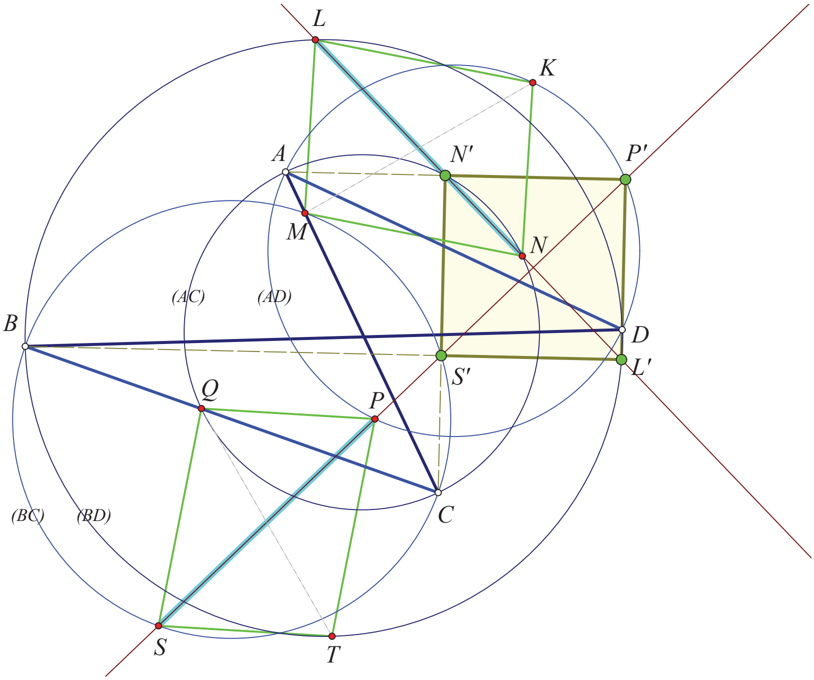

Figure 6. Square defined by two equal and orthogonal diagonals $\{LN, PS\}$

For this, extend the diagonal LN to cut again circle (BD) at L' and the diagonal SP to cut again the circle (AD) at P'. The angles

$$\widehat{BL'D} = \widehat{AP'D} = \pi/2$$

are right angles and the lines $\{L'L, P'P\}$ are respectively bisecting them. Thus, considering the intersections $N' = AP' \cap LL'$ and $S' = BL' \cap SP'$, we easily see that $\{L', D, P'\}$ are collinear and $L'P'N'S'$ is a square. It follows also that $S' \in (BC)$ and $N' \in (AC)$. This implies that the side lines of the square pass respectively through $\{A, B, C, D\}$, and its vertices are viewing the four segments (BC, AD) and (AC, BD) under a right angle, i.e., the square $P'N'S'L'$ belongs to the pair of adjoint squares which *leave out* the segments (AB, CD). ■

Figure 7 gives an overview of the structures underlying the construction of the two *adjoint squares "right viewing"* the couple of pairs of segments $\{(BC, AD), (AC, BD)\}$ and *leaving out* the pair (AB, CD). Their sides extended pass through $\{A, B, C, D\}$, and their right angles or supplements or verticals view the segments $\{(BC, AD), (AC, BD)\}$. Analogous configurations result for the other pairs of *adjoint* squares leaving out correspondingly the pairs of segments (BC, AD) and (AC, BD).

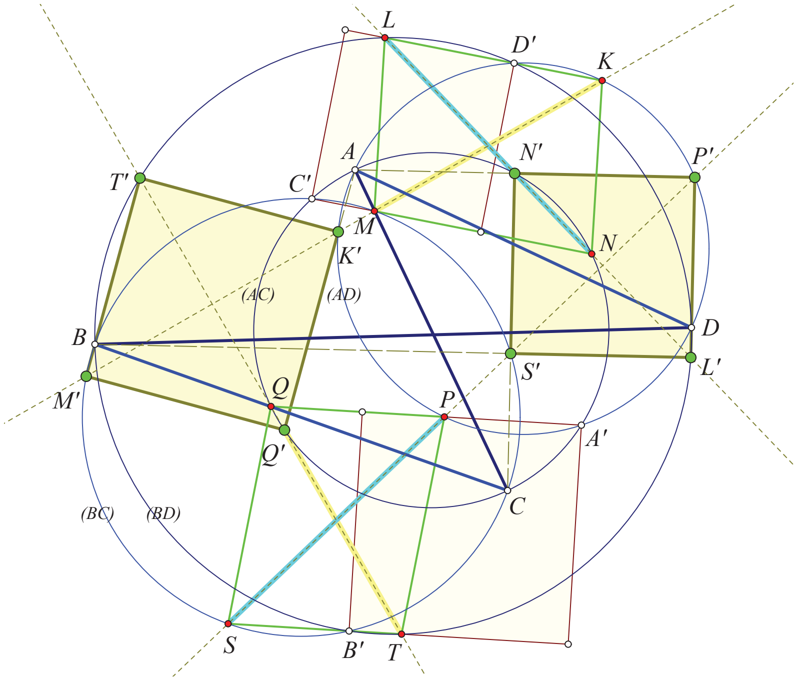

Figure 7. Two *adjoint* squares $\{P'N'D'L', K'T'M'Q'\}$

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