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Short note

A note on rings in which each element is a sum of two idempotents

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Abstract. Let R be a ring in which each element of R is a sum of two idempotents. In this short paper, we exhibit that if R is isomorphic to the direct product of two rings R_1 and R_2, then the characteristic of R_1 is not necessarily two.

1 Introduction

Throughout this paper, R is an associative ring with unity. The notation $R = R_1 + R_2$ means R_1 and R_2 are subrings of R and, for every $r \in R$, there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that $r = r_1 + r_2$. Many authors studied problems and relationships among properties of R_1, R_2 and R in the literature. Recently, a general method of constructing rings which are sums of two subrings was invented by Koşan and Zemlicka [2].

In [3, Proposition 6.1], the authors gave a structure theorem for the rings for which every element is a sum of two idempotents, though some partial results were obtained in [1]: every element of R is a sum of two idempotents if and only if $R \cong R_1 \times R_2$, where $\operatorname{char}(R_1) = 2$ and every element of R_1 is a sum of two idempotents, and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s.

One may note that an element $a \in R$ is called idempotent if $a^2 = a$. In this note, we produce a counterexample for [3, Proposition 6.1]. We exhibit the following observation.

Theorem. *There exists a ring R with the following properties.*

(i) *Each element of the ring is the sum of two idempotent elements.*

(ii) *For any decomposition $R = R_1 \times R_2$, the characteristic of R_1 is different from two.*

2 Proof of the theorem

Proof. Consider

$$R := \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}.$$

The following is easy to check.

(i) R is a (commutative) ring of characteristic three under the addition and the multiplication of matrices modulo three.

(ii) We have

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

i.e., each element of R is the sum of two idempotents.

(iii) Given any decomposition $R \cong R_1 \times R_2$ of the ring R, the characteristic of R_1 cannot be two because the order of R is nine and its characteristic is three.

This can be seen as follows. Each ring of characteristic two has even order, and the least order of a ring of characteristic two is two. Therefore, a ring of characteristic two has at least one non-zero element, say, a such that $2a = 0$. If R_1 is a ring of characteristic two, then R must contain a non-zero element, say, $b = (a, 0)$ such that $2b = 0$. But a ring of order nine cannot contain a non-zero element b such that $2b = 0$. In other words, if $R \cong R_1 \times R_2$ and the characteristic of R_1 is two, then R must contain an isomorphic copy of R_1, but a ring of order nine cannot contain an isomorphic copy of a ring of even order. Therefore, the characteristic of R_1 cannot be two in this case.

Thus, the proof is complete. ■

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References

[1] Y. Hirano and H. Tominaga, Rings in which every element is the sum of two idempotents. *Bull. Austral. Math. Soc.* **37** (1988), no. 2, 161–164 Zbl 0688.16015 MR 930784

[2] M. T. Koşan and J. Žemlička, On finite dimensional algebras which are sums of two subalgebras. *Int. Electron. J. Algebra* **26** (2019), 131–144 Zbl 1469.16044 MR 3985913

[3] Z. Ying, T. Koşan, and Y. Zhou, Rings in which every element is a sum of two tripotents. *Canad. Math. Bull.* **59** (2016), no. 3, 661–672 Zbl 1373.16067 MR 3563747

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