J. Eur. Math. Soc. 26, 4441–4490 (2024) DOI 10.4171/JEMS/1349

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A gradient flow approach to the Boltzmann equation

Received March 29, 2021

Abstract. We show that the spatially homogeneous Boltzmann equation evolves as the gradient flow of the entropy with respect to a suitable geometry on the space of probability measures which takes the collision process into account. This gradient flow structure allows to give a new proof for the convergence of Kac's random walk to the homogeneous Boltzmann equation, exploiting the stability of gradient flows.

Keywords. Boltzmann equation, spatially homogeneous, gradient flow, entropy, Kac system, propagation of chaos

1. Introduction

Since the pioneering work of Otto [30] it has been known that many diffusion equations can be cast as gradient flows of entropy functionals on the space of probability measures. The relevant geometry is given by the L^2-Wasserstein distance. This approach has been used for a variety of equations as a powerful tool in the study of the trend to equilibrium, stability questions and construction of solutions. In each case – as a direct consequence of the gradient flow structure – the driving entropy functional is non-increasing along the solution. One of the most emblematic dissipative evolution equations is the Boltzmann equation modelling the evolution of a dilute gas under elastic collisions of the particles, and Boltzmann's famous H-theorem asserts that the entropy is non-increasing along its solutions. However, uncovering a gradient flow structure for this equation has been an open problem since [30].

In this article we provide a solution and give a characterization of the spatially homogeneous Boltzmann equation as a gradient flow of the entropy. The crucial new insight is the identification of a novel geometry on the space of probability measures that takes the collision process between particles into account. Our main motivation to consider this gradient structure stems from the Kac program, in particular the propagation of chaos for Kac's stochastic many particle systems and its convergence to the homogeneous Boltzmann equation. We provide a new proof of this result by exhibiting a gradient

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Mathematics Subject Classification (2020): Primary 35Q20; Secondary 60K35, 82C40

flow structure also for the Kac system and showing that it Γ-converges to our gradient structure for the Boltzmann equation in the spirit of Sandier–Serfaty [33].

1.1. Homogeneous Boltzmann equation and gradient flow structure

We consider the spatially homogeneous Boltzmann equation

$$\partial_t f = Q(f), \tag{1.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}_+$ is a probability density and Q denotes the Boltzmann collision operator given by

$$Q(f) = \int_{\mathbb{R}^d} \int_{S^{d-1}} [f' f'_* - f f_*] B(v - v_*, \omega) \, \mathrm{d}v_* \, \mathrm{d}\omega. \tag{1.2}$$

Here B is the collision kernel and v, v_* and v', v'_* denote the pre- and post-collisional velocities respectively which are related according to

$$v' = v - \langle v - v_*, \omega\rangle \omega, \quad v'_* = v_* + \langle v - v_*, \omega\rangle \omega, \quad \omega \in S^{d-1}, \tag{1.3}$$

and we will often use the notation $f = f(v)$, $f_* = f(v_*)$, $f' = f(v')$, $f'_* = f(v'_*)$. We consider regularized collision kernels with cutoff, more precisely, we assume that $B(v - v_*, \omega)$ is bounded away from zero and comparable to $(1 + |v - v_*|^2)^{\gamma/2}$ for some $\gamma \in (-\infty, 1]$; see Assumption 2.1 for more details.

Boltzmann's H-theorem asserts that the entropy $\mathcal{H}(f) = \int f \log f$ is non-increasing along solutions to the Boltzmann equation, more precisely, we have $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f_t) = -D(f_t) \leq 0$, where

$$D(f_t) = \frac{1}{4} \int \log \frac{f' f'_*}{f f_*} (f' f'_* - f f_*) B(v - v_*, \omega) \, \mathrm{d}\omega \, \mathrm{d}v_* \, \mathrm{d}v. \tag{1.4}$$

Let us now give a heuristic description of the gradient flow structure of the Boltzmann equation. We recall that the gradient flow of a function E on a Riemannian manifold M is given as $\dot{x}_t = -\nabla E(x_t) = -K_{x_t} DE(x_t)$ with DE being the differential of E and $K_x : T_x^* M \to T_x M$ the canonical map from the cotangent to the tangent space induced by the Riemannian metric.

For the Boltzmann equation we formally take the manifold to be the set $\mathcal{P}(\mathbb{R}^d)$ of probability densities on $\mathbb{R}^d$ and the driving functional to be the entropy $\mathcal{H}$. Its differential $D\mathcal{H}(f)$ at f is given as $\log f = \frac{\delta \mathcal{H}}{\delta f}$ in the sense that for any tangent vector, i.e. a function s with $\int s(v) \, \mathrm{d}v = 0$, we have $\lim_{\varepsilon \to 0} \varepsilon^{-1} [\mathcal{H}(f + \varepsilon s) - \mathcal{H}(f)] = D\mathcal{H}(f)[s] = \int \log f(v) s(v) \, \mathrm{d}v$. Identifying the gradient flow structure of the Boltzmann equation requires identifying the right geometry on the set $\mathcal{P}(\mathbb{R}^d)$ given in terms of a suitable map K. This is achieved by introducing the *Onsager operator* $\mathcal{K}_f^B$ given by

$$\mathcal{K}_f^B \varphi(v) = -\int \bar{\nabla}\varphi \Lambda(f) B(v - v_*, \omega) \, \mathrm{d}v_* \, \mathrm{d}\omega. \tag{1.5}$$

Here we have set $\bar{\nabla}\varphi = \varphi' + \varphi'_* - \varphi - \varphi_*$ and $\Lambda(f)$ is shorthand for $\Lambda(ff_*, f'f'_*)$, where $\Lambda(s,t) = (s-t)/(\log s - \log t)$ denotes the logarithmic mean. Now the Boltzmann equation can be written as

$$\partial_t f = Q(f) = -\mathcal{K}_f^B D\mathcal{H}(f),$$

giving the desired gradient flow structure.

This gradient flow interpretation of the Boltzmann equation can also be expressed by the following *variational characterization*. Denoting by $\langle\cdot,\cdot\rangle_f$ the Riemannian metric at f we find for any curve (f_t) of probability densities that

$$\mathcal{H}(f_T) - \mathcal{H}(f_0) = \int_0^T \langle\nabla\mathcal{H}(f_t), \partial_t f\rangle_{f_t}\,\mathrm{d}t \geq -\frac{1}{2}\int_0^T [|\nabla\mathcal{H}(f_t)|^2_{f_t} + |\partial_t f|^2_{f_t}]\,\mathrm{d}t. \tag{1.6}$$

Moreover, equality holds if and only if $\partial_t f = -\nabla\mathcal{H}(f_t)$, i.e. (f_t) is the gradient flow of the entropy, hence the solution to the Boltzmann equation. In this sense, the Boltzmann equation is a steepest descent flow decreasing the entropy *as fast as possible*.

Our first main result is a rigorous implementation of this variational characterization. To this end we replace the formal norm of the gradient and the speed of the curve with suitable notions. Note that

$$|s|_f^2 = \int \varphi\mathcal{K}_f^B\varphi = \frac{1}{4}\int |\bar{\nabla}\varphi|^2\Lambda(f)B(v-v_*,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v$$

with φ such that $\mathcal{K}_f^B\varphi = s$ and where we have symmetrized over v, v_*, v', v'_*. In particular, the dissipation (1.4) takes the role of norm of the gradient, i.e. $|\nabla\mathcal{H}(f)|_f^2 = \int \log f\,\mathcal{K}_f^B \log f = D(f)$.

In order to define the notion of speed of a curve $(f_t)_t$, we first consider the equation

$$\partial_t f(v) = \mathcal{K}_{f_t}^B\psi_t(v) = -\int \bar{\nabla}\psi_t\Lambda(f)B(v-v_*,\omega)\,\mathrm{d}v_*\,\mathrm{d}\omega. \tag{1.7}$$

We perform a change of variables, setting $U_t(v,v_*,\omega) = \bar{\nabla}\psi_t\Lambda(f)B(v-v_*,\omega)$ so that (1.7) becomes linear in (f,U) and reads for all test functions φ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\varphi f_t = \frac{1}{4}\int\bar{\nabla}\varphi U_t. \tag{1.8}$$

This will be called *collision rate equation* since U governs the evolution of the density f by prescribing the rate at which collisions happen between the particles. Now, the quantity $\int_0^T |\partial_t f|_f^2\,\mathrm{d}t$ will be replaced by the action

$$\mathcal{A}_T(f) := \inf\left\{\frac{1}{4}\int_0^T\int\frac{|U_t|^2}{\Lambda(f_t)B}\,\mathrm{d}t\right\}, \tag{1.9}$$

where the infimum is over all $(U_t)_t$ satisfying the collision rate equation (1.8). See Section 3 for the precise construction where we study (1.8) and (1.9) in a natural measure-valued setting. Under Assumption 2.1 on B we then have the following variational characterization; see Theorem 4.3 below.

Theorem 1.1. *For any curve* $(f_t)_{t\in[0,T]}$ *of probability densities with* $\mathcal{H}(f_0) < \infty$ *and bounded moment of order* $2 + \max(0, \gamma)$ *we have*

$$J_T(f) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \frac{1}{2}\int_0^T D(f_t)\,\mathrm{d}t + \frac{1}{2}\mathcal{A}_T(f) \geq 0.$$

Moreover, $J_T(f) = 0$ *if and only if* $(f_t)_t$ *is a solution to the homogeneous Boltzmann equation starting from* f_0.

We remark that this result can be recast in the framework of gradient flows in metric spaces as developed in [1]. In particular, it is possible to construct the Riemannian distance $\mathcal{W}_B$ on $\mathcal{P}(\mathbb{R}^d)$ associated with the Onsager operator $\mathcal{K}^B$. We explore this point of view in the appendix.

We will also discuss a generalization of the previous theorem giving variational characterizations of the Boltzmann equation in terms of so-called *generalized gradient structures*. To this end one considers a pair of primal and dual dissipation potentials $\mathcal{R}(f, \varphi)$ and $\mathcal{R}^*(f, \xi)$ that are convex conjugate in the second variable. Then similar to (1.6) we find formally for any curve (f_t) of densities that

$$\begin{aligned}\mathcal{H}(f_T) - \mathcal{H}(f_0) &= \int_0^T \langle D\mathcal{H}(f_t), \partial_t f\rangle\,\mathrm{d}t\\ &\geq -\int_0^T [\mathcal{R}(f_t, \partial_t f) + \mathcal{R}^*(f_t, -D\mathcal{H}(f_t))]\,\mathrm{d}t.\end{aligned}$$

Equality is attained if and only if

$$\partial_t f = D_\xi \mathcal{R}^*(f, -D\mathcal{H}(f)). \tag{1.10}$$

Hence the latter evolution is characterized as minimizer of the functional

$$\mathcal{L}_T(f) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \int_0^T [\mathcal{R}(f_t, \partial_t f) + \mathcal{R}^*(f_t, -D\mathcal{H}(f_t))]\,\mathrm{d}t. \tag{1.11}$$

Under suitable compatibility assumptions on $\mathcal{R}$ and $\mathcal{H}$, the resulting evolution (1.10) is indeed the Boltzmann equation. One choice for $\mathcal{R}(f, \partial_t f)$ and $\mathcal{R}^*(f, -D\mathcal{H}(f))$ are the quadratic expressions

$$\tfrac{1}{2}|\partial_t f|_f^2 = \tfrac{1}{2}\langle \partial_t f, \mathcal{K}_f^B \partial_t f\rangle \quad \text{and} \quad \tfrac{1}{2}|\nabla\mathcal{H}(f)|_f^2 = \tfrac{1}{2}\langle D\mathcal{H}(f), \mathcal{K}_f^B D\mathcal{H}(f)\rangle$$

by which we recover the gradient flow structure already discussed. One compelling motivation for such generalized gradient structures comes from the fact that in many situations they arise naturally from the analysis of large deviations for an underlying microscopic particle system whose limiting behavior is described by (1.10). Namely, the functional $\mathcal{L}_T$ appears as the rate function for large deviations on the path level; see e.g. [26] for an in depth discussion. In the construction of the generalized gradient structure, we follow the approach of [31], where such structures have been analyzed in detail in the context of jump processes.

In the present setting we obtain the following result. Fix a pair of even, lower semi-continuous convex conjugate functions $\Psi, \Psi^* : \mathbb{R} \to [0,\infty)$ with $\Psi(0) = \Psi^*(0) = 0$ and a 1-homogeneous concave function $\theta : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that the compatibility condition

$$(\Psi^*)'(\log s - \log t)\theta(s,t) = s - t \quad \forall s,t > 0$$

holds (see Assumption 4.4 for additional assumptions on Ψ^* and θ). Set

$$\mathcal{R}(f,U) := \frac{1}{4}\int \Psi\Big(\frac{U}{\theta(f)B}\Big)\theta(f)B \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega,$$
$$\mathcal{D}_{\Psi^*}(f) := \mathcal{R}^*(f,-DH(f)) := \frac{1}{4}\int \Psi^*(-\bar{\nabla}\log f)\theta(f)B\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

where we have set $\theta(f) := \theta(ff_*, f'f_*')$. Then we have (see Theorem 4.6 below):

Theorem 1.2. *For any curve (f_t) of probability densities with $\mathcal{H}(f_0) < \infty$ and bounded moment of order $2 + \max(0,\gamma)$ and (U_t) such that the collision rate equation (1.8) holds we have*

$$\mathcal{L}_T(f,U) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \int_0^T [D_{\Psi^*}(f_t) + \mathcal{R}(f_t,U_t)]\,\mathrm{d}t \geq 0. \tag{1.12}$$

Moreover, $\mathcal{L}_T(f,U) = 0$ if and only if (f_t) is a solution to the homogeneous Boltzmann equation and $U_t = (ff_* - f'f_*')B$.

This generalized gradient structure encompasses in particular the previous quadratic structure by choosing $\theta = \Lambda$ (the logarithmic mean) and $\Psi(\xi) = \Psi^*(\xi) = \frac{1}{2}|\xi|^2$. Another particular choice of interest is

$$\theta(s,t) = \sqrt{st}, \quad \Psi^*(\xi) = 4(\cosh(\xi/2) - 1).$$

This particular variational structure seems to have been explicitly identified for the first time by Grmela, see e.g. [18, Eq. (A7)]. For jump processes a similar structure is connected with the large deviations on the path level for the empirical measure of a growing number of independent particles; see e.g. [31]. Here, the resulting structure can be related, at least formally, to the large deviations of the Kac particle system that we describe below.

1.2. Consistency for Kac's random walk

A central motivation for considering the gradient flow structure just described is to give a new proof of the convergence of Kac's random walk to the solution of the spatially homogeneous Boltzmann equation. Kac introduced his random walk in the seminal work [21] as a probabilistic model for N colliding particles. It is a continuous time Markov chain on the set $\mathcal{X}_N$ of N velocities with fixed momentum and energy,

$$\mathcal{X}_N := \Big\{(v_1,\ldots,v_N) \in \mathbb{R}^{dN} \,\Big|\, \sum_{i=1}^N v_i = 0,\ \sum_{i=1}^N |v_i|^2 = Nd\Big\}.$$

In each step, two uniformly chosen particles i, j collide, i.e. $\boldsymbol{v}$ is updated to $R_{ij}^{\omega}\boldsymbol{v} = (v_1, \dots, v_i', \dots, v_j', \dots, v_N)$ where $v_i' = v_i - \langle v_i - v_j, \omega\rangle\omega$ and $v_j' = v_j + \langle v_i - v_j, \omega\rangle\omega$ with a random collision parameter $\omega \in S^{d-1}$ distributed according to $B(v_i - v_j, \cdot)$. The rate is chosen in such a way that on average, N collisions occur per unit of time. More precisely, the generator of the Markov chain is given by

$$Af(\boldsymbol{v}) = \frac{1}{2N}\int_{S^{d-1}} \sum_{i,j} [f(R_{ij}^{\omega}\boldsymbol{v}) - f(\boldsymbol{v})]B(v_i - v_j, \omega)\,\mathrm{d}\omega. \tag{1.13}$$

The Markov chain is reversible with respect to the Hausdorff measure π_N on $\mathcal{X}_N$. If we denote by μ_t^N the law of the Markov chain starting from μ_0^N, then its density f_t^N with respect to π_N satisfies Kac's master equation $\partial_t f_t^N = Af_t^N$.

A natural way to study the convergence of Kac's random walk to the Boltzmann equation is via its empirical measures $L_N(\boldsymbol{v}) = \frac{1}{N}\sum_{i=1}^N \delta_{v_i} \in \mathcal{P}(\mathbb{R}^d)$. We will show the following:

Theorem 1.3. *Let B satisfy Assumption 2.1. For each N let $(\mu_t^N)_{t\ge 0}$ be the law of Kac's random walk starting from μ_0^N and denote by $c_t^N := (L_N)_\#\mu_t^N \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ the law of its empirical measures. Assume that μ_0^N is well-prepared for some $\nu_0 = f_0\mathcal{L} \in \mathcal{P}(\mathbb{R}^d)$ with $\mathcal{H}(\nu_0) < \infty$ (if $\gamma > 0$ assume in addition finite fourth moment of ν_0) in the sense that as $N \to \infty$,*

$$c_0^N \rightharpoonup \delta_{\nu_0}, \quad \frac{1}{N}\mathcal{H}(\mu_0^N|\pi_N) \to \mathcal{H}(\nu_0|M).$$

Assume further that for some $p > 2 + \max(0, \gamma)$,

$$\sup_N \langle \mathcal{E}_p^N, \mu_0^N\rangle < \infty, \quad \mathcal{E}_p^N(\boldsymbol{v}) := \frac{1}{N}\sum_{i=1}^N |v_i|^p.$$

Then, for all $t > 0$, as $N \to \infty$ we have

$$c_t^N \rightharpoonup \delta_{\nu_t}, \quad \frac{1}{N}\mathcal{H}(\mu_t^N|\pi_N) \to \mathcal{H}(\nu_t|M), \tag{1.14}$$

where $\nu_t = f_t\mathcal{L}$ and f_t is the unique solution to the spatially homogeneous Boltzmann equation with initial datum f_0.

Here $\mathcal{H}(\cdot|\pi_N)$ denotes the relative entropy with respect to π_N and $\mathcal{H}(\cdot|M)$ the relative entropy with respect to the standard Gaussian density M in $\mathbb{R}^d$. Note that the well-preparedness assumption is satisfied for instance if the initial velocities are independent, i.e. $\mu_0^N = \nu_0^{\otimes N}$. An important feature of Kac's model is the *propagation of chaos*: if the initial distribution of velocities is asymptotically independent as $N \to \infty$ then the same holds for all times. One way of making this precise is the convergence (1.14), which is usually called *entropic propagation of chaos*. This is motivated by the fact that for a true product measure we have $\mathcal{H}(\nu^{\otimes N}) = N \cdot \mathcal{H}(\nu)$.

We point out that the previous theorem is well-known even for a larger class of collision kernels; see the references below. The contribution we make here is to provide a new angle of attack on this problem by exploiting the gradient flow structure. We will use the stability of gradient flows following the approach of Sandier–Serfaty [33]. It turns out that Kac's random walk is the gradient flow of the entropy $\mathcal{H}(\cdot|\pi_N)$ in $\mathcal{P}(\mathcal{X}_N)$ equipped with a suitable geometry, as we shall make precise in Section 5.1. In particular, the energy dissipation identity

$$J_T^N(\mu^N) = \mathcal{H}(\mu_t^N|\pi_N) - \mathcal{H}(\mu_0^N|\pi_N) + \frac{1}{2}\int_0^T D^N(\mu_t^N)\,\mathrm{d}t + \frac{1}{2}\mathcal{A}_T^N(\mu^N) = 0 \qquad (1.15)$$

holds, where D^N is the dissipation of $\mathcal{H}(\cdot|\pi_N)$ along the master equation and $\mathcal{A}_T^N(\mu^N)$ is the action. This is based on results for general Markov chains and jump processes in [15, 23, 25]. To obtain the desired convergence to the Boltzmann equation it is sufficient together with some compactness to prove convergence (in fact only lim inf estimates) for the constituent elements of the gradient flow structure, the entropy, dissipation and the action, which allows one to pass to the limit in (1.15).

1.3. Connection to the literature

For an overview of results for the spatially homogeneous Boltzmann equation, we refer to the review by Desvillettes [13]. Modifications of the Wasserstein geometry have been studied by Maas [23] and Mielke [25] who found gradient flow structures for finite Markov chains and reaction-diffusion equations. The gradient flow structure for the homogeneous Boltzmann equation obtained here is related to the discrete framework of reaction equations in [25]. Formally, the homogeneous Boltzmann equation could be seen as a binary reaction equation with a continuum of species indexed by the velocity. Recently, a gradient flow characterization for the homogeneous Landau equation has been given in [10] using a similar approach. Spatially inhomogeneous linear Boltzmann equations have been characterized variationally in [4] using a variant of the energy dissipation identity. The underlying structure is non-quadratic and inspired by large deviations. We also mention the work [5], where the large deviations of a Kac type particle system with conservation of momentum but not of energy have been determined and a corresponding generalized gradient structure for the limiting Boltzmann type equation has been established.

Theorem 1.3 on the convergence of Kac's random walk goes back to Kac [21] who proved an analogue for a simplified model with one-dimensional velocities. The first proof of convergence to the homogeneous Boltzmann equation for the model considered here is due to Sznitman [34]. In both cases more general collision kernels are considered, including in particular the case of hard spheres. Quantitative convergence results in Wasserstein distance were obtained later by Mischler–Mouhot [27], Norris [29], and Cortez–Fontbona [11]. Quantitative estimates for the entropic propagation of chaos in the Kac model, i.e. on the speed of convergence in (1.14), have been given by Carrapatoso [9]. Similar results have been obtained for the Landau equation and other related models; see e.g. [16, 17, 20]. We also mention the discussion of the relation between different quantified notions of chaoticity in [9] and in the work of Hauray and Mischler [19].

1.4. *Organization*

In Section 2 we collect the necessary preliminaries. In Section 3 we introduce the collision rate equation and the action of a curve. The characterization of the Boltzmann equation as entropic gradient flow is obtained in Section 4. In Section 5 we exhibit a gradient flow structure for Kac's random walk and prove its convergence to the Boltzmann equation.

Appendices A, B, and C contain the construction of the distance associated to the Onsager operator, a reformulation of our results in the framework of gradient flows in metric spaces, and a variational approximation scheme for the Boltzmann equation based on the gradient structure.

2. Preliminaries

2.1. *Homogeneous Boltzmann equation, entropy and dissipation*

Let $d \geq 3$. We denote by $\mathscr{P}(\mathbb{R}^d)$ the space of Borel probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence in duality with bounded continuous functions. We denote by $\mathscr{H}(\mu)$ the *Boltzmann–Shannon entropy* defined for $\mu \in \mathscr{P}(\mathbb{R}^d)$ by

$$\mathscr{H}(\mu) = \int f(v) \log f(v) \, \mathrm{d}v$$

provided $\mu = f\mathscr{L}$ is absolutely continuous with respect to Lebesgue measure $\mathscr{L}$ and $\max(f \log f, 0)$ is integrable, otherwise we set $\mathscr{H}(\mu) = +\infty$. We will also write $\mathscr{H}(f)$ if $\mu = f\mathscr{L}$.

For $p \geq 1$, let $\mathscr{P}_p(\mathbb{R}^d) = \{\mu \in \mathscr{P}(\mathbb{R}^d) : \int |v|^p \, \mathrm{d}\mu(v) < \infty\}$ denote the set of probability measures with finite moment of order p. We will write

$$\mathscr{E}_p(\mu) := \int |v|^p \, \mathrm{d}\mu(v). \tag{2.1}$$

For $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ we define by

$$\mathscr{M}(\mu) := \int v \, \mathrm{d}\mu(v), \quad \mathscr{E}(\mu) := \mathscr{E}_2(\mu) = \int |v|^2 \, \mathrm{d}\mu(v) \tag{2.2}$$

the *momentum* and *energy* of μ. For $E > 0$ we let

$$\mathscr{P}_{p,E}(\mathbb{R}^d) := \{\mu \in \mathscr{P}_p(\mathbb{R}^d) : \mathscr{E}_p(\mu) \leq E\}, \tag{2.3}$$

the set of measures with energy less than E. Note that $\mathscr{P}_{p,E}(\mathbb{R}^d)$ is compact for the weak topology. For $m \in \mathbb{R}^d$ and $E > 0$ we let

$$M^{m,E}(v) = \frac{1}{(2\pi E)^{d/2}} \exp\left(-\frac{|v-m|^2}{2E}\right),$$

denote the Maxwellian or Gaussian density distribution with momentum m and energy Ed. The *relative entropy* with respect to $M^{m,E}$ of a probability measure $\mu = f\mathscr{L}$

is defined by

$$\mathcal{H}(\mu|M^{m,E}) = \int f(v)\log\frac{f(v)}{M^{m,E}(v)}\,\mathrm{d}v. \tag{2.4}$$

For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ we have

$$\mathcal{H}(\mu) = \mathcal{H}(\mu|M^{m,E}) - \frac{1}{2}\int \frac{1}{E}|v-m|^2\,\mu(\mathrm{d}v) - \frac{d}{2}\log(2\pi E). \tag{2.5}$$

By Jensen's inequality we have $\mathcal{H}(\cdot|M^{m,E}) \geq 0$. Hence, we see that $\mathcal{H}$ is bounded below on $\mathcal{P}_{2,E}(\mathbb{R}^d)$. Moreover, $\mathcal{H}(\mu) = \mathcal{H}(\mu|M^\mu) + \mathcal{H}(M^\mu)$. Finally, $\mathcal{H}$ is lower semicontinuous on $\mathcal{P}_2(\mathbb{R}^d)$ with respect to weak convergence. This follows from the corresponding property of $\mathcal{H}(\cdot|M^{m,E})$ and lower semicontinuity of moments.

We collect some well-known results on existence and uniqueness and propagation of integrability for the homogeneous Boltzmann equation. We use the notation

$$\langle k\rangle := \sqrt{1+|k|^2}, \quad k \in \mathbb{R}^d.$$

Moreover, we use the weighted L^1 spaces $L^1_s(\mathbb{R}^d) := \{f \in L^1(\mathbb{R}^d) : \int \langle v\rangle^s |f(v)|\,\mathrm{d}v < \infty\}$.

Throughout this article we make the following assumption on the collision kernel.

Assumption 2.1. $B : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}_+$ is measurable, continuous with respect to the first variable, invariant under the transformation (1.3), and there exist constants $\gamma \in (-\infty, 1]$ and $c_B > 0$ such that for all $k \in \mathbb{R}^d$ and $\omega \in S^{d-1}$,

$$c_B^{-1}\langle k\rangle^\gamma \leq B(k,\omega) \leq c_B\langle k\rangle^\gamma. \tag{2.6}$$

Let us recall that typical choices of the collision kernel motivated on physical grounds are $B(k,\omega) = |k|^\gamma b(\alpha)$ with α the angle between k and ω. The assumption above corresponds to an angular cut-off assumption removing the typical singularity in b as well as a regularization near $k = 0$ removing the singularity for $\gamma < 0$ and ensuring boundedness away from zero for $\gamma > 0$.

Theorem 2.2. *Let $f_0 : \mathbb{R}^d \to \mathbb{R}_+$ be such that*

$$\int_{\mathbb{R}^d} \langle v\rangle^2 f_0(v)\,\mathrm{d}v < \infty, \qquad \int f_0(v)\log f_0(v)\,\mathrm{d}v < \infty.$$

If $\gamma > 0$ assume in addition that $f_0 \in L^1_4(\mathbb{R}^d)$. Then there exists a unique non-negative solution $f \in C([0,\infty); L^1(\mathbb{R}^d)) \cap L^\infty((0,\infty); L^1_2(\mathbb{R}^d))$ to the homogeneous Boltzmann equation (1.1) conserving mass, momentum and energy, i.e.

$$\int (1, v, |v|^2) f_t(v)\,\mathrm{d}v = \int (1, v, |v|^2) f_0(v)\,\mathrm{d}v \quad \forall t \geq 0.$$

Moreover, for all $t > 0$,

$$\mathcal{H}(f_t) - \mathcal{H}(f_s) = -\int_0^t D(f_r)\,\mathrm{d}r \leq 0, \tag{2.7}$$

where

$$D(f) := \int_{\mathbb{R}^{2N}}\int_{S^{d-1}} \log\frac{f'f'_*}{ff_*}[f'f'_* - ff_*]B(v-v_*,\omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega. \tag{2.8}$$

Proof. For existence, conservation of mass, momentum, and energy, as well as uniqueness we refer to [3]. The entropy identity (2.7) is proven in [22]. ■

We note that for conventional hard potential kernels of the form $B(k,\omega) = |k|^\gamma b(\theta)$, $\gamma \in (0,1]$, uniqueness of conservative solutions is known assuming only finite energy of the initial datum [28]. For general kernels as considered here we could not retrieve such an improved result in the literature.

The quantity $D(f)$ is called the *entropy dissipation*. More generally, we define the entropy dissipation $D(\mu)$ for a probability measure μ by setting $D(\mu) = D(f)$ provided $\mu = f\mathscr{L}$ is absolutely continuous, and $+\infty$ otherwise.

2.2. *Regularization by convolution*

For $t > 0$, we consider the Maxwellian distribution

$$M_t(v) = \frac{1}{(2\pi t)^{d/2}} \exp\left(\frac{|v|^2}{2t}\right),$$

and note that $\int |v|^2 M_t(v)\,\mathrm{d}v = 2t$. We write $M := M_1$.

For any non-negative $f \in L^1$ with $\|f\|_{L^1} = 1$, $M_t * f$ is C^∞ with the bounds

$$|M_t * f| \le C_t, \quad |\log M_t * f|(v) \le C_t(1+|v|^2), \tag{2.9}$$

for a suitable constant C_t (see for instance [8]).

For fixed $\omega \in S^{d-1}$ we will denote by T_ω the transformation $(v, v_*) \mapsto (v', v'_*)$ with v', v'_* given by (1.3). Note that T_ω is involutive and has unit Jacobian determinant. We will set

$$X = (v, v_*), \quad X' = (v', v'_*) = T_\omega X.$$

By abuse of notation we denote the Maxwellian distribution in $\mathbb{R}^{2d}$ again by M_t. Note that $M_t(X) := M_t(v)M_t(v_*)$. For a function $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ we will set

$$T_\omega F(X) := F(T_\omega X).$$

Convolution behaves well under tensorization. More precisely, if for a function $f : \mathbb{R}^d \to \mathbb{R}$ we set $F = f \otimes f$, i.e. $F(X) = f f_*$, then

$$F * M_t = (f * M_t) \otimes (f * M_t).$$

The following commutation relation with the pre-post-collision change of variables will be crucial. It can be found in [35, Prop. 4]. For the reader's convenience we will give the short proof.

Lemma 2.3. *Let $F : \mathbb{R}^{2d} \to \mathbb{R}$. Then for each $\omega \in S^{d-1}$ and any $t > 0$,*

$$(T_\omega F) * M_t = T_\omega(F * M_t). \tag{2.10}$$

If $F = f f_*$ we have $M_t * (f' f'_*) = (M_t * f)'(M_t * f)'_*$.

Proof. First note that $M_t(T_\omega X) = M_t(X)$, since the relation between pre- and post-collisional velocities is such that $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$. Using also the fact that T_ω is involutive with unit determinant, we find

$$((T_\omega F) * M_t)(X) = \int F(T_\omega Y) M_t(X - Y)\,\mathrm{d}Y = \int F(Y) M_t(X - T_\omega^{-1} Y)\,\mathrm{d}Y$$
$$= \int F(Y) M_t(T_\omega X - Y)\,\mathrm{d}Y = (F * M_t)(T_\omega X). \qquad ■$$

Lemma 2.4. *For any $p \in \mathbb{R}$ and $0 < \delta < 1$ we have*

$$\int_{\mathbb{R}^d} \langle w \rangle^p M_\delta(v - w)\,\mathrm{d}w \le C \langle v \rangle^p$$

for a constant C depending only on $|p|$ and on $m_{|p|}(M) = \int |v|^{|p|} M(v)\,\mathrm{d}v$.

Proof. We use the fact that for any $p \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$,

$$\frac{\langle x \rangle^p}{\langle y \rangle^p} \le 2^{|p|/2} \langle x - y \rangle^{|p|}, \tag{2.11}$$

known as *Peetre's inequality*. (2.11) can be readily checked for $p = 2$. Taking non-negative powers yields (2.11) for $p \ge 0$. Reversing the roles of x and y and taking positive powers yields the statement for $p < 0$.

Now, using (2.11) we can estimate

$$\int \langle w \rangle^p M_\delta(v - w)\,\mathrm{d}w \le 2^{|p|/2} \langle v \rangle^p \int \langle v - w \rangle M_\delta(v - w)\,\mathrm{d}w$$
$$= 2^{|p|/2} \langle v \rangle^p \int (1 + \delta^2 |w|^2)^{|p|/2} M(w)\,\mathrm{d}w,$$

and the claim readily follows. ■

2.3. *Integral functionals on measures*

We provide here basic results on integral functionals on measures that will be often used in the following.

Let $\mathcal{X}$ be locally compact Polish space. We denote by $\mathcal{M}(\mathcal{X}; \mathbb{R}^n)$ the space of vector-valued Borel measures with finite variation on $\mathcal{X}$. It will be endowed with the weak* topology of convergence in duality with $C_0(\mathcal{X}; \mathbb{R}^n)$, i.e. continuous functions vanishing at infinity. Let $f : \mathbb{R}^n \to [0, \infty]$ be a convex, lower semicontinuous, and superlinear and let σ be a non-negative finite Borel measure on $\mathcal{X}$. Define on $\mathcal{M}(\mathcal{X}; \mathbb{R}^n)$ the functional $\mathcal{F}(\cdot|\sigma)$ via

$$\mathcal{F}_f(\gamma|\sigma) = \int_{\mathcal{X}} f\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma}\right) \mathrm{d}\sigma, \tag{2.12}$$

and set $\mathcal{F}_f(\gamma|\sigma) = +\infty$ if λ is not absolutely continuous with respect to γ. Note that the definition is independent of the choice of σ if f is positively 1-homogeneous, i.e. $f(\lambda r) = \lambda f(r)$ for all $r \in \mathbb{R}^n$ and $\lambda \ge 0$. We will write $\mathcal{F}_f(\cdot)$ instead of $\mathcal{F}_f(\cdot|\sigma)$ in this case.

Lemma 2.5. (i) *$\mathcal{F}_f(\cdot|\sigma)$ is convex and sequentially lower semicontinuous with respect to weak* convergence.*

(ii) *Assume that f is 1-homogeneous. If $\mathcal{Y}$ is another locally compact Polish space and $T : \mathcal{X} \to \mathcal{Y}$ is Borel measurable, then $\mathcal{F}_f(T_\# \gamma) \le \mathcal{F}_f(\gamma)$ for all γ, where $\mathcal{F}$ is defined analogously on $\mathcal{M}(\mathcal{Y}; \mathbb{R}^n)$.*

Proof. (i) This is proven in [7, Thm. 3.4.3].

(ii) Let $\bar{\gamma}^i = T_\# \gamma^i$ and $\bar{\sigma} = T_\# \sigma$. Let $(\sigma_y)_{y \in \mathcal{Y}}$ be a disintegration of σ with respect to $\bar{\sigma}$, i.e. each σ_y is a measure on $\mathcal{X}$ such that $y \mapsto \sigma_y(E)$ is Borel measurable for all Borel sets $E \subset \mathcal{X}$, $\sigma_y(E) = \sigma_y(E \cap T^{-1}(y))$ and $\sigma_y(\mathcal{X}) = \sigma(\mathcal{X})$ for all y, and $\sigma(E) = \int \sigma_y(E)\, \mathrm{d}\bar{\sigma}(y)$. Write $\lambda = \rho\sigma$, and note that $\bar{\lambda} = \bar{\rho}\bar{\sigma}$ with $\bar{\rho}(y) := \int \rho(x)\, \sigma_y(\mathrm{d}x)$. Now put $\rho_y(x) = \rho(x)/\bar{\rho}(y)$. Then

$$\begin{aligned}\mathcal{F}_f(T_\# \gamma) &= \int_{\mathcal{Y}} f[\bar{\rho}]\, \mathrm{d}\bar{\sigma} = \int_{\mathcal{Y}} f\left[\int_{\mathcal{X}} \rho_y \,\mathrm{d}\sigma_y \bar{\rho}(y)\right] \bar{\sigma}(\mathrm{d}y)\\ &\le \int_{\mathcal{Y}} \int_{\mathcal{X}} f[\rho_y(x)\bar{\rho}(y)]\, \sigma_y(\mathrm{d}x)\, \bar{\sigma}(\mathrm{d}y) = \int f[\rho]\, \mathrm{d}\sigma = \mathcal{F}_f(\gamma),\end{aligned}$$

where we have used Jensen's inequality due to the convexity and homogeneity of α. ∎

As a first consequence we obtain

Lemma 2.6 (Lower semicontinuity of dissipation). *For any sequence (μ_n) in $\mathcal{P}(\mathbb{R}^d)$ converging weakly to μ we have*

$$D(\mu) \le \liminf_n D(\mu_n). \tag{2.13}$$

Proof. Consider the convex, lower semicontinuous, and 1-homogeneous function $G(s,t) = \frac{1}{4}(t-s)(\log t - \log s)$. For $\mu \in \mathcal{P}(\mathbb{R}^d)$ define non-negative measures $\mu^1, \mu^2 \in \mathcal{M}_+(\Omega)$ by

$$\mu^1(\mathrm{d}v, \mathrm{d}v_*, \mathrm{d}\omega) := B(v - v_*, \omega)\, \mu(\mathrm{d}v)\, \mu(\mathrm{d}v_*)\, \mathrm{d}\omega, \quad \mu^2 := T_\# \mu^1,$$

where T is the change of variables $(v, v_*, \omega) \mapsto (T_\omega(v, v_*), \omega)$ between pre- and post-collisional variables defined in (1.3). We note that

$$D(\mu) = \mathcal{G}(\mu^1, \mu^2) := \int G\left(\frac{\mathrm{d}\mu^1}{\mathrm{d}\sigma}, \frac{\mathrm{d}\mu^2}{\mathrm{d}\sigma}\right) \mathrm{d}\sigma,$$

where σ is any measure such that $\mu^1, \mu^2 \ll \sigma$. Note that by Assumption 2.1 on the collision kernel B, the weak convergence of μ_n to μ implies the weak* convergence of μ_n^i to μ^i in $\mathcal{M}(\Omega)$ for $i = 1, 2$. Now the claim follows immediately from Lemma 2.5. ∎

3. Collision rate equation and action

In this section, we rigorously define the notion of speed of a curve $(f_t)_t$ associated to the formal Onsager operator $\mathcal{K}^B$. In the next subsection we study the collision rate equation

(1.8) in a measure-valued framework replacing f_t with probability measures μ_t and U_t with a family of signed measures on $\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}$. In Section 3.2 we study the action functional (1.9) on measures and define the action of a curve.

3.1. The collision rate equation

Let us set

$$\Omega = \mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}$$

and denote by $\mathcal{M}(\Omega)$ the space of signed Borel measures with finite variation on Ω equipped with the weak* topology in duality with continuous functions vanishing at infinity. Recall that $\mathcal{P}(\mathbb{R}^d)$ denotes the space of Borel probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence in duality with bounded continuous functions.

We define solutions to the collision rate equation in the following way.

Definition 3.1 (Collision rate equation). We denote by $\mathcal{CRE}_T$ the set of all pairs $(\mu, \mathcal{U})$ satisfying the following conditions:

(i) $\mu : [0, T] \to \mathcal{P}(\mathbb{R}^d)$ is weakly continuous;

(ii) $(\mathcal{U}_t)_{t\in[0,T]}$ is a Borel family of measures in $\mathcal{M}(\Omega)$;

(iii) $\int_0^T |\mathcal{U}_t|(Y)\, \mathrm{d}t < \infty$;

(iv) for any $\varphi \in C_b(\mathbb{R}^d)$ we have, in the sense of distributions,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \,\mathrm{d}\mu_t = \frac{1}{4} \int \bar{\nabla}\varphi \,\mathrm{d}\mathcal{U}_t. \tag{3.1}$$

Moreover, we will denote by $\mathcal{CRE}_T(\bar{\mu}_0, \bar{\mu}_1)$ the set of pairs $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ satisfying in addition $\mu_0 = \bar{\mu}_0,\ \mu_1 = \bar{\mu}_1$.

Note that the integrability condition (iii) ensures that the right hand side in (iv) is well-defined. The measures $\mathcal{U}_t$ will be called *collision rates*.

Remark 3.2. If $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$, then for any $\varphi \in C_b(\mathbb{R}^d)$ and $0 \le t_0 \le t_1 \le T$ we have

$$\int \varphi \,\mathrm{d}\mu_{t_1} - \int \varphi \,\mathrm{d}\mu_{t_0} = \frac{1}{4} \int_{t_0}^{t_1} \int \bar{\nabla}\varphi \,\mathrm{d}\mathcal{U}_t \,\mathrm{d}t. \tag{3.2}$$

This follows readily from (iv) together with the continuity of $t \mapsto \mu_t$ in (i).

The curve $(\mu_t)_{t\in[0,T]}$ is also absolutely continuous with respect to the total variation norm. Indeed, from (3.2) we infer

$$\left| \int \varphi \,\mathrm{d}(\mu_{t_1} - \mu_{t_0}) \right| \le |\varphi|_\infty \int_{t_0}^{t_1} |\mathcal{U}_t|(\Omega)\,\mathrm{d}t,$$

and hence $\|\mu_{t_1} - \mu_{t_0}\|_{\mathrm{TV}} \le \int_{t_0}^{t_1} |\mathcal{U}_t|\,\mathrm{d}t$. Moreover, the distribution $\partial_t \mu_t$ on $[0, T] \times \mathbb{R}^d$ is actually a signed measure with total variation bounded by $\int_0^T |\mathcal{U}_t|(\Omega)\,\mathrm{d}t$.

Remark 3.3. The continuity equation can sometimes be tested against more general test functions. For instance, let $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ and let $\mathcal{U}$ satisfy the stronger integrability condition

$$\int_0^T \int [\langle v\rangle^p + \langle v_*\rangle^p] \,\mathrm{d}|\mathcal{U}_t| \,\mathrm{d}t < \infty \tag{3.3}$$

for some $p > 0$. Then (3.2) holds for all $\varphi : \mathbb{R}^d \to \mathbb{R}$ continuous and satisfying the growth condition $|\varphi(v)| \le c\langle v\rangle^p$. This follows immediately by approximation with functions in C_b and the trivial estimate $\langle v'\rangle^p + \langle v'_*\rangle^p \le C_p(\langle v\rangle^p + \langle v_*\rangle^p)$. If μ_t has density f_t with respect to Lebesgue measure, we infer as above that

$$\left|\int \langle v\rangle^p \varphi(v)(f_{t_1}(v) - f_{t_0}(v)) \,\mathrm{d}v\right| \le C|\varphi|_\infty \int_{t_0}^{t_1} \int [\langle v\rangle^p + \langle v_*\rangle^p] \,\mathrm{d}|\mathcal{U}_t| \,\mathrm{d}t,$$

and hence $t \mapsto \langle v\rangle^p f_t$ is absolutely continuous in L^1.

Next, we note that being a solution to the collision rate equation is invariant under Maxwellian regularization.

Given $\mu \in \mathcal{P}(\mathbb{R}^d)$, we define its convolution with the Maxwellian M as usual as the measure $\mu * M \in \mathcal{P}(\mathbb{R}^d)$ given by

$$(\mu * M)(\mathrm{d}v) = \int_{\mathbb{R}^d} M(v - w)\,\mu(\mathrm{d}w)\,\mathrm{d}v.$$

Given $\mathcal{U} \in \mathcal{M}(\mathbb{R}^{2d} \times S^{d-1})$ we define its convolution $\mathcal{U} * M$ with the Maxwellian M in $\mathbb{R}^{2d}$ as the measure given by

$$(\mathcal{U} * M)(\mathrm{d}X, \mathrm{d}\omega) = \int_{\mathbb{R}^{2d}} M(X - Y)\mathcal{U}(\mathrm{d}Y, \mathrm{d}\omega)\,\mathrm{d}X.$$

Lemma 3.4. *Let* $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ *and set* $\mu_t^\delta := M_\delta * \mu_t$ *and* $\mathcal{U}_t^\delta := M_\delta * \mathcal{U}_t$ *for* $\delta \ge 0$ *and* $t \in [0, T]$. *Then* $(\mu^\delta, \mathcal{U}^\delta) \in \mathcal{CRE}_T$.

Proof. Fix a test function φ and set $\Phi(X) := \varphi(v) + \varphi(v_*)$. Then, using (2.10), we find (dropping δ in the notation)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi \,\mathrm{d}(\mu_t * M) = \frac{\mathrm{d}}{\mathrm{d}t}\int (\varphi * M)\,\mathrm{d}\mu_t &= \int \bar\nabla(\varphi * M)\,\mathrm{d}\mathcal{U}_t \\
&= \int [(\Phi * M)(T_\omega X) - (\Phi * M)(X)]\,\mathrm{d}\mathcal{U}_t(X, \omega) \\
&= \int [((T_\omega\Phi) * M)(X) - (\Phi * M)(X)]\,\mathrm{d}\mathcal{U}_t(X, \omega) \\
&= \int [\Phi(T_\omega X) - \Phi(X)]\,\mathrm{d}(\mathcal{U}_t * M)(X, \omega) = \int \bar\nabla\varphi \,\mathrm{d}(\mathcal{U}_t * M),
\end{aligned}$$

which shows that

$$(\mu * M, \mathcal{U} * M) \in \mathcal{CRE}_T. \qquad \blacksquare$$

3.2. *The action functional*

Let us first recall the definition of the *logarithmic mean* $\Lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Lambda(s,t) = \int_0^1 s^\alpha t^{1-\alpha} \, \mathrm{d}\alpha = \frac{s-t}{\log s - \log t}, \tag{3.4}$$

the latter expression being valid for positive $s \neq t$. Note that Λ is concave and positively homogeneous, i.e. $\Lambda(\alpha s, \alpha t) = \alpha\Lambda(s,t)$ for all $\alpha \geq 0$. Moreover, it is easy to check that

$$\Lambda(s,t) \leq \frac{s+t}{2} \quad \forall s,t \geq 0. \tag{3.5}$$

Given a function $f : \mathbb{R}^d \to \mathbb{R}_+$ we will often write

$$\Lambda(f)(v, v_*, \omega) = \Lambda(f f_*, f' f'_*).$$

We can now define a function $\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to [0,\infty]$ by setting

$$\alpha(s,t,u) := \begin{cases} \frac{u^2}{4\Lambda(s,t)}, & \Lambda(s,t) \neq 0, \\ 0, & \Lambda(s,t) = 0 \text{ and } u = 0, \\ +\infty, & \Lambda(s,t) = 0 \text{ and } u \neq 0. \end{cases} \tag{3.6}$$

The function α is lower semicontinuous, convex and positively homogeneous, i.e. for all $u \in \mathbb{R}$, $s,t \geq 0$, and $r > 0$ we have $\alpha(rs,rt,ru) = r\alpha(s,t,u)$. Indeed, this is easily checked using homogeneity and concavity of Λ and the convexity of the function $(u,y) \mapsto u^2/y$ on $\mathbb{R} \times (0,\infty)$.

We will now define an action functional on pairs of measures $(\mu, \mathcal{U})$ where $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\mathcal{U} \in \mathcal{M}(\Omega)$, generalizing (1.9). For later reference, we work first in a more general setting.

We consider the integral functional associated with the function α on the space $\mathcal{M}(X;\mathbb{R}^3)$ of vector-valued Borel measures with finite variation on a locally compact Polish space X as defined in (2.12), i.e. we set

$$\mathcal{F}_\alpha(\lambda) := \int \alpha\left(\frac{\mathrm{d}\lambda^1}{\mathrm{d}|\lambda|}, \frac{\mathrm{d}\lambda^2}{\mathrm{d}|\lambda|}, \frac{\mathrm{d}\lambda^3}{\mathrm{d}|\lambda|}\right) \mathrm{d}|\lambda|, \tag{3.7}$$

where $|\lambda|$ denotes the variation of λ.

Definition 3.5 (Action). For $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\mathcal{U} \in \mathcal{M}(\Omega)$ the *action* is defined by

$$\mathcal{A}(\mu, \mathcal{U}) := \mathcal{F}_\alpha(\mu^1, \mu^2, \mathcal{U}), \tag{3.8}$$

where μ^1, μ^2 are non-negative measures in $\mathcal{M}_+(\Omega)$ given by

$$\mu^1(\mathrm{d}v, \mathrm{d}v_*, \mathrm{d}\omega) := B(v - v_*, \omega)\, \mu(\mathrm{d}v)\, \mu(\mathrm{d}v_*)\, \mathrm{d}\omega, \quad \mu^2 := T_\# \mu^1, \tag{3.9}$$

where T is the change of variables $(v, v_*, \omega) \mapsto (T_\omega(v, v_*), \omega)$ between pre- and post-collisional variables defined in (1.3).

If the measure μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$ on $\mathbb{R}^d$, the next lemma shows that we recover (1.9). For this we denote by $\mathcal{B} \in \mathcal{M}(\Omega)$ the measure given by

$$\mathcal{B}(\mathrm{d}v, \mathrm{d}v_*, \mathrm{d}\omega) = B(v - v_*, \omega)\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega.$$

Lemma 3.6. *Let $\mu = f\mathcal{L} \in \mathcal{P}(\mathbb{R}^d)$ and $\mathcal{U} \in \mathcal{M}(\Omega)$ be such that $\mathcal{A}(\mu, \mathcal{U}) < \infty$. Then there exists a Borel function $U : \Omega \to \mathbb{R}$ such that $\mathcal{U} = U\Lambda(f)\mathcal{B}$ and*

$$\mathcal{A}(\mu, \mathcal{U}) = \frac{1}{4}\int |U(v, v_*, \omega)|^2 \Lambda(f) B(v - v_*, \omega)\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega. \tag{3.10}$$

Proof. Note that $\mu^i = \rho^i \mathcal{B},\ i = 1, 2$, with

$$\rho^1(v, v_*, \omega) = f(v) f(v_*) \quad \text{and} \quad \rho^2(v, v_*, \omega) = f(v') f(v'_*).$$

Choose $\sigma \in \mathcal{M}(\Omega)$ such that $\mathcal{B} = h\sigma$ and $\mathcal{U} = \widetilde{U}\sigma$ are both absolutely continuous with respect to σ and denote by $\widetilde{\rho}^i$ the density of μ^i with respect to σ. Now by homogeneity of α,

$$\mathcal{A}(\mu, \mathcal{U}) = \int \alpha(\widetilde{\rho}^1, \widetilde{\rho}^2, \widetilde{U})\, \mathrm{d}\sigma < \infty. \tag{3.11}$$

Let $A \subset \Omega$ be such that $\int_A \Lambda(\rho^1, \rho^2)\, \mathrm{d}\mathcal{B} = 0$. Homogeneity of Λ yields

$$0 = \int_A \Lambda(\rho^1, \rho^2)\, \mathrm{d}\mathcal{B} = \int_A \Lambda(\widetilde{\rho}^1, \widetilde{\rho}^2)\, \mathrm{d}\sigma,$$

i.e. $\Lambda(\widetilde{\rho}^1, \widetilde{\rho}^2) = 0$ σ-a.e. on A. Now the finiteness of the integral in (3.11) implies that $\widetilde{U} = 0$ σ-a.e. on A. Thus $|\mathcal{U}|(A) = 0$ and hence $\mathcal{U}$ is absolutely continuous with respect to the measure $\Lambda(f)\mathcal{B}$. Formula (3.10) now follows immediately from the homogeneity of α. ∎

In view of the previous lemma, given a pair of functions $f : \mathbb{R}^d \to \mathbb{R}_+$ and $U : \Omega \to \mathbb{R}$ we will define their action via $\mathcal{A}(f, U) := \mathcal{A}(\mu, \mathcal{U})$ with $\mu = f\mathcal{L}$ and $\mathcal{U} = U\Lambda(f)\widetilde{\mathcal{B}}$.

Next, we establish lower semicontinuity of the action with respect to convergence of μ and $\mathcal{U}$.

Lemma 3.7 (Lower semicontinuity of the action). *Assume that $\mu_n \rightharpoonup \mu$ weakly in $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{U}_n \rightharpoonup^* \mathcal{U}$ weakly* in $\mathcal{M}(\Omega)$. Then*

$$\mathcal{A}(\mu, \mathcal{U}) \leq \liminf_n \mathcal{A}(\mu_n, \mathcal{U}_n).$$

Proof. Note that by Assumption 2.1 on the collision kernel B, the weak convergence of μ_n to μ implies the weak* convergence of μ_n^i to μ^i in $\mathcal{M}(\Omega)$ for $i = 1, 2$. Now the claim follows immediately from Lemma 2.5. ∎

The next estimate will be useful at several points in this paper. For later reference, we formulate it in the general context of (3.7).

Lemma 3.8 (Integrability estimate). *For any Borel function* $\Psi : X \to \mathbb{R}_+$*, any* $\lambda \in \mathcal{M}(X;\mathbb{R}^3)$ *with* $\mathcal{F}_\alpha(\lambda) < \infty$ *and with* λ^1, λ^2 *non-negative measures we have*

$$\int \Psi \,\mathrm{d}|\lambda^3| \le \sqrt{2\mathcal{F}_\alpha(\lambda)}\Big(\int \Psi^2 \,\mathrm{d}(\lambda^1 + \lambda^2)\Big)^{1/2}. \tag{3.12}$$

Proof. Let us write $\lambda^i = \rho^i|\lambda|$. Since $\mathcal{F}_\alpha(\lambda)$ is finite, the set $A = \{\alpha(\rho^1, \rho^2, \rho^3) = \infty\}$ has zero measure with respect to $|\lambda|$. We can now estimate

$$\begin{aligned}\int \Psi \,\mathrm{d}|\lambda^3| \le \int \Psi|\rho^3|\,\mathrm{d}|\lambda| &= 2\int_{A^c} \Psi\sqrt{\Lambda(\rho^1,\rho^2)}\sqrt{\alpha(\rho^1,\rho^2,\rho^3)}\,\mathrm{d}|\lambda| \\ &\le 2\Big(\int \alpha(\rho^1,\rho^2,\rho^3)\,\mathrm{d}|\lambda|\Big)^{1/2}\Big(\int_{A^c} \Psi^2\Lambda(\rho^1,\rho^2)\,\mathrm{d}|\lambda|\Big)^{1/2} \\ &\le \sqrt{2\mathcal{F}_\alpha(\lambda)}\Big(\int \Psi^2\,\mathrm{d}(\lambda^1+\lambda^2)\Big)^{1/2},\end{aligned}$$

where the last inequality follows from the estimate (3.5). ∎

Corollary 3.9. *Let* $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ *be such that* $A := \int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t)\,\mathrm{d}t$ *and* $E := \int_0^T \mathcal{E}_{2p+\gamma_+}(\mu_t)\,\mathrm{d}t$ *are finite for some* $p > 0$ *where* $\gamma_+ = \max(\gamma, 0)$*. Then the integrability condition* (3.3) *is satisfied, more precisely*

$$\int_0^T \int [\langle v\rangle^p + \langle v_*\rangle^p]\,\mathrm{d}|\mathcal{U}_t|\,\mathrm{d}t \le \sqrt{AC_BC_{p,\gamma}E}.$$

Proof. Let $\mu^i, \mathcal{U} \in \mathcal{M}(\Omega\times[0,T])$ be given by $\mathrm{d}\mu^i = \mathrm{d}\mu^i_t\,\mathrm{d}t$ and $\mathrm{d}\mathcal{U} = \mathrm{d}\mathcal{U}_t\,\mathrm{d}t$ and note that

$$\int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t)\,\mathrm{d}t = \int_0^T \mathcal{F}_\alpha(\mu^1_t, \mu^2_t, \mathcal{U}_t)\,\mathrm{d}t = \mathcal{F}_\alpha(\mu^1, \mu^2, \mathcal{U}).$$

Then one concludes by Lemma 3.8, choosing $\Psi(v, v_*, \omega, t) = \langle v\rangle^p + \langle v_*\rangle^p$. ∎

Note that for a given curve $(\mu_t)_{t\in[0,T]}$ there will be several compatible collision rates $(\mathcal{U}_t)_t$ such that $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$. For instance, when $\mathcal{V}_t$ is symmetric under the transformation $(v, v_*, \omega) \mapsto (v', v'_*, \omega)$ we have $\int \bar\nabla\varphi\,\mathrm{d}\mathcal{V}_t = 0$ for any test function φ. Hence, $(\mu, \mathcal{U}+\mathcal{V}) \in \mathcal{CRE}_T$ whenever $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$. Thus, we define the action of a curve as the minimal action of all compatible collision rates.

Definition 3.10 (Action of a curve). Given a curve $(\mu_t)_{t\in[0,T]}$ in $\mathcal{P}(\mathbb{R}^d)$ its *action* is defined by

$$\mathcal{A}_T(\mu) := \inf\Big\{\int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t)\,\mathrm{d}t : (\mu, \mathcal{U}) \in \mathcal{CRE}_T\Big\}. \tag{3.13}$$

If there is no $\mathcal{U}$ with $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$, we set $\mathcal{A}_T(\mu) = +\infty$.

The next result shows that under additional control on the energy of the curve, the infimum above is attained by an optimal collision rate.

Proposition 3.11 (Optimal collision rate). *Let $(\mu_t)_{t\in[0,T]}$ be a curve in $\mathscr{P}(\mathbb{R}^d)$ such that*

$$\mathcal{A}_T(\mu) < \infty, \quad E := \int_0^T \mathcal{E}_2(\mu_t)\,\mathrm{d}t < \infty. \tag{3.14}$$

Then there exists a family $(\mathcal{U}_t)_t$ with $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ attaining the infimum in (3.13).

Proof. Let $(\mathcal{U}^n_t)_t$ be a minimizing sequence of collision rates for (3.13) and define the measures $\mathcal{U}^n \in \mathcal{M}(\Omega \times [0,T])$ given by $\mathrm{d}\mathcal{U}^n = \mathrm{d}\mathcal{U}^n_t\,\mathrm{d}t$. By Lemma 3.8, for every measurable function Ψ on $\mathbb{R}^{2d} \times S^{d-1} \times [0,T]$ we have

$$\begin{aligned}\sup_n \int \Psi\,\mathrm{d}|\mathcal{U}^n| \\ \leq \sqrt{2A}\left(\int (\Psi^2 + \Psi^2 \circ T) B(v - v_*, \omega)\,\mathrm{d}\omega\,\mathrm{d}\mu_t(v)\,\mathrm{d}\mu_t(v_*)\,\mathrm{d}t\right)^{1/2},\end{aligned} \tag{3.15}$$

with $A = \sup_n \int_0^T \mathcal{A}(\mu_t, \mathcal{U}^n_t)\,\mathrm{d}t < \infty$. Choosing $\Psi = \mathbf{1}_{\Omega\times I}$ and using Assumption 2.1, we obtain $|\mathcal{U}^n|(\Omega \times I) \leq 2\sqrt{C_B A E \cdot \mathcal{L}(I)}$. Hence, $\mathcal{U}^n$ has uniformly bounded variation and up to extracting a subsequence we have $\mathcal{U}^n \rightharpoonup^* \mathcal{U}$ in $\mathcal{M}(\Omega \times [0,T])$. Moreover, we see that $\mathcal{U}$ can be disintegrated with respect to Lebesgue measure on $[0,T]$ and we can write $\mathcal{U} = \int_0^T \mathcal{U}_t\,\mathrm{d}t$ for a Borel family $(\mathcal{U}_t)$ still satisfying (iii) in Definition 3.1.

To see that $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$, it suffices to show that for any test functions $a \in C([0,T])$ and $\varphi \in C_b(\mathbb{R}^d)$ we have

$$\int a(t)\bar{\nabla}\varphi\,\mathrm{d}\mathcal{U}^n_t\,\mathrm{d}t \xrightarrow{n\to\infty} \int a(t)\bar{\nabla}\varphi\,\mathrm{d}\mathcal{U}_t\,\mathrm{d}t. \tag{3.16}$$

This follows from a straightforward argument, approximating $\bar{\nabla}\varphi$ with compactly supported continuous functions Ω once we establish the following tightness estimate for $\mathcal{U}^n$: Denoting by B_R the ball of radius R in $\mathbb{R}^{2d}$ and $M_R := B_R^c \times S^{d-1} \times [0,T]$ we have

$$\begin{aligned}|\mathcal{U}^n|(M_R) &\leq 2\sqrt{AC_B}\left(\int_0^T \int_{B^c_{R/2}} [\langle v\rangle^\gamma + \langle v_*\rangle^\gamma]\,\mathrm{d}\mu_t(v)\,\mathrm{d}\mu_t(v_*)\,\mathrm{d}t\right)^{1/2} \\ &\leq 2\frac{\sqrt{AC_B E}}{\sqrt{R}},\end{aligned}$$

which goes to zero uniformly in n as $R \to \infty$. This estimate follows again from (3.15), noting that if (v, v_*) or (v', v'_*) lies outside B_R, then (v, v_*) lies outside of $B_{R/2}$, and further using the estimate $\int_{\{|v|\geq R\}} \langle v\rangle^\gamma\,\mathrm{d}\mu_t(v) \leq \int \frac{\langle v\rangle^2}{R}\,\mathrm{d}\mu_t(v)$, since $\gamma \leq 1$, and the upper bound on the energy in (3.14). Finally, we conclude that $\int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t)\,\mathrm{d}t = \mathcal{A}(\mu)$ noting that $\int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t)\,\mathrm{d}t = \mathcal{F}_\alpha(\mu^1, \mu^2, \mathcal{U})$ and using lower semicontinuity of $\mathcal{F}_\alpha$. ∎

4. Variational characterization of the homogeneous Boltzmann equation

In this section we establish the variational characterization of the homogeneous Boltzmann equation, stated in Theorem 1.1. The crucial ingredient is a chain rule allowing one to take derivatives of the entropy along suitable curves of finite action.

Recall that $\mathcal{E}_p(\mu)$ denotes the p-moment of μ.

Proposition 4.1 (Chain rule). *Let $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ with $(\mu_t)_t \subset \mathcal{P}_p(\mathbb{R}^d)$ be such that $\mathcal{H}(\mu_t)$ is finite for some $t \in [0, T]$, and suppose that*

$$E := \int_0^T \mathcal{E}_p(\mu_t)\,\mathrm{d}t < \infty,$$

where $p = 2 + \max(\gamma, 0)$, and

$$\int_0^T \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)}\,\mathrm{d}t < \infty, \qquad \int_0^T \sqrt{D(\mu_t)}\sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)}\,\mathrm{d}t < \infty. \tag{4.1}$$

Then $\mathcal{H}(\mu_t) < \infty$ for all $t \in [0, T]$, and

$$\mathcal{H}(\mu_t) - \mathcal{H}(\mu_s) = \int_s^t \frac{1}{4} \int_{\{\Lambda(f_r>0\}} \bar{\nabla} \log f_r \,\mathrm{d}\mathcal{U}_r \,\mathrm{d}r \quad \forall 0 \le s \le t \le T, \tag{4.2}$$

where f_r is the density of μ_r. In particular, the map $t \mapsto \mathcal{H}(\mu_t)$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\mu_t) = \frac{1}{4}\int \bar{\nabla} \log f_t \,\mathrm{d}\mathcal{U}_t \quad \textit{for a.e. } t. \tag{4.3}$$

Note that assumption (4.1) and Lemma 3.6 imply that for a.e. t, μ_t is absolutely continuous with a density f_t, and $\mathcal{U}_t$ is absolutely continuous with a density $U_t\Lambda(f_t)B$, in particular the set of (v, v_*, ω) where $\Lambda(f_t) = 0$ is negligible for $\mathcal{U}_t$. Hence the right hand side in (4.2) is well-defined since $f, f_*, f', f'_* > 0$ on $\{\Lambda(f) > 0\}$. More precisely, this and similar integrals in what follows will be understood implicitly to be taken over the set $\{\Lambda(f_t) > 0)\}$.

As a preparatory result we establish the following continuity property of the action and dissipation under Maxwellian convolution.

Lemma 4.2. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\mathcal{U} \in \mathcal{M}(\Omega)$ be such that $\mathcal{A}(\mu, \mathcal{U}) < \infty$ and $D(\mu) < \infty$. Let $\mu^\delta = M_\delta * \mu$ and $\mathcal{U}^\delta = M_\delta * \mathcal{U}$ denote convolutions with the Maxwellian. Then*

$$\lim_{\delta\to 0} \mathcal{A}(\mu^\delta, \mathcal{U}^\delta) = \mathcal{A}(\mu, \mathcal{U}), \quad \lim_{\delta\to 0} D(\mu^\delta) = D(\mu). \tag{4.4}$$

Moreover, there is a constant C depending only on γ and c_B from Assumption 2.1 such that

$$\mathcal{A}(\mu^\delta, \mathcal{U}^\delta) \le C\mathcal{A}(\mu, \mathcal{U}), \quad D(\mu^\delta) \le CD(\mu) \quad \forall \delta > 0. \tag{4.5}$$

In the proofs of the last two results, we took inspiration from [10] to treat the case of unbounded kernels B, namely in the usage of the Peetre inequality and the following version of the dominated convergence theorem (see e.g. [32, Chap. 4, Thm. 17]), termed the *extended dominated convergence theorem*: Let $(R_\delta)_{\delta>0}$ and $(I_\delta)_{\delta>0}$ be families of measurable functions on a measure space X with $I_\delta \ge 0$ and let R, I be measurable. Assume that R^δ, I^δ converge pointwise to R, I respectively, $|R^\delta| \le I^\delta$ a.e., and $\lim_{\delta\to 0} \int_X I^\delta = \int_X I$. Then also $\lim_{\delta\to 0} \int_X R^\delta = \int_X R$.

Proof of Lemma 4.2. We first prove (4.4), (4.5) for the dissipation D. Let $\mu = f\,\mathrm{d}v$ and $\mathcal{U} = U\,\mathrm{d}X\,\mathrm{d}\omega$ and put $F(X) = ff_*$. Similarly let $f^\delta = M_\delta * f$, $F^\delta = M_\delta * F$ and $U^\delta = M_\delta * U$ be the respective densities of μ^δ, $\mu^\delta \otimes \mu^\delta_*$ and $\mathcal{U}^\delta$. Now,

$$D(\mu^\delta) = \int B\Lambda(f^\delta)|\bar{\nabla} f^\delta|^2\,\mathrm{d}X\,\mathrm{d}\omega = \int BG(F^\delta, T_\omega F^\delta)\,\mathrm{d}X\,\mathrm{d}\omega =: \int L_1^\delta\,\mathrm{d}X\,\mathrm{d}\omega,$$

where $G(x,y) = (x-y)(\log x - \log y)$ is convex. Note that $L_1^\delta(X,\omega)$ converges pointwise to $L(X,\omega) := B\cdot G(F, T_\omega F)$ as $\delta \to 0$, and $\int L\,\mathrm{d}X\,\mathrm{d}\omega = D(\mu)$. From the commutation relation of Lemma 2.3 and Jensen's inequality we infer the majorant

$$L_1^\delta \le B\cdot (M_\delta * [G(F, T_\omega F)]) =: L_2^\delta.$$

Obviously also $L_2^\delta \to L$ pointwise as $\delta \to 0$. To prove the continuity (4.4) it suffices by the extended dominated convergence theorem to show that $\int L_2^\delta\,\mathrm{d}X\,\mathrm{d}\omega \to \int L\,\mathrm{d}X\,\mathrm{d}\omega$. But by self-adjointness of convolution we have

$$\begin{aligned}\int L_2^\delta\,\mathrm{d}X\,\mathrm{d}\omega &= \int B\cdot (M_\delta * [G(F, T_\omega F)])\,\mathrm{d}X\,\mathrm{d}\omega\\ &= \int (M_\delta * B)\cdot G(F, T_\omega F)\,\mathrm{d}X\,\mathrm{d}\omega := \int L_3^\delta\,\mathrm{d}X\,\mathrm{d}\omega,\end{aligned}$$

and again $L_3^\delta \to L$. Now, by Assumption 2.1 and Lemma 2.4 we have

$$\begin{aligned}(M_\delta * B)(X,\omega) &\le c_B \int \langle v - v_* - (w - w_*)\rangle^\gamma M_\delta(w) M_\delta(w_*)\,\mathrm{d}w\,\mathrm{d}w_*\\ &\le C_\gamma c_B \langle v - v_*\rangle^\gamma \le C_\gamma c_B^2 B(X,\omega).\end{aligned}$$

Hence, we have a majorant $L_3^\delta \le CL$, and dominated convergence yields $\int L_2^\delta\,\mathrm{d}X\,\mathrm{d}\omega = \int L_3^\delta\,\mathrm{d}X\,\mathrm{d}\omega \to \int L\,\mathrm{d}X\,\mathrm{d}\omega$ as desired. Note that the previous argument also yields the bound (4.5).

To prove the corresponding claims for the action $\mathcal{A}$, we proceed in the same way, writing

$$\mathcal{A}(\mu^\delta, \mathcal{U}^\delta) = \int B^{-1}\frac{|U^\delta|^2}{\Lambda(F^\delta, T_\omega F^\delta)}\,\mathrm{d}X\,\mathrm{d}\omega,$$

and use convexity of the function $(u,r,s) \mapsto |u|^2/\Lambda(r,s)$ and also, in the last step, the bound $M^\delta * B^{-1} \le CB^{-1}$. ∎

Proof of Proposition 4.1. Note that by (4.1) and Lemma 3.6 we have $\mu_r = f_r\,\mathrm{d}v$, $\mathcal{U}_r = U_r\,\mathrm{d}X\,\mathrm{d}\omega$ for a.e. r and suitable densities f_r, U_r. We will now proceed in several steps.

Step 1: Regularization. We will perform a three-fold regularization procedure. First, we regularize the curve by convolution with the Maxwellian. For $\delta > 0$ we set $\mu_t^\delta = M_\delta * \mu_t$ and $\mathcal{U}_t^\delta = M_\delta * \mathcal{U}_t$. Then we perform a convolution in time. For a standard mollifier η on $\mathbb{R}$ supported in $[-1,1]$ and $\lambda > 0$ we define

$$\mu_t^{\delta,\lambda} = \int \eta(t')\mu^\delta_{t-\lambda t'}\,\mathrm{d}t', \quad \mathcal{U}_t^{\delta,\lambda} = \int \eta(t')\mathcal{U}^\delta_{t-\lambda t'}\,\mathrm{d}t'.$$

(For this the curves are assumed to be extended trivially by μ_0^δ, $\mathcal{U}_0^\delta$ on $[-\lambda, 0]$ and similarly on $[T, T+\lambda]$.) By Lemma 3.4 we have $(\mu^\delta, \mathcal{U}^\delta) \in \mathcal{CRE}_T$ and by linearity of the collision rate equation also $(\mu^{\delta,\lambda}, \mathcal{U}^{\delta,\lambda}) \in \mathcal{CRE}_T$.

Finally, let g be a probability density in $\mathcal{P}_{2,E}(\mathbb{R}^d)$ such that

$$|\log g(v)| \le C\langle v\rangle \tag{4.6}$$

for some constant C (for instance, choose $g(v)$ proportional to $e^{-\alpha|v|}$ for suitable $\alpha > 0$). Then we set, for $\varepsilon > 0$, $\mu^{\delta,\lambda,\varepsilon} := (1+\varepsilon)^{-1}(\mu^{\delta,\lambda} + \varepsilon g\mathcal{L})$ and $\mathcal{U}^{\delta,\lambda,\varepsilon} = (1+\varepsilon)^{-1}\mathcal{U}^{\delta,\lambda}$, and note that $(\mu^{\delta,\lambda,\varepsilon}, \mathcal{U}^{\delta,\lambda,\varepsilon}) \in \mathcal{CRE}_T$. Let f^δ, U^δ denote the densities of $\mu^\delta, \mathcal{U}^\delta$, and similarly for λ and ε.

Step 2: Estimates for the regularized curve. Note that the time-integrated p-moment of μ_r^δ is bounded as

$$\int_0^T \mathcal{E}_p(\mu_r^{\delta,\lambda,\varepsilon})\,\mathrm{d}r \le E \tag{4.7}$$

with $p = 2 + \max(\gamma, 0)$ for all $\delta, \lambda, \varepsilon > 0$.

Next, we look at the behavior of the action and dissipation under regularization. From Lemma 4.2 we have

$$\mathcal{A}(\mu^\delta, \mathcal{U}^\delta) \le C\mathcal{A}(\mu, \mathcal{U}), \quad D(\mu^\delta) \le CD(\mu). \tag{4.8}$$

A similar convexity argument gives

$$\int_0^T \mathcal{A}(\mu_r^{\delta,\lambda}, \mathcal{U}_r^{\delta,\lambda})\,\mathrm{d}r \le C\int_0^T \mathcal{A}(\mu_r, \mathcal{U}_r)\,\mathrm{d}r. \tag{4.9}$$

Taking into account Corollary 3.9 and (4.7) we obtain

$$\int_0^T \int [\langle v\rangle + \langle v_*\rangle]\,\mathrm{d}|\mathcal{U}_r^{\delta,\lambda}|\,\mathrm{d}r \le C, \tag{4.10}$$

uniformly in $\delta, \lambda > 0$.

Step 3: Integrated chain rule for regularized curve. Now, we claim that

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{H}(\mu_r^{\delta,\lambda,\varepsilon}) = \int_{\mathbb{R}^d} \log f_r^{\delta,\lambda,\varepsilon}\,\partial_r f_r^{\delta,\lambda,\varepsilon} = \frac{1}{4}\int_\Omega \bar\nabla \log f_r^{\delta,\lambda,\varepsilon}\,U_r^{\delta,\lambda,\varepsilon}, \tag{4.11}$$

where the integral over Ω is with respect to the measure $\mathrm{d}X\,\mathrm{d}\omega$. Indeed, to justify the first identity in (4.11) we use convexity of $r \mapsto r\log r$ and (4.6) to estimate

$$\begin{aligned}\frac{1}{h}|f_{r+h}^{\delta,\lambda,\varepsilon}\log f_{r+h}^{\delta,\lambda,\varepsilon} - f_r^{\delta,\lambda,\varepsilon}\log f_r^{\delta,\lambda,\varepsilon}| &\le C\langle v\rangle\frac{1}{h}|f_{r+h}^{\delta,\lambda,\varepsilon} - f_r^{\delta,\lambda,\varepsilon}|\\ &\le C\langle v\rangle\|\eta'\|_\infty\int_0^T (f_t^\delta + \varepsilon g)\,\mathrm{d}t.\end{aligned}$$

Since $(f_t)_t$ has uniformly bounded time-integrated second moment, by dominated convergence we can take the time derivative inside the integral. The second identity in (4.11) follows by applying the collision rate equation using (4.6) and (4.10); see Remark 3.3.

Integrating (4.11) between s and t we obtain

$$\mathcal{H}(f_t^{\delta,\lambda,\varepsilon}) - \mathcal{H}(f_s^{\delta,\lambda,\varepsilon}) = \int_s^t \frac{1}{4}\int_\Omega \bar{\nabla}\log f_r^{\delta,\lambda,\varepsilon}\, U_r^{\delta,\lambda,\varepsilon}\,\mathrm{d}r. \tag{4.12}$$

Step 4: Passing to the limit. We will now pass to the limit in (4.12) to obtain (4.2) letting $\lambda \to 0$, $\varepsilon \to 0$ and $\delta \to 0$ in this order. Consider first the right hand side.

(a) *RHS,* $\lambda \to 0$. Using the bound $|\log f_r^{\delta,\lambda,\varepsilon}| \le c(\delta,\varepsilon)\langle v\rangle$ ensured by (4.6) which is uniform in λ for fixed δ,ε and the integrability condition (4.10) for $U^{\delta,\lambda}$, we can pass to the limit as $\lambda \to 0$ and obtain

$$(1+\varepsilon)^{-1}\int_s^t \frac{1}{4}\int_\Omega \bar{\nabla}\log(f_r^\delta + \varepsilon g)\, U_r^\delta\,\mathrm{d}r. \tag{4.13}$$

(b) *RHS,* $\varepsilon \to 0$. We use the estimate (dropping time parameter r in notation)

$$\begin{aligned}
|\bar{\nabla}\log(f^\delta+\varepsilon g)\,U^\delta| &\le \sqrt{|\bar{\nabla}\log(f^\delta+\varepsilon g)|^2\Lambda(f^\delta+\varepsilon g)B}\cdot\sqrt{\frac{|U^\delta|^2}{B\Lambda(f^\delta+\varepsilon g)}}\\
&\le \sqrt{|\bar{\nabla}\log(f^\delta+\varepsilon g)|\cdot|(f^\delta+\varepsilon g)((f^\delta)_*+\varepsilon g_*)-((f^\delta)'+\varepsilon g')((f^\delta)'_*+\varepsilon g'_*)|B}\\
&\quad\cdot\sqrt{\frac{|U^\delta|^2}{B\Lambda(f^\delta)}}\\
&\le \sqrt{C(\langle v\rangle^2+\langle v_*\rangle^2)\langle v-v_*\rangle^\gamma}\\
&\quad\cdot\sqrt{|(f^\delta+\varepsilon g)((f^\delta)_*+\varepsilon g_*)-((f^\delta)'+\varepsilon g')((f^\delta)'_*+\varepsilon g'_*)|}\sqrt{\frac{|U^\delta|^2}{B\Lambda(f^\delta)}}.
\end{aligned} \tag{4.14}$$

Here, in the second inequality we have used the definition of Λ and the monotonicity of the logarithmic mean. In the third inequality we have used the bound (2.9) and Assumption 2.1. By the moment assumptions on f and Lemma 4.2, one readily checks that the right hand side in (4.14) is integrable on $[0,T]\times\Omega$ and its integral converges to that of the same expression with $\varepsilon = 0$. Thus, the extended dominated convergence theorem allows us to pass to the limit as $\varepsilon \to 0$ in (4.13) and obtain

$$\int_s^t \frac{1}{4}\int \bar{\nabla}\log f_r^\delta U_r^\delta\,\mathrm{d}r. \tag{4.15}$$

(c) *RHS,* $\delta \to 0$. Note that $\bar{\nabla}\log f_r^\delta\, U_r^\delta$ converges pointwise to $\bar{\nabla}\log f_r\, U_r$ as $\delta \to 0$ at every r where the densities of $\mu_r, \mathcal{U}_r$ exist. To pass to the limit in the integral over Ω we use the majorant (dropping the time parameter r in notation)

$$|\bar{\nabla}\log f^\delta\, U^\delta| \le \frac{1}{2}|\bar{\nabla}\log f^\delta|^2\Lambda(f^\delta)B + \frac{1}{2}\frac{|U^\delta|^2}{B\Lambda(f^\delta)} =: \frac{1}{2}(I_1^\delta + I_2^\delta).$$

Obviously $I_1^\delta \to I_1^0$ and $I_2^\delta \to I_2^0$ pointwise, where I_1^0 and I_2^0 are the corresponding expressions with f^δ, U^δ replaced by f, U. By Lemma 4.2, we also have

$$\int I_1^\delta = D(f^\delta) \to D(f) = \int I_1^0, \quad \int I_2^\delta = \mathcal{A}(f^\delta, U^\delta) \to \mathcal{A}(f,U) = \int I_2^0$$

as $\delta \to 0$ for a.e. $r \in [0, T]$. Thus by the extended dominated convergence theorem we can pass to the limit in the space integral in (4.15).

Finally, to pass to the limit in the time integral, we use the already established almost everywhere in time convergence of the space integral and exhibit a similar majorant using Lemma 4.2:

$$\int \bar{\nabla} \log f_r^\delta \, U_r^\delta \, \mathrm{d}r \le \left(\int I_1^\delta\right)^{1/2} \left(\int I_2^\delta\right)^{1/2} \le C \sqrt{D(\mu_r)} \sqrt{\mathcal{A}(\mu_r, \mathcal{U}_r)}.$$

Recall that the last expression is integrable by assumption.

(d) *LHS.* Let us now show convergence of the left hand side of (4.12). Appealing to the bound (4.6) for g we obtain the estimate

$$|\mathcal{H}(f_t^{\delta,\lambda,\varepsilon}) - \mathcal{H}(f_t^{\delta,\varepsilon})| \le C \int \langle v \rangle |f_{t-\lambda t'}^{\delta,\varepsilon} - f_t^{\delta,\varepsilon}| \eta(t') \, \mathrm{d}t',$$

and we can pass to the limit as $\lambda \to 0$ by the continuity of $t \mapsto \langle v \rangle f_t^{\delta,\varepsilon}$ in L^1; see Remark 3.3. The bound (2.9) allows us to pass to the limit as $\varepsilon \to 0$ and we are left with $\mathcal{H}(f_t^\delta) - \mathcal{H}(f_s^\delta)$. Assume first that $\mathcal{H}(\mu_s)$ is finite. Recall that entropy is decreasing along the Ornstein–Uhlenbeck semigroup and lower semicontinuous. As $\delta \to 0$ we thus infer that $\mathcal{H}(f_t^\delta)$ increases to $\mathcal{H}(\mu_t)$. Hence, $\mathcal{H}(f_t^\delta) - \mathcal{H}(f_s^\delta)$ converges to $\mathcal{H}(f_t) - \mathcal{H}(f_s)$ and $\mathcal{H}(\mu_t)$ is finite due to the boundedness of the right hand side of (4.2) in the limit. Since by assumption there exists s with $\mathcal{H}(\mu_s) < \infty$, this shows that $\mathcal{H}(\mu_t) < \infty$ for all $t \in [0, T]$ and (4.2) is established.

Finally, using the estimate

$$\frac{1}{4} \int \bar{\nabla} \log f_r \, \mathrm{d}\mathcal{U}_r \le \sqrt{D(\mu_r)} \sqrt{\mathcal{A}(\mu_r, \mathcal{U}_r)}, \tag{4.16}$$

obtained just as before for f_r^δ, we see that $t \mapsto \mathcal{H}(\mu_t)$ is absolutely continuous and (4.3) follows. ∎

We can now prove the variational characterization of the homogeneous Boltzmann equation as the gradient flow of the entropy. For convenience we rephrase the statement here.

By a *solution* to the homogeneous Boltzmann equation we mean a family $(f_t)_{t \ge 0}$ of probability densities with $f \in C([0,\infty); L^1(\mathbb{R}^d)) \cap L^\infty([0,\infty); L^1_2(\mathbb{R}^d))$ such that for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, in the sense of distributions,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \varphi f_t = -\frac{1}{4} \int \bar{\nabla}\varphi (f' f'_* - f f_*) B(v - v_*, \omega) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega. \tag{4.17}$$

Theorem 4.3. *Let $(f_t)_{t \in [0,T]}$ be a curve of probability densities in $\mathcal{P}_p(\mathbb{R}^d)$ such that*

$$\mathcal{H}(f_0) < \infty, \qquad \int_0^T \mathcal{E}_p(f_t) \, \mathrm{d}t < \infty \tag{4.18}$$

with $p = 2 + \max(\gamma, 0)$. Then

$$J_T(f) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \frac{1}{2} \int_0^T D(f_t) \, \mathrm{d}t + \frac{1}{2} \mathcal{A}_T(f) \ge 0.$$

Moreover, $J_T(f) = 0$ if and only if $(f_t)_t$ is a solution to the homogeneous Boltzmann equation satisfying the integrability assumptions (4.18) *and*

$$\int_0^T D(f_t)\,\mathrm{d}t < \infty. \tag{4.19}$$

Assuming finite entropy and energy (and finite fourth moment for $\gamma > 0$) of the initial datum f_0, Theorem 2.2 gives existence and uniqueness of a solution $(f_t)_t$ to the homogeneous Boltzmann equation. It satisfies (4.18) and (2.7); in particular, (4.19) holds. Thus, there is actually only one curve such that $J_T(f) = 0$, namely the unique solution to the Boltzmann equation.

Proof of Theorem 4.3. Let $(f_t)_{t\in[0,T]}$ be a curve satisfying (4.18). To show $J_T(f) \geq 0$ we can assume that $\mathcal{A}_T(f) < \infty$ and $\int_0^T D(f_t)\,\mathrm{d}t < \infty$, since otherwise $J_T(f) = +\infty$. Let $(\mathcal{U}_t)_t$ be optimal collision rates given by Proposition 3.11. But then $J_T(f) \geq 0$ follows immediately from Proposition 4.1 and the estimate (4.16).

We now show that any solution (f_t) to the Boltzmann equation satisfying (4.18) and (4.19) satisfies $J_T(f) = 0$. Setting $\mu_t = f_t\mathcal{L}$ and

$$\mathcal{U}_t = -\bar{\nabla}\log f_t \Lambda(f_t)\mathcal{B} = -[(f')_t(f'_*)_t - f_t(f_*)_t]\mathcal{B},$$

we see by (4.17) that $(\mu, \mathcal{U})$ belongs to $\mathcal{CRE}$. Moreover, $\mathcal{A}(\mu_t, \mathcal{U}_t) = D(f_t)$ and thus by (4.19) we can apply the chain rule (4.2) to obtain

$$\mathcal{H}(f_T) - \mathcal{H}(f_0) = -\int_0^T D(f_t)\,\mathrm{d}t = -\frac{1}{2}\int_0^T D(f_t)\,\mathrm{d}t - \frac{1}{2}\mathcal{A}_T(\mu),$$

i.e. $J_T(f) = 0$.

Conversely, let us show that any curve $(f_t)_t$ with $J_T(f) = 0$ is a solution to the Boltzmann equation satisfying (4.19). From (4.18) we find that $\mathcal{H}(\mu_t) < \infty$ for all t and that $\mathcal{A}_T(f) < \infty$ and (4.19) holds. By Proposition 3.11 there exists a family $\mathcal{U}_t$ with $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ such that $\int_0^T \mathcal{A}(f_t, \mathcal{U}_t)\,\mathrm{d}t = \mathcal{A}_T(f)$, in particular $t \mapsto f_t$ is continuous in L^1; see Remark 3.2. By Lemma 3.6 the measure $\mathcal{U}_t$ has a density $U_t\Lambda(f_t)B$. From the chain rule (4.2) and the Cauchy–Schwarz and Young inequalities we infer that

$$\begin{aligned}
\mathcal{H}(f_T) - \mathcal{H}(f_0) &= \int_0^T \frac{1}{4}\int \bar{\nabla}\log f_r U_r \Lambda(f_r)B\,\mathrm{d}r \\
&\geq -\int_0^T \left[\sqrt{\frac{1}{4}\int |\bar{\nabla}\log f_r|^2\Lambda(f_r)B}\sqrt{\frac{1}{4}\int |U_r|^2\Lambda(f_r)B}\right]\mathrm{d}r \\
&\geq -\frac{1}{2}\int_0^T \left[\frac{1}{4}\int |\bar{\nabla}\log f_r|^2\Lambda(f_r)B + \frac{1}{4}\int |U_r|^2\Lambda(f_r)B\right]\mathrm{d}r \\
&= -\frac{1}{2}\int_0^T D(f_r)\,\mathrm{d}r - \frac{1}{2}\mathcal{A}_T(f).
\end{aligned}$$

Since $J_T(f) = 0$, we see that the two inequalities have to be identities. This implies that $\int_0^T \int |U_r + \bar{\nabla}\log f_r|^2\Lambda(f_r)B\,\mathrm{d}t = 0$, hence $U_r = -\bar{\nabla}\log f_r$ for a.e. r and a.e. (v, v_*, ω)

with $\Lambda(f_r)(v, v_*, \omega) > 0$. Thus, the collision rate equation for $(\mu, \mathcal{U})$ turns into the distributional formulation of the Boltzmann equation. ■

The results obtained in this section can be recast in the framework of gradient flows in metric spaces. The action functional gives rise to a distance $\mathcal{W}_B$ on $\mathcal{P}_{p,E}(\mathbb{R}^d)$ and the Boltzmann equation is characterized as the gradient flow of $\mathcal{H}$ (i.e. a curve of maximal slope) in the metric space $(\mathcal{P}_{p,E}(\mathbb{R}^d), \mathcal{W}_B)$. We refer to the appendix for a discussion of this point of view, in particular to Corollary B.3.

Generalized gradient structures

We will now briefly discuss possible generalizations of the variational characterization of the Boltzmann equation above using generalized gradient structures. Such structures arise naturally from the large deviation behavior of an underlying microscopic stochastic system; see the discussion in the introduction. We refer the reader e.g. to [26] and the references therein. Here we aim to indicate how the characterization in Theorem 4.3 can be generalized to non-quadratic gradient structures. A very detailed discussion of generalized gradient structures in the case of jump processes has recently been performed in [31]. We will mainly follow their terminology and constructions and adapt them to the present case of the Boltzmann equation. In comparison, we will impose more restrictive conditions on the gradient structures we consider in order to simplify the presentation while still encompassing the main examples we are interested in; see Example 4.7.

Let us fix a *dual dissipation density* Ψ^* and a *flux density map* θ as follows.

Assumption 4.4. (1) The function $\Psi^* : \mathbb{R} \to [0, \infty)$ is convex, differentiable, superlinear and even, with $\Psi^*(0) = 0$.

(2) The function $\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous, concave, not identically 0 and it satisfies

- symmetry: $\theta(r, s) = \theta(s, r)$ for all $s, r \in [0, \infty)$;
- positive 1-homogeneity: $\theta(\lambda r, \lambda s) = \lambda\theta(r, s)$ for all $r, s \in [0, \infty)$ and $\lambda \geq 0$;
- behavior at 0: $\theta(0, t) = 0$ for all $t \in [0, \infty)$.

(3) In addition we have

- compatibility: $(\Psi^*)'(\log s - \log t)\theta(s, t) = s - t$ for all $s, t > 0$;
- there exists a convex lower semicontinuous function $G_{\Psi^*} : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$\tfrac{1}{4}\Psi^*(\log t - \log s)\theta(s, t) = G_{\Psi^*}(s, t) \quad \forall s, t > 0,$$

and $G_{\Psi^*}(s, t) = 0$ if and only if $s = t$.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be the convex conjugate of Ψ^* and note that it is strictly convex, strictly increasing, superlinear and even, with $\Psi(0) = 0$.

Given $\mu \in \mathscr{P}(\mathbb{R}^d)$, we define the measure $\nu_\mu \in \mathcal{M}_+(\Omega)$ via

$$\nu_\mu := \theta\left(\frac{\mathrm{d}\mu^1}{\mathrm{d}\sigma}, \frac{\mathrm{d}\mu^2}{\mathrm{d}\sigma}\right)\sigma, \tag{4.20}$$

where μ^1, μ^2 are given by (3.9) and σ is any measure such that $\mu^1, \mu^2 \ll \sigma$. Due to the 1-homogeneity of θ the definition is independent of σ. Note that if μ is absolutely continuous with respect to Lebesgue measure with density f, then $\mathrm{d}\nu_\mu = \theta(ff_*, f'f'_*)B\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$.

We can now define the primal and dual dissipation potentials.

Definition 4.5. Given measures $\mu \in \mathscr{P}(\mathbb{R}^d)$, $\mathcal{U} \in \mathcal{M}(\Omega)$ we define

$$\mathcal{R}(\mu, \mathcal{U}) := \frac{1}{4}\int_\Omega \Psi\left(\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\nu_\mu}\right)\mathrm{d}\nu_\mu, \tag{4.21}$$

provided $\mathcal{U} \ll \nu_\mu$, and $\mathcal{R}(\mu, \mathcal{U}) = +\infty$ else. Given moreover a measurable function $\xi : \Omega \to \mathbb{R}$ we define

$$\mathcal{R}^*(\mu, \xi) := \frac{1}{4}\int_\Omega \Psi^*(\xi)\,\mathrm{d}\nu_\mu. \tag{4.22}$$

Finally, we define

$$D_{\Psi^*}(\mu) := \int_\Omega G_{\Psi^*}(ff_*, f'f'_*)B\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \tag{4.23}$$

provided μ is absolutely continuous with density f, and set $D_{\Psi^*}(\mu) = +\infty$ otherwise.

The functional D_{Ψ^*} takes over the role of the entropy dissipation and is formally given by

$$D_{\Psi^*}(\mu) = \mathcal{R}^*(\mu, -\bar{\nabla}\log f)$$

provided μ has density f.

Note that the primal dissipation potential can be rewritten as the integral functional $\mathcal{R}(\mu, \mathcal{U}) = \mathcal{F}_\beta(\mu^1, \mu^2, \mathcal{U})$ using the notation (2.12), with the function β defined by

$$\beta(s,t,u) := \begin{cases} \frac{1}{4}\Psi(\frac{u}{\theta(s,t)})\theta(s,t), & \theta(s,t) \neq 0, \\ 0, & \theta(s,t) = 0 \text{ and } u = 0, \\ +\infty, & \theta(s,t) = 0 \text{ and } u \neq 0. \end{cases}$$

Since $\beta : [0,\infty) \times [0,\infty) \times \mathbb{R} \to [0,\infty]$ is again convex and lower semicontinuous, an analogue to Lemma 3.7 shows that $\mathcal{R}$ is convex and lower semicontinuous with respect to weak convergence of μ and weak* convergence of $\mathcal{U}$. Similarly the assumptions on G_{Ψ^*} guarantee that D_{Ψ^*} is convex and lower semicontinuous with respect to weak convergence.

We can now formulate the variational characterization of the Boltzmann equation.

Theorem 4.6. *Let* $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ *with* $(\mu_t)_t \subset \mathcal{P}_p(\mathbb{R}^d)$ *be such that*

$$\mathcal{H}(\mu_0) < \infty, \qquad \int_0^T \mathcal{E}_p(\mu_t)\,\mathrm{d}t < \infty \tag{4.24}$$

with $p = 2 + \max(\gamma, 0)$*. Then*

$$\mathcal{L}_T(\mu, \mathcal{U}) := \mathcal{H}(\mu_T) - \mathcal{H}(\mu_0) + \int_0^T [D_{\Psi^*}(\mu_t) + \mathcal{R}(\mu_t, \mathcal{U}_t)]\,\mathrm{d}t \geq 0. \tag{4.25}$$

Moreover, $\mathcal{L}_T(\mu, \mathcal{U}) = 0$ *if and only if* μ_t *has density* f_t *with* $(f_t)_t$ *a solution to the homogeneous Boltzmann equation.*

Proof. We can follow with small modifications the proof of Proposition 4.1 and Theorem 4.3. Let us highlight the main steps.

First note that by the convex duality of Ψ and Ψ^*, for $s, t > 0$ we have the estimate

$$\begin{aligned}
\frac{1}{4}|(\log t - \log s)w| &= \frac{1}{4}\left|(\log s - \log t)\frac{w}{\theta(s,t)}\right|\theta(s,t) \\
&\leq \frac{1}{4}\Psi\left(\frac{w}{\theta(s,t)}\right)\theta(s,t) + \frac{1}{4}\Psi^*(\log s - \log t)\theta(s,t) \\
&= \frac{1}{4}\Psi\left(\frac{w}{\theta(s,t)}\right)\theta(s,t) + G_{\Psi^*}(s,t).
\end{aligned} \tag{4.26}$$

Moreover, we have equality,

$$\frac{1}{4}(\log t - \log s)w = -\frac{1}{4}\Psi\left(\frac{w}{\theta(s,t)}\right)\theta(s,t) - G_{\Psi^*}(s,t), \tag{4.27}$$

if and only if $w = (\Psi^*)'(\log s - \log t)\theta(s,t)$ and hence by Assumption 4.4 if and only if $w = s - t$.

To prove (4.25) we can assume that $\int_0^T [D_{\Psi^*}(\mu_t) + \mathcal{R}(\mu_t, \mathcal{U}_t)]\,\mathrm{d}t < \infty$ since otherwise the estimate holds trivially. Then arguing as in Lemma 3.6 we infer for a.e. t that μ_t and $\mathcal{U}_t$ have densities f_t and $U_t = W_t\theta(f_t)B$ respectively, where $\theta(f) := \theta(ff_*, f', f'_*)$. In particular, the set of v, v_*, ω where $\theta(f_t) = 0$ is negligible for $\mathcal{U}_t$.

The first step is then to establish the chain rule

$$\mathcal{H}(\mu_t) - \mathcal{H}(\mu_s) = \int_s^t \frac{1}{4}\int \bar{\nabla}\log f_r\,\mathrm{d}\mathcal{U}_r\,\mathrm{d}r \quad \forall 0 \leq s \leq t \leq T. \tag{4.28}$$

According to the previous comment, the integral can be restricted to the set $\{\theta(f_r) > 0\}$ and is thus well-defined. One argues as in the proof of Proposition 4.1 by regularization and replaces $\frac{1}{2}\mathcal{A}(\mu, \mathcal{U})$ and $\frac{1}{2}D(\mu)$ with $\mathcal{R}(\mu, \mathcal{U})$ and $D_{\Psi^*}(\mu)$. The estimate (4.26) yields the necessary majorants. The essential properties of $\mathcal{A}$ and D used in the proof were convexity and lower semicontinuity which still hold under our assumptions for $\mathcal{R}$ and D_{Ψ^*}.

From (4.28) we obtain (4.25) by estimating as in (4.26) on the set $\{\theta(f) > 0\}$:

$$\begin{aligned}\frac{1}{4}\bar{\nabla}\log f \cdot U &= -\frac{1}{4}\theta(f)(-\bar{\nabla} f)B\frac{U}{\theta(f)B} \\ &\geq -\frac{1}{4}\Psi\left(\frac{U}{\theta(f)B}\right)\theta(f)B - \frac{1}{4}\Psi^*(-\bar{\nabla} f)\theta(f)B.\end{aligned}$$

Now, assume that equality is attained in (4.25). Then for a.e. t and a.e. on $\{\theta(f_t) > 0\}$ we must have

$$U = (\Psi^*)'(-\bar{\nabla} f)\theta(f)B = [ff_* - f'f'_*]B.$$

On the set where $\theta(f_t) = 0$ and hence $U_t = 0$ we must have $G_{\Psi^*}(ff_*, f'f'_*) = 0$. But the assumptions on G_{Ψ^*} and θ then imply that on $\{\theta(f) = 0\}$ we have $ff_* = f'f'_* = 0$ and thus again $U = ff_* - f'f'_*$, i.e. (f_t) is a solution to the Boltzmann equation. Conversely, a solution to the Boltzmann equation leads to equality in (4.25). ∎

Example 4.7. Finally, let us highlight two examples of generalized gradient structures compatible with our Assumption 4.4.

- *Quadratic gradient structure:* Choosing

$$\theta = \Lambda, \quad \Psi^*(r) = \Psi(r) = \tfrac{1}{2}r^2,$$

  with Λ the logarithmic mean defined in (3.4), we recover the framework considered in the previous section and the gradient flow characterization of Theorem 4.3. Namely, $D_{\Psi^*}(\mu) = \frac{1}{2}D(\mu)$ and $\mathcal{R}(\mu, \mathcal{U}) = \frac{1}{2}\mathcal{A}(\mu, \mathcal{U})$ yields the action functional defined in Definition 3.5.

- cosh *structure:* Let us set

$$\theta(s,t) = \sqrt{st}, \quad \Psi^*(\xi) = 4(\cosh(\xi/2) - 1).$$

  Then we obtain

$$\Psi(s) = 2s\log\left(\frac{s + \sqrt{s^2+4}}{2}\right) - \sqrt{s^2+4} + 4,$$

  as well as

$$G_{\Psi^*}(s,t) = \tfrac{1}{4}\Psi^*(\log t - \log s)\theta(s,t) = \tfrac{1}{2}(\sqrt{s} - \sqrt{t})^2.$$

Let us mention that generalized gradient structures involving cosh such as the latter example arise naturally in the context of large deviations for jump processes. Namely, the associated functional $\mathcal{L}_T$ from (4.25) is the path-level large deviation rate functional for the empirical measure of N independent copies of the process in the limit $N \to \infty$; see for instance [26, 31]. In the present setting, the second structure in the above examples can formally be related to the large deviations of the Kac process considered in the next section. However, to the best of our knowledge no full large deviation principle for this classical Kac system with conservation of momentum and energy has been established yet. In [5] a large deviation principle for a Kac type system with only conservation of momentum is established, which gives rise to a cosh-type generalized gradient structure for the corresponding limiting Boltzmann type equation. A similar structure has also been used in [4] to give a variational characterization of linear spatially inhomogeneous Boltzmann equations.

5. Consistency with Kac's random walk

In this section we give a new proof of the convergence of Kac's random walk to the solution of the spatially homogeneous Boltzmann equation (see Theorem 1.3), exploiting the fact that both evolutions have a gradient flow structure. We recall from Section 1.2 that Kac's random walk is the continuous time Markov chain on

$$\mathcal{X}_N := \Big\{(v_1,\dots,v_N)\in\mathbb{R}^{dN}\;\Big|\;\sum_{i=1}^N v_i = 0,\ \sum_{i=1}^N |v_i|^2 = Nd\Big\}$$

with generator

$$Af(\boldsymbol{v}) = \frac{1}{N}\int_{S^{d-1}}\sum_{i<j}[f(R_{ij}^\omega\boldsymbol{v}) - f(\boldsymbol{v})]B(v_i - v_j,\omega)\,\mathrm{d}\omega, \tag{5.1}$$

where $R_{ij}^\omega\boldsymbol{v} = (v_1,\dots,v_i',\dots,v_j',\dots,v_N)$, with $v_i' = v_i - \langle v_i - v_j,\omega\rangle\omega$ and $v_j' = v_j + \langle v_i - v_j,\omega\rangle\omega$. Let us denote by π_N the normalized Hausdorff measure on $\mathcal{X}_N$ and note that the Markov chain is reversible with respect to π_N. Denoting by μ_t^N the law of the chain starting from μ_0^N, its density f_t^N with respect to π_N satisfies Kac's master equation

$$\partial_t f_t^N = Af_t^N. \tag{5.2}$$

We recall the following result. For $\boldsymbol{v}\in\mathbb{R}^{Nd}$ and $p\geq 1$ we set

$$\mathcal{E}_p^N(\boldsymbol{v}) := \frac{1}{N}\sum_{i=1}^N |v_i|^p.$$

Lemma 5.1 (Propagation of moments for Kac's random walk, [27, Lem. 5.3]). *Let μ_0^N be an initial condition with $\langle\mathcal{E}_p^N,\mu_0^N\rangle = \int\mathcal{E}_p^N\,\mathrm{d}\mu_0^N < \infty$. Then the law $(\mu_t^N)_{t\geq 0}$ of Kac's random walk satisfies*

$$\sup_{t\geq 0}\langle\mathcal{E}_p^N,\mu_t^N\rangle \leq \max\{C_p,\langle\mathcal{E}_p^N,\mu_0^N\rangle\}$$

for some constant C_p depending only on p.

We will first detail the gradient flow structure of the master equation.

5.1. Gradient flow structure

Kac's random walk possesses the structure of a gradient flow in $\mathcal{P}(\mathcal{X}_N)$ of the relative entropy $\mathcal{H}(\cdot|\pi_N)$ with respect to a suitable geometry on $\mathcal{P}(\mathcal{X}_N)$ as we shall now describe.

For general Markov chains on finite state spaces a gradient flow structure has been discovered in [23, 25]. Here we briefly show how to extend this result to the present case of the continuous state space $\mathcal{X}_N$. The construction is similar to that in Section 3; see also [15]. Let us stress, however, that for the purpose of showing consistency with the Boltzmann equation it will only be important to know that the solution $(f_t)_t$ to (5.2) satisfies the energy identity $J_T^N(f) = 0$; see (5.5) below.

We introduce a jump kernel on $\mathcal{X}_N$ by setting

$$J(\boldsymbol{v}, \mathrm{d}\boldsymbol{u}) = \frac{1}{2N} \int_{S^{d-1}} \sum_{i,j=1}^{N} \delta_{R_{ij}^{\omega}\boldsymbol{v}}(\mathrm{d}\boldsymbol{u}) B(v_i - v_j, \omega)\, \mathrm{d}\omega.$$

Given a probability measure $\mu \in \mathcal{P}(\mathcal{X}_N)$ we define $\mu^1, \mu^2 \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ via

$$\mathrm{d}\mu^1(\boldsymbol{v}, \boldsymbol{u}) = J(\boldsymbol{v}, \mathrm{d}\boldsymbol{u})\, \mathrm{d}\mu(\boldsymbol{v}), \qquad \mathrm{d}\mu^2(\boldsymbol{v}, \boldsymbol{u}) = J(\boldsymbol{u}, \mathrm{d}\boldsymbol{v})\, \mathrm{d}\mu(\boldsymbol{u}). \tag{5.3}$$

For a pair $(\mu, \mathcal{V})$ with $\mu \in \mathcal{P}(\mathcal{X}_N)$ and $\mathcal{V} \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ we define the action

$$\mathcal{A}^N(\mu, \mathcal{V}) := 2\mathcal{F}_\alpha(\mu^1, \mu^2, \mathcal{V}),$$

where $\mathcal{F}_\alpha$ is defined in (3.7). We define a distance on $\mathcal{P}(\mathcal{X}_N)$ by setting

$$\mathcal{W}_N(\mu_0, \mu_1)^2 := \inf_{\mu, \mathcal{V}} \int_0^1 \mathcal{A}^N(\mu_t, \mathcal{V}_t)\, \mathrm{d}t,$$

where the infimum is taken over all curves $(\mu_t)_{t\in[0,1]}$ connecting μ_0 to μ_1 and all $(\mathcal{V}_t)_{t\in[0,1]}$ subject to the continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{X}_N} \varphi\, \mathrm{d}\mu_t - \frac{1}{2} \int_{\mathcal{X}_N^2} [\varphi(\boldsymbol{u}) - \varphi(\boldsymbol{v})]\, \mathrm{d}\mathcal{V}_t(\boldsymbol{v}, \boldsymbol{u}) = 0, \qquad \forall \varphi \in C_b(\mathcal{X}_N).$$

It follows from the results in [15, Thm. 4.4, Prop. 4.3], by considering J as a jump kernel on the ambient space $\mathbb{R}^{dN}$, that $\mathcal{W}_N$ defines a distance and that the infimum in the definition is attained by an optimal pair $(\mu, \mathcal{V})$. For a curve $(\mu_t)_{t\in[0,T]}$ in $\mathcal{P}(\mathcal{X}_N)$ we define its action by

$$\mathcal{A}_T^N(\mu) := \inf \left\{ \int_0^T \mathcal{A}_N(\mu_t, \mathcal{V}_t)\, \mathrm{d}t \right\},$$

where the infimum is taken over all $(\mathcal{V}_t)_t$ such that $(\mu, \mathcal{V})$ satisfy the continuity equation. There exists an optimal $\mathcal{V}$ attaining the infimum; see [15, Prop. 4.3]. In fact, for a.e. t, $\mathcal{A}^N(\mu_t, \mathcal{V}_t)$ equals the metric derivative of the curve with respect to $\mathcal{W}_N$. We define the *entropy dissipation* of $\mu \in \mathcal{P}(\mathcal{X}_N)$ by

$$D^N(\mu) = \frac{1}{4N} \int_{\mathcal{X}_N} \int_{S^{d-1}} \sum_{i,j} [f(R_{ij}^{\omega}\boldsymbol{v}) - f(\boldsymbol{v})] \times [\log f(R_{ij}^{\omega}\boldsymbol{v}) - \log f(\boldsymbol{v})] B(v_i - v_j, \omega)\, \mathrm{d}\omega\, \mathrm{d}\pi_N(\boldsymbol{v})$$

provided $\mu = f\pi_N$, and we set $D^N(\mu) = +\infty$ if μ is not absolutely continuous. Note that along any solution f_t to the master equation (5.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f_t|\pi_N) = -D^N(f_t). \tag{5.4}$$

Proposition 5.2. *For any curve $(\mu_t)_{t\in[0,T]}$ in $\mathcal{P}(\mathcal{X}_N)$ with $\mathcal{H}(\mu_0|\pi_N) < \infty$ we have*

$$J_T^N(\mu) = \mathcal{H}(\mu_T|\pi_N) - \mathcal{H}_N(\mu_0|\pi_N) + \frac{1}{2}\int_0^T D^N(\mu_t)\,\mathrm{d}t + \frac{1}{2}\mathcal{A}_T^N(\mu) \geq 0. \tag{5.5}$$

Moreover, $J_T^N(\mu) = 0$ if and only if $\mu_t = f_t\pi_N$ where f_t solves (5.2).

Proof. We will focus on showing that any solution $(\mu_t)_t$ to the master equation (5.2) satisfies $J_T^N(\mu) = 0$, since this will be used in what follows. The other statements can be obtained by following a line of reasoning as in Section 4, namely establishing a chain rule for the entropy analogous to Proposition 4.1 via a regularization argument (in fact, the situation is much simpler due to linearity of the master equation).

Let $\mu_t = f_t\pi^N$ be a solution to the master equation (5.2). Then the couple $(\mu_t, \mathcal{V}_t)$ solves the continuity equation if we choose

$$\mathrm{d}\mathcal{V}_t(\boldsymbol{v},\boldsymbol{u}) = \Psi_t(\boldsymbol{v},\boldsymbol{u})\Lambda(f_t(\boldsymbol{v}), f_t(\boldsymbol{u}))\, J(\boldsymbol{v},\mathrm{d}\boldsymbol{u})\,\pi^N(\mathrm{d}v)$$

with $\Psi_t(\boldsymbol{v},\boldsymbol{u}) = \log f_t(\boldsymbol{u}) - \log f_t(\boldsymbol{v})$. Note moreover that

$$\mathcal{A}(\mu_t, \mathcal{V}_t) = D_N(\mu_t).$$

Thus, integrating (5.4) yields $J_T(\mu) = 0$. ∎

5.2. *Convergence to the Boltzmann equation*

In this section we will give a new proof that the distribution of the empirical measure of N particles evolving by Kac's random walk converges to the solution of the homogeneous Boltzmann equation as $N \to \infty$. For convenience let us recall the setup and the convergence statement.

Consider the map assigning to a configuration in $\mathcal{X}_N$ its empirical measure

$$L_N : \mathcal{X}_N \to \mathcal{P}(\mathbb{R}^d), \quad \boldsymbol{v} \mapsto \frac{1}{N}\sum_{i=1}^N \delta_{v_i}.$$

Let us set

$$\mathcal{P}_*(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \mathcal{M}(\mu) = 0,\ \mathcal{E}(\mu) = d\},$$

the set of probability measures with zero momentum and energy d (recall (2.2)). Note that for any $\boldsymbol{v} \in \mathcal{X}_N$ we have $L_N\boldsymbol{v} \in \mathcal{P}_*(\mathbb{R}^d)$. Let us denote by $M = M^{0,d}$ the standard Maxwellian distribution and by $\mathcal{H}(\mu|M)$ the relative entropy (see (2.4)). We consider $\mathcal{P}_*(\mathbb{R}^d)$ as a subset of $\mathcal{P}(\mathbb{R}^d)$ equipped with the topology of weak convergence.

Theorem 5.3. *For each N let $(\mu_t^N)_{t\geq 0}$ be the law of Kac's random walk starting from μ_0^N and let $c_t^N := (L_N)_\# \mu_t^N$ be the law of the empirical measures. Assume that μ_0^N is well-prepared for some $\nu_0 = f_0 \mathcal{L} \in \mathcal{P}_*(\mathbb{R}^d)$ (if $\gamma > 0$ assume further $\mathcal{E}_4(\nu_0) < \infty$) with $\mathcal{H}(\nu_0|M) < \infty$ in the sense that as $N \to \infty$,*

$$c_0^N \rightharpoonup \delta_{\nu_0}, \quad \frac{1}{N}\mathcal{H}(\mu_0^N|\pi_N) \to \mathcal{H}(\nu_0|M).$$

Assume further that for some $p > 2 + \max(\gamma, 0)$,

$$\sup_N \langle \mathcal{E}_p^N, \mu_0^N \rangle < \infty.$$

Then, for all $t > 0$, as $N \to \infty$ we have

$$c_t^N \rightharpoonup \delta_{\nu_t}, \quad \frac{1}{N}\mathcal{H}(\mu_t^N|\pi_N) \to \mathcal{H}(\nu_t|M), \tag{5.6}$$

where $\nu_t = f_t \mathcal{L}$ and f_t is the unique solution to the spatially homogeneous Boltzmann equation with initial datum f_0.

The strategy of the proof will be to pass to the limit in the variational formulation of the master equation and obtain the variational formulation of the Boltzmann equation. The key ingredient to this will be to establish lim inf estimates relating the entropy, dissipation and action for the Kac walk and the Boltzmann equation. Although the proofs of the latter might seem long, the core argument is rather simple and boils down to the lower semicontinuity of integral functionals stated in Lemma 2.5. A non-trivial additional ingredient that we develop is a probabilistic representation result that allows us to view certain curves in $\mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ as superpositions of curves in $\mathcal{P}_*(\mathbb{R}^d)$; see Proposition 5.5.

Let us now first give the proof of the convergence theorem. Afterwards we will develop the necessary ingredients.

Proof of Theorem 5.3. By Proposition 5.2 we know that $(\mu_t^N)_{t\geq 0}$ satisfies

$$\mathcal{H}(\mu_T^N|\pi_N) - \mathcal{H}(\mu_0^N|\pi_N) + \frac{1}{2}\int_0^T D^N(\mu_t^N)\,\mathrm{d}t + \frac{1}{2}\mathcal{A}_T^N(\mu) = 0. \tag{5.7}$$

Together with the convergence of $\mathcal{H}(\mu_0^N|\pi_N)/N$ this implies in particular

$$\sup_N \frac{1}{N}\mathcal{A}_T^N(\mu^N) < \infty.$$

The compactness result of Lemma 5.4 then shows that for every continuous curve $(c_t)_{t\geq 0}$ in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ with c_t concentrated on $\mathcal{P}_*(\mathbb{R}^d) \cap \mathcal{P}_{p,E}(\mathbb{R}^d)$ for all t and suitable $E > 0$, up to a subsequence we have $c_t^N \rightharpoonup c_t$ weakly for all t. A priori, c_t is not a Dirac measure. However, by the superposition principle of Proposition 5.5 the curve $(c_t)_{t\in[0,T]}$ can be represented as $c_t = (e_t)_\# \Theta$ for a probability measure Θ on $C([0,T];\mathcal{P}(\mathbb{R}^d))$. Thanks to the lim inf inequalities for the entropy, dissipation and action given by (5.28), (5.14) and (5.13), dividing by N in (5.7) and passing to lim inf we obtain

$$\int\left[\mathcal{H}(\eta_T)-\mathcal{H}(\eta_0)+\frac{1}{2}\int_0^T D(\eta_t)\,\mathrm{d}t+\frac{1}{2}\mathcal{A}_T(\eta)\right]\mathrm{d}\Theta(\eta)\leq 0, \tag{5.8}$$

using also the fact that $\mathcal{H}(\eta|M)=\mathcal{H}(\eta)+\mathcal{H}(M)$ for $\eta\in\mathcal{P}_*(\mathbb{R}^d)$ and that $\eta_0,\eta_T\in\mathcal{P}_*(\mathbb{R}^d)$ for Θ-a.e. η. By Theorem 4.3 the integrand is non-negative. Thus we have in fact equality in (5.8) and we infer that Θ is concentrated on gradient flow curves $(\eta_t)_t$, i.e. satisfying $J_T(\eta)=0$. Since Θ-a.s. $\eta_0=\nu_0$ and the unique gradient flow curve starting from ν_0 is given by $\nu_t=f_t\mathcal{L}$ with f_t the solution to the Boltzmann equation with initial datum f_0, we infer that $c_t=(e_t)_\#\Theta=\delta_{\nu_t}$ for all t and the convergence of c_t^N to δ_{ν_t} holds for the full sequence. Finally, we prove (5.6). From the previous discussion we retain that

$$\begin{aligned}0\geq\liminf_N\frac{1}{N}J_T^N(\mu^N)-J_T(\nu)&=\liminf_N\frac{1}{N}\mathcal{H}(\mu_T^N|\pi_N)-\mathcal{H}(\nu_T|M)\\&+\frac{1}{2}\left[\liminf_N\frac{1}{N}\int_0^T D_N(\mu_t^N)\,\mathrm{d}t+\mathcal{A}_T^N(\mu^N)-\int_0^T D(\nu_t)\,\mathrm{d}t+\mathcal{A}_T(\nu)\right]\geq 0.\end{aligned}$$

Using again (5.28), (5.13), (5.14), we infer that we have equality:

$$\liminf_N\frac{1}{N}\mathcal{H}(\mu_t^N|\pi_N)=\mathcal{H}(\nu_t|M).$$

Since by the same argument this must hold for any subsequence, we conclude the convergence (5.6) for the full sequence. ∎

We now develop the ingredients to the previous proof. We will first show that any sequence of curves in $\mathcal{P}(\mathcal{X}_N)$ with uniformly bounded action after passing to the empirical measure admits a limit curve in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$. Then we will give a representation of this curve as a superposition of curves in $\mathcal{P}(\mathbb{R}^d)$ and establish lim inf inequalities for the action and dissipation of the limit curve. Finally, we prove the lim inf inequality for the entropy.

5.2.1. Convergence to a limit curve.

Lemma 5.4. *Let $(\mu_t^N)_{t\in[0,T]}$ be a sequence of curves in $\mathcal{P}(\mathcal{X}_N)$ such that*

$$\sup_N\frac{1}{N}\mathcal{A}_T^N(\mu^N)<\infty, \tag{5.9}$$

and for some $p'>2+\max(\gamma,0)$,

$$\sup_N\sup_{t\in[0,T]}\langle\mathcal{E}_p^N,\mu_t^N\rangle<\infty. \tag{5.10}$$

Put $c_t^N=(L_N)_\#\mu_t^N$. Then there exists a continuous curve $(c_t)_{t\in[0,T]}$ in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ such that up to a subsequence we have $c_t^N\rightharpoonup c_t$ weakly and c_t is concentrated on $\mathcal{P}_*(\mathbb{R}^d)$ for all $t\in[0,T]$.

Proof. We consider the set $\mathcal{P}_{2,E}(\mathbb{R}^d)$ of probability measures with energy less than E, with $E = d$ (recall (2.3)). Recall that $\mathcal{P}_{2,E}(\mathbb{R}^d)$ is compact with respect to weak convergence, hence also $\mathcal{P}(\mathcal{P}_{2,E}(\mathbb{R}^d))$ is compact. On $\mathcal{P}_{2,E}(\mathbb{R}^d)$, weak convergence is equivalent to convergence of the first moment, or convergence in the L^1-Wasserstein distance W_1. Let us denote by $\widetilde{W}_1$ the L^1-Wasserstein distance on $\mathcal{P}(\mathcal{P}_{2,E}(\mathbb{R}^d))$ induced by the L^1-Wasserstein distance W_1 on $\mathcal{P}_{2,E}(\mathbb{R}^d)$. Since W_1 is bounded on $\mathcal{P}_{2,E}(\mathbb{R}^d)$, the space $(\mathcal{P}(\mathcal{P}_{2,E}(\mathbb{R}^d)), \widetilde{W}_1)$ is compact. We claim that

$$\mathcal{W}_N(\mu_s^N, \mu_t^N) \geq \frac{C}{\sqrt{N}} W_{1,d}(\mu_s^N, \mu_t^N) \geq C\sqrt{N}\, \widetilde{W}_1(c_s^N, c_t^N) \tag{5.11}$$

for some universal constant $C > 0$. Indeed, to prove the first inequality we view μ_s^N, μ_t^N as measures on $\mathbb{R}^{Nd}$ equipped with the distance $d(\boldsymbol{v}, \boldsymbol{u}) = \sum_i |v_i - u_i|$ and let $W_{1,d}$ denote the L^1-Wasserstein distance induced by d. Then the estimate follows from [15, Prop. 4.5] once we note that $\int_{\mathbb{R}^{Nd}} d(\boldsymbol{v}, \boldsymbol{u})^2 J(\boldsymbol{v}, \mathrm{d}\boldsymbol{u}) \leq CN$, where C depends on the moment bound in (5.10). The second inequality follows from the fact that the map L_N is $1/N$-Lipschitz from $(\mathbb{R}^{Nd}, d)$ to $(\mathcal{P}(\mathbb{R}^d), W_1)$. Together with (5.11), (5.9) implies that the curves $(c_t^N)_t$ are uniformly equicontinuous in $\mathcal{P}(\mathcal{P}_{2,E}(\mathbb{R}^d))$ with respect to the distance $\widetilde{W}_1$. Thus, the Arzelà–Ascoli theorem implies that there exists a continuous curve $(c_t)_{t\in[0,T]}$ in $\mathcal{P}(\mathcal{P}_{2,E}(\mathbb{R}^d))$ such that up to extraction of a subsequence, $c_t^N \rightharpoonup c_t$ weakly for all $t \in [0, T]$.

Finally, assume in addition (5.10) and let us show that c_t is concentrated on $\mathcal{P}_*(\mathbb{R}^d)$ for all t. We need to show $c_t(\{\mathcal{M} = 0, \mathcal{E} = d\}) = 1$. Since $c_t^N(\{\mathcal{M} = 0\}) = 1$ and $\mathcal{M}$ is continuous on $\mathcal{P}_{2,E}(\mathbb{R}^d)$, and hence $\{\mathcal{M} = 0\}$ is closed, the weak convergence $c_t^N \rightharpoonup c_t$ implies that $c_t(\{\mathcal{M} = 0\}) = 1$. It remains to show that $c_t(\{\mathcal{E} = d\}) = 1$. Since c_t is concentrated on $\mathcal{P}_{2,E}(\mathbb{R}^d) = \{\mathcal{E} \leq d\}$, it suffices to show that $\langle \mathcal{E}, c_t\rangle = \lim_N \langle \mathcal{E}, c_t^N\rangle = d$. Set $\mathcal{E}_p(\eta) := \int |v|^p \,\mathrm{d}\eta(v)$. Then (5.10) implies that for any t,

$$\sup_N \langle \mathcal{E}_p, c_t^N\rangle < \infty. \tag{5.12}$$

Note that $\mathcal{E}_2 = \mathcal{E}$. Since by Jensen's inequality we have $\mathcal{E}_2(\nu)^{p/2} \leq \mathcal{E}_p(\nu)$, (5.12) readily shows that $\mathcal{E}_2$ is uniformly integrable with respect to c_t^N. Moreover, $\sup_N c_t^N(\{\mathcal{E}_{2+\varepsilon} \geq R\}) \to 0$ as $R \to \infty$ for $\varepsilon < p - 2$ and $\mathcal{E}_2$ is continuous on $\{\mathcal{E}_{2+\varepsilon} \leq R\}$. Thus we obtain the desired convergence $\langle \mathcal{E}_2, c_t\rangle = \lim_N \langle \mathcal{E}_2, c_t^N\rangle$ (see e.g. [1, Prop. 5.1.10]). ■

5.2.2. *Superposition principle and limits for the action and dissipation*

Proposition 5.5 (Superposition principle for the limit curve). *Let $(\mu_t^N)_{t\in[0,T]}$ be a sequence of curves in $\mathcal{P}(\mathcal{X}_N)$ satisfying (5.9) and (5.10), put $c_t^N = (L_N)_\# \mu_t^N$, and let $(c_t)_{t\in[0,T]}$ be the limit curve of Lemma 5.4. Then there exists a Borel probability measure Θ on $C([0, T]; \mathcal{P}(\mathbb{R}^d))$ and a Borel family $(\mathcal{U}_t^\eta)_{t\in[0,T],\, \eta\in\mathcal{P}(\mathbb{R}^d)}$ of measures such that the following hold:*

- $c_t = (e_t)_\# \Theta$ *for all* $t \in [0, T]$,

- *for Θ-a.e. curve $(\eta_t)_{t\in[0,T]}$, the pair $(\eta_t, \mathcal{U}_t^{\eta_t})_{t\in[0,T]}$ belongs to $\mathcal{CRE}_T$, and $\eta_t \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ for $p = 2 + \max(\gamma, 0)$ and suitable $E > 0$ and all $t \in [0, T]$.*

Proposition 5.6 (lim inf inequality for action and dissipation). *In the setting of Proposition 5.5 we have*

$$\liminf_N \frac{1}{N} \mathcal{A}_T^N(\mu^N) \geq \int \mathcal{A}_T(\eta)\,\mathrm{d}\Theta(\eta), \tag{5.13}$$

$$\liminf_N \frac{1}{N} \int_0^T D^N(\mu_t^N)\,\mathrm{d}t \geq \int \left[\int_0^T D(\eta_t)\,\mathrm{d}t\right] \mathrm{d}\Theta(\eta), \tag{5.14}$$

where $\mathcal{A}_T(\eta)$, $D(\eta)$ are the action and dissipation defined in (3.13), (2.8).

In order to prove Proposition 5.5, we will describe curves in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d)$ as curves in $\mathcal{P}(\mathbb{R}^\infty)$ by choosing a countable number of coordinates given by integrals against test functions. This allows one to employ a superposition principle for solutions to the continuity equation over $\mathbb{R}^\infty$ by Ambrosio and Trevisan [2]. Let us briefly recall this result.

Consider $\mathbb{R}^\infty = \mathbb{R}^\mathbb{N}$, let $p_i : \mathbb{R}^\infty \to \mathbb{R}$ be the natural projections for $i \in \mathbb{N}$ and let $\pi_n = (p_1, \dots, p_n) : \mathbb{R}^\infty \to \mathbb{R}^n$. Equip $\mathbb{R}^\infty$ with the separable and complete distance

$$d_\infty(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |p_n(x) - p_n(y)|\}.$$

In a similar way, $C([0, T]; \mathbb{R}^\infty)$ can be equipped with a separable and complete distance. We denote by $AC_w([0, T]; \mathbb{R}^\infty)$ the subset of $C([0, T]; \mathbb{R}^\infty)$ consisting of all γ such that $p_i \circ \gamma \in AC([0, T]; \mathbb{R})$ for all i.

A function $F : \mathbb{R}^\infty \to \mathbb{R}$ is called *smooth cylindrical* if it is of the form

$$F(x) = \psi(p_1(x), \dots, p_n(x))$$

for some $\psi \in C_b^1(\mathbb{R}^n)$ and $n \in \mathbb{N}$. Its gradient $\nabla F : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined by

$$\nabla F(x) = \big(\partial_1 \psi(\pi_n(x)), \dots, \partial_n \psi(\pi_n(x)), 0, 0, \dots\big).$$

Then we have the following representation result.

Theorem 5.7 ([2, Thm. 7.1]). *Let $\boldsymbol{b} : (0, T) \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ be a Borel vector field and let $(\nu_t)_{t\in(0,T)}$ be a family of Borel probability measures on $\mathbb{R}^\infty$ continuous in duality with smooth cylinder functions satisfying*

$$\int_0^T |p_i(\boldsymbol{b}_t)|\,\mathrm{d}\nu_t\,\mathrm{d}t < \infty \quad \forall i \in \mathbb{N}, \tag{5.15}$$

and in the sense of distributions in $(0, T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int F\,\mathrm{d}\nu_t = \int (\boldsymbol{b}_t, \nabla F)\,\mathrm{d}\nu_t \quad \forall F \text{ smooth cylindrical.} \tag{5.16}$$

Then there exists a Borel probability measure $\boldsymbol{\lambda}$ on $C([0, T]; \mathbb{R}^\infty)$ satisfying $(e_t)_\# \boldsymbol{\lambda} = \nu_t$ for all t, concentrated on $\gamma \in AC_w([0, T], \mathbb{R}^\infty)$ solving the ODE $\dot\gamma = \boldsymbol{b}_t(\gamma)$ a.e. in $(0, T)$.

Proof of Proposition 5.5. We will proceed in three steps. Starting from a solution to the discrete continuity equation over $\mathcal{X}_N$ we pass to the empirical measure and obtain a limiting family of collision rates $\mathcal{U}_t^\eta$. Then, by choosing integrals against a collection of test functions as coordinates, we describe the limiting curve c via a continuity equation over $\mathbb{R}^\infty$ with a vector field determined by the collision rates $\mathcal{U}_t^\eta$. Finally, we apply the superposition principle for $\mathbb{R}^\infty$ and see that the resulting random curve in $\mathbb{R}^\infty$ is indeed the coordinate description of a random curve (η_t) in $\mathcal{P}(\mathbb{R}^d)$ solving the collision rate equation driven by the rates $\mathcal{U}_t^{\eta_t}$.

Step 1: Limiting collision rate. Recall from Section 5.1 that we can choose measures $\mathcal{V}_t^N \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ such that $\mathcal{A}_T^N(\mu^N) = \int_0^T \mathcal{A}_N(\mu_t^N, \mathcal{V}_t^N)\,\mathrm{d}t$. Let us define the measures $\mathcal{V}^N := \mathcal{V}_t^N\,\mathrm{d}t$ and $\mu^{N,k} := \mu_t^{N,k}\,\mathrm{d}t$, $k = 1,2$, in $\mathcal{M}(\mathcal{X}^N \times \mathcal{X}^N \times [0,T])$. Note that by the structure of the jump kernel J, for any $(\boldsymbol{v}, \boldsymbol{u})$ in the support of $\mu_t^{N,1}, \mu_t^{N,2}$ with $\boldsymbol{v} \neq \boldsymbol{u}$, there exist unique (i,j,ω) with $1 \le i < j \le N$, $\omega \in S^{d-1}$ such that $\boldsymbol{u} = R_{ij}^\omega(\boldsymbol{v})$ (when $\boldsymbol{v} = \boldsymbol{u}$, we pick $i = j$ and ω at random). We push forward $\mathcal{V}^N, \mu^{N,k}$ by the map $(\boldsymbol{v},\boldsymbol{u}) \mapsto (L_N(\boldsymbol{v}), L_N(\boldsymbol{u}), v_i, v_j, \omega)$ with i,j,ω as above. This defines measures $\gamma^N, \beta^{N,k}$ on $\mathcal{P}_{p,E}(\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2 \times S^{d-1} \times [0,T]$. We find that

$$
\begin{aligned}
\mathrm{d}\beta^{N,1}(\eta,\eta',v,v_*,\omega,t) &= \frac{N}{2}\,\delta_{\eta^{N,v,v_*,\omega}}(\mathrm{d}\eta')\,B(v-v_*,\omega)\,\eta(\mathrm{d}v)\,\eta(\mathrm{d}v_*)\,\mathrm{d}\omega\,\mathrm{d}c_t^N(\eta)\,\mathrm{d}t \\
&= \frac{N}{2}\,\delta_{\eta^{N,v,v_*,\omega}}(\mathrm{d}\eta')\,\mathrm{d}\eta^1(v,v_*,\omega)\,\mathrm{d}c_t^N(\eta)\,\mathrm{d}t, \qquad (5.17)\\
\mathrm{d}\beta^{N,2}(\eta,\eta',v,v_*,\omega,t) &= \frac{N}{2}\,\delta_{\eta'^{N,T_\omega^{-1}(v,v_*),\omega}}(\mathrm{d}\eta)\,B(v-v_*,\omega) \\
&\quad\cdot \mathrm{d}(T_\omega)_\# \eta'^{\otimes 2}(v,v_*)\,\mathrm{d}\omega\,\mathrm{d}c_t^N(\eta')\,\mathrm{d}t \\
&= \frac{N}{2}\,\delta_{\eta'^{N,T_\omega^{-1}(v,v_*),\omega}}(\mathrm{d}\eta)\,\mathrm{d}\eta'^2(v,v_*,\omega)\,\mathrm{d}c_t^N(\eta')\,\mathrm{d}t, \qquad (5.18)
\end{aligned}
$$

where we set $\eta^{N,v,v_*,\omega} = \eta + \frac{1}{N}(\delta_{v'} + \delta_{v'_*} - \delta_v - \delta_{v_*})$ with v, v_*, v', v'_* related via (1.3); recall that $c_t^N = (L_N)_\# \mu_t^N$ and recall the notation (3.9). To see this, note that $L_N(\boldsymbol{u}) = L_N(\boldsymbol{v})^{N,v_i,v_j,\omega}$ if $\boldsymbol{u} = R_{ij}^\omega(\boldsymbol{v})$ and we can write

$$
\sum_{i,j=1}^N f(v_i,v_j) = N^2 \int f(v,v_*)\,L_N(\boldsymbol{v})(\mathrm{d}v)\,L_N(\boldsymbol{v})(\mathrm{d}v_*).
$$

To obtain the expression for $\beta^{N,2}$, note further that if $\boldsymbol{v} = R_{i,j}^\omega(\boldsymbol{u})$, then we have $(v_i,v_j) = T_\omega(u_i,u_j)$.

From the weak convergence of c_t^N to c_t for all t granted by Lemma 5.4, we infer that as $N \to \infty$ we have $\frac{2}{N}\beta^{N,k} \rightharpoonup \beta^k$ in duality with C_b where

$$
\mathrm{d}\beta^k(\eta,\eta',v,v_*,\omega,t) = \delta_\eta(\mathrm{d}\eta')\,\mathrm{d}\eta^k(v,v_*,\omega)\,\mathrm{d}c_t(\eta)\,\mathrm{d}t. \qquad (5.19)
$$

From Lemma 2.5 (ii) we infer that

$$
\mathcal{F}_\alpha\left(\frac{2}{N}\beta^{N,1}, \frac{2}{N}\beta^{N,2}, \frac{2}{N}\gamma^N\right) \le \frac{2}{N}\mathcal{F}_\alpha(\mu^{N,1}, \mu^{N,2}, \mathcal{V}^N) = \frac{1}{N}\mathcal{A}_T^N(\mu^N),
$$

and the last expression is bounded by assumption. From Lemma 3.8 we infer as in the proof of Proposition 3.11 that $\frac{2}{N}\gamma^N$ has uniformly bounded variation and hence converges weakly* up to a further subsequence to a limit γ. By lower semicontinuity and homogeneity of $\mathcal{F}_\alpha$ we find

$$\mathcal{F}_\alpha(\beta^1, \beta^2, \gamma) \le \liminf_N \frac{1}{N}\mathcal{A}_T^N(\mu^N). \tag{5.20}$$

As in Lemma 3.6 we infer from finiteness of the left hand side that γ is absolutely continuous with respect to the measure $L := \delta_\eta(\mathrm{d}\eta')\,\Lambda(\eta^1,\eta^2)\,c_t(\mathrm{d}\eta)\,\mathrm{d}t$, where $\Lambda(\eta^1,\eta^2) := \Lambda(\frac{\mathrm{d}\eta^1}{\mathrm{d}\sigma}, \frac{\mathrm{d}\eta^2}{\mathrm{d}\sigma})\,\mathrm{d}\sigma$ for any σ such that $\eta^1,\eta^2 \ll \sigma$. Hence there exists a Borel function $U : \mathcal{P}_{p,E}(\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2 \times S^{d-1} \times [0,T] \to \mathbb{R}$ such that $\gamma = UL$ and we can write

$$\mathrm{d}\gamma(\eta,\eta',v,v_*,\omega,t) = \delta_\eta(\mathrm{d}\eta')\,\mathrm{d}\mathcal{U}_t^\eta(v,v_*,\omega)\,\mathrm{d}c_t(\eta)\,\mathrm{d}t, \tag{5.21}$$

where $(\mathcal{U}_t^\eta)_{\eta,t}$ is the Borel family of measures defined by

$$\mathrm{d}\mathcal{U}_t^\eta(v,v_*,\omega) = U(\eta,\eta,v,v_*,\omega,t)\,\mathrm{d}\Lambda(\eta^1,\eta^2)(v,v_*,\omega).$$

Note further that

$$\mathcal{F}_\alpha(\gamma,\beta^1,\beta^2) = \int_0^T \int \mathcal{A}(\eta,\mathcal{U}_t^\eta)\,\mathrm{d}c_t(\eta)\,\mathrm{d}t. \tag{5.22}$$

Step 2: Continuity equation in $\mathbb{R}^\infty$. We now describe the curve (c_t) as an evolution in $\mathcal{P}(\mathbb{R}^\infty)$. Fix a countable collection $\{f_i\}_{i\in\mathbb{N}}$ of functions that is dense (with respect to uniform convergence) in the set of 1-Lipschitz functions on $\mathbb{R}^d$ vanishing at 0. Define a map $I : \mathcal{P}_{p,E}(\mathbb{R}^d) \to \mathbb{R}^\infty$ by setting

$$I(\eta) := (\langle f_1,\eta\rangle, \langle f_2,\eta\rangle, \ldots),$$

and write $I^m = \pi_m \circ I$. Note that I is injective and continuous with respect to the distance W_1 on $\mathcal{P}_{p,E}(\mathbb{R}^d)$ by Kantorovich duality. $I(\mathcal{P}_{p,E}(\mathbb{R}^d))$ is closed in $\mathbb{R}^\infty$, since $(\mathcal{P}_{p,E}(\mathbb{R}^d), W_1)$ is compact, and $I^{-1} : I(X) \to \mathcal{P}_{p,E}(\mathbb{R}^d)$ is continuous with respect to W_1.

We define a curve $(\nu_t)_{t\in[0,T]}$ via $\nu_t := I_\# c_t$ and note that it is continuous in duality with smooth cylinder functions by continuity of $t \mapsto c_t$. We define a Borel vector field $\boldsymbol{b} : (0,T) \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ via

$$\boldsymbol{b}_t^i(x) = \begin{cases} \frac{1}{4}\int \bar\nabla f_i\,\mathrm{d}\mathcal{U}_t^\eta, & x = I(\eta) \in I(\mathcal{P}_{p,E}(\mathbb{R}^d)),\\ 0, & x \notin I(\mathcal{P}_{p,E}(\mathbb{R}^d)). \end{cases} \tag{5.23}$$

We claim that $(\nu, \boldsymbol{b})$ satisfies the continuity equation in $\mathbb{R}^\infty$, i.e. (5.15), (5.16). Indeed, (5.15) follows from (5.22) and (5.20) with Corollary 3.9. To show (5.16), fix a smooth cylinder function $F(\boldsymbol{x}) = \psi(p_1(x),\ldots,p_n(x))$ and $a \in C_c^\infty(0,T)$. From the continuity equation for $(\mu_t^N, \mathcal{V}_t^N)$ we obtain, after passing to the empirical measure,

$$\int_0^T a'(t)\int F\circ I\,\mathrm{d}c_t^N\,\mathrm{d}t = -\frac{1}{2}\int a(t)[F(I(\eta^{N,v,v_*,\omega})) - F(I(\eta))]\,\mathrm{d}\gamma^N.$$

Note that $F(I(\eta^{N,v,v_*,\omega})) - F(I(\eta)) = \frac{1}{N}\sum_i \partial_i\psi(I^m(\eta))\bar{\nabla} f_i(v,v_*,\omega) + o(1)$. We infer from the convergence of c_t^N to c_t and of $\frac{2}{N}\gamma^N$ to γ and (5.21) that

$$\begin{aligned}\int_0^T a'(t)\int F\circ I \,\mathrm{d}c_t\,\mathrm{d}t &= -\frac{1}{4}\int_0^T a(t)\int\sum_i \partial_i\psi(I^m(\eta))\bar{\nabla} f_i \,\mathrm{d}\mathcal{U}_t^\eta\,\mathrm{d}c_t(\eta)\,\mathrm{d}t\\ &= -\int_0^T a(t)\int\langle \boldsymbol{b}_t,\nabla F\rangle\,\mathrm{d}\nu_t\,\mathrm{d}t,\end{aligned}\tag{5.24}$$

which is (5.16).

Step 3: Probabilistic representation. By Theorem 5.7 there exists a Borel probability measure $\boldsymbol{\lambda}$ on $C([0,T];\mathbb{R}^\infty)$ concentrated on the solutions $\gamma\in AC_w([0,T];\mathbb{R}^\infty)$ to the ODE $\dot\gamma = \boldsymbol{b}_t(\gamma)$ such that $(e_t)_\#\boldsymbol{\lambda} = \nu_t$ for all t. Since ν_t is concentrated on the closed set $I(\mathcal{P}_{p,E}(\mathbb{R}^d))$ for all t, we find that $\boldsymbol{x}_t\in I(\mathcal{P}_{p,E}(\mathbb{R}^d))$ for all $t\in[0,T]$ and $\boldsymbol{\lambda}$ a.e. γ. Thus we can set $\Theta = \iota_\#\boldsymbol{\lambda}$, where ι maps $\gamma\in C([0,T];\mathbb{R}^\infty)$ to $I^{-1}\circ\gamma\in C([0,T];\mathcal{P}_{p,E}(\mathbb{R}^d))$. It remains to check that Θ has the desired properties.

Since $\nu_t = I_\# c_t$ we immediately get $(e_t)_\#\Theta = c_t$ for all t. Further, since for fixed i we have $\langle f_i,\iota(\gamma)\rangle = \pi_i(\gamma)$, we see by (5.23) that $t\mapsto\langle f_i,\eta_t\rangle$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f_i,\eta_t\rangle = +\frac{1}{4}\int\bar{\nabla} f_i\,\mathrm{d}\mathcal{U}_t^{\eta_t}\quad\text{for a.e. } t\in(0,T)\text{ and }\Theta\text{-a.e. }\eta.\tag{5.25}$$

From (5.22), (5.20) and Corollary 3.9 we deduce that the integrability condition (3.3) holds (with $p=1$). This allows us to extend (5.25) to all Lipschitz f. Hence for Θ-a.e. curve η we conclude that $t\mapsto(\eta_t,\mathcal{U}_t^{\eta_t})$ belongs to $\mathcal{CRE}_T^E$. ∎

Proof of Proposition 5.6. We recall from (5.20) and (5.22) that

$$\int_0^T\int\mathcal{A}(\eta,\mathcal{U}_t^\eta)\,\mathrm{d}c_t(\eta)\,\mathrm{d}t\le\liminf_N\frac{1}{N}\mathcal{A}_T^N(\mu^N).\tag{5.26}$$

We obtain a lim inf estimate for the dissipation in a similar fashion. We note that

$$D^N(\mu_t^N) = 2\mathcal{G}(\mu^{N,1},\mu^{N,2}),$$

where $\mathcal{G}$ is the integral functional defined in the proof of Lemma 2.6. From Lemma 2.5 we obtain

$$\begin{aligned}\liminf_N\int_0^T\frac{1}{N}D^N(\mu_t^N)\,\mathrm{d}t &\ge\liminf_N\mathcal{G}\left(\frac{2}{N}\beta^{N,1},\frac{2}{N}\beta^{N,2}\right)\\ &\ge\mathcal{G}(\beta^1,\beta^2)\\ &=\int_0^T\int D(\eta)\,\mathrm{d}c_t(\eta)\,\mathrm{d}t,\end{aligned}\tag{5.27}$$

where we recall the definition of $\beta^{N,k}$ and β^k from (5.17)–(5.19). By Proposition 5.5 we can then rewrite (5.26) and (5.27) as (5.13) and (5.14), noting that Θ-a.e. curve $(\eta_t)_t$ satisfies $\mathcal{A}_T(\eta)\le\int_0^T\mathcal{A}(\eta_t,\mathcal{U}_t^{\eta_t})\,\mathrm{d}t$. ∎

5.2.3. Limit for the relative entropy.

Proposition 5.8 (lim inf inequality for the entropy). *Let $(\mu^N)_N$ be a sequence of measures in $\mathcal{P}(\mathcal{X}_N)$ such that $c^N = (L_N)_\# \mu^N$ converges weakly to $c \in \mathcal{P}(\mathcal{P}_{p,E}(\mathbb{R}^d))$. Then*

$$\liminf_N \frac{1}{N} \mathcal{H}(\mu^N | \pi_N) \geq \int \mathcal{H}(\eta | M) \, \mathrm{d}c(\eta). \tag{5.28}$$

To prove this result, we will rely on ideas from large deviation theory. Namely, we will exploit the fact that the empirical measure of independent Gaussian distributed points in $\mathbb{R}^d$ satisfies a large deviation principle and that this implies a Γ-lim inf inequality for the relative entropy with respect to the law of this empirical measure. Then we will conclude by relating the entropy with respect to π_N to the entropy with respect to the product Gaussian distribution. Let us briefly explain the concepts we will be using. For background on large deviation theory we refer to [12].

Let $\mathcal{X}$ be a Polish space and equip the set $\mathcal{P}(\mathcal{X})$ of Borel probability measures with the weak topology. Let $I : \mathcal{X} \to [0, \infty]$ be a lower semicontinuous function. A sequence $(m_N)_N$ of measures in $\mathcal{P}(\mathcal{X})$ is said to satisfy a *large deviation principle* with *rate function I* (and speed N) if for any open set O and any closed set C in $\mathcal{X}$ their probabilities are asymptotically controlled as

$$\liminf_N \frac{1}{N} \log m_N(O) \geq - \inf_{x \in O} I(x), \quad \limsup_N \frac{1}{N} \log m_N(C) \leq - \inf_{x \in C} I(x).$$

If the second inequality holds only for all compact sets C, we speak of a *weak large deviation upper bound*. This weak upper bound is equivalent to a Γ-lim inf inequality for the relative entropy:

Lemma 5.9 ([24, Thm. 3.5] (P1)⇔(H2)). *(m_N) satisfies a weak large deviation upper bound with rate function I and speed N if and only if for any sequence (μ_N) in $\mathcal{P}(\mathcal{X})$ converging to μ we have*

$$\liminf_N \frac{1}{N} \mathcal{H}(\mu_N | m_N) \geq \int_{\mathcal{X}} I \, \mathrm{d}\mu.$$

We will also use the following disintegration principle for the relative entropy, which can be verified by a direct computation.

Let $\mathcal{Y}$ be a further Polish space, μ, m two probability measures on $\mathcal{X}$, and $T : \mathcal{X} \to \mathcal{Y}$ be a Borel map. Let $\mu(\cdot \,|\, T = y)$ and $m(\cdot \,|\, T = y)$ denote the disintegration of μ and m with respect to T, i.e. $\mu(\cdot \,|\, T = y)$ are probability measures concentrated on $T^{-1}(y)$ such that for any measurable set $A \subset \mathcal{X}$, $y \mapsto \mu(A \,|\, T = y)$ is measurable, and

$$\mu(A) = \int_{\mathcal{Y}} \mu(A \,|\, T = y) \, \mathrm{d}T_\# \mu(y),$$

and similarly for m. Then

$$\mathcal{H}(\mu | m) = \mathcal{H}(T_\# \mu | T_\# m) + \int_{\mathcal{Y}} \mathcal{H}(\mu(\cdot \,|\, T = y) | m(\cdot \,|\, T = y)) \, \mathrm{d}T_\# \mu(y). \tag{5.29}$$

Since the relative entropy is non-negative, we have in particular

$$\mathcal{H}(\mu|m) \geq \mathcal{H}(T_\# \mu | T_\# m). \tag{5.30}$$

Proof of Proposition 5.8. (i) Let $\gamma_N \in \mathcal{P}(\mathbb{R}^{Nd})$ denote the distribution of N independent standard d-dimensional Gaussian vectors, i.e. γ_N has density

$$g_N(v_1, \dots, v_N) = (2\pi)^{-Nd/2} \exp\Big(-\sum_{i=1}^N \frac{|v_i|^2}{2}\Big)$$

with respect to Lebesgue measure on $\mathbb{R}^{Nd}$. Note that π_N is obtained by conditioning γ_N to $\mathcal{X}_N \subset \mathbb{R}^{dN}$, i.e.

$$\pi_N = \gamma_N(\cdot \mid \mathcal{M}^N = 0, \mathcal{E}^N = d) = \frac{g_N}{\int_{\mathcal{X}_N} g_N \, \mathrm{d}\pi_N} \pi_N,$$

with $\mathcal{M}^N(\boldsymbol{v}) = (1/N) \sum_i v_i$ and $\mathcal{E}^N(\boldsymbol{v}) = (1/N) \sum_i |v_i|^2$. This follows immediately from g_N being constant on $\mathcal{X}_N$.

(ii) We now claim that the analog of (5.28) holds for γ_N: if $\tilde{\mu}^N$ is a sequence in $\mathcal{P}(\mathbb{R}^{Nd})$ such that $c^N = (L_N)_\# \tilde{\mu}^N$ converges weakly to c, then

$$\liminf_N \frac{1}{N} \mathcal{H}(\tilde{\mu}^N | \gamma_N) \geq \int \mathcal{H}(\eta | M) \, \mathrm{d}c(\eta). \tag{5.31}$$

Setting $m_N := (L_N)_\# \gamma_N$ we infer from (5.30) that $\mathcal{H}(\tilde{\mu}^N | \gamma_N) \geq \mathcal{H}(c^N | m_N)$. Thus, it suffices to show that

$$\liminf_N \frac{1}{N} \mathcal{H}(c^N | m_N) \geq \int \mathcal{H}(\eta | M) \, c(\mathrm{d}\eta). \tag{5.32}$$

By Sanov's theorem on large deviations for empirical measures [12, Thm. 6.2.10], m_N satisfies a large deviation principle with rate function $\mathcal{H}(\cdot | M)$ on $\mathcal{P}(\mathbb{R}^d)$ equipped with the weak topology. Thus, (5.32) follows from Lemma 5.9.

(iii) Finally, we will conclude by relating $\mathcal{H}(\cdot | \gamma_N)$ and $\mathcal{H}(\cdot | \pi_N)$. For $m \in \mathbb{R}^d$, $E > 0$, define $\Psi_{m,E} : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ by $\Psi_{m,E}(\boldsymbol{v}) = (\sqrt{E}\, v_1 + m, \dots, \sqrt{E}\, v_n + m)$. Let $Q_N = (\mathcal{M}^N, \mathcal{E}^N)_\# \gamma_N$ in $\mathcal{P}(\mathbb{R}^d \times [0, \infty))$ be the distribution of momentum and energy under γ_N. We have $\gamma_N(\cdot \mid \mathcal{M}^N = m, \mathcal{E}^N = E) = (\Psi_{m,E/d})_\# \pi_N$ as in (i). Hence γ_N disintegrates as $\gamma_N = \int (\Psi_{m,E/d})_\# \pi_N \, \mathrm{d}Q_N(m, E)$. Define a map $\boldsymbol{\Psi} : \mathcal{P}(\mathcal{X}_N) \to \mathcal{P}(\mathbb{R}^{Nd})$ via

$$\boldsymbol{\Psi}(\mu) = \int (\Psi_{m,E/d})_\# \mu \, \mathrm{d}Q_N(m, E).$$

Note that $(\mathcal{M}^N, \mathcal{E}^N)_\# \boldsymbol{\Psi}(\mu) = Q_N$. Thus, the disintegration formula (5.29) with $T = (\mathcal{M}^N, \mathcal{E}^N)$ gives

$$\mathcal{H}(\boldsymbol{\Psi}(\mu) | \gamma_N) = \int \mathcal{H}((\Psi_{m,E/d})_\# \mu | (\Psi_{m,E/d})_\# \pi_N) \, \mathrm{d}Q_N(m, E) = \mathcal{H}(\mu | \pi_N),$$

where the last equality follows from (5.30) and $\Psi_{m,E}$ being bijective. Since $(L_N)_\# \mu^N \rightharpoonup c$ implies $(L_N)_\# \boldsymbol{\Psi}(\mu^N) \rightharpoonup c$, we can now deduce (5.28) from (5.31). ∎

Appendix A. The collision distance

In this section, we present a new type of distance between probability measures on $\mathbb{R}^d$ which is formally the Riemannian distance associated to the Onsager operator $\mathcal{K}^B$ (see (1.5)). The Riemannian distance $\mathcal{W}_B$ between two probability densities f_0, f_1 is formally given as

$$\mathcal{W}_B(f_0, f_1)^2 = \inf\left\{\frac{1}{4}\int_0^1 \int |\bar{\nabla}\psi_t|^2 \Lambda(f_t) B(v - v_*, \omega)\, \mathrm{d}\omega\, \mathrm{d}v_*\, \mathrm{d}v\, \mathrm{d}t\right\}, \tag{A.1}$$

where the infimum runs over all curves of densities $t \mapsto f_t$ connecting f_0 to f_1 and all functions $\psi : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ related via

$$\partial_t f_t(v) + \int \bar{\nabla}\psi_t \Lambda(f_t) B(v - v_*, \omega)\, \mathrm{d}\omega\, \mathrm{d}v_* = 0. \tag{A.2}$$

Note that the definition of $\mathcal{W}_B$ resembles the dynamic formulation of the L^2-Wasserstein distance, known as the Benamou–Brenier formula [6]. Here, the *collision rate equation* (A.2) takes over the role of the usual continuity equation.

The distance $\mathcal{W}_B$ will be constructed by relaxing the minimization problem above to a measure-valued framework and by minimizing the action as defined in Section 3 over curves connecting two given probability measures via the collision rate equation.

In this section, we will relax the assumptions on the collision kernel and require:

Assumption A.1. $B : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}_+$ is measurable, invariant under the transformation (1.3), $k \mapsto B(k, \omega)$ is continuous for a.e. ω and there exist constants $\gamma \in (-\infty, 1]$ and $c_B > 0$ such that

$$\int_{S^{d-1}} B(k, \omega)\, \mathrm{d}\omega \le c_B \langle k\rangle^\gamma \quad \forall k \in \mathbb{R}^d. \tag{A.3}$$

The following result will allow us to extract subsequential limits from sequences of solutions to the collision rate equation with uniform action and moment bounds.

Given $p \ge 1$ and $E > 0$ we will write

$$\mathcal{CRE}_T^{p,E} := \{(\mu, \mathcal{U}) \in \mathcal{CRE}_T : \mu_t \in \mathcal{P}_{p,E}(\mathbb{R}^d)\ \forall t \in [0, T]\}.$$

Proposition A.2 (Compactness of solutions with bounded action and moments). *Let $(\mu^n, \mathcal{U}^n)$ be a sequence in $\mathcal{CRE}_T^{p,E}$ with $p \ge 2$ such that*

$$\sup_n \int_0^T \mathcal{A}(\mu_t^n, \mathcal{U}_t^n)\, \mathrm{d}t < \infty. \tag{A.4}$$

Then there exists a couple $(\mu, \mathcal{U}) \in \mathcal{CRE}_T^{p,E}$ such that up to extraction of a subsequence,

$$\begin{aligned} &\mu_t^n \rightharpoonup \mu_t && \text{weakly in } \mathcal{P}(\mathbb{R}^d) \text{ for all } t \in [0, T],\\ &\mathcal{U}^n \rightharpoonup^* \mathcal{U} && \text{weakly* in } \mathcal{M}(\Omega \times [0, T]). \end{aligned}$$

Moreover, along this subsequence we have

$$\int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t)\,\mathrm{d}t \le \liminf_n \int_0^T \mathcal{A}(\mu_t^n, \mathcal{U}_t^n)\,\mathrm{d}t.$$

Proof. Thanks to the uniform bounds on action and moments, we can proceed verbatim as in the proof of Proposition 3.11 to obtain existence of a Borel family $(\mathcal{U}_t)_{t\in[0,T]}$ satisfying (iii) of Definition 3.1 such that $\mathcal{U}_t^n\,\mathrm{d}t$ converges weakly* to $\mathcal{U}_t\,\mathrm{d}t$ and the convergence (3.16) holds. By a further argument based on (3.15), we can approximate the indicator function $\mathbf{1}_{(t_0,t_1)}$ for any $0 \le t_0 < t_1 \le T$ by functions $a \in C([0,T])$ and obtain, for any $\xi \in C_b(\mathbb{R}^d)$,

$$\int_{t_0}^{t_1} \int \bar{\nabla}\xi\,\mathrm{d}\mathcal{U}_t^n\,\mathrm{d}t \xrightarrow{n\to\infty} \int_{t_0}^{t_1} \int \bar{\nabla}\xi\,\mathrm{d}\mathcal{U}_t\,\mathrm{d}t. \tag{A.5}$$

Finally, we show existence of a limiting curve $(\mu_t)_{t\in[0,T]}$. Since $\mathcal{P}_{p,E}(\mathbb{R}^d)$ is compact with respect to weak convergence, after extraction of another subsequence we can assume that $\mu_0^n \rightharpoonup \mu_0$ weakly for some $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Using this, the convergence (A.5) and the collision rate equation in the form (3.2) we infer that μ_t^n converges weakly to some probability measure μ_t for every $t \in [0,T]$, and $(\mu, \mathcal{U})$ satisfies (3.2). In particular, $t \mapsto \mu_t$ is weakly continuous and hence $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$. By lower semicontinuity of moments, we infer $\mathcal{E}_p(\mu_t) \le E$ for all t. The lower semicontinuity statement follows from Lemma 2.5 by noting that $\int_0^T \mathcal{A}(\mu_t^n, \mathcal{U}_t^n)\,\mathrm{d}t = \mathcal{F}_\alpha(\mu^{n,1}, \mu^{n,2}, \mathcal{U}^n)$ with $\mu^{n,k} = \mu_t^{n,k}\,\mathrm{d}t$. ∎

Given $p \ge 2$ and $E > 0$ we now define the following distance:

Definition A.3 (Distance). For $\mu_0, \mu_1 \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ we define

$$\mathcal{W}_B(\mu_0,\mu_1)^2 := \inf\left\{\int_0^1 \mathcal{A}(\mu_t, \mathcal{U}_t)\,\mathrm{d}t : (\mu,\mathcal{U}) \in \mathcal{CRE}_1^{p,E}(\mu_0,\mu_1)\right\}, \tag{A.6}$$

with the convention that $\mathcal{W}_B(\mu_0,\mu_1) = +\infty$ if the set over which the infimum is taken is empty.

Remark A.4. In the same way one could construct an (a priori smaller) extended distance on the full space $\mathcal{P}(\mathbb{R}^d)$ by dropping the moment constraint and minimize over $\mathcal{CRE}_1$ instead of $\mathcal{CRE}_1^{p,E}$. We will not explore this here. We stress that $\mathcal{W}_B$ defined as above depends implicitly on the choice of p and E.

Let us give an equivalent characterization of the infimum in (A.6).

Lemma A.5. *For any $T > 0$ and $\mu_0, \mu_1 \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ we have*

$$\mathcal{W}_B(\mu_0,\mu_1) = \inf\left\{\int_0^T \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)}\,\mathrm{d}t : (\mu,\mathcal{U}) \in \mathcal{CRE}_T^{p,E}(\mu_0,\mu_1)\right\}.$$

Proof. This follows from a standard reparametrization argument. See [1, Lem. 1.1.4] or [14, Thm. 5.4] for details in similar situations. ∎

The next result shows that the infimum in the definition above is in fact a minimum.

Proposition A.6. *Let $\mu_0, \mu_1 \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ be such that $W := \mathcal{W}_B(\mu_0, \mu_1)$ is finite. Then the infimum in* (A.6) *is attained by a curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^E(\mu_0, \mu_1)$ satisfying $\mathcal{A}(\mu_t, \mathcal{U}_t) = W^2$ for a.e. $t \in [0, 1]$.*

Proof. Existence of a minimizing curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^E(\mu_0, \mu_1)$ follows immediately by the direct method taking into account Proposition A.2. Invoking Lemma A.5 and Jensen's inequality we see that this curve satisfies

$$\int_0^1 \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)}\, \mathrm{d}t \geq W = \left(\int_0^1 \mathcal{A}(\mu_t, \mathcal{U}_t)\, \mathrm{d}t\right)^{1/2} \geq \int_0^1 \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)}\, \mathrm{d}t.$$

Hence we must have $\mathcal{A}(\mu_t, \mathcal{U}_t) = W^2$ for a.e. $t \in [0, 1]$. ■

We have the following properties of the function $\mathcal{W}_B$.

Theorem A.7. *$\mathcal{W}_B$ defines an (extended) distance on $\mathcal{P}_{p,E}(\mathbb{R}^d)$. The topology, it induces is stronger than the weak topology and bounded sets with respect to $\mathcal{W}_B$ are weakly compact. Moreover, the map $(\mu_0, \mu_1) \mapsto \mathcal{W}_B(\mu_0, \mu_1)$ is lower semicontinuous with respect to weak convergence. For each $\tau \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ the set $\mathcal{P}_\tau := \{\mu \in \mathcal{P}_{p,E}(\mathbb{R}^d) : \mathcal{W}_B(\mu, \tau) < \infty\}$ equipped with the distance $\mathcal{W}_B$ is a complete geodesic space.*

Here, we call a function $d : X \times X \to [0, \infty]$ an *extended distance* on the set X if it is symmetric, satisfies the triangle inequality and vanishes precisely on the diagonal.

Proof of Theorem A.7. Symmetry of $\mathcal{W}_B$ is obvious from the fact that $\alpha(w, \cdot, \cdot) = \alpha(-w, \cdot, \cdot)$. Equation (3.2) shows that two curves in $\mathcal{CRE}_1^{p,E}$ can be concatenated to obtain a curve in $\mathcal{CRE}_2^{p,E}$. Hence the triangle inequality follows easily using Lemma A.5. To see that $\mathcal{W}_B(\mu_0, \mu_1) > 0$ whenever $\mu_0 \neq \mu_1$ assume that $\mathcal{W}_B(\mu_0, \mu_1) = 0$ and choose a minimizing curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^{p,E}(\mu_0, \mu_1)$. Then we must have $\mathcal{A}(\mu_t, \mathcal{U}_t) = 0$ and hence $\mathcal{U}_t = 0$ for a.e. $t \in (0, 1)$. From the continuity equation in the form (3.2) we infer $\mu_0 = \mu_1$.

The compactness assertion and lower semicontinuity of $\mathcal{W}_B$ follow immediately from Proposition A.2. These in turn imply that the topology induced by $\mathcal{W}_B$ is stronger than the weak one.

Let us now fix $\tau \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ and let $\mu_0, \mu_1 \in \mathcal{P}_\tau$. By the triangle inequality we have $\mathcal{W}_B(\mu_0, \mu_1) < \infty$ and hence Proposition A.6 yields existence of a minimizing curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^{p,E}(\mu_0, \mu_1)$. The curve $t \mapsto \mu_t$ is then a constant speed geodesic in $\mathcal{P}_\tau$ since it satisfies

$$\mathcal{W}_B(\mu_s, \mu_t) = \int_s^t \sqrt{\mathcal{A}(\mu_r, \mathcal{U}_r)}\, \mathrm{d}r = (t - s)\mathcal{W}_B(\mu_0, \mu_1) \quad \forall 0 \leq s \leq t \leq 1.$$

To show completeness, let $(\mu^n)_n$ be a Cauchy sequence in $\mathcal{P}_\tau$. In particular, the sequence is bounded with respect to $\mathcal{W}_B$ and we can find a subsequence (still indexed by n) and $\mu^\infty \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ such that $\mu^n \rightharpoonup \mu^\infty$ weakly. Invoking lower semicontinuity of $\mathcal{W}_B$ and the Cauchy condition we infer that $\mathcal{W}_B(\mu^n, \mu^\infty) \to 0$ as $n \to \infty$ and $\mu^\infty \in \mathcal{P}_\tau$. ■

It is not yet clear when precisely the distance $\mathcal{W}_B$ is finite. However, it is easily seen to be finite along solutions to the Boltzmann equation: if f_t is a solution according to Theorem 2.2 and we set $\mu_t = f_t \mathcal{L}$ and

$$\mathcal{U}_t = \bar{\nabla} \log f_t \, \Lambda(f_t)\mathcal{B} = [(f')_t(f'_*)_t - f_t(f_*)_t]\mathcal{B},$$

then $(\mu, \mathcal{U}) \in \mathcal{CRE}^E$ and we have $\mathcal{A}(\mu_t, \mathcal{U}_t) = D(\mu_t)$. Thus,

$$\begin{aligned}\mathcal{W}_B(\mu_0, \mu_T) &\le \int_0^T \sqrt{D(\mu_t)}\,\mathrm{d}t \\ &\le \sqrt{T}\Big(\int_0^T D(\mu_t)\,\mathrm{d}t\Big)^{1/2} \\ &= \sqrt{T}\sqrt{\mathcal{H}(\mu_0) - \mathcal{H}(\mu_T)}.\end{aligned}$$

The following result shows that the distance $\mathcal{W}_B$ can be bounded from below by the L^1-Wasserstein distance. Recall that the L^1-Wasserstein distance is defined for $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ by

$$W_1(\mu_0, \mu_1) := \inf_\pi \int |x - y|\, \pi(\mathrm{d}x, \mathrm{d}y),$$

where the infimum is taken over all probability measures $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ whose first and second marginals are μ_0 and μ_1 respectively.

Proposition A.8. *Let $p \ge 2 + \max(\gamma, 0)$. For any $\mu_0, \mu_1 \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ we have*

$$W_1(\mu_0, \mu_1) \le \sqrt{2c_B E}\, \mathcal{W}_B(\mu_0, \mu_1).$$

Proof. We can assume that $\mathcal{W}_B(\mu_0, \mu_1) < \infty$. Take a minimizing curve $(\mu, \mathcal{U})$ in $\mathcal{CRE}_1^{p,E}(\mu_0, \mu_1)$ and let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a bounded 1-Lipschitz function. This implies that $|\bar{\nabla}\varphi| \le 2|v - v_*|$. Taking into account Remark 3.3 and using Lemma 3.8, we estimate

$$\begin{aligned}\Big|\int \varphi\,\mathrm{d}\mu_1 - \int \varphi\,\mathrm{d}\mu_0\Big| &= \frac{1}{4}\Big|\int_0^1 \int \bar{\nabla}\varphi\,\mathrm{d}\mathcal{U}_t\,\mathrm{d}t\Big| \\ &\le \frac{1}{2}\int_0^1 \int |v - v_*|\,\mathrm{d}|\mathcal{U}_t|(v, v_*, \omega)\,\mathrm{d}t \\ &\le \Big(\int_0^1 \mathcal{A}(\mu_t, \mathcal{U}_t)\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^1 \int [|v|^2 + |v_*|^2]B(v - v_*, \omega)\,\mu_t(\mathrm{d}v)\,\mu_t(\mathrm{d}v_*)\,\mathrm{d}t\Big)^{1/2} \\ &\le \sqrt{2c_B E}\,\mathcal{W}_B(\mu_0, \mu_1).\end{aligned}$$

Here we have also used (A.3) and the fact that μ_t has p-moment less than E in the last inequality. Taking the supremum over all bounded 1-Lipschitz functions φ yields the claim by Kantorovich–Rubinstein duality (see [36, Thm. 5.10, 5.16]). ■

We now give a characterization of absolutely continuous curves with respect to $\mathcal{W}_B$. See (B.1) and (B.2) for the definition of absolutely continuous curves and their metric derivative.

Proposition A.9 (Metric velocity). *A curve $(\mu_t)_{t\in[0,T]}$ in $\mathscr{P}_{p,E}(\mathbb{R}^d)$ is absolutely continuous with respect to $\mathcal{W}_B$ if and only if there exists a Borel family $(\mathcal{U}_t)_{t\in[0,T]}$ such that $(\mu,\mathcal{U}) \in \mathcal{CRE}_T^{p,E}$ and*

$$\int_0^T \sqrt{\mathcal{A}(\mu_t,\mathcal{U}_t)}\,\mathrm{d}t < \infty.$$

In this case, the metric derivative is bounded as $|\dot\mu|^2(t) \le \mathcal{A}(\mu_t,\mathcal{U}_t)$ for a.e. $t\in[0,T]$. Moreover, there exists a unique Borel family $\tilde{\mathcal{U}}_t$ with $(\mu,\tilde{\mathcal{U}}) \in \mathcal{CRE}_T^{p,E}$ such that

$$|\dot\mu|^2(t) = \mathcal{A}(\mu_t,\tilde{\mathcal{U}}_t) \quad \textit{for a.e. } t\in[0,T]. \tag{A.7}$$

Proof. The proof follows from the very same arguments as in [14, Thm. 5.17]. ∎

We can describe the optimal velocity measures $\tilde{\mathcal{U}}_t$ appearing in the preceding proposition in more detail. We define T_μ to be the set of all $\mathcal{U}\in\mathcal{M}(\Omega)$ such that $\mathcal{A}(\mu,\mathcal{U})<\infty$ and $\mathcal{A}(\mu,\mathcal{U}) \le \mathcal{A}(\mu,\mathcal{U}+\boldsymbol{\eta})$ for all $\boldsymbol{\eta}\in\mathcal{M}(\Omega)$ satisfying

$$\frac14\int_\Omega \bar\nabla\xi\,\mathrm{d}\boldsymbol{\eta} = 0 \quad \forall \xi\in C_c^\infty(\Omega).$$

Corollary A.10. *Let $(\mu,\mathcal{U})\in\mathcal{CRE}_T^{p,E}$ be such that the curve $t\mapsto\mu_t$ is absolutely continuous with respect to $\mathcal{W}_B$. Then $\mathcal{U}$ satisfies (A.7) if and only if $\mathcal{U}_t\in T_{\mu_t}$ for a.e. $t\in[0,T]$.*

If μ is absolutely continuous with respect to Lebesgue measure $\mathcal{L}$ we can give an explicit description of T_μ. Recall that $\mathcal{B}\in\mathcal{M}(\Omega)$ is the measure given by $\mathrm{d}\mathcal{B}(v,v_*,\omega) = B(v-v_*,\omega)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$.

Proposition A.11. *Let $\mu = fm \in \mathscr{P}_{p,E}(\mathbb{R}^d)$. Then $\mathcal{U}\in T_\mu$ if and only if $\mathcal{U} = U\Lambda(f)\mathcal{B}$ is absolutely continuous with respect to the measure $\Lambda(f)\mathcal{B}$ and*

$$U \in \overline{\{\bar\nabla\varphi : \varphi\in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\Lambda(f)\mathcal{B})} =: T_f.$$

Proof. If $\mathcal{A}(\mu,\mathcal{U})$ is finite we infer from Lemma 3.6 that $\mathcal{U} = U\Lambda(f)\mathcal{B}$ for some density $U:\Omega\to\mathbb{R}$, and $\mathcal{A}(\mu,\mathcal{U}) = \|U\|^2_{L^2(\Lambda(f)\mathcal{B})}$. Now the optimality condition in the definition of T_μ is equivalent to

$$\|U\|_{L^2(\Lambda(f)\mathcal{B})} \le \|U+V\|_{L^2(\Lambda(f)\mathcal{B})} \quad \forall V\in N_f,$$

where

$$N_f := \left\{ V\in L^2(\Lambda(f)\mathcal{B}) : \int \bar\nabla\xi V\Lambda(f)\mathcal{B} = 0 \;\forall\xi\in C_c^\infty(\mathbb{R}^d)\right\}.$$

This implies the assertion after noting that N_f is the orthogonal complement of T_f in L^2. ∎

In the light of the formal Riemannian interpretation of the distance $\mathcal{W}_B$ one should view T_μ as the tangent space at the measure μ. This is reminiscent of Otto's Riemannian interpretation of the L^2-Wasserstein space [30].

Appendix B. Metric gradient flow

In this section, we recast the variational characterization of Section 4 in the language of the theory of gradient flows in metric spaces. Let us briefly recall the basics of that theory. For a detailed account we refer the reader to [1].

Let (X, d) be a complete metric space and let $E : X \to (-\infty, \infty]$ be a function with proper domain, i.e. the set $D(E) := \{x : E(x) < \infty\}$ is non-empty.

A curve $(x_t)_{t\in(a,b)}$ in (X, d) is called *p-absolutely continuous* for $p \geq 1$ if there exists $m \in L^p((a, b))$ such that

$$d(x_s, x_t) \leq \int_s^t m(r)\,\mathrm{d}r \quad \forall a \leq s \leq t \leq b. \tag{B.1}$$

In this case we write $x \in AC^p((a, b); (X, d))$. For $p = 1$ we simply drop p from the notation. Similarly, one defines locally p-absolutely continuous curves. For a locally absolutely continuous curve the *metric derivative* defined by

$$|\dot{x}|(t) := \lim_{h\to 0} \frac{d(x_{t+h}, x_t)}{|h|} \tag{B.2}$$

exists for a.e. t and is the minimal m in (B.1) (see [1, Thm. 1.1.2]).

The following notion plays the role of the modulus of the gradient in a metric setting.

Definition B.1 (Strong upper gradient). A function $g : X \to [0, \infty]$ is called a *strong upper gradient* of E if for any $x \in AC((a, b); (X, d))$ the function $g \circ x$ is Borel and

$$|E(x_s) - E(x_t)| \leq \int_s^t g(x_r)|\dot{x}|(r)\,\mathrm{d}r \quad \forall a \leq s \leq t \leq b.$$

Note that by the definition of strong upper gradient, and Young's inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we find that for all $s \leq t$,

$$E(x_t) - E(x_s) + \frac{1}{2}\int_s^t [g(x_r)^2 + |\dot{x}|^2(r)]\,\mathrm{d}r \geq 0.$$

Definition B.2 (Curve of maximal slope). A locally 2-absolutely continuous curve $(x_t)_{t\in(0,\infty)}$ is called a *curve of maximal slope* of E with respect to its strong upper gradient g if $t \mapsto E(x_t)$ is non-increasing and

$$E(x_t) - E(x_s) + \frac{1}{2}\int_s^t [g(x_r)^2 + |\dot{x}|^2(r)]\,\mathrm{d}r \leq 0 \quad \forall 0 < s \leq t. \tag{B.3}$$

We say that a curve of maximal slope *starts from* $x_0 \in X$ if $\lim_{t\searrow 0} x_t = x_0$.

Equivalently, we can require equality in (B.3). If a strong upper gradient g of E is fixed we also call a curve of maximal slope of E (relative to g) a *gradient flow curve*.

Finally, we define the (descending) *metric slope* of E as the function $|\partial E| : D(E) \to [0, \infty]$ given by

$$|\partial E|(x) = \limsup_{y\to x} \frac{\max\{E(x) - E(y), 0\}}{d(x, y)}. \tag{B.4}$$

The metric slope is in general only a weak upper gradient E (see [1, Thm. 1.2.5]). In our application to the homogeneous Boltzmann equation, we will show that the square root of the dissipation D provides a strong upper gradient for the entropy $\mathcal{H}$.

Let us assume that $p \geq 2 + \max(\gamma, 0)$ and that the collision kernel B satisfies Assumption 2.1. Then we have the following

Corollary B.3 (Boltzmann equation as curve of maximal slope). *$\sqrt{D}$ is a strong upper gradient for $\mathcal{H}$ on $(\mathcal{P}_{p,E}(\mathbb{R}^d), \mathcal{W}_B)$. Moreover, for any $\mu_0 \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ with $\mathcal{H}(\mu_0) < \infty$, the curves of maximal slope of $\mathcal{H}$ with respect to the strong upper gradient $\sqrt{D}$ starting from μ_0 are precisely the solutions to the Boltzmann equation satisfying* (4.19).

Proof. Let $(\mu_r)_r$ be an absolutely continuous curve such that

$$\int_s^t \sqrt{D(\mu_r)}\, |\dot{\mu}|(r)\, \mathrm{d}r < \infty.$$

This implies that μ_r has a density f_r (and hence by Lemma 3.6, $\mathcal{U}_r$ has a density U_r) for a.e. r. We can also assume that one of the measures μ_s, μ_t has finite entropy, say μ_s. Then Proposition 4.1 together with the estimate (4.16) immediately implies that $\sqrt{D}$ is a strong upper gradient. Theorem 4.3 gives the identification of curves of maximal slope. ■

Appendix C. Variational approximation scheme

In this section, we consider a time-discrete variational approximation scheme for the homogeneous Boltzmann equation. Recall that we make Assumption 2.1 on the collision kernel B and let $p \geq 2 + \max(\gamma, 0)$. The scheme can be interpreted as the implicit Euler scheme for the gradient flow equation. Given a time step $\tau > 0$ and an initial datum $\mu_0 \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ with $\mathcal{H}(\mu_0) < \infty$, we consider a sequence $(\mu_n^\tau)_n$ in $\mathcal{P}_{p,E}(\mathbb{R}^d)$ defined recursively via

$$\mu_0^\tau = \mu_0, \quad \mu_n^\tau \in \operatorname*{argmin}_\nu \Big[\mathcal{H}(\nu) + \frac{1}{2\tau}\mathcal{W}_B(\nu, \mu_{n-1}^\tau)^2\Big]. \tag{C.1}$$

Then we build a discrete gradient flow trajectory as the piecewise constant interpolation $(\bar{\mu}_t^\tau)_{t\geq 0}$ given by

$$\bar{\mu}_0^\tau = \mu_0, \quad \bar{\mu}_t^\tau = \mu_n^\tau \quad \text{if } t \in ((n-1)\tau, n\tau]. \tag{C.2}$$

Then we have the following result.

Theorem C.1. *For any $\tau > 0$ and $\mu_0 \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ with $\mathcal{H}(\mu_0) < \infty$ the variational scheme* (C.1) *admits a solution $(\mu_n^\tau)_n$. As $\tau \to 0$, for any family of discrete solutions there exists a sequence $\tau_k \to 0$ and a locally 2-absolutely continuous curve $(\mu_t)_{t\geq 0}$ such that*

$$\bar{\mu}_t^{\tau_k} \rightharpoonup \mu_t \quad \forall t \in [0, \infty). \tag{C.3}$$

Moreover, any such limit curve is a gradient flow of the entropy, i.e. a solution to the Boltzmann equation satisfying (4.19).

With the knowledge that the Boltzmann equation in our setting has a unique solution (assuming in addition $\mathcal{E}_4(\mu_0) < \infty$) if $\gamma > 0$), we obtain convergence of $\bar{\mu}_t^\tau$ to the solution to the Boltzmann equation for any sequence of time steps $\tau \to 0$.

With the work we have done so far, Theorem C.1 follows basically from standard general results for metric gradient flows where (C.1) is known as the minimizing movement scheme (see [1, Sec. 2.3]). We need one small additional ingredient relating the dissipation D to the metric slope $|\partial\mathcal{H}|$ of the entropy in the metric space $(\mathcal{P}_{p,E}(\mathbb{R}^d), \mathcal{W}_B)$. Recall (B.4) for the definition of the metric slope. We consider its sequentially lower semicontinuous envelope, or *relaxed slope* $|\partial^-\mathcal{H}|$, given by

$$|\partial^-\mathcal{H}|(\mu) = \inf\Big\{\liminf_{n\to\infty} |\partial\mathcal{H}(\mu_n) : \mu_n \rightharpoonup \mu, \sup_n \{\mathcal{W}_B(\mu_n,\mu), \mathcal{H}(\mu_n)\} < \infty\Big\}.$$

Lemma C.2. *For any $\mu \in \mathcal{P}_{p,E}(\mathbb{R}^d)$ with $\mathcal{H}(\mu) < \infty$ we have $\sqrt{D(\mu)} \le |\partial^-\mathcal{H}(\mu)|$. In particular, $|\partial^-\mathcal{H}(\mu)|$ is a strong upper gradient for $\mathcal{H}$.*

Proof. Let f be the density of μ and consider the solution (f_t) to the homogeneous Boltzmann equation with initial datum f. Set $\mu_t = f_t\mathcal{L}$ and observe that

$$\begin{aligned} D(f) &\le \lim_{t\searrow 0} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu_t)}{t} = \lim_{t\searrow 0} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu_t)}{\mathcal{W}_B(\mu_t,\mu)} \frac{\mathcal{W}_B(\mu_t,\mu)}{t} \\ &\le |\partial\mathcal{H}(\mu)|\,|\dot{\mu}|(0) \le |\partial\mathcal{H}(\mu)|\sqrt{D(\mu)}. \end{aligned}$$

Thus, $\sqrt{D(\mu)} \le |\partial\mathcal{H}(\mu)|$ for any such μ. The claim follows immediately from the lower semicontinuity of D (Lemma 2.6). ∎

Proof of Theorem C.1. We verify that the present situation is consistent with the abstract setting considered in [1, Sec. 2].

We consider the metric space $(\mathcal{P}_{\mu_0}, \mathcal{W}_B)$ and endow it with the weak topology σ. By Theorem A.7, $(\mathcal{P}_{\mu_0}, \mathcal{W}_B)$ is complete, $\mathcal{W}_B$ is lower semicontinuous with respect to σ and induces a stronger topology. Recall from Section 2 that the entropy $\mathcal{H}$ is bounded below on $\mathcal{P}_{p,E}(\mathbb{R}^d)$ and lower semicontinuous with respect to weak convergence. Moreover, $\mathcal{P}_{p,E}(\mathbb{R}^d)$ is compact with respect to weak convergence. Thus, the conditions in [1, Assumption 2.1 (a)–(c)] are satisfied.

Existence of a solution to the variational scheme (C.1) and of a subsequential limit curve $(\mu_t)_t$ now follows from [1, Cor. 2.2.2, Prop. 2.2.3]. Moreover, [1, Thm. 2.3.2] shows that the limit curve is a curve of maximal slope for the strong upper gradient $|\partial^-\mathcal{H}|$, i.e.

$$\frac{1}{2}\int_0^t [|\dot{\mu}|^2(r) + |\partial^-\mathcal{H}(\mu_r)|^2]\,dr + \mathcal{H}(\mu_t) \le \mathcal{H}(\mu_0).$$

Thus, by Lemma C.2, it is also a curve of maximal slope for the strong upper gradient $\sqrt{D}$. ∎

Acknowledgments. The author is grateful for the remarks and suggestions of the referees on a previous version of the manuscript. He wishes to thank André Schlichting for fruitful discussions on this work and related topics.

Funding. This research was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through SFB 1060 2017/2 - 211504053 and SFB 1283/2 2021- 317210226.

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