Florian Ivorra · Sophie Morel

The four operations on perverse motives

Abstract. Let k be a field of characteristic zero with a fixed embedding $\sigma : k \hookrightarrow \mathbf{C}$ into the field of complex numbers. Given a k-variety X, we use the triangulated category of étale motives with rational coefficients on X to construct an abelian category $\mathscr{M}(X)$ of perverse mixed motives. We show that over $\mathrm{Spec}(k)$ the category obtained is canonically equivalent to the usual category of Nori motives and that the derived categories $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ are equipped with the four operations of Grothendieck (for morphisms of quasi-projective k-varieties) as well as nearby and vanishing cycles functors and a formalism of weights.

In particular, as an application, we show that many classical constructions done with perverse sheaves, such as intersection cohomology groups or Leray spectral sequences, are motivic and therefore compatible with Hodge theory. This recovers and strengthens work by Zucker, Saito, Arapura and de Cataldo–Migliorini and provides an arithmetic proof of the pureness of intersection cohomology with coefficients in a geometric variation of Hodge structures.

Keywords. Motives, perverse sheaves, sheaf operations

Contents

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Introduction

Let k be a field of characteristic zero with a fixed embedding $\sigma : k \hookrightarrow \mathbf{C}$ into the field of complex numbers. A k-variety is a separated k-scheme of finite type. Unless otherwise specified, we will only consider quasi-projective k-varieties.

In the present work, we construct the four operations of Grothendieck on the derived categories of perverse Nori motives. In order to combine the tools provided by Ayoub [5,6] and Beilinson [16,17] in a most efficient way, we define the abelian category of perverse Nori motives on a given k-variety as a byproduct of the triangulated category of constructible étale motives on the same variety. Over the base field the category obtained still coincides with the usual category of Nori motives but now, as we show, it is possible to equip the derived categories of these abelian categories with the four operations of Grothendieck as well as nearby and vanishing cycles functors. However, we leave the construction of the tensor product and internal Hom operations on these categories to a later paper.

In particular, as an application, we show that many classical constructions done with perverse sheaves, such as intersection cohomology groups or Leray spectral sequences, are motivic and therefore compatible with Hodge theory. This recovers and strengthens works by Zucker [77], Saito [67], Arapura [2] and de Cataldo–Migliorini [26]. Moreover it provides an arithmetic proof via reduction to positive characteristic and the Weil conjectures of the pureness of the Hodge structure on intersection cohomology with coefficients in a geometric variation of Hodge structures.

Conjectural picture and some earlier works. Before going into more detail about the content of this paper, let us discuss perverse motives from the perspective of perverse sheaves and recall parts of the conjectural picture and related earlier works.

For someone interested in perverse sheaves, perverse motives can be thought of as perverse sheaves of geometric origin. However, the classical definition of these perverse sheaves as a full subcategory of the category of all perverse sheaves is not entirely satisfactory. Indeed, this category contains too many morphisms and consequently, as we take the kernels and cokernels of morphisms which should not be considered, too many objects. For example, perverse sheaves of geometric origin should define mixed Hodge modules and therefore any morphism between them should also be a morphism of mixed Hodge modules. Therefore, one expects the category of perverse motives/perverse sheaves of geometric origin to be an abelian category endowed with a faithful – but not full – exact functor into the category of perverse sheaves.

According to Grothendieck, there should exist a $\mathbf{Q}$-linear abelian category $\mathbf{MM}(k)$ whose objects are called mixed motives. Given an embedding $\sigma : k \hookrightarrow \mathbf{C}$, the category $\mathbf{MM}(k)$ should come with a faithful exact functor

$$\mathbf{MM}(k) \to \mathbf{MHS}$$

to the category of (polarizable) mixed $\mathbf{Q}$-Hodge structures $\mathbf{MHS}$, called the realization functor. The mixed Hodge structure on the i-th Betti cohomology group $H^i(X)$ of a given k-variety X should come via the realization functor from a mixed motive $H^i_\mathbf{M}(X)$. The appealing beauty of this picture lies in the expected properties of this category, in particular, the conjectural relations between extension groups and algebraic cycles (see e.g. Jannsen [49]), or the relation with period rings and motivic Galois groups (see e.g. Ayoub's survey [11]).

As part of Grothendieck's more general cohomological program, the category $\mathbf{MM}(k)$ should underlie a system of coefficients. For any k-variety X, there should exist an abelian category $\mathbf{MM}(X)$ of mixed motives along with a realization functor into the category of mixed Hodge modules (or simply of sheaves of $\mathbf{Q}$-vector spaces) on the associated analytic space X^{an}, and their derived categories should satisfy a formalism of (adjoint) triangulated functors

$$\mathrm{D}^{\mathrm{b}}(\mathbf{MM}(X)) \underset{f^\mathbf{M}_*}{\overset{f^*_\mathbf{M}}{\longleftrightarrow}} \mathrm{D}^{\mathrm{b}}(\mathbf{MM}(Y)) \underset{f^\mathbf{M}_!}{\overset{f^!_\mathbf{M}}{\longleftrightarrow}} \mathrm{D}^{\mathrm{b}}(\mathbf{MM}(X)),$$

a formalism which has been at the heart of Grothendieck's approach to every cohomology theory. Then, for a k-variety $a : X \to \operatorname{Spec} k$, the motive $H^i_\mathbf{M}(X)$ would be given as the i-th cohomology of the image under $a^\mathbf{M}_*$ of a complex of mixed motives $\mathbf{Q}^\mathbf{M}_X$ that should realize to the standard constant sheaf $\mathbf{Q}_X$ on X^{an}. Grothendieck was looking for abelian categories modeled after the categories of constructible sheaves, but as pointed out by Beilinson and Deligne one could/should also look for categories modeled after perverse sheaves (see e.g. Deligne [28]).

Many attempts have been made to carry out Grothendieck's program at least partially but unconditionally.

The most successful attempt at constructing the triangulated category of mixed motives (that is, conjecturally, the derived category of $\mathbf{MM}(X)$) stems from Morel–Voevodsky's stable homotopy theory of schemes. The best candidate so far is the triangulated category $\mathbf{DA}_{\mathrm{ct}}(X)$ of constructible étale motivic sheaves (with rational coefficients) extensively studied by Ayoub [5, 6, 8]. The theory developed in [5, 6] provides these categories with the Grothendieck four operations and, as shown by Voevodsky [75], Chow groups of smooth algebraic k-varieties can be computed as extension groups in the category $\mathbf{DA}_{\mathrm{ct}}(k)$.

On the abelian side, Nori has constructed a candidate for the abelian category of mixed motives over k. The construction of Nori's abelian category $\mathbf{HM}(k)$ is tannakian in essence, and since it is a category of comodules over some Hopf algebra, it comes with a built-in motivic Galois group. Moreover, any Nori motive has a canonical weight filtration and Arapura has shown in [3, Theorem 6.4.1] that the full subcategory of pure motives coincides with the semisimple abelian category defined by André [1] using motivated algebraic cycles (see also Huber and Müller-Stach [44, Proposition 10.2.1]). More generally, attempts have been made to define Nori motives over k-varieties. Arapura has defined a constructible variant in [3] and the first author a perverse variant in [47]. However, the Grothendieck four operations have not been constructed (at least in their full extent) in those contexts. For example in [3], the direct image functor is only available for structural morphisms or projective morphisms and no extraordinary inverse image is defined.

Note that the two different attempts should not be unrelated. One expects the triangulated category $\mathbf{DA}_{\mathrm{ct}}(X)$ to possess a special t-structure (called the motivic t-structure) whose heart should be the abelian category of mixed motives. This is a very deep conjecture, even for $X = \operatorname{Spec} k$, which implies for example the Lefschetz and Künneth type standard conjectures (see Beilinson [15]). As of now, the extension groups in Nori's abelian category of mixed motives are known to be related to algebraic cycles only very loosely.

Nonetheless, striking unconditional relations between the two different approaches have been obtained. In particular, in [22], Gallauer–Choudhury have shown that the motivic Galois group constructed by Ayoub [9, 10] using the triangulated category of étale motives is isomorphic to the motivic Galois group obtained by Nori's construction.

Content of this paper. Let us now describe more precisely the content of our paper. Given a k-variety X, consider the bounded derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{Q})$ of sheaves of $\mathbf{Q}$-vector spaces with algebraically constructible cohomology on the analytic space X^{an} associated with the base change of X along σ and the category of perverse sheaves $\mathscr{P}(X)$ which is the heart of the self-dual perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{Q})$ introduced by Beilinson–Bernstein–Deligne [18]. Let $\mathbf{DA}_{\mathrm{ct}}(X)$ be the triangulated category of constructible étale motivic sheaves (with rational coefficients) which is a full triangulated subcategory of the $\mathbf{Q}$-linear counterpart of the stable homotopy category of schemes, $\mathbf{SH}(X)$, introduced by

Morel and Voevodsky (see [59, 74] and Jardine [50]). This category has been extensively studied by Ayoub [5, 6, 8] and comes with a realization functor

$$\mathrm{Bti}_X^* : \mathbf{DA}_{\mathrm{ct}}(X) \to \mathrm{D}_c^b(X, \mathbf{Q})$$

(see Ayoub [7]) and thus, by composing with the perverse cohomology functor, with a homological functor ${}^p H_{\mathscr{P}}^0$ with values in $\mathscr{P}(X)$.

The category of perverse motives considered in the present paper is defined (see [47]) as the universal factorization

$$\mathbf{DA}_{\mathrm{ct}}(X) \xrightarrow{{}^p H_{\mathscr{M}}^0} \mathscr{M}(X) \xrightarrow{\mathrm{rat}_X^{\mathscr{M}}} \mathscr{P}(X)$$

of ${}^p H_{\mathscr{P}}^0$, where $\mathscr{M}(X)$ is an abelian category, ${}^p H_{\mathscr{M}}^0$ is a homological functor and $\mathrm{rat}_X^{\mathscr{M}}$ is a faithful exact functor. This kind of universal construction goes back to Freyd and is recalled in Section 1. As we see in Section 6, ℓ-adic perverse sheaves can also be used to defined the category of perverse motives (see Definition 6.3 and Proposition 6.11).

Given a morphism of k-varieties $f : X \to Y$, the four functors

$$\mathrm{D}_c^b(X, \mathbf{Q}) \underset{f_*^{\mathscr{P}}}{\overset{f_{\mathscr{P}}^*}{\leftrightarrows}} \mathrm{D}_c^b(Y, \mathbf{Q}) \underset{f_{\mathscr{P}}^!}{\overset{f_!^{\mathscr{P}}}{\leftrightarrows}} \mathrm{D}_c^b(X, \mathbf{Q}) \tag{0.1}$$

were developed by Verdier [72] (see also Kashiwara–Schapira's book [52]) on the model of the theory developed by Artin, Grothendieck et al. [4] for étale and ℓ-adic sheaves. The nearby and vanishing cycles functors

$$\Psi_g : \mathrm{D}_c^b(X_\eta, \mathbf{Q}) \to \mathrm{D}_c^b(X_\sigma, \mathbf{Q}), \quad \Phi_g : \mathrm{D}_c^b(X, \mathbf{Q}) \to \mathrm{D}_c^b(X_\sigma, \mathbf{Q})$$

associated with a morphism $g : X \to \mathbf{A}_k^1$ were constructed by Grothendieck [30] (here X_η denotes the generic fiber and X_σ the special fiber). By a theorem of Gabber, the functors $\psi_g := \Psi_g[-1]$ and $\phi_g := \Phi_g[-1]$ are t-exact for the perverse t-structures and thus induce exact functors

$$\psi_g : \mathscr{P}(X_\eta) \to \mathscr{P}(X_\sigma), \quad \phi_g : \mathscr{P}(X) \to \mathscr{P}(X_\sigma). \tag{0.2}$$

In this work, we prove that $\mathscr{M}(k)$ is canonically equivalent to the abelian category $\mathbf{HM}(k)$ of Nori motives (see Proposition 2.11) and that the four operations (0.1) (for morphisms of quasi-projective k-varieties) and the functors (0.2) can be canonically lifted along the functors

$$\mathrm{D}^b(\mathscr{M}(X)) \xrightarrow{\mathrm{rat}_X^{\mathscr{M}}} \mathrm{D}^b(\mathscr{P}(X)) \xrightarrow{\mathrm{real}} \mathrm{D}_c^b(X, \mathbf{Q}),$$

where "real" is the realization functor of [18, Section 3.1.9], which has been shown to be an equivalence by Beilinson [17], to (adjoint) triangulated functors

$$\mathrm{D}^b(\mathscr{M}(X)) \underset{f_*^{\mathscr{M}}}{\overset{f_{\mathscr{M}}^*}{\leftrightarrows}} \mathrm{D}^b(\mathscr{M}(Y)) \underset{f_{\mathscr{M}}^!}{\overset{f_!^{\mathscr{M}}}{\leftrightarrows}} \mathrm{D}^b(\mathscr{M}(X))$$

and to exact functors

$$\psi_f^{\mathscr{M}} : \mathscr{M}(X_\eta) \to \mathscr{M}(X_\sigma), \quad \phi_f^{\mathscr{M}} : \mathscr{M}(X) \to \mathscr{M}(X_\sigma).$$

Relying on Ayoub's work [5, 6] and on the compatibility of the Betti realization with the four operations, our strategy consists in establishing enough of the formalism to show that the categories $D^b(\mathscr{M}(X))$ underlie a stable homotopical 2-functor in the sense of [5] (see Theorem 5.1), so that the rest of the formalism is obtained from [5, 6]. The existence of the direct image by a closed immersion or the inverse image by a smooth morphism are obtained immediately via the universal property (see Section 2). However, to construct the inverse image by a closed immersion (see Section 4), we need to develop analogues, for étale motives, of the functors and gluing exact sequences obtained by Beilinson [16]. This is done in Section 3 and uses derivators, and the logarithmic specialization system of Ayoub [6, 8]. The proof of the main theorem is carried out in Section 5 and the most important step is the proof of the existence of the direct image by the projection of the affine line $\mathbf{A}_X^1$ onto its base X (see Proposition 5.2). We conclude this section by the aforementioned applications to intersection cohomology and Leray spectral sequences.

In Section 6, we show that perverse motives can also be defined using ℓ-adic perverse sheaves and that they admit a notion of weights. We deduce the existence of the weight filtration from the properties of Bondarko's Chow weight structure and from the Weil conjectures (cf. Deligne [27, Théorème 1]). Then, using the strict support decomposition of pure objects to reduce to the case of a point, we show that the category of pure objects of a given weight is semisimple. As an application, we get the existence of a weight structure on the derived category of $\mathscr{M}(X)$ and an arithmetic proof of Zucker's theorem [77, Theorem, p. 416] for geometric variations of Hodge structures (see Theorem 6.28 and Corollary 6.29).

1. Categorical preliminaries

Let us recall in this section a few universal constructions related to abelian and triangulated categories. They date back to Freyd's construction of the abelian hull of an additive category, see his paper [35] and have been considered in many different forms in various works (see e.g. Verdier [73], Krause [57], Prest [63] and Barbieri-Viale and Prest [14]).

Let $\mathcal{S}$ be an additive category. Let $\mathbf{Mod}(\mathcal{S})$ be the category of right $\mathcal{S}$-modules, that is, the category of additive functors from $\mathcal{S}^{\mathrm{op}}$ to the category $\mathbf{Ab}$ of abelian groups. The category $\mathbf{Mod}(\mathcal{S})$ is abelian and a sequence of right $\mathcal{S}$-modules

$$0 \to F' \to F \to F'' \to 0$$

is exact if and only if for every $s \in \mathcal{S}$ the sequence of abelian groups

$$0 \to F'(s) \to F(s) \to F''(s) \to 0$$

is exact.

A right $\mathcal{S}$-module F is said to be *of finite presentation* if there exist objects s, t in $\mathcal{S}$ and an exact sequence

$$\mathcal{S}(-, s) \to \mathcal{S}(-, t) \to F \to 0$$

in $\mathbf{Mod}(\mathcal{S})$.

Definition 1.1. Let $\mathcal{S}$ be an additive category. We denote by $\mathbf{R}(\mathcal{S})$ the full subcategory of $\mathbf{Mod}(\mathcal{S})$ consisting of right $\mathcal{S}$-modules of finite presentation.

The category $\mathbf{R}(\mathcal{S})$ is an additive category with cokernels (the cokernel of a morphism of right $\mathcal{S}$-modules of finite presentation is of finite presentation) and the Yoneda functor

$$h_{\mathcal{S}} : \mathcal{S} \to \mathbf{R}(\mathcal{S})$$

is a fully faithful additive functor. Recall that, given a morphism $t \to s$ in $\mathcal{S}$, a morphism $r \to t$ is called a *pseudo-kernel* if the sequence

$$\mathcal{S}(-, r) \to \mathcal{S}(-, t) \to \mathcal{S}(-, s)$$

is exact in $\mathbf{Mod}(\mathcal{S})$. The category $\mathbf{R}(\mathcal{S})$ is abelian if and only if $\mathcal{S}$ has pseudo-kernels (see [35, Theorem 1.4] and [57, Lemma 2.2]). It also satisfies the following universal property.

Proposition 1.2 ([57, Universal Property 2.1 and Lemma 2.6]). *Let $\mathcal{S}$ be an additive category. Let $\mathcal{A}$ be an additive category with cokernels and $F : \mathcal{S} \to \mathcal{A}$ be an additive functor. Then there exists, up to a natural isomorphism, a unique right exact functor $\mathbf{R}(\mathcal{S}) \to \mathcal{A}$ that extends F. Moreover, if $\mathcal{S}$ and $\mathcal{A}$ admit pseudo-kernels, then this functor is exact if and only if F preserves pseudo-kernels.*

Note that the construction can be dualized so that there is a universal way to add kernels to an additive category. One simply sets $\mathbf{L}(\mathcal{S}) := \mathbf{R}(\mathcal{S}^{\mathrm{op}})^{\mathrm{op}}$. The two constructions can be combined to add both cokernels and kernels at the same time. Let $\mathcal{S}$ be an additive category and let

$$\mathbf{A}^{\mathrm{ad}}(\mathcal{S}) := \mathbf{L}(\mathbf{R}(\mathcal{S})).$$

Then the functor $h : \mathcal{S} \to \mathbf{A}^{\mathrm{ad}}(\mathcal{S})$ is a fully faithful additive functor and $\mathbf{A}^{\mathrm{ad}}(\mathcal{S})$ is an abelian category which enjoys the following universal property (this is Freyd's abelian hull).

Proposition 1.3. *Let $\mathcal{A}$ be an abelian category and $F : \mathcal{S} \to \mathcal{A}$ be an additive functor. Then there exists, up to a natural isomorphism, a unique exact functor $\mathbf{A}^{\mathrm{ad}}(\mathcal{S}) \to \mathcal{A}$ that extends F.*

Note also that the category $\mathbf{A}^{\mathrm{ad}}(\mathcal{S})$ is canonically equivalent to $\mathbf{R}(\mathbf{L}(\mathcal{S}))$.

This construction can be used to provide an alternative description of Nori's category (see Barbieri-Viale and Prest [14]). Let $\mathcal{Q}$ be a quiver, $\mathcal{A}$ be an abelian category and $T : \mathcal{Q} \to \mathcal{A}$ be a representation. Let $\mathbf{P}(\mathcal{Q})$ be the path category and $\mathbf{P}(\mathcal{Q})^{\oplus}$ be its additive completion obtained by adding finite direct sums. Then, up to natural isomorphisms, we have a commutative diagram

$$\mathcal{Q} \longrightarrow \mathbf{P}(\mathcal{Q})^{\oplus} \longrightarrow \mathbf{A}^{\mathrm{ad}}(\mathbf{P}(\mathcal{Q})^{\oplus}) =: \mathbf{A}^{\mathrm{qv}}(\mathcal{Q})$$

where ϱ_T is an additive functor and ρ_T an exact functor. The kernel of ρ_T is a thick subcategory of $\mathbf{A}^{\mathrm{qv}}(\mathcal{Q})$ and we define the abelian category $\mathbf{A}^{\mathrm{qv}}(\mathcal{Q}, T)$ to be the quotient of $\mathbf{A}^{\mathrm{qv}}(\mathcal{Q})$ by this kernel. By construction, the functor ρ_T has a canonical factorization

$$\mathbf{A}^{\mathrm{qv}}(\mathcal{Q}) \xrightarrow{\pi_T} \mathbf{A}^{\mathrm{qv}}(\mathcal{Q}, H) \xrightarrow{r_T} \mathcal{A}$$

where π_T is an exact functor and r_T is a faithful exact functor. If we denote by $\overline{T}$ the composition of the representation $\mathcal{Q} \to \mathbf{A}^{\mathrm{qv}}(\mathcal{Q})$ and the functor $\pi_T : \mathbf{A}^{\mathrm{qv}}(\mathcal{Q}) \to \mathbf{A}^{\mathrm{qv}}(\mathcal{Q}, T)$, it provides a canonical factorization of T:

$$\mathcal{Q} \xrightarrow{\overline{T}} \mathbf{A}^{\mathrm{qv}}(\mathcal{Q}, T) \xrightarrow{r_T} \mathcal{A}$$

where $\overline{T}$ is a representation and r_T is a faithful exact functor. It is easy to see that the above factorization is universal among all factorizations of T of the form

$$\mathcal{Q} \xrightarrow{R} \mathcal{B} \xrightarrow{s} \mathcal{A}$$

where $\mathcal{B}$ is an abelian category, R is a representation and s is a faithful exact functor. In particular, whenever Nori's construction is available, e.g. if $\mathcal{A}$ is Noetherian, Artinian and has finite-dimensional Hom-groups over $\mathbf{Q}$ (see [47]), then the category $\mathbf{A}^{\mathrm{qv}}(\mathcal{Q}, T)$ is equivalent to Nori's abelian category associated with the quiver representation T.

Let us consider the case when $\mathcal{Q}$ is an additive category and T is an additive functor. Then, up to natural isomorphisms, we have a commutative diagram

$$\mathcal{Q} \longrightarrow \mathbf{A}^{\mathrm{ad}}(\mathcal{Q})$$

where T^* is an exact functor. The kernel of T^* is a thick subcategory of $\mathbf{A}^{\mathrm{ad}}(\mathcal{Q})$ and we define the abelian category $\mathbf{A}^{\mathrm{ad}}(\mathcal{Q}, T)$ to be the quotient of $\mathbf{A}^{\mathrm{ad}}(\mathcal{Q})$ by this kernel.

Lemma 1.4. *Let $\mathcal{Q}$ and $\mathcal{A}$ be additive categories. Then, for every additive functor $T : \mathcal{Q} \to \mathcal{A}$, the categories $\mathbf{A}^{\mathrm{qv}}(\mathcal{Q}, T)$ and $\mathbf{A}^{\mathrm{ad}}(\mathcal{Q}, T)$ are canonically equivalent.*

Proof. To see this, it suffices to check that the factorization

$$\mathcal{Q} \to \mathbf{A}^{\mathrm{ad}}(\mathcal{Q}, T) \to \mathcal{A}$$

satisfies the universal property that defines $\mathbf{A}^{\mathrm{qv}}(\mathcal{Q}, T)$. Consider a factorization of the representation T of the quiver $\mathcal{Q}$,

$$\mathcal{Q} \xrightarrow{R} \mathcal{B} \xrightarrow{s} \mathcal{A},$$

where $\mathcal{B}$ is an abelian category, R is a representation and s is a faithful exact functor. Since s is faithful, R must be an additive functor. Therefore, we get a commutative diagram (up
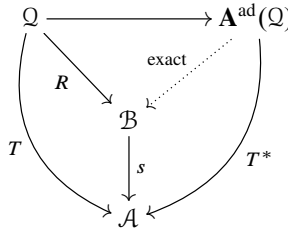
to natural isomorphisms)

The exactness and faithfulness of s imply that the above diagram can be further completed to a commutative diagram (up to natural isomorphisms)

This shows the desired universal property. ∎

Let us finally consider the special case when $\mathcal{S}$ is a triangulated category. In that case the additive category $\mathcal{S}$ has pseudo-kernels and pseudo-cokernels, in particular, the category $\mathbf{A}^{\mathrm{tr}}(\mathcal{S}) := \mathbf{R}(\mathcal{S})$ is an abelian category.[1] The Yoneda embedding $h_{\mathcal{S}} : \mathcal{S} \to \mathbf{A}^{\math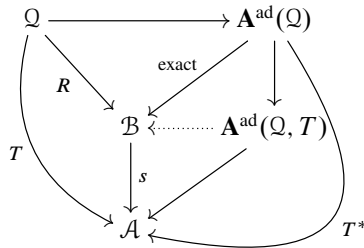rm{tr}}(\mathcal{S})$ is a homological functor and is universal for this property (see [62, Theorem 5.1.18]). In particular, if $\mathcal{A}$ is an abelian category, any homological functor $H : \mathcal{S} \to \mathcal{A}$ admits a canonical factorization

$$\mathcal{S} \xrightarrow{h_{\mathcal{S}}} \mathbf{A}^{\mathrm{tr}}(\mathcal{S}) \xrightarrow{\rho_H} \mathcal{A}$$

where ρ_H is an exact functor. This factorization of H is universal among all such factorizations.

The kernel of ρ_H is a thick subcategory of $\mathbf{A}^{\mathrm{tr}}(\mathcal{S})$ and we define the abelian category $\mathbf{A}^{\mathrm{tr}}(\mathcal{S}, H)$ to be the quotient of $\mathbf{A}^{\mathrm{tr}}(\mathcal{S})$ by this kernel. By construction, the functor ρ_H has a canonical factorization

$$\mathbf{A}^{\mathrm{tr}}(\mathcal{S}) \xrightarrow{\pi_H} \mathbf{A}^{\mathrm{tr}}(\mathcal{S}, H) \xrightarrow{r_H} \mathcal{A}$$

where π_H is an exact functor and r_H is a faithful exact functor. Setting $H_{\mathcal{S}} := \pi_H \circ h_{\mathcal{S}}$ provides a canonical factorization of H:

$$\mathcal{S} \xrightarrow{H_{\mathcal{S}}} \mathbf{A}^{\mathrm{tr}}(\mathcal{S}, H) \xrightarrow{r_H} \mathcal{A}$$

where $H_{\mathcal{S}}$ is a homological functor and r_H a faithful exact functor. It is easy to see that the above factorization is universal among all factorizations of H of the form

$$\mathcal{S} \xrightarrow{L} \mathcal{B} \xrightarrow{s} \mathcal{A}$$

where L is a homological functor and s is a faithful exact functor.

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[1]This is the abelian category denoted by $A(\mathcal{S})$ in Neeman's book [62, Chapter V].

We can also see the triangulated category $\mathcal{S}$ simply as a quiver (resp. an additive category) and the homological functor $H : \mathcal{S} \to \mathcal{A}$ simply as a representation (resp. an additive functor). In particular, we have at our disposal the universal factorizations of the representation H:

$$\mathcal{S} \to \mathbf{A}^{\mathrm{qv}}(\mathcal{S}, H) \to \mathcal{A}$$

and

$$\mathcal{S} \to \mathbf{A}^{\mathrm{ad}}(\mathcal{S}, H) \to \mathcal{A}$$

where the arrows on the right are exact and faithful functors.

Lemma 1.5. *Let $\mathcal{S}$ be a triangulated category, $\mathcal{A}$ be an abelian category and $H : \mathcal{S} \to \mathcal{A}$ be a homological functor. Then the three abelian categories $\mathbf{A}^{\mathrm{qv}}(\mathcal{S}, H)$, $\mathbf{A}^{\mathrm{ad}}(\mathcal{S}, H)$ and $\mathbf{A}^{\mathrm{tr}}(\mathcal{S}, H)$ are canonically equivalent.*

Proof. We have seen in Lemma 1.4 that $\mathbf{A}^{\mathrm{qv}}(\mathcal{S}, H)$ and $\mathbf{A}^{\mathrm{ad}}(\mathcal{S}, H)$ are canonically equivalent. Let us prove that so do $\mathbf{A}^{\mathrm{ad}}(\mathcal{S}, H)$ and $\mathbf{A}^{\mathrm{tr}}(\mathcal{S}, H)$. It suffices to check that the factorization

$$\mathcal{S} \to \mathbf{A}^{\mathrm{tr}}(\mathcal{S}, H) \to \mathcal{A}$$

satisfies the universal property that defines $\mathbf{A}^{\mathrm{ad}}(\mathcal{S}, H)$. Consider a factorization of the additive functor H,

$$\mathcal{Q} \xrightarrow{R} \mathcal{B} \xrightarrow{s} \mathcal{A},$$

where $\mathcal{B}$ is an abelian category, R is an additive functor and s is a faithful exact functor. Since s is faithful, R must be homological. Therefore, we get a commutative diagram (up to natural isomorphisms)
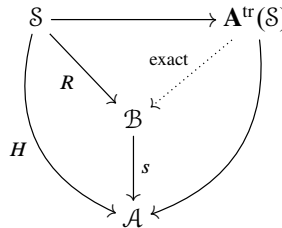
The exactness and faithfulness of s imply that the above diagram can be further completed to a commutative diagram (up to natural isomorphisms)
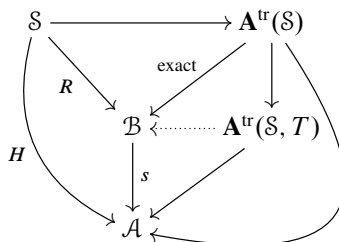
This shows the desired universal property. ∎

2. Perverse motives

We fix a field k that admits an embedding $\sigma : k \to \mathbf{C}$. Unless otherwise specified, we will only consider quasi-projective k-varieties in this article.

2.1. Definition

Let X be a quasi-projective k-variety. We denote by X^{an} the complex analytic space associated with the base change of X along σ, by $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X, \mathbf{Q})$ the category of complexes of sheaves of $\mathbf{Q}$-vector spaces on X^{an} with bounded algebraically constructible cohomology, by $\mathscr{P}(X)$ the heart of the perverse t-structure on $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X, \mathbf{Q})$ introduced by Beilinson–Bernstein–Deligne [18, Section 2] for the self-dual perversity and by $\mathbf{DA}_{\mathrm{ct}}(X)$ the triangulated category of constructible étale motivic sheaves with rational coefficients (see for example Ayoub [8, Section 3]). By [23, Theorem 16.2.18], this last category is equivalent to the category of constructible Beilinson motives studied by Cisinski and Déglise [23], and the equivalence commutes with the operations we will consider later (direct and inverse images and tensor product). So we will refer to Ayoub's articles or to the book [23], as convenient.

To construct the abelian category of perverse motives $\mathscr{M}(X)$ used in the present work, we take $\mathcal{S}$ to be the triangulated category $\mathbf{DA}_{\mathrm{ct}}(X)$ and H to be the homological functor $^{p}H_{\mathscr{P}}^{0}$ obtained by composing the Betti realization

$$\mathrm{Bti}_{X}^{*} : \mathbf{DA}_{\mathrm{ct}}(X) \to \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X, \mathbf{Q})$$

constructed by Ayoub [7] and the perverse cohomology functor $^{p}H^{0} : \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X, \mathbf{Q}) \to \mathscr{P}(X)$.

Definition 2.1. Let X be a k-variety. The abelian category of *perverse motives* is the abelian category

$$\mathscr{M}(X) := \mathbf{A}^{\mathrm{tr}}(\mathcal{S}, H) = \mathbf{A}^{\mathrm{tr}}(\mathbf{DA}_{\mathrm{ct}}(X), {}^{p}H_{\mathscr{P}}^{0}).$$

By construction the functor $^{p}H_{\mathscr{P}}^{0}$ has a factorization

$$\mathbf{DA}_{\mathrm{ct}}(X) \xrightarrow{{}^{p}H_{\mathscr{M}}^{0}} \mathscr{M}(X) \xrightarrow{\mathrm{rat}_{X}^{\mathscr{M}}} \mathscr{P}(X)$$

where $\mathrm{rat}_{X}^{\mathscr{M}}$ is a faithful exact functor and $^{p}H_{\mathscr{M}}^{0}$ is a homological functor. Let us recall the two consequences (denoted by **P1** and **P2** below) of the universal property of the factorization

$$\mathbf{DA}_{\mathrm{ct}}(X) \xrightarrow{{}^{p}H_{\mathscr{M}}^{0}} \mathscr{M}(X) \xrightarrow{\mathrm{rat}_{X}^{\mathscr{M}}} \mathscr{P}(X)$$
$$\Downarrow \rho_{X}$$
$$^{p}H_{\mathscr{P}}^{0}$$

Property **P1** below is proved in [47, Proposition 6.6]. A proof of property **P2** was given by Ivorra–Yamazaki [48, Proposition 2.5].

P1. For every commutative diagram

$$
\begin{array}{ccc}
\mathbf{DA}_{\mathrm{ct}}(X) & \xrightarrow{{}^{p}H^0_{\mathscr{P}}} & \mathscr{P}(X) \\
{\scriptstyle F}\downarrow & \Downarrow_\alpha & \downarrow{\scriptstyle G} \\
\mathbf{DA}_{\mathrm{ct}}(Y) & \xrightarrow[{}^{p}H^0_{\mathscr{P}}]{} & \mathscr{P}(Y)
\end{array}
$$

where F is a triangulated functor, G is an exact functor and $\alpha : G \circ {}^{p}H^0_{\mathscr{P}} \to {}^{p}H^0_{\mathscr{P}} \circ F$ is an invertible natural transformation, there exists a commutative diagram

$$
\begin{array}{ccccc}
\mathbf{DA}_{\mathrm{ct}}(X) & \longrightarrow & \mathscr{M}(X) & \longrightarrow & \mathscr{P}(X) \\
{\scriptstyle F}\downarrow & \Downarrow_\beta & \downarrow{\scriptstyle E} & \Downarrow_\gamma & \downarrow{\scriptstyle G} \\
\mathbf{DA}_{\mathrm{ct}}(Y) & \longrightarrow & \mathscr{M}(Y) & \longrightarrow & \mathscr{P}(Y)
\end{array}
$$

where E is an exact functor and $\beta : E \circ {}^{p}H^0_{\mathscr{M}} \to {}^{p}H^0_{\mathscr{M}} \circ F$, $\gamma : G \circ \mathrm{rat}^{\mathscr{M}}_X \to \mathrm{rat}^{\mathscr{M}}_Y \circ E$ are invertible natural transformations such that the diagram

is commutative.

P2. Let

be a commutative diagram in which F_1, F_2 are triangulated functors, G_1, G_2 are exact functors, $\alpha_1 : G_2 \circ {}^{p}H^0_{\mathscr{P}} \to {}^{p}H^0_{\mathscr{P}} \circ F_1$, $\alpha_2 : G_2 \circ {}^{p}H^0_{\mathscr{P}} \to {}^{p}H^0_{\mathscr{P}} \circ F_2$ are invertible natural transformations and $\lambda : F_1 \to F_2$, $\mu : G_1 \to G_2$ are natural transformations. Let

$$
\begin{array}{ccccc}
\mathbf{DA}_{\mathrm{ct}}(X) & \longrightarrow & \mathscr{M}(X) & \longrightarrow & \mathscr{P}(X) \\
{\scriptstyle F_1}\downarrow & \Downarrow_{\beta_1} & \downarrow{\scriptstyle E_1} & \Downarrow_{\gamma_1} & \downarrow{\scriptstyle G_1} \\
\mathbf{DA}_{\mathrm{ct}}(Y) & \longrightarrow & \mathscr{M}(Y) & \longrightarrow & \mathscr{P}(Y)
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{DA}_{\mathrm{ct}}(X) & \longrightarrow & \mathscr{M}(X) & \longrightarrow & \mathscr{P}(X) \\
{\scriptstyle F_2}\downarrow & \Downarrow_{\beta_2} & \downarrow{\scriptstyle E_2} & \Downarrow_{\gamma_2} & \downarrow{\scriptstyle G_2} \\
\mathbf{DA}_{\mathrm{ct}}(Y) & \longrightarrow & \mathscr{M}(Y) & \longrightarrow & \mathscr{P}(Y)
\end{array}
$$

be commutative diagrams given in property **P1**, then there exists a unique natural transformation $\theta : E_1 \to E_2$ such that the diagram

$$
\begin{array}{ccccc}
\mathbf{DA}_{\mathrm{ct}}(X) & \xrightarrow{\;^p H^0_{\mathscr{M}}\;} & \mathscr{M}(X) & \xrightarrow{\;\mathrm{rat}^{\mathscr{M}}_X\;} & \mathscr{P}(X) \\
\Big\| \quad \searrow^{F_1} & \Downarrow \beta_1 \quad \;^p H^0_{\mathscr{M}} & \;\;\; \searrow^{E_1} & \Downarrow \gamma_1 \quad \mathrm{rat}^{\mathscr{M}} & \quad \searrow^{G_1} \\
& \mathbf{DA}_{\mathrm{ct}}(Y) & \xrightarrow{\qquad} & \mathscr{M}(Y) \xrightarrow{\;\mathrm{rat}^{\mathscr{M}}_Y\;} & \mathscr{P}(Y) \\
\Downarrow \lambda & \;^p H^0_{\mathscr{M}} & \Downarrow \theta & \quad \mathrm{rat}^{\mathscr{M}}_X & \Downarrow \mu \\
\mathbf{DA}_{\mathrm{ct}}(X) & \searrow \;^p H^0_{\mathscr{M}} \; \mathscr{M}(X) & & \mathscr{P}(X) & \\
\quad \searrow^{F_2} & \Downarrow \beta_2 \quad \;^p H^0_{\mathscr{M}} & \;\;\; \searrow^{E_2} & \Downarrow \gamma_2 \quad \mathrm{rat}^{\mathscr{M}}_Y & \quad \searrow^{G_2} \\
& \mathbf{DA}_{\mathrm{ct}}(Y) & \xrightarrow{\qquad} & \mathscr{M}(Y) \xrightarrow{\qquad} & \mathscr{P}(Y)
\end{array}
$$

is commutative.

2.2. *Lifting of 2-functors*

As in [5, Section 1.1], in this work, we only consider strict 2-categories. However, as in loc.cit., 2-functors are not necessarily strict (see also [29]).

Let (Sch/k) be the category of quasi-projective k-varieties and $\mathcal{C}$ be a subcategory of (Sch/k). Properties **P1** and **P2** can be used to lift (covariant or contravariant) 2-functors. Indeed, let $\mathsf{F} : \mathcal{C} \to \mathfrak{TR}$ be a 2-functor (say covariant to fix the notation), where $\mathfrak{TR}$ is the 2-category of triangulated categories, such that $\mathsf{F}(X) = \mathbf{DA}_{\mathrm{ct}}(X)$ for every k-variety X in $\mathcal{C}$. Similarly, let $\mathfrak{Ab}$ be the 2-category of abelian categories, and let $\mathsf{G} : \mathcal{C} \to \mathfrak{Ab}$ be a 2-functor such that $\mathsf{G}(X) = \mathscr{P}(X)$ for every k-variety X in $\mathcal{C}$ and $\mathsf{G}(f)$ is exact for every morphism f in $\mathcal{C}$.

We have forgetful functors from $\mathfrak{TR}$ and $\mathfrak{Ab}$ to the 2-category of additive categories. Assume that $(\Theta, \alpha) : \mathsf{F} \to \mathsf{G}$ is a 1-morphism of 2-functors, where we see F and G as 2-functors into the 2-category of additive categories via these forgetful functors, such that $\Theta_X = \;^p H^0_{\mathscr{P}}$ for every $X \in \mathcal{C}$ and α_f is invertible for every morphism f in $\mathcal{C}$.

Let $f : X \to Y$ be a morphism in $\mathcal{C}$. By applying **P1** to the square

$$
\begin{array}{ccc}
\mathbf{DA}_{\mathrm{ct}}(X) & \xrightarrow{\;^p H^0_{\mathscr{P}}\;} & \mathscr{P}(X) \\
{\scriptstyle \mathsf{F}(f)} \Big\downarrow & \Downarrow \alpha_f & \Big\downarrow {\scriptstyle \mathsf{G}(f)} \\
\mathbf{DA}_{\mathrm{ct}}(Y) & \xrightarrow[\;^p H^0_{\mathscr{P}}\;]{} & \mathscr{P}(Y)
\end{array}
$$

we get a commutative diagram

$$
\begin{array}{ccccc}
\mathbf{DA}_{\mathrm{ct}}(X) & \longrightarrow & \mathscr{M}(X) & \longrightarrow & \mathscr{P}(X) \\
{\scriptstyle \mathsf{F}} \Big\downarrow & \Downarrow \beta_f & {\scriptstyle \mathsf{E}(f)} \Big\downarrow \Downarrow \gamma_f & & \Big\downarrow {\scriptstyle \mathsf{G}(f)} \\
\mathbf{DA}_{\mathrm{ct}}(Y) & \longrightarrow & \mathscr{M}(Y) & \longrightarrow & \mathscr{P}(Y)
\end{array}
$$

where $\mathsf{E}(f)$ is an exact functor and β_f, γ_f are invertible natural transformations such that the diagram

$$
\begin{array}{ccc}
\mathbf{DA}_{\mathrm{ct}}(X) & =\!=\!=\!=\!=\!=\!=\!=\!= & \mathbf{DA}_{\mathrm{ct}}(X)
\end{array}
$$
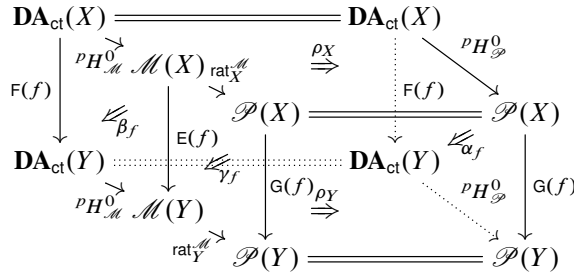
is commutative. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathcal{C}$. By applying **P2** to the commutative diagram
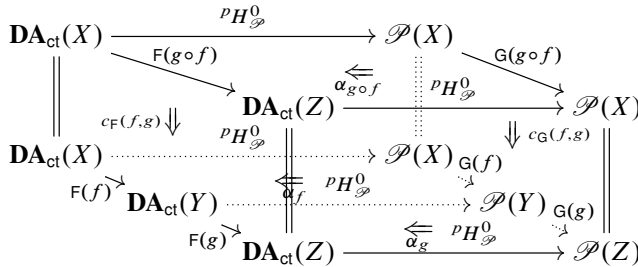
there exists a unique natural transformation $c_{\mathsf{E}}(f,g) : \mathsf{E}(g \circ f) \to \mathsf{E}(g) \circ \mathsf{E}(f)$ that fits into the commutative diagram
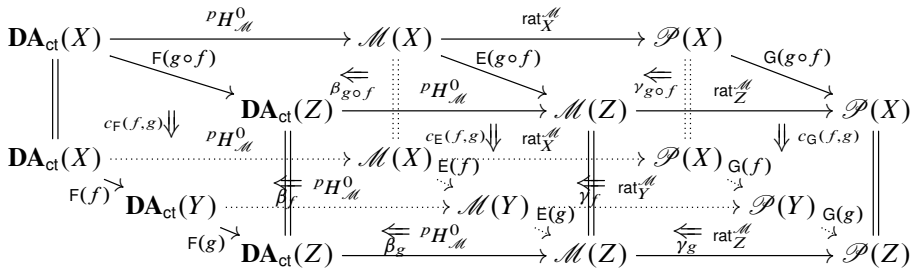
Using the uniqueness and the fact that the functors $\mathrm{rat}_X^{\mathscr{M}}$, for X in $\mathcal{C}$, are faithful it is easy to see that the morphisms c_{E} satisfy the cocycle condition. Hence $\mathsf{E} : \mathcal{C} \to \mathfrak{Ab}$ is a 2-functor and $({}^p H_{\mathscr{M}}^0, \beta)$, $(\mathrm{rat}^{\mathscr{M}}, \gamma)$ are 1-morphisms of 2-functors, where again we see the 2-functors as functors into the 2-category of additive categories. As 1- and 2-morphisms in $\mathfrak{Ab}$ are the same as 1- and 2-morphisms in the 2-category of additive categories, the morphism $(\mathrm{rat}^{\mathscr{M}}, \gamma)$ is also a 1-morphism of 2-functors $\mathcal{C} \to \mathfrak{Ab}$.

2.3. Betti realization of étale motives

Let $f : X \to Y$ be a morphism of quasi-projective k-varieties. Recall that the category $D_c^b(X, \mathbf{Q})$ is equivalent to the derived category of the abelian category of perverse sheaves on X via the realization functor constructed in [18, Section 3.1.9] (it is known to be an equivalence by [17, Theorem 1.3]). In particular, the four (adjoint) functors

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,\mathbf{Q}) \xleftarrow[\;f^{\mathscr{P}}_{*}\;]{\;f^{*}_{\mathscr{P}}\;} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y,\mathbf{Q}) \xleftarrow[\;f^{!}_{\mathscr{P}}\;]{\;f^{\mathscr{P}}_{!}\;} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,\mathbf{Q})$$

can be seen as functors between the derived categories of perverse sheaves (for their construction in terms of perverse sheaves see [17, 69]).

Let $\mathrm{Bti}^{*}_{X} : \mathbf{DA}_{\mathrm{ct}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,\mathbf{Q})$ be the realization functor of [7]. If $f : X \to Y$ is a morphism of quasi-projective k-varieties, by construction, there is an invertible natural transformation

$$\theta_f : f^{*}_{\mathscr{P}} \circ \mathrm{Bti}^{*}_{Y} \to \mathrm{Bti}^{*}_{X} \circ f^{*}$$

(see [7, Proposition 2.4]). Let θ be the collection of these natural transformations, then $(\mathrm{Bti}^{*},\theta)$ is a morphism of stable homotopical 2-functors in the sense of [7, Definition 3.1]. Following the notation in [7], we denote by

$$\gamma_f : \mathrm{Bti}^{*}_{Y} \circ f_{*} \to f^{\mathscr{P}}_{*} \circ \mathrm{Bti}^{*}_{X}, \quad \rho_f : f^{\mathscr{P}}_{!} \circ \mathrm{Bti}^{*}_{X} \to \mathrm{Bti}^{*}_{Y} \circ f_{!}, \quad \xi_f : \mathrm{Bti}^{*}_{X} \circ f^{!} \to f^{!}_{\mathscr{P}} \circ \mathrm{Bti}^{*}_{Y}$$

the induced natural transformations. By [7, Théorème 3.19] these transformations are invertible.

2.4. Direct images under affine and quasi-finite morphisms

Let $^{\mathrm{QAff}}(\mathrm{Sch}/k)$ be the subcategory of (Sch/k) with the same objects but in which we only retain the morphisms that are quasi-finite and affine. By [18, Corollaire 4.1.3], for such a morphism $f : X \to Y$, the functors

$$f^{\mathscr{P}}_{*}, f^{\mathscr{P}}_{!} : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,\mathbf{Q}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y,\mathbf{Q})$$

are t-exact for the perverse t-structures. In particular, they induce exact functors between categories of perverse sheaves and by applying property **P1** to the canonical transformation $\gamma_f : \mathrm{Bti}^{*}_{Y} \circ f_{*} \to f^{\mathscr{P}}_{*} \circ \mathrm{Bti}^{*}_{X}$, we get a commutative diagram

$$
\begin{array}{ccccc}
\mathbf{DA}_{\mathrm{ct}}(X) & \longrightarrow & \mathscr{M}(X) & \longrightarrow & \mathscr{P}(X) \\
{\scriptstyle f_{*}}\downarrow & {\scriptstyle \gamma^{\mathbf{DA}}_{f}}\nearrow\!\!\!\!\!\Downarrow & {\scriptstyle f^{\mathscr{M}}_{*}}\downarrow & {\scriptstyle \gamma^{\mathscr{M}}_{f}}\nearrow\!\!\!\!\!\Downarrow & \downarrow{\scriptstyle f^{\mathscr{P}}_{*}} \\
\mathbf{DA}_{\mathrm{ct}}(Y) & \longrightarrow & \mathscr{M}(Y) & \longrightarrow & \mathscr{P}(Y)
\end{array}
$$

where $f^{\mathscr{M}}_{*}$ is an exact functor and $\gamma^{\mathbf{DA}}_{f}, \gamma^{\mathscr{M}}_{f}$ are invertible natural transformations such that the diagram

$$
\begin{array}{c}
\end{array}
$$

is commutative. Moreover, since the natural transformations γ_f are compatible with the composition of morphisms (that is, with the connection 2-isomorphisms), Section 2.2 provides a 2-functor

$$^{\mathrm{QAff}}\mathsf{H}_*^{\mathscr{M}} : {}^{\mathrm{QAff}}(\mathrm{Sch}/k) \to \mathfrak{T}\mathfrak{R}$$

with $^{\mathrm{QAff}}\mathsf{H}_*^{\mathscr{M}}(X) = \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ and such that $({}^{p}H_{\mathscr{M}}^{0}, \gamma^{\mathbf{DA}})$ and $(\mathrm{rat}^{\mathscr{M}}, \gamma^{\mathscr{M}})$ are 1-morphisms of 2-functors. For every affine and quasi-finite morphism $f : X \to Y$ we have a natural transformation

$$\gamma_f^{\mathscr{M}} : \mathrm{rat}_Y^{\mathscr{M}} f_*^{\mathscr{M}} \to f_*^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}}$$

compatible with the composition of morphisms.

2.5. *Inverse image by a smooth morphism*

Let $f : X \to Y$ be a smooth morphism of k-varieties. Then Ω_f is a locally free $\mathscr{O}_X$-module of finite rank. Let d_f its rank (which is constant on each connected component of X). Then d_f is the relative dimension of f (see Grothendieck's EGA IV, more precisely [39, (17.10.2)]) and if $g : Y \to Z$ is a smooth morphism, then $d_{g \circ f} = d_g + d_f$ with the obvious abuse of notation (see [39, (17.10.3)]). By [18, Section 4.2.4], the functor

$$f_{\mathscr{P}}^*[d_f] : \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(Y, \mathbf{Q}) \to \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X, \mathbf{Q})$$

is t-exact for the perverse t-structures. In particular, it induces an exact functor between the categories of perverse sheaves and by applying property **P1** to the canonical transformation $\theta_f : f_{\mathscr{P}}^* \circ \mathrm{Bti}_Y^* \to \mathrm{Bti}_X^* \circ f^*$, we get a commutative diagram

$$
\begin{array}{ccccc}
\mathbf{DA}_{\mathrm{ct}}(Y) & \longrightarrow & \mathscr{M}(Y) & \longrightarrow & \mathscr{P}(Y) \\
{\scriptstyle f^*[d_f]}\downarrow & {\scriptstyle \Vert \theta_f^{\mathbf{DA}}} \downarrow & & {\scriptstyle \Vert \theta_f^{\mathscr{M}}} \downarrow {\scriptstyle f_{\mathscr{P}}^*[d_f]} \\
\mathbf{DA}_{\mathrm{ct}}(X) & \longrightarrow & \mathscr{M}(X) & \longrightarrow & \mathscr{P}(X)
\end{array}
$$

where the functor in the middle $f_{\mathscr{M}}^*[d_f]$ is an exact functor and $\theta_f^{\mathbf{DA}}, \theta_f^{\mathscr{M}}$ are invertible natural transformations such that the diagram

is commutative.

Remark 2.2. Note that $f_{\mathscr{M}}^* A$, given A in $\mathscr{M}(Y)$, is not yet defined. We define the function $f_{\mathscr{M}}^*$ by setting $f_{\mathscr{M}}^* := (f_{\mathscr{M}}^*[d_f])[-d_f]$.

Let $^{\mathrm{Liss}}(\mathrm{Sch}/k)$ be the subcategory of (Sch/k) with the same objects but having as morphisms only the smooth morphisms of k-varieties. Since the natural transformations θ_f are compatible with the composition of morphisms (that is, with the connection 2-isomorphisms), Section 2.2 provides a contravariant 2-functor

$$^{\mathrm{Liss}}\mathsf{H}^*_{\mathcal{M}} : {}^{\mathrm{Liss}}(\mathrm{Sch}/k) \to \mathfrak{T}\mathfrak{R}$$

with $^{\mathrm{Liss}}\mathsf{H}^*_{\mathcal{M}}(X) = \mathrm{D}^{\mathrm{b}}(\mathcal{M}(X))$ and such that $(^{p}H^0_{\mathcal{M}}, \theta^{\mathbf{DA}})$ and $(\mathrm{rat}^{\mathcal{M}}, \theta^{\mathcal{M}})$ are 1-morphisms of 2-functors. For every smooth morphism $f : X \to Y$ we have a natural transformation

$$\theta_f^{\mathcal{M}} : f^*_{\mathcal{P}}\mathrm{rat}^{\mathcal{M}}_Y \to \mathrm{rat}^{\mathcal{M}}_X f^*_{\mathcal{M}}$$

compatible with the composition of morphisms.

2.6. Exchange structure

Let us denote by

$$^{\mathrm{Imm}}\mathsf{H}^{\mathcal{M}}_* : {}^{\mathrm{Imm}}(\mathrm{Sch}/k) \to \mathfrak{T}\mathfrak{R}$$

the restriction of the 2-functor obtained in Section 2.4 to the subcategory $^{\mathrm{Imm}}(\mathrm{Sch}/k)$ of (Sch/k) with the same objects but having as morphisms only closed immersions of k-varieties. Exchange structures are defined in [5, Définition 1.2.1].

Proposition 2.3. *The exchange structure* Ex^*_* *on* $(^{\mathrm{Liss}}\mathsf{H}^*_{\mathcal{P}}, {}^{\mathrm{Imm}}\mathsf{H}^{\mathcal{P}}_*)$ *with respect to carte-sian squares can be lifted to an exchange structure on the pair* $(^{\mathrm{Liss}}\mathsf{H}^*_{\mathcal{M}}, {}^{\mathrm{Imm}}\mathsf{H}^{\mathcal{M}}_*)$.

Proof. The proposition is a simple application of property **P2**. Consider a cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{\;i'\;} & Y' \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
X & \xrightarrow{\;i\;} & Y
\end{array}
$$

such that i is a closed immersion and f is a smooth morphism (more generally, i need not be a closed immersion and can be any quasi-finite affine morphism). Note that, since the morphism $i'^*\Omega^1_f \to \Omega^1_{f'}$ is an isomorphism, f and f' have the same relative dimension d. Let $\mathrm{Ex}^*_* : f^*i_* \to i'_*f'^*$ and $^{\mathcal{P}}\mathrm{Ex}^*_* : f^*_{\mathcal{P}}i^{\mathcal{P}}_* \to i'^{\mathcal{P}}_* f'^*_{\mathcal{P}}$ the exchange 2-isomorphisms in **DA**$(-)$ and $\mathrm{D}^{\mathrm{b}}(\mathcal{P}(-))$. We have to construct a 2-isomorphism

$$
\mathrm{D}^{\mathrm{b}}(\mathcal{M}(X)) \xRightarrow[\quad i'^{\mathcal{M}}_* f'^*_{\mathcal{M}} \quad]{\quad f^*_{\mathcal{M}} i^{\mathcal{M}}_* \quad}{}^{\mathcal{M}}\mathrm{Ex}^*_* \; \mathrm{D}^{\mathrm{b}}(\mathcal{M}(Y'))
$$

which is compatible with $^{\mathcal{P}}\mathrm{Ex}^*_*$ via the 2-isomorphisms $\gamma_g^{\mathcal{M}}, \gamma_{g'}^{\mathcal{M}}$ and $\theta_f^{\mathcal{M}}, \theta_{f'}^{\mathcal{M}}$. This amounts to constructing a 2-isomorphism

$$
\mathcal{M}(X) \xRightarrow[\quad i'^{\mathcal{M}}_* f'^*_{\mathcal{M}}[d] \quad]{\quad f^*_{\mathcal{M}}[d] i^{\mathcal{M}}_* \quad}{}^{\mathcal{M}}\mathrm{Ex}^*_*[d] \; \mathcal{M}(Y')
$$

such that

$$
\begin{array}{c}
\xymatrix{
\mathscr{M}(X) \ar[r]^{\mathrm{rat}_X^{\mathscr{M}}} & \mathscr{P}(X)
}
\end{array}
$$
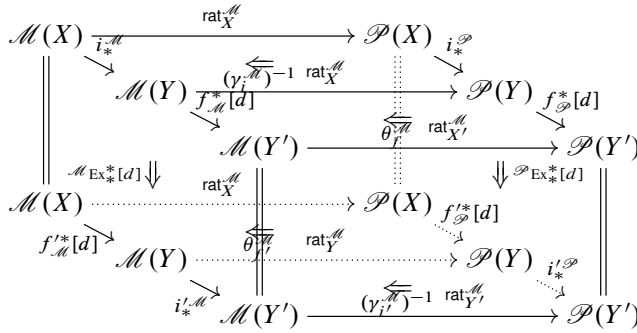
is commutative. Note that such a 2-isomorphism is necessarily unique since the functors $\mathrm{rat}_S^{\mathscr{M}}$, for S a k-variety, are faithful. For the same reason they will also be compatible with the horizontal and vertical compositions of mixed squares (see [5, Définition 1.21]). The proposition follows from property **P2** applied to the commutative diagram
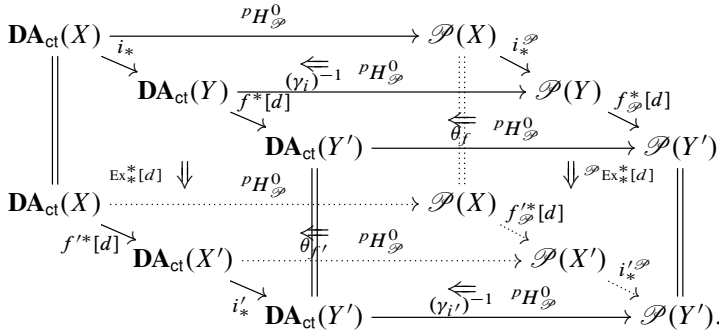
The commutativity of this diagram follows from the compatibility of the Betti realization with the exchange structures. ∎

Remark 2.4. The application of property **P2** ensures that the two exchange structures $\mathscr{M}\mathrm{Ex}_*^*$ and $\mathscr{P}\mathrm{Ex}_*^*$ are compatible with the canonical 2-isomorphisms $\theta^{\mathscr{M}}$. That is, the diagram

$$
\begin{array}{ccc}
f_{\mathscr{P}}^* \mathrm{rat}_Y^{\mathscr{M}} i_*^{\mathscr{M}} \xrightarrow{\ \gamma_i^{\mathscr{M}}\ } f_{\mathscr{P}}^* i_*^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} \xrightarrow{\ \mathscr{P}\mathrm{Ex}_*^*\ } i_*'^{\mathscr{P}} f_{\mathscr{P}}'^* \mathrm{rat}_X^{\mathscr{M}} \\
\downarrow^{\theta_f^{\mathscr{M}}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow^{\theta_{f'}^{\mathscr{M}}} \\
\mathrm{rat}_{Y'}^{\mathscr{M}} f_{\mathscr{M}}^* i_*^{\mathscr{M}} \xrightarrow{\ \mathscr{M}\mathrm{Ex}_*^*\ } \mathrm{rat}_{Y'}^{\mathscr{M}} i_*'^{\mathscr{M}} f_{\mathscr{M}}'^* \xrightarrow{\ \gamma_{i'}^{\mathscr{M}}\ } i_*'^{\mathscr{P}} \mathrm{rat}_{X'}^{\mathscr{M}} f_{\mathscr{M}}'^*
\end{array}
$$

is commutative. This follows from the faithfulness of the functors $\mathrm{rat}^{\mathscr{M}}$ after applying the shift functor $(-)[d]$.

2.7. Adjunction

Let $f : X \to Y$ be an affine and étale morphism. In that case the exact functors $f_* : \mathscr{P}(X) \to \mathscr{P}(Y)$ and $f_! : \mathscr{P}(X) \to \mathscr{P}(Y)$ are respectively right and left adjoint to the

exact functor $f_{\mathscr{P}}^* : \mathscr{P}(Y) \to \mathscr{P}(X)$. We can use property **P2** to lift these adjunctions to the functors $f_!^{\mathscr{M}}, f_{\mathscr{M}}^*, f_*^{\mathscr{M}}$.

Proposition 2.5. *Let $f : X \to Y$ be an affine and étale morphism.*

(1) *There exist unique natural transformations* $\mathrm{Id} \to f_*^{\mathscr{M}} f_{\mathscr{M}}^*$ *and* $f_{\mathscr{M}}^* f_*^{\mathscr{M}} \to \mathrm{Id}$ *such that the squares*

$$
\begin{array}{ccc}
\mathrm{rat}_Y^{\mathscr{M}} & \longrightarrow & \mathrm{rat}_Y^{\mathscr{M}} f_*^{\mathscr{M}} f_{\mathscr{M}}^* \\
\downarrow & & \downarrow \\
f_*^{\mathscr{P}} f_{\mathscr{P}}^* \mathrm{rat}_Y^{\mathscr{M}} & \longrightarrow & f_*^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} f_{\mathscr{M}}^*
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{rat}_X^{\mathscr{M}} f_{\mathscr{M}}^* f_*^{\mathscr{M}} & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} \\
\uparrow & & \uparrow \\
f_{\mathscr{P}}^* \mathrm{rat}_Y^{\mathscr{M}} f_*^{\mathscr{M}} & \longrightarrow & f_{\mathscr{P}}^* f_*^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}}
\end{array}
$$

are commutative. With these natural transformations, the functors $(f_{\mathscr{M}}^*, f_*^{\mathscr{M}})$ form a pair of adjoint functors.

(2) *There exist unique natural transformations* $\mathrm{Id} \to f_{\mathscr{M}}^* f_!^{\mathscr{M}}$ *and* $f_!^{\mathscr{M}} f_{\mathscr{M}}^* \to \mathrm{Id}$ *such that the squares*

$$
\begin{array}{ccc}
\mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} f_{\mathscr{M}}^* f_!^{\mathscr{M}} \\
\downarrow & & \uparrow \\
f_{\mathscr{P}}^* f_!^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & f_{\mathscr{P}}^* \mathrm{rat}_Y^{\mathscr{M}} f_!^{\mathscr{M}}
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{rat}_Y^{\mathscr{M}} f_!^{\mathscr{M}} f_{\mathscr{M}}^* & \longrightarrow & \mathrm{rat}_Y^{\mathscr{M}} \\
\uparrow & & \uparrow \\
f_!^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} f_{\mathscr{M}}^* & \longleftarrow & f_!^{\mathscr{P}} f_{\mathscr{P}}^* \mathrm{rat}_X^{\mathscr{M}}
\end{array}
$$

are commutative. With these natural transformations, the functors $(f_!^{\mathscr{M}}, f_{\mathscr{M}}^*)$ form a pair of adjoint functors.

Proof. Just as for Proposition 2.3, the proof is a simple application of property **P2**. The details are left to the reader. ∎

2.8. Duality

The result in this subsection will be used in the proof of Proposition 5.3. Let $\mathbb{D}_X^{\mathscr{P}}$ be the duality functor for perverse sheaves and $\varepsilon_X^{\mathscr{P}} : \mathrm{Id} \to \mathbb{D}_X^{\mathscr{P}} \circ \mathbb{D}_X^{\mathscr{P}}$ be the canonical 2-isomorphism. Recall that, given a smooth morphism $f : X \to Y$ of relative dimension d, there is a canonical 2-isomorphism

$$
\varepsilon_f^{\mathscr{P}} : \mathbb{D}_X^{\mathscr{P}} \circ f_{\mathscr{P}}^*(-)(d)[d] \to f_{\mathscr{P}}^*(-)[d] \circ \mathbb{D}_Y^{\mathscr{P}}.
$$

Proposition 2.6. *Let X, Y be k-varieties and $f : X \to Y$ be a smooth morphism of relative dimension d.*

(1) *There exist a contravariant exact functor $\mathbb{D}_X^{\mathscr{M}} : \mathscr{M}(X) \to \mathscr{M}(X)$ and 2-isomorphisms*

$$
\nu_X^{\mathscr{M}} : \mathbb{D}_X^{\mathscr{P}} \circ \mathrm{rat}_X^{\mathscr{M}} \to \mathrm{rat}_X^{\mathscr{M}} \circ \mathbb{D}_X^{\mathscr{M}}, \quad \varepsilon_X^{\mathscr{M}} : \mathrm{Id} \to \mathbb{D}_X^{\mathscr{M}} \circ \mathbb{D}_X^{\mathscr{M}}
$$

such that the diagram

$$\mathbb{D}_X^{\mathscr{P}} \circ \mathbb{D}_X^{\mathscr{P}} \circ \mathrm{rat}_X^{\mathscr{M}}$$

$$\varepsilon_X^{\mathscr{P}} \qquad \nu_X^{\mathscr{M}}$$

$$\mathrm{rat}_X^{\mathscr{M}} \qquad \mathbb{D}_X^{\mathscr{P}} \circ \mathrm{rat}_X^{\mathscr{M}} \circ \mathbb{D}_X^{\mathscr{M}}$$

$$\varepsilon_X^{\mathscr{M}} \qquad \nu_X^{\mathscr{M}}$$

$$\mathrm{rat}_X^{\mathscr{M}} \circ \mathbb{D}_X^{\mathscr{M}} \circ \mathbb{D}_X^{\mathscr{M}}$$

is commutative.

(2) *There exists a 2-isomorphism*

$$\varepsilon_f^{\mathscr{M}} : \mathbb{D}_X^{\mathscr{M}} \circ f_{\mathscr{M}}^*(-)(d)[d] \to f_{\mathscr{M}}^*(-)[d] \circ \mathbb{D}_Y^{\mathscr{M}}$$

such that the diagram

$$\mathbb{D}_X^{\mathscr{P}} \circ f_{\mathscr{P}}^*(-)(d)[d] \circ \mathrm{rat}_Y^{\mathscr{M}} \xrightarrow{\varepsilon_f^{\mathscr{P}}} f_{\mathscr{P}}^*(-)[d] \circ \mathbb{D}_Y^{\mathscr{P}} \circ \mathrm{rat}_Y^{\mathscr{M}}$$

$$\theta_f^{\mathscr{M}} \qquad \qquad \nu_Y^{\mathscr{M}}$$

$$\mathbb{D}_X^{\mathscr{P}} \circ \mathrm{rat}_X^{\mathscr{M}} \circ f_{\mathscr{M}}^*(-)(d)[d] \qquad f_{\mathscr{P}}^*(-)[d] \circ \mathrm{rat}_Y^{\mathscr{M}} \circ \mathbb{D}_Y^{\mathscr{M}}$$

$$\nu_X^{\mathscr{M}} \qquad \qquad \theta_f^{\mathscr{M}}$$

$$\mathrm{rat}_X^{\mathscr{M}} \circ \mathbb{D}_X^{\mathscr{M}} \circ f_{\mathscr{M}}^*(-)(d)[d] \xrightarrow{\varepsilon_f^{\mathscr{M}}} \mathrm{rat}_X^{\mathscr{M}} \circ f_{\mathscr{M}}^*(-)[d] \circ \mathbb{D}_Y^{\mathscr{M}}$$

is commutative.

Proof. Again, the proof is a simple application of property **P2**, once we know the existence and properties of the Verdier duality functor on motives. We give references for these properties and leave the rest of the details to the reader.

By [5, Théorème 2.3.75] and [6, Section 4.5] (see also [23, Théorème 3 and Théorème 7 in the introduction]), the categories $\mathbf{DA}_{\mathrm{ct}}(X)$ are symmetric monoidal closed and we have Verdier duality functors $\mathbb{D}_X$ such that, for $f : X \to Y$ a morphism of quasi-projective k-varieties, there is a canonical isomorphism $f^* \circ \mathbb{D}_Y \simeq \mathbb{D}_X \circ f^!$. If f is smooth of relative dimension d, the functor $f^!$ is defined in [5, Section 1.5.3] as the composition $\mathrm{Th}(\Omega_f) \circ f^*$, where $\mathrm{Th}(\Omega_f)$ is the Thom equivalence associated with the locally free $\mathscr{O}_X$-module Ω_f. As Ω_f has rank d, we get an isomorphism $f^! \simeq f^*(d)[2d]$ by [8, Corollaire 2.14] (see also [23, Section A.5.1, property 4]). Hence, we get an isomorphism $f^*[d] \circ \mathbb{D}_Y \simeq \mathbb{D}_X \circ f^*(d)[d]$. Moreover, the Betti realization functors Bti_X are symmetric monoidal unital (see [7, Remarque 3.3]), and they commute with internal Homs on constructible objects by [7, Théorème 3.19]; so it commutes with the Verdier duality functor on constructible objects, as that functor is constructed using the four operations and the internal Hom (see for example [23, Theorem A.5.2]). The last crucial observation is that Verdier duality on $\mathrm{D}_c^b(X, \mathbf{Q})$ restricts to an exact contravariant functor on the

subcategory of perverse sheaves (see for example the beginning of [18, Section 4]). ∎

2.9. Perverse motives as a stack

Let S be a quasi-projective k-variety. Let us denote by AffEt/S the category of affine étale schemes over S endowed with the étale topology. As in [70, Tag 02XU], the 2-functor

$$\mathsf{AffEt}/X \to \mathfrak{Ab}, \quad U \mapsto \mathscr{M}(U), \quad u \mapsto u^*_{\mathscr{M}},$$

can be turned into a fibered category $\mathscr{M} \to \mathsf{AffEt}/S$ such that the fiber over an object U of AffEt/X is the category $\mathscr{M}(U)$.

Proposition 2.7. *The fibered category $\mathscr{M} \to \mathsf{AffEt}/S$ is a stack for the étale topology.*

Proof. Let U be a k-variety, I be a finite set and $\mathscr{U} = (u_i : U_i \to U)_{i \in I}$ be a covering of U by affine and étale morphisms. If $J \subseteq I$ is a nonempty subset of I, we denote by U_J the fiber product of the U_j, $j \in J$, over U and by $u_J : U_J \to U$ the induced morphism. Given an object $A \in \mathscr{M}(U)$, and $k \in \mathbf{Z}$, we set

$$C^k(A, \mathscr{U}) := \begin{cases} 0 & \text{if } k < 0, \\ \bigoplus_{J \subseteq I, |J|=k+1} (u_J)^{\mathscr{M}}_* (u_J)^*_{\mathscr{M}} A & \text{if } k \geq 0. \end{cases}$$

We make $C^\bullet(A, \mathscr{U})$ into a complex using the alternating sum of the maps obtained from the unit of the adjunction in Proposition 2.5. The unit of this adjunction also provides a canonical morphism $A \to C^\bullet(A, \mathscr{U})$ in $\mathrm{C}^{\mathrm{b}}(\mathscr{M}(U))$. This morphism induces a quasi-isomorphism on the underlying complex of perverse sheaves and so is a quasi-isomorphism itself since the forgetful functor to the derived category of perverse sheaves is conservative.

By [70, Tag 0268], to prove the proposition we have to show the following:

(1) if U is an object in AffEt/S and A, B are objects in $\mathscr{M}(U)$, then the presheaf $(V \to vU) \mapsto \mathrm{Hom}_{\mathscr{M}(V)}(v^*_{\mathscr{M}} A, v^*_{\mathscr{M}} B)$ on AffEt/U is a sheaf for the étale topology;

(2) for any covering $\mathscr{U} = (u_i : U_i \to U)_{i \in I}$ of the site AffEt/S, any descent datum is effective.

We already know that similar assertions are true for perverse sheaves by [18, Proposition 3.2.2, Théorème 3.2.4]. Let $\mathscr{U} = (u_i : U_i \to U)_{i \in I}$ be a covering in the site AffEt/S. Given $i, j \in I$, we denote by $u_{ij} : U_{ij} := U_i \times_U U_j \to U$ the fiber product and by $p_{ij} : U_{ij} \to U_i$, $p_{ji} : U_{ij} \to U_j$ the projections.

Let us first prove (1). Let A, B be objects in $\mathscr{M}(U)$ and K, L be their underlying perverse sheaves. Consider the canonical commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(A, B) & \longrightarrow & \prod_{i \in I} \mathrm{Hom}((u_i)^*_{\mathscr{M}} A, (u_i)^*_{\mathscr{M}} B) & \underset{\longrightarrow}{\longrightarrow} & \prod_{i,j \in I} \mathrm{Hom}((u_{ij})^*_{\mathscr{M}} A, (u_{ij})^*_{\mathscr{M}} B) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(K, L) & \longrightarrow & \prod_{i \in I} \mathrm{Hom}((u_i)^*_{\mathscr{P}} K, (u_i)^*_{\mathscr{P}} L) & \underset{\longrightarrow}{\longrightarrow} & \prod_{i,j \in I} \mathrm{Hom}((u_{ij})^*_{\mathscr{P}} K, (u_{ij})^*_{\mathscr{P}} L) \end{array}$$

The lower row is exact and the vertical arrows are injective. We only have to check that the upper row is exact at the middle term. Let c be an element in $\prod_{i \in I} \mathrm{Hom}((u_i)^*_{\mathscr{M}} A, (u_i)^*_{\mathscr{M}} B)$ which belongs to the equalizer of the two maps on the right-hand side. Then it defines (by adjunction) a morphism c_0 and a morphism c_1 such that the square

$$
\begin{array}{ccc}
C^0(A, \mathscr{U}) & \xrightarrow{d^0} & C^1(A, \mathscr{U}) \\
c_0 \downarrow & & \downarrow c_1 \\
C^0(B, \mathscr{U}) & \xrightarrow{d^0} & C^1(B, \mathscr{U})
\end{array}
\tag{2.1}
$$

is commutative. Since $A \to C^{\bullet}(A, \mathscr{U})$ and $B \to C^{\bullet}(B, \mathscr{U})$ are quasi-isomorphisms, A is the kernel of the upper map in (2.1) and B is the kernel of the lower map. Hence, c_0 and c_1 induce a morphism $A \to B$ in $\mathscr{M}(U)$ which maps to c.

Now we prove (2). Consider a descent datum. In other words, consider, for every $i \in I$, an object A_i in $\mathscr{M}(U_i)$ and, for every $i, j \in I$, an isomorphism $\phi_{ij} : (p_{ij})^*_{\mathscr{M}} A_i \to (p_{ji})^*_{\mathscr{M}} A_j$ in $\mathscr{M}(U_{ij})$ satisfying the usual cocycle condition. Let A be the kernel of the map

$$
\bigoplus_{i \in I} (u_i)^{\mathscr{M}}_* A_i \to \bigoplus_{i,j \in I} (u_i)^{\mathscr{M}}_* (p_{ij})^{\mathscr{M}}_* (p_{ij})^*_{\mathscr{M}} A_i = \bigoplus_{i,j \in I} (u_{ij})^{\mathscr{M}}_* (p_{ij})^*_{\mathscr{M}} A_i
$$

given on $(u_i)^{\mathscr{M}}_* A_i$ by the difference of the maps obtained by composing the morphism induced by adjunction

$$
(u_i)^{\mathscr{M}}_* A_i \to (u_i)^{\mathscr{M}}_* (p_{ij})^{\mathscr{M}}_* (p_{ij})^*_{\mathscr{M}} A_i
$$

with either the identity or the isomorphism ϕ_{ij}. Using the fact that descent data on perverse sheaves are effective, it is easy to see that A makes the given descent datum effective. ∎

2.10. A simpler generating quiver

Let X be a k-variety. Consider the quiver $\mathbf{Pairs}^{\mathrm{eff}}_X$ defined as follows. A vertex in $\mathbf{Pairs}^{\mathrm{eff}}_X$ is a triple $(a : Y \to X, Z, n)$ where $a : Y \to X$ is morphism of k-varieties, Z is a closed subscheme of Y and $n \in \mathbf{Z}$ is an integer.

- Let (Y_1, Z_1, i) and (Y_2, Z_2, i) be vertices in $\mathbf{Pairs}^{\mathrm{eff}}_X$. Then every morphism of X-schemes $f : Y_1 \to Y_2$ such that $f(Z_1) \subseteq Z_2$ defines an edge

$$
f : (Y_1, Z_1, i) \to (Y_2, Z_2, i).
\tag{2.2}
$$

- For every vertex $(a : Y \to X, Z, i)$ in $\mathbf{Pairs}^{\mathrm{eff}}_X$ and every closed subscheme $W \subseteq Z$, we have an edge

$$
\partial : (a : Y \to X, Z, i) \to (az : Z \to X, W, i - 1)
\tag{2.3}
$$

where $z : Z \hookrightarrow Y$ is the closed immersion.

The quiver $\mathbf{Pairs}_X^{\mathrm{eff}}$ admits a natural representation in $\mathrm{D}_c^b(X, \mathbf{Q})$. If $c = (a : Y \to X, Z, i)$ is a vertex in the quiver $\mathbf{Pairs}_X^{\mathrm{eff}}$ and $u : U \hookrightarrow Y$ is the inclusion of the complement of Z in Y, then we set

$$B(c) := a_!^{\mathscr{P}} u_*^{\mathscr{P}} K_U[-i]$$

where K_U is the dualizing complex of U.

Remark 2.8. There is a difference between the representation ${}^pH^0 \circ \bar{B}$ used here and the representation used in [47, 7.2–7.4] (see [47, Remark 7.8]). In loc.cit. the relative dualizing complex $u_{\mathscr{P}}^! a_{\mathscr{P}}^! \mathbf{Q}_X$ is used instead of the absolute dualizing complex K_U. If X is smooth, then the two different choices lead to equivalent categories.

On vertices the representation B is defined as follows. Let $c_1 := (a_1 : Y_1 \to X, Z_1, i)$ and $c_2 := (a_2 : Y_2 \to X, Z_2, i)$ be vertices in $\mathbf{Pairs}_X^{\mathrm{eff}}$ and $f : c_1 \to c_2$ be an edge of type (2.2). The morphism f maps Z_1 to Z_2 and therefore $U := f^{-1}(U_2)$ is contained in U_1. Let $u : U \hookrightarrow U_1$ be the open immersion. Then we have a morphism

$$f_!^{\mathscr{P}} u_{1*}^{\mathscr{P}} K_{U_1} \xrightarrow{\mathrm{adj.}} f_!^{\mathscr{P}} (u_1 u)_*^{\mathscr{P}} K_U \to u_{2*}^{\mathscr{P}} f_!^{\mathscr{P}} K_U = u_{2*}^{\mathscr{P}} f_!^{\mathscr{P}} f_{\mathscr{P}}^! K_{U_2} \xrightarrow{\mathrm{adj.}} u_{2*}^{\mathscr{P}} K_{U_2}$$

where the arrow in the middle is given by the exchange morphism. By taking the image of this morphism under $a_{2!}[-i]$, we get a morphism

$$B(f) : B(c_1) := a_{1!}^{\mathscr{P}} u_{1*}^{\mathscr{P}} K_{U_1}[-i] \to B(c_2) := a_{2!}^{\mathscr{P}} u_{2*}^{\mathscr{P}} K_{U_2}[-i].$$

Let $c = (Y \xrightarrow{a} X, Z, i)$ be a vertex in $\mathbf{Pairs}_X^{\mathrm{eff}}$, and $W \subseteq Z$ be a closed subset. Consider the commutative diagram

where v, v_Y, j are the open immersions, z the closed immersion and a, b the structural morphisms. The localization triangle

$$(z_V)_!^{\mathscr{P}} (z_V)_{\mathscr{P}}^! \to \mathrm{Id} \to j_*^{\mathscr{P}} j_{\mathscr{P}}^* \xrightarrow{+1},$$

applied to $K_{Y \setminus W}$, provides a morphism

$$j_*^{\mathscr{P}} K_U \to (z_V)_!^{\mathscr{P}} K_V[1].$$

As z and z_V are closed immersions, applying $(v_Y)_*$, yields a morphism

$$u_*^{\mathscr{P}} K_U \to z_!^{\mathscr{P}} v_*^{\mathscr{P}} K_V[1].$$

Applying $a_![-i]$, we obtain a morphism

$$B(\partial) : B(c) := a_!^{\mathscr{P}} u_*^{\mathscr{P}} K_U[-i] \to B(az : Z \to X, Z, i-1) := b_!^{\mathscr{P}} v_*^{\mathscr{P}} K_V[1-i].$$

The category of perverse Nori motives considered in [47] is defined as follows.

Definition 2.9. Let X be a k-variety. The category of *effective perverse Nori motives* is the abelian category

$$\mathscr{N}^{\mathrm{eff}}(X) := \mathbf{A}^{\mathrm{qv}}(\mathbf{Pairs}_X^{\mathrm{eff}}, {}^p H^0 \circ B).$$

Recall that the category $\mathscr{M}(X)$ can also be obtained by considering $\mathbf{DA}_{\mathrm{ct}}(X)$ simply as a quiver, that is, it is canonically equivalent to the abelian category $\mathbf{A}^{\mathrm{qv}}(\mathbf{DA}_{\mathrm{ct}}(X), {}^p H^0 \circ \mathrm{Bti}_X^*)$ (see Lemma 1.5). The Grothendieck six operations formalism constructed in [5,6] and its compatibility with its topological counterpart on the triangulated categories $D_c^b(X, \mathbf{Q})$ shown in [7] imply that the quiver representation B can be lifted via the realization functor Bti_X^* to a quiver representation

$$\bar{B} : \mathbf{Pairs}_X^{\mathrm{eff}} \to \mathbf{DA}_{\mathrm{ct}}(X).$$

In particular, since the diagram

$$\mathbf{DA}_{\mathrm{ct}}(X) \xrightarrow{\mathrm{Bti}_X^*} D_c^b(X, \mathbf{Q})$$

$$\bar{B} \uparrow \qquad \nearrow B$$

$$\mathbf{Pairs}_X^{\mathrm{eff}}$$

is commutative (up to natural isomorphisms), there exists a canonical faithful exact functor

$$\mathscr{N}^{\mathrm{eff}}(X) \to \mathscr{M}(X). \tag{2.4}$$

Let us explain now how Tate twists can be defined in the categories $\mathscr{N}^{\mathrm{eff}}(X)$ and $\mathscr{M}(X)$. In the category $\mathbf{DA}_{\mathrm{ct}}(X)$, the Tate twist $(-)(1)$ is defined to be the endofunctor $\mathrm{Th}(\mathscr{O}_X)(-)[-2]$ where $\mathrm{Th}(\mathscr{O}_X)$ is the Thom equivalence associated with the trivial locally free sheaf $\mathscr{O}_X$ (see [5, Section 1.5.3]). This construction, being compatible with the usual Tate twist via the Betti realization, induces an exact functor $(-)(1)$ on the category $\mathscr{M}(X)$. Note that this functor is an equivalence by construction.

In the category $\mathscr{N}^{\mathrm{eff}}(X)$ Tate twists can be defined using the following observation: if S is a k-variety, $q : \mathbf{G}_{\mathrm{m},S} \to S$ is the structural morphism and $v : V \hookrightarrow \mathbf{G}_{\mathrm{m},S}$ is the complement of the unit section, then $q_! v_* v^* q^! K = K(1)[1]$ for every $K \in D_c^b(S, \mathbf{Q})$. In particular, if $Q : \mathbf{Pairs}_X^{\mathrm{eff}} \to \mathbf{Pairs}_X^{\mathrm{eff}}$ is the morphism of quivers which maps (Y, Z, n) to $(\mathbf{G}_{\mathrm{m},Y}, \mathbf{G}_{\mathrm{m},Z} \cup Y, n+1)$ (here Y is embedded in $\mathbf{G}_{\mathrm{m},Y}$ via the unit section), then one has a natural isomorphism between $B(Q(Y, Z, n))$ and $B(Y, Z, n)(1)$. As a consequence, the Tate twist on the category of effective perverse Nori motives can be defined as the exact functor induced by the morphism of quivers Q (and the usual Tate twist).

This last construction does not yield an equivalence and one defines $\mathscr{N}(X)$ to be the category obtained from $\mathscr{N}^{\mathrm{eff}}(X)$ by inverting the Tate twists (see [47, Section 7.6] for

details). By construction, the category $\mathbf{HM}(k)$ of Nori motives (see [32]) coincides with $\mathcal{N}(k)$.

Lemma 2.10. *The functor* (2.4) *extends to a faithful exact*

$$\mathcal{N}(X) \to \mathcal{M}(X). \tag{2.5}$$

Proof. To prove the lemma it is enough to observe that there is a natural isomorphism in $\mathbf{DA}_{\mathrm{ct}}(X)$ between $\bar{B}(Q(Y, Z, n))$ and $\bar{B}(Y, Z, n)(1)$. ∎

Proposition 2.11. *The category $\mathcal{M}(k)$ is canonically equivalent to the abelian category $\mathbf{HM}(k)$ of Nori motives. More precisely, the functor* (2.5) *is an equivalence when $X = \mathrm{Spec}(k)$.*

Proof. (See also Barbieri-Viale, Huber and Prest [13, Proposition 4.12].) Consider the triangulated functor $\mathcal{R}_{\mathrm{N},s} : \mathbf{DA}_{\mathrm{ct}}(X) \to \mathrm{D}^{\mathrm{b}}(\mathbf{HM}(k))$ constructed by Choudhury–Gallauer [22, Proposition 7.12]. Up to a natural isomorphism, the diagram

$$
\begin{array}{ccc}
 & \overset{\mathrm{Bti}_k^*}{\overbrace{\qquad\qquad\qquad}} & \\
\mathbf{DA}_{\mathrm{ct}}(k) \xrightarrow[\mathcal{R}_{\mathrm{N},s}]{} & \mathrm{D}^{\mathrm{b}}(\mathbf{HM}(k)) \xrightarrow[\text{forgetful}]{} & \mathrm{D}^{\mathrm{b}}(\mathbf{Q})
\end{array}
$$

is commutative. In particular, it provides a factorization of the cohomological functor $H^0 \circ \mathrm{Bti}_k^*$,

$$\mathbf{DA}_{\mathrm{ct}}(k) \xrightarrow{H^0 \circ \mathcal{R}_{\mathrm{N},s}} \mathbf{HM}(k) \xrightarrow{\text{forgetful}} \mathbf{Vec}(\mathbf{Q}).$$

This implies the existence of a canonical faithful exact functor $\mathcal{M}(k) \to \mathbf{HM}(k)$ such that

$$
\begin{array}{ccc}
 & \overset{H^0 \circ \mathcal{R}_{\mathrm{N},s}}{\overbrace{\qquad\qquad\qquad}} & \\
\mathbf{DA}_{\mathrm{ct}}(k) \xrightarrow[{}^p H^0_{\mathcal{M}}]{} & \mathcal{M}(k) \longrightarrow & \mathbf{HM}(k)
\end{array}
$$

is commutative up to a natural isomorphism. Using the universal properties, it is easy to see that it is a quasi-inverse to (2.5). ∎

The following conjecture seems reasonable and reachable via our current technology.

Conjecture 2.12. *Let X be a smooth k-variety. Let $\mathcal{N}(X)$ be the category of perverse motives constructed in [47] and*

$$\mathsf{RL}_X^{\mathcal{N}} : \mathbf{DA}_{\mathrm{ct}}(X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{N}(X))$$

be the triangulated functor constructed in [46]. Then the Betti realization Bti_X^* is isomorphic to the composition

$$\mathbf{DA}_{\mathrm{ct}}(X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{N}(X)) \xrightarrow{\text{forgetful}} \mathrm{D}^{\mathrm{b}}(\mathscr{P}(X)) \xrightarrow{\text{real}} \mathrm{D}^{\mathrm{b}}_c(X, \mathbf{Q}).$$

If Conjecture 2.12 holds then the same proof as the one of Proposition 2.11 implies the following.

Conjecture 2.13. *Let X be a smooth k-variety. Then the functor (2.5) is an equivalence.*

3. Unipotent nearby and vanishing cycles

In [16], Beilinson has given an alternative construction of unipotent vanishing cycles functors for perverse sheaves and has used it to explain a gluing procedure for perverse sheaves (see [16, Proposition 3.1]). In this section, our main goal is to obtain similar results for perverse Nori motives. Later on, the vanishing cycles functors for perverse Nori motives will play a crucial role in the construction of the inverse image functor (see Section 4).

Given the way the abelian categories of perverse Nori motives are constructed from the triangulated categories of étale motives, our first step is to carry out Beilinson's constructions for perverse sheaves within the categories of étale motives or analytic motives (the latter categories being equivalent to the classical unbounded derived categories of sheaves of **Q**-vector spaces on the associated analytic spaces). This is done in Sections 3.2 and 3.4. Our starting point is the logarithmic specialization system constructed by Ayoub [6]. However, by working in triangulated categories instead of abelian categories as Beilinson did, one has to face the classical functoriality issues, one of the major drawbacks of triangulated categories. To avoid these problems and ensure that all our constructions are functorial we will rely heavily on the fact that the triangulated categories of motives underlie a triangulated derivator.

Only then, using the compatibility with the Betti realization, will we be able to obtain in Section 3.5 the desired functors for perverse Nori motives.

3.1. Reminder on derivators

Let us recall some features of triangulated (also called stable) derivators $\mathbb{D}$ needed in the construction of the motivic unipotent vanishing cycles functor and the related exact triangles. For the general theory, originally introduced by Grothendieck [40], we refer to Ayoub [5, 6], Cisinski–Neeman [24], Groth [37, 38] and Maltsiniotis [58].

We will assume that our derivator $\mathbb{D}$ is defined over all small categories. In our applications, the derivators considered will be of the form $\mathbb{D} := \mathbf{DA}(S, -)$ for some k-variety S. Given a functor $\rho : A \to B$, we denote by

$$\rho^* : \mathbb{D}(B) \to \mathbb{D}(A), \quad \rho_* : \mathbb{D}(A) \to \mathbb{D}(B), \quad \rho_\sharp : \mathbb{D}(A) \to \mathbb{D}(B)$$

the structural functor and its right and left adjoint. Note that in the literature on derivators, the notation $\rho_!$ is used instead of $\rho_\sharp$. We follow here the notation used in [5, 6].

Notation. We let $\mathbf{e}$ be the punctual category reduced to one object and one morphism. Given a small category A, we denote by $p_A : A \to \mathbf{e}$ the projection functor, and if a is an

object in A, we denote by $a : \mathbf{e} \to A$ the functor that maps the unique object of $\mathbf{e}$ to a. Given $n \in \mathbb{N}$, we let $\underline{\mathbf{n}}$ be the category

$$n \leftarrow \cdots \leftarrow 1 \leftarrow 0.$$

If one thinks of functors in $\mathsf{Hom}(A^{\mathrm{op}}, \mathbb{D}(\mathbf{e}))$ as diagrams, then an object in $\mathbb{D}(A)$ can be thought as a "coherent diagram". Indeed, every object M in $\mathbb{D}(A)$ has an underlying diagram called its A-*skeleton* and defined to be the functor $A^{\mathrm{op}} \to \mathbb{D}(\mathbf{e})$ which maps an object a in A to the object a^*M of $\mathbb{D}(\mathbf{e})$. This construction gives the A-skeleton functor

$$\mathbb{D}(A) \to \mathsf{Hom}(A^{\mathrm{op}}, \mathbb{D}(\mathbf{e}))$$

which is not an equivalence in general (coherent diagrams are richer than simple diagrams). We say that $M \in \mathbb{D}(A)$ is a *coherent lifting* of a given diagram of shape A if its A-skeleton is isomorphic to the given diagram.

We will not give here the definition of a stable derivator (see e.g. [5, Definition 2.1.34] or [38, Section 1 and Definition 4.1]), but instead recall a few properties which will be constantly used.

(**1**) Let $\rho : A \to B$ be a functor and b be an object in B. Denote by $j_{A/b} : A/b \to A$ and $j_{b \backslash A} : b \backslash A \to A$ be the canonical functors where A/b and $b \backslash A$ are respectively the slice and coslice categories. The exchange 2-morphisms (given by adjunction)

$$b^* \rho_* \to (p_{A/b})_* j_{A/b}^*, \quad (p_{b \backslash A})_\sharp j_{b \backslash A}^* \to b^* \rho_\sharp$$

are invertible (see [5, Définition 2.1.34] or the base change axiom **Der 3** of [24, Définition 1.11]).

(**2**) If a small category A admits an initial object o (resp. a final object o), then the 2-morphism $o^* \to (p_A)_\sharp$ (resp. the 2-morphism $(p_A)_* \to o^*$) is invertible too (see [5, Corollaire 2.1.40]).

(**3**) Let A and B be small categories. Given an object $a \in A$ we denote by $a : B \to A \times B$ the functor which maps $b \in B$ to the pair (a, b). The A-skeleton of an object M in $\mathbb{D}(A \times B)$ is defined to be the functor $A^{\mathrm{op}} \to \mathbb{D}(B)$ which maps an object a in A to the object a^*M of $\mathbb{D}(B)$. This construction gives the A-skeleton functor

$$\mathbb{D}(A \times B) \to \mathsf{Hom}(A^{\mathrm{op}}, \mathbb{D}(B)).$$

This functor is conservative. Moreover if $A = \underline{\mathbf{1}}$, it is full and essentially surjective. (See axioms **Der 2** and **Der 5** of [24, Definition 1.11].)

We denote by $\square = \underline{\mathbf{1}} \times \underline{\mathbf{1}}$ the category

$$
\begin{array}{ccc}
(1,1) & \longleftarrow & (0,1) \\
\uparrow & & \uparrow \\
(1,0) & \longleftarrow & (0,0)
\end{array}
\tag{3.1}
$$

We denote by $\ulcorner$ the full subcategory of $\square$ that does not contain the object $(0,0)$ and by $i_{\ulcorner} : \ulcorner \to \square$ the inclusion functor. We denote by $(-,1) : \underline{1} \to \ulcorner$ the fully faithful functor which maps 0 to $(0,1)$ and 1 to $(1,1)$. Similarly we denote by $\llcorner$ the full subcategory of $\square$ that does not contain the object $(1,1)$ and $i_{\llcorner} : \llcorner \to \square$ the inclusion functor. We denote by $(0,-) : \underline{1} \to \llcorner$ the fully faithful functor that maps 0 and 1 respectively to $(0,0)$ and $(0,1)$

An object M in $\mathbb{D}(\square)$ is said to be *cocartesian* (resp. *cartesian*) if the canonical morphism $(i_{\ulcorner})_\sharp (i_{\ulcorner})^* M \to M$ (resp. $M \to (i_{\ulcorner})_* (i_{\ulcorner})^* M$) is an isomorphism. Since $\mathbb{D}$ is stable, a square M in $\mathbb{D}(\square)$ is cartesian if and only if it is cocartesian.

Let $\square\!\square$ be the category

$$
\begin{array}{ccccc}
(2,1) & \longleftarrow & (1,1) & \longleftarrow & (0,1) \\
\uparrow & & \uparrow & & \uparrow \\
(2,0) & \longleftarrow & (1,0) & \longleftarrow & (0,0)
\end{array}
\tag{3.2}
$$

There are three natural ways to embed $\square$ in $\square\!\square$, and an object $M \in \mathbb{D}(\square\!\square)$ is said to be cocartesian if the squares in $\mathbb{D}(\square)$ obtained by pullback along those embeddings are cocartesian. A coherent triangle is a cocartesian object $M \in \mathbb{D}(\square\!\square)$ such that $(0,1)^* M$ and $(2,0)^* M$ are zero. For such an object, we have a canonical isomorphism $(0,0)^* M \simeq (2,1)^* M[1]$ and the induced sequence

$$
(2,1)^* M \to (1,1)^* M \to (1,0)^* M \to (2,1)^* M[1]
\tag{3.3}
$$

is an exact triangle in $\mathbb{D}(\mathbf{e})$.

One of the main advantages of working in a stable derivator is the possibility to associate with a coherent morphism $M \in \mathbb{D}(\underline{1})$ functorially a coherent triangle. Let us briefly recall the construction of this triangle. Let U be the full subcategory of $\square\!\square$ that does not contain $(0,0)$ and $(1,0)$. Denote by $v : \underline{1} \to U$ the functor that maps 0 and 1 respectively to $(1,1)$ and $(2,1)$ and by $u : U \to \square\!\square$ the inclusion functor. The image under the functor

$$
u_\sharp v_* : \mathbb{D}(A \times \underline{1}) \to \mathbb{D}(A \times \square\!\square)
$$

of a coherent morphism M in $\mathbb{D}(A \times \underline{1})$ is a coherent triangle. Using properties **(1)**–**(2)** recalled above, we see that (3.3) provides an exact triangle

$$
1^* M \to 0^* M \to \mathsf{Cof}(M) \to 1^* M[1]
\tag{3.4}
$$

where the cofiber functor Cof is defined by

$$
\mathsf{Cof} := (1,0)^* u_\sharp v_* : \mathbb{D}(\underline{1}) \to \mathbb{D}(\mathbf{e}).
\tag{3.5}
$$

Using properties **(1)**–**(3)** recalled above, it is easy to see that this functor is also given by

$$
\mathsf{Cof} = (0,0)^* (i_{\ulcorner})_\sharp (-,1)_*.
$$

In the exact triangle (3.4), the canonical morphism $0^* M \to \mathsf{Cof}(M)$ is the $\underline{1}$-skeleton of the coherent morphism $(1,-)^* u_\sharp v_* M$ where $(1,-) : \underline{1} \to \square\!\square$ is the fully faithful functor

that maps 0 and 1 respectively to $(1, 0)$ and $(1, 1)$. Note that we have an isomorphism of functors

$$(1, -)^* u_\sharp v_* M \simeq (0, -)^* (i_\lrcorner)^* (i_\ulcorner)_\sharp (-, 1)_* : \mathbb{D}(\underline{\mathbf{1}}) \to \mathbb{D}(\underline{\mathbf{1}}).$$

Similarly the boundary morphism $\mathsf{Cof}(M) \to 1^* M[1]$ is the $\underline{\mathbf{1}}$-skeleton of the coherent morphism $(-, 0)^* u_\sharp v_* M$ where $(-, 0) : \underline{\mathbf{1}} \to \square$ is the fully faithful functor that maps 0 and 1 respectively to $(0, 0)$ and $(1, 0)$.

The construction of the cofiber functor Cof and the cofiber triangle (3.4) can be dualized to get a fiber functor Fib and a fiber triangle. Let us recall the following lemma.

Lemma 3.1. *Let $M \in \mathbb{D}(\square)$. Then we have a morphism of exact triangles*

$$
\begin{array}{ccccccc}
\mathsf{Fib}((-, 1)^* M) & \longrightarrow & (1, 1)^* M & \longrightarrow & (0, 1)^* M & \xrightarrow{+1} & \\
\downarrow{\scriptstyle u} & & \downarrow & & \downarrow & & \\
\mathsf{Fib}((-, 0)^* M) & \longrightarrow & (1, 0)^* M & \longrightarrow & (0, 0)^* M & \xrightarrow{+1} &
\end{array}
$$

which is functorial in M. Furthermore, M is cartesian if and only if the canonical morphism

$$u : \mathsf{Fib}((-, 1)^* M) \to \mathsf{Fib}((-, 0)^* M)$$

is an isomorphism.

Proof. The first statement follows from the fact that we have functorially defined fibers, as we just recalled. The second statement is Proposition 15.1.10 of Moritz Groth's unpublished book *Introduction to the theory of derivators*. Let us recall its proof. We have a commutative cube

$$
\begin{array}{ccc}
(1, 1)^* M & \longrightarrow & (0, 1)^* M \\
& \searrow & & \searrow \\
& (1, 0)^* M & \longrightarrow & (0, 0)^* M \\
\downarrow & & \downarrow & \\
\mathsf{Fib}((-, 1)^* M) & \dashrightarrow & 0 & \\
& {\scriptstyle u}\searrow & & \searrow \\
& \mathsf{Fib}((-, 0)^* M) & \longrightarrow & 0
\end{array}
$$

The front and back squares are cartesian by definition of an exact triangle. By [38, Proposition 4.6], the top square is cartesian if and only if the bottom square is cartesian. But the top square is M and the bottom square is cartesian if and only if u is an isomorphism, hence the result. ∎

There is also a functorial version of the octahedron axiom in $\mathbb{D}$ (see e.g. [37, proof of Theorem 4.20] or [38, proof of Theorem 4.15]), that is, there is a functor $\mathbb{D}(\underline{\mathbf{2}}) \to \mathbb{D}(O)$ which associates to a coherent sequence of morphisms a coherent octahedron diagram. Here the category $O \subseteq \underline{\mathbf{4}} \times \underline{\mathbf{2}}$ is the full subcategory that does not contain the objects $(4, 0)$ and $(0, 2)$. In other words, O is the category

$$(4, 2) \longleftarrow (3, 2) \longleftarrow (2, 2) \longleftarrow (1, 2)$$
$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$
$$(4, 1) \longleftarrow (3, 1) \longleftarrow (2, 1) \longleftarrow (1, 1) \longleftarrow (0, 1)$$
$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$
$$(3, 0) \longleftarrow (2, 0) \longleftarrow (1, 0) \longleftarrow (0, 0)$$
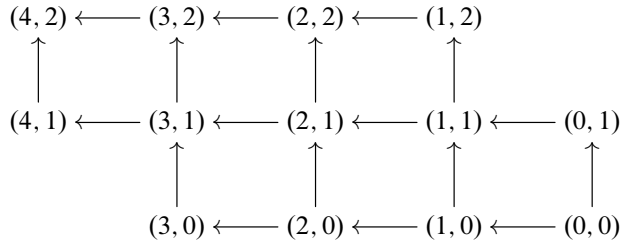
Let W be the full subcategory of O that does not contain the objects $(1, 1)$, $(2, 1)$, $(3, 1)$, $(0, 0)$, $(1, 0)$ and $(2, 0)$. Denote by $\omega : \underline{2} \to W$ the fully faithful functor which maps 0, 1 and 2 respectively on $(2, 2)$, $(3, 2)$ and $(4, 2)$ and by $w : W \to O$ the inclusion functor. The octahedron diagram functor is defined to be the functor

$$w_\sharp \omega_* : \mathbb{D}(\underline{2}) \to \mathbb{D}(O).$$

Denote by $\mathsf{sm} : \underline{1} \to \underline{2}$ the fully faithful functor that maps 0 and 1 respectively to 0 and 1 and by $\mathsf{fm} : \underline{1} \to \underline{2}$ the fully faithful functor that maps 0 and 1 respectively to 1 and 2. Denote also by $\mathsf{cm} : \underline{1} \to \underline{2}$ the functor which maps 0 and 1 respectively to 0 and 2. Consider the fully faithful functor $\mathsf{fsq} : \square \to O$ which maps the square (3.1) to the square

$$(4, 2) \longleftarrow (3, 2)$$
$$\uparrow \qquad\qquad \uparrow$$
$$(4, 1) \longleftarrow (3, 1)$$
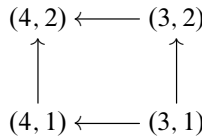
Similarly we denote by $\mathsf{ssq} : \square \to O$ (resp. $\mathsf{csq} : \square \to O$) the fully faithful functor which maps the square (3.1) to the square

$$(3, 2) \longleftarrow (2, 2) \qquad (\text{resp. } (4, 2) \longleftarrow (2, 2)$$
$$\uparrow \qquad\qquad \uparrow \qquad\qquad\qquad\quad \uparrow \qquad\qquad \uparrow$$
$$(3, 0) \longleftarrow (2, 0) \qquad\qquad\qquad (4, 1) \longleftarrow (2, 1))$$
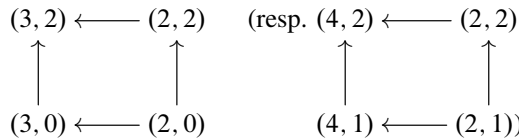
We have the following lemma.

Lemma 3.2. *We have canonical isomorphisms*

$$\mathsf{fsq}^* w_\sharp \omega_* \simeq (i_\ulcorner)_\sharp (-, 1)_* \mathsf{fm}^*, \quad \mathsf{ssq}^* w_\sharp \omega_* \simeq (i_\ulcorner)_\sharp (-, 1)_* \mathsf{sm}^*$$

and

$$\mathsf{csq}^* w_\sharp \omega_* \simeq (i_\ulcorner)_\sharp (-, 1)_* \mathsf{cm}^*.$$

Proof. Let $i : \ulcorner \to W$ be the fully faithful functor that maps $(0, 1)$, $(1, 0)$ and $(1, 1)$ respectively to $(3, 2)$, $(4, 1)$ and $(4, 2)$. Since $\omega \circ \mathsf{fm} = i \circ (-, 1)$, we get a natural transformation $i^* \omega_* \to (0, 1)_* \mathsf{fm}^*$. Using properties **Der1–3**, it is easy to see that this natural transformation is invertible. Similarly, since $w \circ i = \mathsf{fsq} \circ i_\ulcorner$, there is a natural transformation

$(i_{\mathsf{r}})_\sharp i^* \to \mathsf{fsq}^* w_\sharp$. Again, using properties **Der1–3**, we see that it is invertible. This provides invertible natural transformations

$$(i_{\mathsf{r}})_\sharp i^* \omega_* \longrightarrow (i_{\mathsf{r}})_\sharp (-, 1)_* \mathsf{fm}^*$$

$$\downarrow$$

$$\mathsf{fsq}^* w_\sharp \omega_*$$

The other invertible natural transformations are constructed similarly. ∎

In particular, it follows from Lemma 3.2 that $(3, 1)^* w_\sharp \omega_*$ is isomorphic to $\mathsf{Cof} \circ \mathsf{fm}^*$, $(2, 0)^* w_\sharp \omega_*$ is isomorphic to $\mathsf{Cof} \circ \mathsf{sm}^*$ and $(2, 1)^* w_\sharp \omega_*$ is isomorphic to $\mathsf{Cof} \circ \mathsf{cm}^*$. Since the inverse image of $w_\sharp \omega_*$ along the fully faithful functor $\square \to O$ that maps the square (3.1) to the square

$$\begin{array}{ccc} (3, 1) & \longleftarrow & (2, 1) \\ \uparrow & & \uparrow \\ (3, 0) & \longleftarrow & (2, 0) \end{array}$$

is a cocartesian square, by Lemma 3.2 and [5, Définition 2.1.34], we get a natural exact triangle

$$\mathsf{Cof}(\mathsf{fm}^*(-)) \to \mathsf{Cof}(\mathsf{cm}^*(-)) \to \mathsf{Cof}(\mathsf{sm}^*(-)) \xrightarrow{+1} . \tag{3.6}$$

Let us recall [5, Lemma 1.4.8]. Note that the functors $j^* : \mathbf{DA}(X, \mathfrak{I}) \to \mathbf{DA}(U, \mathfrak{I})$ and $j_* : \mathbf{DA}(U, \mathfrak{I}) \to \mathbf{DA}(X, \mathfrak{I})$ used below are induced by the functoriality of the categories of presheaves on diagrams of schemes (see [6, Section 4.5] for details).

Lemma 3.3. *Let $\mathfrak{I}$ be a small category and $j : U \hookrightarrow X$ be an open immersion. Assume that we have an exact triangle*

$$M \to j_* j^* M \to C(M) \xrightarrow{+1}$$

for every given object $M \in \mathbf{DA}(X, \mathfrak{I})$. Then for every morphism $\alpha : M \to N$ in $\mathbf{DA}(X, \mathfrak{I})$ there exists a unique morphism $C(M) \to C(N)$ such that the square

$$\begin{array}{ccc} C(M) & \longrightarrow & M[1] \\ {\scriptstyle C(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha[1]} \\ C(N) & \longrightarrow & N[1] \end{array}$$

is commutative. Moreover, the whole diagram

$$\begin{array}{ccccccc} M & \longrightarrow & j_* j^* M & \longrightarrow & C(M) & \longrightarrow & M[1] \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle j_* j^* \alpha} & & \downarrow{\scriptstyle C(\alpha)} & & \downarrow{\scriptstyle \alpha[1]} \\ N & \longrightarrow & j_* j^* N & \longrightarrow & C(N) & \longrightarrow & N[1] \end{array}$$

is commutative.

Note that in loc.cit. the lemma is stated only in the case $\mathfrak{I} = \mathbf{e}$. However, its proof works in the more general situation considered here.

We will need the following technical lemma.

Lemma 3.4. *Let $\mathfrak{I}$ be a small category and $f : Y \to X$ be a morphism of separated k-schemes of finite type. There exists a functor $\Delta_f^* : \mathbf{DA}(X, \mathfrak{I}) \to \mathbf{DA}(X, \underline{\mathbf{1}} \times \mathfrak{I})$ such that, for every $M \in \mathbf{DA}(X, \mathfrak{I})$, the $\underline{\mathbf{1}}$-skeleton of $\Delta_f^*(M)$ is $M \to f_* f^* M$.*

Proof. Consider the diagram of k-varieties $(\mathscr{F}, \underline{\mathbf{1}} \times \mathfrak{I}) : \underline{\mathbf{1}} \times \mathfrak{I} \to (\mathrm{Sch}/k)$ that maps $(0, i)$ to Y and $(1, i)$ to X, and the canonical morphisms of diagrams of k-varieties

$$(\mathscr{F}, \underline{\mathbf{1}} \times \mathfrak{I}) \xrightarrow{\beta} (X, \underline{\mathbf{1}} \times \mathfrak{I})$$
$$\downarrow{\alpha}$$
$$(X, \mathfrak{I})$$

The functor $\Delta_f^* := \beta_* \alpha^*$ satisfies the desired property. ∎

Remark 3.5. Assume $\mathfrak{I} = \mathbf{e}$. Given M in $\mathbf{DA}(X)$, we have an exact triangle

$$M \to j_* j^* M \to \mathrm{Cof}(\Delta_j^*(M)) \xrightarrow{+1}$$

functorial in M. It follows from [5, Lemme 1.4.8] that the functor $i_! i^!(-)[1]$ is isomorphic to $\mathrm{Cof} \circ \Delta_j^*(-)$

Similarly we will need the following lemma. Its proof is completely similar to the one of Lemma 3.4 and will be omitted.

Lemma 3.6. *Let $\mathfrak{I}$ be a small category and $f : Y \to X$ be a smooth morphism of separated k-schemes of finite type. There exists a functor $\Delta_f^! : \mathbf{DA}(X, \mathfrak{I}) \to \mathbf{DA}(X, \mathfrak{I} \times \underline{\mathbf{1}})$ such that for every $A \in \mathbf{DA}(X, \mathfrak{I})$ the $\underline{\mathbf{1}}$-skeleton of $\Delta_f^!(A)$ is $f_\sharp f^* A \to A$.*

Remark 3.7. Assume $\mathfrak{I} = \mathbf{e}$. Given M in $\mathbf{DA}(X)$, as in Remark 3.5, we have an exact triangle

$$j_! j^! M \to M \to \mathrm{Cof}(\Delta_j^!(M)) \xrightarrow{+1}$$

functorial in M. It follows from (the dual statement of) [5, Lemme 1.4.8] that the functor $i_* i^*(-)$ is isomorphic to $\mathrm{Cof} \circ \Delta_j^!(-)$.

3.2. Motivic unipotent vanishing cycles functor

Let $f : X \to \mathbf{A}_k^1$ be a morphism of k-varieties. We consider the following diagram of k-varieties:

$$
\begin{array}{ccccc}
X_\eta & \longrightarrow & X & \longleftarrow & X_\sigma \\
\downarrow{f_\eta} & \square & \downarrow{f} & \square & \downarrow{f_\sigma} \\
\mathbf{G}_{\mathrm{m},k} & \xrightarrow{j} & \mathbf{A}_k^1 & \xleftarrow{i} & \mathrm{Spec}(k)
\end{array}
$$

where i denotes the zero section of $\mathbf{A}_k^1$ and j the open immersion of the complement. We denote also by i the closed immersion of the special fiber X_σ in X and by j the open immersion of the generic fiber X_η in X. Let Log_f be the logarithmic specialization system constructed in [6, Section 3.6] (see also [8, pp. 103–109]). It is defined by

$$\mathrm{Log}_f := \chi_f((-) \otimes f_\eta^* \mathscr{L}og^\vee) =: i^* j_*((-) \otimes f_\eta^* \mathscr{L}og^\vee)$$

where $\mathscr{L}og^\vee$ is the commutative associative unitary algebra in $\mathbf{DA}(\mathbf{G}_{\mathrm{m},k})$ constructed in [6, Définition 3.6.29] (see also [8, Définition 11.6]). The monodromy triangle

$$\mathbf{Q}(0) \to \mathscr{L}og^\vee \xrightarrow{N} \mathscr{L}og^\vee(-1) \xrightarrow{+1} \tag{3.7}$$

(see [6, Corollaire 3.6.21] or [8, (116)]) in the triangulated category $\mathbf{DA}(\mathbf{G}_{\mathrm{m},k})$ induces an exact triangle

$$\chi_f(-) \to \mathrm{Log}_f(-) \to \mathrm{Log}_f(-)(-1) \xrightarrow{+1} .$$

To construct the motivic unipotent vanishing cycles functor, we shall use the fact that the $\mathbf{1}$-skeleton functor

$$\mathbf{DA}(\mathbf{A}_k^1, \mathbf{1}) \to \mathrm{Hom}(\mathbf{1}^{\mathrm{op}}, \mathbf{DA}(\mathbf{A}_k^1))$$

is full and essentially surjective. This allows us to choose an object $\mathscr{L}$ in $\mathbf{DA}(\mathbf{A}_k^1, \mathbf{1})$ that lifts the morphism $\mathbf{Q}(0) \to j_* \mathscr{L}og^\vee$ obtained as the composition of the adjunction morphism $\mathbf{Q}(0) \to j_* \mathbf{Q}(0)$ and the image under j_* of the unit $\mathbf{Q}(0) \to \mathscr{L}og^\vee$ of the commutative associative unitary algebra $\mathscr{L}og^\vee$. Moreover, using the monodromy triangle (3.7), we can fix an isomorphism between $\mathscr{L}og^\vee(-1)$ and the cofiber of $j^* \mathscr{L}$ such that the diagram

$$
\begin{array}{ccccc}
\mathbf{Q}(0) & \longrightarrow & \mathscr{L}og^\vee & \longrightarrow & \mathscr{L}og^\vee(-1) \xrightarrow{+1} \\
\| & & \| & & \downarrow \\
\mathbf{Q}(0) & \longrightarrow & \mathscr{L}og^\vee & \longrightarrow & \mathrm{Cof}(j^*\mathscr{L}) \xrightarrow{+1}
\end{array}
$$

is commutative.

Consider the object $\mathscr{Q} := \Delta_j^*(\mathscr{L})$ in $\mathbf{DA}(\mathbf{A}_k^1, \square)$ obtained by applying the functor Δ_j^* of Lemma 3.4. Its $\square$-skeleton is the commutative square

$$
\begin{array}{ccc}
\mathbf{Q}(0) & \longrightarrow & j_* \mathscr{L}og^\vee \\
\downarrow & & \| \\
j_* \mathbf{Q}(0) & \longrightarrow & j_* \mathscr{L}og^\vee
\end{array}
$$

Let $\llcorner$ be the full subcategory of $\square$ that does not contain $(0, 1)$. Denote by $i_\llcorner : \llcorner \to \square$ the inclusion and by $p_{\square, \llcorner} : \square \to \llcorner$ the unique functor which is the identity on $\llcorner$ and maps $(0, 1)$ to $(0, 0)$. Consider the functor

$$\Theta_f(-) := (p_{\square, \llcorner})^* (i_\llcorner)^* \Delta_j^* ((p_{\mathbf{1}})^*(-) \otimes f^* \mathscr{L}) : \mathbf{DA}(X) \to \mathbf{DA}(X, \square).$$

By construction, $\Theta_f(-)$ is a coherent lifting of the commutative square

$$
\begin{array}{ccc}
\mathrm{Id}(-) & \longrightarrow & j_*(j^*(-) \otimes f_\eta^* \mathscr{L}og^\vee) \\
\downarrow & & \| \\
j_* j^*(-) & \longrightarrow & j_*(j^*(-) \otimes f_\eta^* \mathscr{L}og^\vee)
\end{array}
$$

By pulling back along the closed immersion $i : X_\sigma \hookrightarrow X$ we get the functor

$$
i^* \Theta_f(-) : \mathbf{DA}(X) \to \mathbf{DA}(X_\sigma, \square)
$$

which is a coherent lifting of the commutative square

$$
\begin{array}{ccc}
i^*(-) & \longrightarrow & \mathsf{Log}_f(j^*(-)) \\
\downarrow & & \| \\
\chi_f(j^*(-)) & \longrightarrow & \mathsf{Log}_f(j^*(-))
\end{array}
$$

Let $(-, 1) : \underline{\mathbf{1}} \to \square$ be the fully faithful functor that maps 0 and 1 respectively to $(0, 1)$ and $(1, 1)$. In particular, the $\underline{\mathbf{1}}$-skeleton of $(-, 1)^* i^* \Theta_f(-)$ is the morphism

$$
i^*(-) \to \mathsf{Log}_f(j^*(-)).
$$

Definition 3.8. The motivic unipotent vanishing cycles functor $\Phi_f : \mathbf{DA}(X) \to \mathbf{DA}(X_\sigma)$ is defined as the composition of $(-, 1)^* i^* \Theta_f(-)$ and the cofiber functor:

$$
\Phi_f := \mathsf{Cof} \circ (-, 1)^* i^* \Theta_f(-).
$$

By construction, we get a natural transformation $can : \mathsf{Log}_f(-) \circ j^* \to \Phi_f(-)$ and an exact triangle

$$
i^* \to \mathsf{Log}_f(-) \circ j^* \xrightarrow{can} \Phi_f \xrightarrow{+1} . \tag{3.8}
$$

We also get a natural transformation

$$
var : \Phi_f(-) \to \mathsf{Log}_f(j^*(-))(-1)
$$

such that $var \circ can = N$. Indeed, let $(-, 0) : \underline{\mathbf{1}} \to \square$ be the fully faithful functor that maps 0 and 1 respectively to $(0, 0)$ and $(1, 0)$.

The chosen isomorphism between $\mathscr{L}og^\vee(-1)$ and the cofiber of $j^* \mathscr{L}$ induces an isomorphism between $\mathsf{Log}_f(j^*(-))(-1)$ and the cofiber of $(-, 0)^* i^* \Theta_f(-)$ such that the diagram

$$
\begin{array}{ccccccc}
\chi_f(j^*(-)) & \longrightarrow & \mathsf{Log}_f(j^*(-)) & \xrightarrow{N} & \mathsf{Log}_f(j^*(-))(-1) & \xrightarrow{+1} & \\
\| & & \| & & \uparrow & & \\
\chi_f(j^*(-)) & \longrightarrow & \mathsf{Log}_f(j^*(-)) & \longrightarrow & \mathsf{Cof}((-, 0)^* i^* \Theta_f(-)) & \xrightarrow{+1} &
\end{array}
$$

is commutative. On the other hand, the canonical morphism $(-, 1)^* \Theta_f(-) \to (-, 0)^* \Theta_f(-)$ in $\mathbf{DA}(X, \underline{\mathbf{1}})$ induces a commutative diagram

$$
\begin{array}{ccccc}
\chi_f(j^*(-)) & \longrightarrow & \mathrm{Log}_f(j^*(-)) & \longrightarrow & \mathrm{Cof}((-,0)^* i^* \Theta_f(-)) \xrightarrow{+1} \\
\uparrow & & \| & & \uparrow \\
i^*(-) & \longrightarrow & \mathrm{Log}_f(j^*(-)) & \xrightarrow{can} & \Phi_f(-) \xrightarrow{\ +1\ }
\end{array}
$$

By applying the coherent triangle functor $u_\sharp v_*$ to the object $i^* \Theta_f(-)$ of the category $\mathbf{DA}(X_\sigma, \square) = \mathbf{DA}(X_\sigma, \underline{\mathbf{1}} \times \underline{\mathbf{1}})$, we get a functor

$$
\mathbf{DA}(X) \to \mathbf{DA}(X_\sigma, \underline{\mathbf{1}} \times \boxdot)
$$

which is a coherent lifting of the commutative diagram

$$
\begin{array}{ccc}
i^*(-) \longrightarrow \mathrm{Log}_f(j^*(-)) \longrightarrow 0 \\
\downarrow \quad \searrow \chi_f(j^*(-)) \xrightarrow{can} \mathrm{Log}_f(j^*(-)) \longrightarrow 0 \\
0 \quad \Big\downarrow \quad \Phi_f(-) \xrightarrow[var]{N} i^*(-)[1] \\
\searrow 0 \longrightarrow \mathrm{Log}_f(j^*(-))(-1) \longrightarrow \chi_f(j^*(-))[1]
\end{array}
$$

The category $\underline{\mathbf{1}} \times \boxdot$ is given by

$$
\begin{array}{ccc}
(1,2,1) \longleftarrow (1,1,1) \longleftarrow (1,0,1) \\
\uparrow \quad \nwarrow \quad \uparrow \quad \nwarrow \quad \uparrow \quad \nwarrow \\
(0,2,1) \longleftarrow (0,1,1) \longleftarrow (0,1,0) \\
(1,2,0) \longleftarrow (1,0,1) \longleftarrow (1,0,0) \quad \uparrow \\
\nwarrow \quad \uparrow \quad \nwarrow \quad \uparrow \quad \nwarrow \\
(0,2,0) \longleftarrow (0,1,0) \longleftarrow (0,0,0)
\end{array}
$$

and we consider the functor $\mathsf{sq} : \square \to \underline{\mathbf{1}} \times \boxdot$ which maps (3.1) to the square

$$
\begin{array}{ccc}
(1,0,1) & \longleftarrow & (1,0,0) \\
\uparrow & & \uparrow \\
(0,1,0) & \longleftarrow & (0,0,0)
\end{array}
$$

inside $\underline{\mathbf{1}} \times \boxdot$. In the next subsection, we will be mainly focusing on the functor

$$
\mathsf{sq}^* u_\sharp v_* i^* \Theta_f : \mathbf{DA}(X) \to \mathbf{DA}(X, \square)
$$

which is a coherent lifting of the commutative square

$$
\begin{array}{ccc}
\Phi_f(-) & \longrightarrow & i^*(-)[1] \\
{\scriptstyle var}\downarrow & & \downarrow \\
\mathrm{Log}_f(j^*(-))(-1) & \longrightarrow & \chi_f(j^*(-))[1]
\end{array}
$$

Remark 3.9. The square $\mathsf{sq}^* u_\sharp v_* i^* \Theta_f$ is cartesian. This can be deduced from the basic properties of cartesian squares (for example from [38, Proposition 4.6]).

3.3. Maximal extension functor

Let us now construct Beilinson's maximal extension functor Ξ_f (see [16]) and the related exact triangles in the triangulated categories of étale motives. This will be essential when proving Theorem 3.15 and for gluing perverse motives. By applying the coherent triangle functor $u_\sharp v_*$ to the object $\Theta_f(-)$ in $\mathbf{DA}(X, \Box) = \mathbf{DA}(X, \underline{\mathbf{1}} \times \underline{\mathbf{1}})$, we get a functor

$$u_\sharp v_* \Theta_f : \mathbf{DA}(X) \to \mathbf{DA}(X, \underline{\mathbf{1}} \times \Box)$$

which is a coherent lifting of the commutative diagram
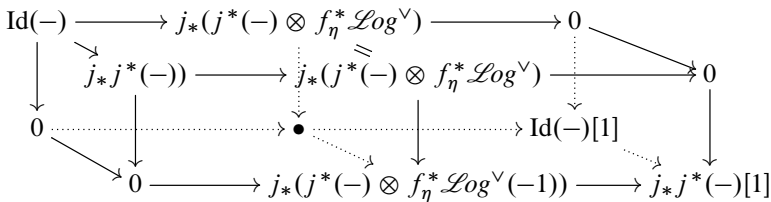
(Here $\bullet$ is some motive which we do not need to specify.) The category $\underline{\mathbf{1}} \times \Box$ is given by
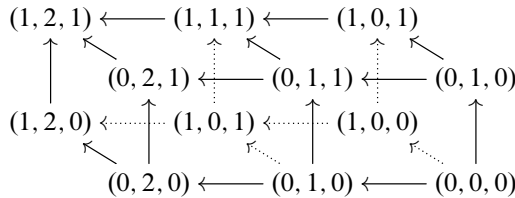
Let $\llcorner$ be the full subcategory of $\Box$ that does not contain $(1, 1)$. Then $\underline{\mathbf{1}} \times \llcorner$ is the category

(3.9)
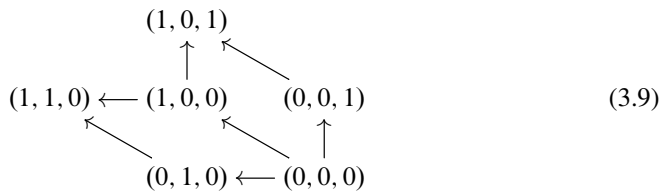
We denote by $\alpha : \underline{\mathbf{1}} \times \llcorner \to \Box \times \underline{\mathbf{1}}$ the functor which maps (3.9) to
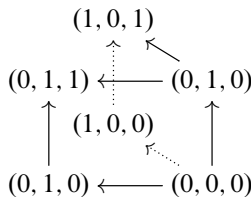
Then $\alpha^* u_\sharp v_* \Theta_f(-) : \mathbf{DA}(X) \to \mathbf{DA}(X, \underline{\mathbf{1}} \times \lrcorner)$ is a coherent lifting of the commutative diagram

$$
\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
j_*(j^*(-) \otimes f_\eta^* \mathscr{L}og^\vee) \longrightarrow & 0 & \longrightarrow \mathrm{Id}(-)[1] \\
& \searrow N & \downarrow \\
& j_*(j^*(-) \otimes f_\eta^* \mathscr{L}og^\vee(-1)) \to & j_* j^*(-)[1]
\end{array}
$$

and $(0 \times \mathrm{Id}_\lrcorner)^* \alpha^* u_\sharp v_* \Theta_f(-) = (i_\lrcorner)^* \mathsf{sq}^* u_\sharp v_* \Theta_f(-)$. Let $\beta : \underline{\mathbf{1}} \times \lrcorner \to \underline{\mathbf{1}} \times \underline{\mathbf{1}} \times \lrcorner$ be the fully faithful functor defined by

$$
\begin{aligned}
(0,0,0) &\mapsto (0,0,0,0) & (1,0,0) &\mapsto (0,1,0,0), \\
(0,0,1) &\mapsto (0,0,0,1) & (1,0,1) &\mapsto (0,1,0,1), \\
(0,1,0) &\mapsto (0,0,1,0) & (1,1,0) &\mapsto (1,1,1,0).
\end{aligned}
$$

The $\underline{\mathbf{1}} \times \lrcorner$-skeleton of the functor

$$
\Sigma_f(-) := \beta^* \circ \Delta_j^! \circ \alpha^* u_\sharp v_* \Theta_f(-) : \mathbf{DA}(X) \to \mathbf{DA}(X, \underline{\mathbf{1}} \times \lrcorner)
$$

is now the commutative diagram

$$
\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
j_!(j^*(-) \otimes f_\eta^* \mathscr{L}og^\vee) \longrightarrow & 0 & \longrightarrow \mathrm{Id}(-)[1] \\
& \searrow & \downarrow \\
& j_*(j^*(-) \otimes f_\eta^* \mathscr{L}og^\vee(-1)) \to & j_* j^*(-)[1]
\end{array}
$$

where the nonzero diagonal morphism is obtained via the canonical morphism $j_! \to j_*$ and the monodromy operator. Note that

$$
(0 \times \mathrm{Id}_\lrcorner)^* \Sigma_f = (0 \times \mathrm{Id}_\lrcorner)^* \alpha^* u_\sharp v_* \Theta_f(-) = (i_\lrcorner)^* \mathsf{sq}^* u_\sharp v_* \Theta_f(-).
$$

In particular, we have canonical isomorphisms $(0,0,0)^* \Sigma_f(-) = j_* j^*(-)[1]$ and $(0,0,1)^* \Sigma_f(-) = \mathrm{Id}(-)[1]$.

Definition 3.10. Let $\Xi_f : \mathbf{DA}(X) \to \mathbf{DA}(X)$ be the functor defined by

$$
\Xi_f(-) := (1,0)^* \mathsf{Cof}(\Sigma_f(-)).
$$

We also define $\Omega_f : \mathbf{DA}(X) \to \mathbf{DA}(X)$ to be the functor

$$
\Omega_f(-) := (1,1)^*(i_\lrcorner)_* \mathsf{Cof}(\Sigma_f(-)).
$$

By construction, we have an exact triangle

$$
\Omega_f(-) \to \Xi_f(-) \oplus (0,1)^* \mathsf{Cof}(\Sigma_f(-)) \to (0,0)^* \mathsf{Cof}(\Sigma_f(-)) \xrightarrow{+1} . \qquad (3.10)
$$

Since the canonical morphisms

$$\mathrm{Id}(-)[1] = (0,0,1)^*\Sigma_f(-) \to (0,1)^*\mathrm{Cof}(\Sigma_f(-))$$

and

$$j_*j^*(-)[1] = (0,0,0)^*\Sigma_f(-) \to (0,0)^*\mathrm{Cof}(\Sigma_f(-))$$

are isomorphisms, the exact triangle (3.10) can be rewritten as

$$\Omega_f(-) \to \Xi_f(-) \oplus \mathrm{Id}(-)[1] \to j_*j^*(-)[1] \xrightarrow{+1} .$$

On the other hand, we have an exact triangle

$$(1,1,0)^*\Sigma_f(-) \to (0,1,0)^*\Sigma_f(-) \to \Xi_f(-) \xrightarrow{+1}$$

that is an exact triangle

$$j_!(j^*(-) \otimes f_\eta^*\mathscr{L}og^\vee) \to j_*(j^*(-) \otimes f_\eta^*\mathscr{L}og^\vee(-1)) \to \Xi_f(-) \xrightarrow{+1} . \qquad (3.11)$$

Proposition 3.11. *There are exact triangles*

$$i_*\mathrm{Log}_f(j^*(-)) \to \Xi_f \to j_*j^*(-)[1] \xrightarrow{+1} \qquad (3.12)$$

and

$$j_!j^*(-)[1] \to \Xi_f \to i_*\mathrm{Log}_f(j^*(-))(-1) \xrightarrow{+1} . \qquad (3.13)$$

Proof. Let us first construct (3.12) using the functorial version of the octahedron axiom (see Section 3.1). Recall that by definition

$$\Xi_f(-) := (1,0)^*\mathrm{Cof}(\Sigma_f(-)) = \mathrm{Cof}((-,1,0)^*\Sigma_f(-)).$$

Let us set

$$\Sigma'_f(-) := \Delta_j^! \circ \alpha^* u_\sharp v_* \Theta_f(-) : \mathbf{DA}(X) \to \mathbf{DA}(X, \underline{\mathbf{1}} \times \underline{\mathbf{1}} \times \lrcorner)$$

so that $\Sigma_f(-) = \beta^*\Sigma'_f(-)$. Now let $\gamma : \underline{\mathbf{2}} \to \underline{\mathbf{1}} \times \underline{\mathbf{1}} \times \lrcorner$ be the fully faithful functor that maps 0, 1 and 2 respectively to $(0,0,1,0)$, $(0,1,1,0)$ and $(1,1,1,0)$. Recall that $\mathsf{cm} : \underline{\mathbf{1}} \to \underline{\mathbf{2}}$ is the fully faithful functor that maps 0 and 1 respectively to 0 and 2. Then $\beta \circ (-,1,0) = \gamma \circ \mathsf{cm}$. In particular,

$$(-,1,0)^*\Sigma_f(-) = \mathsf{cm}^*\gamma^*\Sigma'_f(-).$$

Using the exact triangle (3.6) given by the functorial octahedron axiom, we get an exact triangle

$$\mathrm{Cof}(\mathsf{fm}^*\gamma^*\Sigma'_f(-)) \to \Xi_f(-) \to \mathrm{Cof}(\mathsf{sm}^*\gamma^*\Sigma'_f(-)) \xrightarrow{+1} .$$

However, by construction, we have an exact triangle

$$j_!(j^*(-) \otimes f_\eta^*\mathscr{L}og^\vee) \to j_*(j^*(-) \otimes f_\eta^*\mathscr{L}og^\vee) \to \mathrm{Cof}(\mathsf{fm}^*\gamma^*\Sigma'_f(-)) \xrightarrow{+1} .$$

Using Remark 3.7, we see that $\mathsf{Cof}(\mathsf{fm}^* \gamma^* \Sigma'_f(-))$ is isomorphic to

$$i_* \mathsf{Log}_f(j^*(-)) := i_* i^* j_*(j^*(-) \otimes f_\eta^* \mathscr{L}og^\vee).$$

On the other hand, $\mathsf{sm}^* \gamma^* \Sigma'_f(-) = (0, 1, -)^* u_\sharp v_* \Theta_f(-)$, so that we get an isomorphism

$$\mathsf{Cof}(\mathsf{fm}^* \gamma^* \Sigma'_f(-)) = (0, 0, 0)^* u_\sharp v_* \Theta_f(-) = j_* j^*(-)[1].$$

This constructs the exact triangle (3.12). Consider now the localization triangle

$$j_! j^* \Xi_f(-) \to \Xi_f(-) \to i_* i^* \Xi_f(-) \xrightarrow{+1} .$$

To obtain (3.13) it is enough to check that $j^* \Xi_f(-)$ is isomorphic to $j^*(-)[1]$ and that $i^* \Xi_f(-)$ is isomorphic to $\mathsf{Log}_f(j^*(-))$. The first isomorphism is obtained by applying j^* to (3.12) and the second isomorphism is obtained by applying i^* to (3.11). ∎

Proposition 3.12. *There are exact triangles*

$$i_* \mathsf{Log}_f(j^*(-)) \to \Omega_f \to \mathsf{Id}(-)[1] \xrightarrow{+1} \quad\quad (3.14)$$

and

$$j_! j^*(-)[1] \to \Omega_f \to i_* \Phi_f(-) \xrightarrow{+1} . \quad\quad (3.15)$$

Proof. Using (3.12), the exact triangle (3.14) is obtained by applying Lemma 3.1 to the cartesian square $(i_\lrcorner)_* \mathsf{Cof}(\Sigma_f(-))$.

Since $j^* i_* = 0$, (3.14) provides an isomorphism between $j^* \Omega_f(-)$ and $j^*(-)[1]$. Now, consider the localization triangle

$$j_! j^* \Omega_f \to \Omega_f \to i_* i^* \Omega_f(-) \xrightarrow{+1} .$$

To construct (3.15), it is enough to obtain an isomorphism between $i^* \Omega_f(-)$ and $\Phi_f(-)$. By definition

$$i^* \Omega_f(-) = (1, 1)^* (i_\lrcorner)_* \mathsf{Cof}(i^* \Sigma_f(-)).$$

However, since $i^* j_! = 0$, the canonical morphism

$$(0 \times \mathsf{Id}_\lrcorner)^* i^* \Sigma_f(-) \to \mathsf{Cof}(i^* \Sigma_f(-))$$

is an isomorphism. Given that

$$(0 \times \mathsf{Id}_\lrcorner)^* \Sigma_f = (0 \times \mathsf{Id}_\lrcorner)^* \alpha^* u_\sharp v_* \Theta_f(-) = (i_\lrcorner)^* \mathsf{sq}^* u_\sharp v_* \Theta_f(-),$$

we get isomorphisms

$$(1, 1)^* (i_\lrcorner)_* (0 \times \mathsf{Id}_\lrcorner)^* i^* \Sigma_f(-) \xrightarrow{\;\simeq\;} (1, 1)^* (i_\lrcorner)_* \mathsf{Cof}(i^* \Sigma_f(-)) = i^* \Omega_f(-)$$

$$\|$$

$$(1, 1)^* (i_\lrcorner)_* (i_\lrcorner)^* \mathsf{sq}^* u_\sharp v_* \Theta_f(-)$$

By Remark 3.9, the canonical morphism

$$\Phi_f(-) = (1, 1)^* \mathsf{sq}^* u_\sharp v_* \Theta_f(-) \to (1, 1)^* (i_\lrcorner)_* (i_\lrcorner)^* \mathsf{sq}^* u_\sharp v_* \Theta_f(-)$$

is an isomorphism. This concludes the proof. ∎

3.4. Betti realization

Let X be a complex algebraic variety. Let $\mathbf{AnDA}(X)$ be the triangulated category of analytic motives. This category is obtained as the special case of the category $\mathbf{SH}_{\mathfrak{M}}^{\mathrm{an}}(X)$ considered in [7] when the stable model category $\mathfrak{M}$ is taken to be the category of unbounded complexes of $\mathbf{Q}$-vector spaces with its projective model structure. Recall that the canonical triangulated functor

$$i_X^* : \mathbf{D}(X) \to \mathbf{AnDA}(X) \tag{3.16}$$

is an equivalence of categories (see [7, Théorème 1.8]). Here $\mathbf{D}(X)$ denotes the (unbounded) derived category of sheaves of $\mathbf{Q}$-vector spaces on the associated analytic space X^{an}. The functor

$$\mathsf{An}_X : (\mathrm{Sm}/X) \to (\mathrm{AnSm}/X^{\mathrm{an}})$$

which maps a smooth X-scheme Y to the associated X^{an}-analytic space Y^{an} induces a triangulated functor

$$\mathsf{An}_X^* : \mathbf{DA}(X) \to \mathbf{AnDA}(X). \tag{3.17}$$

The Betti realization Bti_X^* of [7] is obtained as the composition of (3.17) and a quasi-inverse to (3.16).

Let $\mathscr{L}og_{\mathscr{P}}^\vee$ be the image under the Betti realization of the motive $\mathscr{L}og^\vee$ and consider the specialization system it defines,

$$\mathsf{Log}_f^{\mathscr{P}}(-) := i_{\mathscr{P}}^* j_*^{\mathscr{P}}((-) \otimes (f_\eta)_{\mathscr{P}}^* \mathscr{L}og_{\mathscr{P}}^\vee) : \mathbf{D}(X_\eta) \to \mathbf{D}(X_\sigma).$$

Recall that in Section 3.2 we fixed an object $\mathscr{L}$ in $\mathbf{DA}(\mathbf{A}_k^1, \underline{\mathbf{1}})$ that lifts the morphism $\mathbf{Q}(0) \to j_* \mathscr{L}og^\vee$ obtained as the composition of the adjunction morphism $\mathbf{Q}(0) \to j_* \mathbf{Q}(0)$ and the image under j_* of the unit $\mathbf{Q}(0) \to \mathscr{L}og^\vee$ of the commutative associative unitary algebra $\mathscr{L}og^\vee$.

Let $\mathscr{L}_{\mathscr{P}}$ be the image in $\mathbf{D}(X, \underline{\mathbf{1}})$ of $\mathscr{L}$. Using this object, we can perform the same constructions as in Sections 3.2 and 3.3 using the derivator $\mathbf{D}(X, -)$ to obtain functors

$$\Xi_f^{\mathscr{P}}(-), \Omega_f^{\mathscr{P}}(-) : \mathbf{D}(X) \to \mathbf{D}(X) \quad \text{and} \quad \Phi_f^{\mathscr{P}}(-) : \mathbf{D}(X) \to \mathbf{D}(X_\sigma)$$

and four exact triangles: the two triangles

$$i_*^{\mathscr{P}} \mathsf{Log}_f^{\mathscr{P}} j_{\mathscr{P}}^* \to \Xi_f^{\mathscr{P}} \to j_*^{\mathscr{P}} j_{\mathscr{P}}^*[1] \xrightarrow{+1}, \quad j_!^{\mathscr{P}} j_{\mathscr{P}}^*[1] \to \Xi_f^{\mathscr{P}} \to i_*^{\mathscr{P}} \mathsf{Log}_f^{\mathscr{P}}(-1) \xrightarrow{+1}$$

and the two triangles

$$i_*^{\mathscr{P}} \mathsf{Log}_f^{\mathscr{P}} j_{\mathscr{P}}^* \to \Omega_f^{\mathscr{P}} \to \mathrm{Id}[1] \xrightarrow{+1}, \quad j_!^{\mathscr{P}} j_{\mathscr{P}}^*[1] \to \Omega_f^{\mathscr{P}} \to i_*^{\mathscr{P}} \Phi_f^{\mathscr{P}} \xrightarrow{+1}.$$

Moreover, we have canonical natural transformations

$$\mathrm{Bti}^* \circ \mathsf{Log}_f \to \mathsf{Log}_f^{\mathscr{P}} \circ \mathrm{Bti}^*, \quad \mathrm{Bti}^* \circ \Phi_f \to \Phi_f^{\mathscr{P}} \circ \mathrm{Bti}^*$$

and

$$\mathrm{Bti}^* \circ \Xi_f \to \Xi_f^{\mathscr{P}} \circ \mathrm{Bti}^*, \quad \mathrm{Bti}^* \circ \Omega_f \to \Omega_f^{\mathscr{P}} \circ \mathrm{Bti}^*$$

which are isomorphisms when applied to constructible motives (see [7, Théorème 3.9]) and are also compatible with the various exact triangles.

As proved in [7, Théorème 4.9], the Betti realization is compatible with the (total) nearby cycles functors for constructible motives. In this subsection, we will need the compatibility of the Betti realization with the unipotent nearby cycles functors.

Lemma 3.13. *The functor* $\mathsf{Log}_f^{\mathscr{P}}(-)$ *is isomorphic to the unipotent nearby cycles functor* $\psi_f^{\mathrm{un}}(-)$.

Let $e : \mathbf{C} \to \mathbf{C}^\times$, $z \mapsto \exp(z)$, be the universal cover of the punctured complex plane $\mathbf{C}^\times$. The group of deck transformations is identified with $\mathbf{Z}$ by mapping $k \in \mathbf{Z}$ to the deck transformation $z \mapsto z + 2i\pi k$.

Let $\mathscr{E}_n$ be the unipotent rational local system on $\mathbf{C}^\times$ of rank $n + 1$ with (nilpotent) monodromy given by one Jordan block of maximal size. It underlies a variation of $\mathbf{Q}$-mixed Hodge structures described e.g. by Saito [66, Section 1.1].

Let us recall the description of this local system and relate it to Ayoub's logarithmic motive $\mathscr{Log}_n^\vee$. The following description was given by Saito [65, 2.3. Remark]. Let $\mathscr{E}_n$ be the subsheaf of $e_* \mathbf{Q}_\mathbf{C}$ annihilated by $(T - \mathrm{Id})^{n+1}$ where T is the automorphism of $e_* \mathbf{Q}_\mathbf{C}$ induced by the deck transformation corresponding to $1 \in \mathbf{Z}$. The restriction of T to $\mathscr{E}_n$ is unipotent and we denote by $N = \log T$ the associated nilpotent endomorphism.

The sheaf $\mathscr{E}_n$ is a local system on $\mathbf{C}^\times$ of rank $n + 1$. Let $(\mathscr{E}_n)_1$ be its fiber over 1. We have an inclusion

$$(\mathscr{E}_n)_1 \subseteq (e_* \mathbf{Q}_\mathbf{C})_1 = \prod_{k \in \mathbf{Z}} (\mathbf{Q}_\mathbf{C})_{2i\pi k} = \prod_{k \in \mathbf{Z}} \mathbf{Q}.$$

Note that the automorphism T acts by mapping a sequence $(a_k)_{k \in \mathbf{Z}}$ to $(a_{k+1})_{k \in \mathbf{Z}}$. Let τ_n be the element in $(\mathscr{E}_n)_1$ given by $\tau_n = (k^n/n!)_{k \in \mathbf{Z}}$. The family $(1, \tau_1, \dots, \tau_n)$ is a basis of $(\mathscr{E}_n)_1$ such that $T(\tau_r) = \sum_{k=0}^r \tau_k/(r-k)!$ for every $r \in [\![1, n]\!]$. The matrix with respect to the basis $(1, \tau_1, \dots, \tau_n)$ of the unipotent endomorphism T of $(\mathscr{E}_n)_1$ is thus given by $\sum_{k=0}^n (J_n)^k/k!$ where J_n is the nilpotent Jordan block of size $n + 1$ and therefore N is given by the Jordan block J_n in the basis $(1, \tau_1, \dots, \tau_n)$.

The multiplication $e_* \mathbf{Q}_\mathbf{C} \otimes e_* \mathbf{Q}_\mathbf{C} \to e_* \mathbf{Q}_\mathbf{C}$ induces a morphism of local systems $\mathscr{E}_k \otimes \mathscr{E}_\ell \to \mathscr{E}_{k+\ell}$. In particular, for $n \in \mathbb{N}^*$, we have a canonical morphism $\mathscr{E}_1^{\otimes n} \to \mathscr{E}_n$ which defines a morphism

$$\mathrm{Sym}^n \mathscr{E}_1 \to \mathscr{E}_n. \tag{3.18}$$

If $\tau := \tau_1$, then $\tau_n = \tau^n/n!$ and the above description of $\mathscr{E}_n$ implies that (3.18) is an isomorphism.

Let us consider the Kummer natural transformation $e_K : \mathrm{Id}(-)(-1)[-1] \to \mathrm{Id}(-)$ in Betti cohomology (see [6, Définition 3.6.22]). By [69, proof of Lemma 5.1], the local system $\mathscr{E}_1$ fits into an exact triangle

$$\mathbf{Q}(-1)[-1] \xrightarrow{e_K} \mathbf{Q} \to \mathscr{E}_1 \xrightarrow{+1} .$$

By [7, Théorème 3.19] the Betti realization is compatible with the Kummer transformation (for constructible motives). In particular, we have a natural isomorphism $\mathrm{Bti}^* \mathscr{K} \to \mathscr{E}_1$

where $\mathscr{K} \in \mathbf{DA}(\mathbf{G}_{\mathrm{m}})$ is the motivic Kummer extension, that is, the cone of the Kummer natural transformation for étale motives (see [6, Lemme 3.6.28]). Since the Betti realization Bti^{*} is a symmetric monoidal functor, it induces an isomorphism

$$\mathrm{Bti}^{*}\mathscr{L}og_{n}^{\vee} = \mathrm{Bti}^{*}\mathrm{Sym}^{n}\mathscr{K} \xrightarrow{\simeq} \mathrm{Sym}^{n}\mathrm{Bti}^{*}\mathscr{K} \xrightarrow{\simeq} \mathrm{Sym}^{n}\mathscr{E}_{1} \xrightarrow{(3.18)} \mathscr{E}_{n}$$

for every $n \in \mathbb{N}$. Therefore, we get an isomorphism

$$\mathscr{L}og_{\mathscr{P}}^{\vee} := \mathrm{Bti}^{*}\mathscr{L}og^{\vee} \xrightarrow{\simeq} \mathscr{E} \qquad (3.19)$$

where $\mathscr{E}$ is the ind-local system given by $\mathscr{E} = \mathrm{colim}_{n \in \mathbb{N}^{\times}} \mathscr{E}_{n}$.

Let $K \in \mathrm{D}_{c}^{b}(X, \mathbf{Q})$, the unipotent nearby cycles functor ψ_{f}^{un} is given by

$$\psi_{f}^{\mathrm{un}}(K) = i_{\mathscr{P}}^{*}j_{*}^{\mathscr{P}}(K \otimes (f_{\eta})_{\mathscr{P}}^{*}\mathscr{E})$$

(see [65, (2.3.3)], Beilinson [16] or Reich [64]). With this description, Lemma 3.13 is an immediate consequence of (3.19).

Corollary 3.14. *The functors*

$${}^{p}\mathsf{Log}_{f}^{\mathscr{P}}(-) := \mathsf{Log}_{f}^{\mathscr{P}}(-)[-1], \quad {}^{p}\Phi_{f}^{\mathscr{P}}(-) := \Phi_{f}^{\mathscr{P}}(-)[-1], \quad {}^{p}\Xi_{f}^{\mathscr{P}}(-) := \Xi_{f}^{\mathscr{P}}(-)[-1]$$

and

$${}^{p}\Omega_{f}^{\mathscr{P}}(-) := \Omega_{f}^{\mathscr{P}}(-)[-1]$$

are t-exact for the perverse t-structure.

Proof. Since the functor $\psi_{f}^{\mathrm{un}}(-)[-1]$ is t-exact for the perverse t-structure, the corollary is an immediate consequence of Lemma 3.13 and the exact triangles relating the various functors. ∎

3.5. Application to perverse motives

Now, we can apply the universal property of the categories of perverse motives to obtain four exact functors

$${}^{p}\mathsf{Log}_{f}^{\mathscr{M}}(-) : \mathscr{M}(X_{\eta}) \to \mathscr{M}(X_{\sigma}), \quad {}^{p}\Phi_{f}^{\mathscr{M}}(-) : \mathscr{M}(X) \to \mathscr{M}(X_{\sigma})$$

and

$${}^{p}\Xi_{f}^{\mathscr{M}}(-) : \mathscr{M}(X) \to \mathscr{M}(X), \quad {}^{p}\Omega_{f}^{\mathscr{M}}(-) : \mathscr{M}(X) \to \mathscr{M}(X).$$

Moreover, we have four canonical exact sequences obtained from the exact triangles relating the four functors used in the construction: two exact sequences

$$0 \to i_{*}^{\mathscr{M}}{}^{p}\mathsf{Log}_{f}^{\mathscr{M}}(j_{\mathscr{M}}^{*}(-)) \to {}^{p}\Xi_{f}^{\mathscr{M}} \to j_{*}^{\mathscr{M}}j_{\mathscr{M}}^{*}(-) \to 0$$

and

$$0 \to j_{!}^{\mathscr{M}}j_{\mathscr{M}}^{*}(-) \to {}^{p}\Xi_{f}^{\mathscr{M}} \to i_{*}^{\mathscr{M}}{}^{p}\mathsf{Log}_{f}^{\mathscr{M}}(-)(-1) \to 0,$$

as well as two exact sequences

$$0 \to i_*^{\mathscr{M}} \, {}^p \mathsf{Log}_f^{\mathscr{M}} (j_{\mathscr{M}}^*(-)) \to {}^p \Omega_f^{\mathscr{M}}(-) \to \mathrm{Id}(-) \to 0 \qquad (3.20)$$

and

$$0 \to j_!^{\mathscr{M}} \, j_{\mathscr{M}}^*(-) \to {}^p \Omega_f^{\mathscr{M}}(-) \to i_*^{\mathscr{M} \, p} \Phi_f^{\mathscr{M}}(-) \to 0.$$

These four functors and the associated exact sequences are compatible with the various functors and exact triangles constructed in Sections 3.2–3.4.

Now we can prove the following theorem.

Theorem 3.15. *Let* $i : Z \hookrightarrow X$ *be a closed immersion of* k*-varieties. Then the functor*

$$i_*^{\mathscr{M}} : \mathscr{M}(Z) \to \mathscr{M}(X)$$

is fully faithful and its essential image is the kernel, denoted by $\mathscr{M}_Z(X)$, *of the exact functor*

$$j_{\mathscr{M}}^* : \mathscr{M}(X) \to \mathscr{M}(U)$$

where $j : U \hookrightarrow X$ *is the open immersion of the complement of* Z *in* X.

We first consider the case of the immersion of a special fiber.

Lemma 3.16. *Let* X *be a* k*-variety and* $f : X \to \mathbf{A}_k^1$ *be a morphism. Let* $i : X_\sigma \hookrightarrow X$ *be the closed immersion of the special fiber in* X *and* Z *be a closed subscheme of* X_σ. *Then the exact functor*

$$i_*^{\mathscr{M}} : \mathscr{M}_Z(X_\sigma) \to \mathscr{M}_Z(X)$$

is an equivalence of categories.

Proof. We may assume $Z = X_\sigma$. Indeed, let $u : X \setminus Z \hookrightarrow X$ and $v : X_\sigma \setminus Z \hookrightarrow X_\sigma$ be the open immersion. By Proposition 2.3 applied to the cartesian square

$$
\begin{array}{ccc}
X_\sigma \setminus Z & \xrightarrow{\ i'\ } & X \setminus Z \\
{\scriptstyle v} \downarrow & \square & \downarrow {\scriptstyle u} \\
X_\sigma & \xrightarrow{\ i\ } & X
\end{array}
$$

we get an isomorphism $u_{\mathscr{M}}^* i_*^{\mathscr{M}} \simeq i_*^{\prime \mathscr{M}} v_{\mathscr{M}}^*$. Since the functor $i_*^{\prime \mathscr{M}}$ is conservative (it is faithful exact), we see that an object A in $\mathscr{M}(X_\sigma)$ belongs to $\mathrm{Ker}\, v_{\mathscr{M}}^*$ if and only if $i_*^{\mathscr{M}} A$ belongs to $\mathrm{Ker}\, u_{\mathscr{M}}^*$. Hence, it is enough to show that

$$i_*^{\mathscr{M}} : \mathscr{M}(X_\sigma) \to \mathscr{M}_{X_\sigma}(X)$$

is an equivalence.

Let us show that the functor ${}^p \Phi_f^{\mathscr{M}}$ is a quasi-inverse. Let X_η be the generic fiber and $j : X_\eta \hookrightarrow X$ be the open immersion. The exact triangle (3.8) provides an isomorphism of endomorphisms of $\mathbf{DA}(X_\sigma)$ between $i^* i_*$ and $\Phi_f[-1] i_*$. By composing with

the isomorphism of functors $i^* i_* \to \mathrm{Id}$, we get an isomorphism of functors between the identity of $\mathbf{DA}(X_\sigma)$ and $^p \Phi_f[-1] i_*$.

Similarly, we get an isomorphism between the identity of $\mathrm{D}(X_\sigma, \mathbf{Q})$ and the functor $\Phi_f^{\mathscr{P}}[-1] i_*^{\mathscr{P}}$. Since these isomorphisms are compatible with the Betti realization, property **P2** ensures that $^p \Phi_f^{\mathscr{M}} i_*^{\mathscr{M}}$ is isomorphic to the identity functor of the category $\mathscr{M}(X_\sigma)$.

An isomorphism between the identity of $\mathscr{M}_{X_\sigma}(X) := \mathrm{Ker}\, j_{\mathscr{M}}^*$ and $i_{\mathscr{M}}^* {}^p \Phi_f$ is provided by the exact sequences

$$0 \to i_*^{\mathscr{M}} {}^p \mathsf{Log}_f^{\mathscr{M}}(j_{\mathscr{M}}^*(-)) \to {}^p \Omega_f^{\mathscr{M}}(-) \to \mathrm{Id}(-) \to 0$$

and

$$0 \to j_!^{\mathscr{M}} j_{\mathscr{M}}^*(-) \to {}^p \Omega_f^{\mathscr{M}}(-) \to i_*^{\mathscr{M}} {}^p \Phi_f^{\mathscr{M}}(-) \to 0$$

(the first terms vanish for objects in the kernel of $j_{\mathscr{M}}^*$). This concludes the proof. ∎

Proof of Theorem 3.15. Using Proposition 2.7, we may assume that X is an affine scheme. Let U be the open complement of Z in X and let $f_1, \ldots, f_r$ be elements in $\mathscr{O}(X)$ such that $U = D(f_1) \cup \cdots \cup D(f_r)$. Let $Z_{r+1} = X$ and set $Z_k = Z_{k+1} \setminus D(f_k)$ for $k \in [\![1, r]\!]$. Let $i_k : Z_k \hookrightarrow Z_{k+1}$ be the closed immersion. We have $Z_1 = Z$ and $i = i_r \circ i_{r-1} \circ \cdots \circ i_1$, so that the functor $i_*^{\mathscr{M}} : \mathscr{M}(Z) \to \mathscr{M}_Z(X)$ is obtained as the composition

$$\mathscr{M}(Z) \xrightarrow{(i_1)_*^{\mathscr{M}}} \mathscr{M}_Z(Z_2) \xrightarrow{(i_2)_*^{\mathscr{M}}} \mathscr{M}_Z(Z_3) \to \cdots \to \mathscr{M}_Z(Z_k) \xrightarrow{(i_k)_*^{\mathscr{M}}} \mathscr{M}_Z(X).$$

By Lemma 3.16, all these functors are equivalences. This concludes the proof. ∎

4. Inverse images

The purpose of this section is to extend the (contravariant) 2-functor $^{\mathrm{Liss}}\mathsf{H}_{\mathscr{M}}^*$ constructed in Section 2.5 to a (contravariant) 2-functor

$$\mathsf{H}_{\mathscr{M}}^* : (\mathrm{Sch}/k) \to \mathfrak{TR}, \quad X \mapsto \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)), \quad f \mapsto f_{\mathscr{M}}^*.$$

To do this, we first use the vanishing cycles functor to show that the (covariant) 2-functor $^{\mathrm{Imm}}\mathsf{H}_*^{\mathscr{M}}$ admits a global left adjoint $^{\mathrm{Imm}}\mathsf{H}_{\mathscr{M}}^*$ (we recall that a global left adjoint is unique up to unique isomorphism and refer to [5, Définition 1.1.18] for the definition). Then, we show that the 2-functors $^{\mathrm{Liss}}\mathsf{H}_{\mathscr{M}}^*$ and $^{\mathrm{Imm}}\mathsf{H}_{\mathscr{M}}^*$ can be glued into a 2-functor $\mathsf{H}_{\mathscr{M}}^*$.

4.1. Inverse image by a closed immersion

By [5, Proposition 1.1.17], to show that $^{\mathrm{Imm}}\mathsf{H}_*^{\mathscr{M}}$ admits a global left adjoint $^{\mathrm{Imm}}\mathsf{H}_{\mathscr{M}}^*$ it suffices to show that for every closed immersion $i : Z \hookrightarrow X$ the functor $i_*^{\mathscr{M}}$ admits a left adjoint; this in turn is proved in Proposition 4.2.

Theorem 4.1. *Let $i : Z \hookrightarrow X$ be a closed immersion. Then the functor*

$$i_*^{\mathcal{M}} : \mathrm{D}^{\mathrm{b}}(\mathcal{M}(Z)) \to \mathrm{D}^{\mathrm{b}}(\mathcal{M}(X))$$

is fully faithful and its essential image is the kernel, denoted by $\mathrm{D}^{\mathrm{b}}_Z(\mathcal{M}(X))$, of the exact functor

$$j_{\mathcal{M}}^* : \mathrm{D}^{\mathrm{b}}(\mathcal{M}(X)) \to \mathrm{D}^{\mathrm{b}}(\mathcal{M}(U))$$

where $j : U \hookrightarrow X$ is the open immersion of the complement of Z in X.

Proof. We know that the essential image of $i_*^{\mathcal{M}} : \mathrm{D}^{\mathrm{b}}(\mathcal{M}(Z)) \to \mathrm{D}^{\mathrm{b}}(\mathcal{M}(X))$ is contained in $\mathrm{D}^{\mathrm{b}}_Z(\mathcal{M}(X))$ by Theorem 3.15. We now want to prove that the functor $i_*^{\mathcal{M}} : \mathrm{D}^{\mathrm{b}}(\mathcal{M}(Z)) \to \mathrm{D}^{\mathrm{b}}_Z(\mathcal{M}(X))$ is an equivalence of categories. Note that the obvious t-structure on $\mathrm{D}^{\mathrm{b}}(\mathcal{M}(X))$ induces a t-structure on $\mathrm{D}^{\mathrm{b}}_Z(\mathcal{M}(X))$, whose heart is the thick abelian subcategory $\mathcal{M}_Z(X)$ of $\mathcal{M}(X)$. By Theorem 3.15, the functor $i_*^{\mathcal{M}} : \mathcal{M}(Z) \to \mathcal{M}(X)$ induces an equivalence of categories $\mathcal{M}(Z) \to \mathcal{M}_Z(X)$. So, by [17, Lemma 1.4], the functor $i_*^{\mathcal{M}} : \mathrm{D}^{\mathrm{b}}(\mathcal{M}(Z)) \to \mathrm{D}^{\mathrm{b}}_Z(\mathcal{M}(X))$ is an equivalence of categories if and only if, for any A, B in $\mathcal{M}_Z(X)$ and $i \geq 1$, and any class $u \in \mathrm{Ext}^i_{\mathcal{M}(X)}(A, B)$, there exists a monomorphism $B \hookrightarrow B'$ in $\mathcal{M}_Z(X)$ such that the image of u in $\mathrm{Ext}^i_{\mathcal{M}(X)}(A, B')$ is 0.

Suppose that $j : V \hookrightarrow X$ is an affine open immersion, that A is an object of $\mathcal{M}(X)$ and that B is an object of $\mathcal{M}(V)$. Let $i \geq 1$. Then we have

$$\mathrm{Ext}^i_{\mathcal{M}(X)}(A, j_*^{\mathcal{M}} B) = \mathrm{Ext}^i_{\mathcal{M}(V)}(j_{\mathcal{M}}^* A, B)$$

by Proposition 2.5, and, if $u \in \mathrm{Ext}^i_{\mathcal{M}(V)}(j_{\mathcal{M}}^* A, B)$ and $B \hookrightarrow B'$ is a monomorphism of $\mathcal{M}(V)$ such that the image of u in $\mathrm{Ext}^i_{\mathcal{M}(V)}(j_{\mathcal{M}}^* A, B')$ is 0, then the image in $\mathrm{Ext}^i_{\mathcal{M}(X)}(A, j_*^{\mathcal{M}} B')$ of the element of $\mathrm{Ext}^i_{\mathcal{M}(X)}(A, j_*^{\mathcal{M}} B)$ corresponding to u is also 0. Applying this to an open cover $j_1 : U_1 \hookrightarrow X, \dots, j_n : U_n \hookrightarrow X$ of X by affine subsets and using the fact that the canonical map $B \to \bigoplus_{r=1}^n (j_r)_*^{\mathcal{M}} (j_r)_{\mathcal{M}}^* B$ given by Proposition 2.5 is a monomorphism for every object B of $\mathcal{M}(X)$, we reduce to the case where X is affine.

If X is affine, then, as in the proof of Theorem 3.15, we write $i = i_r \circ \cdots \circ i_1$, where $Z_1 = Z$, $Z_{r+1} = X$, and, for every $k \in \{1, \dots, r\}$, $i_k : Z_k \to Z_{k+1}$ is the immersion of the complement of an open set of the form $D(f)$, with $f \in \mathcal{O}(Z_{k+1})$. It suffices to show that each $(i_k)_*^{\mathcal{M}} : \mathrm{D}^{\mathrm{b}}(\mathcal{M}(Z_k)) \to \mathrm{D}^{\mathrm{b}}_{Z_k}(\mathcal{M}(Z_{k+1}))$ is an equivalence of categories. So we may assume that there exists $f \in \mathcal{O}(X)$ such that i is the immersion of the complement of $D(f)$. In that case, we showed in the proof of Lemma 3.16 that the trivial derived functor of the exact functor ${}^p\Phi_f^{\mathcal{M}} : \mathcal{M}(X) \to \mathcal{M}(Z)$ induces a quasi-inverse of $i_*^{\mathcal{M}} : \mathrm{D}^{\mathrm{b}}(\mathcal{M}(Z)) \to \mathrm{D}^{\mathrm{b}}_Z(\mathcal{M}(X))$. ∎

Proposition 4.2. *Let $i : Z \hookrightarrow X$ be a closed immersion. Then the functor*

$$i_*^{\mathcal{M}} : \mathrm{D}^{\mathrm{b}}(\mathcal{M}(Z)) \to \mathrm{D}^{\mathrm{b}}(\mathcal{M}(X))$$

admits a left adjoint.

Proof. By Theorem 4.1, it suffices to show that the inclusion functor

$$D_Z^b(\mathscr{M}(X)) \to D^b(\mathscr{M}(X)) \tag{4.1}$$

admits a left adjoint $C^\bullet$. Let $j : U \hookrightarrow X$ be the open immersion of the complement of Z in X. Let us first assume that U is affine. In that case, given A in $C^b(\mathscr{M}(X))$, we define $C^\bullet(A)$ as the mapping cone of the canonical morphism $j_!^{\mathscr{M}} j_{\mathscr{M}}^* A \to A$ given by Proposition 2.5. This construction induces a triangulated functor $C^\bullet : D^b(\mathscr{M}(X)) \to D^b(\mathscr{M}(X))$ and there is a canonical exact triangle $j_!^{\mathscr{M}} j_{\mathscr{M}}^* A \to A \to C^\bullet(A) \to j_!^{\mathscr{M}} j_{\mathscr{M}}^* A[1]$, which shows that $C^\bullet$ takes its values in the full subcategory $D_Z^b(\mathscr{M}(X))$. Let $B \in D_Z^b(\mathscr{M}(X))$. Using the long exact sequence associated with this triangle and Proposition 2.5 which ensures that

$$\mathrm{Hom}_{D^b(\mathscr{M}(X))}(j_!^{\mathscr{M}} j_{\mathscr{M}}^* A, B[n]) = \mathrm{Hom}_{D^b(\mathscr{M}(U))}(j_{\mathscr{M}}^* A, j_{\mathscr{M}}^* B[n]) = 0,$$

we get a functorial isomorphism

$$\mathrm{Hom}_{D^b(\mathscr{M}(X))}(C^\bullet(A), B) \xrightarrow{\sim} \mathrm{Hom}_{D^b(\mathscr{M}(X))}(A, B)$$

as desired.

In the general case, the adjoint $C^\bullet$ can be constructed by considering a finite set I and an affine open covering $\mathscr{U} = (j_i : U_i \to U)_{i \in I}$. For every $J \subseteq I$, let j_J be the inclusion $\bigcap_{i \in J} U_i \hookrightarrow X$. We define an exact functor $C^\bullet : \mathscr{M}(X) \to C^b(\mathscr{M}(X))$ in the following way. Let A be an object of $\mathscr{M}(X)$. We set

$$C^i(A) = \begin{cases} 0 & \text{if } i \geq 1, \\ A & \text{if } i = 0, \\ \bigoplus_{I \subset \{1,\ldots,r\}, |J|=-i} (j_J)_!^{\mathscr{M}} (j_J)_{\mathscr{M}}^* A & \text{if } i \leq -1. \end{cases}$$

The differential of $C^\bullet(A)$ is an alternating sum of maps given by Proposition 2.5. Then the left adjoint of $D_Z^b(\mathscr{M}(X)) \to D^b(\mathscr{M}(X))$ is the functor sending $A^\bullet$ to the total complex of $C^\bullet(A^\bullet)$. ∎

Let Z be a closed immersion such that the open immersion $j : U \hookrightarrow X$ of the complement of Z in X is affine. It follows from the proof of Proposition 4.2 that we have a canonical exact triangle

$$j_!^{\mathscr{M}} j_{\mathscr{M}}^* \to \mathrm{Id} \to i_*^{\mathscr{M}} i_{\mathscr{M}}^* \xrightarrow{+1} .$$

Moreover, the diagram

$$\begin{array}{ccccc}
j_!^{\mathscr{P}} j_{\mathscr{P}}^* \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & i_*^{\mathscr{P}} i_{\mathscr{P}}^* \mathrm{rat}_X^{\mathscr{M}} \xrightarrow{+1} \\
\downarrow{\theta_j^{\mathscr{M}}} & & \parallel & & \downarrow{\theta_i^{\mathscr{M}}} \\
j_!^{\mathscr{P}} \mathrm{rat}_U^{\mathscr{M}} j_{\mathscr{M}}^* & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & i_*^{\mathscr{P}} \mathrm{rat}_Z^{\mathscr{M}} i_{\mathscr{M}}^* \xrightarrow{+1} \\
\downarrow{\rho_j^{\mathscr{M}}} & & \parallel & & \downarrow{(\gamma_i^{\mathscr{M}})^{-1}} \\
\mathrm{rat}_X^{\mathscr{M}} j_!^{\mathscr{M}} j_{\mathscr{M}}^* & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} i_*^{\mathscr{M}} i_{\mathscr{M}}^* \xrightarrow{+1}
\end{array} \tag{4.2}$$

is commutative (the morphisms in the second row are those obtained by adjunction from $\theta_i^{\mathcal{M}}$ and the inverse of $\theta_j^{\mathcal{M}}$).

Lemma 4.3. *Let* $i : Z \to X$ *be a closed immersion. Then the natural transformation*

$$\theta_i^{\mathcal{M}} : i_{\mathscr{P}}^* \circ \mathrm{rat}_X^{\mathcal{M}} \to \mathrm{rat}_Z^{\mathcal{M}} \circ i_{\mathcal{M}}^*$$

is invertible.

Proof. The statement is local on X, so we may assume that X is affine. Then, as in the proof of Theorem 4.1, we can write $i = i_r \circ \cdots \circ i_1$, where each i_s is a closed immersion with affine complement. Using the compatibility of the 2-morphisms $\theta_i^{\mathcal{M}}$ with the composition of morphisms in $^{\mathrm{Imm}}(\mathrm{Sch}/k)$ we may assume that the open immersion $j : U \hookrightarrow X$ of the complement of Z in X is affine. Then, our assertion follows from (4.2) and the conservativity of the functor $i_*^{\mathscr{P}}$. ∎

4.2. Gluing of the pullback 2-functors

Let us fix a global left adjoint $^{\mathrm{Imm}}\mathrm{H}_{\mathcal{M}}^*$ of $^{\mathrm{Imm}}\mathrm{H}_*^{\mathcal{M}}$. To be able to glue the 2-functors $^{\mathrm{Imm}}\mathrm{H}_{\mathcal{M}}^*$ and $^{\mathrm{Liss}}\mathrm{H}_{\mathcal{M}}^*$ using [5, Théorème 1.3.1], it suffices to construct, for every commutative square

$$
\begin{array}{ccc}
X' & \xrightarrow{\;i'\;} & Y' \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\;i\;} & Y
\end{array}
\tag{4.3}
$$

such that i, i' are closed immersions and f, f' smooth morphisms, a 2-isomorphism

$$i'^*_{\mathcal{M}} \circ f_{\mathcal{M}}^* \xrightarrow{\;\simeq\;} f'^*_{\mathcal{M}} \circ i_{\mathcal{M}}^* \tag{4.4}$$

and prove that these 2-isomorphisms define an exchange structure, that is, they are compatible with the horizontal and vertical composition of commutative squares (see [5, Définition 1.2.1]).

Dg enhancements. For the general theory of dg categories we refer to Drinfeld [31], Keller [54, 55] or Toën [71]. Let $\mathcal{A}$ be an abelian category. We denote by $\mathcal{C}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{A})$ the dg category of bounded complexes of objects of $\mathcal{A}$ and by $\mathcal{D}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{A})$ the dg quotient of $\mathcal{C}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{A})$ by the subcategory of acyclic bounded complexes (for a simple construction of the dg quotient see [31, Section 3.1]). The bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ of $\mathcal{A}$ is the homotopy category of the dg category $\mathcal{D}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{A})$. We let $\mathrm{rep}(\mathcal{D}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{B}))$ be the category of dg quasi-functors from $\mathcal{D}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{A})$ to $\mathcal{D}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{B})$ (this category is denoted by $\mathcal{T}(\mathcal{D}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}_{\mathrm{dg}}^{\mathrm{b}}(\mathcal{B}))$ in Vologodsky's paper [76]). Let us recall the following particular case of [76, Theorem 1].

Proposition 4.4. *Let* $\mathcal{A}, \mathcal{B}$ *be abelian categories and* $F, G \in r(\mathcal{A}, \mathcal{B})$ *be dg quasi-functors. Assume that the induced triangulated functors* $F, G : \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{B})$ *are* t*-exact for the classical* t*-structures. Then* F, G *are respectively canonically isomorphic to the functors induced by the exact functors* $H^0 F : \mathcal{A} \to \mathcal{B}, H^0 G : \mathcal{A} \to \mathcal{B}$ *and the canonical*

map

$$\mathrm{Hom}_{\mathrm{rep}(\mathcal{D}^b_{dg}(\mathcal{A}), \mathcal{D}^b_{dg}(\mathcal{B}))}(F, G) \to \mathrm{Hom}_{\mathrm{Fct}(\mathcal{A}, \mathcal{B})}(H^0 F, H^0 G)$$

is an isomorphism.

A triangulated functor $\mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{B})$ is said to be *dg enhanced* if it is induced by some dg quasi-functor in $\mathrm{rep}(\mathcal{D}^b_{dg}(\mathcal{A}), \mathcal{D}^b_{dg}(\mathcal{B}))$. Note that a composition of dg enhanced functors is also dg enhanced.

Remark 4.5. Let $i : Z \hookrightarrow X$ be a closed immersion and $f : X \to Y$ be a smooth morphism of quasi-projective k-varieties. By construction the triangulated functors $i_*^{\mathcal{M}}$ and $f_{\mathcal{M}}^*$ are dg enhanced. This is also the case of the triangulated functor

$$i_{\mathcal{M}}^* : \mathrm{D}^b(\mathcal{M}(X)) \to \mathrm{D}^b(\mathcal{M}(Z)).$$

Indeed, let $j : U \hookrightarrow X$ be the open immersion of the complement of Z in X and fix a finite open covering of U by affine open subsets. Let $\mathcal{D}^b_{dg,Z}(\mathcal{M}(X))$ be the dg full subcategory of $\mathcal{D}^b_{dg}(\mathcal{M}(X))$ formed by the complexes that belong to $\mathrm{D}^b_Z(\mathcal{M}(X))$. We then have dg functors

$$\mathcal{D}^b_{dg}(\mathcal{M}(Z)) \xrightarrow{i_*^{\mathcal{M}}} \mathcal{D}^b_{dg,Z}(\mathcal{M}(X)) \xleftarrow{C^\bullet} \mathcal{D}^b_{dg}(\mathcal{M}(X)) \tag{4.5}$$

where $C^\bullet$ is the dg functor constructed (using the given open covering of U by affine open subsets) in the proof of Proposition 4.2. Since the dg functor on the left is a quasi-equivalence, the diagram (4.5) defines a quasi-functor from $\mathcal{D}^b_{dg}(\mathcal{M}(X))$ to $\mathcal{D}^b_{dg}(\mathcal{M}(Z))$ that induces the triangulated functor $i_{\mathcal{M}}^*$.

Gluing of the 2-functors. Let us now start with the construction of the 2-isomorphisms (4.4).

Step 1. When the square (4.3) is cartesian, the 2-isomorphism (4.4) is obtained by considering the exchange structure $^{\mathcal{M}}\mathrm{Ex}_*^*$ on the pair $(^{\mathrm{Imm}}\mathrm{H}_*^{\mathcal{M}}, {}^{\mathrm{Liss}}\mathrm{H}_*^{*\mathcal{M}})$ obtained in Proposition 2.3 (in this exchange structure, all squares are cartesian). By applying [5, Proposition 1.2.5], we get an exchange structure $^{\mathcal{M}}\mathrm{Ex}^{**}$ on the pair $(^{\mathrm{Imm}}\mathrm{H}_{\mathcal{M}}^*, {}^{\mathrm{Liss}}\mathrm{H}_{\mathcal{M}}^*)$ for the class of cartesian squares (4.3). The uniqueness in loc.cit. implies that this exchange structure lifts the trivial exchange structure on $(^{\mathrm{Imm}}\mathrm{H}_{\mathcal{P}}^*, {}^{\mathrm{Liss}}\mathrm{H}_{\mathcal{P}}^*)$ given by the connection 2-isomorphisms of the 2-functor $\mathrm{H}_{\mathcal{P}}^*$. In particular, the conservativity of the functors $\mathrm{rat}_X^{\mathcal{M}} : \mathrm{D}^b(\mathcal{M}(X)) \to \mathrm{D}^b(\mathcal{P}(X))$ implies that $^{\mathcal{M}}\mathrm{Ex}^{**}$ is an iso-exchange.

Step 2. Let us consider a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & Y \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} \\ & & S \end{array} \tag{4.6}$$

in which i is a closed immersion and f, g are smooth morphisms. As preparation for the construction of the 2-isomorphism (4.4), we first construct a 2-isomorphism

$$i_{\mathcal{M}}^* \circ f_{\mathcal{M}}^* \to g_{\mathcal{M}}^*. \tag{4.7}$$

To do this, observe that if d is the relative dimension of g, then the triangulated functors $i_{\mathcal{M}}^* \circ f_{\mathcal{M}}^*[d]$ and $g_{\mathcal{M}}^*[d]$ are t-exact for the classical t-structures. This is a vanishing statement that can be checked after application of the functor $\mathrm{rat}_X^{\mathcal{M}}$ and, for perverse sheaves, it follows from [18, Section 4.2.4] since $g_{\mathcal{P}}^*$ and $i_{\mathcal{P}}^* \circ f_{\mathcal{P}}^*$ are isomorphic. Moreover both functors are dg enhanced by Remark 4.5.

By Proposition 4.4, to construct (4.7), it is enough to construct a 2-isomorphism

$$i_{\mathcal{M}}^* \circ f_{\mathcal{M}}^*[d] \to g_{\mathcal{M}}^*[d] \tag{4.8}$$

where both functors are exact functors from $\mathcal{M}(S)$ to $\mathcal{M}(X)$. Therefore, it suffices to prove the following proposition.

Proposition 4.6. *Consider the commutative diagram* (4.6). *Let A be an object in $\mathcal{M}(S)$ and let K be its underlying perverse sheaf. Then the canonical morphism of perverse sheaves $i_{\mathcal{P}}^* \circ f_{\mathcal{P}}^*[d](K) \to g_{\mathcal{P}}^*[d](K)$ lies in the image of the injective morphism*

$$\mathrm{Hom}_{\mathcal{M}(X)}(i_{\mathcal{M}}^* \circ f_{\mathcal{M}}^*[d](A), g_{\mathcal{M}}^*[d](A)) \to \mathrm{Hom}_{\mathcal{P}(X)}(i_{\mathcal{P}}^* \circ f_{\mathcal{P}}^*[d](K), g_{\mathcal{P}}^*[d](K)). \tag{4.9}$$

Remark 4.7. Note that the map (4.9) is obtained via the functor $\mathrm{rat}_X^{\mathcal{M}}$ using the invertible natural transformations $f_{\mathcal{P}}^* \circ \mathrm{rat}_S^{\mathcal{M}} \to \mathrm{rat}_Y^{\mathcal{M}} \circ f_{\mathcal{M}}^*$, $g_{\mathcal{P}}^* \circ \mathrm{rat}_S^{\mathcal{M}} \to \mathrm{rat}_X^{\mathcal{M}} \circ g_{\mathcal{M}}^*$ and $i_{\mathcal{P}}^* \circ \mathrm{rat}_Y^{\mathcal{M}} \to \mathrm{rat}_X^{\mathcal{M}} \circ i_{\mathcal{M}}^*$ which have been previously constructed.

Proof. Step (a). Consider a commutative diagram
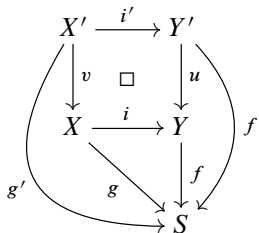
where i is a closed immersion, f, g are smooth morphisms and u is an étale morphism. By Step 1, we have a natural transformation $i_{\mathcal{M}}'^* \circ u_{\mathcal{M}}^* \to v_{\mathcal{M}}^* \circ i_{\mathcal{M}}^*$ that lifts the corresponding natural transformation in the derived category of perverse sheaves. Assume the proposition is true for the diagram (4.6). Then, the morphism $i_{\mathcal{P}}^* \circ f_{\mathcal{P}}^*[d](K) \to g_{\mathcal{P}}^*[d](K)$ lifts to a morphism $i_{\mathcal{M}}^* \circ f_{\mathcal{M}}^*[d](A) \to g_{\mathcal{M}}^*[d](A)$. By applying $v_{\mathcal{M}}^*$ to this lift we obtain a morphism $i_{\mathcal{M}}'^* \circ f_{\mathcal{M}}'^*[d](A) \to g_{\mathcal{M}}'^*[d](A)$ that lifts the morphism $i_{\mathcal{P}}'^* \circ f_{\mathcal{P}}'^*[d](K) \to g_{\mathcal{M}}'^*[d](K)$. This shows, in particular, that if the proposition is true for the diagram (4.6) then it is also true for the diagram
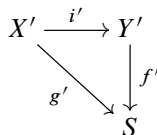
Step (b). Let $\mathcal{Y} = (Y_\alpha)_{\alpha \in I}$ be a finite Zariski open covering of Y and consider for every $\alpha \in I$ the commutative diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\;i_\alpha\;} & Y_\alpha \\ {\scriptstyle v_\alpha} \downarrow & \square & \downarrow {\scriptstyle u_\alpha} \\ X & \xrightarrow{\;i\;} & Y \\ & {\scriptstyle g} \searrow & \downarrow {\scriptstyle f} \\ & & S \end{array}$$
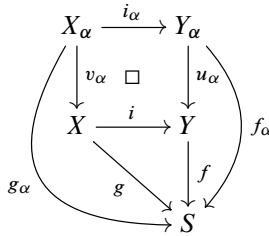
where u_α is the open immersion of Y_α in Y. Note that the canonical morphism of perverse sheaves $i_{\mathscr{P}}^* \circ f_{\mathscr{P}}^*[d](K) \to g_{\mathscr{P}}^*[d](K)$ is obtained by gluing the morphism $i_{\alpha,\mathscr{P}}^* \circ f_{\alpha,\mathscr{P}}^*[d](K) \to g_{\alpha,\mathscr{P}}^*[d](K)$ along the Zariski open covering $\mathscr{X} = (X_\alpha)_{\alpha \in I}$ of X. Hence it follows from Step (a) and Proposition 2.7 that the proposition is true for the diagram (4.6) if and only if it is true for the diagrams

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\;i_\alpha\;} & Y_\alpha \\ & {\scriptstyle g_\alpha} \searrow & \downarrow {\scriptstyle f_\alpha} \\ & & S \end{array}$$
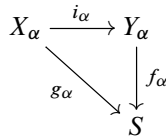
Step (c). By Step (b) the problem is local on Y for the Zariski topology. Since both Y and X are smooth over S, we may assume that there exists a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & Y \\ {\scriptstyle v} \downarrow & \square & \downarrow {\scriptstyle u} \\ \mathbf{A}_S^d & \longrightarrow & \mathbf{A}_S^{d+c} \\ & \searrow & \downarrow {\scriptstyle f} \\ & & S \end{array}$$
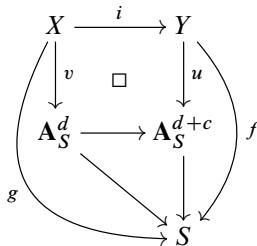
where u is an étale morphism. Using Step (a) and induction, we are reduced to proving the proposition in the case

$$\begin{array}{ccc} \mathbf{A}_S^d & \xrightarrow{\;s\;} & \mathbf{A}_S^{d+1} \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle p} \\ & & S \end{array}$$
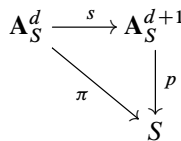
where p and π are the projections and s is the zero section. By considering the factorization

$$\begin{array}{ccc} \mathbf{A}_S^d & \xrightarrow{\;s\;} & \mathbf{A}_S^{d+1} \\ {\scriptstyle \pi} \searrow & & \downarrow {\scriptstyle p} \\ & \mathbf{A}_S^d & \\ & \downarrow {\scriptstyle \pi} & \\ & S & \end{array}$$
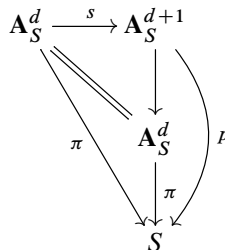
and observing that the functors $\pi^*_{\mathscr{M}}[d]$, $\pi^*_{\mathscr{P}}[d]$ are exact, we may further assume that $d = 0$.

Step (d). It remains to prove the proposition in the case of the diagram

$$\begin{array}{ccc} S & \xrightarrow{s} & \mathbf{A}^1_S \\ & \diagdown\diagdown & \downarrow{\scriptstyle p} \\ & & S \end{array}$$

where s is the zero section and p is the projection. Let $f : \mathbf{A}^1_S = \mathbf{A}^1 \times S \to \mathbf{A}^1$ be the first projection and $a : \mathbf{G}_m \times S \to \mathbf{A}^1 \times S$ be the inclusion. We set $q = p \circ a$. Given a motive $B \in \mathscr{M}(\mathbf{A}^1_S)$, consider the connecting morphism $B \to s_*^{\mathscr{M}} \, {}^p\mathsf{Log}^{\mathscr{M}}_f (a^*_{\mathscr{M}}(B))[1]$ in $\mathrm{D}^b(\mathscr{M}(\mathbf{A}^1_S))$ obtained from the exact sequence (3.20). By adjunction, we get a morphism $s^*_{\mathscr{M}}(B) \to {}^p\mathsf{Log}^{\mathscr{M}}_f (a^*_{\mathscr{M}}B)[1]$ in the category $\mathrm{D}^b(\mathscr{M}(S))$. Taking B to be the perverse motive $B = p^*_{\mathscr{M}}[1]A$, we get after a shift a morphism

$$s^*_{\mathscr{M}} \circ p^*_{\mathscr{M}}(A) \to {}^p\mathsf{Log}^{\mathscr{M}}_f (q^*_{\mathscr{M}}[1](A))$$

in $\mathrm{D}^b(\mathscr{M}(S))$. As both objects are concentrated in degree zero, the above morphism is actually a morphism in the abelian category $\mathscr{M}(S)$. Moreover, it is an isomorphism since it is on the underlying perverse sheaves. Moreover, we know that the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{M}(S)}({}^p\mathsf{Log}^{\mathscr{M}}_f (q^*_{\mathscr{M}}[1](A)), A) & \longrightarrow & \mathrm{Hom}_{\mathscr{P}(S)}({}^p\mathsf{Log}^{\mathscr{P}}_f (q^*_{\mathscr{P}}[1](K)), K) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Hom}_{\mathscr{M}(S)}(s^*_{\mathscr{M}} \circ p^*_{\mathscr{M}}(A), A) & \longrightarrow & \mathrm{Hom}_{\mathscr{P}(S)}(s^*_{\mathscr{P}} \circ p^*_{\mathscr{P}}(K), K) \end{array}$$

is commutative. Hence, to conclude, it suffices to show that the canonical morphism of perverse sheaves

$${}^p\mathsf{Log}^{\mathscr{P}}_f (q^*_{\mathscr{P}}[1](K)) \to K \tag{4.10}$$

lifts to a morphism ${}^p\mathsf{Log}^{\mathscr{M}}_f (q^*_{\mathscr{M}}[1](A)) \to A$ in the abelian category $\mathscr{M}(S)$. By construction of the exact functors ${}^p\mathsf{Log}^{\mathscr{M}}_f$ and $q^*_{\mathscr{M}}[1]$, this is an application of property **P2**, since (4.10) is the Betti realization of a natural transformation

$$\mathsf{Log}_f (q^*(-)) \to \mathrm{Id}$$

in the triangulated category of étale motives on S. ∎

Lemma 4.8. *Consider a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{s} & Z \\ & \diagdown{\scriptstyle h} & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ & & & S & \end{array}$$

in which i, s are closed immersions and f, g, h are smooth morphisms. Then the diagram

$$i^*_{\mathscr{M}} \circ s^*_{\mathscr{M}} \circ f^*_{\mathscr{M}} \longrightarrow i^*_{\mathscr{M}} \circ g^*_{\mathscr{M}} \longrightarrow h^*_{\mathscr{M}}$$
$$\simeq \Big\downarrow \qquad \qquad \qquad$$
$$(s \circ i)^*_{\mathscr{M}} \circ f^*_{\mathscr{M}} \qquad \qquad$$

is commutative.

Proof. The lemma follows from the analogous statement for perverse sheaves. Indeed, let d be the relative dimension of h. It suffices to show that the diagram

$$i^*_{\mathscr{M}} \circ s^*_{\mathscr{M}} \circ f^*_{\mathscr{M}}[d] \longrightarrow i^*_{\mathscr{M}} \circ g^*_{\mathscr{M}}[d] \longrightarrow h^*_{\mathscr{M}}[d]$$
$$\simeq \Big\downarrow \qquad \qquad \qquad$$
$$(s \circ i)^*_{\mathscr{M}} \circ f^*_{\mathscr{M}}[d] \qquad \qquad$$

is commutative. Since all functors in this diagram are dg enhanced and t-exact for the classical t-structures, by Proposition 4.4 it suffices to check the commutativity of the diagram induced on the hearts. This can be done on the underlying perverse sheaves. ∎

Step 3. To construct the 2-isomorphisms (4.4) in the general case, we can decompose the commutative square (4.3) as follows:

$$\begin{array}{ccccc}
& & i' & & \\
& \nearrow & & \searrow & \\
X' & \xrightarrow{i'''} & X \times_Y Y' & \xrightarrow{i''} & Y' \\
& \searrow_{f'} & \Big\downarrow_{f''} \quad \square & & \Big\downarrow_{f} \\
& & X & \xrightarrow{i} & Y
\end{array}$$

where i'', i''' are closed immersions and f'' is a smooth morphism. Then, using the iso-exchange constructed in Step 1, the 2-isomorphism of Step 2 and the connection 2-isomorphisms of the 2-functor $^{\mathrm{Imm}}\mathrm{H}^*_{\mathscr{M}}$ we get (4.4) as the composition

$$i'^*_{\mathscr{M}} \circ f^*_{\mathscr{M}} \xrightarrow{\simeq} i'''^*_{\mathscr{M}} \circ i''^*_{\mathscr{M}} \circ f^*_{\mathscr{M}} \xrightarrow{\simeq} i'''^*_{\mathscr{M}} \circ f''^*_{\mathscr{M}} \circ i^*_{\mathscr{M}} \xrightarrow{\simeq} f'^*_{\mathscr{M}} \circ i^*_{\mathscr{M}}.$$

Lemma 4.9. *Let*

$$\begin{array}{ccccc}
X' & \xrightarrow{i'} & Y' & \xrightarrow{s} & Z \\
\Big\downarrow_{f'} & \square & \Big\downarrow_{f} & \nearrow_{g} & \\
X & \xrightarrow{i} & Y & &
\end{array}$$

be a commutative diagram of morphisms of k-varieties in which g, f are smooth and i, s are closed immersions. Consider the commutative diagram

$$\begin{array}{ccccc}
& & s \circ i' & & \\
& \nearrow & & \searrow & \\
X' & \xrightarrow{s'} & X \times_Y Z & \xrightarrow{i''} & Z \\
& \searrow_{f'} & \Big\downarrow_{g'} \quad \square & & \Big\downarrow_{g} \\
& & X & \xrightarrow{i} & Y
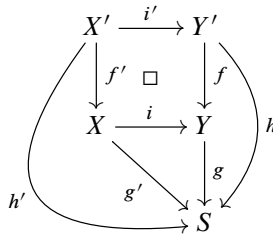\end{array}$$

Then the following diagram is commutative:

$$(s \circ i')^*_{\mathcal{M}} \circ g^*_{\mathcal{M}} \xrightarrow{\simeq} i'^*_{\mathcal{M}} \circ s^*_{\mathcal{M}} \circ g^*_{\mathcal{M}} \longrightarrow i'^*_{\mathcal{M}} \circ f^*_{\mathcal{M}} \longrightarrow f'^*_{\mathcal{M}} \circ i^*_{\mathcal{M}}$$

$$(i'' \circ s')^*_{\mathcal{M}} \circ g^*_{\mathcal{M}} \xrightarrow{\simeq} s'^*_{\mathcal{M}} \circ i''^*_{\mathcal{M}} \circ g^*_{\mathcal{M}} \longrightarrow s'^*_{\mathcal{M}} \circ g'^*_{\mathcal{M}} \circ i^*_{\mathcal{M}} \longrightarrow f'^*_{\mathcal{M}} \circ i^*_{\mathcal{M}}$$

Proof. By adjunction, it is enough to show that the diagram

$$s^*_{\mathcal{M}} \circ g^*_{\mathcal{M}} \circ i^{\mathcal{M}}_* A \longrightarrow f^*_{\mathcal{M}} \circ i^{\mathcal{M}}_* A \longrightarrow i'^{\mathcal{M}}_* \circ f'^*_{\mathcal{M}} A$$

$$s^*_{\mathcal{M}} \circ i''^{\mathcal{M}}_* \circ g'^*_{\mathcal{M}} A \longrightarrow i'^{\mathcal{M}}_* \circ s'^*_{\mathcal{M}} \circ g'^*_{\mathcal{M}} A$$

is commutative for every object A in $D^b(\mathcal{M}(X))$. Since all the entries of the above diagram are dg enhanced and t-exact functors up to a shift by the relative dimension d of f, by Proposition 4.4 it suffices to check the commutativity of the diagram induced on the hearts. Again, this can be done on the underlying perverse sheaves. ∎

Lemma 4.10. *Consider a commutative diagram*

in which i, i' are closed immersions and all other morphisms are smooth. Then the diagram

$$i'^*_{\mathcal{M}} \circ f^*_{\mathcal{M}} \circ g^*_{\mathcal{M}} \longrightarrow f'^*_{\mathcal{M}} \circ i^*_{\mathcal{M}} \circ g^*_{\mathcal{M}} \longrightarrow f'^*_{\mathcal{M}} \circ g'^*_{\mathcal{M}}$$

$$i'^*_{\mathcal{M}} \circ h^*_{\mathcal{M}} \longrightarrow h'^*_{\mathcal{M}}$$

is commutative.

Proof. Let d be the relative dimension of h'. It is enough to check that the diagram

$$i'^*_{\mathcal{M}} \circ f^*_{\mathcal{M}} \circ g^*_{\mathcal{M}}[d] \longrightarrow f'^*_{\mathcal{M}} \circ i^*_{\mathcal{M}} \circ g^*_{\mathcal{M}}[d] \longrightarrow f'^*_{\mathcal{M}} \circ g'^*_{\mathcal{M}}[d]$$

$$i'^*_{\mathcal{M}} \circ h^*_{\mathcal{M}}[d] \longrightarrow h'^*_{\mathcal{M}}[d]$$

is commutative. Since all functors in this diagram are dg enhanced and t-exact for the

classical t-structures, by Proposition 4.4 it suffices to check the commutativity of the diagram induced on the hearts. Once again, this can be done on the underlying perverse sheaves. ∎

Proposition 4.11. *The* 2-*isomorphisms* (4.4) *define an exchange structure, i.e., they are compatible with the horizontal and vertical compositions of commutative squares.*

Proof. • *Horizontal composition of squares.* Consider a commutative diagram

$$
\begin{array}{ccccc}
Z' & \xrightarrow{\ s'\ } & X' & \xrightarrow{\ i'\ } & Y' \\
{\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ s\ } & X & \xrightarrow{\ i\ } & Y
\end{array}
\tag{4.11}
$$

in which i, s, i', s' are closed immersions and f, f', f'' are smooth morphisms. We have to prove that the diagram

$$
\begin{array}{ccc}
(i' \circ s')^*_{\mathscr{M}} \circ f^*_{\mathscr{M}} & \xrightarrow{\hspace{5cm}} & f''^*_{\mathscr{M}} \circ (i \circ s)^*_{\mathscr{M}} \\
{\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\
s'^*_{\mathscr{M}} \circ i'^*_{\mathscr{M}} \circ f^*_{\mathscr{M}} & \xrightarrow{} s'^*_{\mathscr{M}} \circ f'^*_{\mathscr{M}} \circ i^*_{\mathscr{M}} \xrightarrow{} & f''^*_{\mathscr{M}} \circ s^*_{\mathscr{M}} \circ i^*_{\mathscr{M}}
\end{array}
$$

is commutative. Let us decompose (4.11) the following ways:

$$
\begin{array}{ccccccccc}
Z' & \longrightarrow & Z \times_X X' & \longrightarrow & X' & \longrightarrow & X \times_Y Y' & \longrightarrow & Y' \\
& \searrow & \downarrow & \square & \downarrow & & \square & & \swarrow \\
& & Z & \longrightarrow & X & \longrightarrow & Y & &
\end{array}
\tag{4.12}
$$

and

$$
\begin{array}{ccccc}
Z' & \longrightarrow & Z \times_Y Y' & \longrightarrow & Y' \\
& \searrow & \downarrow & \square & \downarrow \\
& & Z & \longrightarrow & Y
\end{array}
$$

Since $Z \times_X (X \times_Y Y') = Z \times_Y Y'$ we can rewrite the portion

$$
\begin{array}{ccccc}
Z \times_X X' & \longrightarrow & X' & \longrightarrow & X \times_Y Y' \\
\downarrow & \square & \downarrow & & \swarrow \\
Z & \longrightarrow & X & &
\end{array}
$$

of the diagram (4.12) as

$$
\begin{array}{ccccc}
Z \times_X X' & \longrightarrow & Z \times_Y Y' & \longrightarrow & X \times_Y Y' \\
& \searrow & \downarrow & \square & \downarrow \\
& & Z & \longrightarrow & X
\end{array}
$$

Therefore the desired compatibility is a consequence of Proposition 2.3, Lemma 4.9 and Lemma 4.8.

• *Vertical composition of squares.* Consider a commutative diagram

$$
\begin{array}{ccc}
X'' & \xrightarrow{\ i''\ } & Y'' \\
\ \downarrow{\scriptstyle g'} & & \ \downarrow{\scriptstyle g} \\
X' & \xrightarrow{\ i'\ } & Y' \\
\ \downarrow{\scriptstyle f'} & & \ \downarrow{\scriptstyle f} \\
X & \xrightarrow{\ i\ } & Y
\end{array}
\tag{4.13}
$$

in which i, i', i'' are closed immersions and f, g, f', g' are smooth morphisms. We have to prove that the diagram

$$
\begin{array}{ccc}
i''^{*}_{\mathscr{M}} \circ (f \circ g)^{*}_{\mathscr{M}} & \xrightarrow{\hspace{5cm}} & (f' \circ g')^{*}_{\mathscr{M}} \circ i^{*}_{\mathscr{M}} \\
\downarrow & & \ \downarrow{\scriptstyle \simeq} \\
i''^{*}_{\mathscr{M}} \circ g^{*}_{\mathscr{M}} \circ f^{*}_{\mathscr{M}} \longrightarrow g'^{*}_{\mathscr{M}} \circ i'^{*}_{\mathscr{M}} \circ f^{*}_{\mathscr{M}} \longrightarrow g'^{*}_{\mathscr{M}} \circ f'^{*}_{\mathscr{M}} \circ i^{*}_{\mathscr{M}}
\end{array}
$$

is commutative. We can refine (4.13) to the following commutative diagrams:

$$
\begin{array}{ccccccc}
X'' & \longrightarrow & X' \times_{Y'} Y'' & \longrightarrow & X \times_Y Y'' & \longrightarrow & Y'' \\
& & \downarrow & \square & \downarrow & \square & \downarrow \\
& X' & \longrightarrow & X \times_Y Y' & \longrightarrow & Y' \\
& & & & \downarrow & \square & \downarrow \\
& & & & X & \longrightarrow & Y
\end{array}
$$

or

$$
\begin{array}{ccccc}
X'' & \longrightarrow & X \times_Y Y'' & \longrightarrow & Y'' \\
& & \downarrow & \square & \downarrow \\
& X & \longrightarrow & Y
\end{array}
$$

The desired compatibility is now a consequence of Proposition 2.3 and Lemmas 4.10 and 4.8. ∎

5. Main theorem

In Section 3.5, we have shown that the unipotent nearby and vanishing cycles functors can be defined at the level of perverse Nori motives.

Our goal is to prove that the four operations (0.1) can be lifted to the derived categories of perverse Nori motives. To obtain these various functors

$$
\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)) \underset{f_*^{\mathscr{M}}}{\overset{f_{\mathscr{M}}^{*}}{\rightleftarrows}} \mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y)) \underset{f_!^{\mathscr{M}}}{\overset{f_{\mathscr{M}}^{!}}{\rightleftarrows}} \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))
\tag{5.1}
$$

(and their compatibility relations) with the least effort, we have chosen to follow Ayoub's approach developed in [5] around the notion of stable homotopical 2-functor, which encompasses in a small package all the ingredients needed to build the rest of the formalism.

5.1. Statement of the theorem

As before, (Sch/k) denotes the category of quasi-projective k-varieties. Recall that a contravariant 2-functor

$$\mathsf{H}^* : (\mathrm{Sch}/k) \to \mathfrak{T}\mathfrak{R}$$

is a called a *stable homotopical* 2-*functor* (see [5, Définition 1.4.1]) when the following six properties are satisfied:

(1) $\mathsf{H}(\varnothing) = 0$ (that is, $\mathsf{H}(\varnothing)$ is the trivial triangulated category).

(2) For every morphism $f : X \to Y$ in (Sch/k), the functor $f^* : \mathsf{H}(Y) \to \mathsf{H}(X)$ admits a right adjoint. Furthermore for every immersion i the counit $i^* i_* \to \mathrm{Id}$ is invertible.

(3) For every smooth morphism $f : X \to Y$ in (Sch/k), the functor $f^* : \mathsf{H}(Y) \to \mathsf{H}(X)$ admits a left adjoint $f_\sharp$. Furthermore, for every cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{\ g'\ } & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{\ g\ } & Y
\end{array}
$$

with f smooth, the exchange 2-morphism $f'_\sharp g'^* \to g^* f_\sharp$ is invertible.

(4) If $j : U \to X$ is an open immersion in (Sch/k) and $i : Z \to X$ is the closed immersion of the complement, then the pair (j^*, i^*) is conservative.

(5) If $p : \mathbf{A}^1_X \to X$ is the canonical projection, then the unit morphism $\mathrm{Id} \to p_* p^*$ is invertible.

(6) If s is the zero section of the canonical projection $p : \mathbf{A}^1_X \to X$, then

$$p_\sharp s_* : \mathsf{H}(X) \to \mathsf{H}(X)$$

is an equivalence of categories.

The main theorem of [5] says that these data can be expanded into a complete formalism of the four operations (see [5, Scholie 1.4.2]).

Theorem 5.1. *The contravariant* 2-*functor* $\mathsf{H}^*_{\mathscr{M}}$ *constructed in Section* 4 *is a stable homotopical* 2-*functor in the sense of* [5, Définition 1.4.1]*, and* $(\mathrm{rat}^{\mathscr{M}}, \theta^{\mathscr{M}})$ *is a morphism of stable homotopical* 2-*functors.*

In particular, we can apply [5, Scholie 1.4.2] to get the functors (5.1). The next subsection is devoted to the proof of Theorem 5.1, and the reader will find some applications of the main theorem in Section 5.4.

5.2. Proof of the main theorem 5.1

We start by showing the existence of the direct image functor. The most important step is the proof of the existence of the direct image by the projection of the affine line $\mathbf{A}_Y^1$ onto its base Y.

Proposition 5.2. *For every morphism $f : X \to Y$ in* (Sch$/k$)*, the functor*

$$f_{\mathscr{M}}^* : \mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y)) \to \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$$

admits a right adjoint $f_*^{\mathscr{M}}$. Moreover,

(1) *if $i : Z \to X$ is a closed immersion, the counit of the adjunction $i_{\mathscr{M}}^* i_*^{\mathscr{M}} \to \mathrm{Id}$ is invertible;*

(2) *the natural transformation*

$$\gamma_f^{\mathscr{M}} : \mathrm{rat}_Y^{\mathscr{M}} f_*^{\mathscr{M}} \to f_*^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}}$$

obtained from $\theta_f^{\mathscr{M}}$ by adjunction is invertible;

(3) *if $p : \mathbf{A}_X^1 \to X$ is the canonical projection, then the unit morphism $\mathrm{Id} \to p_*^{\mathscr{M}} p_{\mathscr{M}}^*$ is invertible.*

Proof. In the proof, all products are fiber products over the base field k and $\mathbf{A}^1$ is the affine line over k.

Step 1. Suppose first that f is a closed immersion. Then $f_{\mathscr{M}}^*$ admits $f_*^{\mathscr{M}}$ as a right adjoint by construction of $f_{\mathscr{M}}^*$, we know item (2) by Lemma 4.3, and item (1) is true by (2) and by conservativity of $\mathrm{rat}_X^{\mathscr{M}}$.

Step 2. Now we consider the case where f is the projection morphism $p : X := \mathbf{A}_Y^1 \to Y$. As before, if we can prove that $p_{\mathscr{M}}^*$ admits a right adjoint satisfying (2), then item (3) will follow automatically.

We consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbf{A}^1 \times Y & \xleftarrow{\ q_2\ } & \mathbf{A}^1 \times \mathbf{A}^1 \times Y & \longleftarrow & U \times Y \\
\ \downarrow{\scriptstyle p} & {\scriptstyle \mathrm{Id}} & \ \downarrow{\scriptstyle q_1} & {\scriptstyle j} & \\
Y & \xleftarrow{\ p\ } & \mathbf{A}^1 \times Y & \xleftarrow{\ \mathrm{Id}\ } & \mathbf{A}^1 \times Y
\end{array}
$$

where $q_1 = \mathrm{Id}_{\mathbf{A}^1} \times p$, q_2 is the product of the projection $\mathbf{A}^1 \to \mathrm{Spec}\, k$ and of $\mathrm{Id}_{\mathbf{A}^1 \times Y}$, i is the product of the diagonal morphism of $\mathbf{A}^1$ and of Id_Y, and j is the complementary open inclusion. We also denote by $s : Y \to \mathbf{A}^1 \times Y$ the zero section of p. By the smooth base change theorem (or a direct calculation), the base change map $p_{\mathscr{P}}^* p_*^{\mathscr{P}} \to q_{1*}^{\mathscr{P}} q_{2\mathscr{P}}^*$ is an isomorphism, so we get a functorial isomorphism $p_*^{\mathscr{P}} \simeq s_{\mathscr{P}}^* p_{\mathscr{P}}^* p_*^{\mathscr{P}} \to s_{\mathscr{P}}^* q_{1*}^{\mathscr{P}} q_{2\mathscr{P}}^*$.

Let K be a perverse sheaf on Y. Then $L := q_{2\mathscr{P}}^* K[1]$ is perverse, and we have $i_{\mathscr{P}}^* L = K[1]$, so we get an exact sequence of perverse sheaves on $\mathbf{A}^1 \times Y$:

$$0 \to i_*^{\mathscr{P}} K \to j_!^{\mathscr{P}} j_{\mathscr{P}}^* L \to L \to 0.$$

Applying the functor $q_{1*}^{\mathscr{P}}$ and using the fact that $q_1 \circ i = \mathrm{Id}_{\mathbf{A}^1 \times Y}$, we get an exact triangle

$$q_{1*}^{\mathscr{P}} q_{2\mathscr{P}}^* K \to K \to q_{1*}^{\mathscr{P}} j_!^{\mathscr{P}} j_{\mathscr{P}}^* L \xrightarrow{+1} .$$

We claim that $q_{1*}^{\mathscr{P}} j_!^{\mathscr{P}} j_{\mathscr{P}}^* L$ is perverse. Indeed, this complex is concentrated in perverse degrees -1 and 0 by [18, Théorème 4.1.1 and Section 4.2.4]. So we just need to prove that $M := {}^{\mathrm{p}}\mathrm{H}^{-1} q_{1*}^{\mathscr{P}} j_!^{\mathscr{P}} j_{\mathscr{P}}^* L$ is equal to 0. By [18, Corollaire 4.2.6.2], the adjunction morphism $q_{1\mathscr{P}}^* M[1] \to j_!^{\mathscr{P}} j_{\mathscr{P}}^* L$ is injective; we denote its quotient by N. Then, as $q_1 \circ i = \mathrm{Id}_{\mathbf{A}^1 \times Y}$ and $i_{\mathscr{P}}^* j_!^{\mathscr{P}} = 0$, we have $i_{\mathscr{P}}^* N = M[2]$. But i is the complement of an open affine embedding, so $i_{\mathscr{P}}^*$ is of perverse cohomological amplitude $[-1, 0]$ by [18, Corollaire 4.1.10], hence $M = 0$.

Finally, we get an exact sequence of perverse sheaves on $\mathbf{A}^1 \times Y$:

$$0 \to {}^{\mathrm{p}}\mathrm{H}^0 q_{1*}^{\mathscr{P}} q_{2\mathscr{P}}^* K \to K \to q_{1*}^{\mathscr{P}} j_!^{\mathscr{P}} j_{\mathscr{P}}^* q_{2\mathscr{P}}^* K[1] \to {}^{\mathrm{p}}\mathrm{H}^1 q_{1*}^{\mathscr{P}} q_{2\mathscr{P}}^* K \to 0.$$

Consider the functors $\mathsf{F}_{\mathscr{P}}, \mathsf{G}_{\mathscr{P}} : \mathscr{P}(\mathbf{A}^1 \times Y) \to \mathrm{D}^{\mathrm{b}}(\mathscr{P}(\mathbf{A}^1 \times Y))$ defined by

$$\mathsf{F}_{\mathscr{P}}(K) := K, \quad \mathsf{G}_{\mathscr{P}}(K) := q_{1*}^{\mathscr{P}} j_!^{\mathscr{P}} j_{\mathscr{P}}^* q_{2\mathscr{P}}^* K[1].$$

We have just proved that these functors are t-exact (of course, this is obvious for the first one) and that there is a functorial exact triangle

$$q_{1*}^{\mathscr{P}} q_{2\mathscr{P}}^* \to \mathsf{F}_{\mathscr{P}} \to \mathsf{G}_{\mathscr{P}} \xrightarrow{+1} .$$

The functors $\mathsf{F}_{\mathscr{P}}$ and $\mathsf{G}_{\mathscr{P}}$ are defined in terms of the four operations. The existence of these operations in the categories $\mathbf{DA}_{\mathrm{ct}}(-)$ and the compatibility of the Betti realization with the four operations (see [7, Théorème 3.19]) imply by the universal property of the categories of perverse motives that there exist

- two exact functors

$$\mathsf{F}_{\mathscr{M}}, \mathsf{G}_{\mathscr{M}} : \mathscr{M}(\mathbf{A}^1 \times Y) \to \mathscr{M}(\mathbf{A}^1 \times Y),$$

- a natural transformation $\mathsf{F}_{\mathscr{M}} \to \mathsf{G}_{\mathscr{M}}$,
- two invertible natural transformations

$$\mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}} \circ \mathsf{F}_{\mathscr{M}} \to \mathsf{F}_{\mathscr{P}} \circ \mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}}, \quad \mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}} \circ \mathsf{G}_{\mathscr{M}} \to \mathsf{G}_{\mathscr{P}} \circ \mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}}$$

such that the diagram

$$
\begin{array}{ccc}
\mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}} \circ \mathsf{F}_{\mathscr{M}} & \longrightarrow & \mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}} \circ \mathsf{G}_{\mathscr{M}} \\
\downarrow & & \downarrow \\
\mathsf{F}_{\mathscr{P}} \circ \mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}} & \longrightarrow & \mathsf{G}_{\mathscr{P}} \circ \mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}}
\end{array}
$$

is commutative.

Given a complex $M^\bullet$ of perverse motives on $X = \mathbf{A}^1 \times Y$, let $\mathsf{H}_{\mathscr{M}}(M^\bullet)$ be the mapping fiber of the morphism $\mathsf{F}_{\mathscr{M}}(M^\bullet) \to \mathsf{G}_{\mathscr{M}}(M^\bullet)$ of complexes of perverse motives on X. We get a triangulated functor

$$\mathsf{H}_{\mathscr{M}} : \mathsf{D}^{\mathrm{b}}(\mathscr{M}(\mathbf{A}^1 \times Y)) \to \mathsf{D}^{\mathrm{b}}(\mathscr{M}(\mathbf{A}^1 \times Y)),$$

and the Betti realization of $\mathsf{H}_{\mathscr{M}}$ is isomorphic to $q_{1*}^{\mathscr{P}} q_2^{*\mathscr{P}}$.

We now define a functor

$$p_\bullet^{\mathscr{M}} := s_{\mathscr{M}}^* \mathsf{H}_{\mathscr{M}}(-) : \mathsf{D}^{\mathrm{b}}(\mathscr{M}(\mathbf{A}^1 \times Y)) \to \mathsf{D}^{\mathrm{b}}(\mathscr{M}(Y)).$$

By construction of $p_\bullet^{\mathscr{M}}$, we have an invertible natural transformation

$$\mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}} p_\bullet^{\mathscr{M}} \to p_*^{\mathscr{P}} \mathrm{rat}_{\mathbf{A}^1 \times Y}^{\mathscr{M}}.$$

Note also the following useful fact. We denote by $f : \mathbf{A}^1 \times Y \to \mathbf{A}^1$ the first projection and by $a : \mathbf{G}_m \times Y \to \mathbf{A}^1 \times Y$ the inclusion. Then applying $s_{\mathscr{M}}^*$ to the connecting map in the exact sequence (3.20) in Section 3.5, we get a natural transformation

$$s_{\mathscr{M}}^* \to {}^p\mathrm{Log}_f^{\mathscr{M}} a_{\mathscr{M}}^*[1],$$

whose composition with the functor $\mathsf{H}_{\mathscr{M}}$ is invertible. Indeed, we can check this last statement after applying the functors $\mathrm{rat}_Y^{\mathscr{M}}$, and then this follows from the exact triangle $q_{1*}^{\mathscr{P}} q_2^{*\mathscr{P}} \to \mathsf{F}_{\mathscr{P}} \to \mathsf{G}_{\mathscr{P}} \xrightarrow{+1}$ and the fact that the composition of the natural transformation $s_{\mathscr{P}}^* \to {}^p\mathrm{Log}_f^{\mathscr{P}} a_{\mathscr{P}}^*[1]$ and of the functor $q_{1*}^{\mathscr{P}} q_2^{*\mathscr{P}} \simeq p_{\mathscr{P}}^* p_*^{\mathscr{P}} \simeq \mathbf{Q}_{\mathbf{A}^1} \boxtimes p_*^{\mathscr{P}}$ is invertible. As the functor ${}^p\mathrm{Log}_f^{\mathscr{M}} a_{\mathscr{M}}^*$ is exact, we get an isomorphism from $p_\bullet^{\mathscr{M}}$ to the mapping cone of the morphism of exact functors ${}^p\mathrm{Log}_f^{\mathscr{M}} a_{\mathscr{M}}^* \circ \mathsf{F}_{\mathscr{M}} \to {}^p\mathrm{Log}_f^{\mathscr{M}} a_{\mathscr{M}}^* \circ \mathsf{G}_{\mathscr{M}}$.

Let us prove that the functor $p_\bullet^{\mathscr{M}}$ is right adjoint to the functor $p_{\mathscr{M}}^*$. Let $\eta_{\mathscr{P}} : \mathrm{Id} \to p_*^{\mathscr{P}} p_{\mathscr{P}}^*$ and $\delta_{\mathscr{P}} : p_{\mathscr{P}}^* p_*^{\mathscr{P}} \to \mathrm{Id}$ be the unit and the counit of the adjunction between $p_{\mathscr{P}}^*$ and $p_{\mathscr{P}}^*$. It suffices to lift $\eta_{\mathscr{P}}$ and $\delta_{\mathscr{P}}$ to natural transformations $\eta_{\mathscr{M}} : \mathrm{Id} \to p_\bullet^{\mathscr{M}} p_{\mathscr{M}}^*$ and $\delta_{\mathscr{M}} : p_{\mathscr{M}}^* p_\bullet^{\mathscr{M}} \to \mathrm{Id}$ such that the two natural transformations

$$p_{\mathscr{M}}^* \xrightarrow{p_{\mathscr{M}}^* \eta_{\mathscr{M}}} p_{\mathscr{M}}^* p_\bullet^{\mathscr{M}} p_{\mathscr{M}}^* \xrightarrow{\delta_{\mathscr{M}} p_{\mathscr{M}}^*} p_{\mathscr{M}}^*$$

and

$$p_\bullet^{\mathscr{M}} \xrightarrow{\eta_{\mathscr{M}} p_\bullet^{\mathscr{M}}} p_\bullet^{\mathscr{M}} p_{\mathscr{M}}^* p_\bullet^{\mathscr{M}} \xrightarrow{p_\bullet^{\mathscr{M}} \delta_{\mathscr{M}}} p_\bullet^{\mathscr{M}}$$

are isomorphisms and the first one is the identity (see [69, Section 3.1]). Note that the fact that these natural transformations are isomorphisms will follow automatically from the conservativity of the functors $\mathrm{rat}_X^{\mathscr{M}}$.

We first construct $\eta_{\mathscr{M}}$. Let us first show that $\mathsf{G}_{\mathscr{M}} \circ p_{\mathscr{M}}^* = 0$. As the functors $\mathrm{rat}_X^{\mathscr{M}}$ are conservative, it suffices to prove that $\mathsf{G}_{\mathscr{P}} \circ p_{\mathscr{P}}^* = 0$. Let $k : U \to \mathbf{A}^1 \times \mathbf{A}^1$ be the open immersion (remember that U is the complement of the diagonal in $\mathbf{A}^1 \times \mathbf{A}^1$), so that $j = k \times \mathrm{Id}_Y$, and let $\pi : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ be the first projection, so that $q_1 = \pi \times \mathrm{Id}_Y$. Then

$$\mathsf{G}_{\mathscr{P}} \circ p_{\mathscr{P}}^* = q_{1*}^{\mathscr{P}} j_!^{\mathscr{P}} j_{\mathscr{P}}^* q_{2\mathscr{P}}^* p_{\mathscr{M}}^*[1] \simeq q_{1*}^{\mathscr{P}}((k_!^{\mathscr{P}} \mathbf{Q}_U) \boxtimes -)[1] \simeq (\pi_*^{\mathscr{P}} k_!^{\mathscr{P}} \mathbf{Q}_U) \boxtimes (-)[1],$$

so it suffices to show that

$$\pi_*^{\mathscr{P}} k_!^{\mathscr{P}} \mathbf{Q}_U = 0.$$

Let $\Delta : \mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{A}^1$ be the diagonal embedding. Then we have an exact triangle

$$k_! \mathbf{Q}_U \to \mathbf{Q}_{\mathbf{A}^1 \times \mathbf{A}^1} \to \Delta_* \mathbf{Q}_{\mathbf{A}^1} \xrightarrow{+1},$$

so, applying $\pi_*^{\mathscr{P}}$, we get an exact triangle

$$\pi_*^{\mathscr{P}} k_!^{\mathscr{P}} \mathbf{Q}_U \to \mathbf{Q}_{\mathbf{A}^1} \xrightarrow{\mathrm{Id}} \mathbf{Q}_{\mathbf{A}^1} \xrightarrow{+1},$$

and this implies the desired result.

Now that we know that $\mathsf{G}_{\mathscr{P}} \circ p_{\mathscr{M}}^* = 0$, we get $\mathsf{H}_{\mathscr{M}} \circ p_{\mathscr{M}}^* = p_{\mathscr{M}}^*$, hence $p_{\bullet}^{\mathscr{M}} p_{\mathscr{M}}^* = s_{\mathscr{M}}^* \circ \mathsf{H}_{\mathscr{M}} \circ p_{\mathscr{M}}^* = s_{\mathscr{M}}^* p_{\mathscr{M}}^*$, and we take for $\eta_{\mathscr{M}} : \mathrm{Id} \to p_{\bullet}^{\mathscr{M}} p_{\mathscr{M}}^*$ the inverse of the connection isomorphism $s_{\mathscr{M}}^* p_{\mathscr{M}}^* \xrightarrow{\sim} \mathrm{Id}$.

Next we construct $\delta_{\mathscr{M}}$. First we define a functor

$$q_{1\bullet}^{\mathscr{M}} : \mathrm{D}^{\mathrm{b}}(\mathscr{M}(\mathbf{A}^1 \times Y)) \to \mathrm{D}^{\mathrm{b}}(\mathscr{M}(\mathbf{A}^1 \times \mathbf{A}^1 \times Y))$$

in the same way as $p_{\bullet}^{\mathscr{M}}$, that is, we consider the commutative diagram

$$
\begin{array}{ccccc}
\mathbf{A}^1 \times \mathbf{A}^1 \times Y & \xleftarrow{r_2} & \mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1 \times Y & \xleftarrow{J} & \mathbf{A}^1 \times U \times Y \\
\downarrow{q_1} & \overset{\mathrm{Id}}{\longleftarrow} & \downarrow{r_1} & \overset{I}{\longleftarrow} & \\
\mathbf{A}^1 \times Y & \xleftarrow{q_1} & \mathbf{A}^1 \times \mathbf{A}^1 \times Y & \xleftarrow{\mathrm{Id}} & \mathbf{A}^1 \times \mathbf{A}^1 \times Y
\end{array}
$$

where $r_1 = \mathrm{Id}_{\mathbf{A}^1} \times q_1$, $r_2 = \mathrm{Id}_{\mathbf{A}^1} \times q_2$, $I = \mathrm{Id}_{\mathbf{A}^1} \times i$ and $J = \mathrm{Id}_{\mathbf{A}^1} \times j$, and we set $t = \mathrm{Id}_{\mathbf{A}^1} \times s : \mathbf{A}^1 \times Y \to \mathbf{A}^1 \times \mathbf{A}^1 \times Y$. Then the functors $\mathsf{F}'_{\mathscr{P}}, \mathsf{G}'_{\mathscr{P}}$ from $\mathrm{D}^{\mathrm{b}}(\mathscr{P}(\mathbf{A}^1 \times \mathbf{A}^1 \times Y))$ to itself defined by $\mathsf{F}'_{\mathscr{P}} = \mathrm{Id}$ and $\mathsf{G}'_{\mathscr{P}} = r_{1*}^{\mathscr{P}} J_!^{\mathscr{P}} J_{\mathscr{P}}^* r_{2\mathscr{P}}^* [1]$ are t-exact and we have a natural transformation $\mathsf{F}'_{\mathscr{P}} \to \mathsf{G}'_{\mathscr{P}}$. As before, we can lift these functors and the transformation to endofunctors $\mathsf{F}'_{\mathscr{M}} \to \mathsf{G}'_{\mathscr{M}}$ of $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(\mathbf{A}^1 \times \mathbf{A}^1 \times Y))$. We denote by $\mathsf{H}'_{\mathscr{M}}$ the mapping fiber of $\mathsf{F}'_{\mathscr{M}} \to \mathsf{G}'_{\mathscr{M}}$, and we set $q_{1\bullet}^{\mathscr{M}} = t_{\mathscr{M}}^* \circ \mathsf{H}'_{\mathscr{M}}$. Also, if we denote by $f' : \mathbf{A}^1 \times \mathbf{A}^1 \times Y \to \mathbf{A}^1$ the second projection and by a' the injection of $\mathbf{A}^1 \times \mathbf{G}_m \times Y$ into $\mathbf{A}^1 \times \mathbf{A}^1 \times Y$, we get as above an invertible natural transformation from $q_{1\bullet}^{\mathscr{M}}$ to the mapping cone of the morphism ${}^p\mathsf{Log}_{f'}^{\mathscr{M}}(a')_{\mathscr{M}}^* \circ \mathsf{F}'_{\mathscr{M}} \to {}^p\mathsf{Log}_{f'}^{\mathscr{M}}(a')_{\mathscr{M}}^* \circ \mathsf{G}'_{\mathscr{M}}$ of exact functors.

Let us show that the base change isomorphism $p_{\mathscr{P}}^* p_*^{\mathscr{P}} \xrightarrow{\sim} q_{1*}^{\mathscr{P}} q_{2\mathscr{P}}^*$ lifts to a morphism $p_{\mathscr{M}}^* p_{\bullet}^{\mathscr{M}} \to q_{1\bullet}^{\mathscr{M}} q_{2\mathscr{M}}^*$ (which will automatically be an isomorphism). We have invertible natural transformations $\mathsf{F}'_{\mathscr{P}} \circ q_{2\mathscr{P}}^* \simeq q_{2\mathscr{P}}^* \circ \mathsf{F}_{\mathscr{P}}$ and $\mathsf{G}'_{\mathscr{P}} \circ q_{2\mathscr{P}}^* \simeq q_{2\mathscr{P}}^* \circ \mathsf{G}_{\mathscr{P}}$. As all the functors involved are t-exact up to the same shift, the transformations lift to natural transformations $\mathsf{F}'_{\mathscr{M}} \circ q_{2\mathscr{M}}^* \simeq q_{2\mathscr{M}}^* \circ \mathsf{F}_{\mathscr{M}}$ and $\mathsf{G}'_{\mathscr{M}} \circ q_{2\mathscr{M}}^* \simeq q_{2\mathscr{M}}^* \circ \mathsf{G}_{\mathscr{M}}$, and induce an invertible natural transformation $\mathsf{H}'_{\mathscr{M}} \circ q_{2\mathscr{M}}^* \simeq q_{2\mathscr{M}}^* \circ \mathsf{H}_{\mathscr{M}}$. Composing on the left with $t_{\mathscr{M}}^*$ and using the connection isomorphism $t_{\mathscr{M}}^* q_{2\mathscr{M}}^* \simeq p_{\mathscr{M}}^* s_{\mathscr{M}}^*$, we get the desired isomorphism $p_{\mathscr{M}}^* p_{\bullet}^{\mathscr{M}} \xrightarrow{\sim} q_{1\bullet}^{\mathscr{M}} q_{2\mathscr{M}}^*$.

Composing this isomorphism with the unit of the adjunction $(i_{\mathscr{M}}^*, i_*^{\mathscr{M}})$ and using the connection isomorphism $i_{\mathscr{M}}^* q_{2\mathscr{M}}^* \simeq \mathrm{Id}$ gives a natural transformation $p_{\mathscr{M}}^* p_{\bullet}^{\mathscr{M}} \to q_{1\bullet}^{\mathscr{M}} i_*^{\mathscr{M}}$.

It remains to show that the isomorphism $q_{1*}^{\mathscr{P}} i_*^{\mathscr{P}} \simeq \mathrm{Id}$ lifts to a natural transformation $q_{1\bullet}^{\mathscr{M}} i_*^{\mathscr{M}} \to \mathrm{Id}$. First we note that the functors

$$^p\mathsf{Log}_{f'}^{\mathsf{B}}(a')_{\mathscr{P}}^* r_{1*}^{\mathscr{P}} r_2^{*\mathscr{P}} i_*^{\mathscr{P}}[1] \qquad (5.2)$$

and

$$^p\mathsf{Log}_{f'}^{\mathsf{B}}(a')_{\mathscr{P}}^* r_{1*}^{\mathscr{P}} J_!^{\mathscr{P}} J_{\mathscr{P}}^* r_2^{*\mathscr{P}} i_*^{\mathscr{P}}[1]$$

are t-exact and the counit of the adjunction $(J_!^{\mathscr{P}}, J_{\mathscr{P}}^*)$ induces a natural transformation from the second one to the first one. Hence, the functor (5.2) induces an exact endofunctor $\mathsf{H}''_{\mathscr{M}}$ of $\mathscr{M}(\mathbf{A}^1 \times Y)$, together with a natural transformation $^p\mathsf{Log}^{\mathscr{M}}(a')_{\mathscr{M}}^* \circ \mathsf{G}'_{\mathscr{M}} \circ i_*^{\mathscr{M}} \to \mathsf{H}''_{\mathscr{M}}$. But we also have an invertible natural transformation of t-exact functors

$$\begin{aligned}
\mathrm{Id}_{\mathsf{D}^b(\mathscr{P}(\mathbf{A}^1 \times Y))} &\simeq t_{\mathscr{P}}^* q_1^{*\mathscr{P}} q_{1*}^{\mathscr{P}} i_*^{\mathscr{P}} \quad \text{(connection isomorphisms)} \\
&\xrightarrow{\sim} t_{\mathscr{P}}^* r_{1*}^{\mathscr{P}} r_{2\mathscr{P}}^* i_*^{\mathscr{P}} \quad \text{(base change)} \\
&\xrightarrow{\sim} {}^p\mathsf{Log}_{f'}^{\mathsf{B}}(a')_{\mathscr{P}}^* r_{1*}^{\mathscr{P}} r_{2\mathscr{P}}^* i_*^{\mathscr{P}}[1] \quad \text{(by Section 3.5 as above)}
\end{aligned}$$

and all the maps in it are defined in the categories $\mathbf{DA}_{\mathrm{ct}}(-)$, so it induces an invertible natural transformation $\mathrm{Id}_{\mathscr{M}(\mathbf{A}^1 \times Y)} \xrightarrow{\sim} \mathsf{H}''_{\mathscr{M}}$. Composing it with $^p\mathsf{Log}^{\mathscr{M}}(a')_{\mathscr{M}}^* \circ \mathsf{G}'_{\mathscr{M}} \circ i_*^{\mathscr{M}} \to \mathsf{H}''_{\mathscr{M}}$ and using the isomorphism from $q_{1\bullet}^{\mathscr{M}}$ to the mapping fiber of $^p\mathsf{Log}^{\mathscr{M}}(a')_{\mathscr{M}}^* \circ \mathsf{F}'_{\mathscr{M}} \to {}^p\mathsf{Log}^{\mathscr{M}}(a')_{\mathscr{M}}^* \circ \mathsf{G}'_{\mathscr{M}}$, we finally get the desired natural transformation $\delta_{\mathscr{M}}$:

$$p_{\mathscr{M}}^* p_\bullet^{\mathscr{M}} \xrightarrow{\sim} q_{1\bullet}^{\mathscr{M}} q_2^{*\mathscr{M}} \to q_{1\bullet}^{\mathscr{M}} i_*^{\mathscr{M}} \to {}^p\mathsf{Log}^{\mathscr{M}}(a')_{\mathscr{M}}^* \circ \mathsf{G}'_{\mathscr{M}} \circ i_*^{\mathscr{M}} \to \mathsf{H}''_{\mathscr{M}} \simeq \mathrm{Id}_{\mathsf{D}^b(\mathscr{M}(\mathbf{A}^1 \times Y))}.$$

Finally, we check that the natural transformation

$$p_{\mathscr{M}}^* \xrightarrow{p_{\mathscr{M}}^* \eta_{\mathscr{M}}} p_{\mathscr{M}}^* p_\bullet^{\mathscr{M}} p_{\mathscr{M}}^* \xrightarrow{\delta_{\mathscr{M}} p_{\mathscr{M}}^*} p_{\mathscr{M}}^*$$

is the identity. The two functors $p_{\mathscr{M}}^* p_\bullet^{\mathscr{M}} p_{\mathscr{M}}^*[1]$ and $p_{\mathscr{M}}^*[1]$ are exact and equal to the derived functor of their $\mathbf{H}^0$, and the natural transformations

$$(p_{\mathscr{M}}^* \eta_{\mathscr{M}})^{-1}, \delta_{\mathscr{M}} p_{\mathscr{M}}^* : p_{\mathscr{M}}^* p_\bullet^{\mathscr{M}} p_{\mathscr{M}}^*[1] \to p_{\mathscr{M}}^*[1]$$

are also defined by extending their action on the $\mathbf{H}^0$'s, so it suffices to check that they are equal on these $\mathbf{H}^0$'s. But this follows from the analogous result for the category of perverse sheaves.

Step 3. We can now use the Brown Representability Theorem to see that the proposition is true more generally if f is the projection $p : X := E \to Y$ of a vector bundle E on Y. Indeed, given a k-variety S, let $\mathrm{Ind}(\mathscr{M}(S))$ be the abelian category of Ind-objects of $\mathscr{M}(S)$ and consider the bounded derived category $\mathsf{D}^b(\mathscr{M}(S))$ as a full subcategory of the unbounded derived category $\mathsf{D}(\mathrm{Ind}(\mathscr{M}(S)))$ (see e.g. Kashiwara–Schapira [53, Theorem 15.3.1]). As the morphism $p : E \to Y$ is smooth, the functor $p_{\mathscr{M}}^*$ extends to a triangulated functor $\mathsf{L} : \mathsf{D}(\mathrm{Ind}(\mathscr{M}(Y))) \to \mathsf{D}(\mathrm{Ind}(\mathscr{M}(E)))$. By the Brown Representability Theorem (see e.g. Neeman [61, Theorem 4.1] or [62, Theorem 8.4.4]), the functor L admits a right adjoint $\mathsf{R} : \mathsf{D}(\mathrm{Ind}(\mathscr{M}(E)) \to \mathsf{D}(\mathrm{Ind}(\mathscr{M}(Y)))$.

To prove that $p_{\mathscr{M}}^*$ admits a right adjoint $p_*^{\mathscr{M}}$, it suffices to check that, given $M \in \mathsf{D}^b(\mathscr{M}(E))$, the object $\mathsf{R}(M)$ belongs to the subcategory $\mathsf{D}^b(\mathscr{M}(Y))$. This can be checked

on a finite Zariski open covering of Y that trivializes E and thus follows from the case of a projection $\mathbf{A}_Y^1 \to Y$ proved in Step 2. That $p_*^{\mathscr{M}}$ satisfies (2) can again be checked on a finite Zariski open covering of Y that trivializes E and we conclude using step 2.

Step 4. By Steps 1 and 3, the conclusion of the proposition is true if f is an affine morphism. Indeed, if f is affine, then we can write $f = p \circ i$, where i is a closed immersion and $p : E \to Y$ is a vector bundle on Y.

Step 5. We now consider the case of an arbitrary morphism $f : X \to Y$ in (Sch/k). By Jouanolou's trick [51], there exists a vector bundle $E \to X$ and an affine E-torsor $p : \widetilde{X} \to X$. As p is affine, we know the proposition for p by step 3. Moreover, the unit $\mathrm{Id} \to p_*^{\mathscr{M}} p_{\mathscr{M}}^*$ is an isomorphism; indeed, it suffices to show this after restricting to an open covering of X, so we may assume that the morphism p is isomorphic to the second projection $\mathbf{A}^n \times X \to X$, and then the result follows from item (3) of the proposition. As the unit of the adjunction $(p_{\mathscr{M}}^*, p_*^{\mathscr{M}})$ is an isomorphism, the left adjoint $p_{\mathscr{M}}^*$ is fully faithful.

Let $g = f \circ p$. As $\widetilde{X}$ is affine, the morphism g is affine. Also, we show as before that the unit $\mathrm{Id} \to p_*^{\mathscr{P}} p_{\mathscr{P}}^*$ is an isomorphism, so we get an isomorphism $f_*^{\mathscr{P}} \xrightarrow{\sim} f_*^{\mathscr{P}} p_*^{\mathscr{P}} p_{\mathscr{P}}^* \simeq g_*^{\mathscr{P}} p_{\mathscr{P}}^*$. We set $f_*^{\mathscr{M}} = g_*^{\mathscr{M}} p_{\mathscr{M}}^*$; by the calculation we just did, this satisfies condition (2). It remains to show that $f_*^{\mathscr{M}}$ is right adjoint to $f_{\mathscr{M}}^*$. Let $K \in \mathrm{Ob}\, \mathrm{D}^{\mathrm{b}}\mathscr{M}(Y)$ and $L \in \mathrm{Ob}\, \mathrm{D}^{\mathrm{b}}\mathscr{M}(X)$. Then we have isomorphisms

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}\mathscr{M}(Y)}(K, f_*^{\mathscr{M}} L) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}\mathscr{M}(Y)}(K, g_*^{\mathscr{M}} p_{\mathscr{M}}^* L) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}\mathscr{M}(\widetilde{X})}(g_{\mathscr{M}}^* K, p_{\mathscr{M}}^* L)$$

$$\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}\mathscr{M}(\widetilde{X})}(p_{\mathscr{M}}^* f_{\mathscr{M}}^* K, p_{\mathscr{M}}^* L),$$

and the last group is isomorphic to $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}\mathscr{M}(X)}(f_{\mathscr{M}}^* K, L)$ since $p_{\mathscr{M}}^*$ is fully faithful. ∎

Proposition 5.3. *For every smooth morphism $f : X \to Y$ in (Sch/k) the functor*

$$f_{\mathscr{M}}^* : \mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y)) \to \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$$

admits a left adjoint $f_\sharp$. Moreover,

(1) *the natural transformation*

$$f_\sharp^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} \to \mathrm{rat}_Y^{\mathscr{M}} f_\sharp^{\mathscr{M}}$$

obtained from $\theta_f^{\mathscr{M}}$ by adjunction is invertible;

(2) *for every cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{\ g'\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\ g\ } & Y \end{array}$$

with f smooth, the exchange 2-morphism $f_\sharp'^{\mathscr{M}} g_{\mathscr{M}}'^* \to g_{\mathscr{M}}^* f_\sharp^{\mathscr{M}}$ is invertible.

Proof. Assertion (2) is an immediate consequence of (1) since the functor $\mathrm{rat}_Y^{\mathscr{M}}$ is conservative.

We deduce the proposition from Proposition 5.2 using Verdier duality. Let $f : X \to Y$ be a smooth morphisms of relative dimension d. Note that $f_{\mathscr{P}}^*$ has a left adjoint given by

$$f_{\sharp}^{\mathscr{P}} := \mathbb{D}_Y^{\mathscr{P}} f_*^{\mathscr{P}}(-d)[-2d]\mathbb{D}_X^{\mathscr{P}}.$$

Therefore we similarly set $f_{\sharp}^{\mathscr{M}} := \mathbb{D}_Y^{\mathscr{M}} f_*^{\mathscr{M}}(-d)[-2d]\mathbb{D}_X^{\mathscr{M}}$.

Let A be an object in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ and B be an object in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y))$. Then Propositions 5.2 and 2.6 provide isomorphisms

$$
\begin{aligned}
\mathrm{Hom}(f_{\sharp}^{\mathscr{M}} A, B) &\simeq \mathrm{Hom}(\mathbb{D}_Y^{\mathscr{M}} B, f_*^{\mathscr{M}}(-d)[-2d]\mathbb{D}_X^{\mathscr{M}} A) \\
&\simeq \mathrm{Hom}(f_{\mathscr{M}}^*(d)[2d]\mathbb{D}_Y^{\mathscr{M}} B, \mathbb{D}_X^{\mathscr{M}} A) \\
&\simeq \mathrm{Hom}(A, \mathbb{D}_X^{\mathscr{M}} f_{\mathscr{M}}^*(d)[2d]\mathbb{D}_Y^{\mathscr{M}} B) \simeq \mathrm{Hom}(A, f_{\mathscr{M}}^* B).
\end{aligned}
$$

This shows that $(f_{\sharp}^{\mathscr{M}}, f_{\mathscr{M}}^*)$ is a pair of adjoint functors. Note that the counit $^{\mathscr{M}}\delta_{\sharp}^*$ of the adjunction is given by the composition

$$f_{\sharp}^{\mathscr{M}} f_{\mathscr{M}}^* \xrightarrow{\varepsilon_f^{\mathscr{M}}} \mathbb{D}_Y^{\mathscr{M}} f_*^{\mathscr{M}} f_{\mathscr{M}}^* \mathbb{D}_Y^{\mathscr{M}} \xrightarrow{^{\mathscr{M}}\eta_*^*} (\mathbb{D}_Y^{\mathscr{M}})^2 \xrightarrow{(\varepsilon_Y^{\mathscr{M}})^{-1}} \mathrm{Id} \tag{5.3}$$

and the unit $^{\mathscr{M}}\eta_{\sharp}^*$ by the composition

$$\mathrm{Id} \xrightarrow{\varepsilon_X^{\mathscr{M}}} (\mathbb{D}_X^{\mathscr{M}})^2 \xrightarrow{^{\mathscr{M}}\delta_*^*} \mathbb{D}_X^{\mathscr{M}} f_{\mathscr{M}}^* f_*^{\mathscr{M}} \mathbb{D}_X^{\mathscr{M}} \xrightarrow{\varepsilon_f^{\mathscr{M}}} f_{\mathscr{M}}^* f_{\sharp}^{\mathscr{M}}. \tag{5.4}$$

To show that the morphism

$$f_{\sharp}^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} \xrightarrow{^{\mathscr{M}}\eta_{\sharp}^*} f_{\sharp}^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} f_{\mathscr{M}}^* f_{\sharp}^{\mathscr{M}} \xrightarrow{(\theta_f^{\mathscr{M}})^{-1}} f_{\sharp}^{\mathscr{P}} f_{\mathscr{P}}^* \mathrm{rat}_Y^{\mathscr{M}} f_{\sharp}^{\mathscr{M}} \xrightarrow{^{\mathscr{P}}\delta_{\sharp}^*} \mathrm{rat}_Y^{\mathscr{M}} f_{\sharp}^{\mathscr{M}}$$

is invertible, it is enough to check that it is equal to the morphism

$$f_{\sharp}^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} \xrightarrow{(\nu_X^{\mathscr{M}})^{-1}} \mathbb{D}_Y^{\mathscr{P}} f_*^{\mathscr{P}}(-d)[-2d]\mathrm{rat}_X^{\mathscr{M}} \mathbb{D}_X^{\mathscr{M}}$$

$$\downarrow{\gamma_f^{\mathscr{M}}}$$

$$\mathbb{D}_Y^{\mathscr{P}} \mathrm{rat}_Y^{\mathscr{M}} f_*^{\mathscr{M}}(-d)[-2d]\mathbb{D}_X^{\mathscr{M}} \xrightarrow{\nu_Y^{\mathscr{M}}} \mathrm{rat}_Y^{\mathscr{M}} f_{\sharp}^{\mathscr{M}}$$

where $\gamma_f^{\mathscr{M}}$ is the invertible natural transformation of Proposition 5.2. Using the expressions of $^{\mathscr{M}}\delta_{\sharp}^*$ and $^{\mathscr{M}}\eta_{\sharp}^*$ given in (5.3) and (5.4), this follows directly from Proposition 2.6 (1, 2), which ensures that the diagram

$$
\begin{array}{ccc}
\mathbb{D}_X^{\mathscr{P}} f_{\mathscr{P}}^*[d]\mathbb{D}_Y^{\mathscr{P}} \mathrm{rat}_Y^{\mathscr{M}} & \xrightarrow{\quad\quad\varepsilon_f^{\mathscr{P}}\quad\quad} & (\mathbb{D}_X^{\mathscr{P}})^2 f_{\mathscr{P}}^*(d)[d]\mathrm{rat}_Y^{\mathscr{M}} \\
{\scriptstyle \nu_Y^{\mathscr{M}}}\big\uparrow & {\scriptstyle \varepsilon_f^{\mathscr{M}}}\quad & \big\downarrow{\scriptstyle (\varepsilon_X^{\mathscr{P}})^{-1}} \\
\mathbb{D}_X^{\mathscr{P}} f_{\mathscr{P}}^*[d]\mathrm{rat}_Y^{\mathscr{M}} \mathbb{D}_Y^{\mathscr{M}} & f_{\mathscr{P}}^*(d)[d](\mathbb{D}_Y^{\mathscr{P}})^2\mathrm{rat}_Y^{\mathscr{M}} \xrightarrow{(\varepsilon_Y^{\mathscr{P}})^{-1}} f_{\mathscr{P}}^*(d)[d]\mathrm{rat}_Y^{\mathscr{M}} & \big\uparrow{\scriptstyle (\varepsilon_Y^{\mathscr{M}})^{-1}} \\
{\scriptstyle \varepsilon_f^{\mathscr{P}}}\big\downarrow & {\scriptstyle \nu_Y^{\mathscr{M}}}\quad & \\
f_{\mathscr{P}}^*(d)[d]\mathbb{D}_Y^{\mathscr{P}} \mathrm{rat}_Y^{\mathscr{M}} \mathbb{D}_Y^{\mathscr{M}} & \xrightarrow{\quad\quad\nu_Y^{\mathscr{M}}\quad\quad} & f_{\mathscr{P}}^*(d)[d]\mathrm{rat}_Y^{\mathscr{M}} (\mathbb{D}_Y^{\mathscr{M}})^2
\end{array}
$$

is commutative. ∎

The pair $(j_{\mathscr{M}}^*, i_{\mathscr{M}}^*)$ is conservative, since so is $(j_{\mathscr{P}}^*, i_{\mathscr{P}}^*)$. This follows from the existence of the isomorphisms $\theta_j^{\mathscr{M}}$, $\theta_i^{\mathscr{M}}$ and the fact that $\mathrm{rat}_X^{\mathscr{M}}$ is a conservative functor.

To finish the proof of Theorem 5.1, it remains to check that if s is the zero section of $p : \mathbf{A}_X^1 \to X$, then $p_{\sharp}^{\mathscr{M}} s_*^{\mathscr{M}}$ is an equivalence of categories. By construction

$$p_{\sharp}^{\mathscr{M}} s_*^{\mathscr{M}} = \mathbb{D}_X^{\mathscr{M}} p_*^{\mathscr{M}}(-1)[-2] \mathbb{D}_{\mathbf{A}_X^1}^{\mathscr{M}} s_*^{\mathscr{M}}.$$

Note that the isomorphism $\mathbb{D}_{\mathbf{A}_X^1}^{\mathscr{P}} s_*^{\mathscr{P}} \simeq s_*^{\mathscr{P}} \mathbb{D}_X^{\mathscr{P}}$ exists in the category of (constructible) étale motives. Therefore, the compatibility of the Betti realization with the four operations (see [7, Théorème 3.19]) implies by the universal property of the categories of perverse motives that this isomorphism lifts to an isomorphism $\mathbb{D}_{\mathbf{A}_X^1}^{\mathscr{M}} s_*^{\mathscr{M}} \simeq s_*^{\mathscr{M}} \mathbb{D}_X^{\mathscr{M}}$. As a consequence, we get an isomorphism

$$p_{\sharp}^{\mathscr{M}} s_*^{\mathscr{M}} \simeq \mathbb{D}_X^{\mathscr{M}} p_*^{\mathscr{M}} s_*^{\mathscr{M}}(-1)[-2] \mathbb{D}_{\mathbf{A}_X^1}^{\mathscr{M}} \simeq \mathrm{Id}(-1)[-2].$$

This shows that $p_{\sharp}^{\mathscr{M}} s_*^{\mathscr{M}}$ is an equivalence of categories and concludes the proof of Theorem 5.1.

5.3. Complement to the main theorem

The following proposition complements Theorem 5.1.

Proposition 5.4. *Let $f : X \to Y$ be a morphism of quasi-projective k-varieties. Then the natural transformations*

$$\xi_f^{\mathscr{M}} : \mathrm{rat}_X^{\mathscr{M}} f_{\mathscr{M}}^! \to f_{\mathscr{P}}^! \mathrm{rat}_Y^{\mathscr{M}}, \quad \rho_f^{\mathscr{M}} : f_!^{\mathscr{P}} \mathrm{rat}_X^{\mathscr{M}} \to \mathrm{rat}_Y^{\mathscr{M}} f_!^{\mathscr{M}}$$

are invertible.

Proof. By [7, Théorème 3.4], it just remains to check that $\xi_i^{\mathscr{M}}$ is invertible if $i : Z \hookrightarrow X$ is a closed immersion. Let $j : U \hookrightarrow X$ be the open immersion of the complement of Z in X. Then we have a commutative diagram

$$
\begin{array}{ccccccc}
i_*^{\mathscr{P}} i_{\mathscr{P}}^! \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & j_*^{\mathscr{P}} j_{\mathscr{P}}^* \mathrm{rat}_X^{\mathscr{M}} & \xrightarrow{+1} & \\
\big\uparrow {\xi_i^{\mathscr{M}}} & & \big\| & & \big\uparrow {(\theta_j^{\mathscr{M}})^{-1}} & & \\
i_*^{\mathscr{P}} \mathrm{rat}_Z^{\mathscr{M}} i_{\mathscr{M}}^! & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & j_*^{\mathscr{P}} \mathrm{rat}_U^{\mathscr{M}} j_{\mathscr{M}}^* & \xrightarrow{+1} & \\
\big\uparrow {\gamma_i^{\mathscr{M}}} & & \big\| & & \big\uparrow {\gamma_j^{\mathscr{M}}} & & \\
\mathrm{rat}_X^{\mathscr{M}} i_*^{\mathscr{M}} i_{\mathscr{M}}^! & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} & \longrightarrow & \mathrm{rat}_X^{\mathscr{M}} j_*^{\mathscr{M}} j_{\mathscr{M}}^* & \xrightarrow{+1} &
\end{array}
$$

which implies that the image of $\xi_i^{\mathscr{M}}$ under $i_*^{\mathscr{M}}$ is invertible since all the other morphisms are. Therefore $\xi_i^{\mathscr{M}}$ is also invertible. ∎

5.4. Some consequences

In this subsection, we draw some immediate consequences of the main theorem 5.1.

Geometric local systems are motivic. A $\mathbf{Q}$-local system $\mathscr{L}$ on a quasi-projective k-variety X will be called *geometric* if there exists a smooth proper morphism $g : Z \to X$ such that $\mathscr{L} = R^i g_* \mathbf{Q}$ for some $i \in \mathbf{Z}$. We will say that $\mathscr{L}$ is *motivic* if there exists an object L in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ such that $\mathscr{L}$ and $\mathrm{rat}_X^{\mathscr{M}}(L)$ are isomorphic in the category $\mathrm{D}^{\mathrm{b}}(\mathscr{P}(X))$.

Corollary 5.5. *A geometric $\mathbf{Q}$-local system $\mathscr{L}$ on a quasi-projective k-variety X is motivic.*

Proof. If the local system $\mathscr{L}$ is geometric, there exists a smooth proper morphism $g : Z \to X$ such that $\mathscr{L} = R^i g_* \mathbf{Q}_Z$ for some $i \in \mathbf{Z}$. Then $\mathscr{L}$ is the image under the functor $\mathrm{rat}_X^{\mathscr{M}}$ of the perverse motive ${}^{\mathrm{c}}\mathrm{H}^i(g_*^{\mathscr{M}} \mathbf{Q}_Z^{\mathscr{M}})$, where ${}^{\mathrm{c}}\mathrm{H}^i$ is the cohomological functor associated with the constructible t-structure (see below). ∎

Remark 5.6. In this remark, we denote by H^i the standard cohomology functors on the category $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$. Let $\mathscr{L}$ be a geometric $\mathbf{Q}$-local system on a smooth quasi-projective variety of (pure) dimension d and choose a smooth proper morphism $g : Z \to X$ and an integer j such that $\mathscr{L} = R^j g_* \mathbf{Q}_Z$. As Z is smooth and g is proper and smooth, the constructible sheaves $R^i g_* \mathbf{Q}_Z$ are all $\mathbf{Q}$-local systems on X. Hence the complexes $(R^i g_* \mathbf{Q}_Z)[d]$ are perverse sheaves and therefore $(R^i g_* \mathbf{Q}_Z)[d] = {}^p\mathrm{H}^{d+i} g_* \mathbf{Q}_Z$ for every $i \in \mathbf{Z}$. In particular, $\mathscr{L}[d] = {}^p\mathrm{H}^{j+d} g_* \mathbf{Q}_Z$ and it follows that $\mathscr{L}[d]$ is the image under $\mathrm{rat}_X^{\mathscr{M}}$ of the perverse motivic sheaf $A := \mathrm{H}^{j+d}(g_*^{\mathscr{M}} \mathbf{Q}_Z^{\mathscr{M}})$.

Intersection cohomology. The four operations formalism allows the definition of a motivic avatar of intersection complexes. In particular, intersection cohomology groups with coefficients in geometric systems are motivic. More precisely:

Corollary 5.7. *Let X be an irreducible quasi-projective k-variety and $\mathscr{L}$ be a $\mathbf{Q}$-local system on a smooth dense open subscheme of X. If $\mathscr{L}$ is motivic (in particular if $\mathscr{L}$ is geometric), then the intersection cohomology group $IH^i(X, \mathscr{L})$, for $i \in \mathbf{Z}$, is canonically the Betti realization of a Nori motive over k.*

Proof. Let d be the dimension of X and $\mathscr{L}$ be a $\mathbf{Q}$-local system on a smooth dense open subscheme U of X. Since $\mathscr{L}$ is motivic, there exists an object $L \in \mathrm{D}^{\mathrm{b}}(\mathscr{M}(U))$ such that $\mathscr{L}$ is isomorphic to $\mathrm{rat}_U^{\mathscr{M}}(L)$. Since $\mathscr{L}[d]$ is a perverse sheaf on U and $\mathrm{rat}_U^{\mathscr{M}}$ is conservative, the complex $L[d]$ is a perverse motivic sheaf on U that is belongs to $\mathscr{M}(U)$. Then, with the notation of Definition 6.19 below, the intersection complex

$$\mathrm{IC}_X(\mathscr{L}) := \mathrm{Im}({}^p\mathrm{H}^0 j_!^{\mathscr{P}} \mathscr{L}[d] \to {}^p\mathrm{H}^0 j_*^{\mathscr{P}} \mathscr{L}[d])$$

is canonically isomorphic to the image under $\mathrm{rat}_X^{\mathscr{M}}$ of the perverse motivic sheaf $j_{!*}^{\mathscr{M}} L[d]$ $:= \mathrm{Im}(\mathrm{H}^0(j_!^{\mathscr{M}} L[d]) \to \mathrm{H}^0(j_*^{\mathscr{M}} L[d]))$. This implies $IH^i(X, \mathscr{L}) := \mathbf{H}^{i-d}(X, \mathrm{IC}_X(\mathscr{L}))$

is the Betti realization of the Nori motive $H^{i-d}(\pi_*^{\mathcal{M}} j_{!*}^{\mathcal{M}} L[d])$ where $\pi : X \to \operatorname{Spec} k$ is the structural morphism. ∎

This shows, in particular, that intersection cohomology groups carry a natural Hodge structure. If X is a smooth projective curve, and $\mathscr{L}$ underlies a polarizable variation of Hodge structure, then the Hodge structure on the intersection cohomology groups was constructed by Zucker [77, (7.2) Theorem, (11.6) Theorem]. In general, it follows from Saito's work on mixed Hodge modules [67], and a different proof has been given by de Cataldo [25]. We consider the weights in the next section (see Theorem 6.28 and Corollary 6.29).

Leray spectral sequences. Let $f : X \to Y$ be a morphism of quasi-projective k-varieties and $\mathscr{L}$ be a **Q**-local system on X. Then we can associate with it two Leray spectral sequences in Betti cohomology: the classical one

$$\mathbf{H}^r(Y, R^s f_* \mathscr{L}) \Longrightarrow H^{r+s}(X, \mathscr{L})$$

and the perverse one

$$\mathbf{H}^r(Y, {}^p H^s f_* \mathscr{L}) \Longrightarrow H^{r+s}(X, \mathscr{L}).$$

The main theorem of Arapura [2] shows that if $\mathscr{L} = \mathbf{Q}_X$ is the constant local system on X and the morphism f is projective, then the classical Leray sequence is motivic, that is, it is the realization of a spectral sequence in the abelian category of Nori motives over k (see precisely [2, Theorem 3.1]).

This property is still true without the projectivity assumption and also more generally if the local system $\mathscr{L}$ is geometric:

Corollary 5.8. *If the local system $\mathscr{L}$ is motivic (in particular if it is geometric), then the classical Leray spectral sequence and the perverse Leray spectral sequence are spectral sequences of Nori motives over k.*

Proof. The result follows from the functoriality of the direct image functors. ∎

In particular, the Leray spectral sequences are spectral sequences of (polarizable) mixed Hodge structures. The compatibility of the classical Leray spectral sequence result in Hodge theory was already proved by Zucker [77] when X is a curve and more generally, for both spectral sequences, by Saito [67] if $\mathscr{L}$ underlies an admissible variation of mixed Hodge structures. This result has been recovered by de Cataldo and Migliorini [26] with different techniques.

Nearby cycles. The theory developed here also shows that nearby cycles functors applied to perverse motives produce Nori motives.

Corollary 5.9. *Let X be a quasi-projective k-variety, $f : X \to \mathbf{A}_k^1$ a flat morphism with smooth generic fiber X_η and $\mathscr{L}$ be a **Q**-local system on X_η. If $\mathscr{L}$ is motivic (in particular if it is geometric), then, for every point $x \in X_\sigma(k)$ and every $i \in \mathbf{Z}$, the Betti cohomology $H^i(\Psi_f(\mathscr{L})_x)$ of the nearby fiber is canonically a Nori motive over k.*

Proof. The nearby cycles functor $\psi_f := \Psi_f[-1]$ is t-exact for the perverse t-structure. Since it exists in the triangulated category of constructible étale motives (see [6, Section 3.5]) and the Betti realization is compatible with the nearby cycles functor by [7, Theorem 4.9], the universal property ensures the existence of an exact functor $\psi_f^{\mathscr{M}}$: $\mathscr{M}(X_\eta) \to \mathscr{M}(X_\sigma)$ and an invertible natural transformation $\mathrm{rat}_{X_\sigma}^{\mathscr{M}} \psi_f^{\mathscr{M}} \simeq \psi_f \, \mathrm{rat}_{X_\eta}^{\mathscr{M}}$.

Let d be the dimension of the generic fiber X_η. Since $\mathscr{L}$ is motivic, there exists an object L in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X_\eta))$ such that $\mathscr{L}$ and $\mathrm{rat}_{X_\eta}^{\mathscr{M}}(L)$ are isomorphic. As $\mathscr{L}[d]$ is perverse and $\mathrm{rat}_{X_\eta}^{\mathscr{M}}$ is conservative, the complex $L[d]$ belongs to $\mathscr{M}(X_\eta)$. So we conclude that $H^i(\Psi_f(\mathscr{L})_x)$ is the Betti realization of the Nori motive $H^{i+1-d}(x^* \psi_f^{\mathscr{M}} L[d])$. ∎

Exponential motives. The perverse motives introduced in the present paper and their stability under the four operations could also be used in the study of exponential motives as introduced in Fresán—Jossen's book [34]. Indeed, recall that Kontsevich and Soibelman [56] define an exponential mixed Hodge structure as a mixed Hodge module A on the complex affine line $\mathbf{A}_{\mathbf{C}}^1$ such that $p_* A = 0$ where $p : \mathbf{A}_{\mathbf{C}}^1 \to \mathrm{Spec}(\mathbf{C})$ is the projection. Their definition can be mimicked in the motivic context and the abelian category of exponential Nori motives can be defined as the full subcategory of $\mathscr{M}(\mathbf{A}_k^1)$ formed by the objects which have no global cohomology.

Constructible t-structure. Let us conclude by a possible comparison with Arapura's construction from [3]. Let X be a k-variety and consider the following full subcategories of $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$:

$$^c\mathrm{D}^{\leq 0} := \{A \in \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)) : H^k(x_{\mathscr{M}}^* A) = 0, \; \forall x \in X, \; \forall k > 0\},$$

$$^c\mathrm{D}^{\geq 0} := \{A \in \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)) : H^k(x_{\mathscr{M}}^* A) = 0, \; \forall x \in X, \; \forall k < 0\}.$$

As in Saito [67, 4.6. Remarks] (see also [3, Theorem C.0.12]), we can check that these categories define a t-structure on $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$.

Let $^{\mathrm{ct}}\mathscr{M}(X)$ be the heart of this t-structure. Then the functor $\mathrm{rat}_X^{\mathscr{M}}$ induces a faithful exact functor from $^{\mathrm{ct}}\mathscr{M}(X)$ into the abelian category of constructible sheaves of $\mathbf{Q}$-vector spaces on X. Then, using the universal property of the category of constructible motives $\mathcal{M}(X, \mathbf{Q})$ constructed by Arapura in [3], we get a faithful exact functor $\mathcal{M}(X, \mathbf{Q}) \to {}^{\mathrm{ct}}\mathscr{M}(X)$. Is this functor an equivalence? If $X = \mathrm{Spec}\, k$, then both categories are equivalent to the abelian category of Nori motives, so this functor is an equivalence.

6. Weights

In this section, we will use results on motives and weight structures from Bondarko's and Hébert's papers [21, 41]. To apply these references directly in our context, we will make use of the fact that if S is a Noetherian scheme of finite dimension, then Ayoub's category $\mathbf{DA}_{\mathrm{ct}}(S)$ is canonically equivalent to the category of constructible Beilinson motives studied in Cisinski and Déglise's book [23]. This follows from [23, Theorem 16.2.18] and

will henceforth be used without further comment. (Note also that, though the authors of [21, 41] have chosen to use Beilinson's motives, étale motives could have been used.)

6.1. Continuity of the abelian hull

Remember that, in Neeman's book [62, Chapter 5], there are four constructions of the abelian hull of a triangulated category. The first one gives a lax 2-functor from the 2-category of triangulated categories to that of abelian categories, but the other three constructions give strict 2-functors. If we use the fourth construction, which Neeman calls $D(\mathcal{S})$ (see [62, Definition 5.2.1]), then the following proposition is immediate.

Proposition 6.1. *Let $\mathcal{S}$ be a triangulated category, and suppose that we have an equivalence of triangulated categories $\mathcal{S} \xrightarrow{\sim} 2\text{-}\varinjlim_{i \in I} \mathcal{S}_i$, where I is a small filtered category. Then the canonical functor*

$$\mathbf{A}^{\mathrm{tr}}(\mathcal{S}) \to 2\text{-}\varinjlim_{i \in I} \mathbf{A}^{\mathrm{tr}}(\mathcal{S}_i)$$

is an equivalence of abelian categories.

6.2. Étale realization and ℓ-adic perverse Nori motives

Let S be a Noetherian excellent scheme finite-dimensional scheme, and let ℓ be a prime number invertible over S; we assume that S is a $\mathbf{Q}$-scheme. (By [45, Exposé XVIII-A], the hypotheses above imply Hypothesis 5.1 in Ayoub [8].) Under this hypothesis, Ayoub has constructed an étale ℓ-adic realization functor on $\mathbf{DA}_{\mathrm{ct}}(S)$, compatible with pullbacks.

Theorem 6.2 (see [8, Section 5]). *Denote by $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(S, \mathbf{Q}_\ell)$ the category of constructible ℓ-adic complexes on S. Then we have a triangulated functor $\mathfrak{R}_S^{\text{ét}} : \mathbf{DA}_{\mathrm{ct}}(S) \to \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(S, \mathbf{Q}_\ell)$ for every S and, for every morphism $f : S \to S'$ with S' satisfying the same hypotheses as S, we have an invertible natural transformation*

$$\theta_f : f^* \circ \mathfrak{R}_Y^{\text{ét}} \to \mathfrak{R}_X^{\text{ét}} \circ f^*.$$

Using results of Gabber [36] (see also Fargues [33, Sections 4 and 5]), we can construct an abelian category $\mathscr{P}(S, \mathbf{Q}_\ell)$ of ℓ-adic perverse sheaves on S, satisfying all the usual properties. In particular, we get a perverse cohomology functor ${}^p H_\ell^0 := {}^p H^0 \circ \mathfrak{R}_S^{\text{ét}} : \mathbf{DA}_{\mathrm{ct}}(S) \to \mathscr{P}(S, \mathbf{Q}_\ell)$.

Definition 6.3. Let S be as above. The abelian category of ℓ-*adic perverse motives* on S is the abelian category

$$\mathscr{M}(S)_\ell := \mathbf{A}^{\mathrm{tr}}(\mathbf{DA}_{\mathrm{ct}}(S), {}^p H_\ell^0).$$

By construction, the functor ${}^p H_\ell^0$ has a factorization

$$\mathbf{DA}_{\mathrm{ct}}(S) \xrightarrow{{}^p H_{\mathscr{M}}^0} \mathscr{M}(S)_\ell \xrightarrow{\mathrm{rat}_{S,\ell}^{\mathscr{M}}} \mathscr{P}(S, \mathbf{Q}_\ell)$$

where $\mathrm{rat}_{S,\ell}^{\mathscr{M}}$ is a faithful exact functor and ${}^p H_{\mathscr{M}}^0$ is a homological functor.

By the universal property of $\mathcal{M}(S)_\ell$, we also get pullback functors between these categories as soon as the pullback functor between the categories of ℓ-adic complexes preserves the category of perverse sheaves.

We will use the following important fact: If we fix a base field k of characteristic zero and only consider schemes that are quasi-projective over k, then the main theorem (Section 5.1) stays true for the categories $\mathcal{M}(S)_\ell$. Of course, we have to replace $\mathrm{D}^{\mathrm{b}}_{\mathrm{ct}}(S)$ and the Betti realization functor by $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S, \mathbf{Q}_\ell)$ and the étale realization functor in all the statements. Indeed, the proof of the main theorem, and of the statements that it uses, still works if we use the ℓ-adic étale realization instead of the Betti realization. The only result that requires a slightly different proof is Lemma 3.13: we have to show that the ℓ-adic realization of Ayoub's logarithmic motive $\mathscr{L}og_n^\vee$ is the local system used in Beilinson's construction of the unipotent nearby cycle functor (see Beilinson [16, Sections 1.1 and 1.2] or [60, Definition 5.2.1]). As in the proof of Lemma 3.13, it suffices to check this for $n = 1$, and then it follows from [8, Lemma 11.22].

6.3. Mixed horizontal perverse sheaves

Let k be a field and S be a k-scheme of finite type. Suppose that k is finitely generated over its prime field. We also fix a prime number ℓ invertible over S. The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(S, \mathbf{Q}_\ell)$ of mixed horizontal $\mathbf{Q}_\ell$-complexes and its perverse t-structure with heart $\mathscr{P}_{\mathrm{m}}(S, \mathbf{Q}_\ell)$ (the category of mixed horizontal ℓ-adic perverse sheaves on S) were constructed by Huber [42] (see also [60, Section 2]). We recall the definition briefly and refer to [42, 60] for the details. First we consider the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(S, \mathbf{Q}_\ell)$ of horizontal complexes on S, which is by definition the 2-colimit of the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}, \mathbf{Q}_\ell)$ where $\mathscr{X}$ runs over all flat finite type models of X over regular subalgebras A of k that are of finite type over $\mathbf{Z}$ and have k as their fraction field. There is an obvious functor $\eta^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(S, \mathbf{Q}_\ell) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S, \mathbf{Q}_\ell)$, which is triangulated and conservative, and a perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(S, \mathbf{Q}_\ell)$ that is characterized by the fact that η^* is t-exact. Also, the functor η^* is fully faithful on the heart of this t-structure [60, Proposition 2.6.2].

We say that an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(S, \mathbf{Q}_\ell)$ is *mixed* if it extends to a complex K on a model $\mathscr{X}$ of X as before such that all the (ordinary) cohomology sheaves of K are successive extensions of punctually pure sheaves in the sense of Deligne [27]. The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(S, \mathbf{Q}_\ell)$ of mixed horizontal complexes is the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(S, \mathbf{Q}_\ell)$ whose objects are mixed complexes. The perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(S, \mathbf{Q}_\ell)$ restricts to a t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(S, \mathbf{Q}_\ell)$, whose heart is the category $\mathscr{P}_{\mathrm{m}}(S, \mathbf{Q}_\ell)$ of mixed horizontal perverse sheaves; this last category is a full subcategory of the heart of the perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(S, \mathbf{Q}_\ell)$, so η^* induces a fully faithful functor $\mathscr{P}_{\mathrm{m}}(S, \mathbf{Q}_\ell) \to \mathscr{P}(S, \mathbf{Q}_\ell)$.

Now we want to show that the realization functor $\mathrm{rat}^{\mathcal{M}}_{S,\ell} : \mathcal{M}(S)_\ell \to \mathscr{P}(S, \mathbf{Q}_\ell)$ factors through the fully faithful functor $\mathscr{P}_{\mathrm{m}}(S, \mathbf{Q}_\ell) \to \mathscr{P}(S, \mathbf{Q}_\ell)$.

We have a continuity theorem for the categories of étale motives, proved by Ayoub [12, Corollaire 1.A.3], [8, Corollaire 3.22] and by Cisinski–Déglise [23, Proposition 15.1.6]).

Theorem 6.4. *Let S be a Noetherian scheme of finite dimension. Suppose that we have $S = \varprojlim_{i \in I} S_i$, where all the S_i are finite-dimensional Noetherian schemes and all the transition maps $S_i \to S_j$ are affine. Then the canonical functor $2\text{-}\varinjlim_{i \in I} \mathbf{DA}_{ct}(S_i) \to \mathbf{DA}_{ct}(S)$ is an equivalence of monoidal triangulated categories.*

Using the definition of mixed horizontal ℓ-adic complexes, we immediately get the following corollary.

Corollary 6.5. *Let S and ℓ be as at the beginning of this subsection. Then the étale realization functor $\mathbf{DA}_{ct}(S) \to D_c^b(S, \mathbf{Q}_\ell)$ factors through a functor $\mathbf{DA}_{ct}(S) \to D_h^b(S, \mathbf{Q}_\ell)$.*

Corollary 6.6. *With the notation of the previous corollary, the essential image of the functor $\mathbf{DA}_{ct}(S) \to D_h^b(S, \mathbf{Q}_\ell)$ is contained in the full subcategory $D_m^b(S, \mathbf{Q}_\ell)$. In particular, the perverse cohomology functor ${}^p H_\ell^0 : \mathbf{DA}_{ct}(S) \to \mathscr{P}(S, \mathbf{Q}_\ell)$ factors through the subcategory $\mathscr{P}_m(S, \mathbf{Q}_\ell)$.*

Proof. This follows from the facts that $\mathbf{DA}_{ct}(S)$ is generated by the Tate twists of motives of smooth S-schemes (see [23, Definition 15.1.1 and Proposition 15.1.4]) and that mixed horizontal complexes are preserved by direct images and Tate twists (see [42, Proposition 3.2] for direct images; the stability by Tate twists is easy). ∎

Corollary 6.7. *The essential image of the realization functor $\mathrm{rat}_{S,\ell}^{\mathscr{M}} : \mathscr{M}(S)_\ell \to \mathscr{P}(S, \mathbf{Q}_\ell)$ is contained in the subcategory $\mathscr{P}_m(S, \mathbf{Q}_\ell)$.*

We will also denote the resulting faithful exact functor $\mathscr{M}(S) \to \mathscr{P}_m(S, \mathbf{Q}_\ell)$ by $\mathrm{rat}_{S,\ell}^{\mathscr{M}}$.

Remark 6.8. Suppose that k is not necessarily finitely generated over its prime field. We define $D_m^b(S, \mathbf{Q}_\ell)$ as the 2-colimit of the categories $D_m^b(S', \mathbf{Q}_\ell)$ for S' a model of S over a finitely generated subfield of k. This category inherits a perverse t-structure from the perverse t-structures on the $D_m^b(S', \mathbf{Q}_\ell)$, whose heart we denote by $\mathscr{P}_m(S, \mathbf{Q}_\ell)$. The obvious functor $\mathscr{P}_m(S, \mathbf{Q}_\ell) \to \mathscr{P}(S, \mathbf{Q}_\ell)$ is only exact faithful in general (not necessarily fully faithful), but the perverse cohomology functor ${}^p H_\ell^0 : \mathbf{DA}_{ct}(S) \to \mathscr{P}(S, \mathbf{Q}_\ell)$ still factors through this functor as in Corollary 6.6 (by Theorem 6.4), so we get a faithful exact realization functor $\mathscr{M}(S)_\ell \to \mathscr{P}_m(S, \mathbf{Q}_\ell)$.

6.4. Continuity for perverse Nori motives

Like the triangulated category of motives, the category of perverse Nori motives satisfies a continuity property.

Proposition 6.9. *Let S be a scheme and ℓ be a prime number satisfying the conditions of Section 6.2. Assume that $S = \varprojlim_{i \in I} S_i$, where $(S_i)_{i \in I}$ is a directed projective system of schemes satisfying the same conditions as S, and in which the transition maps are affine. Also assume that the pullback by any transition map $S_i \to S_j$ preserves the category of perverse sheaves, and that there exists $a \in \mathbf{Z}$ such that if $f_i : S \to S_i$ is the canonical map, then $f_i^*[a]$ preserves the category of perverse sheaves for every $i \in I$. Under these*

hypotheses, the functors $f_i^*[a]$ induce a functor

$$2\text{-}\varinjlim_{i \in I} \mathscr{M}(S_i)_\ell \to \mathscr{M}(S)_\ell,$$

and this functor is full and essentially surjective.

If moreover the canonical exact functor $2\text{-}\varinjlim_{i \in I} \mathscr{P}(S_i, \mathbf{Q}_\ell) \to \mathscr{P}(S, \mathbf{Q}_\ell)$ induced by the $f_i^*[a]$ is faithful, then the canonical functor

$$2\text{-}\varinjlim_{i \in I} \mathscr{M}(S_i)_\ell \to \mathscr{M}(S)_\ell$$

is an equivalence of abelian categories.

Proof. This follows from Proposition 6.1 and Theorem 6.4. ∎

Corollary 6.10. *Let S and ℓ be as above, and suppose that S is integral. If η is the generic point of S, the canonical exact functor*

$$2\text{-}\varinjlim_{U} \mathscr{M}(U)_\ell \to \mathscr{M}(\eta)_\ell,$$

where the colimit is taken over all nonempty affine open subschemes of S and where the image of $K_U \in \mathrm{Ob}\, \mathscr{M}(U)_\ell$ is $K_{U,\eta}[-\dim S]$, is an equivalence of categories.

Proof. By Proposition 6.9, it suffices to check that the similar functor

$$2\text{-}\varinjlim_{U} \mathscr{P}(U, \mathbf{Q}_\ell) \to \mathscr{P}(\eta, \mathbf{Q}_\ell)$$

is faithful. Let K be an object of $2\text{-}\varinjlim_{U} \mathscr{P}(U, \mathbf{Q}_\ell)$ whose image in $\mathscr{P}(\eta, \mathbf{Q}_\ell)$ is 0, and let U be a nonempty open affine subscheme of S such that K comes from an object K' of $\mathscr{P}(U, \mathbf{Q}_\ell)$. After shrinking U (which does not change K), we may assume that $K'[-\dim S]$ is a local system. Then the condition $K'_\eta[-\dim S] = 0$ implies that this local system is zero, so $K = 0$. ∎

6.5. Comparison of the different categories of perverse Nori motives

In the next proposition, we compare the ℓ-adic definition of perverse motives with the one used previously and obtained via the Betti realization.

Proposition 6.11. *Suppose that k is a field of characteristic zero and that S is quasi-projective over k. Write ρ_ℓ for the canonical exact functor $\mathbf{A}^{\mathrm{tr}}(\mathbf{DA}_{\mathrm{ct}}(S)) \to \mathscr{P}(S, \mathbf{Q}_\ell)$ induced by ${}^p H_\ell^0$. If σ is an embedding of k into $\mathbf{C}$, then we also have an exact functor ρ_σ from $\mathbf{A}^{\mathrm{tr}}(\mathbf{DA}_{\mathrm{ct}}(S))$ to $\mathscr{P}(S)$ induced by ${}^p H^0$. Then:*

(1) *If ℓ, ℓ' are two prime numbers, then $\mathrm{Ker}\,\rho_\ell = \mathrm{Ker}\,\rho_{\ell'}$. In particular, we get a canonical equivalence of abelian categories $\mathscr{M}(S)_\ell = \mathscr{M}(S)_{\ell'}$.*

(2) *If $\sigma : k \to \mathbf{C}$ is an embedding, then* $\operatorname{Ker} \rho_\ell = \operatorname{Ker} \rho_\sigma$. *In particular, we get a canonical equivalence of abelian categories* $\mathscr{M}(S)_\ell = \mathscr{M}(S)$.

Proof. We first treat the case $S = \operatorname{Spec} k$. If k can be embedded in $\mathbf{C}$, then (2) follows from Huber's construction of mixed realizations [43], and (1) follows from (2). In the general case, (1) follows from the case where k can be embedded in $\mathbf{C}$ and from Proposition 6.9, applied to the family of subfields of k that can be embedded in $\mathbf{C}$.

Now we treat the case of a general k-scheme S. As in the first case, (1) follows from (2) and from Proposition 6.9. So suppose that we have an embedding $\sigma : k \to \mathbf{C}$. We prove the desired result by induction on the dimension of S. The case $\dim S = 0$ has already been treated, so we may assume that $\dim S > 0$ and the result is known for all schemes of lower dimension. We denote by $M \mapsto [M]$ the canonical functor $\mathbf{DA}_{\mathrm{ct}}(S) \to \mathbf{A}^{\mathrm{tr}}(\mathbf{DA}_{\mathrm{ct}}(S))$; as $\mathbf{DA}_{\mathrm{ct}}(S)$ is a triangulated category, this is a fully faithful functor. Let X be an object of $\mathbf{A}^{\mathrm{tr}}(\mathbf{DA}_{\mathrm{ct}}(S))$. By construction of $\mathbf{A}^{\mathrm{tr}}(\mathbf{DA}_{\mathrm{ct}}(S))$, there exists a morphism $N \to M$ in $\mathbf{DA}_{\mathrm{ct}}(S)$ such that X is the cokernel of $[N] \to [M]$. Then $\rho_\ell(X)$ is the cokernel of ${}^p H_\ell^0(N) \to {}^p H_\ell^0(M)$, so $\rho_\ell(X) = 0$ if and only if ${}^p H_\ell^0(N) \to {}^p H_\ell^0(M)$ is surjective. Similarly, $\rho_\sigma(X) = 0$ if and only if ${}^p H^0(N) \to {}^p H^0(M)$ is surjective. We can check these conditions on a Zariski open covering of S, so we may assume that S is affine. Choose a nonempty smooth open subset U of S such that the restrictions to U of $\rho_\ell(M)$, $\rho_\ell(N)$, $\rho_\sigma(M)$ and $\rho_\sigma(N)$ are all locally constant sheaves placed in degree $-\dim S$. As S is affine, after shrinking U we may assume that U is the complement of the vanishing set of a nonzero function $f \in \mathcal{O}(S)$. By [16, Proposition 3.1], $\rho_\ell(N) \to \rho_\ell(M)$ is surjective if and only if both $\rho_\ell(N)_{|U} \to \rho_\ell(M)_{|U}$ and ${}^p \Phi_f^{\mathscr{M}} \rho_\ell(N) \to {}^p \Phi_f^{\mathscr{M}} \rho_\ell(M)$ are, which is equivalent to the surjectivity of $\rho_\ell(N_{|U}) \to \rho_\ell(M_{|U})$ and $\rho_\ell(\Phi_f N) \to \rho_\ell(\Phi_f M)$. We have a similar statement for ρ_σ. As $\dim(S - U) < \dim S$, we can use the induction hypothesis to reduce to the case $S = U$. It suffices to check the result on an étale cover of S, so we may assume that S has a rational point x. Let $i : x \to S$ be the obvious inclusion. As $\rho_\ell(N)[-\dim S]$ and $\rho_\ell(M)[-\dim S]$ are locally constant sheaves on S, the morphism $\rho_\ell(N) \to \rho_\ell(M)$ is surjective if and only if $\rho_\ell(i^* N[-\dim S]) \to \rho_\ell(i^* M[-\dim S])$ is, and similarly for ρ_σ. So we are reduced to the result on the scheme x, which we have already treated. ∎

Corollary 6.12. *Let k be a field of characteristic zero and S a quasi-projective scheme over k. We have a canonical $\mathbf{Q}$-linear abelian category $\mathscr{M}(S)$ of perverse Nori motives, together with a cohomological functor ${}^p H^0_{\mathscr{M}} : \mathbf{DA}_{\mathrm{ct}}(S) \to \mathscr{M}(S)$, with a ℓ-adic realization functor $\mathrm{rat}_{S,\ell}^{\mathscr{M}} : \mathscr{M}(S) \to \mathscr{P}(S, \mathbf{Q}_\ell)$ for every prime number ℓ, with a Betti realization functor $\mathrm{rat}_{S,\sigma}^{\mathscr{M}} : \mathscr{M}(S) \to \mathscr{P}(S)$ for every embedding $\sigma : k \to \mathbf{C}$, and it has a formalism of the four operations, duality, unipotent nearby and vanishing cycles compatible with all these operations.*

We fix a field k of characteristic zero and a quasi-projective scheme S over k. We first define weights via the ℓ-adic realizations.

Definition 6.13. Let $w \in \mathbf{Z}$. Let K be an object of $\mathscr{M}(S)$. We say that K is *of weight* $\leq w$ (resp. $\geq w$) if $\mathrm{rat}_{S,\ell}^{\mathscr{M}}(K) \in \mathrm{Ob}(\mathscr{P}_{\mathrm{m}}(S, \mathbf{Q}_\ell))$ is of weight $\leq w$ (resp. $\geq w$) for every prime number ℓ. We say that K is *pure of weight* w if it is both of weight $\leq w$ and of weight $\geq w$.

In Proposition 6.18, we will give a more intrinsic definition of weights that does not use the realization functors.

Definition 6.14. A *weight filtration* on an object K of $\mathscr{M}(S)$ is an increasing filtration $W_\bullet K$ on K such that $W_i K = 0$ for i small enough, $W_i K = K$ for i large enough, and $W_i K / W_{i-1} K$ is pure of weight i for every $i \in \mathbf{Z}$.

The next result follows immediately from the similar result in the categories of mixed horizontal perverse sheaves (see Huber [42, Proposition 3.4 and Lemma 3.8]).

Proposition 6.15. *Let K, L be objects of $\mathscr{M}(S)$, and let $w \in \mathbf{Z}$.*

(1) *If K is of weight $\leq w$ (resp. $\geq w$), so is every subquotient of K.*

(2) *If K is of weight $\leq w$ and L is of weight $\geq w + 1$, then $\mathrm{Hom}_{\mathscr{M}(S)}(K, L) = 0$.*

Recall that if A and B are objects of an abelian category endowed with increasing filtrations $(F_i A)_{i \in \mathbf{Z}}$ and $(F_i B)_{i \in \mathbf{Z}}$, then a morphism $u : A \to B$ is called *compatible* (resp. *strictly compatible*) with the filtrations if, for every $i \in \mathbf{Z}$, we have $u(F_i A) \subset F_i B$ (resp. $u(F_i A) = u(A) \cap F_i B$).

Corollary 6.16. *A weight filtration on an object of $\mathscr{M}(S)$ is unique if it exists, and morphisms of $\mathscr{M}(S)$ are strictly compatible with weight filtrations. In particular, if an object of $\mathscr{M}(S)$ has a weight filtration, then so do all its subquotients.*

6.6. Application of Bondarko's weight structures

Let S be as in the previous subsection. We will now make use of Bondarko's Chow weight structure on $\mathbf{DA}_{\mathrm{ct}}(S)$. Let $\mathrm{Chow}(S)$ be the full subcategory of $\mathbf{DA}_{\mathrm{ct}}(S)$ whose objects are direct factors of finite direct sums of objects of the form $f_! \mathbf{Q}_X(d)[2d]$, with $f : X \to S$ a proper morphism from a smooth k-scheme X to S and $d \in \mathbf{Z}$. Then, as shown by Hébert [41, Theorem 3.3], and also by Bondarko [21, Theorem 2.1.1], there exists a unique weight structure on $\mathbf{DA}_{\mathrm{ct}}(S)$ with heart $\mathrm{Chow}(S)$ (see [41, Definition 1.5] or [21, Definition 1.2.1] for the definition of a weight structure).

In particular, for every object K of $\mathbf{DA}_{\mathrm{ct}}(S)$, there exists an exact triangle $A \to K \to B \xrightarrow{+1}$ (not unique) such that A (resp. B) is a direct factor of a successive extension of objects of $\mathrm{Chow}(S)[i]$ with $i \leq 0$ (resp. $i \geq 1$).

Proposition 6.17. *Every object of $\mathscr{M}(S)$ has a weight filtration. Moreover, if $S = \mathrm{Spec}\, k$ and σ is an embedding of k in $\mathbf{C}$, then the notion of weights of Definition 6.13 coincides with that of [44, Section 10.2.2].*

Proof. We first prove that if M is an object of Chow(S), then ${}^{p}H^{0}_{\mathscr{M}}(M)$ is pure of weight 0 in our sense, and also in the sense of [44, Section 10.2.2] if $S = \operatorname{Spec} k$ with k embeddable in **C**. The second statement is actually an immediate consequence of [44, Definition 10.2.4] and of the motivic Chow lemma (see for example [41, Lemma 3.1]). To prove the first statement, by definition of the weights on $\mathscr{P}_{m}(S, \mathbf{Q}_{\ell})$, we may assume that k is finitely generated over **Q**; then the statement follows immediately from [18, Stabilités 5.1.14] (see [42, first remark on p. 116]).

Then we note that every object of $\mathscr{M}(S)$ is a quotient of an object of the form ${}^{p}H^{0}_{\mathscr{M}}(M)$ for $M \in \operatorname{Ob}(\mathbf{DA}_{ct}(S))$ (because this is true for objects of $\mathbf{A}^{tr}(\mathbf{DA}_{ct}(S))$). So it suffices to prove the result for objects in the essential image of ${}^{p}H^{0}_{\mathscr{M}}$. Let $M \in \operatorname{Ob}(\mathbf{DA}_{ct}(S))$, and let $K = {}^{p}H^{0}_{\mathscr{M}}(M)$. Let $w \in \mathbf{Z}$. By the first part of the proof, if M is a direct factor of a successive extension of objects of Chow(S)$[i]$ with $i \leq w$ (resp. $i \geq w + 1$), then ${}^{p}H^{0}_{\mathscr{M}}(M)$ is of weight $\leq w$ (resp. $\geq w + 1$) in our sense, and also in the sense of [44] when this applies. In general, using the Chow weight structure of Bondarko, we can find an exact triangle $A \to M \to B \xrightarrow{+1}$ such that A (resp. B) is a direct factor of a successive extension of objects of Chow(S)$[i]$ with $i \leq w$ (resp $i \geq w + 1$). Applying ${}^{p}H^{0}_{\mathscr{M}}$, we get an exact sequence ${}^{p}H^{0}_{\mathscr{M}}(A) \to K \to {}^{p}H^{0}_{\mathscr{M}}(B)$, with ${}^{p}H^{0}_{\mathscr{M}}(A)$ of weight $\leq w$ and ${}^{p}H^{0}_{\mathscr{M}}(B)$ of weight $\geq w + 1$. If we set $W_{w}K = \operatorname{Im}({}^{p}H^{0}_{\mathscr{M}}(A) \to K)$, then $W_{w}K$ is of weight $\leq w$ and $K/W_{w}K$ is of weight $\geq w + 1$. This defines a weight filtration on K. ∎

Weights and the related weight filtration so far have been defined and constructed for perverse motives via the ℓ-adic realizations. As we shall see now, we can also define weights more directly. Let $\mathbf{DA}_{ct}(S)_{w \leq i}$ be the full subcategory of $\mathbf{DA}_{ct}(S)$ whose objects are direct factors of successive extensions of objects of Chow(S)$[w]$ with $w \leq i$, and consider the abelian category

$$\mathscr{M}(S)_{w \leq i} := \mathbf{A}^{ad}(\mathbf{DA}_{ct}(S)_{w \leq i}, {}^{p}H^{0}_{\ell})$$

for some prime number ℓ. It follows from Proposition 6.11 that this category, up to an equivalence, does not depend on ℓ. Indeed, the universal property provides a commutative diagram (up to isomorphisms of functors)

where I is the inclusion and J, ϱ_{ℓ} are exact functors. As by construction $\mathscr{M}(S)_{w \leq i} := \mathbf{A}^{ad}(\mathbf{DA}_{ct}(S)_{w \leq i})/\operatorname{Ker} \varrho_{\ell}$, it suffices to show that $\operatorname{Ker} \varrho_{\ell}$ is independent of ℓ. Let A be an object in $\mathbf{A}^{ad}(\mathbf{DA}_{ct}(S)_{w \leq i})$. Since A belongs to $\operatorname{Ker} \varrho_{\ell}$ if and only if $J(A)$ belongs to $\operatorname{Ker} \rho_{\ell}$, our claim follows from Proposition 6.11.

The inclusion $\mathbf{DA}_{\mathrm{ct}}(S)_{w\leq i} \subseteq \mathbf{DA}_{\mathrm{ct}}(S)$ induces a faithful exact functor

$$u_i : \mathscr{M}(S)_{w\leq i} \to \mathscr{M}(S).$$

Let K be an object in $\mathscr{M}(S)$. Given an object $(L, \alpha : u_i(L) \to K)$ in the slice category $\mathscr{M}(S)_{w\leq i}/K$, we can consider the subobject $\mathrm{Im}\,\alpha$ of K and define $\mathcal{W}_i K$ to be the union of all such subobjects in K, that is, we set

$$\mathcal{W}_i K := \operatorname*{colim}_{(L,\alpha)\in\mathscr{M}(S)_{w\leq i}/K} \mathrm{Im}\,\alpha.$$

This construction is functorial in K (and moreover using the inclusion of $\mathbf{DA}_{\mathrm{ct}}(S)_{w\leq i}$ in $\mathbf{DA}_{\mathrm{ct}}(S)_{w\leq i+1}$ it is easy to see that it defines a filtration on K).

Proposition 6.18. *Let $K \in \mathscr{M}(S)$. Then $W_i K = \mathcal{W}_i K$ for every integer $i \in \mathbf{Z}$.*

Proof. As observed in the proof of Proposition 6.17, if M belongs to $\mathbf{DA}_{\mathrm{ct}}(S)_{w\leq i}$, then $^p H^0_{\mathscr{M}}(M)$ is of weight $\leq i$. Hence, the functor u_i takes its values in the abelian subcategory of $\mathscr{M}(S)$ formed by the objects of weight $\leq i$. As a consequence, for (L, α) in the slice category $\mathscr{M}(S)_{w\leq i}/K$, the subobject $\mathrm{Im}\,\alpha$ of K is of weight $\leq i$ and therefore $\mathcal{W}_i K \subseteq W_i K$.

Conversely, there exists an epimorphism $e : {}^p H^0_{\mathscr{M}}(M) \twoheadrightarrow K$ where M belongs to $\mathbf{DA}_{\mathrm{ct}}(S)$. By construction

$$W_i\, {}^p H^0_{\mathscr{M}}(M) := \mathrm{Im}\big({}^p H^0_{\mathscr{M}}(A) \to {}^p H^0_{\mathscr{M}}(M)\big) \subseteq \mathcal{W}_i\, {}^p H^0_{\mathscr{M}}(M)$$

where A is an object of $\mathbf{DA}_{\mathrm{ct}}(S)_{w\leq i}$ that fits in an exact triangle $A \to M \to B \xrightarrow{+1}$ such that B is a direct factor of a successive extension of objects of $\mathrm{Chow}(S)[w]$ with $w \geq i+1$. Therefore, since the weight filtration on K is the induced filtration (see Corollary 6.16), we get

$$W_i K = e(W_i\, {}^p H^0_{\mathscr{M}}(M)) \subseteq e(\mathcal{W}_i\, {}^p H^0_{\mathscr{M}}(M)) \subseteq \mathcal{W}_i K.$$

This concludes the proof. ∎

6.7. The intermediate extension functor

Recall the definition of the intermediate extension functor, which already appeared in the proof of Corollary 5.7.

Definition 6.19. Let $j : S \to T$ be a quasi-finite morphism between quasi-projective k-schemes. We define a functor $j_{!*}^{\mathscr{M}} : \mathscr{M}(S) \to \mathscr{M}(T)$ by

$$j_{!*}^{\mathscr{M}}(K) = \mathrm{Im}(\mathrm{H}^0(j_!^{\mathscr{M}} K) \to \mathrm{H}^0(j_*^{\mathscr{M}} K)).$$

Note that, as j is quasi-finite, the functor $j_!^{\mathscr{M}}$ is right exact and the functor $j_*^{\mathscr{M}}$ is left exact. In particular, the functor $j_{!*}^{\mathscr{M}}$ preserves injective and surjective morphisms, but it is not exact in general.

Proposition 6.20. *Let $j : S \to T$ be an open immersion, and let $w \in \mathbf{Z}$. If $K \in \operatorname{Ob} \mathcal{M}(S)$ is of weight $\leq w$ (resp. of weight $\geq w$, resp. pure of weight w), so is $j_{!*}^{\mathcal{M}} K$.*

Also, the functor $j_{!*}^{\mathcal{M}}$ is exact on the full abelian subcategory of objects that are pure of weight w.

Proof. It suffices to show these statement for mixed ℓ-adic perverse sheaves. The first statement follows from [18, Corollaire 5.3.2] (more precisely, if j is not affine, it follows from [18, Stabilités 5.1.14 and Proposition 5.3.1]). The second statement follows from [60, Corollary 9.4]. ∎

6.8. Pure objects

Let us start with the definition of objects with strict support on a given closed subscheme.

Definition 6.21. Let Z be a closed integral subscheme of S, and denote the immersion $Z \to S$ by i. We say that an object K of $\mathcal{M}(S)$ *has strict support* Z if $K_{|S-Z} = 0$ and, for every nonempty open subset $j : U \to Z$, the adjunction morphism $K \to (ij)_*^{\mathcal{M}} (ij)_{\mathcal{M}}^* K$ is injective and induces an isomorphism between K and $(ij)_{!*}^{\mathcal{M}} (ij)_{\mathcal{M}}^* K$.

Remark 6.22. For example, if $K_{|S-Z} = 0$ and if there exists a smooth dense open subset $j : U \to Z$ such that $\operatorname{rat}_U^{\mathcal{M}} (K_{|U})[-\dim U]$ (or any $\operatorname{rat}_{U,\ell}^{\mathcal{M}}(K_{|U})[-\dim U]$ for some prime number ℓ) is locally constant and $K_Z = j_{!*}^{\mathcal{M}}(K_{|U})$, then K has strict support Z. Indeed, this follows from the similar result for perverse sheaves, which follows from [18, Lemme 4.3.2] (note that the proof of this result does not use the hypothesis that L is irreducible).

Proposition 6.23 (cf. [18, Théorème 5.3.8]). *Let K be an object of $\mathcal{M}(S)$, and suppose that K is pure of some weight. Then we can write $K = \bigoplus_Z K_Z$, where the sum is over all integral closed subschemes Z of S, each K_Z is an object of $\mathcal{M}(S)$ with strict support Z, and $K_Z = 0$ for all but finitely many Z.*

Proof. We prove the result by Noetherian induction on S. If $\dim S = 0$, there is nothing to prove. Suppose that $\dim S \geq 1$, and let $j : U \to S$ be a nonempty open affine subset of S. After shrinking U, we may assume that U is smooth and that $\operatorname{rat}_S^{\mathcal{M}}(K)[-\dim U]$ is a locally constant sheaf on U. Let w be the weight of K. Then [60, Corollary 9.4] implies that $j_*^{\mathcal{M}} j_{\mathcal{M}}^* K / j_{!*}^{\mathcal{M}} j_{\mathcal{M}}^* K$ is of weight $\geq w + 1$, so the adjunction morphism $K \to j_*^{\mathcal{M}} j_{\mathcal{M}}^* K$ factors through a morphism $K \to j_{!*}^{\mathcal{M}} j_{\mathcal{M}}^* K$. Similarly, the adjunction morphism $j_!^{\mathcal{M}} j_{\mathcal{M}}^* K \to K$ factors through a morphism $j_{!*}^{\mathcal{M}} j_{\mathcal{M}}^* \to K$. By definition of $j_{!*}^{\mathcal{M}}$, the composition $j_{!*}^{\mathcal{M}} j_{\mathcal{M}}^* K \to K \to j_{!*}^{\mathcal{M}} j_{\mathcal{M}}^* K$ is the identity of $j_{!*}^{\mathcal{M}} j_{\mathcal{M}}^* K$. So we have $K = j_{!*}^{\mathcal{M}} j_{\mathcal{M}}^* K \oplus L$, with $j_{\mathcal{M}}^* L = 0$. The first summand has strict support $\overline{U}$ by the remark above, and $L_{|U} = 0$, so the conclusion follows from the induction hypothesis applied to $L_{|S-U}$. ∎

Theorem 6.24. *Let S be as before, and let $w \in \mathbf{Z}$. Let $\mathcal{M}(S)_w$ be the full abelian subcategory of $\mathcal{M}(S)$ whose objects are motives that are pure of weight w. Then $\mathcal{M}(S)_w$ is semisimple.*

Proof. By Proposition 6.23, we may assume that S is integral, and it suffices to prove the result for the full subcategory $\mathscr{M}(S)_w^0$ of objects in $\mathscr{M}(S)_w$ with strict support S itself.

Let η be the generic point of S. By Corollary 6.10, we have a full and essentially surjective exact functor (given by the restriction morphisms) $2\text{-}\varinjlim_U \mathscr{M}(U) \to \mathscr{M}(\eta)$, where the limit is over the projective system of nonempty affine open subsets U of S. For such a U, we denote by $\mathscr{M}(U)_w^0$ the full subcategory of $\mathscr{M}(U)$ whose objects are motives that are pure of weight w and have strict support U. By Proposition 6.17, the functor above induces a full and essentially surjective functor $2\text{-}\varinjlim_U \mathscr{M}(U)_w \to \mathscr{M}(\eta)_w$, and by Proposition 6.23 this is turn gives a full and essentially surjective functor $2\text{-}\varinjlim_U \mathscr{M}(U)_w^0 \to \mathscr{M}(\eta)_w$. Moreover, if $j : U \to S$ is a nonempty open subset, then the exact functor $j_{\mathscr{M}}^* : \mathscr{M}(S)_w^0 \to \mathscr{M}(U)_w^0$ is an equivalence of categories, because it has a quasi-inverse, given by $j_{!*}^{\mathscr{M}}$. So we deduce that the restriction functor $\mathscr{M}(S)_w^0 \to \mathscr{M}(\eta)_w$ is full and essentially surjective. But this functor is also faithful, because the analogous functor on categories of ℓ-adic perverse sheaves is faithful. So $\mathscr{M}(S)_w^0 \to \mathscr{M}(\eta)_w$ is an equivalence of categories, which means that we just need to show the theorem in the case $S = \eta$, i.e., if S is the spectrum of a field.

Now suppose that $S = \operatorname{Spec} k$. Then, by Proposition 6.9, $\mathscr{M}(k)_w = 2\text{-}\varinjlim_{k'} \mathscr{M}(k')_w$, where the limit is over all the subfields k' of k that are finitely generated over $\mathbf{Q}$. So it suffices to show the theorem for k finitely generated over $\mathbf{Q}$. But then we can embed k into $\mathbf{C}$, and the conclusion follows from [44, Theorem 10.2.7]. ∎

Definition 6.25. Let K be an object of $\mathrm{D}^{\mathrm{b}}\mathscr{M}(X)$ and $w \in \mathbf{Z}$. We say that K is *of weight* $\leq w$ (resp. *of weight* $\geq w$, resp. *pure of weight* w) if, for every $i \in \mathbf{Z}$, the perverse motive $\mathsf{H}^i K$ is of weight $\leq w + i$ (resp. of weight $\geq w + i$, resp. pure of weight $w + i$).

Corollary 6.26. *Let* K, L *be objects of* $\mathscr{M}(S)$. *If* K *and* L *are pure of respective weights* i *and* j, *then* $\operatorname{Ext}^r_{\mathscr{M}(S)}(A, B) = 0$ *if* $i < j + r$.

Proof. By Lemma 4.5 of Saito [68], this follows from the existence of the weight filtration and the fact that it is strictly compatible with morphisms of $\mathscr{M}(S)$, and from the semisimplicity of pure objects of $\mathscr{M}(S)$. ∎

Corollary 6.27. (1) *There exists a unique weight structure (see* [21, *Definition* 1.2.1]) *on* $\mathrm{D}^{\mathrm{b}}\mathscr{M}(S)$ *whose heart is the full subcategory of complexes of weight* 0.

(2) *Let* K, L *be objects of* $\mathrm{D}^{\mathrm{b}}\mathscr{M}(S)$ *and* $w \in \mathbf{Z}$. *If* K *is of weight* $\leq w$ *and* L *is of weight* $> w$, *then* $\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}\mathscr{M}(S)}(K, L) = 0$.

(3) *The weight structure of* (1) *is transversal to the canonical* t-*structure on* $\mathrm{D}^{\mathrm{b}}\mathscr{M}(S)$ *in the sense of Bondarko* [20, *Definition* 1.2.2].

(4) *If* $K \in \operatorname{Ob}\mathrm{D}^{\mathrm{b}}\mathscr{M}(S)$ *is pure of some weight, then* $K \simeq \bigoplus_{i \in \mathbf{Z}} \mathsf{H}^i K[-i]$.

Proof. To prove (1), we apply [19, part II of Theorem 4.3.2] to the triangulated category $\mathrm{D}^{\mathrm{b}}\mathscr{M}(S)$ and the full subcategory $\mathscr{A}$ of complexes of weight 0. This subcategory is stable by finite coproducts and direct summands, and it generates $\mathrm{D}^{\mathrm{b}}\mathscr{M}(S)$. Indeed, to prove the second statement, it suffices to show that the triangulated subcategory generated by $\mathscr{A}$

contains $\mathscr{P}(S)$; but every perverse motive is a successive extension of pure perverse motives (by the existence of the weight filtration), and if K is a pure perverse motive, then some shift of K is an $\mathscr{A}$. By [19, Theorem 4.3.2], there exists a weight structure on $\mathrm{D}^b\mathscr{M}(S)$ with heart $\mathscr{A}$ if and only if, for any objects K, L of $\mathscr{A}$ and every integer $n > 0$, we have $\mathrm{Hom}_{\mathrm{D}^b\mathscr{M}(S)}(K, L) = 0$. As the functor Hom is cohomological in each variable, we may assume that K and L are concentrated in one degree, so that there exist objects A and B that are pure of respective weights i and j such that $K = A[-i]$ and $L = B[-j]$. Then $\mathrm{Hom}_{\mathrm{D}^b\mathscr{M}(S)}(K, L[n]) = \mathrm{Ext}^{n+i-j}_{\mathscr{M}(S)}(A, B)$ is zero by Corollary 6.26.

We prove (2). We have $\mathrm{Hom}_{\mathrm{D}^b\mathscr{M}(S)}(K, L) = \mathrm{Hom}_{\mathrm{D}^b\mathscr{M}(S)}(K[-w], L[-w])$. As $K[-w]$ is of weight ≤ 0 and $L[-w]$ is of weight ≥ 1, the statement follows from [19, Proposition 1.3.3 (1)].

Item (3) follows immediately from the existence of the weight filtration on objects of $\mathscr{M}(S)$.

We prove (4). Let w be the weight of K. Let $i \in \mathbf{Z}$. Then $\tau_{\leq i} K$ and $\tau_{>i} K$ are pure of weight w, so $\mathrm{Hom}_{\mathrm{D}^b\mathscr{M}(S)}(\tau_{>i} K, \tau_{\leq i} K[1]) = 0$ by (3), so the exact triangle

$$\tau_{\leq i} K \to K \to \tau_{>i} K \xrightarrow{+1}$$

splits. This implies the statement. ∎

Theorem 6.28. *Let $f : X \to S$ be a proper morphism of quasi-projective k-varieties with X irreducible. Let $j : U \to X$ be an open immersion, and K be a perverse motive on U. If K is pure of weight w, then $\mathrm{H}^i(f_*^{\mathscr{M}} j_{!*}^{\mathscr{M}} K)$ is a motivic perverse sheaf that is pure of weight $w + i$.*

Proof. Let us say that $L \in \mathrm{D}^b(\mathscr{M}(S))$ is pure of weight w if $\mathrm{H}^i L$ is pure of weight $w + i$ for every $i \in \mathbf{Z}$. For such an L, by Corollary 6.12, it follows from the Weil conjectures proved by Deligne [27] that $f_*^{\mathscr{M}} L$ is pure of weight w (see [42, remark after Definition 3.3]). Hence, Proposition 6.20 ensures that $f_*^{\mathscr{M}} j_{!*}^{\mathscr{M}} K$ is pure of weight w. This gives the conclusion. ∎

In particular, this provides (for geometric variations of Hodge structures) an arithmetic proof of Zucker's theorem [77, Theorem p. 416] via reduction to positive characteristic and to the Weil conjectures [27, Théorème 1]. More generally, in higher dimension:

Corollary 6.29. *Let k be a field embedded into $\mathbf{C}$. Let X be an irreducible proper k-variety and $\mathscr{L}$ be a $\mathbf{Q}$-local system on a smooth dense open subscheme U of X of the form $\mathscr{L} = R^w g_* \mathbf{Q}_V$ where $g : V \to U$ is a smooth proper morphism and w is an integer. Then the intersection cohomology group $IH^i(X, \mathscr{L})$, for $i \in \mathbf{Z}$, is canonically the Betti realization of a Nori motive over k which is pure of weight $i + w$. In particular, $IH^i(X, \mathscr{L})$ carries a canonical pure Hodge structure of weight $i + w$.*

Proof. Let d be the dimension of X, $\pi : X \to \mathrm{Spec}(k)$ be the structural morphism and j be the inclusion of U in X. As in Corollary 5.7, $IH^i(X, \mathscr{L})$ is the Betti realization of the Nori motive $\mathrm{H}^{i-d}(\pi_*^{\mathscr{M}} j_{!*}^{\mathscr{M}} \mathrm{H}^{w+d}(g_*^{\mathscr{M}} \mathbf{Q}_V^{\mathscr{M}}))$, which is pure of weight $w + i$ by Theorem 6.28. ∎

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