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Filtered instanton Floer homology and the homology cobordism group

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Abstract. For any $s \in [-\infty, 0]$ and oriented homology 3-sphere Y, we introduce a homology cobordism invariant $r_s(Y) \in (0, \infty]$. The values $\{r_s(Y)\}$ are included in the critical values of the $SU(2)$-Chern–Simons functional of Y, and we show a negative definite cobordism inequality and a connected sum formula for r_s. As applications, we obtain several new results on the homology cobordism group. First, we give infinitely many homology 3-spheres which cannot bound any definite 4-manifold. Next, we show that if the 1-surgery of S^3 along a knot has the Frøyshov invariant negative, then all positive $1/n$-surgeries along the knot are linearly independent in the homology cobordism group. In another direction, we use $\{r_s\}$ to define a filtration on the homology cobordism group which is parametrized by $[0, \infty]$. Moreover, we compute an approximate value of r_s for the hyperbolic 3-manifold obtained by $1/2$-surgery along the mirror of the knot 5_2.

Keywords. Homology cobordism group, instanton Floer homology, Chern–Simons functional, definite 4-manifold, knot concordance group

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1. Introduction

The study of the structure of the 3-dimensional homology cobordism group $\Theta^3_{\mathbb{Z}}$ is one of the central topics in low-dimensional topology. One of the motivations is a relation to the triangulation problem of topological manifolds. In 1985, Galewski–Stern [25] and Matumoto [42] proved that a topological n-manifold M with $n \geq 5$ admits a triangulation if and only if a certain cohomology class $\delta(M) \in H^5(M; \operatorname{Ker}\mu)$ satisfies $\delta(M) = 0$, where $\mu\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$ is the Rokhlin homomorphism. Since there is no essential difference between PL and smooth categories for 3- and 4-manifolds, $\Theta^3_{\mathbb{Z}}$ is isomorphic to its PL version. On the other hand, the n-dimensional PL version of homology cobordism group is known to be trivial for $n \neq 3$ ([34]). Also, Freedman's result [20] implies that the topological version of the 3-dimensional homology cobordism group is trivial.

Various gauge theories and Floer theories have been developed and used to improve the understanding of $\Theta^3_{\mathbb{Z}}$. In the 1980s, Donaldson [10] applied Yang–Mills gauge theory to 4-dimensional topology and proved the diagonalization theorem. The diagonalization theorem and its extension due to Furuta [23] imply that the Poincaré sphere has infinite order in $\Theta^3_{\mathbb{Z}}$. Fintushel–Stern [15] and Furuta [24] developed Yang–Mills gauge theory for orbifolds with cylindrical ends to prove that $\Theta^3_{\mathbb{Z}}$ contains $\mathbb{Z}^\infty$ as a subgroup. On the other hand, Manolescu [40] disproved the triangulation conjecture using Seiberg–Witten Floer theory. Recently, Dai–Hom–Stoffregen–Truong [8] proved the existence of a $\mathbb{Z}^\infty$-summand in $\Theta^3_{\mathbb{Z}}$ using involutive Heegaard–Floer theory.

In this paper, we interpret the work of [15, 24] in Yang–Mills gauge theory in terms of instanton Floer homology, and introduce a family $\{r_s\}$ of real-valued homology cobordism invariants for any homology 3-sphere.

1.1. The invariants r_s

Let Y be an oriented homology 3-sphere. In [12], Donaldson defined an obstruction class $[\theta]$ (denoted by D_1 in [12]) lying in the first instanton cohomology group of Y such that $[\theta] \neq 0$ implies the non-existence of any negative definite 4-manifold with boundary Y. On the other hand, Fintushel–Stern [16] defined filtered versions of the instanton cohomology group $\{I^*_{[r,r+1]}(Y)\}_{r\in\mathbb{R}}$ such that their filtrations are given by a perturbed Chern–Simons functional. Here, one can see that Fintushel–Stern's method actually enables us to define a cohomology group $I^*_{[s,r]}(Y)$ for an arbitrary interval $[s,r]$, and the obstruction class $[\theta]$ is well-defined in $I^1_{[s,r]}(Y)$ for any $r \in (0,\infty]$ $(:= \mathbb{R}_{>0} \cup \{\infty\})$ and $s \in [-\infty, 0]$ $(:= \mathbb{R}_{\leq 0} \cup \{-\infty\})$. Therefore, it is natural to ask whether $[\theta]$ vanishes in $I^1_{[s,r]}(Y)$ for a given Y and interval $[s,r]$. In light of this observation, we define

$$r_s(Y) := \sup\{r \in (0,\infty] \mid [\theta] = 0 \in I^1_{[s,r]}(Y)\}$$

for any oriented homology 3-sphere Y and $s \in [-\infty, 0]$. A more precise definition of r_s is stated in Definition 3.2. Such a quantitative construction in Floer homology has appeared in several Floer theories including Hamiltonian Floer homology [18, 19] and embedded contact homology [33].

Our main theorem is stated as follows.

Theorem 1.1. *The values $\{r_s(Y)\}_{s\in[-\infty,0]}$ are homology cobordism invariants of Y. Moreover, the invariants $\{r_s\}_{s\in[-\infty,0]}$ satisfy the following properties:*

(1) *If there exists a negative definite cobordism W with $\partial W = Y_1 \amalg -Y_2$, then*

$$r_s(Y_2) \leq r_s(Y_1) \quad \textit{for any } s \in [-\infty, 0].$$

Moreover, if W is simply connected and $r_s(Y_1) < \infty$, then

$$r_s(Y_2) < r_s(Y_1).$$

(2) *If $r_s(Y) < \infty$, then $r_s(Y)$ is a critical value of the Chern–Simons functional of Y.*

(3) *If $s_1 \leq s_2$, then $r_{s_1}(Y) \geq r_{s_2}(Y)$.*

(4) *The inequality*

$$r_s(Y_1 \# Y_2) \geq \min\{r_{s_1}(Y_1) + s_2, r_{s_2}(Y_2) + s_1\}$$

holds for any $s, s_1, s_2 \in (-\infty, 0]$ with $s = s_1 + s_2$.

Recently, Daemi [6] introduced a family $\{\Gamma_Y(i)\}_{i\in\mathbb{Z}}$ of real-valued homology cobordism invariants. Since the $\Gamma_Y(i)$ are also defined by using instanton Floer theory and satisfy the properties (1) and (2) in Theorem 1.1, it is natural to ask whether the $\Gamma_Y(i)$ are related to our $r_s(Y)$. Roughly speaking, our invariants $\{r_s(Y)\}_{s\in[-\infty,0]}$ can be seen as a one-parameter family including $\Gamma_{-Y}(1)$. More precisely, we prove the following equality.

Theorem 1.2. *For any oriented homology* 3*-sphere* Y*,*

$$r_{-\infty}(Y) = \Gamma_{-Y}(1).$$

As consequences of Theorem 1.2 and results in [6], we can understand a relationship between r_s and the Frøyshov invariant $h\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ ([21]), and obtain infinitely many examples with non-trivial r_s. Note that $r_s(S^3) = \infty$ for any $s \in [-\infty, 0]$, and so we say that $r_s(Y)$ is *non-trivial* if $r_s(Y) < \infty$.

Corollary 1.3. *The inequality* $r_{-\infty}(Y) < \infty$ *holds if and only if* $h(Y) < 0$*. In particular, if* $h(Y) < 0$*, then* $r_s(Y)$ *is finite for any* $s \in [-\infty, 0]$.

Let $\Sigma(a_1, \dots, a_n)$ denote the Seifert homology 3-sphere corresponding to a tuple $(a_1, \dots, a_n)$ of pairwise coprime integers, and let $R(a_1, \dots, a_n)$ be an odd integer introduced by Fintushel–Stern [14].

Corollary 1.4. *If* $R(a_1, \dots, a_n) > 0$*, then for any* $s \in [-\infty, 0]$*,*

$$r_s(-\Sigma(a_1, \dots, a_n)) = \frac{1}{4a_1 \cdots a_n} \quad \textit{and} \quad r_s(\Sigma(a_1, \dots, a_n)) = \infty.$$

For instance, it is known that $R(p, q, pqk - 1) = 1$ for any coprime $p, q > 1$ and $k \in \mathbb{Z}_{>0}$. Here, one might ask whether r_s is constant for any Y. We show that the answer is negative. Indeed, the connected sum formula for r_0 in Theorem 1.1 and the above corollaries imply that any $Y_k := 2\Sigma(2, 3, 5) \# (-\Sigma(2, 3, 6k + 5))$ $(k \in \mathbb{Z}_{>0})$ satisfies $r_0(Y_k) = \frac{1}{24(6k+5)} < \infty$, while $r_{-\infty}(Y_k) = \infty$ because $h(Y_k) = 1$.

1.2. Topological applications

Next, we show topological applications of r_s, which include several new results on the homology cobordism group $\Theta^3_{\mathbb{Z}}$ and the knot concordance group $\mathcal{C}$.

1.2.1. Homology 3*-spheres with no definite bounding.* We call a 4-manifold *definite* if it is positive definite or negative definite. It is well-known that the Frøyshov invariant [21, 22] and the Heegaard–Floer correction term [45] are obstructions to the existence of a positive definite bounding or a negative definite bounding. However, there has been no invariant which is an obstruction to the existence of both positive and negative definite boundings. Our invariant $r_s(Y)$ is the first example of such an obstruction. We have the following theorem.

Theorem 1.5. *There exist infinitely many homology* 3*-spheres* $\{Y_k\}_{k=1}^{\infty}$ *which cannot bound any definite* 4*-manifold. Moreover, we can choose them linearly independent in* $\Theta^3_{\mathbb{Z}}$.

Indeed, we can take $\{Y_k\}_{k=1}^{\infty} := \{2\Sigma(2,3,5) \# (-\Sigma(2,3,6k+5))\}_{k=1}^{\infty}$ as a concrete example for Theorem 1.5. We will show that $r_0(Y_k) < \infty$ and $r_0(-Y_k) < \infty$. Here, we note that if a homology 3-sphere Y is Seifert or obtained by a knot surgery, then Y bounds a definite 4-manifold. In addition, the existence of a definite bounding is invariant under homology cobordism. Therefore, we have the following corollaries.

Corollary 1.6. *For any $k \in \mathbb{Z}_{>0}$, the homology cobordism class*

$$[2\Sigma(2,3,5) \# (-\Sigma(2,3,6k+5))]$$

does not contain any Seifert homology 3-sphere.

The existence of such a homology 3-sphere was first proved by Stoffregen [56] using Pin(2)-monopole Floer homology. On the other hand, our proof is based on Yang–Mills instanton theory.

Corollary 1.7. *For any $k \in \mathbb{Z}_{>0}$, no representative of* $[2\Sigma(2,3,5) \# (-\Sigma(2,3,6k+5))]$ *is obtained by a knot surgery.*

1.2.2. Linear independence of $1/n$-surgeries. In [15, 24], Fintushel–Stern and Furuta proved that for any coprime integers $p, q > 1$, the Seifert homology 3-spheres $\{\Sigma(p,q,pqn-1)\}_{n=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$. We note that $\Sigma(p,q,pqn-1) = -S^3_{1/n}(T_{p,q})$, where $T_{p,q}$ is the (p,q)-torus knot and $S^3_{1/n}(K)$ denotes the $1/n$-surgery along a knot K in S^3. From this viewpoint, we generalize the above results as follows.

Theorem 1.8. *For any knot K in S^3, if $h(S^3_1(K)) < 0$, then $\{S^3_{1/n}(K)\}_{n=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$.*

Theorem 1.8 gives a huge number of linearly independent families in $\Theta^3_{\mathbb{Z}}$. In fact, there exist infinitely many hyperbolic knots and satellite knots with $h(S^3_1(K)) < 0$. As hyperbolic examples, we can take the mirrors K_k^* of the 2-bridge knots K_k $(k \in \mathbb{Z}_{>0})$ corresponding to the rational numbers $\frac{2}{4k-1}$. (These K_k are often called *twist knots*. See Figure 2 in Section 5.2.)

Corollary 1.9. *For any $k \in \mathbb{Z}_{>0}$, the homology 3-spheres $\{S^3_{1/n}(K_k^*)\}_{n=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$.*

As satellite examples, we can take the $(2,q)$-cable of any knot K (denoted by $K_{2,q}$) with odd $q \geq 3$.

Corollary 1.10. *For any knot K in S^3 and odd integer $q \geq 3$, the homology 3-spheres $\{S^3_{1/n}(K_{2,q})\}_{n=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$.*

1.2.3. Linear independence of Whitehead doubles. In this paper, we consider the subgroup $\mathcal{T}$ in the knot concordance group $\mathcal{C}$ generated by topologically slice knots. The group $\mathcal{T}$ has been thorougly studied via several gauge theories, Floer theories [1, 7, 9, 13, 27,30–32,38,39,41,44,46–49] and Khovanov homology theory [50] as in the case of $\Theta^3_{\mathbb{Z}}$. However, the structure of $\mathcal{T}$ is still mysterious.

Here we focus on the positively-clasped Whitehead double $D(K)$ of a knot K. Since $D(K)$ has trivial Alexander polynomial, $D(K)$ lies in $\mathcal{T}$, namely $D(K)$ is topologically slice [20]. There is a famous conjecture about the Whitehead doubles:

Conjecture 1.11 ([35, Problem 1.38]). As elements of the knot concordance group $\mathcal{C}$, the equality $[D(K)] = 0$ holds if and only if $[K] = 0$.

Motivated by this conjecture, Hedden–Kirk [27] conjectured that the map

$$D: \mathcal{C} \to \mathcal{C}, \quad [K] \mapsto [D(K)],$$

preserves the linear independence, and they proved that the conjecture holds for the family $\{T_{2,2^n-1}\}_{n=2}^{\infty}$, that is, the Whitehead doubles $\{D(T_{2,2^n-1})\}_{n=2}^{\infty}$ are linearly independent in $\mathcal{C}$.

We refine their result as follows.

Theorem 1.12. *For any coprime $p, q > 1$, the Whitehead doubles $\{D(T_{p,np+q})\}_{n=0}^{\infty}$ are linearly independent in $\mathcal{C}$.*

Corollary 1.13. *The Whitehead doubles $\{D(T_{2,2n-1})\}_{n=2}^{\infty}$ are linearly independent in $\mathcal{C}$.*

Note that Hedden–Kirk's results were extended to more general satellite knots in [49], and our technique enables us to extend a result in [49]. Moreover, our approach can be used to see the linear independence of $D(K)$ for a certain family of twisted knots K.

1.3. Additional structures on $\Theta^3_{\mathbb{Z}}$ and Ker h

Using involutive Heegaard–Floer theory, Hendricks, Hom and Lidman [28] introduced a poset filtration on $\Theta^3_{\mathbb{Z}}$ and re-proved the existence of a $\mathbb{Z}^\infty$-subgroup of $\Theta^3_{\mathbb{Z}}$. Moreover, for the knot concordance group, such filtrations coming from Heegaard–Floer theory are also given in [26, 54]. Inspired by these work, we give a $[0, \infty]$-filtration of $\Theta^3_{\mathbb{Z}}$ using our invariant r_s, which can be used to re-prove that Fintushel–Stern's and Furuta's sequence $\{\Sigma(p, q, pqk - 1)\}_{k=1}^{\infty}$ is linearly independent in $\Theta^3_{\mathbb{Z}}$ for any pair (p, q) of coprime integers. Since $r_s(Y)$ coincides with a critical value of the $SU(2)$-Chern–Simons functional of Y, our filtration has a flavor of geometry.

More precisely, for any $r \in [0, \infty]$, we consider the set

$$\Theta^3_{\mathbb{Z},r} := \left\{[Y] \in \Theta^3_{\mathbb{Z}} \;\middle|\; \min\{r_0(Y), r_0(-Y)\} \geq r\right\}.$$

Then it follows from the connected sum formula for r_0 that $\Theta^3_{\mathbb{Z},r}$ is a subgroup of $\Theta^3_{\mathbb{Z}}$. Moreover, by definition, it is obvious that if $r \geq r'$, then $\Theta^3_{\mathbb{Z},r} \subset \Theta^3_{\mathbb{Z},r'}$. In particular, $\Theta^3_{\mathbb{Z},0} = \Theta^3_{\mathbb{Z}}$. For this filtration, we prove that any quotient group is infinitely generated.

Theorem 1.14. *For any $r \in (0, \infty]$, the quotient group $\Theta^3_{\mathbb{Z}}/\Theta^3_{\mathbb{Z},r}$ contains $\mathbb{Z}^\infty$ as a subgroup.*

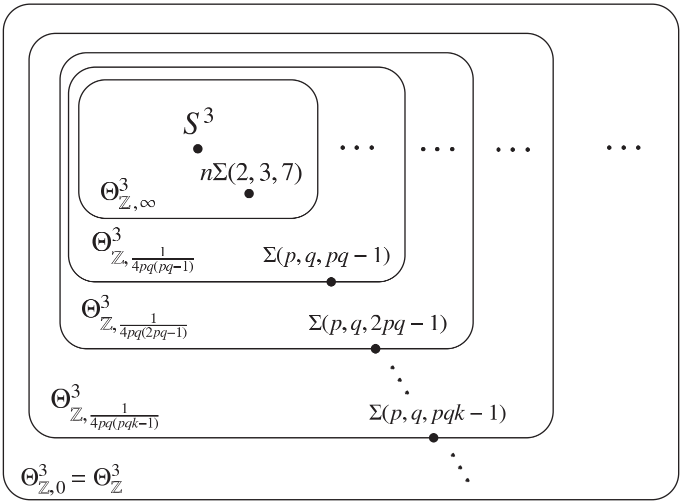

Fig. 1. A schematic picture of the filtration $\{\Theta^3_{\mathbb{Z},r}\}$.

Figure 1 gives a schematic picture of the filtration $\{\Theta^3_{\mathbb{Z},r}\}$. Since $\Theta^3_{\mathbb{Z},r}$ is a subgroup for any $r \in [0,\infty]$, it is easy to see from the figure that $\{\Sigma(p,q,pqk-1)\}_{k=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$. Here, we note that the smallest subgroup $\Theta^3_{\mathbb{Z},\infty}$ is infinitely generated. In fact, it is proved by Hendricks–Hom–Lidman [28] that $\{S^3_{-1}(T_{2,4n+1})\}_{n=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$. Moreover, it is not hard to see that $S^3_{-1}(T_{2,4n+1})$ bounds both a positive definite 4-manifold and a negative definite 4-manifold, and hence Theorem 1.1 (1) gives $r_0(S^3_{-1}(T_{2,4n+1})) = r_0(-S^3_{-1}(T_{2,4n+1})) = \infty$. Note that any positive knot bounds a null-homologous disk in $B^4 \# k\mathbb{C}P^2$ for sufficiently large k ([5]). Therefore, we have

$$\Theta^3_{\mathbb{Z},\infty} \supset \mathrm{span}_{\mathbb{Z}}\{[S^3_{-1}(T_{2,4n+1})]\}_{n=1}^{\infty} \cong \mathbb{Z}^{\infty},$$

where $\mathrm{span}_{\mathbb{Z}}$ is the $\mathbb{Z}$-linear span in $\Theta^3_{\mathbb{Z}}$. We pose the following fundamental questions about the filtration $\{\Theta^3_{\mathbb{Z},r}\}_{r\in[0,\infty]}$.

Question 1.15. Which subquotient $\Theta^3_{\mathbb{Z},r'}/\Theta^3_{\mathbb{Z},r}$ is infinitely generated?

As another approach to studying $\Theta^3_{\mathbb{Z}}$, we use the value

$$s_\infty(Y) := \sup\{s \in [-\infty,0] \mid r_s(Y) = \infty\}$$

to introduce a pseudometric on $\mathrm{Ker}(h\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z})$. The pseudometric induces a metric on the quotient group $\mathrm{Ker}\, h/\Theta^3_{\mathbb{Z},\infty}$. For more details, see Section 6.1.

1.4. Computations for a hyperbolic 3-manifold

Finally, we discuss the relation between our invariants $\{r_s\}$ and geometric structures on homology 3-spheres. While it is proved by Myers [43] that any homology cobordism class contains a hyperbolic representative, we also know that there exist infinitely many homology cobordism classes containing no Seifert representative, as discussed in Section 1.2. As the next step, it is natural to ask whether $\Theta^3_{\mathbb{Z}}$ is generated by Seifert homology 3-spheres or not. Recently, Hendricks, Hom, Stoffregen, and Zemke [29] proved that the homology cobordism class $[S^3_1(T^*_{6,7} \# T^*_{6,7} \# T_{6,13} \# T^*_{2,3:2,5})]$ is not contained in Θ^3_S, where $T_{2,3:2,5}$ is the $(2,5)$-cable of $T_{2,3}$ and Θ^3_S denotes the subgroup of $\Theta^3_{\mathbb{Z}}$ generated by Seifert homology 3-spheres. Namely, they proved $\Theta^3_{\mathbb{Z}} \supsetneq \Theta^3_S$.

Here, we mention that $S^3_1(T^*_{6,7} \# T^*_{6,7} \# T_{6,13} \# T^*_{2,3:2,5})$ is not Seifert but a graph manifold. (The proof will be given in Appendix A.) Hence, the following question still remains open.

Question 1.16. Is the group $\Theta^3_{\mathbb{Z}}$ generated by graph homology 3-spheres?

Let Θ^3_G denote the subgroup of $\Theta^3_{\mathbb{Z}}$ generated by all graph homology 3-spheres, and then Question 1.16 is equivalent to whether the equality $\Theta^3_{\mathbb{Z}} = \Theta^3_G$ holds or not. Here we note that critical values of the $SU(2)$-Chern–Simons functional of graph 3-manifolds are rational [2], and hence the image $r_s(\Theta^3_G)$ is included in $\mathbb{Q}_{>0} \cup \{\infty\}$ for any $s \in [-\infty, 0]$. Therefore, we have the following proposition.

Proposition 1.17. *If there exists a homology* 3*-sphere* Y *and* $s \in [-\infty, 0]$ *such that* $r_s(Y)$ *is finite and irrational, then* $[Y] \notin \Theta^3_G$.

From the viewpoint of Proposition 1.17, we try to calculate $\{r_s\}$ for the $1/2$-surgery along the knot 5^*_2, where 5^*_2 is the mirror of the knot 5_2 in Rolfsen's knot table. Note that 5_2 is K_2 as a twist knot, $S^3_1(5^*_2) \cong -\Sigma(2,3,11)$ and that $S^3_{1/2}(5^*_2)$ is a hyperbolic 3-manifold (see [4]). These facts imply that the value $r_s(S^3_{1/2}(5^*_2))$ is finite and might be irrational. Moreover, using the computer, we get the following result.

Theorem 1.18. *The numerical approximation*

$$r_s(S^3_{1/2}(5^*_2)) \approx 0.0017648904\ 7864885113\ 0739625897\ 0947779330\ 4925308209$$

holds for any $s \in [-\infty, 0]$*, where the error is at most* 10^{-50}.

It is an open problem whether there exists a 3-manifold whose $SU(2)$-Chern–Simons functional has an irrational critical value. Note that the decimal in Theorem 1.18 has no repetition. Therefore, we have the following conjecture.

Conjecture 1.19. The value $r_s(S^3_{1/2}(5^*_2))$ is an irrational number.

If Conjecture 1.19 is true, then it follows from Proposition 1.17 that $[S^3_{1/2}(5^*_2)]$ is not contained in Θ^3_G.

Organization

The paper is organized as follows. In Section 2, we give a review of filtered instanton homology. In Section 3, we introduce the invariants r_s using notions of Section 2, and establish several basic properties of r_s. In particular, Theorem 1.1 will be proved in this section. Section 4 is devoted to discussing the relation between r_s and Daemi's $\Gamma_Y(k)$. In Section 5, we prove all assertions stated in Section 1.2. In Section 6, we discuss additional structures on $\Theta^3_{\mathbb{Z}}$ and $\operatorname{Ker} h$ by using r_s. In Section 7, we explain how to compute an approximate value of $r_s(S^3_{1/2}(5^*_2))$.

2. Review of filtered instanton Floer homology

Throughout this paper, all manifolds are assumed to be smooth, compact, orientable and oriented, and diffeomorphisms are orientation-preserving unless otherwise stated. In this section, we review the definition of filtered instanton Floer homology. For instanton Floer homology, see [12, 17]. For the filtered version of instanton Floer homology, see [16].

2.1. Preliminaries

2.1.1. Chern–Simons functional. For a homology 3-sphere Y, we denote the product $SU(2)$ bundle by P_Y, and the product connection on P_Y by θ. In addition, we denote

- $\mathcal{A}(Y) :=$ the set of $SU(2)$-connections on P_Y,
- $\mathcal{A}^{\text{flat}}(Y) :=$ the set of $SU(2)$-flat connections on P_Y,
- $\widetilde{\mathcal{B}}(Y) := \mathcal{A}(Y)/\operatorname{Map}_0(Y, SU(2))$,
- $\widetilde{R}(Y) := \mathcal{A}^{\text{flat}}(Y)/\operatorname{Map}_0(Y, SU(2))$,
- $R(Y) := \mathcal{A}^{\text{flat}}(Y)/\operatorname{Map}(Y, SU(2))$,

where $\operatorname{Map}(Y, SU(2))$ (resp. $\operatorname{Map}_0(Y, SU(2))$) is the set of smooth functions (resp. smooth functions of mapping degree 0), and the right action of $\operatorname{Map}(Y, SU(2))$ on $\mathcal{A}(Y)$ is given by $a \cdot g := g^{-1}dg + g^{-1}ag$. Note that the action preserves the flatness of a for any g. Also, we write $\widetilde{\mathcal{B}}^*(Y)$, $\widetilde{R}^*(Y)$ and $R^*(Y)$ respectively for the subsets of $\widetilde{\mathcal{B}}(Y)$, $\widetilde{R}(Y)$ and $R(Y)$ whose stabilizers are constants in $\{\pm I_2\}$. The elements in $\widetilde{\mathcal{B}}^*(Y)$ and $\widetilde{R}^*(Y)$ are called *irreducible connections*. When the stabilizer of an $SU(2)$-connection is larger than $\{\pm I_2\}$, the connection is said to be *reducible*. The *Chern–Simons functional on* $\mathcal{A}(Y)$ is the map $cs_Y : \mathcal{A}(Y) \to \mathbb{R}$ defined by

$$cs_Y(a) := \frac{1}{8\pi^2}\int_Y \operatorname{Tr}\bigl(a \wedge da + \tfrac{2}{3} a \wedge a \wedge a\bigr).$$

It is known that $cs_Y(a \cdot g) - cs_Y(a) = \deg(g)$ holds for $g \in \operatorname{Map}(Y, SU(2))$, where $\deg(g)$ is the mapping degree of g. Therefore, $cs_Y(a \cdot g) = cs_Y(a)$ for any $g \in \operatorname{Map}_0(Y, SU(2))$, and hence cs_Y descends to a map $\widetilde{\mathcal{B}}(Y) \to \mathbb{R}$. We denote it by the same notation cs_Y. Moreover, we use the notations Λ_Y and Λ^*_Y for $cs_Y(\widetilde{R}(Y))$ and $cs_Y(\widetilde{R}^*(Y))$, respectively. Note that the set Λ_Y is locally finite, that is, $[m, m+1] \cap \Lambda_Y$ is a finite set for any $m \in \mathbb{R}$. For example, one can see that $\Lambda_{S^3} = \mathbb{Z}$ and $\Lambda^*_{S^3} = \emptyset$. Set

$$\mathbb{R}_Y := \mathbb{R} \setminus \Lambda_Y$$

for any oriented homology 3-sphere Y.

2.1.2. Perturbations of cs_Y. Roughly speaking, the instanton Floer homology of Y is the Morse homology associated to $cs_Y : \widetilde{\mathcal{B}}^*(Y) \to \mathbb{R}$, where the set of critical points is $\widetilde{R}^*(Y)$. However, $\widetilde{R}^*(Y)$ does not satisfy non-degeneracy in general, and hence we need to perturb cs_Y so that $\widetilde{R}^*(Y)$ becomes non-degenerate. In this paper, we use several classes of perturbations of cs_Y introduced in [17, Section (1b)] and [3, Section 5.5.1].

For any $d \in \mathbb{Z}_{>0}$ and fixed $l \gg 2$, consider the set of orientation-preserving embeddings of d solid tori into Y,

$$\mathcal{F}_d := \left\{ (f_i : S^1 \times D^2 \hookrightarrow Y)_{1 \le i \le d} \right\},$$

and denote by $C^l(SU(2)^d, \mathbb{R})_{\mathrm{ad}}$ the set of adjoint invariant C^l functions on $SU(2)^d$. Then the *set of perturbations* is defined by

$$\mathcal{P}(Y) := \bigcup_{d \in \mathbb{N}} \mathcal{F}_d \times C^l(SU(2)^d, \mathbb{R})_{\mathrm{ad}}.$$

Fix a 2-form $d\mathcal{S}$ on D^2 supported in the interior of D^2 with $\int_{D^2} d\mathcal{S} = 1$. Then, for any $\pi = (f, h) \in \mathcal{P}(Y)$, we can define the *$\pi$-perturbed Chern–Simons functional* $cs_{Y,\pi} : \widetilde{\mathcal{B}}^*(Y) \to \mathbb{R}$ by

$$cs_{Y,\pi}(a) = cs_Y(a) + \int_{x \in D^2} h(\mathrm{Hol}_{f_1(-,x)}(a), \dots, \mathrm{Hol}_{f_d(-,x)}(a))\, d\mathcal{S}, \tag{1}$$

where $\mathrm{Hol}_{f_i(-,x)}(a)$ is the holonomy around the loop $t \mapsto f_i(t, x)$ for each $i \in \{1, \dots, d\}$. We denote $\|h\|_{C^l}$ by $\|\pi\|$ and the second term of the right-hand side in (1) by h_f.

2.1.3. Gradient of $cs_{Y,\pi}$. We next consider the gradient of $cs_{Y,\pi}$. Fix a Riemannian metric g_Y on Y. For $i \in \{1, \dots, d\}$, let $\iota_i : SU(2) \to SU(2)^d$ denote the i-th inclusion, and set $h_i := h \circ \iota_i : SU(2) \to \mathbb{R}$. Then, identifying $\mathfrak{su}(2)$ with its dual by the Killing form, we can regard the derivative h_i' as a map $h_i' : SU(2) \to \mathfrak{su}(2)$.

Using the value of the holonomy around each loop $\{f_i(s, x) \mid s \in S^1\}$, we obtain a section $\mathrm{Hol}_{f_i(s,x)}(a)$ of the bundle $\mathrm{Aut}\, P_Y$ over $\mathrm{Im}\, f_i$. Sending the section $\mathrm{Hol}_{f_i(s,x)}(a)$ by the bundle map induced by $h_i' : \mathrm{Aut}\, P_Y \to \mathrm{ad}\, P_Y$, we obtain a section $h_i'(\mathrm{Hol}_{f_i(s,x)}(a))$ of $\mathrm{ad}\, P_Y$ over $\mathrm{Im}\, f_i$.

We now describe the gradient-line equation of $cs_{Y,\pi}$ with respect to the L^2-metric:

$$\frac{\partial}{\partial t} a_t = \mathrm{grad}_a\, cs_{Y,\pi} = *_{g_Y} \Big(F(a_t) + \sum_{i=1}^{d} h_i'(\mathrm{Hol}(a_t)_{f_i(s,x)})(f_i)_* \mathrm{pr}_2^* \, d\mathcal{S} \Big), \tag{2}$$

where pr_2 is the second projection $S^1 \times D^2 \to D^2$, $*_{g_Y}$ is the Hodge star operator and $F(a)$ denotes the curvature of a connection a. We denote $\mathrm{pr}_2^* d\mathcal{S}$ by η. We set

$$\widetilde{R}(Y)_\pi := \left\{ a \in \widetilde{\mathcal{B}}(Y) \;\middle|\; F(a) + \sum_{i=1}^{d} h_i'(\mathrm{Hol}(a)_{f_i(s,x)})(f_i)_* \eta = 0 \right\},$$

$$\widetilde{R}^*(Y)_\pi := \widetilde{R}(Y)_\pi \cap \widetilde{\mathcal{B}}^*(Y).$$

The solutions of (2) correspond to connections A over $Y \times \mathbb{R}$ which satisfy

$$F^+(A) + \pi(A)^+ = 0, \tag{3}$$

where

- the 2-form $\pi(A)$ is given by

$$\sum_{i=1}^{d} h_i'(\mathrm{Hol}(A)_{\tilde{f}_i(t,x,s)})(\tilde{f}_i)_*(\mathrm{pr}_1^*\eta),$$

- pr_1 is the projection from $(S^1 \times D^2) \times \mathbb{R}$ to $S^1 \times D^2$,
- the superscript $+$ is $\frac{1}{2}(1 + *)$ where $*$ is the Hodge star operator with respect to the product metric on $Y \times \mathbb{R}$,
- the map $\tilde{f}_i \colon S^1 \times D^2 \times \mathbb{R} \to Y \times \mathbb{R}$ is $f_i \times \mathrm{Id}$.

We introduce the spaces $M^Y(a,b)_\pi$ of trajectories for given $a, b \in \widetilde{R}^*(Y)_\pi$. Fix a positive integer $q \geq 3$. Let $A_{a,b}$ be an $SU(2)$-connection on $Y \times \mathbb{R}$ satisfying $A_{a,b}|_{Y \times (-\infty,1]} = p^*a$ and $A_{a,b}|_{Y \times [1,\infty)} = p^*b$ where p is the projection $Y \times \mathbb{R} \to Y$. We define

$$M^Y(a,b)_\pi := \{A_{a,b} + c \mid c \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_q} \text{ with } (3)\}/\mathcal{G}(a,b), \tag{4}$$

where the gauge group $\mathcal{G}(a,b)$ is given by

$$\mathcal{G}(a,b) := \{g \in \mathrm{Aut}(P_{Y\times\mathbb{R}}) \subset \mathrm{End}(\mathbb{C}^2)_{L^2_{q+1,\mathrm{loc}}} \mid g^*A_{a,b} - A_{a,b} \in L^2_q\}.$$

Here the space $L^2_{q+1,\mathrm{loc}}$ consists of the sections which are L^2_{q+1} on each compact set in $Y \times \mathbb{R}$, and $g^*A_{a,b}$ denotes the pull-back of the connection $A_{a,b}$ by g. The group $\mathcal{G}(a,b)$ acts on $\{A_{a,b} + c \mid c \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_q} \text{ with } (3)\}$ via pull-backs of connections. Since

$$\|g^*A_{a,b} - A_{a,b}\|_{L^2_q(Y\times[n,n+1])} \to 0 \quad \text{as } n \to \pm\infty,$$

g lies in the stabilizer $\{\pm 1\}$ of a and b asymptotically on both ends respectively. When we define $M^Y(a,\theta)_{\pi,\delta}$, we use the $L^2_{q,\delta}$-norm instead of the L^2_q-norm. The definition of $L^2_{q,\delta}$-norm is given later in (5). The space $\mathbb{R}$ acts on $M^Y(a,b)_\pi$ by translation.

2.1.4. Classes of perturbations. We also use several classes of perturbations. If the cohomology groups defined by the complex given in [52, (12)] satisfy $H^i_{\pi,a} = 0$ for all $[a] \in \widetilde{R}(Y)_\pi \setminus \{[g^*\theta] \mid g \in \mathrm{Map}(Y, SU(2))\}$ for a given π, we call π a *non-degenerate perturbation*. If π satisfies the following conditions for a fixed small number $\delta > 0$ and g_Y, we call π a *regular perturbation*:

- The linearization

$$d^+_A + d\pi^+_A \colon \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_q} \to \Omega^+(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{q-1}}$$

of the left-hand side of (3) is surjective for $a, b \in \widetilde{R}^*(Y)_\pi$ and $[A] \in M^Y(a,b)_\pi$.

- The linearization
$$d_A^+ + d\pi_A^+ : \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{q,\delta}} \to \Omega^+(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{q-1,\delta}}$$
of the left-hand side of (3) is surjective for $a \in \widetilde{R}^*(Y)_\pi$ and $[A] \in M^Y(a,\theta)_\pi$.

The relevant norms are given by
$$\|f\|^2_{L^2_q} := \sum_{j=0}^{q} \int_{Y\times\mathbb{R}} |\nabla^j_{A_{a,b}} f|^2$$
and
$$\|f\|^2_{L^2_{q,\delta}} := \sum_{j=0}^{q} \int_{Y\times\mathbb{R}} e^{\delta\tilde{\tau}} |\nabla^j_{A_a} f|^2 \tag{5}$$
for $f \in \Omega^i(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)$ with compact support, where

- $A_{a,b}$ and $A_{a,\theta}$ are fixed connections as above,
- $|-|$ is the product metric on $Y \times \mathbb{R}$,
- q is an integer greater than 2,
- $\tilde{\tau}: Y \times \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\tilde{\tau}(y,t) = t$ if $t > 1$ and $\tilde{\tau}(y,t) = 0$ if $t < -1$.

Here the spaces $M^Y(a,b)_\pi$ and $M^Y(a,\theta)_{\pi,\delta}$ are given in (4) in Section 2.1.3.

Next, we will introduce a class of small perturbations which we actually use. In order to explain this, we follow a method introduced in [16].

Definition 2.1. Let Y be a homology 3-sphere and g be a Riemannian metric on Y. For $\epsilon > 0$, we define a class of perturbations $\mathcal{P}^\epsilon(Y,g)$ as the subset of $\mathcal{P}(Y)$ consisting of elements $\pi = (f,h)$ which satisfy

(1) $|h_f(a)| < \epsilon$ for all $a \in \widetilde{\mathcal{B}}(Y)$,

(2) $\|\mathrm{grad}_g\, h_f(a)\|_{L^4} < \epsilon/2$, $\|\mathrm{grad}_g\, h_f(a)\|_{L^2} < \epsilon/2$ for all $a \in \widetilde{\mathcal{B}}(Y)$.

If necessary, for a non-degenerate regular perturbation $\pi = (f,h)$, we can assume h is smooth (see [52, Section 8]).

For $r, s \in \mathbb{R}_Y \cup \{-\infty\}$ and a fixed Riemannian metric g, we define a class of perturbations $\mathcal{P}(Y,r,s,g)$ in the following way. Let $\{R_\alpha\}$ be the connected components of $R^*(Y)$. Let U_α be a neighborhood of R_α in $\mathcal{B}(Y)$ with respect to the C^∞-topology such that $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$ and $\{U_\alpha\}$ is a covering of $R^*(Y)$. We take all lifts of U_α with respect to $\mathrm{pr} : \widetilde{\mathcal{B}}_Y \to \mathcal{B}_Y$. Since $\mathrm{Map}(Y, SU(2))/\mathrm{Map}_0(Y, SU(2))$ is isomorphic to $\mathbb{Z}$, we denote all lifts by $\{U^i_\alpha\}_{i\in\mathbb{Z}}$. In addition, we impose the following conditions on U^i_α:

- If $a \in U^i_\alpha$, $|cs(a) - cs(R_\alpha)| < \min\{d(r,\Lambda_Y)/8, d(s,\Lambda_Y)/8\}$, where
$$d(r,\Lambda_Y) := \min\{|r-a| \in \mathbb{R}_{>0} \mid a \in \Lambda_Y\}.$$
- U^i_α has no reducible connections.

Note that, for any element $\rho \in \widetilde{R}(Y)$, we have unique α and $i \in \mathbb{Z}$ such that $\rho \in U^i_\alpha$.

By the Uhlenbeck compactness theorem, we can take a sufficiently small real number $\epsilon_1(Y, g, \{U_\alpha\}) > 0$ satisfying the following condition:

$$\text{If } a \in \mathcal{B}^*(Y) \text{ and } \|F(a)\|_{L^2} \leq \epsilon_1(Y, g, \{U_\alpha\}), \text{ then } a \in U_\alpha \text{ for some } \alpha. \tag{6}$$

Definition 2.2. Set

$$\epsilon_1(Y, g) := \frac{1}{2} \sup_{\{U_\alpha\}} \epsilon_1(Y, g, \{U_\alpha\}),$$

where $\{U_\alpha\}$ runs over all coverings of $\{R_\alpha\}$ as above.

We also use the notation $\lambda_Y := \min\{|a-b| \mid a, b \in \Lambda_Y \text{ with } a \neq b\}$. Then we define a class of perturbations which we will use later.

Definition 2.3. For a given $r \in \mathbb{R}_Y$, $s \in [-\infty, \infty)$ and a metric g, we define

$$\epsilon(Y, r, s, g) := \begin{cases} \min\{\epsilon_1(Y, g), d(s, \Lambda_Y)/8, d(r, \Lambda_Y)/8, \lambda_Y/32\} & \text{if } s \in \mathbb{R}_Y, \\ \min\{\epsilon_1(Y, g), d(r, \Lambda_Y)/8, \lambda_Y/32\} & \text{if } s \in \Lambda_Y \end{cases}$$

and

$$\mathcal{P}(Y, r, s, g) := \mathcal{P}^{\epsilon(Y,r,s,g)}(Y, g) \subset \mathcal{P}(Y).$$

With the use of $\mathcal{P}(Y, r, s, g)$, we have the following fundamental properties of the values of the perturbed Chern–Simons functional.

Lemma 2.4. *Given $r \in \mathbb{R}_Y$, $s \in \mathbb{R}$, and $\pi \in \mathcal{P}(Y, r, s, g)$, for any $a \in \widetilde{R}_\pi(Y)$ one has*

(1) $|cs_\pi(a) - r| > \frac{3}{4} d(r, \Lambda_Y)$,

(2) $|cs_\pi(a) - s| > \frac{3}{4} d(s, \Lambda_Y)$ *if* $s \in \mathbb{R}_Y$,

(3) $|cs_\pi(a) - s + \frac{1}{2}\lambda_Y| > \frac{3}{4} d(s - \lambda_Y, \Lambda_Y)$ *if* $s \in \Lambda_Y$.

Proof. We only show (1). Due to the choice of perturbations, for each $a \in \widetilde{R}_\pi(Y)$ one can find $\rho \in \widetilde{R}(Y)$ satisfying $a \in U_\rho$, which leads to

$$\begin{aligned} |cs_\pi(a) - r| &\geq |cs(\rho) - r| - |cs_\pi(a) - cs(a)| - |cs(a) - cs(\rho)| \\ &> d(r, \Lambda_Y) - \tfrac{1}{8} d(r, \Lambda_Y) - \tfrac{1}{8} d(r, \Lambda_Y) = \tfrac{3}{4} d(r, \Lambda_Y). \end{aligned}$$

The proofs of (2) and (3) are essentially the same as that of (1). ∎

2.2. *Instanton Floer homology*

In this subsection, we give the definition of the filtration of instanton Floer (co)homology by using the technique in [16]. First, we give the definition of $\mathbb{Z}$-graded instanton Floer homology. Let Y be a homology S^3 and fix a Riemannian metric g_Y on Y. Fix a non-degenerate regular perturbation $\pi \in \mathcal{P}(Y)$. Roughly speaking, instanton Floer homology is infinite-dimensional Morse homology with respect to

$$cs_{Y,\pi} \colon \widetilde{\mathcal{B}}^*(Y) \to \mathbb{R}. \tag{7}$$

Floer defined ind: $\widetilde{R}^*(Y)_\pi \to \mathbb{Z}$, called the *Floer index*. The (co)chains of instanton Floer homology are defined by

$$CI_i(Y) := \mathbb{Z}\{a \in \widetilde{R}^*(Y)_\pi \mid \mathrm{ind}(a) = i\} \quad (\text{resp. } CI^i(Y) := \mathrm{Hom}(CI_i(Y), \mathbb{Z})).$$

The (co)boundary maps $\partial\colon CI_i(Y) \to CI_{i-1}(Y)$ ($\delta\colon CI^i(Y) \to CI^{i+1}(Y)$) are given by

$$\partial(a) := \sum_{b \in \widetilde{R}^*(Y)_\pi,\, \mathrm{ind}(b)=i-1} \#(M^Y(a,b)_\pi/\mathbb{R})b \quad (\delta := \partial^*),$$

where $M^Y(a,b)_\pi$ is the space of trajectories of $cs_{Y,\pi}$ from a to b.

Remark 2.5. Originally, instanton Floer homology is modeled on infinite-dimensional Morse homology with respect to the functional

$$cs_{Y,\pi}\colon \mathcal{B}^*(Y) := \mathcal{A}^*(Y)/\mathrm{Map}(Y, SU(2)) \to S^1. \tag{8}$$

If we use $\mathcal{B}(Y)$, the Floer indices take values in $\mathbb{Z}/8\mathbb{Z}$. For our purpose, we will use the $\mathbb{R}$-valued Chern–Simons functional, that is, we consider (7) instead of (8). On $\widetilde{\mathcal{B}}^*(Y)$, the Floer indices take values in $\mathbb{Z}$. So, we obtain a $\mathbb{Z}$-graded chain complex. The original instanton chain group was given by

$$C_i(Y) := \mathbb{Z}\{a \in \widetilde{R}^*(Y)_\pi/\mathbb{Z} \mid \mathrm{ind}(a) = i\}$$

for each $i \in \mathbb{Z}/8\mathbb{Z}$. The following isomorphism gives a relation between the original instanton Floer homology and $\mathbb{Z}$-graded instanton Floer homology of Y:

$$H_*(CI_j(Y), \partial) \xrightarrow{\cong} H_*(C_i(Y), \partial)$$

for $i \in \mathbb{Z}/8\mathbb{Z}$, $j \in \mathbb{Z}$ with $j \equiv i \bmod 8$.

We review how to give orientations of $M^Y(a,b)_\pi/\mathbb{R}$. For a given homology 3-sphere Y, a non-degenerate perturbation π, $a \in \widetilde{R}^*_\pi(Y)$ and any oriented compact 4-manifold X bounded by $-Y$, we define a configuration space

$$\mathcal{B}(a, X) := \{A_a + c \mid c \in \Omega^1(X^*) \otimes \mathfrak{su}(2)_{L^2_q}\}/\mathcal{G}(a, X^*), \tag{9}$$

where X^* denotes $X \cup Y \times [0,\infty)$ with a product Riemannian metric on the end and $\mathcal{G}(a, X^*)$ is the gauge group defined analogously to the cylindrical case as above. Our convention for the orientations is the same as that in [12, Section 5.4]. If we choose a connection a as the product connection θ, we need to use the weighted Sobolev norm (5) to define the space $\mathcal{B}(\theta, X)$. To give orientations of the spaces $M^Y(a,b)_\pi/\mathbb{R}$, we use a real line bundle

$$\mathbb{L}_a \to \mathcal{B}(a, X) \tag{10}$$

for $a \in \widetilde{R}^*(Y)_\pi$, which is called the *determinant line bundle*. This bundle is defined as a determinant line bundle of a family of operators $d_A^* + d_A^+$ parametrized by $A \in \mathcal{B}(a, X)$.

For details, see [12, Section 5.4]. It is shown there that the bundle $\mathbb{L}_a \to \mathcal{B}(a, X)$ is trivial. Define

$$\mathbb{L}_X := \bigwedge^{\max}(H^0(X;\mathbb{R}) \oplus H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})).$$

If we fix an orientation of the orientation bundle

$$\lambda_{a,X} := \mathbb{L}_a \otimes \mathbb{L}_X \tag{11}$$

associated with a, one can define an orientation of $M^Y(a,b)_\pi$ in the following way. By gluing connections and operators, one can obtain a continuous map

$$\mathfrak{gl}\colon M^Y(a,b)_\pi \times \mathcal{B}(b, X) \to \mathcal{B}(a, X).$$

Furthermore, we obtain a bundle isomorphism whose restriction to the fiber over $(A, B) \in M^Y(a,b)_\pi \times \mathcal{B}(b, X)$ is given by

$$\widetilde{\mathfrak{gl}}\colon (\mathrm{Det}(TM^Y(a,b)_\pi) \otimes \mathbb{L}_b)|_{(A,B)} \to \mathfrak{gl}^*\mathbb{L}_a|_{(A,B)},$$

where $TM^Y(a,b)_\pi$ is the tangent bundle of $M^Y(a,b)_\pi$. We have two orientations of $M^Y(a,b)_\pi$: an orientation of $\mathrm{Det}(TM^Y(a,b)_\pi)$ such that $\widetilde{\mathfrak{gl}}$ is an orientation-preserving map and one coming from the $\mathbb{R}$-translation. The consistency of these orientations gives the sign of the differential. This definition does not depend on the choice of (A, B) and bump functions which are used to construct the map $\mathfrak{gl}$. Moreover, one can see that $\partial^2 = 0$ as in the case of Morse homology for finite-dimensional manifolds.

The instanton Floer (co)homology $I_*(Y)$ (resp. $I^*(Y)$) is defined by

$$I_*(Y) := \mathrm{Ker}\,\partial/\mathrm{Im}\,\partial \quad (\text{resp. } I^*(Y) := \mathrm{Ker}\,\delta/\mathrm{Im}\,\delta).$$

If we take another data of perturbations, Riemannian metric and orientations of $\lambda_{a,X}$, then the corresponding chain complexes are chain homotopy equivalent to each other. Therefore the isomorphism classes of the groups $CI_*(Y)$ and $CI^*(Y)$ are well-defined.

2.3. *Filtered instanton Floer homology*

In this section, we introduce filtered instanton Floer homology which refines Fintushel–Stern's Floer homology introduced in [16].

We recall $\Lambda_Y = cs_Y(\widetilde{R}(Y))$, $\Lambda_Y^* = cs_Y(\widetilde{R}^*(Y))$ and $\mathbb{R}_Y = \mathbb{R} \setminus \Lambda_Y$. For $r \in \mathbb{R}_Y$, we define the filtered instanton (co)homology $I_*^{[s,r]}(Y)$ (resp. $I^*_{[s,r]}(Y)$) using ϵ-perturbations.

Definition 2.6. We fix $s \in [-\infty, 0]$. For a given $r \in \mathbb{R}_Y$, metric g on Y, a non-degenerate regular perturbation $\pi \in \mathcal{P}(Y, r, s, g)$ and orientations on line bundles $\lambda_{a,X}$, the (co)chains of filtered instanton Floer (co)homology are defined by

$$CI_i^{[s,r]}(Y,\pi) := \begin{cases} \mathbb{Z}\{[a] \in \widetilde{R}^*(Y)_\pi \mid \mathrm{ind}(a) = i,\ s < cs_{Y,\pi}(a) < r\} & \text{if } s \in \mathbb{R}_Y, \\ \mathbb{Z}\{[a] \in \widetilde{R}^*(Y)_\pi \mid \mathrm{ind}(a) = i,\ s - \lambda_Y/2 < cs_{Y,\pi}(a) < r\} & \text{if } s \in \Lambda_Y, \end{cases}$$

$$CI^i_{[s,r]}(Y,\pi) := \mathrm{Hom}(CI_i^{[s,r]}(Y,\pi), \mathbb{Z}),$$

where $\lambda_Y := \min\{|a-b| \mid a \neq b,\ a,b \in \Lambda_Y\}$. The (co)boundary maps

$$\partial^{[s,r]}\colon CI_i^{[s,r]}(Y,\pi) \to CI_{i-1}^{[s,r]}(Y,\pi) \quad (\text{resp. } \delta^r\colon CI^i_{[s,r]}(Y) \to CI^{i+1}_{[s,r]}(Y))$$

are given by the restriction of ∂ to $CI_i^{[s,r]}(Y)$ (resp. $\delta^{[s,r]} := (\partial^{[s,r]})^*$).

Then one can see $(\partial^{[s,r]})^2 = 0$.

Definition 2.7. The *filtered instanton Floer (co)homology* $I_*^{[s,r]}(Y)$ (resp. $I^*_{[s,r]}(Y)$) is defined by

$$I_*^{[s,r]}(Y) := \operatorname{Ker} \partial^{[s,r]} / \operatorname{Im} \partial^{[s,r]} \quad (\text{resp. } I^*_{[s,r]}(Y) := \operatorname{Ker} \delta^{[s,r]} / \operatorname{Im} \delta^{[s,r]}).$$

Although the isomorphism class of $CI_i^{[s,r]}(Y,\pi)$ depends on the choice of π, the chain homotopy type is an invariant of Y. Thus, we omit π in the notation for Floer (co)homology groups. The following lemma provides well-definedness of our invariants $I_*^{[s,r]}(Y)$ and $I^*_{[s,r]}(Y)$.

Lemma 2.8. *Fix $s \in [-\infty, 0]$, $r \in \mathbb{R}_Y$ with $s \leq 0 \leq r$, two Riemannian metrics g and g' on Y, non-degenerate regular perturbations π, π' in $\mathcal{P}(Y,r,s,g)$ and orientations of orientation bundles for $\widetilde{R}^*(Y)_\pi$ and $\widetilde{R}^*(Y)_{\pi'}$ respectively. If we choose two elements π and π' in $\mathcal{P}(Y,r,s,g)$ and $\mathcal{P}(Y,r,s,g')$, then there exists a chain homotopy equivalence between $CI_i^{[s,r]}(Y,\pi)$ and $CI_i^{[s,r]}(Y,\pi')$, the instanton chain complexes with respect to π (resp. π').*

Proof. Fix the following data:

- Fix a Riemannian metric $g_\#$ on $Y \times \mathbb{R}$ which coincides with $g + dt^2$ on $Y \times (-\infty,-1]$ and with $g' + dt^2$ on $Y \times [1,\infty)$.
- Fix a regular perturbation $\pi_\#$ on $Y \times \mathbb{R}$ which coincides with π on $Y \times (-\infty,-1]$ and with π' on $Y \times [1,\infty)$ such that

$$\|\pi_\#(A)\|_{L^2(Y\times[-1,1])} < \min\{\epsilon(Y,r,s,g), \epsilon(Y,r,s,g')\}.$$

(In Section 2.1.4, we gave the definition of regular perturbations for the product perturbations. In the general case, we also define regular perturbations using a certain surjectivity condition. For more details, see [12].)

Then, by considering the moduli space $M^Y(a,b)_{\pi_\#}$ under the assumption $\operatorname{ind}(a) - \operatorname{ind}(b) = 0$, we obtain compact 0-dimensional manifolds. The orientation comes as in the case of cobordism maps which we will introduce in Section 2.4. Thus, we have a map

$$\mu_{\pi,\pi'}\colon CI_i^{[s,r]}(Y,\pi) \to CI_i^{[s,r]}(Y,\pi')$$

defined by

$$a \mapsto \sum_{b\colon \operatorname{ind}(a)-\operatorname{ind}(b)=0} \#M^Y(a,b)_{\pi_\#}.$$

Similarly, we have a map $\mu_{\pi',\pi}\colon CI_i^{[s,r]}(Y,\pi') \to CI_i^{[s,r]}(Y,\pi)$. One can check that $\mu_{\pi,\pi'}$ and $\mu_{\pi',\pi}$ are chain maps. Moreover, $\mu_{\pi',\pi}\mu_{\pi,\pi'}$ and $\mu_{\pi,\pi'}\mu_{\pi',\pi}$ are chain homotopic to the identity, by the same argument as in [16]. This completes the proof. ∎

Lemma 2.9. *For $r, r' \in \mathbb{R}_Y$, $s, s' \in \mathbb{R}$ with $s \le s' \le 0 \le r \le r'$, there exists a chain map*

$$i_{[s,r]}^{[s',r']}\colon CI_i^{[s,r]}(Y) \to CI_i^{[s',r']}(Y).$$

The map $i_{[s,r]}^{[s',r']}$ satisfies the following conditions:

(1) *The chain homotopy class of $i_{[s,r]}^{[s',r']}$ does not depend on the choice of additional data.*

(2) *If we take two triples of numbers (r, r', r''), (s, s', s'') with $s \le s' \le s'' \le 0 \le r \le r' \le r''$, then*
$$i_{[s',r']}^{[s,r]} \circ i_{[s'',r'']}^{[s',r']} = i_{[s'',r'']}^{[s,r]}$$
as induced maps on cohomology, where $i_{[s',r']}^{[s,r]}$, $i_{[s'',r'']}^{[s',r']}$ and $i_{[s'',r'']}^{[s,r]}$ are duals of $i_{[s,r]}^{[s',r']}$, $i_{[s',r']}^{[s'',r'']}$ and $i_{[s,r]}^{[s'',r'']}$.

(3) *If $[r, r'], [s, s'] \subset \mathbb{R} \setminus \Lambda_Y^*$, then the map $i_{[s,r]}^{[s',r']}$ gives a chain homotopy equivalence.*

Proof. First, we give the construction of $i_{[s,r]}^{[s',r']}$. One can take a non-degenerate regular perturbation π satisfying $\pi \in \mathcal{P}(Y, r, s, g) \cap \mathcal{P}(Y, r', s', g)$. This gives a natural map $i_{[s,r]}^{[s',r']}\colon CI_i^{[s,r]}(Y,\pi) \to CI_i^{[s',r']}(Y,\pi)$ by setting

$$i_{[s,r]}^{[s',r']}(a) := \begin{cases} a & \text{if } a \in CI_i^{[s',r']}(Y,\pi), \\ 0 & \text{otherwise.} \end{cases}$$

This yields a chain map. The proof of (1) is similar to the proof of independence from the choice of π; (2) is obvious, because we can take a non-degenerate regular perturbation

$$\pi \in \mathcal{P}(Y, r, s, g) \cap \mathcal{P}(Y, r', s', g) \cap \mathcal{P}(Y, r'', s'', g).$$

To prove (3), suppose that $[r, r'], [s, s'] \subset \mathbb{R} \setminus \Lambda_Y^*$. We take a sequence of non-degenerate regular perturbations $\{\pi_n\} \subset \mathcal{P}(Y, r, s, g) \cap \mathcal{P}(Y, r', s', g)$ such that $\|\pi_n\| \to 0$. We show that the following maps are bijective for sufficiently large n:

$$i_n\colon \{a \in \widetilde{R}(Y)_{\pi_n} \mid s < cs_{\pi_n}(a) < r\} \to \{a \in \widetilde{R}(Y)_{\pi_n} \mid s' < cs_{\pi_n}(a) < r'\}.$$

Suppose there is a sequence $\{n_k\}$ of positive integers such that $n_k \to \infty$ as $k \to \infty$ and i_{n_k} is not bijective for any $k \in \mathbb{Z}_{>0}$. Then we can take a sequence $\{b_k\}$ with $s < cs_{\pi_k}(b_k) < s'$ and $r < cs_{\pi_k}(b_k) < r'$. Using Uhlenbeck's compactness theorem, we bound $\|g_k^* b_k\|_{L_k^2(Y)} \le C_k$ for some gauge transformations. By taking a subsequence, we can take a limit connection b_∞. Since a reducible connection is isolated for a small perturbation π_{n_k}, we can assume that b_∞ is irreducible. Since $\|\pi_{n_k}\| \to 0$, b_∞ satisfies $F(b_\infty) = 0$ and $cs(b_\infty) \in [s, s'] \cup [r, r']$. This gives a contradiction. Therefore, i_k is a bijection for sufficiently large k. This completes the proof. ∎

2.4. Cobordism maps

First, let us fix the convention about oriented cobordisms. We use the *outward normal first* convention. For example,

$$\partial(Y \times [0,1]) \cong -\partial([0,1] \times Y) = \{0\} \times Y \amalg (-\{1\} \times Y)$$

for an oriented 3-manifold Y, where $\cong$ denotes orientation-preserving diffeomorphism. In this section, we review the cobordism maps for filtered instanton chain complexes. These maps have already been considered in [16]. We fix $s_j \in [-\infty, 0]$ for $1 \leq j \leq m$ and put $s = \sum_{1 \leq j \leq m} s_j$. Let Y^- be the finite disjoint union of oriented homology 3-spheres Y_j^- for $1 \leq j \leq m$, Y^+ an oriented homology sphere and W a negative definite connected cobordism with $\partial W = Y^+ \amalg (-Y^-)$ and $b_1(W) = 0$. We assume $s_j \in \mathbb{R}_{Y_j^-}$ for $1 \leq j \leq m$. First, we fix the following data related to ∂W:

- a Riemannian metric g on $\partial W = Y^+ \amalg -Y^-$ and
$$r \in \mathbb{R}_{Y^+} \cap \bigcap_{j=1}^{m} \mathbb{R}_{Y_j^-},$$
- non-degenerate regular perturbations $\pi^+ \in \mathcal{P}(Y^+, r, s, g_{Y^+})$ and $\pi_j^- \in \mathcal{P}(Y_j^-, r, r - s + s_j, g_{Y_j^-})$ for $1 \leq j \leq m$,
- for any $a \in R^*(Y^+)_{\pi^+}$ and $b_j \in R^*(Y_j^-)_{\pi_j^-}$, orientations of $\mathbb{L}_a$ and $\mathbb{L}_{b_j}$ for $1 \leq j \leq m$.

Using the above data, one can define filtered Floer chain complexes $(C_*^{[s,r]}(Y^+), \partial^r)$ and $(C_*^{[s,r]}(Y_j^-), \partial^r)$. Let us denote by W^* the end-cylindrical 4-manifold given by

$$Y^+ \times \mathbb{R}_{\leq 0} \cup W \cup Y^- \times \mathbb{R}_{\geq 0}.$$

We fix an orientation of W^* which agrees with the orientations on $Y^+ \times \mathbb{R}_{\leq 0}$ and $Y^- \times \mathbb{R}_{\geq 0}$, and a Riemannian metric g_{W^*} on W^* which coincides with the product metric of g and the standard metric of $\mathbb{R}$ on $Y^+ \times \mathbb{R}_{\leq 0} \amalg Y^- \times \mathbb{R}_{\geq 0}$. For $\mathfrak{a} \in R(Y^+)_{\pi^+}$ and $\mathfrak{b} = (b_j) \in \prod_{1 \leq j \leq m} R(Y_j^-)_{\pi_j^-}$, we can define the ASD moduli space

$$M(\mathfrak{a}, W^*, \mathfrak{b}) := \{A_{\mathfrak{a},\mathfrak{b}} + c \mid c \in \Omega^1(W^*) \otimes \mathfrak{su}(2)_{L^2_q}, (*)\}/\mathcal{G}(\mathfrak{a}, W^*, \mathfrak{b}),$$

where

$$F^+(A_{\mathfrak{a},\mathfrak{b}} + c) + \pi_W^+(A_{\mathfrak{a},\mathfrak{b}} + c) = 0,$$

$A_{\mathfrak{a},\mathfrak{b}}$ is an $SU(2)$-connection on W^* whose restrictions to the ends $Y^+ \times \mathbb{R}_{\leq -1} \cup Y^- \times \mathbb{R}_{\geq 1}$ coincide with the pull-backs of $\mathfrak{a}$ and $\mathfrak{b}$, and the group $\mathcal{G}(\mathfrak{a}, W^*, \mathfrak{b})$ is given similarly to the product case. If we take a limit connection θ, we use the weighted norm with a small positive weight as in the case of $Y \times \mathbb{R}$. The part π_W is a perturbation on W^* satisfying the following *conditions* $(**)$:

- The perturbation π_W coincides with π_j^- on $Y_j^- \times \mathbb{R}_{\geq 0}$ for any j and with π^+ on $Y^+ \times \mathbb{R}_{\leq 0}$.

- For $a \in \Omega^1(W)_{L^2_q}$,

$$\|\pi_W^+(a)\|_{L^2} < \frac{1}{8}\min_j \left\{ \begin{matrix} d(r, \Lambda_{Y_j^-}), \lambda_{Y_j^-}, d(r-s+s_j, \Lambda_{Y_j^-}), \\ d(r, \Lambda_{Y^+}), \lambda_{Y^+}, d(s, \Lambda_{Y^+}), d(s_j, \Lambda_{Y_j^-}) \end{matrix} \right\}.$$

- For any irreducible element $A \in M(\mathfrak{a}, W^*, \mathfrak{b})$,

$$d_A^+ + d(\pi_W^+)_A \colon \Omega^1(W^*) \otimes \mathfrak{su}(2)_{L^2_q} \to \Omega^+(W^*) \otimes \mathfrak{su}(2)_{L^2_{q-1}}$$

  is surjective, where $d(\pi_W^+)_A$ is the linearization of π_W^+. If $\mathfrak{a}$ or $\mathfrak{b}$ contains the reducible connection θ, we need to consider the weighted norm as in the case of $Y \times \mathbb{R}$.

Now we explain how to give an orientation of $M(\mathfrak{a}, W^*, \mathfrak{b})$. Let X_j^- and X^+ be compact oriented 4-manifolds with $\partial(X_j^-) = Y_j^-$ and $\partial(X^+) = Y^+$. Then we obtain a continuous map

$$\mathfrak{gl} \colon M(\mathfrak{a}, W^*, \mathfrak{b}) \times \prod_{j=1}^m \mathcal{B}(b_j, Y_j^-) \to \mathcal{B}\Big(a, W \cup \bigcup_{j=1}^m X_j^-\Big)$$

by gluing connections using cut-off functions. This gives a bundle map

$$\widetilde{\mathfrak{gl}} \colon \operatorname{Det} TM(\mathfrak{a}, W^*, \mathfrak{b}) \otimes \bigotimes_{j=1}^m \mathbb{L}_{b_j} \to \mathfrak{gl}^* \mathbb{L}_a.$$

We fix the orientation of $\operatorname{Det} TM(\mathfrak{a}, W^*, \mathfrak{b})$ so that $\widetilde{\mathfrak{gl}}$ is orientation-preserving with respect to the orientations induced from the orientations of the orientation bundles. Then by computing the 0-dimensional part of $M(\mathfrak{a}, W^*, \mathfrak{b})$, we get a map

$$CW_i^{[s,r]} \colon CI_i^{[s,r]}(Y^+) \to \bigoplus_{\substack{\sum_j l_j = i \\ 0 \le l_j \le i}} \bigotimes_{j=1}^m CI_{l_j}^{[s,r]}(Y_j^-)$$

for $i \in \mathbb{Z}$. In this paper, we use only the cases of $m = 1$ and $m = 2$. In particular, for $m = 2$, we also use the map

$$\widetilde{CW}_i^{[s,r]} \colon CI_i^{[s,r]}(Y^+) \to CI_i^{[s,r]}(Y_1^-) \oplus CI_i^{[s,r]}(Y_2^-)$$

defined via the 0-dimensional moduli spaces $M(\mathfrak{a}, W^*, (b, \theta))$ and $M(\mathfrak{a}, W^*, (\theta, b))$. (We use the weighted norm here.)

The following is the key lemma of this paper. Roughly speaking, it implies that the cobordism maps are filtered.

Lemma 2.10. (1) *Suppose $m = 1$. Let $r \in \mathbb{R}_{Y^+} \cap \mathbb{R}_{Y_1^-}$, $s \in [-\infty, r)$, $\pi^+ \in \mathcal{P}(Y^+, r, s, g)$ and $\pi_1^- \in \mathcal{P}(Y_1^-, r, s, g_1^-)$. Let $\mathfrak{a} = a \in R(Y^+)_{\pi^+}$ and $\mathfrak{b} = b \in R(Y_1^-)_{\pi_1^-}$. Suppose that $M(\mathfrak{a}, W^*, \mathfrak{b}) \neq \emptyset$ for some perturbation π_W satisfying $(**)$, and $a \in C_*^{[s,r]}(Y^+)$. Then $cs_{\pi_1^-}(b) < r$.*

(2) *Suppose* $m=2$. *Let* $r \in \mathbb{R}_{Y^+} \cap \mathbb{R}_{Y_1^-} \cap \mathbb{R}_{Y_2^-}$ *and* $s, s_1, s_2 \in [-\infty, r)$ *with* $s = s_1 + s_2$, $r - s_1 \in \mathbb{R}_{Y_2^-}$ *and* $r - s_2 \in \mathbb{R}_{Y_1^-}$ *and choose perturbations* $\pi^+ \in \mathcal{P}(Y^+, r, r', g)$, $\pi_1^- \in \mathcal{P}(Y_1^-, r - s_2, s_1, g_1^-)$ *and* $\pi_1^- \in \mathcal{P}(Y_2^-, r - s_1, s_2, g_2^-)$. *Let* $a \in R(Y^+)_{\pi^+}$ *and* $\mathfrak{b} = (b_1, b_2) \in R(Y_1^-)_{\pi_1^-} \times R(Y_2^-)_{\pi_2^-}$. *If* $M(\mathfrak{a}, W^*, \mathfrak{b}) \neq \emptyset$ *for some perturbation* π_W *satisfying* $(**)$, $s < cs_{\pi^+}(a) < r$ *and* $s_1 < cs_{\pi_1^-}(b_1) < r - s_2$, *then* $cs_{\pi_2^-}(b_2) < r - s_1$.

(3) *Under the same assumption as in Lemma* 2.4 (2), *the following holds. Let* $a \in R(Y^+)_{\pi^+}$ *and* $b_2 \in R(Y_2^-)_{\pi_2^-}$. *Suppose that* $M(a, W^*, (\theta, b_2)) \neq \emptyset$ *for some perturbation* π_W *satisfying* $(**)$, *and* $s < cs_{\pi^+}(a) < r$. *Then* $cs_{\pi_2^-}(b_2) < r - s_1$.

(4) *Under the same assumption as in Lemma* 2.4 (2), *additionally assume that* $s_1 + s_2 \in \mathbb{R}_{Y^+}$. *Let* $a \in R(Y^+)_{\pi^+}$ *and* $\mathfrak{b} = (b_1, b_2) \in R(Y_1^-)_{\pi_1^-} \times R(Y_2^-)_{\pi_2^-}$. *If* $M(\mathfrak{a}, W^*, \mathfrak{b}) \neq \emptyset$ *for some perturbation* π_W *satisfyng* $(**)$, $b_1 \in C^{[s_1, r-s_2]}(Y_1^-)$ *and* $b_2 \in C^{[s_2, r-s_1]}(Y_2^-)$, *then* $cs_{\pi^+}(a) > s_1 + s_2$.

(5) *Under the same assumption as in Lemma* 2.4 (2), *additionally assume that* $s_1 + s_2 \in \mathbb{R}_{Y^+}$. *Let* $a \in R(Y^+)_{\pi^+}$ *and* $b_2 \in R(Y_2^-)_{\pi_2^-}$. *If* $M(a, W^*, (\theta, b_2)) \neq \emptyset$ *for some perturbation* π_W *satisfying* $(**)$, *and* $b_2 \in C^{[s_2, r-s_1]}(Y_2^-)$, *then* $cs_{\pi^+}(a) > s_1 + s_2$.

Proof. First, let us show (1). By Lemma 2.4, we have

$$|r - cs_{\pi_1^-}(b)| > \tfrac{3}{4} d(r, \Lambda_{Y^-}).$$

If $r - cs_{\pi_1^-}(b) > \frac{3}{4} d(r, \Lambda_{Y^-})$, then this is the conclusion, so assume

$$r - cs_{\pi_1^-}(b) < -\tfrac{3}{4} d(r, \Lambda_{Y^-}). \tag{12}$$

Let A be an element in $M(a, W^*, b)$. We set $A_+ = A|_{\partial(Y^+ \times \mathbb{R}_{\leq 0})}$ and $A_- = A|_{\partial(Y^- \times \mathbb{R}_{\geq 0})}$. Since A is a flow of grad cs_{π^+} on $Y^+ \times \mathbb{R}_{\leq 0}$,

$$cs_{\pi^+}(a) \geq cs_{\pi^+}(A_+).$$

We also have $cs_{\pi_1^-}(A_-) \geq cs_{\pi_1^-}(b)$ by the same argument. Moreover, we see that

$$\begin{aligned} &cs_{\pi_1^-}(A_-) - cs_{\pi^+}(A_+) \\ &\qquad = (cs_{\pi_1^-}(A_-) - cs(A_-)) - (cs_{\pi^+}(A_+) - cs(A_+)) + cs(A_-) - cs(A_+) \\ &\qquad \leq \max\left\{\frac{1}{4} d(r, \Lambda_{Y^+}), \frac{1}{4} d(r, \Lambda_{Y^-})\right\} - \frac{1}{8\pi^2} \int_W \mathrm{Tr}(F(A) \wedge F(A)). \end{aligned}$$

Here, the second term is bounded by $\frac{1}{8} \min\{d(r, \Lambda_{Y^+}), d(r, \Lambda_{Y_1^-})\}$ because

$$\begin{aligned} -\frac{1}{8\pi^2} \int_W \mathrm{Tr}(F(A) \wedge F(A)) &= -\frac{1}{8\pi^2} \int_W \mathrm{Tr}\big((F^+(A) + F^-(A)) \wedge (F^+(A) + F^-(A))\big) \\ &= -\frac{1}{8\pi^2} \int_W \mathrm{Tr}(\pi_W^+(A) \wedge \pi_W^+(A)) + \frac{1}{8\pi^2} \int_W \mathrm{Tr}(F^-(A) \wedge * F^-(A)) \\ &\leq \frac{1}{8} \min\{d(r, \Lambda_{Y^+}), d(r, \Lambda_{Y^-})\} - \frac{1}{8\pi^2} \|F^-(A)\|^2_{L^2(W)} \end{aligned}$$

by the choice of π_W. Therefore,

$$cs_{\pi_1^-}(A_-) - cs_{\pi^+}(A_+) \le \tfrac{3}{8}\max\{d(r,\Lambda_{Y^+}), d(r,\Lambda_{Y_1^-})\}.$$

On the other hand,

$$cs_{\pi^+}(a) < -\tfrac{3}{4}d(r,\Lambda_{Y^+}) + r$$

by Lemma 2.4. Now we have

$$cs_{\pi_1^-}(b) < r - \tfrac{3}{4}d(r,\Lambda_{Y^+}) + \tfrac{3}{8}\max\{d(r,\Lambda_{Y^+}), d(r,\Lambda_{Y_1^-})\}. \tag{13}$$

Combining (12) and (13), we get

$$0 < -\tfrac{3}{4}d(r,\Lambda_{Y^+}) - \tfrac{3}{4}d(r,\Lambda_{Y_1^-}) + \tfrac{3}{8}\max\{d(r,\Lambda_{Y^+}), d(r,\Lambda_{Y_1^-})\}.$$

This gives a contradiction.

Next, we show (2). Using Lemma 2.4, we have

$$|r - s_1 - cs_{\pi_2^-}(b_2)| > \tfrac{3}{4}d(r - s_1, \Lambda_{Y_2^-}).$$

As above, we assume

$$r - s_1 - cs_{\pi_2^-}(b_2) < -\tfrac{3}{4}d(r - s_1, \Lambda_{Y_2^-}).$$

Suppose $A \in M(\mathfrak{a}, W^*, \mathfrak{b})$, $A_+ = A|_{Y^+\times\mathbb{R}_{\le 0}}$, $A_-^1 = A|_{Y_1^-\times\mathbb{R}_{\ge 0}}$ and $A_-^2 = A|_{Y_2^-\times\mathbb{R}_{\ge 0}}$. Then

$$\begin{aligned}-r + \tfrac{3}{4}d(r,\Lambda_{Y^+}) + s_1 + \tfrac{3}{4}d(s_1,\Lambda_{Y_1^-}) + cs_{\pi_2^-}(b_2)&\\ &\le -cs_{\pi^+}(A_+) + cs_{\pi_1^-}(A_-^1) + cs_{\pi_2^-}(A_-^2)\\ &\le \tfrac{1}{2}\max\{d(r,\Lambda_{Y^+}), d(s_1,\Lambda_{Y_1^-}), d(r - s_1,\Lambda_{Y_2^-})\}\end{aligned}$$

by the same argument. These give a contradiction.

Let us show (3). By using Lemma 2.4, we have

$$|r - s_1 - cs_{\pi_2^-}(b_2)| > \tfrac{3}{4}d(r - s_1, \Lambda_{Y_2^-}).$$

As above, we assume

$$r - s_1 - cs_{\pi_2^-}(b_2) < -\tfrac{3}{4}d(r - s_1, \Lambda_{Y_2^-}).$$

By a similar discussion, since $cs_{\pi_1^-}(\theta) = 0$, we have

$$\begin{aligned}-r + \tfrac{3}{4}d(r,\Lambda_{Y^+}) + s_1 + \tfrac{3}{4}d(s_1,\Lambda_{Y_1^-}) + cs_{\pi_2^-}(b_2)&\\ &\le \tfrac{1}{2}\max\{d(r,\Lambda_{Y^+}), d(s_1,\Lambda_{Y_1^-}), d(r - s_1,\Lambda_{Y_2^-})\}.\end{aligned}$$

This gives a contradiction.

Next, we show (4). By using Lemma 2.4, we have

$$|cs_{\pi^+}(a) - s| > \tfrac{3}{4} d(s, \Lambda_{Y^+}).$$

As above, we assume

$$cs_{\pi^+}(a) - s < -\tfrac{3}{4} d(s, \Lambda_{Y^+}).$$

For $A \in M(\mathfrak{a}, W^*, \mathfrak{b})$, $A_+ = A|_{Y^+ \times \mathbb{R}_{\leq 0}}$, $A_-^1 = A|_{Y_1^- \times \mathbb{R}_{\geq 0}}$ and $A_-^2 = A|_{Y_2^- \times \mathbb{R}_{\geq 0}}$, one has

$$\begin{aligned} s_1 + \tfrac{3}{4} d(s_1, \Lambda_{Y_1^-}) + s_2 + \tfrac{3}{4} d(s_2, \Lambda_{Y_2^-}) + cs_{\pi^+}(a) \\ \leq \tfrac{1}{2} \max \{d(s, \Lambda_{Y^+}), d(s_1, \Lambda_{Y_1^-}), d(s_2, \Lambda_{Y_2^-})\}. \end{aligned}$$

These give a contradiction. The proof of (5) is similar to that of (4). ∎

Fintushel–Stern proved the following lemma. This is a corollary of Lemma 2.10 under the assumption $m = 1$.

Lemma 2.11. *For $r \in \mathbb{R}_{Y^+} \cap \mathbb{R}_{Y^-}$,*

$$CW^{[s,r]} \partial^{Y^+}_{[s,r]} = \partial^{Y^-}_{[s,r]} CW^{[s,r]}.$$

We denote the induced map of $CW^{[s,r]}$ (resp. $CW_{[s,r]}$) on instanton Floer (co)homology by $IW^{[s,r]}$ (resp. $IW_{[s,r]}$).

2.5. *Obstruction class*

In this section, we give a refinement of D_1 that appeared in Donaldson's book [12], which computes gradient flows of Chern–Simons functionals between irreducible critical points and the product connection. We define a filtered version of D_1.

Let Y be an oriented homology sphere. For $r \in \mathbb{R}_Y \cap [0, \infty]$ and $s \in [-\infty, 0]$, we now define an invariant in $I^1_{[s,r]}(Y)$. A version of this invariant is defined in the third author's paper [57].

Definition 2.12. We define a homomorphism $\theta^{[s,r]}_Y \colon CI^{[s,r]}_1(Y) \to \mathbb{Z}$ by

$$\theta^{[s,r]}_Y([a]) := \#(M^Y(a, \theta)_{\pi,\delta}/\mathbb{R}). \tag{14}$$

As in [12, Section 3.3.1] and [21, Section 2.1], we use the weighted $L^2_{q,\delta}$ norm in (5) for $M^Y(a, \theta)_{\pi,\delta}$ to use Fredholm theory. (In [12, Section 3.3.1] and [21, Section 2.1], θ is denoted by D_1 or δ.) By the same discussion as in the proof of $(\delta^{[s,r]})^2 = 0$, we can show $\delta^{[s,r]}(\theta^{[s,r]}_Y) = 0$. Therefore we get the class $[\theta^{[s,r]}_Y] \in I^1_{[s,r]}(Y)$.

The class $[\theta^{[s,r]}_Y]$ does not depend on the small perturbation or the metric. The proof is similar to the proof for the original $[\theta]$.

Lemma 2.13. *Let Y_1 and Y_2 be oriented homology spheres. Suppose that there is an oriented negative definite cobordism W with $H^1(W;\mathbb{R}) = 0$ and $\partial W = Y_1 \amalg (-Y_2)$. For $r \in \mathbb{R}_{Y_1} \cap \mathbb{R}_{Y_2}$, we have*

$$IW_{[s,r]}[\theta_{Y_2}^{[s,r]}] = c(W)[\theta_{Y_1}^{[s,r]}], \quad \text{where} \quad c(W) := \# H_1(W;\mathbb{Z}).$$

Proof. We first consider the case of $H^1(W;\mathbb{Z}) = 0$. First we fix Riemannian metrics g_i on Y_i, non-degenerate regular perturbations $\pi_i \in \mathcal{P}(Y_i, r, g_i)$ for $i = 1, 2$, and orientations of $\lambda_{a,X}$ for every critical point a. To consider the cobordism map induced by W, we fix a perturbation π_W satisfying conditions $(**)$. For fixed $a \in \widetilde{R}(Y_1)_{\pi_1}$ satisfying $cs_{\pi_1}(a) < r$ and $\operatorname{ind}(a) = 1$, we consider the end-cylindrical manifold W^* and the moduli space $M(a, W^*, \theta)$ as above. We can choose a perturbation π_W such that $M(a, W^*, \theta)$ has the structure of a 1-manifold. There is a natural orientation on $M(a, W^*, \theta)$ induced by the orientations of $\lambda_{a,X}$. We now describe the end of a compactification of $M(a, W^*, \theta)$.

By dimension counting and instanton gluing, we have two maps: $\mathfrak{gl}_1$ from

$$\Big(\bigcup_{b \in \widetilde{R}^*(Y_1)_{\pi_1},\, \operatorname{ind}(b)=0} M^{Y_1}(a,b)_{\pi_1}/\mathbb{R} \times M(b, W^*, \theta) \cup M^{Y_1}(a,\theta)/\mathbb{R} \times M(\theta, W^*, \theta) \Big) \times (0,\infty)$$

to $M(a, W^*, \theta)$, and

$$\mathfrak{gl}_2\colon \Big(\bigcup_{c \in \widetilde{R}^*(Y_2)_{\pi_2},\, \operatorname{ind}(c)=1} M(a, W^*, c) \times M^{Y_2}(c,\theta)_{\pi_2}/\mathbb{R} \Big) \times (-\infty, 0) \to M(a, W^*, \theta).$$

These are diffeomorphisms onto their images. Also, the complement of the union of their images is compact.

Claim 2.14. *By the definition of the orientations of $M^{Y_1}(a,b)_{\pi_1}/\mathbb{R}$, $M^{Y_1}(a,\theta)_{\pi_1}/\mathbb{R}$, and $M^{Y_2}(c,\theta)_{\pi_2}/\mathbb{R}$, the maps $\mathfrak{gl}_1$ and $\mathfrak{gl}_2$ are orientation-preserving.*

More details about orientations of the moduli spaces are explained in the proof of Claim 2.14 written after the proof of Lemma 2.13. Using the maps $\mathfrak{gl}_1$ and $\mathfrak{gl}_2$, one can compactify $M(a, W^*, \theta)$. The end of the compactified moduli space is the disjoint union of three types of oriented points:

$$\bigcup_{b \in \widetilde{R}^*(Y_1)_{\pi_1},\, \operatorname{ind}(b)=0} M^{Y_1}(a,b)_{\pi_1}/\mathbb{R} \times M(b, W^*, \theta),$$

$$M^{Y_1}(a,\theta)/\mathbb{R} \times M(\theta, W^*, \theta),$$

$$- \bigcup_{c \in \widetilde{R}^*(Y_2)_{\pi_2},\, \operatorname{ind}(c)=1} M(a, W^*, c) \times M^{Y_2}(c,\theta)_{\pi_2}/\mathbb{R}.$$

Here we follow the orientations convention in [12, Section 5.4]. Since the first homology of W^* is zero and the formal dimension of $M(\theta, W^*, \theta)$ is -3, there is no reducible connection except for θ in $M(\theta, W^*, \theta)$. So the space $M(\theta, W^*, \theta)$ has just one point.

When the space $M^Y(a,b)_\pi/\mathbb{R}$ is non-empty, we have $cs_{\pi_1}(a) > cs_{\pi_2}(b)$. Therefore, the first case can be written

$$\bigcup_{b\in\widetilde{R}^*(Y_1)_{\pi_1},\, \mathrm{ind}(b)=0,\, cs_{\pi_1}(b)<r} M^{Y_1}(a,b)_{\pi_1}/\mathbb{R}\times M(b,W^*,\theta).$$

If we can write the third case as

$$-\bigcup_{c\in\widetilde{R}^*(Y_2)_{\pi_2},\, \mathrm{ind}(c)=1,\, cs_{\pi_2}(c)<r} M(a,W^*,c)\times M(c,\theta)_{\pi_2}/\mathbb{R},$$

these computation imply that

$$\delta^r(n^r)(a) + \theta^{[s,r]}_{Y_1}(a) = CW^r\theta^{[s,r]}_{Y_2}(a) \tag{15}$$

for any $a \in CI^{[s,r]}_1(Y_1)$. If $M(a,W^*,c)$ is non-empty, then by the use of Lemma 2.10 (1) and $cs_{\pi_1}(a) < r$, we get

$$cs_{\pi_2}(c) < r.$$

Next, we handle the case of $H^1(W;\mathbb{R}) = 0$. We need to consider the transversality of the moduli space $M(\theta, W^*, \theta)$. As explained in [6, (2.16)] and [11], there are two types of reducible flat connections and we can take a perturbation π_W so that $M(\theta, W^*, \theta)$ is a finite set. Moreover, Donaldson [11] showed that the signs of the points are the same. One can count the points and see that $\#M(\theta, W^*, \theta) = \#H_1(W;\mathbb{Z})$. Using the above perturbation, we have

$$\delta^r(n^r)(a) + c(W)\theta^{[s,r]}_{Y_1}(a) = CW^r\theta^{[s,r]}_{Y_2}(a),$$

where $c(W) = \#H_1(W;\mathbb{Z})$. ■

Proof of Claim 2.14. We discuss the orientations of the moduli spaces. Let us consider the restrictions

$$M^{Y_1}(a,b)_{\pi_1}/\mathbb{R}\times M(b,W^*,\theta)\times(0,\infty)\to M(a,W^*,\theta),$$
$$M(a,W^*,c)\times M^{Y_2}(c,\theta)_{\pi_2}/\mathbb{R}\times(-\infty,0)\to M(a,W^*,\theta)$$

of the maps $\mathfrak{gl}_1$ and $\mathfrak{gl}_2$, respectively, where $b \in \widetilde{R}^*(Y_1)_{\pi_1}$ with $\mathrm{ind}(b) = 0$ and $c \in \widetilde{R}^*(Y_2)_{\pi_2}$ with $\mathrm{ind}(c) = 1$. We just give a sketch of the proof for these maps being orientation-preserving. Note that the case $b = \theta$ is shown similarly.

(i) We first introduce the configuration spaces $\mathcal{B}^{Y_1}(a,b)$, $\mathcal{B}(b,W^*,\theta)$, $\mathcal{B}(a,W^*,c)$ and $\mathcal{B}^{Y_2}(c,\theta)$ containing the moduli spaces $M^{Y_1}(a,b)$, $M(b,W^*,\theta)$, $M(a,W^*,c)$ and $M^{Y_2}(c,\theta)$, respectively, which are defined similarly to (9).

(ii) As introduced in (10), using the sliced and linearized ASD map $d^*_A + d^+_A + d\pi^+_A$, we have the determinant line bundles $\mathbb{L}(a,b)$, $\mathbb{L}(b,W^*,\theta)$, $\mathbb{L}(a,W^*,c)$ and $\mathbb{L}(c,\theta)$ over $\mathcal{B}^{Y_1}(a,b)$, $\mathcal{B}(b,W^*,\theta)$, $\mathcal{B}(a,W^*,c)$ and $\mathcal{B}^{Y_2}(c,\theta)$, respectively. Let $\Lambda^{Y_1}(a,b)$, $\Lambda(b,W^*,\theta)$, $\Lambda(a,W^*,c)$ and $\Lambda^{Y_2}(c,\theta)$ denote the sets of orientations of $\mathbb{L}(a,b)$, $\mathbb{L}(b,W^*,\theta)$, $\mathbb{L}(a,W^*,c)$ and $\mathbb{L}(c,\theta)$, respectively.

(iii) The (homotopy classes of) pregluing maps

$$\mathcal{B}^{Y_1}(a,b) \times \mathcal{B}(b, W^*, \theta) \to \mathcal{B}(a, W^*, \theta),$$
$$\mathcal{B}(a, W^*, c) \times \mathcal{B}^{Y_2}(c, \theta) \to \mathcal{B}(a, W^*, \theta)$$

give identifications

$$i_1 \colon \Lambda^{Y_1}(a,b) \times_{\mathbb{Z}_2} \Lambda(b, W^*, \theta) \to \Lambda(a, W^*, \theta),$$
$$i_2 \colon \Lambda(a, W^*, c) \times_{\mathbb{Z}_2} \Lambda^{Y_2}(c, \theta) \to \Lambda(a, W^*, \theta),$$

where the $\mathbb{Z}_2$-actions on the set of orientations are non-trivial. See [12, Proposition 5.11] for the details of the construction. A similar argument can be found in [37, Section 20.3].

(iv) We now fix orientations of the bundle λ_{a,X_1} defined in (11) for critical points a of the perturbed Chern–Simons functional of Y_1, where X_1 is a compact 4-manifold bounded by $-Y_1$. Note that we have the canonical homology orientation of W since $b_1(W) = 0$ and $b^+(W) = 0$. Using these data, we associate an element of $\Lambda(a, W^*, \theta)$ with each critical point a by the following discussion: An excision argument for determinant line bundles similar to [12, Proposition 5.11, p. 132] gives an identification

$$\{\text{The set of orientations of } \lambda_{a,X_1}\} \times_{\mathbb{Z}_2} \Lambda(a, W^*, \theta) \to \{\text{The set of orientations of } \lambda_{\theta, X_1 \cup_{Y_2} W}\}. \qquad (16)$$

Here we have used the canonical homology orientation of W. By the definition of $\lambda_{a,X}$, there is a canonical orientation of $\lambda_{\theta, X_1 \cup_{Y_2} W}$. For the orientation of λ_{a,X_1} in (iv), one has the corresponding orientation in $\Lambda(a, W^*, \theta)$ via (16) and the canonical orientation of $\lambda_{\theta, X_1 \cup_{Y_2} W}$. We fix these orientations in $\Lambda(a, W^*, \theta)$ for all a. We also take an element in $\Lambda^{Y_1}(a,b)$ which is compatible with fixed elements in $\Lambda(b, W^*, \theta)$ and $\Lambda(a, W^*, \theta)$ under i_1. Here, for a sufficiently small T_1, one can consider a gluing map

$$\mathfrak{g}'_1 \colon \{[A_1] \in M^{Y_1}(a,b)_{\pi_1} \mid c(A_1) < T_1\} \times M(b, W^*, \theta) \to M(a, W^*, \theta),$$

where $c(A)$ denotes the center of the density function $\|F(A) + \pi(A)\|_{Y \times \{r\}}$ ($r \in \mathbb{R}$) for a finite (perturbed) energy $SU(2)$-connection A on $Y \times \mathbb{R}$.

Let us explain the construction of $\mathfrak{g}'_1$, which implies that $\mathfrak{g}'_1$ is orientation-preserving. For $([A_1],[A_2]) \in M^{Y_1}(a,b)_{\pi_1} \times M(b, W^*, \theta)$ with $c(A_1) \ll 0$, we first fix representatives (A_1, A_2) of $([A_1],[A_2])$ so that the exponential decay estimates [12, Proposition 4.3] are satisfied. In particular, under the assumption $c(A_1) \ll 0$, there exist positive constants C_k and δ such that

$$\|A_1|_{Y_1 \times [s,s+1]} - p^* b\|_{L^2_k(Y_1 \times [s,s+1])} \le C_k e^{\delta(c(A_1) - s)} \quad \text{for } c(A_1) \le s \le 0,$$

where p denotes the projection $Y_1 \times \mathbb{R} \to Y_1$. Similarly, there exist $C'_k > 0$ and $\delta' > 0$ such that

$$\|A_2|_{Y_1 \times [s,s+1]} - p^* b\|_{L^2_k(Y_1 \times [s,s+1])} \le C'_k e^{\delta' s} \quad \text{for } s \le -1.$$

We now define an $SU(2)$-connection $\psi(A_1, A_2)$ on W^* called a pregluing by

$$\psi(A_1, A_2) = \begin{cases} A_1 & \text{on } Y_1 \times \left(-\infty, \frac{3}{4}c(A_1)\right), \\ A_1\psi_{c(A_1)}(-) + (1-\psi_{c(A_1)}(-))p^*b & \text{on } Y_1 \times \left(\frac{3}{4}c(A_1)-1, \frac{1}{2}c(A_1)+\frac{1}{2}\right), \\ (1-\psi_{c(A_1)}(-))p^*b + A_2\psi_{c(A_1)}(-) & \text{on } Y_1 \times \left(\frac{1}{2}c(A_1)-\frac{1}{2}, \frac{1}{4}c(A_1)+1\right), \\ A_2 & \text{on } Y_1 \times \left(\frac{1}{4}c(A_1), 0\right] \cup W \cup Y_2 \times [0,\infty). \end{cases}$$

Here $\psi_s\colon (\frac{3}{4}s-1, \frac{1}{4}s+1) \to \mathbb{R}$ is a smooth cut-off function satisfying

$$\psi_s(t) = \begin{cases} 1 & \text{if } t \in \left(\frac{3}{4}s-1, \frac{3}{4}s\right) \cup \left(\frac{1}{4}s, \frac{1}{4}s+1\right), \\ 0 & \text{if } t \in \left(\frac{1}{2}s-\frac{1}{2}, \frac{1}{2}s+\frac{1}{2}\right) \end{cases}$$

and $|d\psi_s| \leq c/|s|$ for $s \ll 0$, where $c > 0$ is a constant independent of s. It follows from the exponential decay estimates that there exist $\delta'' > 0$ and $C''_k > 0$ such that

$$\|F^+(\psi(A_1, A_2)) + \pi^+(\psi(A_1, A_2))\|_{L^2_{k-1}(W^*)} \leq C''_k e^{\delta'' c(A_1)}.$$

We perturb the connection $\psi(A_1, A_2)$ to obtain a solution by the following argument based on [52, proof of Theorem 9.1, p. 854]. First, for the above (A_1, A_2) with $c(A_1) \ll 0$, we can prove that the operator

$$\mathcal{D}_{\psi(A_1,A_2)} := d^*_{\psi(A_1,A_2)} + d^+_{\psi(A_1,A_2)} + d\pi^+_{\psi(A_1,A_2)}$$

has a right inverse $Q_{\psi(A_1,A_2)}$ with uniform bounds stated in [52, (112), (113)]. Then we apply the implicit function theorem for the perturbed ASD equation $F^+(\psi(A_1, A_2) + a) + \pi^+(\psi(A_1, A_2) + a) = 0$ with a slice $d^*_{\psi(A_1,A_2)}(a) = 0$ and obtain a solution $a(A_1, A_2)$ to these equations. For a sufficiently small T_1, we define the gluing map

$$\mathfrak{g}'_1\colon \{[A_1] \in M^{Y_1}(a, b)_{\pi_1} \mid c(A_1) < T_1\} \times M(b, W^*, \theta) \to M(a, W^*, \theta)$$

by

$$\mathfrak{g}'_1([A_1], [A_2]) = [\psi(A_1, A_2) + a(A_1, A_2)].$$

This map is orientation-preserving by construction. Similarly, under suitable orientations, the map

$$\mathfrak{g}'_2\colon M(a, W^*, c) \times \{[A_2] \in M^{Y_2}(c, \theta)_{\pi_2} \mid c(A_2) > T_2\} \to M(a, W^*, \theta)$$

is defined and orientation-preserving for a sufficiently large T_2.

(v) Now, we orient the moduli spaces $M^{Y_1}(a,b)_{\pi_1}/\mathbb{R}$ and $M^{Y_2}(c,\theta)_{\pi_2}/\mathbb{R}$ so that

$$(M^{Y_1}(a,b)_{\pi_1}/\mathbb{R}) \times \mathbb{R} = M^{Y_1}(a,b)_{\pi_1} \text{ and } (M^{Y_2}(c,\theta)_{\pi_2}/\mathbb{R}) \times \mathbb{R} = M^{Y_2}(c,\theta)_{\pi_2} \tag{17}$$

as oriented ($\mathbb{R}$-equivariant) manifolds, where the orientations of $M^{Y_1}(a, b)_{\pi_1}$ and $M^{Y_2}(c, \theta)_{\pi_2}$ are those in (iv). We now fix the convention of the $\mathbb{R}$-action as

$c(s \cdot A) = c(A) - s$ for $s \in \mathbb{R}$. Here, we identify $\{[A_1] \in M^{Y_1}(a,b)_{\pi_1} \mid c(A_1) < T_1\}$ with $(M^{Y_1}(a,b)_{\pi_1}/\mathbb{R}) \times (-T_1, \infty)$ by sending $[A_1]$ to $([A_1]/\mathbb{R}, -c(A_1))$, which is orientation-preserving due to the fact $c(r \cdot A) = c(A) - r$ for all $r \in \mathbb{R}$. Also, one has a similar identification between $\{[A_2] \in M^{Y_2}(a,b)_{\pi_2} \mid c(A_2) > T_2\}$ and $(M^{Y_2}(a,b)_{\pi_2}/\mathbb{R}) \times (-\infty, -T_2)$. By these identifications, $\mathfrak{g}'_1$ and $\mathfrak{g}'_2$ induce orientation-preserving maps

$$\mathfrak{g}_1\colon M^{Y_1}(a,b)_{\pi_1}/\mathbb{R} \times (-T_1,\infty) \times M(b,W^*,\theta) \to M(a,W^*,\theta),$$
$$\mathfrak{g}_2\colon M(a,W^*,c) \times M^{Y_2}(c,\theta)_{\pi_2}/\mathbb{R} \times (-\infty,-T_2) \to M(a,W^*,\theta)$$

with respect to the orientations fixed in (iv) and (v). Since $\dim M(b,W^*,\theta) = 0$, we also have an identification

$$M^{Y_1}(a,b)_{\pi_1}/\mathbb{R} \times M(b,W^*,\theta) \times (-T_1,\infty) = M^{Y_1}(a,b)_{\pi_1}/\mathbb{R} \times (-T_1,\infty) \times M(b,W^*,\theta)$$

as oriented manifolds. Thus, $\mathfrak{g}_1$ induces an orientation-preserving map

$$\mathfrak{g}_1\colon M^{Y_1}(a,b)_{\pi_1}/\mathbb{R} \times (-T_1,\infty) \times M(b,W^*,\theta) \to M(a,W^*,\theta).$$

This completes the sketch of the proof. ∎

The following property of the class $\theta_Y^{[s,r]}$ is useful when studying the invariants $\{r_s\}$.

Lemma 2.15. *For $s, s' \in \mathbb{R}_Y$ and $r, r' \in \mathbb{R}$ with $s \le s' \le 0 \le r \le r'$,*

$$i^{[s,r]}_{[s',r']}[\theta_Y^{[s',r']}] = [\theta_Y^{[s,r]}].$$

Proof. This property follows from the construction of $i^{[s,r]}_{[s',r']}$ in Lemma 2.9. ∎

3. The invariant r_s

3.1. Definition and invariance

We now introduce a family of invariants of an oriented homology 3-sphere Y. The definition of our invariants uses the birth-death property of our obstruction class $[\theta_Y^{[s,r]}]$ given in the previous section.

Before introducing our invariant $r_s(Y)$, we need to prove the following lemma.

Lemma 3.1. *Let R be a commutative ring with 1. For any homology 3-sphere Y,*

$$\{r \in (0,\infty] \mid 0 = [\theta_Y^{[s,r]} \otimes \mathrm{Id}_R] \in I^1_{[s,r]}(Y;R)\} \neq \emptyset$$

for any $s \in [-\infty, 0]$, where Id_R is the identity map on R.

Proof. Suppose that

$$\{r \in (0,\infty] \mid 0 = [\theta_Y^{[s,r]} \otimes \mathrm{Id}_R] \in I^1_{[s,r]}(Y;R)\} = \emptyset.$$

Then there exists a sequence $r_n \in \mathbb{R}_Y$ with $0 \neq [\theta_Y^{[s,r_n]}] \in I^1_{[s,r_n]}(Y)$ and $0 < r_n \to 0$. We take a sequence of non-degenerate regular perturbations π_n in $\mathcal{P}(Y, g, s, r_n)$ satisfying the following conditions:

- $\|\pi_n\| \to 0$.
- There exists a small neighborhood U of $[\theta] \in \widetilde{\mathcal{B}}^*(Y)$ such that $(h_n)_f|_U = 0$, where $\pi_n = (f, h_n)$.

Note that we can assume the second condition because

$$\mathrm{Ker}(*d_\theta\colon \mathrm{Ker}\, d_\theta^* \subset \Omega^1_Y \otimes \mathfrak{su}(2) \to \mathrm{Ker}\, d_\theta^*) = H^1(Y;\mathbb{R}) \otimes \mathfrak{su}(2) = \{0\}.$$

Since $0 \neq [\theta_Y^{[s,r_n]}]$, one can take a sequence $a_n \in \widetilde{R}(Y)_{\pi_n}$ such that $M^Y(a_n, \theta)_{\pi_n}$ is non-empty for all n and $cs_{\pi_n}(a_n) \to 0$. Because of the choice of perturbations π_n, we have $a_n \notin U$ for each n. We take a sequence A_n in $M^Y(a_n, \theta)_{\pi_n}$. Moreover there is no bubble because the dimension of $M^Y(a_n, \theta)_{\pi_n}$ is 1. Since $\{A_n\}$ has bounded energy and $\mathrm{ind}(a_n) = 1$, there exists a sequence $\{s_j\}$ of real numbers, a subsequence $\{A_{n_j}\}$ of $\{A_n\}$ and gauge transformations $\{g_j\}$ on $Y \times \mathbb{R}$ such that $g_j^* T_{s_j}^* A_{n_j}$ converges to A_∞ on $Y \times \mathbb{R}$, where T_{s_j} is the translation map on $Y \times \mathbb{R}$. We denote the limit connection of A_∞ by a_∞. One can see that $[a_\infty] \neq [\theta]$ because $a_n \notin U$.

On the other hand, we have $cs(a_\infty) = 0$. This implies that A_∞ becomes a flat connection on $Y \times \mathbb{R}$. However, $\lim_{t\to\infty} A_\infty|_{Y\times\{t\}} \cong \theta$. Since the connection θ is isolated in $\widetilde{R}(Y)$, we have $[a_\infty] = [\theta]$. This gives a contradiction. ∎

Definition 3.2. For $s \in [-\infty, 0]$ and a commutative ring R with 1 and an oriented homology 3-sphere Y, we define

$$r_s^R(Y) := \sup\{r \in (0,\infty] \mid 0 = [\theta_Y^{[s,r]} \otimes \mathrm{Id}_R] \in I^1_{[s,r]}(Y;R)\}.$$

We often abbreviate $\theta_Y^{[s,r]} \otimes \mathrm{Id}_R$ to $\theta_Y^{[s,r]}$. By definition, it follows that $r_s^R(Y)$ is invariant under orientation-preserving diffeomorphisms of Y. In addition, obviously $r_s^{\mathbb{Z}}(Y) \leq r_s^{\mathbb{Q}}(Y)$. We focus on $r_s^{\mathbb{Q}}(Y)$ in the most part of this paper, and hence we denote $r_s^{\mathbb{Q}}(Y)$ simply by $r_s(Y)$.

Non-triviality of r_s implies the following:

Theorem 3.3. *Suppose that $r_s(Y) < \infty$ for some s. Then for any metric g on Y, there exists a solution A to the ASD equation on $Y \times \mathbb{R}$ with*

$$\frac{1}{8\pi^2}\|F(A)\|^2_{L^2} = r_s(Y).$$

Proof. We put $r = r_s(Y)$ and take a sequence ϵ_n with $0 < \epsilon_n \to 0$ and a sequence of regular non-degenerate perturbations $\pi_n \in \mathcal{P}(Y, g, r + \epsilon_n, s)$ with $\|\pi_n\| \to 0$. Since $0 \neq [\theta_s^{\epsilon_n + r}]$, we have a sequence $a_n \in \widetilde{R}(Y)_{\pi_n}$ such that $M^Y(a_n, \theta)_{\pi_n}$ is non-empty for all n and $cs_{\pi_n}(a_n) \to r$. We take elements A_n in $M^Y(a_n, \theta)_{\pi_n}$ for each n. There is no bubble because the dimension of $M^Y(a_n, \theta)_{\pi_n}$ is 1. Since $\mathrm{ind}(a_n) = 1$, by the gluing argument, we can conclude that there exists a sequence s_j of real numbers, a subsequence $\{A_{n_j}\}$ of

$\{A_n\}$ and gauge transformations $\{g_j\}$ on $Y \times \mathbb{R}$ such that $\{g_j^* T_{s_j}^* A_{n_j}\}$ converges to A_∞ on $Y \times \mathbb{R}$, where T_{s_j} is the translation map on $Y \times \mathbb{R}$. We can see that

$$\frac{1}{8\pi^2}\|F(A_\infty)\|^2 = \lim_{n\to\infty} cs_{\pi_n}(a_n) = r = r_s(Y).$$

Moreover, A_∞ satisfies $F^+(A_\infty) = 0$. This completes the proof. ■

In the following, we state the fundamental properties of r_s^R.

Lemma 3.4 (Theorem 1.1 (2)). *For any $s \in [-\infty, 0]$ and a homology 3-sphere Y, we have $r_s^R(Y) \in \Lambda_Y^* \cup \{\infty\}$.*

Proof. By using Lemmas 2.15 and 2.9, we obtain the conclusion. ■

In the case of S^3, note that $\Lambda_{S^3}^* = \emptyset$. Therefore, by Lemma 3.4, we have $r_s^R(S^3) = \infty$ for any s.

Lemma 3.5 (Theorem 1.1 (3)). *Let $s \le s'$ be non-positive numbers. Then, for any homology 3-sphere Y, we have $r_{s'}^R(Y) \le r_s^R(Y)$.*

Proof. This is also a corollary of Lemmas 2.15 and 2.9. ■

Using Lemma 2.15, we have the following lemma.

Lemma 3.6. *For any $s \in [-\infty, 0]$ and $r \in \mathbb{R}_Y^{>0} \cup \{\infty\}$, if $r < r_s^R(Y)$, then $[\theta_Y^{[s,r]}] = 0$, where $\mathbb{R}_Y^{>0} := \mathbb{R}_Y \cap (0, \infty)$.*

Proof. By the definition of $r_s^R(Y)$, we can take $r' \in \mathbb{R}_Y^{>0}$ such that $[\theta_Y^{[s,r']}] = 0$ and $r < r' \le r_s^R(Y)$. Then it follows from Lemma 2.15 that

$$[\theta_Y^{[s,r]}] = i_{[s,r']}^{[s,r]}([\theta_Y^{[s,r']}]) = 0. \qquad ■$$

Now we establish an important property of r_s.

Theorem 3.7. *Fix a commutative ring R with 1. Let Y_1 and Y_2 be oriented homology 3-spheres. Suppose that there is an oriented negative definite cobordism W with $H^1(W;\mathbb{R}) = 0$ and $\partial W = Y_1 \amalg -Y_2$. If $c(W) = \#H_1(W;\mathbb{Z})$ is invertible in R, then*

$$r_s^R(Y_2) \le r_s^R(Y_1) \quad \text{for any } s \in [-\infty, 0].$$

Moreover, if $r_s^R(Y_2) = r_s^R(Y_1) < \infty$, then there exist irreducible $SU(2)$-representations ρ_1 and ρ_2 of $\pi_1(Y_1)$ and $\pi_1(Y_2)$ respectively which extend to one of $\pi_1(W)$.

Proof. Suppose that $r_s^R(Y_1) < \infty$. For $\epsilon > 0$ satisfying $\epsilon + r_s^R(Y_1) \notin \Lambda_{Y_2}$, by Lemma 2.13, we get

$$IW_s^{\epsilon + r_s^R(Y_1)}[\theta_{Y_2}^{[s,\epsilon + r_s^R(Y_1)]}] = c(W)\theta_{Y_1}^{[s,\epsilon + r_s^R(Y_1)]}.$$

Since $[\theta_s^{\epsilon + r_s^R(Y_1)}(Y_1)] \neq 0$ for any $\epsilon > 0$ and $c(W)$ is invertible, we have $r_s^R(Y_2) \le r_s^R(Y_1) + \epsilon$. This implies the conclusion.

Suppose that $r := r_s^R(Y_2) = r_s^R(Y_1)$ for some s. Fix Riemannian metrics g_1 and g_2 on Y_1 and Y_2. We take a sequence $r < r_n \to r$ with $r_n \in \mathbb{R}_{Y_1} \cap \mathbb{R}_{Y_2}$, the classes $[\theta_{Y_1}^{[s,r_n]}] \neq 0$ and $[\theta_{Y_2}^{[s,r_n]}] \neq 0$. Then we have sequences of regular perturbations on Y_1 and Y_2 denoted by $\{\pi_n^1\} \subset \mathcal{P}(Y_1, g_1, r_n)$ and $\{\pi_n^2\} \subset \mathcal{P}(Y_2, g_2, r_n)$ satisfying

$$0 \neq [\theta_{Y_1}^{[s,r_n]}] \in CI^1_{[s,r_n]}(Y_1) \quad \text{and} \quad 0 \neq [\theta_{Y_2}^{[s,r_n]}] \in CI^1_{[s,r_n]}(Y_2).$$

Moreover, one can take critical points a_n and b_n of $\{\pi_n^1\}$ and $\{\pi_n^2\}$ and regular perturbations π_W^n on W^* satisfying the following conditions:

- $cs_{\pi_n^1}(a_n) \to r$, $cs_{\pi_n^2}(b_n) \to r$,
- $\|\pi_n^i\| \to 0$ for $i = 1, 2$,
- $\|\pi_W^n\|_{C^1} \to 0$,
- $\theta_Y(a_n) \neq 0$ and $\theta_Y(b_n) \neq 0$.

For all such data, by using (15), we take a_n and b_n satisfying

$$M(a_n, W^*, b_n) \neq \emptyset.$$

Now we choose an element A_n in $M(a_n, W^*, b_n)$ for each n. Since we take regular perturbations, the dimension of $M(a_n, W^*, b_n)$ is 0. Since $\{A_n\}$ has bounded energy and there is no sliding end sequence by a gluing argument, one can take a subsequence $\{A_{n_j}\}$ and gauge transformations $\{g_j\}$ such that $\{g_j^* A_{n_j}\}$ converges on W^*. We denote the limit by A_∞. By the second condition, we can see that the limit points a_∞ and b_∞ are flat connections. Moreover, $cs(a_\infty) = cs(b_\infty) = r$ and $\|F(A_\infty)\|^2_{L^2(Y\times\mathbb{R})} = 0$. Since we can take perturbations so that the reducible flat connections of Y_1 and Y_2 are isolated, we see that a_∞ and b_∞ are irreducible flat connections. Therefore, A_∞ determines an irreducible flat connection on W. This gives a homomorphism $\rho(A_\infty)\colon \pi_1(W) \to SU(2)$. ∎

This result gives the following conclusion.

Corollary 3.8. *The invariants r_s^R are homology cobordism invariants.*

In addition, we also have the following corollary.

Corollary 3.9. *If there exists a negative definite simply connected cobordism with boundary $Y_1 \amalg -Y_2$ and $r_s^R(Y_1) < \infty$, then $r_s^R(Y_2) < r_s^R(Y_1)$.*

Also by Theorem 3.7, for the case of $r_s = r_s^{\mathbb{Q}}$, we have the following.

Theorem 3.10 (Theorem 1.1 (1)). *Let Y_1 and Y_2 be oriented homology 3-spheres. Suppose that there is an oriented negative definite cobordism W with $\partial W = Y_1 \amalg -Y_2$. Then*

$$r_s(Y_2) \leq r_s(Y_1) \quad \text{for any } s \in [-\infty, 0].$$

Moreover, if $H^1(W;\mathbb{R}) = 0$ and $r_s(Y_2) = r_s(Y_1) < \infty$, then there exist irreducible $SU(2)$-representations ρ_1 and ρ_2 of $\pi_1(Y)$ and $\pi_2(Y)$ respectively which extend the same representation of $\pi_1(W)$.

Proof. By surgering out loops representing the free part of $H_1(W;\mathbb{Z})$, without loss of generality, we may assume that $H_1(W;\mathbb{R}) = 0$. Then $c(W) = \#H_1(W;\mathbb{Z})$ is invertible in $\mathbb{Q}$, and hence Theorem 3.7 gives $r_s(Y_2) \le r_s(Y_1)$. The "moreover" assertion directly follows from Theorem 3.7. ∎

Corollary 3.11. *If a homology 3-sphere Y bounds a negative definite 4-manifold, then*

$$r_s(Y) = \infty \quad \textit{for any } s \in [-\infty, 0].$$

Proof. Suppose that Y bounds a negative definite 4-manifold X, and let W denote X with an open 4-ball deleted. Then W is a negative definite 4-manifold with $\partial W = Y \amalg -S^3$. Therefore, by Theorem 3.10, we have $r_s(Y) \ge r_s(S^3) = \infty$ for any $s \in [-\infty, 0]$. ∎

3.2. Connected sum formula

The aim of this subsection is to prove the following connected sum formula for r_s.

Theorem 3.12 (Theorem 1.1 (4)). *Let $s, s_1, s_2 \in (-\infty, 0]$ with $s = s_1 + s_2$. For any homology 3-spheres Y_1 and Y_2, we have*

$$r_s(Y_1 \# Y_2) \ge \min\{r_{s_1}(Y_1) + s_2, r_{s_2}(Y_2) + s_1\}.$$

Before starting the proof, let us fix several additional data to define filtered instanton Floer homology. Fix $s, s_1, s_2 \in (-\infty, 0]$ with $s = s_1 + s_2$ and homology 3-spheres Y_1 and Y_2. Take $r \in \mathbb{R}^{>0}_{Y_1\#Y_2}$ such that $r - s_2 \in \mathbb{R}^{>0}_{Y_1}$ and $r - s_1 \in \mathbb{R}^{>0}_{Y_2}$. Fix Riemannian metrics g_i on Y_i (resp. $g_\#$ on $Y_1 \# Y_2$), non-degenerate regular perturbations $\pi_i \in \mathcal{P}(Y_i, r - s_j, s_i, g_i)$ for $\{i, j\} = \{1, 2\}$ (resp. a non-degenerate regular perturbation $\pi_\# \in \mathcal{P}(Y_1 \# Y_2, r, s, g_\#)$) and orientations on line bundles $\lambda_{a,X}$ with respect to π_1, π_2 and $\pi_\#$. Here, we first suppose that $s_1 \in \mathbb{R}_{Y_1}$, $s_2 \in \mathbb{R}_{Y_2}$ and $s \in \mathbb{R}_{Y_1\#Y_2}$. Next, let us consider a cobordism W with $\partial W = (Y_1 \# Y_2) \amalg -(Y_1 \amalg Y_2)$, which consists of only a single 1-handle. Define $\mathbb{Q}$-vector spaces $C_i^{[s,r]}$ $(i = 0, 1)$ as

$$C_0^{[s,r]} := \begin{array}{c} CI_0^{[s_1,r-s_2]}(Y_1) \otimes_{\mathbb{Q}} CI_0^{[s_2,r-s_1]}(Y_2) \\ \oplus \\ CI_0^{[s_1,r-s_2]}(Y_1) \\ \oplus \\ CI_0^{[s_2,r-s_1]}(Y_2) \end{array}$$

and

$$C_1^{[s,r]} := \begin{array}{c} (CI_1^{[s_1,r-s_2]}(Y_1) \otimes_{\mathbb{Q}} CI_0^{[s_2,r-s_1]}(Y_2)) \oplus (CI_0^{[s_1,r-s_2]}(Y_1) \otimes_{\mathbb{Q}} CI_1^{[s_2,r-s_1]}(Y_2)) \\ \oplus \\ CI_1^{[s_1,r-s_2]}(Y_1) \\ \oplus \\ CI_1^{[s_2,r-s_1]}(Y_2). \end{array}$$

By the discussion of Section 2.4, the above initial data give the maps

$$CW_i^{[s,r]} \oplus \widetilde{CW}_i^{[s,r]} \colon CI_i^{[s,r]}(Y_1 \# Y_2) \to C_i^{[s,r]}$$

for $i = 0$ and $i = 1$. We denote $\mathrm{pr}_j \circ CW_i^{[s,r]} \oplus \widetilde{CW}_i^{[s,r]}$ by $p_j CW_i^{[s,r]}$, where pr_j is the projection to the j-th component of $C_i^{[s,r]}$ for $j \in \{1,2,3\}$. The following lemma is a key to proving the connected sum inequality.

Lemma 3.13. *Suppose that $s_1 \in \mathbb{R}_{Y_1}$, $s_2 \in \mathbb{R}_{Y_2}$ and $s \in \mathbb{R}_{Y_1 \# Y_2}$. The homomorphisms $CW_0^{[s,r]}$ and $CW_1^{[s,r]}$ satisfy the following equalities:*

(1) $p_1 CW_0^{[s,r]} \circ \partial^{[s,r]}_{Y_1 \# Y_2} - (\partial^{[s_1, r-s_2]}_{Y_1} \otimes 1, 1 \otimes \partial^{[s_2, r-s_1]}_{Y_2}) \circ p_1 CW_1^{[s,r]} = 0,$

(2) $p_2 CW_0^{[s,r]} \circ \partial^{[s,r]}_{Y_1 \# Y_2} - \partial^{[s_1, r-s_2]}_{Y_1} \circ p_2 CW_1^{[s,r]} - (0, 1 \otimes \theta^{[s_2, r-s_1]}_{Y_2}) \circ p_1 CW_1^{[s,r]} = 0,$

(3) $p_3 CW_0^{[s,r]} \circ \partial^{[s,r]}_{Y_1 \# Y_2} - \partial^{[s_2, r-s_1]}_{Y_2} \circ p_3 CW_1^{[s,r]} - (\theta^{[s_1, r-s_2]}_{Y_1} \otimes 1, 0) \circ p_1 CW_1^{[s,r]} = 0.$

Proof. First, let us prove (1). For generators $[a] \in CI_1^{[s,r]}(Y_1 \# Y_2)$ and $[b_1] \otimes [b_2] \in CI_0^{[s_1, r-s_2]}(Y_1) \otimes_{\mathbb{Q}} CI_0^{[s_2, r-s_1]}(Y_2)$, we see that the moduli space $M(a, W^*, b_1 \amalg b_2)$ has the structure of an oriented manifold of dimension 1 whose orientation is induced by the orientations of the line bundles $\lambda_{a,X}$. Moreover, by a gluing argument, we obtain the gluing map $\mathfrak{gl}$ from the union of

$$\left(\bigcup_{\substack{[c] \in \widetilde{R}^*(Y_1 \# Y_2)_{\pi_\#}, \, \mathrm{ind}(c)=0 \\ cs_{\pi_\#}([c]) < cs_{\pi_\#}([a])}} M^{Y_1 \# Y_2}(a, c)_{\pi_\#}/\mathbb{R} \times M(c, W^*, b_1 \amalg b_2) \right) \times (0, \infty), \tag{18}$$

$$\left(\bigcup_{\substack{[d] \in \widetilde{R}^*(Y_1)_{\pi_1}, \, \mathrm{ind}(d)=1 \\ cs_{\pi_1}([d]) > cs_{\pi_1}([b_1])}} M(a, W^*, d \amalg b_2) \times M^{Y_1}(d, b_1)_{\pi_1}/\mathbb{R} \right) \times (-\infty, 0), \tag{19}$$

and

$$\left(\bigcup_{\substack{[e] \in \widetilde{R}^*(Y_2)_{\pi_2}, \, \mathrm{ind}(e)=1 \\ cs_{\pi_2}([e]) > cs_{\pi_2}([b_2])}} M(a, W^*, b_1 \amalg e) \times M^{Y_2}(e, b_2)_{\pi_2}/\mathbb{R} \right) \times (-\infty, 0) \tag{20}$$

to $M(a, W^*, b_1 \amalg b_2)$. On the first two components (18) and (19), we can check that $\mathfrak{gl}$ is orientation-preserving as in the case of $Y_2 = \emptyset$. For the third component (20), in general, $\mathfrak{gl}$ changes the orientation by $(-1)^{\mathrm{ind}(b_1)}$. This follows from a standard calculation of index bundles via a gluing argument. In our situation, since $\mathrm{ind}(b_1) = 0$, $\mathfrak{gl}$ is orientation-preserving. So, the oriented boundaries of the compactification of $M(a, W^*, b_1 \amalg b_2)$ are as follows:

$$\bigcup_{\substack{[c] \in \widetilde{R}^*(Y_1 \# Y_2)_{\pi_\#}, \, \mathrm{ind}(c)=0 \\ cs_{\pi_\#}([c]) < cs_{\pi_\#}([a])}} M^{Y_1 \# Y_2}(a, c)_{\pi_\#}/\mathbb{R} \times M(c, W^*, b_1 \amalg b_2),$$

$$- \bigcup_{\substack{[d]\in\widetilde{R}^*(Y_1)_{\pi_1},\,\mathrm{ind}(d)=1 \\ cs_{\pi_1}([d])>cs_{\pi_1}([b_1])}} M(a, W^*, d \amalg b_2) \times M^{Y_1}(d, b_1)_{\pi_1}/\mathbb{R},$$

$$- \bigcup_{\substack{[e]\in\widetilde{R}^*(Y_2)_{\pi_2},\,\mathrm{ind}(e)=1 \\ cs_{\pi_2}([e])>cs_{\pi_2}([b_2])}} M(a, W^*, b_1 \amalg e) \times M^{Y_2}(e, b_2)_{\pi_2}/\mathbb{R}.$$

Claim 3.14. *The following inequalities hold:*

$$cs_{\pi_\#}([c]) > s, \quad cs_{\pi_1}([d]) < r - s_2, \quad cs_{\pi_2}([e]) < r - s_1.$$

Proof. This is just a corollary of Lemma 2.10. ■

By using this lemma, we can regard $[c]$, $[d]$ and $[e]$ above as $[c] \in CI_0^{[s,r]}(Y_1 \# Y_2)$, $[d] \in CI_1^{[s_1,r-s_2]}(Y_1)$ and $[e] \in CI_1^{[s_2,r-s_1]}(Y_2)$ respectively. Thus, we have

$$p_1 CW_0^{[s,r]} \circ \partial^{[s,r]}_{Y_1\#Y_2}([a]) - (\partial^{[s_1,r-s_2]}_{Y_1} \otimes 1, 0) \circ p_1 CW_1^{[s,r]}([a]) \\ - (0, 1 \otimes \partial^{[s_2,r-s_1]}_{Y_2}) \circ p_1 CW_1^{[s,r]}([a]) = 0.$$

Next, we prove (2). For generators $[a] \in CI_1^{[s,r]}(Y_1 \# Y_2)$ and $[b_1] \in CI_0^{[s_1,r-s_2]}(Y_1)$, consider $M(a, W^*, b_1 \amalg \theta_{Y_2})$ as an oriented 1-manifold. Then its ends are the following:

$$\bigcup_{[c]\in\widetilde{R}^*(Y_1\#Y_2)_{\pi_\#},\,\mathrm{ind}(c)=0} M^{Y_1\#Y_2}(a, c)_{\pi_\#}/\mathbb{R} \times M(c, W^*, b_1 \amalg \theta_{Y_2}),$$

$$- \bigcup_{[d]\in\widetilde{R}^*(Y_1)_{\pi_1},\,\mathrm{ind}(d)=1} M(a, W^*, d \amalg \theta_{Y_2}) \times M^{Y_1}(d, b_1)_{\pi_1}/\mathbb{R},$$

$$- \bigcup_{[e]\in\widetilde{R}^*(Y_2)_{\pi_2},\,\mathrm{ind}(e)=1} M(a, W^*, b_1 \amalg e) \times M^{Y_2}(e, \theta_{Y_2})_{\pi_2}/\mathbb{R}.$$

We need to show the following claim.

Claim 3.15. *The following inequalities hold:*

$$cs_{\pi_\#}([c]) > s, \quad cs_{\pi_1}([d]) < r - s_2, \quad cs_{\pi_2}([e]) < r - s_1.$$

Proof. This is also a corollary of Lemma 2.10. ■

Hence, we have

$$p_2 CW_0^{[s,r]} \circ \partial^{[s,r]}_{Y_1\#Y_2}([a]) - \partial^{[s_1,r-s_2]}_{Y_1} \circ p_2 CW_1([a]) \\ - (0, 1 \otimes \theta^{[s_2,r-s_1]}_{Y_2}) \circ p_1 CW_1^{[s,r]}(a) = 0.$$

By the same argument, the third assertion follows from considering the 1-dimensional moduli space $M(a, W^*, \theta_{Y_1} \amalg b_2)$.

Since the proof of (3) is essentially the same as that of (2), we omit it. ■

Next, we define a homomorphism $\partial_C\colon C_1^{[s,r]} \to C_0^{[s,r]}$ by

$$\partial_C = \begin{bmatrix} (\partial_{Y_1}^{[s_1,r-s_2]} \otimes 1, 1 \otimes \partial_{Y_2}^{[s_2,r-s_1]}) & 0 & 0 \\ (0, 1 \otimes \theta_{Y_2}^{[s_2,r-s_1]}) & \partial_{Y_1}^{[s_1,r-s_2]} & 0 \\ (\theta_{Y_1}^{[s_1,r-s_2]} \otimes 1, 0) & 0 & \partial_{Y_2}^{[s_2,r-s_1]} \end{bmatrix}.$$

Lemma 3.16. *For $s_1 \in \mathbb{R}_{Y_1}$, $s_2 \in \mathbb{R}_{Y_2}$ and $s \in \mathbb{R}_{Y_1 \# Y_2}$, we have*

$$\partial_C \circ CW_1^{[s,r]} = CW_0^{[s,r]} \circ \partial_{Y_1 \# Y_2}^{[s,r]}.$$

Proof. By Lemma 3.13, we have

$$\partial_C \circ CW_1^{[s,r]} = \begin{bmatrix} (\partial_{Y_1}^{[s_1,r-s_2]} \otimes 1, 1 \otimes \partial_{Y_2}^{[s_2,r-s_1]}) \circ p_1 CW_1^{[s,r]} \\ (0, 1 \otimes \theta_{Y_2}^{[s_2,r-s_1]}) \circ p_1 CW_1^{[s,r]} + \partial_{Y_1}^{[s_1,r-s_2]} \circ p_2 CW_1^{[s,r]} \\ (\theta_{Y_1}^{[s_1,r-s_2]} \otimes 1, 0) \circ p_1 CW_1^{[s,r]} \partial_{Y_2}^{[s_2,r-s_1]} \circ p_3 CW_1^{[s,r]} \end{bmatrix}$$

$$= \begin{bmatrix} p_1 CW_0^{[s,r]} \circ \partial_{Y_1 \# Y_2}^{[s,r]} \\ p_2 CW_0^{[s,r]} \circ \partial_{Y_1 \# Y_2}^{[s,r]} \\ p_3 CW_0^{[s,r]} \circ \partial_{Y_1 \# Y_2}^{[s,r]} \end{bmatrix} = CW_0^{[s,r]} \circ \partial_{Y_1 \# Y_2}^{[s,r]}. \qquad \blacksquare$$

Lemma 3.17. *For $s_1 \in \mathbb{R}_{Y_1}$, $s_2 \in \mathbb{R}_{Y_2}$ and $s \in \mathbb{R}_{Y_1 \# Y_2}$, there exists a cochain $f \in CI^0_{[s,r]}(Y_1 \# Y_2)$ such that*

$$\theta_{Y_1 \# Y_2}^{[s,r]} + f \circ \partial_{Y_1 \# Y_2}^{[s,r]} - (0, \theta_{Y_1}^{[s_1,r-s_2]}, \theta_{Y_2}^{[s_2,r-s_1]}) \circ CW_1^{[s,r]} = 0.$$

Proof. For a generator $[a] \in CI_1^{[s,r]}(Y_1 \# Y_2)$, consider $M(a, W^*, \theta_{Y_1} \amalg \theta_{Y_2})$ as an oriented 1-manifold; then its ends are the following:

$$\begin{aligned} &M^{Y_1 \# Y_2}(a, \theta_{Y_1 \# Y_2})_{\pi_\#}/\mathbb{R} \times M(\theta_{Y_1 \# Y_2}, W^*, \theta_{Y_1} \amalg \theta_{Y_2}), \\ &\bigcup_{[b] \in \widetilde{R}^*(Y_1 \# Y_2)_{\pi_\#}, \operatorname{ind}(b)=0} M^{Y_1 \# Y_2}(a, b)_{\pi_\#}/\mathbb{R} \times M(b, W^*, \theta_{Y_1} \amalg \theta_{Y_2}), \\ &- \bigcup_{[c] \in \widetilde{R}^*(Y_1)_{\pi_1}, \operatorname{ind}(c)=1} M(a, W^*, c \amalg \theta_{Y_2}) \times M^{Y_1}(c, \theta_{Y_1})_{\pi_1}/\mathbb{R}, \\ &- \bigcup_{[d] \in \widetilde{R}^*(Y_2)_{\pi_2}, \operatorname{ind}(d)=1} M(a, W^*, \theta_{Y_1} \amalg d) \times M^{Y_2}(d, \theta_{Y_2})_{\pi_2}/\mathbb{R}. \end{aligned}$$

Since Y_1 and Y_2 are homology spheres, we see that $M(\theta_{Y_1 \# Y_2}, W^*, \theta_{Y_1} \amalg \theta_{Y_2})$ has just one point. Thus, defining a homomorphism $f\colon CI_0^{[s,r]}(Y_1 \# Y_2) \to \mathbb{Q}$ by

$$[b] \mapsto \#(M(b, W^*, \theta_{Y_1} \amalg \theta_{Y_2})),$$

we have

$$\theta_{Y_1 \# Y_2}^{[s,r]}([a]) + f \circ \partial_{Y_1 \# Y_2}^{[s,r]}([a]) - \theta_{Y_1}^{[s_1,r-s_2]} \circ p_2 CW_1([a]) - \theta_{Y_2}^{[s_2,r-s_1]} \circ p_3 CW_1([a]) = 0.$$

This completes the proof. ■

Theorem 3.18. *Let $s_1 \in \mathbb{R}_{Y_1}$, $s_2 \in \mathbb{R}_{Y_2}$ and $s \in \mathbb{R}_{Y_1 \# Y_2}$. If $\theta_{Y_1}^{[s_1, r-s_2]}$ and $\theta_{Y_2}^{[s_2, r-s_1]}$ are coboundaries, then $\theta_Y^{[s,r]}(Y_1 \# Y_2)$ is also a coboundary.*

Proof. Suppose that $f_i \in CI^0_{[s_i, r-s+s_i]}(Y_i)$ satisfies $f_i \circ \partial_{Y_i}^{[s_i, r-s+s_i]} = \theta_{Y_i}^{[s_i, r-s+s_i]}$ for each $i \in \{1, 2\}$. Then we have a homomorphism

$$f' := (-f_1 \otimes f_2, f_1, f_2) \colon C_0^{[s,r]} \to \mathbb{Q}$$

and the equalities

$$\begin{aligned} f' \circ \partial_C &= (-f_1 \otimes f_2, f_1, f_2) \circ \partial_C \\ &= (*, f_1 \circ \partial_{Y_1}^{[s_1, r-s_2]}, f_2 \circ \partial_{Y_2}^{[s_2, r-s_1]}) \\ &= (*, \theta_{Y_1}^{[s_1, r-s_2]}, \theta_{Y_2}^{[s_2, r-s_1]}), \end{aligned}$$

where

$$\begin{aligned} * = &-\big((f_1 \circ \partial_{Y_1}^{[s_1, r-s_2]}) \otimes f_2, f_1 \otimes (f_2 \circ \partial_{Y_2}^{[s_2, r-s_1]})\big) \\ &+ (0, f_1 \otimes \theta_{Y_2}^{[s_2, r-s_1]}) + (\theta_{Y_1}^{[s_1, r-s_2]} \otimes f_2, 0) = 0. \end{aligned}$$

Therefore, combining it with Lemmas 3.16 and 3.17, we have

$$\begin{aligned} \theta_Y^{[s,r]}(Y_1 \# Y_2) &= -f \circ \partial_{Y_1 \# Y_2}^{[s,r]} + (0, \theta_{Y_1}^{[s_1, r-s_2]}, \theta_{Y_2}^{[s_2, r-s_1]}) \circ CW_1^{[s,r]} \\ &= -f \circ \partial_{Y_1 \# Y_2}^{[s,r]} + f' \circ \partial_C \circ CW_1^{[s,r]} \\ &= (-f + f' \circ CW_0^{[s,r]}) \circ \partial_{Y_1 \# Y_2}^{[s,r]}. \end{aligned}$$ ■

Proof of Theorem 3.12. First, we suppose that $s_1 \in \mathbb{R}_{Y_1}$, $s_2 \in \mathbb{R}_{Y_2}$ and $s \in \mathbb{R}_{Y_1 \# Y_2}$. Without loss of generality, we may require that $0 < r_{s_1}(Y_1) + s_2 \leq r_{s_2}(Y_2) + s_1$. Assume $r_s(Y_1 \# Y_2) < r_{s_1}(Y_1) + s_2$. Then there exists $r \in \mathbb{R}^{>0}_{Y_1 \# Y_2}$ such that $r_s(Y_1 \# Y_2) < r < r_{s_1}(Y_1) + s_2$, $r - s_2 \in \mathbb{R}_{Y_1}$ and $r - s_1 \in \mathbb{R}_{Y_2}$. Lemma 3.6 implies that

$$[\theta_{Y_1}^{[s_1, r-s_2]}] = 0 \quad \text{and} \quad [\theta_{Y_2}^{[s_2, r-s_1]}] = 0.$$

Therefore, by Theorem 3.18, we have $[\theta_{Y_1 \# Y_2}^{[s,r]}] = 0$. This contradicts the assertion $r_s(Y_1 \# Y_2) < r$, and hence $r_s(Y_1 \# Y_2) \geq r_{s_1}(Y_1) + s_2$.

Now, we handle the case of $s_1 \in \Lambda_{Y_1}$, $s_2 \in \Lambda_{Y_2}$ or $s \in \Lambda_{Y_1 \# Y_2}$. Let $\{s_i^n\}_{n \in \mathbb{Z}_{>0}}$ be sequences for $i = 1, 2$ such that $s_1^n \to s_1$, $s_2^n \to s_2$, $s_1^n \leq s_1$, $s_2^n \leq s_2$, $s_1^n + s_2^n$ is a regular value of $Y_1 \# Y_2$, and s_i^n is a regular value of Y_i for $i = 1, 2$. By the choices of $\{s_i^n\}_{n \in \mathbb{Z}_{>0}}$ for $i = 1, 2$, we have

$$r_{s_1^n + s_2^n}(Y_1 \# Y_2) \geq \min\{r_{s_1^n}(Y_1) + s_2^n, r_{s_2^n}(Y_2) + s_1^n\}.$$

For sufficiently large n, we have $r_{s_1^n + s_2^n}(Y_1 \# Y_2) = r_s(Y_1 \# Y_2)$, $r_{s_1^n}(Y_1) = r_{s_1}(Y_1)$ and $r_{s_2^n}(Y_2) = r_{s_2}(Y_2)$. This completes the proof. ■

Remark 3.19. As described in [55, Section 9.4], we have Fukaya's translation of $CI_*(Y_1 \# Y_2)$ into a certain combination of $CI_*(Y_1)$ and $CI_*(Y_2)$. From this viewpoint, we can extend $C_0^{[s,r]}$ and $C_1^{[s,r]}$ to a chain complex C_* such that we have a "projection"

$$CI_*(Y_1 \# Y_2) \twoheadrightarrow C_*.$$

We guess that an alternative proof of Theorem 3.12 can be obtained from a filtered version $C_*^{[s,r]}$ of C_*, and such a proof would be more natural. However, establishing $C_*^{[s,r]}$ requires too many extra arguments, and so we extract a small part of $C_*^{[s,r]}$.

4. Comparison with Daemi's invariants

In this section, we compare our invariants $r_s(Y)$ with Daemi's invariants $\Gamma_Y(k)$.

In [6], Daemi constructed a family $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}}$ of real-valued homology cobordism invariants which has the following properties:

- Let Y_1 and Y_2 be homology 3-spheres, and W a negative definite cobordism with $\partial W = Y_1 \amalg -Y_2$ and $b_1(W) = 0$. Then there exists a constant $\eta(W) \geq 0$ such that

$$\Gamma_{Y_1}(k) \leq \begin{cases} \Gamma_{Y_2}(k) - \eta(W) & \text{if } k > 0, \\ \max\{\Gamma_{Y_2}(k) - \eta(W), 0\} & \text{if } k \leq 0. \end{cases}$$

  Moreover, the constant $\eta(W)$ is positive unless there exists an $SU(2)$-representation of $\pi_1(W)$ whose restrictions to both $\pi_1(Y_1)$ and $\pi_1(Y_2)$ are non-trivial.
- We have $\cdots \leq \Gamma_Y(-1) \leq \Gamma_Y(0) \leq \Gamma_Y(1) \leq \cdots$.
- $\Gamma_Y(k)$ is finite if and only if $2h(Y) \geq k$, where $h(Y)$ is the Frøyshov invariant of Y.

In this section, we prove that our invariant $r_{-\infty}(Y)$ coincides with $\Gamma_{-Y}(1)$.

Theorem 1.4. *For any Y, we have*

$$r_{-\infty}(Y) = \Gamma_{-Y}(1).$$

Since $r_s(Y) \leq r_{-\infty}(Y)$ for any $s \in \mathbb{R}_{<0}$, several facts and calculations for r_s immediately follow from the study of $\Gamma_Y(1)$ in [6]. We also discuss them in this section.

4.1. Review of Daemi's $\Gamma_Y(1)$

Here, we need to compare our notations with [6]. The instanton Floer chain complex depends on the choice of several conventions. For example, our sign convention for the Chern–Simons functional is different from Daemi's (see Table 1). In this section, we consider a fixed homology 3-sphere Y.

First we introduce the coefficient ring

$$\Lambda := \left\{ \sum_{i=1}^{\infty} q_i \lambda^{r_i} \;\middle|\; q_i \in \mathbb{Q},\ r_i \in \mathbb{R},\ \lim_{i\to\infty} r_i = \infty \right\},$$

	Sign of cs	Cylinder	Gradient
[6]	$-$	$\mathbb{R} \times Y$	downward
ours	$+$	$Y \times \mathbb{R}$	downward

Tab. 1. Conventions.

where λ is a formal variable. We have an evaluating function mdeg: $\Lambda \to \mathbb{R}$ defined by

$$\operatorname{mdeg}\Big(\sum_{i=1}^{\infty} q_i \lambda^{r_i}\Big) := \min_{i \in \mathbb{Z}_{>0}} \{r_i \mid q_i \neq 0\}.$$

Fix a non-degenerate regular perturbation π and orientations of $\lambda_{a,X}$. Then a $\mathbb{Z}/8$-graded chain complex $C_*^{\Lambda}(Y)$ over Λ is defined by

$$C_i^{\Lambda}(Y) := C_i(Y) \otimes_{\mathbb{Z}} \Lambda = \Lambda\{[a] \in R^*(Y)_\pi \mid \operatorname{ind}(a) = i\},$$

with differential

$$d^{\Lambda}([a]) := \sum_{\operatorname{ind}(a)-\operatorname{ind}(b)\equiv 1 \bmod 8} \#(M^Y([a],[b])_\pi/\mathbb{R}) \cdot \lambda^{\mathcal{E}(A)}[b],$$

where $[A] \in M^Y([a],[b])_\pi$, $\mathcal{E}(A) := \frac{1}{8\pi^2}\int_{Y\times\mathbb{R}} \operatorname{Tr}((F(A)+\pi(A)) \wedge (F(A)+\pi(A)))$ and $M^Y([a],[b])_\pi$ denotes $M^Y(a,b)_\pi$ for some representatives a and b of $[a]$ and $[b]$ respectively satisfying $\operatorname{ind}(a) - \operatorname{ind}(b) = 1$. Note that we use $\widetilde{R}^*(Y)_\pi$ as a generating set of the Floer chain complex. On the other hand, in Daemi's formulation, the chain group is generated by $R^*(Y)_\pi$. In our notation, a, b are elements of $\widetilde{R}^*(Y)_\pi$ and we let $[a]$ and $[b]$ denote their images in $R^*(Y)_\pi$.

Remark 4.1. In Daemi's formulation, $C_*(Y)$ and $C_i^{\Lambda}(Y)$ are regarded as $\mathbb{Z}/8\mathbb{Z}$-graded chain complexes.

We extend the function mdeg to a function on C_*^{Λ} by

$$\operatorname{mdeg}\Big(\sum_{k=1}^{n} \eta_k [a_k]\Big) = \min_{1\le k\le n} \operatorname{mdeg} \eta_k.$$

In addition, we define the map $D_1 \colon C_1^{\Lambda}(Y) \to \Lambda$ by

$$D_1([a]) = (\# M^Y([a],[\theta])_\pi/\mathbb{R}) \cdot \lambda^{\mathcal{E}(A)},$$

where $M^Y([a],[\theta])_\pi$ denotes $M^Y(a,\theta^i)_\pi$ for some lifts a and θ^i of $[a]$ and $[\theta]$ respectively satisfying $\operatorname{ind}(a) - \operatorname{ind}(\theta^i) = 1$, and $A \in M^Y([a],[\theta])_\pi$. Now, in our conventions, $\Gamma_{-Y}(1)$ is described as

$$\Gamma_{-Y}(1) = \lim_{\|\pi\|\to 0} \Big(\inf_{\substack{\alpha \in C_1^{\Lambda}(Y),\, d^{\Lambda}(\alpha)=0 \\ D_1(\alpha)\neq 0}} \{\operatorname{mdeg}(D_1(\alpha)) - \operatorname{mdeg}(\alpha)\}\Big).$$

4.2. Translating $\Gamma_Y(1)$ into a $\mathbb{Z}$-grading

Following the construction of C_*^Λ, we can define a $\mathbb{Z}$-graded chain complex

$$CI_i^\Lambda(Y) := CI_i(Y) \otimes_{\mathbb{Q}} \Lambda = \Lambda\{a \in \widetilde{R}^*(Y)_\pi \,|\, \mathrm{ind}(a) = i\}$$

with differential

$$\begin{aligned}\partial^\Lambda(a) &:= \sum_{\mathrm{ind}(a)-\mathrm{ind}(b)=1,\, A\in M^Y(a,b)_\pi} \#(M^Y(a,b)_\pi/\mathbb{R}) \cdot \lambda^{\mathcal{E}(A)} b \\ &= \sum_{\mathrm{ind}(a)-\mathrm{ind}(b)=1} \#(M^Y(a,b)_\pi/\mathbb{R}) \cdot \lambda^{cs_\pi(a)-cs_\pi(b)} b.\end{aligned}$$

We can also define the map $\theta^\Lambda \colon CI_1^\Lambda(Y) \to \Lambda$ by

$$\theta^\Lambda(a) = (\#M^Y(a,\theta)_\pi/\mathbb{R}) \cdot \lambda^{\mathcal{E}(A)} = \theta_Y^{[-\infty,\infty]}(a)\lambda^{cs_\pi(a)}.$$

We define a $\mathbb{Z}$-graded version of $\Gamma_{-Y}(1)$ by

$$\widetilde{\Gamma}_{-Y}(1) = \lim_{\|\pi\|\to 0} \Bigg(\inf_{\substack{\alpha\in CI_1^\Lambda(Y),\, \partial^\Lambda(\alpha)=0 \\ \theta^\Lambda(\alpha)\neq 0}} \{\mathrm{mdeg}(\theta^\Lambda(\alpha)) - \mathrm{mdeg}(\alpha)\} \Bigg).$$

Lemma 4.2. $\widetilde{\Gamma}_{-Y}(1) = \Gamma_{-Y}(1)$.

Proof. The maps $\psi_i \colon CI_i^\Lambda(Y) \to C_i^\Lambda(Y)$ $(0 \le i \le 7)$ induced by $\widetilde{R}(Y)_\pi \to R(Y)_\pi$ are Λ-linear isomorphisms such that

- $d^\Lambda \circ \psi_i = \psi_{i-1} \circ \partial^\Lambda$ for each $1 \le i \le 7$,
- $D_1 \circ \psi_1 = \theta^\Lambda$,
- mdeg is preserved by the ψ_i.

These imply that the infimum in the definition of $\widetilde{\Gamma}_{-Y}(1)$ coincides with that of $\Gamma_{-Y}(1)$ for each π, and hence $\widetilde{\Gamma}_{-Y}(1) = \Gamma_{-Y}(1)$. ∎

4.3. Proof of Theorem 1.2

In this subsection, we fix orientations of the line bundles $\lambda_{a,X}$ for non-degenerate regular perturbations.

Lemma 4.3. *Let $\alpha = \sum_{k=1}^n a_k \otimes x_k$ be a chain in $CI_1^{[-\infty,r]}(Y) \otimes \mathbb{Q}$. Then α is a cycle if and only if $\widetilde{\alpha}$ is a cycle of $CI_1^\Lambda(Y)$, where $\widetilde{\alpha} := \sum_{k=1}^n a_k \otimes x_k \lambda^{-cs(a_k)}$.*

Proof. For any generator $b \otimes 1 \in CI_0^{[-\infty,r]}(Y) \otimes \mathbb{Q}$, the coefficient of $b \otimes 1$ in $\partial^{[-\infty,r]}(\alpha) \in CI_1^{[-\infty,r]}(Y) \otimes \mathbb{Q}$ is

$$\sum_{k=1}^n \#(M^Y(a_k,b)_\pi/\mathbb{R}) \cdot x_k,$$

while the coefficient of $b \otimes 1$ in $\partial^\Lambda(\widetilde{\alpha})$ is

$$\sum_{k=1}^{n} \#(M^Y(a_k, b)_\pi/\mathbb{R}) \cdot \lambda^{cs_\pi(a_k)-cs_\pi(b)} \cdot x_k \lambda^{-cs_\pi(a_k)} = \Big(\sum_{k=1}^{n} \#(M^Y(a_k, b)_\pi/\mathbb{R}) \cdot x_k\Big) \cdot \lambda^{-cs_\pi(b)}.$$

This completes the proof. ∎

Lemma 4.4. *For a chain $\alpha = \sum_{k=1}^n a_k \otimes x_k$ in $CI_1^{[-\infty,r]}(Y) \otimes \mathbb{Q}$, we have $\theta_Y^{[-\infty,r]}(\alpha) \neq 0$ if and only if $\theta^\Lambda(\widetilde{\alpha}) \neq 0$. Moreover, if $\theta_Y^{[-\infty,r]}(\alpha) \neq 0$, then for a number $k' \in \{1, \dots, n\}$ with $x_{k'} \neq 0$ and $cs_\pi(a_{k'}) = \max\{cs_\pi(a_k) \mid x_k \neq 0\}$, we have*

$$\mathrm{mdeg}(\theta^\Lambda(\widetilde{\alpha})) - \mathrm{mdeg}(\widetilde{\alpha}) = cs_\pi(a_{k'}) < r.$$

Proof. The proof of the first assertion follows from the same argument as in Lemma 4.3. Moreover, it is easy to see that

$$\mathrm{mdeg}(\widetilde{\alpha}) = \mathrm{mdeg}\Big(\sum_{k=1}^{n} a_k \otimes x_k \lambda^{-cs(a_k)}\Big) = -cs_\pi(a_{k'})$$

and $\mathrm{mdeg}(\theta^\Lambda(\widetilde{\alpha})) = 0$. ∎

Lemma 4.5. *Let $\hat{\alpha} = \sum_{k=1}^n a_k \otimes \eta_k$ be a cycle in $CI_1^\Lambda(Y)$ with $d := \mathrm{mdeg}(\theta^\Lambda(\hat{\alpha})) < \infty$, and x_k the coefficient of $\lambda^{d-cs_\pi(a_k)}$ in η_k $(k = 1, \dots, n)$. Then*

$$\alpha := \sum_{k=1}^{n} a_k \otimes x_k$$

is a cycle in $CI_1(Y) \otimes \mathbb{Q}$ with $\theta_Y^{[-\infty,\infty]}(\alpha) \neq 0$. Moreover, for a number $k' \in \{1, \dots, n\}$ with $x_{k'} \neq 0$ and $cs_\pi(a_{k'}) = \max\{cs_\pi(a_k) \mid x_k \neq 0\}$, the cycle $\widetilde{\alpha}$ satisfies

$$\mathrm{mdeg}(\theta^\Lambda(\widetilde{\alpha})) - \mathrm{mdeg}(\widetilde{\alpha}) = cs_\pi(a_{k'}) \leq \mathrm{mdeg}(\theta^\Lambda(\hat{\alpha})) - \mathrm{mdeg}(\hat{\alpha}).$$

Proof. The coefficient of λ^d in $\theta^\Lambda(\hat{\alpha})$ is equal to

$$\sum_{k=1}^{n} \#(M^Y(a_k, \theta)_\pi/\mathbb{R}) \cdot x_k,$$

which coincides with $\theta_Y^{[-\infty,\infty]}(\alpha)$. Moreover, since $\mathrm{mdeg}(\theta^\Lambda(\hat{\alpha})) = d$, this value is nonzero. Hence $\theta_Y^{[-\infty,\infty]}(\alpha) \neq 0$. In a similar way, we can also verify that for any generator $b \otimes 1 \in CI_0(Y) \otimes \mathbb{Q}$, the coefficient of $b \otimes 1$ in $\partial(\alpha)$ is equal to that of $\lambda^{d-cs_\pi(b)}$ in $\partial^\Lambda(\hat{\alpha})$, and hence $\partial(\alpha) = 0$. Next, it follows from Lemma 4.4 that $\mathrm{mdeg}(\theta^\Lambda(\widetilde{\alpha})) - \mathrm{mdeg}(\widetilde{\alpha}) = cs_\pi(a_{k'})$. Moreover, since $x_{k'} \neq 0$ is the coefficient of $\lambda^{d-cs_\pi(a_{k'})}$ in $\eta_{k'}$, we have

$$\mathrm{mdeg}(\hat{\alpha}) \leq d - cs_\pi(a_{k'}).$$

This gives

$$\mathrm{mdeg}(\theta^{\Lambda}(\hat{\alpha})) - \mathrm{mdeg}(\hat{\alpha}) = d - \mathrm{mdeg}(\hat{\alpha}) \geq cs_{\pi}(a_{k'}). \qquad \blacksquare$$

Proof of Theorem 1.2. Assume that $r_{-\infty}(Y) < \Gamma_{-Y}(1)$, and take $r \in \mathbb{R}_Y$ with $r_{-\infty}(Y) < r < \Gamma_{-Y}(1)$. For any sufficiently small perturbation π, we have $[\theta_Y^{[-\infty,r]}] \neq 0$, and hence there exists a cycle $\alpha = \sum_{k=1}^{n} a_k \otimes x_k$ in $CI_1^{[-\infty,r]}(Y) \otimes \mathbb{Q}$ with $\theta_Y^{[-\infty,r]}(\alpha) \neq 0$. Therefore, it follows from Lemmas 4.3 and 4.4 that there exists a sequence $\{\pi_l\}_{l \in \mathbb{Z}_{>0}}$ of perturbations with $\|\pi_l\| \to 0$ $(l \to \infty)$ such that for each π_l, $CI_1^{\Lambda}(Y)$ has a cycle $\widetilde{\alpha}_l$ with $\theta^{\Lambda}(\widetilde{\alpha}_l) \neq 0$ and $\mathrm{mdeg}(\theta^{\Lambda}(\widetilde{\alpha}_l)) - \mathrm{mdeg}(\widetilde{\alpha}_l) < r$. This gives

$$\Gamma_{-Y}(1) = \lim_{\|\pi\| \to 0} \Big(\inf_{\substack{\alpha \in CI_1^{\Lambda}(Y),\, \partial^{\Lambda}(\alpha)=0 \\ \theta^{\Lambda}(\alpha) \neq 0}} \{\mathrm{mdeg}(\theta^{\Lambda}(\alpha)) - \mathrm{mdeg}(\alpha)\} \Big) \leq r,$$

a contradiction. Hence $r_{-\infty}(Y) \geq \Gamma_{-Y}(1)$.

Conversely, assume that $\Gamma_{-Y}(1) < r_{-\infty}(Y)$, and take $r \in \mathbb{R}_Y$ with $\Gamma_{-Y}(1) < r < r_{-\infty}(Y)$. Then, for any sufficiently small perturbation compatible with r, there exists a cycle $\hat{\alpha} \in CI_1^{\Lambda}(Y)$ with $\theta^{\Lambda}(\hat{\alpha}) \neq 0$ and $\mathrm{mdeg}(\theta^{\Lambda}(\hat{\alpha})) - \mathrm{mdeg}(\hat{\alpha}) < r$. Then, by Lemma 4.5, we obtain a cycle $\alpha = \sum_{k=1}^{n} a_k \otimes x_k$ in $CI_1(Y) \otimes \mathbb{Q}$ and a number $k' \in \{1, \dots, n\}$ such that

(1) $x_{k'} \neq 0$ and $cs_{\pi}(a_{k'}) = \max\{cs_{\pi}(a_k) \mid x_k \neq 0\}$,
(2) $\theta_Y^{[-\infty,\infty]}(\alpha) \neq 0$,
(3) $\mathrm{mdeg}(\theta^{\Lambda}(\widetilde{\alpha})) - \mathrm{mdeg}(\widetilde{\alpha}) = cs_{\pi}(a_{k'}) < r$.

Here, (1) and (3) imply $\alpha \in CI_1^{[-\infty,r]}(Y) \otimes \mathbb{Q}$, and hence (2) implies $[\theta_Y^{[-\infty,r]}] \neq 0$. This gives $r \geq r_{-\infty}(Y)$, a contradiction. Hence $\Gamma_Y(1) \geq r_{-\infty}(Y)$. ∎

4.4. Consequences

Here, we prove some corollaries of Theorem 1.2.

Corollary 1.3. *The inequality $r_{-\infty}(Y) < \infty$ holds if and only if $h(Y) < 0$. In particular, if $h(Y) < 0$, then $r_s(Y)$ is finite for any $s \in [-\infty, 0]$.*

Proof. It is shown in [6] that $h(Y) < 0$ if and only if $\Gamma_{-Y}(1) < \infty$. This fact and Theorem 1.2 give the conclusion. ∎

Recall that $\Sigma(a_1, \dots, a_n)$ denotes the Seifert homology 3-sphere corresponding to a tuple $(a_1, \dots, a_n)$ of pairwise coprime integers and $R(a_1, \dots, a_n)$ is an odd integer introduced by Fintushel–Stern [14].

Corollary 1.4. *If $R(a_1, \dots, a_n) > 0$, then for any $s \in [-\infty, 0]$,*

$$r_s(-\Sigma(a_1, \dots, a_n)) = \frac{1}{4a_1 \cdots a_n}, \quad r_s(\Sigma(a_1, \dots, a_n)) = \infty.$$

Proof. By using Lemma 3.5 and Theorem 1.2, we obtain

$$r_s(-\Sigma(a_1,\dots,a_n)) \le r_{-\infty}(-\Sigma(a_1,\dots,a_n)) = \Gamma_{\Sigma(a_1,\dots,a_n)}(1) = \frac{1}{4a_1\cdots a_n}.$$

Moreover, it is shown in [15, 24] that

$$\min(\Lambda_{-\Sigma(a_1,\dots,a_n)} \cap \mathbb{R}_{>0}) = \frac{1}{4a_1\cdots a_n}.$$

This gives the first equality in Corollary 1.4. The second equality follows from Corollary 3.11 and the fact that $\Sigma(a_1,\dots,a_n)$ bounds a negative definite 4-manifold. ■

Corollary 4.6. *For any positive coprime integers $p, q > 1$ and positive integer k, we have*

$$r_s(-\Sigma(p,q,pqk-1)) = \frac{1}{4pq(pqk-1)}, \quad r_s(\Sigma(p,q,pqk-1)) = \infty.$$

5. Applications

In this section, we prove the theorems stated in Section 1.2.

5.1. Useful lemmas

We first give several lemmas which are useful for computing r_0.

Lemma 5.1. *For any homology 3-spheres Y_1 and Y_2, if $r_0(-Y_1) = r_0(-Y_2) = \infty$, then*

$$r_0(Y_1 \# Y_2) = \min\{r_0(Y_1), r_0(Y_2)\}, \quad r_0(-Y_1 \# -Y_2) = \infty.$$

Proof. The equality $r_0(-Y_1 \# -Y_2) = \infty$ and the inequality

$$r_0(Y_1 \# Y_2) \ge \min\{r_0(Y_1), r_0(Y_2)\}$$

immediately follow from Theorem 3.12. To prove the opposite inequality, we first consider $r_0(Y_1 \# Y_2 \# -Y_2)$. Then, by Corollary 3.8 and Theorem 3.12, we have

$$r_0(Y_1) = r_0(Y_1 \# Y_2 \# -Y_2) \ge \min\{r_0(Y_1 \# Y_2), r_0(-Y_2)\}.$$

Here, since $r_0(Y_1 \# Y_2) \le r_0(-Y_2) = \infty$, we obtain $r_0(Y_1) \ge r_0(Y_1 \# Y_2)$. Similarly, we have $r_0(Y_2) \ge r_0(Y_1 \# Y_2)$. This completes the proof. ■

Corollary 5.2. *Suppose a homology 3-sphere Y satisfies $r_0(Y) < \infty$ and $r_0(-Y) = \infty$. Then, for any $n \in \mathbb{Z}_{>0}$, we have*

$$r_0(n[Y]) = r_0(Y) < \infty, \quad r_0(-n[Y]) = \infty.$$

In particular, Y has infinite order in $\Theta^3_{\mathbb{Z}}$.

Proof. By induction on n, this corollary directly follows from Lemma 5.1. ■

Lemma 5.3. *For any homology* 3*-spheres* Y_1 *and* Y_2*, if* $r_0(Y_1) < \min\{r_0(Y_2), r_0(-Y_2)\}$*, then* $r_0(Y_1 \# -Y_2) = r_0(Y_1)$*.*

Proof. By applying Theorem 3.12 to $Y_1 \# -Y_2$ and $Y_1 \# -Y_2 \# Y_2$, we have

$$r_0(Y_1 \# -Y_2) \geq \min\{r_0(Y_1), r_0(-Y_2)\} = r_0(Y_1),$$
$$r_0(Y_1) = r_0(Y_1 \# -Y_2 \# Y_2) \geq \min\{r_0(Y_1 \# -Y_2), r_0(Y_2)\}.$$

Here, since $r_0(Y_1) < r_0(Y_2)$, we obtain $r_0(Y_1) \geq r_0(Y_1 \# -Y_2)$. ■

Theorem 5.4. *Suppose that a linear combination* $\sum_{k=1}^m n_k[Y_k] \in \Theta^3_{\mathbb{Z}}$ *satisfies*

- $r_0(Y_m) < \min_{1 \leq k < m}\{r_0(Y_k), r_0(-Y_k)\}$*,*
- $r_0(-Y_m) = \infty$*,*
- $n_m > 0$*.*

Then $r_0(\sum_{k=1}^m n_k[Y_k]) = r_0(Y_m) < \infty$*.*

Proof. By assumption, it follows from Corollary 5.2 that $r_0(n_m[Y_m]) = r_0(Y_m)$. Moreover, Theorem 3.12 implies that

$$\min\left\{r_0\Big(\sum_{k=1}^{m-1}(-n_k)[Y_k]\Big), r_0\Big(-\sum_{k=1}^{m-1}(-n_k)[Y_k]\Big)\right\} \geq \min_{1 \leq k < m}\{r_0(Y_k), r_0(-Y_k)\} > r_0(n_m[Y_m]).$$

Therefore, by Lemma 5.3, we have

$$r_0\Big(\sum_{k=1}^{m} n_k[Y_k]\Big) = r_0\Big(n_m[Y_m] - \sum_{k=1}^{m-1}(-n_k)[Y_k]\Big) = r_0(Y_m). \quad ■$$

For a homology 3-sphere Y, set

$$\varepsilon_1(Y) := \inf(\Lambda_Y \cap \mathbb{R}_{>0}) \quad \text{and} \quad \varepsilon_2(Y) := \min\{\varepsilon_1(Y), \varepsilon_1(-Y)\}.$$

Theorem 5.4 is regarded as a generalization of the following theorem due to Furuta.

Corollary 5.5 ([24, Theorem 6.1]). *Let* $Y_1, \ldots, Y_m$ *be homology* 3*-spheres with* $\varepsilon_2(Y_i) > 0$ $(i = 1, \ldots, m)$*. Let* $Y_0 = \Sigma(a_1, \ldots, a_n)$ *be a Seifert homology* 3*-sphere such that* $R(a_1, \ldots, a_n) > 0$ *and* $a_1 \cdots a_n > \varepsilon_2(Y_i)^{-1}$ $(i = 1, \ldots, m)$*. Then*

$$\mathbb{Z}[Y_0] \cap (\mathbb{Z}[Y_1] + \cdots + \mathbb{Z}[Y_m]) = 0 \quad \text{in } \Theta^3_{\mathbb{Z}}.$$

Proof. Note that since $r_0(Y) \in \Lambda_Y \cap \mathbb{R}_{>0}$ in general, we have

$$\min\{r_0(Y_i), r_0(-Y_i)\} \geq \varepsilon_2(Y_i) > \frac{1}{a_1 \cdots a_n} = r_0(-Y_0)$$

for any $i = 1, \ldots, m$, where the last equality follows from Corollary 1.4. Therefore, if

$$n_0[-Y_0] = n_1[Y_1] + n_2[Y_2] + \cdots + n_m[Y_m]$$

and $n_0 > 0$, then it follows from Theorem 5.4 that

$$\frac{1}{a_1 \cdots a_n} = r_0(-Y_0) = r_0\Big(n_0[-Y_0] - \sum_{i=1}^{m} n_i[Y_i]\Big) = r_0(S^3) = \infty,$$

a contradiction. ∎

Corollary 5.6. *Let* $\{Y_k\}_{k=1}^{\infty}$ *be a sequence of homology* 3*-spheres satisfying*

$$\infty > r_0(Y_1) > r_0(Y_2) > \cdots \quad \textit{and} \quad \infty = r_0(-Y_1) = r_0(-Y_2) = \cdots .$$

Then the Y_k*'s are linearly independent in* $\Theta^3_{\mathbb{Z}}$.

Proof. Let $\sum_{k=1}^{m} n_k[Y_k]$ be a linear combination with $n_m \neq 0$. Without loss of generality, we can assume $n_m > 0$. Then $\sum_{k=1}^{m} n_k[Y_k]$ satisfies the hypothesis of Theorem 5.4, and hence

$$r_0\Big(\sum_{k=1}^{m} n_k[Y_k]\Big) = r_0(Y_m) < \infty = r_0(S^3).$$

This implies that $\sum_{k=1}^{m} n_k[Y_k] \neq 0$. ∎

5.2. *Homology 3-spheres with no definite bounding*

In this subsection, we prove Theorem 1.5.

Theorem 1.5. *There exist infinitely many homology* 3*-spheres* $\{Y_k\}_{k=1}^{\infty}$ *which cannot bound any definite* 4*-manifold. Moreover, we can take such* Y_k *so that the* Y_k*'s are linearly independent in* $\Theta^3_{\mathbb{Z}}$.

For any $k \in \mathbb{Z}_{>0}$, let K_k be the knot depicted in Figure 2. Note that K_k is the 2-bridge knot corresponding to the rational number $\frac{2}{4k-1}$. In particular, the first two knots K_1 and K_2 are the left-handed trefoil 3_1 and the knot 5_2 in Rolfsen's knot table [51] respectively.

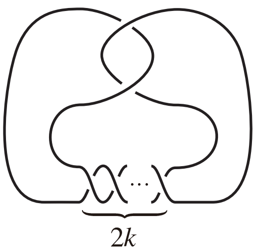

Fig. 2. The knot K_k.

Lemma 5.7. *For any* $k \in \mathbb{Z}_{>0}$, *we have a diffeomorphism*

$$\Sigma(2, 3, 6k-1) \cong S^3_{-1}(K_k).$$

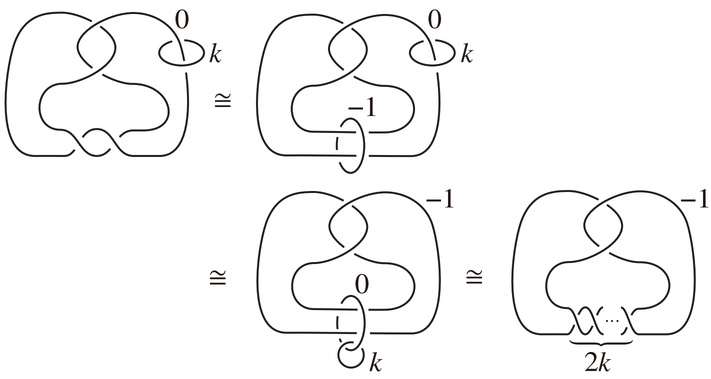

Fig. 3. Kirby calculus for $S^3_{-1/k}(K_1) \cong S^3_{-1}(K_k)$.

Proof. It is well-known that $\Sigma(2,3,6k-1) \cong S^3_{-1/k}(3_1)$, and Figure 3 proves $S^3_{-1/k}(3_1) = S^3_{-1/k}(K_1) \cong S^3_{-1}(K_k)$. ■

While the Frøyshov invariant h is hard to compute in general, we have a nice estimate for (-1)-surgeries on genus 1 knots.

Lemma 5.8 ([21, Lemma 9]). *For any genus* 1 *knot* K*, we have*

$$0 \leq h(S^3_{-1}(K)) \leq 1.$$

Lemma 5.9. *For any* $k \in \mathbb{Z}_{>0}$*, we have* $h(\Sigma(2,3,6k-1)) = 1$.

Proof. The inequality $h(\Sigma(2,3,6k-1)) = -h(-\Sigma(2,3,6k-1)) > 0$ follows from Corollaries 1.3 and 4.6. The inequality $h(\Sigma(2,3,6k-1)) \leq 1$ follows from Lemmas 5.7 and 5.8 and the fact that K_k has genus 1 for any $k \in \mathbb{Z}_{>0}$. ■

Now we prove one of the main theorems.

Proof of Theorem 1.5. We put $Y_k := 2\Sigma(2,3,5) \# (-\Sigma(2,3,6k+5))$ for any $k \in \mathbb{Z}_{>0}$. Then it follows from Corollaries 4.6 and 5.2 and Lemma 5.3 that $r_0(Y_k) = \frac{1}{24(6k+5)} < \infty$. This fact and Corollary 3.11 imply that Y_k cannot bound any negative definite 4-manifold.

Next, since the invariant h is a group homomorphism, Lemma 5.9 gives $h(Y_k) = 1$. This fact and Corollary 1.3 imply that $r_{-\infty}(-Y_k) < \infty$, and hence it follows from Corollary 3.11 that Y_k cannot bound any positive definite 4-manifold.

The linear independence of $\{Y_k\}_{k=1}^{\infty}$ follows from that of $\{\Sigma(2,3,6k-1)\}_{k=1}^{\infty}$. ■

5.3. *Linear independence of* $1/n$*-surgeries*

In this subsection, we prove the theorems stated in Section 1.2.2.

Theorem 1.8. *For any knot* K *in* S^3*, if* $h(S^3_1(K)) < 0$*, then* $\{S^3_{1/n}(K)\}_{n=1}^{\infty}$ *are linearly independent in* $\Theta^3_{\mathbb{Z}}$.

Corollary 1.9. *For any $k \in \mathbb{Z}_{>0}$, the homology 3-spheres $\{S^3_{1/n}(K_k)\}_{n=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$.*

Corollary 1.10. *For any knot K in S^3 and odd integer $q \geq 3$, the homology 3-spheres $\{S^3_{1/n}(K_{2,q})\}_{n=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$.*

Observe that Corollary 1.9 immediately follows from Theorem 1.8 and Lemmas 5.7 and 5.9. On the other hand, Corollary 1.10 follows from Theorem 1.8 and the following two facts. Note that the intersection form of any spin 4-manifold whose boundary is a homology 3-sphere is even, and hence it is non-diagonalizable, namely the intersection form is not isomorphic to $\bigoplus(\pm 1)$.

Theorem 5.10 ([21, Theorem 3]). *If Y bounds a positive definite 4-manifold with non-diagonalizable intersection form, then $h(Y) < 0$.*

Theorem 5.11 ([53, Theorem 1.7]). *For any knot K and odd $q \geq 3$, the homology 3-sphere $S^3_1(K_{2,q})$ bounds a positive definite spin 4-manifold.*

The proof of Theorem 1.8 is obtained by combining Corollary 5.6 with the following theorem.

Theorem 5.12. *For any knot K in S^3, if $h(S^3_1(K)) < 0$, then for any s, we have*

$$\infty > r_s(S^3_1(K)) > r_s(S^3_{1/2}(K)) > \cdots,$$
$$\infty = r_s(-S^3_1(K)) = r_s(-S^3_{1/2}(K)) = \cdots.$$

Proof. For any knot K and $n \in \mathbb{Z}_{>0}$, since $S^3_{1/n}(K)$ bounds a positive definite 4-manifold, we have

$$\infty = r_s(-S^3_1(K)) = r_s(-S^3_{1/2}(K)) = \cdots.$$

Suppose that K satisfies $h(S^3_1(K)) < 0$. Then Corollary 1.3 gives $r_s(S^3_1(K)) < \infty$. For any $n \in \mathbb{Z}_{>0}$, let W_n be the cobordism given by the relative Kirby diagram in Figure 4. It is easy to see that $\partial W_n = S^3_{1/(n+1)}(K) \amalg -S^3_{1/n}(K)$.

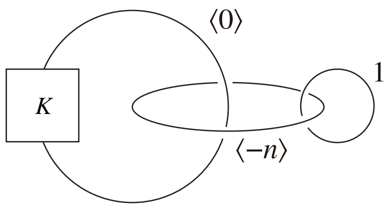

Fig. 4. The cobordism W_n.

Claim 5.13. *The cobordism W_n is positive definite.*

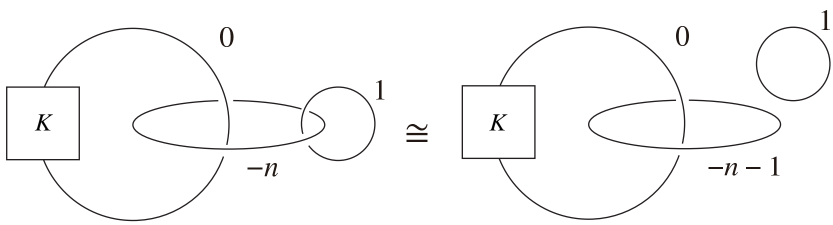

Fig. 5. The 4-manifold X_n.

Proof. Let X_n be a 4-manifold given by the Kirby diagrams in Figure 5, and X'_n a 4-dimensional submanifold of X_n obtained by attaching 2-handles along the 2-component sublink in the left diagram of Figure 5 whose framing is $(0, -n)$. Then we have the diffeomorphism $X_n \cong X'_n \cup_{S^3_{1/n}(K)} W_n$. For a 4-manifold M, let $b_2^+(M)$ (resp. $b_2^-(M)$) denote the number of positive (resp. negative) eigenvalues of the intersection form of M. Then it is easy to check that $b_2^+(X_n) = 2$, $b_2^-(X_n) = 1$ and $b_2^+(X'_n) = b_2^-(X'_n) = 1$. These imply that $b_2(W_n) = b_2^+(W_n) = 1$ and $b_2^-(W_n) = 0$. ■

Claim 5.14. *The cobordism W_n is simply connected.*

Proof. Suppose that the number of crossings in the diagram of Figure 6 is $m + 1$. Then, $\pi_1(S^3_{1/n}(K))$ has the presentation

$$\left\langle x_1, \dots, x_m, y \;\middle|\; \begin{array}{l} x_{i+1} = x_{k_i}^{\varepsilon_i} x_i x_{k_i}^{-\varepsilon_i} \ (i = 1, \dots, m-1), \\ x_m y = y x_1, \ y x_1 = x_1 y, \\ \lambda = 1, \ x_1 y^{-n} = 1 \end{array} \right\rangle,$$

where

- the labels x_1, x_m and y are associated as shown in Figure 6,
- $\varepsilon_i \in \{\pm 1\}$ and $k_i \in \{1, \dots, m\}$ $(i = 1, \dots, m)$,
- λ is a word corresponding to a longitude of K with framing 0.

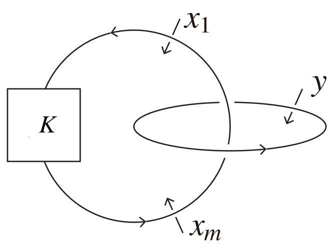

Fig. 6. A surgery link for $S^3_{1/n}(K)$.

Moreover, since the attaching sphere of the (unique) 2-handle of W_n is homotopic to y, $\pi_1(W_n)$ has the presentation

$$\left\langle x_1,\dots,x_m,y \;\middle|\; \begin{array}{l} x_{i+1}=x_{k_i}^{\varepsilon_i}x_ix_{k_i}^{-\varepsilon_i}\ (i=1,\dots,m-1),\\ x_my=yx_1,\ yx_1=x_1y,\\ \lambda=1,\ x_1y^{-n}=1,\ y=1 \end{array}\right\rangle$$
$$\cong\left\langle x_1,\dots,x_m \;\middle|\; \begin{array}{l} x_{i+1}=x_{k_i}^{\varepsilon_i}x_ix_{k_i}^{-\varepsilon_i}\ (i=1,\dots,m-1),\\ x_m=x_1=1,\ \lambda|_{y=1}=1 \end{array}\right\rangle,$$

where $\lambda|_{y=1}$ is the word obtained from λ by substituting 1 for y. Now, by induction, we see that the relations $x_1=\cdots=x_m=1$ hold, and hence $\pi_1(W_n)=1$. ∎

By Claims 5.13 and 5.14, we can apply Corollary 3.9 to all $-W_n$ ($n=1,2,\dots$), and obtain

$$\infty > r_s(S^3_1(K)) > r_s(S^3_{1/2}(K)) > \cdots .$$ ∎

5.4. *Linear independence of Whitehead doubles*

In this subsection, we prove Theorem 1.12.

Theorem 1.12. *For any coprime $p,q>1$, the Whitehead doubles $\{D(T_{p,np+q})\}_{n=0}^{\infty}$ are linearly independent in $\mathcal{C}$.*

Let $\Theta^3_{\mathbb{Q}}$ denote the rational homology cobordism group of rational homology 3-spheres. Then we have a natural group homomorphism

$$\Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}}, \quad [Y]\mapsto [Y]_{\mathbb{Q}},$$

where $[Y]_{\mathbb{Q}}$ is the rational homology cobordism class of Y. We say that the rational homology 3-spheres $\{Y_k\}_{k=1}^{\infty}$ are *linearly independent in* $\Theta^3_{\mathbb{Q}}$ if $\{[Y_k]_{\mathbb{Q}}\}_{k=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Q}}$. Then the invariance of r_s and Corollary 5.6 are naturally generalized in the following sense.

Theorem 5.15. *For any homology 3-sphere Y and $s\in[-\infty,0]$, the value $r_s(Y)$ is invariant under rational homology cobordism. Moreover, if a sequence $\{Y_k\}_{k=1}^{\infty}$ of homology 3-spheres satisfies the assumption of Corollary 5.6, then the Y_k's are linearly independent in $\Theta^3_{\mathbb{Q}}$.*

Next, let K be an oriented knot and $\Sigma(K)$ the double branched cover of S^3 over K. Then it is known that the map

$$\mathcal{C}\to\Theta^3_{\mathbb{Q}}, \quad [K]\mapsto[\Sigma(K)]_{\mathbb{Q}},$$

is well-defined and a group homomorphism. Moreover, for Whitehead doubles, it is also known that $\Sigma(D(K))\cong S^3_{1/2}(K\#-K)$, where $-K$ is orientation-reversed K. In particular, $\Sigma(D(K))$ is a homology 3-sphere and bounds a positive definite 4-manifold. Hence $r_0(-\Sigma(D(K)))=\infty$ for any K. These arguments imply the following.

Lemma 5.16. *For a sequence* $\{K_n\}_{n=1}^{\infty}$ *of oriented knots, if the homology 3-spheres* $\{\Sigma(D(K_n))\}_{n=1}^{\infty}$ *satisfy*

$$\infty > r_0(\Sigma(D(K_1))) > r_0(\Sigma(D(K_2))) > \cdots,$$

then the Whitehead doubles $\{D(K_n)\}_{n=1}^{\infty}$ *are linearly independent in* $\mathcal{C}$.

For any coprime integers $p, q > 1$, we abbreviate $D(T_{p,q})$ to $D_{p,q}$. The proof of Theorem 1.12 is obtained by combining Lemma 5.16 with the following theorem.

Theorem 5.17. *For any coprime integers* $p, q > 1$*, we have*

$$\frac{1}{4pq(2pq-1)} \geq r_s(\Sigma(D_{p,q})) > r_s(\Sigma(D_{p,p+q})) > r_s(\Sigma(D_{p,2p+q})) > \cdots.$$

As another corollary of Theorem 5.17, we also have the following family of linearly independent elements.

Corollary 5.18. *Let a and b be coprime integers with* $1 < a < b$ *and*

$$b = q_0 a + r_0, \quad a = q_1 r_0 + r_1, \ \ldots, \ r_{N-1} = q_{N+1} r_N + 1$$

the sequence derived from the Euclidean algorithm. Then

$$\begin{aligned}\infty &> r_s(\Sigma(D_{r_N, r_N+1})) > r_s(\Sigma(D_{r_N, 2r_N+1})) > \cdots > r_s(\Sigma(D_{r_N, r_{N-1}})) \\ &> r_s(\Sigma(D_{r_{N-1}, r_{N-1}+r_N})) > \cdots > r_s(\Sigma(D_{r_{N-1}, r_{N-2}})) \\ &> \cdots > r_s(\Sigma(D_{a, a+r_0})) > \cdots > r_s(\Sigma(D_{a,b})).\end{aligned}$$

In particular, all of these Whitehead doubles are linearly independent in $\mathcal{C}$.

Now we start to prove Theorem 5.17. Let K be an oriented knot, $\mathcal{D}$ a diagram of K and $x_1, \ldots, x_m$ the arcs of $\mathcal{D}$. Fix a base point in $S^3 \setminus K$, and associate a loop in $S^3 \setminus K$ to each x_i in the usual way. (For instance, see [51, Section 3.D].) Then, for any $n \in \mathbb{Z}$, we have a presentation of $\pi_1(S^3_{1/n}(K))$ in the form

$$\langle x_1, \ldots, x_m \mid R \cup \{\lambda^n x_1 = 1\} \rangle,$$

where R is the set of relations induced from the crossings of $\mathcal{D}$ (in the same way as the Wirtinger presentation), and λ is a word corresponding to a longitude of K with framing 0. (In particular, λ is in the commutator subgroup of $\pi_1(S^3_{1/n}(K))$.)

Next, we consider the *positive crossing change* at a *positive crossing* c, which is a deformation of $\mathcal{D}$ shown in Figure 7. Then we denote the labels of the arcs around c by x_{i_c}, x_{j_c} and $x_{j'_c}$ as shown in Figure 7.

Lemma 5.19. *Suppose that* $\mathcal{D}$ *is deformed into a diagram* $\mathcal{D}'$ *of a knot* K' *by performing positive crossing changes at crossings* $c_1, \ldots, c_l$ *respectively. Then there exists a negative definite cobordism* W *such that* $\partial W = -S^3_{1/n}(K) \amalg S^3_{1/n}(K')$ *and*

$$\pi_1(W) \cong \langle x_1, \ldots, x_m \mid R \cup \{\lambda^n x_1 = 1\} \cup \{x_{i_{c_k}} = x_{j_{c_k}}\}_{k=1}^{l} \rangle.$$

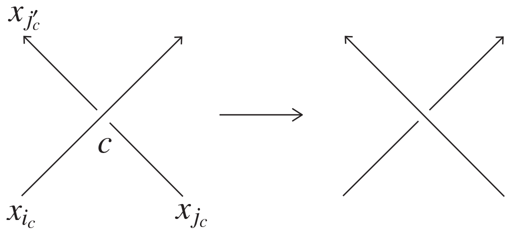

Fig. 7. A positive crossing change.

Proof. We make a relative Kirby diagram $\langle\mathcal{D}\rangle$ from $\mathcal{D}$ in the following way:

- Replace a neighborhood of each crossing c_k with the picture shown in Figure 8. (Then each component except for the original one has a framing.)
- Associate the framing $\langle 1/n\rangle$ to the original component.

Then $\langle\mathcal{D}\rangle$ is a diagram for a cobordism W obtained by attaching l copies of 2-handles to $S^3_{1/n}(K)\times[0,1]$. Moreover, we can verify that

- $\partial W = -S^3_{1/n}(K) \amalg S^3_{1/n}(K')$,
- the intersection form of W is isomorphic to $\bigoplus_{k=1}^{l}(-1)$,
- as a loop, the attaching sphere of the 2-handle near c_k is written as $x_{j_{c_k}}^{-1}x_{i_{c_k}}$.

This completes the proof. ∎

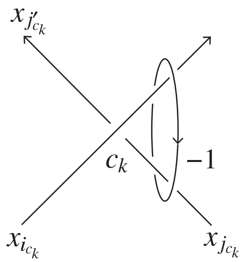

Fig. 8. The 2-handle attached to $S^3_{1/n}(K)\times[0,1]$ near c_k.

Next, for any coprime $p, q > 1$, we consider the homology 3-sphere $\Sigma(D_{p,q})$.

Lemma 5.20. $r_s(\Sigma(D_{p,q})) \leq \frac{1}{4pq(2pq-1)}$.

Proof. Note that $T_{p,q}$ has a diagram $\overline{\Delta_p^q}$ with only positive crossings. (Indeed, the closure of the braid $\Delta_p^q = (\sigma_1\cdots\sigma_{p-1})^q$ with p strands is such a diagram for $T_{p,q}$.) For any knot diagram, there exist finitely many crossings such that after crossing changes at the crossings, the resulting diagram describes the unknot. As a consequence, we have finitely many positive crossings of $\overline{\Delta_p^q}$ such that after positive crossing changes at the crossings, the resulting diagram $(\overline{\Delta_p^q})^U$ is as for the unknot. Now, considering the connected sum of two $\overline{\Delta_p^q}$'s, we have finitely many positive crossings of $\overline{\Delta_p^q}\#\overline{\Delta_p^q}$ such that after positive crossing changes at the crossings, we have the diagram $\overline{\Delta_p^q} \,\#\, (\overline{\Delta_p^q})^U$. Hence, by applying Lemma 5.19, we have a negative definite cobordism with boundary $-S^3_{1/2}(T_{p,q} \# T_{p,q}) \amalg S^3_{1/2}(T_{p,q})$. Therefore, it follows from Theorem 3.10 and Corollary 4.6 that

$$\begin{aligned} r_s(\Sigma(D_{p,q})) &= r_s(S^3_{1/2}(T_{p,q} \# T_{p,q})) \\ &\le r_s(S^3_{1/2}(T_{p,q})) \\ &= r_s(\Sigma(p,q,2pq-1)) = \frac{1}{4pq(2pq-1)}. \end{aligned}$$ ■

Now, let us consider a concrete diagram of $T_{p,p+q} \# T_{p,p+q}$, which is depicted in Figure 9 and denoted by $\mathcal{D}$. Here Δ_H is the braid

$$(\sigma_1 \cdots \sigma_{p-1})(\sigma_1 \cdots \sigma_{p-2}) \cdots (\sigma_1\sigma_2)\sigma_1,$$

which is often called the *half-twist*. In addition, we associate the labels $\{x_k\}_{k=1}^{2p}$ to arcs in D as shown in Figure 9.

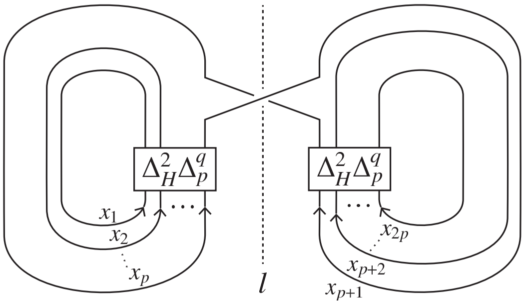

Fig. 9. The diagram $\mathcal{D}$.

Lemma 5.21. *As elements of $\pi_1(S^3 \setminus T_{p,p+q} \# T_{p,p+q})$, all arcs belonging to the left (resp. right) side of the dashed line l in the diagram $\mathcal{D}$ are written as a conjugate of x_k by a word w consisting of $x_1, \ldots, x_p$ (resp. $x_{p+1}, \ldots, x_{2p}$) for some $k \in \{1, \ldots, p\}$ (resp. $k \in \{p+1, \ldots, 2p\}$).*

Proof. We prove the lemma by induction on the place of arcs. Here we first consider the left side of l. Let us start from the bottom of the box $\Delta_H^2\Delta_p^q$. Then, for any $k \in \{1, \ldots, p\}$, the k-th arc from the left is just x_k, and hence these p arcs satisfy the assertion of this lemma.

Next, fix a crossing σ_k in $\Delta_H^2\Delta_p^q$ and assume that all arcs below this σ_k satisfy the assertion of the lemma. Then, since the upper right arc of the σ_k is the same as the bottom left arc (denoted $x_{i_{\sigma_k}}$), it also satisfies the assertion. Moreover, the upper left arc is equal to $x_{i_{\sigma_k}}^{-1} x_{j_{\sigma_k}} x_{i_{\sigma_k}}$, where $x_{j_{\sigma_k}}$ denotes the bottom right arc. Here, by assumption, there exist some $k' \in \{1, \ldots, p\}$ and a word w consisting of $x_1, \ldots, x_p$ such that $x_{j_{\sigma_k}} = w^{-1}x_{k'}w$.

Therefore, we have $x_{i_{\sigma_k}}^{-1} x_{j_{\sigma_k}} x_{i_{\sigma_k}} = (w x_{i_{\sigma_k}})^{-1} x_{k'} (w x_{i_{\sigma_k}})$. Since $x_{i_{\sigma_k}}$ also consists of $x_1, \dots, x_p$, this completes the proof for the left side of the dashed line l.

Similarly, we can prove the lemma for the right side of l. ■

Proof of Theorem 5.17. It is easy to check that

$$\Delta_H = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{p-1}\cdots\sigma_2\sigma_1).$$

In particular,

$$(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-1}^{-1})(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-2}^{-1})\cdots(\sigma_1^{-1}\sigma_2^{-1})\sigma_1^{-1} = \Delta_H^{-1},$$

and hence the positive crossing changes at all crossings in Δ_H give Δ_H^{-1}. Now we perform the positive crossing changes at all crossings in the first Δ_H of both $\Delta_H^2 \Delta_p^q$'s in Figure 9. Then, by Lemma 5.19, we obtain a negative definite cobordism W with boundary $-S^3_{1/2}(T_{p,p+q} \# T_{p,p+q}) \# S^3_{1/2}(T_{p,q} \# T_{p,q})$ such that $\pi_1(W)$ has a presentation of the form shown in Lemma 5.19. Here, we note that the crossing changes at the first p crossings in $\Delta_H^2 \Delta_p^q$ on the left gives the relations

$$x_1 = \cdots = x_p.$$

Similarly, we have $x_{p+1} = \cdots = x_{2p}$. Therefore, by Lemma 5.21, all arcs in the left (resp. right) side of the dashed line l in D are equal to x_1 (resp. x_{p+1}) as elements of $\pi_1(W)$. Moreover, x_{p+1} belongs not only to the right side of l but also to the left side, and hence $x_{p+1} = x_1$.

Now, any two generators in our presentation of $\pi_1(W)$ are equal. Moreover, since λ is in the commutator subgroup, we have $\lambda = 1$, and hence $x_1 = 1$. This gives $\pi_1(W) = 1$. Therefore, by applying Corollary 3.9 to W, we have

$$r_s(\Sigma(D_{p,p+q})) = r_s(S^3_{1/2}(T_{p,p+q} \# T_{p,p+q})) < r_s(S^3_{1/2}(T_{p,q} \# T_{p,q})) = r_s(\Sigma(D_{p,q})).$$

Since q is an arbitrary integer with $q > 1$ and $\gcd(p,q) = 1$, this inequality holds even if we replace q with $kp + q$ for any $k \in \mathbb{Z}_{>0}$. Consequently,

$$r_s(\Sigma(D_{p,q})) > r_s(\Sigma(D_{p,p+q})) > r_s(\Sigma(D_{p,2p+q})) > \cdots.$$

Combining this with Lemma 5.20 completes the proof. ■

6. Additional structures on $\Theta^3_{\mathbb{Z}}$ and Ker h

In this section, we prove Theorems 1.14 and 6.1. Recall that for any $r \in [0, \infty]$, the subgroup $\Theta^3_{\mathbb{Z},r} \subset \Theta^3_{\mathbb{Z}}$ is defined by

$$\Theta^3_{\mathbb{Z},r} := \left\{ [Y] \in \Theta^3_{\mathbb{Z}} \;\middle|\; \min\{r_0(Y), r_0(-Y)\} \geq r \right\}.$$

Theorem 1.14. *For any $r \in (0, \infty]$, the quotient group $\Theta^3_{\mathbb{Z}} / \Theta^3_{\mathbb{Z},r}$ contains $\mathbb{Z}^\infty$ as a subgroup.*

Proof. To prove the theorem, we use the sequence $\{\Sigma(2,3,6k-1)\}_{k=1}^{\infty}$. (In fact, we can replace it with any sequence $\{Y_k\}_{k=1}^{\infty}$ such that $\{r_0(Y_k)\}_{k=1}^{\infty}$ is a decreasing sequence and converges to zero.) Fix $r \in (0, \infty]$. Then, since $r_0(-\Sigma(2,3,6k-1)) = 1/24(6k-1)$ converges to zero, there exists an integer N such that $1/24(6N-1) < r$. Let $[Y]_r$ denote the equivalence class of $[Y]$ in $\Theta^3_{\mathbb{Z}}/\Theta^3_{\mathbb{Z},r}$. We prove that $\{[\Sigma(2,3,6k-1)]_r\}_{k=N}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}/\Theta^3_{\mathbb{Z},r}$.

Assume that $\sum_{k=N}^{M} n_k[\Sigma(2,3,6k-1)]_r = 0$ and $n_M \neq 0$. Without loss of generality, we may assume that $n_M > 0$. Then, by the definition of $\Theta^3_{\mathbb{Z},r}$, we have

$$\min\left\{r_0\Big(\sum_{k=N}^{M} n_k[\Sigma(2,3,6k-1)]\Big), r_0\Big(-\sum_{k=N}^{M} n_k[\Sigma(2,3,6k-1)]\Big)\right\} \geq r.$$

However, Theorem 5.4 implies

$$\begin{aligned} r_0\Big(-\sum_{k=N}^{M} n_k[\Sigma(2,3,6k-1)]\Big) &= r_0(-\Sigma(2,3,6M-1)) \\ &= \frac{1}{24(6M-1)} \leq \frac{1}{24(6N-1)} < r, \end{aligned}$$

a contradiction. ∎

6.1. *A pseudometric on* Ker h

We next consider a pseudometric on Ker h, where $h : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ is the Frøyshov invariant. To define it, set

$$s_\infty(Y) := \sup\{s \in [-\infty, 0] \mid r_s(Y) = \infty\}.$$

As a corollary of the connected sum formula for $\{r_s\}$, we have the following theorem.

Theorem 6.1. $s_\infty(Y_1 \# Y_2) \geq s_\infty(Y_1) + s_\infty(Y_2)$.

Moreover, Corollary 1.3 implies that if $h(Y) = 0$, then $\max\{-s_\infty(Y), -s_\infty(-Y)\} < \infty$. Now we can define a pseudometric on Ker h as

$$d_\infty([Y_1],[Y_2]) := -s_\infty(Y_1 \# (-Y_2)) - s_\infty((-Y_1) \# Y_2).$$

Moreover, the set of elements with $d_\infty([S^3],[Y]) = 0$ coincides with $\Theta^3_{\mathbb{Z},\infty}$.

Theorem 6.2. *The map d_∞ gives a metric on* Ker $h/\Theta^3_{\mathbb{Z},\infty}$*, and the action of* Ker $h/\Theta^3_{\mathbb{Z},\infty}$ *on* (Ker $h/\Theta^3_{\mathbb{Z},\infty}, d_\infty$) *is an isometry. In particular,* Ker $h/\Theta^3_{\mathbb{Z},\infty}$ *is a topological group with respect to the metric topology induced by d_∞.*

Note that $\{\Sigma(2,3,5) \# (-\Sigma(2,3,6k-1))\}_{k\in\mathbb{Z}_{>1}}$ are linearly independent in Ker $h/\Theta^3_{\mathbb{Z},\infty}$, and hence Ker $h/\Theta^3_{\mathbb{Z},\infty}$ contains $\mathbb{Z}^\infty$ as a subgroup. Here, we ask the following question.

Question 6.3. What is the isomorphism type of $(\mathrm{Ker}\, h/\Theta^3_{\mathbb{Z},\infty}, d_\infty)$ as a topological group? In particular, is it a discrete group?

Lemma 6.4. *If $s_\infty(Y) = 0$, then $r_0(Y) = \infty$.*

Proof. Since $s_\infty(Y) = 0$, for any $s < 0$ we have $r_s(Y) = \infty$, and hence $[\theta_Y^{[s,\infty]}] = 0$. Suppose that $0 \in \mathbb{R} \setminus \Lambda_Y^*$. Then we can take $s < 0$ such that $[s, 0] \subset \mathbb{R} \setminus \Lambda_Y^*$. Hence, Lemmas 2.9 and 2.15 give an isomorphism from $I^1_{[0,\infty]}(Y)$ to $I^1_{[s,\infty]}(Y)$ which maps $[\theta_Y^{[0,\infty]}]$ to $[\theta_Y^{[s,\infty]}]$. This implies $[\theta_Y^{[0,\infty]}] = 0$, and hence $r_0(Y) = \infty$.

Next, suppose that $0 \in \Lambda_Y^*$. Then, by the definition of $CI^*_{[0,\infty]}(Y)$, the cohomology group $I^1_{[0,\infty]}(Y)$ and $[\theta_Y^{[0,\infty]}]$ coincide with $I^1_{[-\frac{1}{2}\lambda_Y,\infty]}(Y)$ and $[\theta_Y^{[-\frac{1}{2}\lambda_Y,\infty]}]$ respectively, where $\lambda_Y := \min\{|a-b| \mid a, b \in \Lambda_Y \text{ with } a \neq b\} > 0$. This implies $[\theta_Y^{[0,\infty]}] = 0$, and hence $r_0(Y) = \infty$. ■

Next, we prove Theorem 6.1.

Proof of Theorem 6.1. We may assume $s_\infty(Y_1) + s_\infty(Y_2) \neq -\infty$. For any $s \in (-\infty, s_\infty(Y_1) + s_\infty(Y_2))$, there exist $s_1 \in (-\infty, s_\infty(Y_1))$ and $s_2 \in (-\infty, s_\infty(Y_2))$ such that $s = s_1 + s_2$. For such s_1 and s_2, we have the connected sum formula

$$r_s(Y_1 \# Y_2) \geq \min\{r_{s_1}(Y_1) + s_2, r_{s_2}(Y_2) + s_1\}.$$

Since s_1 and s_2 are in $(-\infty, s_\infty(Y_1))$ and $(-\infty, s_\infty(Y_2))$ respectively, it follows that $r_s(Y_1 \# Y_2) = \infty$. This completes the proof. ■

Now we prove Theorem 6.2. Recall that d_∞ is a function on $\mathrm{Ker}\, h \times \mathrm{Ker}\, h$ defined by

$$d_\infty([Y_1], [Y_2]) := -s_\infty(Y_1 \# -Y_2) - s_\infty(-Y_1 \# Y_2).$$

By Corollary 1.3, the equalities $r_{-\infty}(Y) = r_{-\infty}(-Y) = 0$ hold if and only if $h(Y) = 0$. Moreover, we have $r_{-\infty}(\pm Y) = r_s(\pm Y)$ for sufficiently small $s \in (-\infty, 0]$. These imply that $d_\infty([Y_1], [Y_2])$ is finite for any pair $([Y_1], [Y_2]) \in \mathrm{Ker}\, h \times \mathrm{Ker}\, h$.

Proof of Theorem 6.2. For any $[Y_1], [Y_2] \in \mathrm{Ker}\, h$, the equalities $d_\infty([Y_1], [Y_1]) = 0$ and $d_\infty([Y_1], [Y_2]) = d_\infty([Y_2], [Y_1])$ obviously hold. Suppose that $[Y_1]$, $[Y_2]$ and $[Y_3]$ are three elements of $\mathrm{Ker}\, h$. Then, by Theorem 6.1, we have

$$\begin{aligned} d_\infty([Y_1], [Y_3]) &= -s_\infty(Y_1 \# -Y_3) - s_\infty(-Y_1 \# Y_3) \\ &= -s_\infty(Y_1 \# -Y_3 \# Y_2 \# -Y_2) - s_\infty(-Y_1 \# Y_3 \# Y_2 \# -Y_2) \\ &\leq -s_\infty(Y_1 \# -Y_2) - s_\infty(Y_2 \# -Y_3) - s_\infty(-Y_1 \# Y_2) - s_\infty(-Y_2 \# Y_3) \\ &= d_\infty([Y_1], [Y_2]) + d_\infty([Y_2], [Y_3]). \end{aligned}$$

Therefore d_∞ gives a pseudometric on $\mathrm{Ker}\, h$.

We next prove that d_∞ is well-defined on $\operatorname{Ker} h/\Theta^3_{\mathbb{Z},\infty}$. Let $[Y_1], [Y_2] \in \operatorname{Ker} h$ and $[M_1], [M_2] \in \Theta^3_{\mathbb{Z},\infty}$. Then $s_\infty(M_i) = s_\infty(-M_i) = 0$ $(i = 1, 2)$, and hence Theorem 6.1 implies that

$$\begin{aligned} &d_\infty([Y_1] + [M_1], [Y_2] + [M_2]) \\ &\qquad = -s_\infty(Y_1 \# M_1 \# -Y_2 \# -M_2) - s_\infty(-Y_1 \# -M_1 \# Y_2 \# M_2) \\ &\qquad \le -s_\infty(Y_1 \# -Y_2) - s_\infty(-Y_1 \# Y_2) = d_\infty([Y_1], [Y_2]) \end{aligned}$$

and

$$\begin{aligned} &d_\infty([Y_1], [Y_2]) = -s_\infty(Y_1 \# -Y_2) - s_\infty(-Y_1 \# Y_2) \\ &= -s_\infty(Y_1 \# -Y_2 \# M_1 \# -M_1 \# M_2 \# -M_2) - s_\infty(-Y_1 \# Y_2 \# M_1 \# -M_1 \# M_2 \# -M_2) \\ &\le -s_\infty(Y_1 \# M_1 \# -Y_2 \# -M_2) - s_\infty(-Y_1 \# -M_1 \# Y_2 \# M_2) \\ &= d_\infty([Y_1] + [M_1], [Y_2] + [M_2]). \end{aligned}$$

Therefore, d_∞ is well-defined on $\operatorname{Ker} h/\Theta^3_{\mathbb{Z},\infty}$.

Next, we prove that d_∞ is a metric on $\operatorname{Ker} h/\Theta^3_{\mathbb{Z},\infty}$. It is easy to check that it is a pseudometric. Suppose that $[Y_1], [Y_2] \in \operatorname{Ker} h/\Theta^3_{\mathbb{Z},\infty}$ satisfy $d_\infty([Y_1], [Y_2]) = 0$. Then $s_\infty(Y_1 \# -Y_2) = s_\infty(-Y_1 \# Y_2) = 0$, and these equalities and Lemma 6.4 imply that $r_0(Y_1 \# -Y_2) = r_0(-Y_1 \# Y_2) = \infty$. Therefore, by the definition of $\Theta^3_{\mathbb{Z},\infty}$ we see that $[Y_1] = [Y_2]$ as elements of $\operatorname{Ker} h/\Theta^3_{\mathbb{Z},\infty}$. This proves that d_∞ is a metric on $\operatorname{Ker} h/\Theta^3_{\mathbb{Z},\infty}$.

Finally, we prove that the group operation of $\operatorname{Ker} h/\Theta^3_{\mathbb{Z},\infty}$ is an isometry with respect to d_∞. Indeed,

$$\begin{aligned} d_\infty([Y_1] + [M], [Y_2] + [M]) &= -s_\infty(Y_1 \# M \# -Y_2 \# -M) - s_\infty(-Y_1 \# -M \# Y_2 \# M) \\ &= d_\infty([Y_1], [Y_2]) \end{aligned}$$

for any elements $[Y_1]$, $[Y_2]$ and $[M]$ of $\operatorname{Ker} h/\Theta^3_{\mathbb{Z},\infty}$. This completes the proof. ∎

As a concrete example, we give partial estimates of d_∞ for connected sums of some Seifert homology 3-spheres.

Proposition 6.5. *For any $n \in \mathbb{Z}_{>0}$, we have*

$$d_\infty([S^3], [\Sigma(2,3,6n-1) \# -\Sigma(2,3,6n+5)]) \ge \frac{1}{4(6n-1)(6n+5)}.$$

Proof. The connected sum formula for r_s gives

$$\begin{aligned} &r_s(-\Sigma(2,3,6n+5)) \\ &\quad \ge \min\{r_{s_1}(\Sigma(2,3,6n-1) \# -\Sigma(2,3,6n+5)) + s_2, r_{s_2}(-\Sigma(2,3,6n-1)) + s_1\} \end{aligned}$$

for any $s, s_1, s_2 \in (-\infty, 0]$ with $s = s_1 + s_2$. In particular, if $s_2 = 0$, then

$$\frac{1}{24(6n+5)} \ge \min\left\{r_{s_1}(\Sigma(2,3,6n-1) \# -\Sigma(2,3,6n+5)), \frac{1}{24(6n-1)} + s_1\right\}.$$

Hence if $\frac{1}{24(6n+5)} < \frac{1}{24(6n-1)} + s_1$, then

$$r_{s_1}(\Sigma(2,3,6n-1) \# -\Sigma(2,3,6n+5)) < \infty.$$

Consequently,

$$\begin{aligned} -s_\infty(\Sigma(2,3,6n-1) \# -\Sigma(2,3,6n+5)) &\geq \frac{1}{24(6n-1)} - \frac{1}{24(6n+5)} \\ &= \frac{1}{4(6n-1)(6n+5)}. \end{aligned}$$

Moreover, since $-\Sigma(2,3,6n-1) = S^3_{1/n}(3_1^*)$, we obtain a negative definite 4-manifold with boundary $-\Sigma(2,3,6n-1) \# \Sigma(2,3,6n+5)$ from the cobordism W_n in Section 5.3 with reversed orientation. Therefore, $r_0(-\Sigma(2,3,6n-1) \# \Sigma(2,3,6n+5)) = \infty$ and $s_\infty(-\Sigma(2,3,6n-1) \# \Sigma(2,3,6n+5)) = 0$. This completes the proof. ■

Here we pose the following question:

Question 6.6. Does the equality

$$d_\infty([S^3], [\Sigma(2,3,6n-1) \# -\Sigma(2,3,6n+5)]) = \frac{1}{4(6n-1)(6n+5)}$$

hold?

If the equality holds, then the sequence

$$\{a_n\}_{n=1}^{\infty} := \{[\Sigma(2,3,6n-1) \# -\Sigma(2,3,6n+5)]\}_{n=1}^{\infty}$$

converges to $[S^3]$ in $\operatorname{Ker} h/\Theta_{\mathbb{Z},\infty}$. In particular, we would conclude that the topology on $\operatorname{Ker} h/\Theta_{\mathbb{Z},\infty}$ induced by d_∞ is different from the discrete topology.

7. Computation for a hyperbolic 3-manifold

In this section, we give approximations of the critical values of the Chern–Simons functional on a certain hyperbolic 3-manifold. Moreover, using the computer, we obtain an approximate value of $r_s(Y)$ for this hyperbolic 3-manifold.

7.1. $1/n$-surgery along a knot K

We here review a formula for cs due to Kirk and Klassen [36], and explain our method of computing an approximate value of cs. For a compact manifold M, we define $\mathcal{R}(M) = \operatorname{Hom}(\pi_1(M), SL(2,\mathbb{C}))$ and call it the *$SL(2,\mathbb{C})$-representation space* of M. In this paper, we equip $\mathcal{R}(M)$ with the compact-open topology.

For a knot K in S^3, let $E(K)$ denote the exterior of an open tubular neighborhood of K, and let $\mu, \lambda \in \pi_1(T^2)$ be a meridian and a (preferred) longitude respectively.

Theorem 7.1 ([36, Theorem 4.2]). *Let ρ_0, ρ_1 be $SU(2)$-representations of $\pi_1(S^3_{1/n}(K))$ and $\gamma\colon [s_0, s_1] \to \{\rho \in \mathcal{R}(E(K)) \mid \rho|_{\pi_1(T^2)}$ is completely reducible$\}$ a piecewise smooth path with $\gamma(s_i) = \rho_i$ in $\mathcal{R}(E(K))$. Then*

$$cs(\rho_1) - cs(\rho_0) \equiv 2\int_{s_0}^{s_1} \beta(s)\alpha'(s)\,ds + n(\beta(s_1)^2 - \beta(s_0)^2) \bmod \mathbb{Z}, \tag{21}$$

where $\alpha, \beta\colon [s_0, s_1] \to \mathbb{C}$ are piecewise smooth functions such that the matrices $\gamma(s)(\mu)$, $\gamma(s)(\lambda)$ are simultaneously diagonalized as

$$\begin{bmatrix} e^{2\pi i\alpha(s)} & 0 \\ 0 & e^{-2\pi i\alpha(s)} \end{bmatrix}, \quad \begin{bmatrix} e^{2\pi i\beta(s)} & 0 \\ 0 & e^{-2\pi i\beta(s)} \end{bmatrix},$$

respectively.

Remark 7.2. Kirk and Klassen prowed Theorem 7.1 for a family of $SU(2)$-connections. As written in [36, p. 354], the formula can be extended to the case of $SL(2, \mathbb{C})$. We need to define the smoothness of a path

$$\gamma\colon [s_0, s_1] \to \{\rho \in \mathcal{R}(E(K)) \mid \rho|_{\pi_1(T^2)} \text{ is completely reducible}\}$$

since Stokes' theorem is used in the proof of Theorem 7.1. If we fix a generating system of $\pi_1(E(K))$, the space $\mathcal{R}(E(K))$ can be embedded into $SL(2, \mathbb{C})^N$, where N is the number of generators. If the composite of $\gamma\colon [s_0, s_1] \to \mathcal{R}(E(K))$ and $\mathcal{R}(E(K)) \to SL(2, \mathbb{C})^N$ is piecewise smooth, we call γ a *piecewise smooth path*. For such a path γ on $[s_0, s_1] = \bigcup_j I_j$, a piecewise smooth family of $SL(2, \mathbb{C})$-connections A_s on $E(K)$ is defined by considering the inverse map of the holonomy correspondence. Then we have a smooth connection on $E(K) \times I_j$ for each j, and one can check formula (21).

It is difficult to find a suitable path and compute the above integral in general. For a 2-bridge knot K, the subspace $\mathcal{R}^{\mathrm{irr}}(E(K))$ of the irreducible representations is explicitly described by the Riley polynomial as follows. We first recall that $\pi_1(E(K))$ admits a presentation of the form $\langle x, y \mid wx = yw\rangle$, where w is a certain word in x and y (see [36, p. 358]). For $t \in \mathbb{C} \setminus \{0\}$, $u \in \mathbb{C}$ and $\varepsilon \in \{\pm 1\}$, let $\rho_{t,u,\varepsilon}$ denote the representation of the free group $\langle x, y \mid -\rangle$ of rank 2 given by

$$\rho_{t,u,\varepsilon}(x) = \varepsilon \begin{bmatrix} \sqrt{t} & 1/\sqrt{t} \\ 0 & 1/\sqrt{t} \end{bmatrix}, \quad \rho_{t,u,\varepsilon}(y) = \varepsilon \begin{bmatrix} \sqrt{t} & 0 \\ -\sqrt{t}\,u & 1/\sqrt{t} \end{bmatrix},$$

where $\sqrt{re^{i\theta}} = \sqrt{r}\,e^{i\theta/2}$ for $r \geq 0$ and $-\pi < \theta \leq \pi$. Here, the *Riley polynomial* of K (for the above presentation) is defined by $\phi(t, u) = w_{11} + (1 - t)w_{12} \in \mathbb{Z}[t^{\pm 1/2}, u]$, where w_{ij} is the (i, j)-entry of $\rho_{t,u,\varepsilon}(w)$. Then $\rho_{t,u,\varepsilon}$ gives a representation of $\pi_1(E(K))$ if and only if $\phi(t, u) = 0$. Moreover, any irreducible representation of $\pi_1(E(K))$ is conjugate to $\rho_{t,u,\varepsilon}$ for some t, u and ε.

Here, $\rho_{t,u,\varepsilon}$ is conjugate to an $SU(2)$-representation if and only if $|t| = 1$, $t \neq 1$ and $u \in (t + t^{-1} - 2, 0)$. Note that an $SU(2)$-representation $\rho_{t,u,\varepsilon}$ is $SU(2)$-conjugate to $\rho_{t^{-1},u,\varepsilon}$.

Let us find a path from $\rho_{t_0,u_0,\varepsilon_0}$ to $\rho_{t_1,u_1,\varepsilon_1}$ in

$$\{\rho \in \mathcal{R}^{\mathrm{irr}}(E(K)) \mid \rho|_{\pi_1(T^2)} \text{ is completely reducible}\}.$$

First note that one need not care about ε_i since the right-hand side of (21) is independent of the choice of ε_i. Consider the d-fold branched cover

$$\mathrm{pr}_1\colon \{(t,u) \in (\mathbb{C} \setminus \{0,1\}) \times \mathbb{C} \mid \phi(t,u) = 0\} \to \mathbb{C} \setminus \{0,1\},$$

where $d = \deg_u \phi$. In order to find a path, we first take a path γ from t_0 to t_1 and its lift $\tilde{\gamma}$ satisfying $\mathrm{pr}_2 \circ \tilde{\gamma}(s_j) = u_j$. Since the lift starting from (t_0, u_0) might end at (t_1, u_1') with $u_1' \neq u_1$, one should choose a path γ carefully. We now have $\alpha(s) = \frac{1}{4\pi i} \log \gamma(s)$ with an analytic continuation along γ.

Once the function $u(s)$ satisfying $\tilde{\gamma}(s) = (\gamma(s), u(s))$ is given explicitly, one gets

$$\beta(s) = \frac{1}{2\pi i} \log\,(P^{-1}\rho_{\gamma(s),u(s),\varepsilon}(\lambda)P)_{11},$$

where $P = P(s)$ is a matrix satisfying $(P^{-1}\rho_{\gamma(s),u(s),\varepsilon}(\mu)P)_{11} = e^{2\pi i\alpha(s)}$. We finally integrate $\beta(s)\alpha'(s)$ on $[s_0, s_1]$.

In fact, one can express $u(s)$ explicitly by solving $\phi(t,u) = 0$ when $\deg_u \phi \leq 4$. Here, we should be careful to connect the solutions. For instance, let $\phi(t,u) = t - u^2$. Then we have $u_0(t) = \sqrt{t}$, $u_1(t) = -\sqrt{t}$. In order to find a path from $(i, e^{\pi i/4})$ to $(-i, e^{3\pi i/4})$, we define $\gamma\colon [1/2, 3/2] \to \mathbb{C}$ by $\gamma(s) = e^{s\pi i}$. The lift of γ is obtained by combining u_0 and u_1:

$$\tilde{\gamma}(s) = \begin{cases} (\gamma(s), u_0(\gamma(s))) & \text{if } 1/2 \leq s \leq 1, \\ (\gamma(s), u_1(\gamma(s))) & \text{if } 1 < s \leq 3/2. \end{cases}$$

Remark 7.3. It is difficult to solve $\phi(t,u) = 0$ and $\rho_{t,u,\varepsilon}(\mu\lambda^n) = I_2$ simultaneously. We actually use the A-polynomial $A_K(L,M) \in \mathbb{Z}[L,M]$ of K. Indeed, first solve the one variable equation $A_K(L, L^{-n}) = 0$, and then $t = M^2 = L^{-2n}$. We next solve $\phi(L^{-2n}, u) = 0$ with respect to u.

7.2. *1/2-surgery along the knot* 5_2^*

We actually consider the manifold $S^3_{-1/2}(5_2) = -S^3_{1/2}(5_2^*)$ and multiply the result of computation of $cs(\rho)$ by -1. Recall that $cs_Y(\rho) = -cs_{-Y}(\rho)$.

We first fix the presentation of the group $\pi_1(E(5_2))$ as $\langle x, y \mid [y, x^{-1}]^2 x = y[y, x^{-1}]^2\rangle$, where a meridian and a longitude are expressed as x and $[x, y^{-1}]^2[y, x^{-1}]^2$, respectively. Then the Riley polynomial and A-polynomial of 5_2 are given by

$$\begin{aligned} \phi(t,u) &= -(t^{-2} + t^2)u + (t^{-1} + t)(2 + 3u + 2u^2) - (3 + 6u + 3u^2 + u^3), \\ A_{5_2}(L,M) &= -L^3 - M^{14} + L^2(1 - 2M^2 - 2M^4 + M^8 - M^{10}) \\ &\quad + LM^4(-1 + M^2 - 2M^6 - 2M^8 + M^{10}). \end{aligned}$$

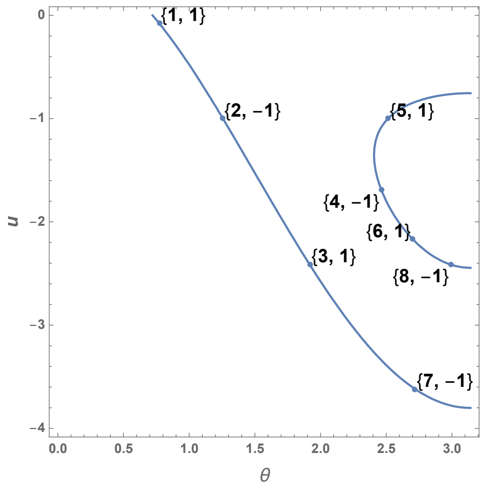

Fig. 10. The eight non-trivial representations of $\pi_1(S^3_{-1/2}(5_2))$ in the quotient space obtained from the non-abelian representations of $\pi_1(E(5_2))$ by identifying $\rho_{t,u,\varepsilon}$ and $\rho_{t,u,-\varepsilon}$, where $t = e^{i\theta}$. The second entry of a label indicates ε.

Here, one sees that there are eight conjugacy classes of non-trivial $SU(2)$-representations of $\pi_1(S^3_{-1/2}(5_2))$ as drawn in Figure 10. Strictly speaking, these are candidates to being representations, coming from a numerical computation. Assume that some of them do not give representations. Then there exists a non-trivial representation ρ close to one of the candidates such that $H^1(S^3_{-1/2}(5_2); \mathfrak{su}(2)_{\mathrm{Ad}\circ\rho}) \neq 0$ since the Casson invariant of $S^3_{-1/2}(5_2)$ is equal to -4 and $|-4| \times 2 = 8$. Here, since ρ is non-abelian, we have $\mathrm{rank}_{\mathbb{R}}\, \partial_1 = 3$ in the chain complex

$$C_*(E(5_2); \mathfrak{su}(2)_{\mathrm{Ad}\circ\rho}) = \begin{cases} \mathbb{R}^3 & \text{if } * = 0, 2, \\ \mathbb{R}^6 & \text{if } * = 1, \\ 0 & \text{otherwise,} \end{cases}$$

obtained from the above presentation of $\pi_1(E(5_2))$. It follows from the Mayer–Vietoris exact sequence and Poincaré–Lefschetz duality that $\mathrm{rank}_{\mathbb{R}}\, \partial_2 \leq 1$. On the other hand, we can see by computer that this inequality does not hold for the eight candidates. Therefore, they correspond to true representations.

The following computation is based on Mathematica. Since $\deg_u \phi = 3$, one gets the explicit solutions $u_1(t), u_2(t), u_3(t)$ of $\phi(t,u) = 0$. We take eight paths as illustrated in Figure 11 and apply Theorem 7.1 to these paths. Note that some paths start from a root of the Alexander polynomial $\Delta_{5_2}(t)$ of 5_2, and for these paths we use [20, Lemma 5.3] to compute integrals. The result of the computation is listed in Table 2.

Recall that $r_s(S^3_1(5_2^*)) = 1/4 \cdot 2 \cdot 3 \cdot 11 \approx 0.00379$. Since 0.00176489 is the only value less than 0.00379 among the eight values, we conclude that $r_s(S^3_{1/2}(5_2^*)) \approx 0.00176489$.

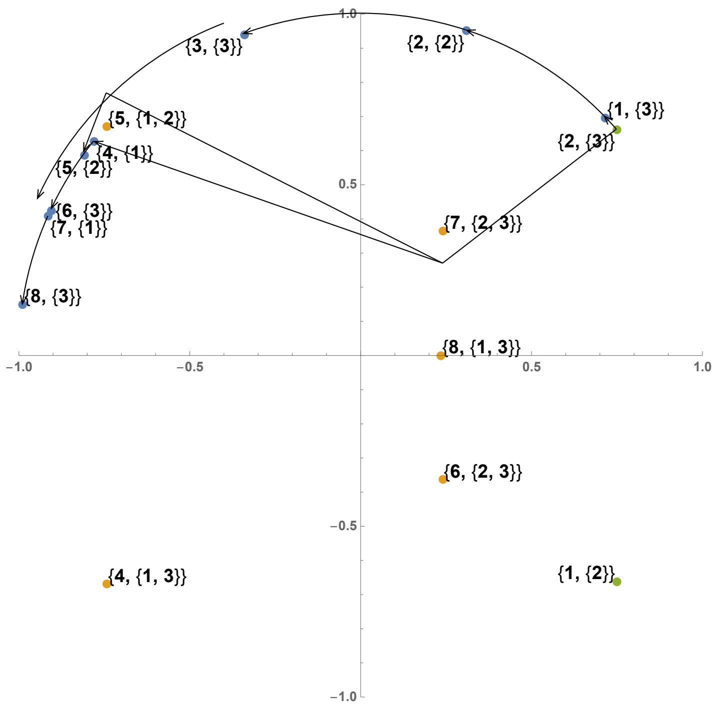

Fig. 11. The blue (resp. green, orange) dots correspond to representations of $\pi_1(S^3_{-1/2}(5_2))$ (resp. the roots of $\Delta_{5_2}(t)$, some branched points). Here the label $\{i, \{j\}\}$ (resp. $\{i, \{j_1, j_2\}\}$) at $t \in \mathbb{C}$ means $\phi(t, u_j(t)) = 0$ (resp. $\phi(t, u_{j_k}(t)) = 0$ for $k = 1, 2$).

	t	u	ε	$-cs$
ρ_1	$0.716932 + 0.697143i$	-0.0755806	1	0.00176489
ρ_2	$0.309017 + 0.951057i$	-1.00000	-1	0.166667
ρ_3	$-0.339570 + 0.940581i$	-2.41421	1	0.604167
ρ_4	$-0.778407 + 0.627759i$	-1.69110	-1	0.388460
ρ_5	$-0.809017 + 0.587785i$	-1.00000	1	0.166667
ρ_6	$-0.905371 + 0.424621i$	-2.16991	1	0.865934
ρ_7	$-0.912712 + 0.408603i$	-3.62043	-1	0.321158
ρ_8	$-0.988857 + 0.148870i$	-2.41421	-1	0.604167

Tab. 2. The values of $-cs$ for the representations of $\pi_1(S^3_{-1/2}(5_2))$. Note that $0.16666\ldots67 \approx 1/6$ and $0.60416\ldots67 \approx 29/48$, where both decimals have 46 digits of 6's in the omitted part.

Moreover, we improve the precision, and get

$$r_s(S^3_{1/2}(5_2^*)) \approx 0.0017648904\ 7864885113\ 0739625897\ 0947779330\ 4925308209$$

for all $s \in [-\infty, 0]$.

Appendix A. Hendricks, Hom, Stoffregen, and Zemke's example

In [29], the authors intensively studied the homology 3-sphere obtained from S^3 by Dehn surgery along the framed knot at the top left in Figure 12. This appendix is devoted to showing that their homology 3-sphere is a graph manifold.

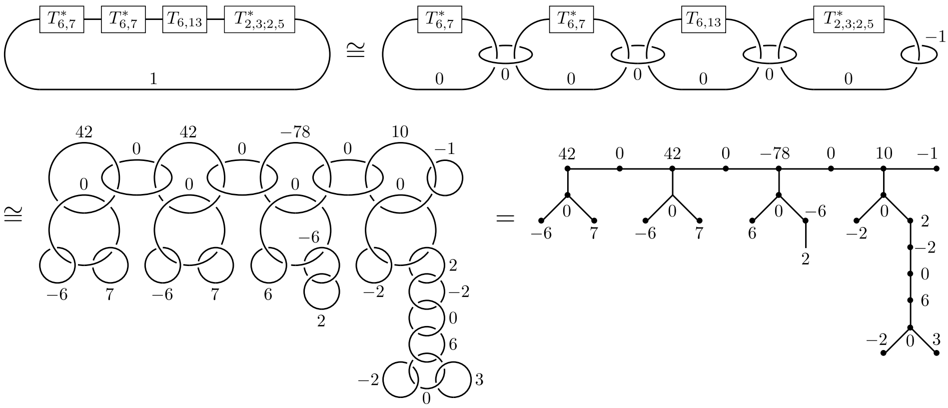

Fig. 12. A diffeomorphism between $S^3_1(2T^*_{6,7} \# T_{6,13} \# T^*_{2,3;2,5})$ and a graph manifold.

The first diffeomorphism in Figure 12 follows from standard Kirby calculus. In order to prove the second diffeomorphism, we consider three 3-manifolds obtained by Dehn surgery along $T^*_{6,7}$, $T_{6,13}$, and the mirror of the $(2, 5)$-cable of $T_{2,3}$, respectively. Here we put framed knots in these 3-manifolds as drawn in thick lines in Figures 13 and 14. Then, regarding Figures 13 and 14 as diffeomorphisms of the exteriors of the framed knots, respectively, one obtains the 3-manifold in Figure 12.

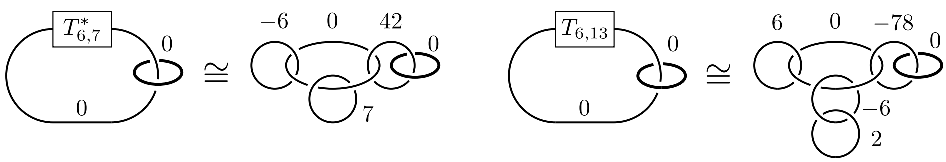

Fig. 13. Diffeomorphisms between 3-manifolds with framed knots. Note that the thick components are not used for surgery.

In Figure 13, the diffeomorphisms between 3-manifolds with framed knots are shown by Kirby calculus including a Rolfsen twist (or the slam-dunk).

In Figure 14, the first diffeomorphism follows from the definition of the $(2, 5)$-cable of $T_{2,3}$. The fourth diffeomorphism is obtained by sliding the 0-framed unknot at the bottom to the one at the top. The rest of the diffeomorphisms are shown by standard Kirby calculus.

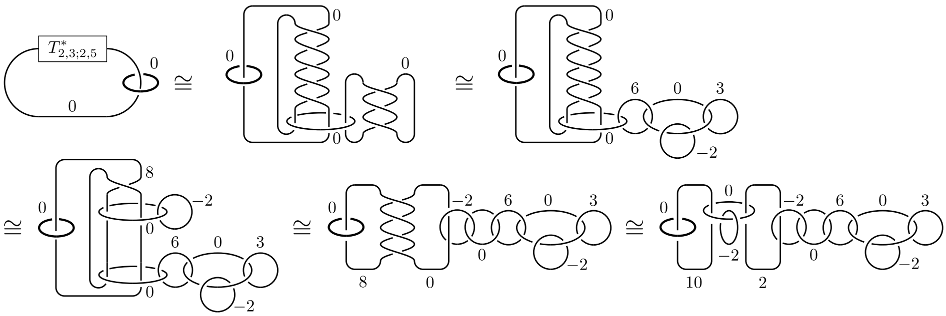

Fig. 14. Diffeomorphisms of 3-manifolds with framed knots.

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