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A variational approach to hyperbolic evolutions and fluid-structure interactions

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Abstract. We show the existence of a weak solution for a system of partial differential equations describing the motion of a flexible solid inside a fluid: A nonlinear, viscoelastic, n-dimensional bulk solid governed by a PDE including inertia is interacting with an incompressible fluid governed by the (n-dimensional) Navier–Stokes equation for $n \geq 2$. The result is the first allowing for large bulk deformations in the regime of long time existence for fluid-structure interactions. The existence is achieved by introducing a novel variational scheme involving two time-scales that allows us to extend the method of minimizing movements to hyperbolic problems involving nonconvex and degenerate energies.

Keywords. Fluid-structure interaction, Navier–Stokes equation, elastodynamics, calculus of variations, minimizing movements

Contents

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1. Introduction

Fluid-structure interaction (FSI) happens whenever the mechanical movements of a solid body and of an adjacent fluid influence each other. Examples of such interaction are numerous, including the blood flow in vessels, the usage of vocal chords during speech or the flying of airplanes. Mathematically, fluid-structure interaction is given by a system of coupled PDEs that describe the motion of a fluid and a solid body exchanging momentum over a common interface; see for example (1.1)–(1.11) below.

Due to its relevance for various applications, fluid-structure interaction has attracted much attention in various fields. For an overview on mathematical progress, see the recent monograph [57]. In the present paper we show the existence of a weak solution to a problem describing a *viscoelastic bulk solid* moving inside and interacting with an incompressible fluid governed by the Navier–Stokes equations.

The majority of the available mathematical literature on fluid-structure interaction concentrates on rigid solid bodies, for which also weak existence theory was developed; see [15, 20, 21, 34, 37, 38, 41, 42, 44–47, 76, 84, 85] and the references therein. Other existence results for weak solutions are for a prescribed geometry, as e.g. in [77]. A very well studied regime is lower-dimensional solids, namely the important case of plates or shells bounding a fluid domain [22–25, 50, 65, 66, 71–74]. Analytical results concerning existence of weak solutions for the interaction of a fluid with an elastic body of the same dimension are few and generally include some condition that restricts the problem to the regime of small deformations; see e.g. [5, 9, 35, 43, 49, 56]. On the other hand, a number of results on strong solutions for elastic solids interacting with fluids are available [4–6, 9–13, 24, 28, 29, 55, 56, 61, 62]. They include existence, uniqueness, (higher) regularity and stability results and rely on linearizations and short times and/or small data analysis.

In this work we systematically develop a *variational methodology for fluid-structure interaction problems* that allows us to prove, for the first time, existence of weak solutions in a general setting, i.e. allowing for both the fluid and the structure to have full dimension and permitting large, unrestricted forces and deformations – a setting which in particular precludes approaches relying on linearization or convexity (see the next subsection for detailed explanations).

Let us point out some important difficulties that appear in fluid-structure interactions. Their mathematical description is given through a *system of mutually coupled PDEs* that is required to satisfy additional *geometric restrictions*. To be more precise, the *common interface between the fluid and the solid is part of the solution* and hence has to be constructed simultaneously. This challenge appears whenever motions of the solid are allowed, and many methods have been developed to overcome such deep nonlinear couplings (see e.g. the pioneering works [28,29,60,61]). On top of that, as in the setting here the solid is allowed to undergo large structural deformations, it is necessary to ensure that the solutions we consider consist of deformations that are known to be injective a priori [36, 52]. This is essential not only in order for the solutions to stay physically meaningful under large deformations, but also for the coupling to the fluid variables to be well-posed. In order to reflect these geometric restrictions, the operator appearing in the equations describing the movement of the solid is necessarily *nonmonotone* and possesses a *nonconvex* domain of definition.

In this work, in order to handle these (highly nonlinear) limitations we develop a methodology that focuses on the underlying *energetic* structure of the PDEs and a *variational* point of view. We believe this methodology has great potential to prove existence and approximation results for large classes of dynamical problems in continuum mechanics. The theory is built upon the following three ideas, each introduced in a separate section:

Sec. 2, **Minimizing movements for fluid-structure interaction**: We introduce an adaptation of De Giorgi's *minimizing movements* [1, 31] scheme. This is used to show the existence of weak solutions to the fluid-structure interaction problem in the simplified *parabolic* case where inertial effects are omitted. Consequently, the fluid motion is quasi-steady satisfying the Stokes equation and the solid evolves along a gradient flow. In more detail, we construct an approximate, time-discretized solution by iteratively solving a *coupled minimization problem*. Each minimization produces the subsequent deformation and thus a new interface to be used in the next time step to separate the fluid and the solid. The new variational approach we introduce is essential, as it can cope both with the nonmonotonicity of the operators involved and the nonconvex nature of the underlying state space. Moreover, used properly, it will also automatically induce the correct coupling conditions between the fluid and solid velocities and the forces at their common interface.

While the minimizing movements method and its variations are commonly used for quasi-static evolution problems in the field of viscoelastic solids (see e.g. [59]), to the best of our knowledge it has never been applied to fluid-structure interaction before.

Sec. 3, **Minimizing movements for hyperbolic evolutions**: The original minimizing movements scheme is restricted to gradient flows. We propose an extension of this scheme, capable of approximating solutions to hyperbolic equations.

The method is built on an approach involving *two time-scales*: the larger *acceleration time-scale* and the smaller *velocity time-scale*. Consequently, the second time derivative is approximated by a double difference quotient with respect to these different scales.

This provides the setting of a minimizing movements iteration for all fixed positive *acceleration scales*. In doing so, we "combine the best of the two worlds", by first using the variational approach to handle all the nonlinearities during the construction of approximate solutions, and then considering the Euler–Lagrange equations to gain respective (hyperbolic) a priori estimates that are uniform with respect to both scales.

The *hyperbolic minimizing movements* scheme introduced here allows one to approximate hyperbolic PDEs. However, here we perform this scheme for nonlinear PDEs motivated from continuum mechanics of solids. For these it is to date unavoidable to include damping terms to recover weak solutions in the limit; consequently, those PDEs are not hyperbolic in a strict sense. See Remark 1.5 for more explanations.

Sec. 4, **Bulk elastic solids coupled to Navier–Stokes equations**: In this section the ideas from the previous two sections are combined to establish the main result of the paper, which is the existence of weak solutions to a problem involving a solid (with inertia) interacting with the *unsteady incompressible Navier–Stokes equations*.

In order to adapt the method to the *Eulerian* description of the fluid by its *velocity* and *pressure* in the time changing domain, we use *Lagrangian approximation of the material time derivative* on the acceleration scale. This is done via the method of characteristics by constructing a *flow map* for short times. Since the flow velocity and the flow map are inextricably linked and both are additionally connected to the changing fluid domain, the main difficulty here is that the three have to be constructed simultaneously. In particular, we introduce a discretized flow map in the discretized variable domain for which we prove various natural regularity and approximability results. This allows for a coupling between the hyperbolic minimizing movements scheme for the (viscous) solid deformation and the respective semi-Lagrangian approximation of the unsteady Navier–Stokes equation.

Some of the ideas in this section are inspired by the works [48, 80] on the Navier–Stokes equations. In particular, in [48] the authors apply minimizing movements to construct solutions to the Navier–Stokes equations. Their approach, however, strongly depends on having a fixed fluid domain.

In sum, we introduce a new viewpoint on fluid-structure interaction and hyperbolic problems in general. In particular, (for the first time) we prove existence for a bulk, large deformations interaction problem between a viscoelastic solid and the incompressible Navier–Stokes equations until the point of self-contact of the fluid boundaries (Theorem 1.2).

1.1. Setup

We consider the following setup for the fluid-structure interaction problem: The fluid together with the elastic structure are both confined to a container $\Omega \subset \mathbb{R}^n$ that is fixed in time. The deformation of the solid is at any instant of time t described by the deformation function $\eta(t) := \eta(t, \cdot) : Q \to \Omega$. Here Q is a given reference configuration of the solid. We assume that both $Q, \Omega \subset \mathbb{R}^n$ are Lipschitz domains. Here, $n \geq 2$ is the dimension of the problem with $n = 2$ corresponding to the planar case and $n = 3$ to the bulk case. The

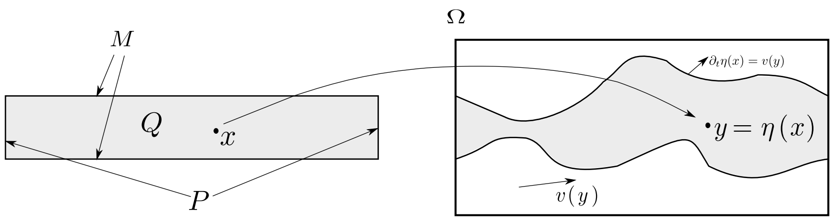

Fig. 1. A scheme of the geometry of the fluid-structure interaction. Left: the reference configuration. Right: the actual configuration at a given time instant t.

fluid variables are defined in the time-dependent domain $\Omega(t) := \Omega \setminus \eta(t, Q)$. The flow of the fluid is determined by its velocity $v(t) : \Omega(t) \to \mathbb{R}^n$ and its pressure $p(t) : \Omega(t) \to \mathbb{R}$. Thus, the solid is described in *Lagrangian* coordinates and the fluid in *Eulerian* coordinates. Observe that a similar configuration for linear elasticity has already been studied in [35]. We refer to Figure 1 for better orientation.

To set up the evolution equation, we will need the basic physical balances to be fulfilled. As we are not modeling any thermal effects, this reduces to the balance of momentum for both the fluid and the structure together with suitable conditions on their mutual boundary, as well as conservation of mass. In the interior these balance equations read, in strong formulation,

$$\rho_s \partial_t^2 \eta + \operatorname{div} \sigma = \rho_s f \circ \eta \qquad \text{in } Q, \tag{1.1}$$

$$\rho_f (\partial_t v + [\nabla v] v) = \nu \Delta v - \nabla p + \rho_f f \qquad \text{on } \Omega(t), \tag{1.2}$$

$$\operatorname{div} v = 0 \qquad \text{on } \Omega(t). \tag{1.3}$$

Here, σ is the *first Piola–Kirchhoff stress tensor* of the solid, ν is the *viscosity constant* of the fluid, ρ_s and ρ_f are the *densities* of the solid and fluid respectively, and f is the actual applied force in the current (Eulerian) configuration. Thus, the fluid is assumed to be Newtonian with the *Navier–Stokes equation* modeling its behavior.

The solid is going to be modeled in the large deformation regime. This means that the strains as well as the rotations within the material may be large so that linearized theories are not applicable. For our purposes, we will restrict our attention further to *hyper-viscoelasticity*. Classically, hyperelasticity (see e.g. [36]) refers to a subclass of models for which the Piola–Kirchhoff stress tensor can be derived from an underlying energy function $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$, i.e. $\sigma_{ij}(x) = \frac{\partial W(F)}{\partial F_{ij}}\big|_{F=\nabla\eta(x)}$. Then the large deformation setting is reflected in the following assumptions:

$$W(RF) = W(F) \quad \text{for all } F \in \mathbb{R}^{n\times n} \text{ and all } R \in SO(n), \tag{1.4}$$

$$W(F) = +\infty \quad \text{if } \det F \le 0, \text{ and } W(F) \to +\infty \text{ if } \det F \to 0_+, \tag{1.5}$$

where the first equation corresponds to material frame-indifference and has to be imposed due to the possibility of large rotations, while the second corresponds to the infinite resistance of the material to infinite compression; see e.g. [7].

It has been realized early on [27] that already the frame-indifference assumption excludes the possibility of the underlying energy being convex in the first gradient, which in turn excludes methods based on monotonicity and calls for a variational approach where more general convexity settings can be utilized (see e.g. [7, 30]). The second condition then makes the domain of the energy a nonconvex set, which is a challenging issue already for static problems (see [8]). In particular, it is open whether under condition (1.5) a minimizer of the energy satisfies the corresponding Euler–Lagrange equation (see [8] for a discussion) and, to the authors' knowledge, this question could only be answered in the affirmative [52] if the energy additionally depends on the second gradient of the deformation, which is the setting of so-called *nonsimple materials*; see Section 1.4.

Yet, in the setting of FSI, the often used, even if potentially unphysical, simplification to drop (1.5) is not admissible as exactly this condition enforces *local invertibility* (and until contact even global invertibility) of the deformation, which is needed for well-posedness of the FSI problem. In linearized settings, for example, this invertibility follows from restricting the strain to be very small. However, in our situation where no such control is available, use of an energy functional satisfying (1.5) is essential.

Thus, following the current state of the art, we will consider a regularized version of classical hyperelasticity. First, as in [52] we will let the energy depend also on the second gradient of the deformation (see Section 1.4). Moreover, we need to consider a material that is *additionally viscous*, i.e. the stress tensor has a viscous part. For our approach, it is advantageous to assume that, analogously to the elastic part, the viscous part can also be derived from an underlying dissipation function (potential). Materials satisfying this assumption are often called *generalized standard materials* [51, 59, 79]; see Section 1.4.

In sum, we will restrict ourselves to materials for which the first Piola–Kirchhoff stress tensor σ can be derived from underlying *energy and dissipation potentials*, i.e.

$$\operatorname{div}\sigma := DE(\eta) + D_2 R(\eta, \partial_t \eta)$$

with E being the energy functional describing the elastic properties, while R is the dissipation functional used to model the viscosity of the solid. Here D denotes the Fréchet derivative and D_2 the Fréchet derivative with respect to the second argument. For the analysis performed in this paper, quite general forms of E and R can be admitted (see Section 1.4 for a list of assumptions). The prototypical examples of the potentials are the following (we will prove that they satisfy the imposed assumptions):

$$R(\eta, \partial_t \eta) := \int_Q |(\nabla \partial_t \eta)^T \nabla \eta + (\nabla \eta)^T (\nabla \partial_t \eta)|^2 \, dx = \int_Q |\partial_t (\nabla \eta^T \nabla \eta)|^2 \, dx, \tag{1.6}$$

$$E(\eta) := \begin{cases} \int_Q \left[\frac{1}{8} |\nabla \eta^T \nabla \eta - I|_{\mathcal{C}} + \frac{1}{(\det \nabla \eta)^a} + \frac{1}{q} |\nabla^2 \eta|^q \right] dx & \text{if } \det \nabla \eta > 0 \text{ a.e. in } Q, \\ +\infty & \text{otherwise}, \end{cases} \tag{1.7}$$

where we use the notation $|\nabla \eta^T \nabla \eta - I|_{\mathcal{C}} := (\mathcal{C}(\nabla \eta^T \nabla \eta - I)) \cdot (\nabla \eta^T \nabla \eta - I)$, with $\mathcal{C}$ being a positive definite tensor of elastic constants, $q > n$ and $a > \frac{qn}{q-n}$.

Notice that in (1.7) the first term corresponds to the Saint Venant–Kirchhoff energy, the second models the resistance of the solids to infinite compression and the last is a regularization term.

Additionally, we impose coupling conditions between η and v on their common interface; namely, we will assume the *continuity of deformation* (i.e. no-slip conditions adapted to the moving domain) as well as *traction on the boundary between the fluid and the solid*. We denote by M the portion of the boundary of Q that is mapped to the contact interface between the fluid and the solid. While Q is only assumed to be a Lipschitz domain, we assume that the pieces of its boundary that belong to M are additionally C^2. The boundary conditions read

$$v(t,\eta(x)) = \partial_t \eta(t,x) \qquad \text{in } [0,T]\times M, \tag{1.8}$$

$$\sigma(t,x)n(x) = \big(\nu\varepsilon v(t,\eta(t,x)) + p(t,\eta(t,x))I\big)\hat{n}(t,\eta(t,x)) \qquad \text{in } [0,T]\times M, \tag{1.9}$$

where $n(x)$ is the unit normal to M, while $\hat{n}(t,\eta(t,x)) := \operatorname{cof}(\nabla\eta(t,x))n(x)$ is the normal transformed to the actual configuration and $\varepsilon v := \nabla v + (\nabla v)^T$ is the symmetrized gradient. Additionally, there are second order Neumann-type zero boundary conditions for the deformation η arising from the second order gradient in its energy.[1]

Finally, we will prescribe Dirichlet boundary conditions on $P := \partial Q \setminus M$:

$$\eta(x,t) = \gamma(x) \quad \text{in } [0,T]\times P \tag{1.10}$$

for some fixed boundary displacement $\gamma : P \to \Omega$. Together with the injectivity of deformations, we will encode this condition in the set $\mathcal{E}$ of admissible deformations (see Remark 1.8 for the precise definition).

We close the system by prescribing initial conditions for $v, \eta, \partial_t\eta$:

$$\begin{aligned} &\eta(0,x) = \eta_0(x) \text{ for } x \in Q, && \partial_t\eta(0,x) = \eta_*(x) \text{ for } x \in Q,\\ &\Omega(0) = \Omega \setminus \eta_0(Q), && v(0,y) = v_0(y) \text{ for all } y \in \Omega(0). \end{aligned} \tag{1.11}$$

1.2. Main result

The final objective of this paper is to prove existence of weak solutions to the system (1.1)–(1.3) subject to the coupling conditions (1.8)–(1.9) and the remaining boundary and initial conditions detailed in the previous subsection.

As is customary in fluid-structure interaction problems (see e.g. [49, 65]), the weak formulation is designed in such a way that the coupling conditions are realized by choosing *well-fitted test functions*. Indeed, we have the following definition:

[1] Specifically, these naturally occur while minimizing the elastic energy and not prescribing boundary values for $\nabla\eta$. This can also be seen as a kind of integrability condition for σ. That is, for σ to be defined as a measure, we need $\langle DE(\eta), \phi_\delta\rangle \to 0$ (for $\delta \to 0$) , where ϕ_δ is a regularized version of $\xi\delta(1-\operatorname{dist}(\cdot,\partial Q)/\delta)^+$ with $\xi \in C_0^\infty(M)$ extended constantly along the normal direction. For our example energy (1.7), this simply reduces to $\partial^2\eta/\partial n^2 = 0$ on M.

Definition 1.1. Let $f \in C^0([0,\infty) \times \Omega)$, $v_0 \in L^2(\Omega)$, $\eta_* \in L^2(Q)$ and $\eta_0 \in \mathcal{E}$ with $\mathcal{E}$ defined in (1.22), such that $v_0 \circ \eta_0 = \eta_*$. We call[2] $\eta : [0,T] \times Q \to \Omega$, $v : [0,T] \times \Omega(t) \to \mathbb{R}^n$ and $p : [0,T] \times \Omega(t) \to \mathbb{R}$, where $\Omega(t) := \Omega \setminus \eta(t,Q)$, a *weak solution to the fluid-structure interaction problem* (1.1)–(1.2) *and* (1.8)–(1.11) if the following holds:

The deformation satisfies $\eta \in L^2([0,T]; W^{2,q}(Q;\mathbb{R}^n)) \cap W^{1,2}([0,T]; W^{1,2}(Q;\mathbb{R}^n))$ with $\eta(0) = \eta_0$ and $\partial_t \eta \in C_w([0,T]; L^2(Q;\mathbb{R}^n))$, while the velocity satisfies $v \in L^2([0,T]; W^{1,2}_{\mathrm{div}}(\Omega(\cdot);\mathbb{R}^n))$ and the pressure satisfies[3] $p \in \mathcal{D}'([0,T] \times \Omega)$ with $\operatorname{supp}(p) \subset [0,T] \times \overline{\Omega(t)}$. Further, for all

$$(\phi,\xi) \in L^2([0,T]; W^{2,q}(Q;\mathbb{R}^n)) \cap W^{1,2}([0,T]; W^{1,2}(Q;\mathbb{R}^n)) \times C^\infty([0,T]; C_0^\infty(\Omega;\mathbb{R}^n))$$

satisfying $\xi(T) = 0$, $\phi(t) = \xi(t) \circ \eta(t)$ on Q, $\phi(t) = 0$ on P for all $t \in [0,T]$, we require

$$\begin{aligned}
\int_0^T [-\rho_s \langle \partial_t \eta, \partial_t \phi \rangle_Q - \rho_s \langle v, \partial_t \xi - v \cdot \nabla \xi \rangle_{\Omega(t)} \\
+ \langle DE(\eta), \phi \rangle + \langle D_2 R(\eta, \partial_t \eta), \phi \rangle + \nu \langle \varepsilon v, \varepsilon \xi \rangle_{\Omega(t)}]\, dt \\
= \int_0^T [\langle p, \operatorname{div} \xi \rangle_{\Omega(t)} + \rho_s \langle f \circ \eta, \phi \rangle_Q + \rho_f \langle f, \xi \rangle_{\Omega(t)}]\, dt \\
- \rho_s \langle \eta_*, \phi(0) \rangle_Q - \rho_f \langle v_0, \xi(0) \rangle_{\Omega(0)}
\end{aligned}$$

and $\partial_t \eta(t) = v(t) \circ \eta(t)$ on M, $\eta(t) \in \mathcal{E}$ and $v(t)|_{\partial\Omega} = 0$ for almost all $t \in [0,T]$.

We can then formulate our main theorem as follows:

Theorem 1.2 (Existence of weak solutions). *Assume that E satisfies Assumption 1.7 and R satisfies Assumption 1.10 given in Section 1.4. Then for any $\eta_0 \in \operatorname{int}(\mathcal{E}) = \mathcal{E} \setminus \partial\mathcal{E}$ (see Remark 1.8) with $E(\eta_0) < \infty$, any $\eta_* \in L^2(Q;\mathbb{R}^n)$, $v_0 \in L^2(\Omega(0);\mathbb{R}^n)$ and any right hand side $f \in C^0([0,\infty) \times \Omega;\mathbb{R}^n)$ there exists a $T > 0$ such that there exists a weak solution to (1.1)–(1.11) on $[0,T)$ according to Definition 1.1. Here either $T = \infty$ or T is the time of the first contact of the boundary of the solid body with either itself or $\partial\Omega$ (i.e. $\eta(T) \in \partial\mathcal{E}$).*

Moreover, the solution satisfies the energy inequality (1.12); for additional regularity of the pressure see (4.12).

Let us remark that the assumptions on the energy and dissipation functionals are in particular satisfied by the model case energies (1.6)–(1.7) (see Section 2.3).

[2] We use standard notation for Bochner spaces over Lebesgue spaces and Sobolev spaces with time-changing domains. The subscript div indicates the respective solenoidal subspace: $W^{1,2}_{\mathrm{div}}(\Omega(t);\mathbb{R}^n) = \{v \in W^{1,2}(\Omega(t);\mathbb{R}^n) \mid \operatorname{div} v = 0\}$.

[3] From the given weak formulation one can deduce some more regularity of the pressure. However, as is known from the theory for fixed domains, regularity of the pressure in time can be obtained merely in a negative Sobolev space. See the estimates in Section 4.2, Step 3b, which show that the pressure is in the respective natural class.

The coupled system satisfies a natural energy inequality:

$$\begin{aligned}
&E(\eta(t)) + \rho_s \int_Q \tfrac{1}{2}|\partial_t \eta(t)|^2\,dx + \rho_f \int_{\Omega(t)} \tfrac{1}{2}|v(t)|^2\,dy \\
&\quad + \int_0^t \left[2R(\eta(s), \partial_t \eta(s)) + \nu \int_{\Omega(s)} |\varepsilon v(s)|^2\,dy \right] ds \\
&\leq E(\eta_0) + \rho_s \int_Q \tfrac{1}{2}|\eta_*|^2\,dx + \rho_f \int_{\Omega(0)} \tfrac{1}{2}|v_0|^2\,dy \\
&\quad + \int_0^t \left[\rho_s \int_Q f(s) \circ \eta(s) \cdot \partial_t \eta(s)\,dx + \rho_f \int_{\Omega(s)} f(s) \cdot v(s)\,dy \right] ds. \qquad (1.12)
\end{aligned}$$

As is usual in evolution equations, this inequality holds as an equality for sufficiently regular solutions, i.e. if $(\partial_t \eta, v)$ can be used as a pair of test functions.

1.3. Outline of the paper

The proof of Theorem 1.2 is based on three main ideas, each of independent interest and each explained in a separate section. Sections 2 and 3 can be read pretty much independently. Section 4, however, relies on both preceding sections. We now describe the content of each section.

Section 2: Minimizing movements for fluid-structure interactions. In this section we consider a reduced parabolic system with the inertial terms omitted:

$$\begin{aligned}
\operatorname{div} \sigma(\eta) &= \rho_s f \circ \eta && \text{in } Q, && (1.13)\\
0 &= \nu \Delta v - \nabla p + \rho_f f && \text{in } \Omega(t), && (1.14)\\
\operatorname{div} v &= 0 && \text{in } \Omega(t), && (1.15)
\end{aligned}$$

together with the same coupling and boundary conditions as before.

We construct a weak solution to (1.13)–(1.15) by an implicit-explicit time-discretization scheme that exploits the *variational structure* of the problem, namely that the stress tensors for both the solid and the fluid have a potential.

Indeed, let us split $[0, T]$ into N time steps of length τ each. Assume, for $k \in \{0, \dots, N-1\}$, that $\eta_k \in \mathcal{E}$ is given and denote $\Omega_k = \Omega \setminus \eta_k(Q)$. We then define η_{k+1}, v_{k+1} to be the solution of the following minimization problem:

$$\operatorname*{argmin}_{\eta, v} E(\eta) + \tau R\big(\eta_k, \tfrac{\eta - \eta_k}{\tau}\big) + \tfrac{\tau \nu}{2} \|\nabla v\|_{\Omega_k}^2 - \tau \rho_s \big\langle f \circ \eta_k, \tfrac{\eta - \eta_k}{\tau} \big\rangle_Q - \tau \rho_f \langle f, v \rangle_{\Omega_k} \qquad (1.16)$$

over all $\eta \in \mathcal{E}$, $v \in W^{1,2}(\Omega_k; \mathbb{R}^n)$ satisfying $\operatorname{div} v = 0$, $v|_{\partial\Omega} = 0$ and the affine coupling condition $v \circ \eta_k = \frac{\eta - \eta_k}{\tau}$ in M. Minimizing with respect to this coupling is essential here: not only does it represent the discretized version of the coupling of velocities (1.8), but it will also result in the equality of tractions (1.9) via the minimization procedure.

Searching for time-discrete approximations of the weak solution to (1.13) alone via a minimization problem similar to the one above is actually well known and heavily used in the mathematics of continuum mechanics of solids (see e.g. [59]). The method is known as the *method of minimizing movements* or, especially in the engineering literature, the *time-incremental problem.* As far as the authors are aware, this method has not been applied to fluid-structure interaction problems before.

The advantage of the variational approach over directly solving the corresponding Euler–Lagrange equations is twofold. Not only do we deal with the nonconvexity of E and the underlying nonconvex space $\mathcal{E}$ in a natural way, but we also automatically gain an *energetic a prori estimate*. Indeed, comparing the value of the functional in (1.16) at (η_{k+1}, v_{k+1}) with its value for $(\eta_k, 0)$ and iterating, we get the following (quantitatively optimal) estimate of energy and dissipation:

$$\underbrace{E(\eta_{k+1})}_{\text{final energy}} + \underbrace{\sum_{l=0}^{k} \tau\Big[R\Big(\eta_l, \tfrac{\eta_{l+1}-\eta_l}{\tau}\Big) + \tfrac{\nu}{2}\|\varepsilon v_{l+1}\|^2_{\Omega_k}\Big]}_{1/2 \text{ of dissipation}}$$
$$\leq \underbrace{E(\eta_0)}_{\text{initial energy}} + \underbrace{\sum_{l=0}^{k} \tau\Big[\rho_s\Big\langle f\circ\eta_l, \tfrac{\eta_{l+1}-\eta_l}{\tau}\Big\rangle_Q + \rho_f\langle f, v_{l+1}\rangle\Big]}_{\text{work from forces}}. \tag{1.17}$$

Starting from this energetic estimate some other analytic tools need to be developed in order to deduce a priori estimates. In particular, a Korn inequaliy (Lemma 2.11) estimating the fluid and solid velocities simultaneously is introduced. The limit equation is then established via weak compactness results, the so-called Minty method (see Proposition 2.23) and a subtle approximation of test functions (see Lemma 2.22). The latter is necessary due to the fact that the fluid domain (the part where the test function is supposed to be solenoidal) is part of the solution. It is an analytic density result that is technically quite involving and as such possibly of independent interest.

Remark 1.3 (Chain rule). If we considered Euler–Lagrange equations instead of the variational problem (1.16), the standard way to obtain a priori estimates would be testing with $\eta_{k+1} - \eta_k$ and v_{k+1}. To obtain a similar estimate, we would then need to use a discrete variant of the chain rule to see that

$$E(\eta_{k+1}) - E(\eta_k) \leq \langle DE(\eta_{k+1}), \eta_{k+1} - \eta_k\rangle.$$

While this inequality is valid if E is convex, it is generally not true otherwise. In particular (see also Section 1.4), convexity of E is ruled out for physical reasons in mechanics of solids.

Section 3: Minimizing movements for hyperbolic evolutions. As we have seen in the previous step, discretizing the parabolic equation via a variational scheme is essential due to the nonconvexities involved. However, once an inertial, or hyperbolic, term is present, comparing values in the minimization problem no longer leads to useful estimates. Instead, having this term, it seems more advantageous to obtain a priori estimates from the Euler–Lagrange equation, which in turn is incompatible with the nonconvexity of the energy again. To overcome this difficulty, we consecutively approximate using *two different time-scales*: the *velocity scale* τ and the *acceleration scale* h. Keeping the acceleration scale fixed at first, we may use the strategy from the previous step and obtain, after passing to the limit $\tau \to 0$, a *continuous-in-time* equation from which a second set of a priori estimates can be seen.

In order to explain the concept, we first illustrate this procedure on the solid alone: We thus consider $\eta : [0, T] \times Q \to \mathbb{R}^n$, evolving according to

$$DE(\eta) + D_2 R(\eta, \partial_t \eta) - f \circ \eta = \rho_s \partial_t^2 \eta \tag{1.18}$$

without any coupling to the fluid, but with otherwise similar initial data to those before.

In order to be able to approximate the equation by a gradient flow we replace the second time derivative $\partial_t^2 \eta$ with a difference quotient and solve what we will call the *time-delayed problem*

$$DE(\eta(t)) + D_2 R(\eta(t), \partial_t \eta(t)) - f \circ \eta(t) = \tfrac{\partial_t \eta(t) - \partial_t \eta(t-h)}{h} \tag{1.19}$$

for some fixed h.

Considered on a short interval of length h, the term $\partial_t \eta(t - h)$ can be seen as fixed given data. Then on this interval the problem is parabolic and can be solved using a similar minimizing movements approximation to what was described before, where for fixed h we pick $\tau \ll h$ and solve a problem similar to

$$\operatorname*{argmin}_{\eta} E(\eta) + \tau R\big(\eta_k, \tfrac{\eta - \eta_k}{\tau}\big) - \tau \rho_s \big\langle f \circ \eta_k, \tfrac{\eta - \eta_k}{\tau}\big\rangle_Q + \frac{1}{2h} \Big\| \tfrac{\eta - \eta_k}{\tau} - \partial_t \eta(\tau k - h) \Big\|^2. \tag{1.20}$$

Upon letting $\tau \to 0$, using the same techniques as before, we then obtain a weak solution to (1.19) on $[0, h]$ which can be used as data on $[h, 2h]$ and so on, until we have derived a solution on $[0, T)$.

Now, as the time-delayed equation (1.19) is continuous, we avoid the problem with the chain rule (see Remark 1.3) and we can test[4] with $\partial_t \eta$ and obtain an energy inequality for the time-delayed problem in the form[5]

$$\begin{aligned} E(\eta(t)) + \rho_s \fint_{t-h}^{t} \tfrac{1}{2} \|\partial_t \eta(s)\|_Q^2 \, ds + \int_0^t 2R(\eta, \partial_t \eta) \, ds \\ \le E(\eta_0) + \tfrac{1}{2} \rho_s \|\eta_*\|_Q^2 + \int_0^t \langle f \circ \eta, \partial_t \eta \rangle_Q \, ds. \end{aligned}$$

[4]In order to guarantee that $\partial_t \eta$ is an admissible test function we have to include a parabolic regularizer in the dissipation functional of the solid. As the resulting estimate is only needed for $h > 0$ and independent of the regularizer, we can choose it in such a way that it vanishes as $h \to 0$.

[5]Here and in the following we use the notation $\fint_a^b = \frac{1}{|b-a|} \int$ for the mean-value integral.

As before, we can then turn this into a uniform a priori estimate, in order to finally let $h \to 0$.

We will discuss this in full rigor and prove existence for solutions to (1.18) in Section 3.

Remark 1.4 (Previous works on related PDEs). While our approach to existence of solutions to hyperbolic evolutions seems to be new, we wish to mention some previous approaches to similar PDEs. However, they either rely on convexity or, more generally, *polyconvexity* (convexity on minors) [33, 70], or use more explicit schemes which do not work well with the injectivity considerations [32]. In particular, notice that, for the solid, the scheme proposed here is fully implicit, which has the advantage that it is ensured that the constructed solution, even in the discrete setting, fulfills all relevant nonlinear constraints at all times, in particular global (almost) injectivity.[6]

Remark 1.5 (Hyperbolic minimizing movements and dissipation). Note that the approximation scheme does work in the purely elastic case, which means without any dissipation (the case $R \equiv 0$). Moreover, the a priori estimates (1.20) are also valid $R \equiv 0$. Hence whenever energy estimates suffice to pass to the limit, the hyperbolic minimizing movements scheme can be used to construct a (weak or measure-valued) solution. A simple PDE example would be the wave equation for which it is easy to see that the scheme produces a solution. But no strategy is known to deal with the nonlinearity in $DE(\eta)$ without resorting to a relaxed concept of solutions such as measure-valued solutions, which is outside the scope of this paper.

Observe that this problem of nonlinearity in the hyperbolic regime is a well known challenge for many open problems. Difficulties already arise with convex energies. Indeed, even for the hyperbolic p-Laplacian equation $\partial_t^2 \eta - \operatorname{div}(|\nabla \eta|^{p-2} \nabla \eta) = 0$ the existence of weak solutions is a long-standing, unsolved problem (see e.g. [2] for a discussion). Only in the case $p = 2$, where the elastic energy is quadratic in highest order (and thus its derivative is linear) existence of solutions is known. But certainly, for any $p \in (1, \infty)$ the hyperbolic minimizing movements would produce a well defined approximation, satisfying a natural a priori estimate. However, without dissipation, the compactness resulting from such an estimate is only enough to obtain measure-valued solutions.

Section 4: Bulk elastic solids coupled to Navier–Stokes equations. In Section 4 we combine the previous two sections and apply the two-time-scale approach to the fluid-structure interaction problem (1.1)–(1.11). The main obstacle here lies in the Eulerian description of the fluid. Here it turns out to be natural to approximate the *material derivative of the fluid velocity* $(\partial_t v + [\nabla v]v)$ by a time-discrete differential quotient. This is done by subsequently introducing a flow map $\Phi_s(t) : \Omega(t) \to \Omega(t+s)$ fulfilling

[6] To be more precise, we will consider the Ciarlet–Nečas condition, which guarantees injectivity up to a set of measure zero and which has the important property of being preserved under the relevant convergence.

$\partial_s \Phi_s(t, y) = v(t + s, \Phi_s(t, y))$ (resp. a discrete version) and $\Phi_0(t, y) = y$ in both the discrete and the time-delayed approximation layers. This means that Φ *transports* the domain of the fluid along with its velocity.

In particular, the fluid analogue of the difference quotient in the time-delayed problem will be a "material difference quotient" in the size of the acceleration scale h, which is essentially of the form

$$\frac{v(t, \Phi_h(t-h, y)) - v(t-h, y)}{h}.$$

As Φ and v are inseparably linked, we need to construct their discrete counterparts alongside each other already in the τ scale. This discrete construction of the highly nonlinear Φ and its subsequent convergence are one of the main additions in the proof and one of the technical centerpieces of the paper.

As in Step 2, an approximation of the final energy estimates (1.12) is obtained only *after the first limit passage* $\tau \to 0$. More precisely, they are again obtained by testing the coupled Euler–Lagrange equation of the continuous-in-time quasi-steady approximation with the fluid and solid velocities. The energy inequality obtained by testing differs from (1.12) by replacing the kinetic energies with their moving averages. From this we can then derive an a priori estimate that allows us to pass to the limit. A particular difficulty is the material derivative of the fluid velocity. Here we need to derive a modified Aubin–Lions lemma in order to obtain stronger convergence for a time-average approximation of u, which is the natural quantity in this context.

Remark 1.6 (Previous variational approaches to the Navier–Stokes equation). While the minimizing movements method seems to be new in the field of fluid-structure interactions, it has been previously used to show existence of solutions to the Navier–Stokes equations. In particular, we want to highlight [48] as an inspiration.[7] There the authors also employ flow maps to obtain the material derivative, but as they work on a fixed domain, they do not need to construct them iteratively but can instead rely on the corresponding existence theory for the Stokes problem. As an indirect consequence, their minimization happens on what we would consider the h-level, which makes it incompatible with our way of handling the solid evolution. Thus, the scheme proposed here is more than an improvement of the numerical scheme [80] which has been developed much earlier.

1.4. Mechanical and analytical restrictions on the energy/dissipation functional

As introduced in Section 1.1, we consider solid materials for which the stress tensor can be determined by prescribing two functionals: *the energy and dissipation functionals*. Materials admitting such modeling fall into the class of so-called *generalized standard materials* [51, 59, 79], together with many other rheological models [59]. In fact, this setting has its roots in modeling plasticity in solids and has later been generalized to

[7] We would also like to mention similar approaches for the compressible case found in [18].

many other processes with internal variables [67]. Nevertheless, the classical rheological models starting with the Kelvin–Voigt and Maxwell model can also be put into this frame, as one can find a suitable dissipation function for them.

Nonetheless, the two functionals cannot be chosen completely freely, but have to comply with certain physical requirements. We summarize these at this point.

As in examples (1.6) and (1.7), for the sake of discussion we will assume that the energy and dissipation functionals each have a density, i.e.

$$E(\eta) = \int_Q e(\nabla\eta, \nabla^2\eta)\,dx, \quad R(\eta, \partial_t\eta) = \int_Q r(\nabla\eta, \partial_t\nabla\eta)\,dx, \tag{1.21}$$

for all smooth vector fields $\eta : Q \to \mathbb{R}^n$.

Here, the energy density depends on the first and second gradients[8] of the deformation, which puts us into the class of so-called nonsimple (or second grade) materials (see the pioneering work [86] as well as [39, 82] for later developments). This is in contrast with classical hyperelasticity, where the energy depends on the first gradient only. In the mechanical literature, the second gradient is usually employed when microstructure is modeled [69] or when localization is present. In the mathematical literature (see e.g. [59]), second grade materials are usually exploited as they allow for control of the deformation in higher order Sobolev spaces. Moreover, the physical restrictions on the energy do not exclude convexity in higher order terms, because the restrictions (1.4) and (1.5) concern the first gradient only.

On top of all that, within the setting of second grade materials it is possible to obtain a uniform lower bound on the Jacobian of the deformation, as has been first realized in [52]. Not only will this result in a meaningful boundary for the fluid domain, but it will also help us to readily switch between Lagrangian and Eulerian descriptions of the solid velocity.

As for the dissipation potential, we will also need it to be independent of the observer, i.e. for all smoothly time-varying proper rotations $R(t)$, and all smooth time-dependent $F : [0, T] \to \mathbb{R}^n$,

$$r(RF, \partial_t(RF)) = r(F, \partial_t F).$$

This restriction implies [3] that r *cannot* depend on $\partial_t\eta$ only but needs to also depend on η. This in turn will require using *fine Korn-type inequalities* [78, 81] to deduce a priori estimates (as already in [68]). We also note that for both physical and analytic reasons, r should be nonnegative and convex in the second variable. We additionally require R to be a quadratic form in its second variable.[9]

Taking into account these, as well as some analytical requirements, we will now detail our setup for the deformation. Throughout the paper, for the elastic energy potential we impose the following assumptions.

[8] The formalism of our proofs naturally also allows for dependence on material and spatial positions x and $\eta(x)$, but the latter dependence is nonphysical and the former does not add much to the discussion. Nevertheless we emphasize that our results also hold for inhomogeneous materials.

[9] See Remark 1.5 for possible relaxations of the assumptions on the dissipation potential.

Assumption 1.7 (Elastic energy). Let $Q, \Omega \subset \mathbb{R}^n$ and $q > n$. Then $E : W^{2,q}(Q;\Omega) \to \overline{\mathbb{R}}$ satisfies:

S1 *Lower bound:* There exists a number $E_{\min} > -\infty$ such that

$$E(\eta) \geq E_{\min} \quad \text{for all } \eta \in W^{2,q}(Q;\Omega).$$

S2 *Lower bound of the determinant:* For any $E_0 > 0$ there is $\epsilon_0 > 0$ such that $\det \nabla\eta \geq \epsilon_0$ for all $\eta \in W^{2,q}(Q;\Omega)$ with $E(\eta) < E_0$.

S3 *Weak lower semicontinuity:* If $\eta_l \rightharpoonup \eta$ in $W^{2,q}(Q;\Omega)$ then $E(\eta) \leq \liminf_{l\to\infty} E(\eta_l)$.

S4 *Coercivity:* All sublevel sets $\{\eta \in \mathcal{E} \mid E(\eta) < E_0\}$ are bounded in $W^{2,q}(Q;\Omega)$.

S5 *Existence of derivatives:* For finite values E has a well defined derivative which we will formally denote by

$$DE : \{\eta \in \mathcal{E} \mid E(\eta) < \infty\} \to (W^{2,q}(Q;\mathbb{R}^n))'.$$

Furthermore, on any sublevel set of E, DE is bounded and continuous with respect to strong $W^{2,q}$ convergence.

S6 *Monotonicity and Minty-type property:* If $\eta_l \rightharpoonup \eta$ in $W^{2,q}(Q;\Omega)$, then

$$\liminf_{l\to\infty} \langle DE(\eta_l) - DE(\eta), (\eta_l - \eta)\psi\rangle \geq 0 \quad \text{for all } \psi \in C_0^\infty(Q;[0,1]).$$

If additionally

$$\limsup_{l\to\infty} \langle DE(\eta_l) - DE(\eta), (\eta_l - \eta)\psi\rangle \leq 0$$

then $\eta_l \to \eta$ in $W^{2,q}(Q;\Omega)$.

Let us briefly elaborate on the above stated assumptions. As elastic energies are generally bounded from below, assumption S1 is a natural one. Similarly, S5 is to be expected as we need to take the derivative of the energy to determine a weak version of the Piola–Kirchhoff stress tensor. Assumptions S3 and S4 are standard in any variational approach as they open up the possibility for using the direct method of the calculus of variations. Assumption S6 effectively means that the energy density has to be convex in the highest gradient (but of course not convex overall) and allows us to get weak solutions and not merely measure-valued ones (as in the case of a solid material in [33]). Finally, S2 is probably the most restricting one and, to the authors' knowledge, necessitates the use of second-grade elasticity, combined with an energy density e which blows up sufficiently fast as $\det F \to 0$ (see [52]). This is, in particular, the case for the model energy (1.7).

Definition 1.8 (Domain of definition). The set of functions in $W^{2,q}(Q;\Omega)$ (and satisfying the Dirichlet boundary condition) used for minimization in (1.16) can be expressed as

$$\mathcal{E} := \left\{ \eta \in W^{2,q}(Q;\Omega) \;\middle|\; E(\eta) < \infty,\ |\eta(Q)| = \int_Q \det \nabla\eta \, dx,\ \eta|_P = \gamma \right\}. \tag{1.22}$$

Here, the finite energy guarantees local injectivity (see assumption S2), and the equality $|\eta(Q)| = \int_Q \det \nabla\eta \, dx$, termed the *Ciarlet–Nečas condition*, has been proposed in [26] and, as proved there, it ensures that any C^1 local homeomorphism is globally injective except for possible touching at the boundary. Working with this equivalent condition has the advantage that it is easily seen to be preserved under weak convergence in $W^{2,q}(Q;\Omega)$.

Remark 1.9. Of particular interest is the topology of $\mathcal{E}$. It is easy to see that this is a closed subset of the affine space $W^{2,q}_{\gamma}(Q;\Omega)$, i.e. $W^{2,q}(Q;\Omega)$ with fixed boundary conditions. As a subset of this topological space it has both interior points (denoted by $\operatorname{int}(\mathcal{E})$) and a boundary $\partial\mathcal{E}$. As we construct our approximate solutions by minimization over $\mathcal{E}$, it is crucial to know if $\eta_k \in \operatorname{int}(\mathcal{E})$ because only then are we allowed to test in all directions and have the full Euler–Lagrange equation we need.

Luckily, however, $\operatorname{int}(\mathcal{E})$ and $\partial\mathcal{E}$ are easily quantifiable. As long as $\det \nabla\eta > 0$, which is true for finite energy, we are able to vary in all directions, if and only if $\eta|_M$ is injective and does not touch $\partial\Omega$. Thus the relevant part of $\partial\mathcal{E}$, i.e. the deformations with finite energy, consists precisely of the η which have a collision.

Finally, for the dissipation functional we have the following assumption:

Assumption 1.10 (Dissipation functional). The dissipation $R : \mathcal{E} \times W^{1,2}(Q;\mathbb{R}^n) \to \mathbb{R}$ satisfies:

R1 *Weak lower semicontinuity:* If $b_l \rightharpoonup b$ in $W^{1,2}$ then

$$\liminf_{l\to\infty} R(\eta, b_l) \geq R(\eta, b).$$

R2 *Homogeneity of degree* 2*:* The dissipation is homogeneous of degree 2 in its second argument, i.e.

$$R(\eta, \lambda b) = \lambda^2 R(\eta, b) \quad \forall \lambda \in \mathbb{R}.$$

In particular, this implies $R(\eta, b) \geq 0$ and $R(\eta, 0) = 0$.

R3 *Energy-dependent Korn-type inequality:* Fix $E_0 > 0$. Then there exists a constant $c_K = c_K(E_0) > 0$ such that for all $\eta \in W^{2,q}(Q;\mathbb{R}^n)$ with $E(\eta) \leq E_0$ and all $b \in W^{1,2}(Q;\mathbb{R}^n)$ with $b|_P = 0$ we have

$$c_K \|b\|^2_{W^{1,2}(Q)} \leq R(\eta, b).$$

R4 *Existence of a continuous derivative:* The derivative $D_2 R(\eta, b) \in (W^{1,2}(Q;\mathbb{R}^n))'$ given by

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} R(\eta, b + \epsilon\phi) =: \langle D_2 R(\eta, b), \phi \rangle$$

exists and is weakly continuous in its two arguments. Due to the homogeneity of degree 2 this in particular implies

$$\langle D_2 R(\eta, b), b \rangle = 2R(\eta, b).$$

Again some remarks are in order. As above, assumption R4 is natural as we need to be able to evaluate the actual stress. Assumption R2, on the other hand, reflects the fact that we are considering viscous dissipation. Assumption R1 is again important from the point of view of the calculus of variations. Assumption R3 is a coercivity assumption in a sense and needs to be stated in this rather weak form to satisfy frame indifference. Indeed, our model dissipation (1.6) satisfies this assumption, as shown e.g. in [68] relying on quite general Korn inequalities due to [78, 81].

1.5. Outlook and further applicability

We would like to highlight that all three parts of the result outlined here present a huge potential for generalizations. In particular, the ideas are easily applicable to different material laws for both the fluid and the solid.

For the fluid, we would like to highlight in particular that modifying the dissipation potential for the fluid could allow for working with non-Newtonian fluids and that by the time of publication there are already first results for compressible fluids [16], extending some of the known FSI results such as [17, 64] to a 3D-3D setting. On the other hand, the hyperbolic minimizing movement scheme could be used to allow for a displacement in all coordinate directions for a 2D-1D fluid-structure interaction [58]. It would be interesting to connect the regimes, studying the limit passage from bulk solids to shells to the FSI setting, as has been done for solids alone e.g. in [40, 63].

For the solid in turn, we remark that this approach is suitable to similarly describe inertial evolution for a large class of materials, extending previous results such as [32, 33]. But additionally the way of dealing with the interplay of Lagrangian and Eulerian representation, developed here for the fluid, might in turn be useful in dealing with similar problems in the solid, such as plasticity or coupling with electro-magnetic effects. In particular, in combination with the variational approach, this has some interesting implications for the study of the (self-)contact problem, as some preliminary results show [19]. This should also have some applications to the same problem in an FSI context (see e.g. [53, 54] or [50]).

A final point we would like to mention is potential numerical applications of our results. While our proofs rely on specifically dealing with the limit of one process as an approximation of another and thus offer no guarantee of convergence of a numerical approximation, we believe that for more specific setups, estimates on convergence rates and thus on approximation quality should be available.

1.6. Notation

Let us detail the notation, some of which we have already used. Throughout the paper we will deal with a number of quantities that may depend on either or both of the parameters τ and h, corresponding to our two time-scales. Whenever this dependence is relevant, i.e. when we are varying the parameter or passing to the corresponding limit, we will indicate it by a superscript, e.g. $\eta^{(\tau)} \to \eta$. There will be no context where both are relevant at the same time, but we note that in Sections 3 and 4, all quantities that depend on τ also depend on the larger scale h. Any such dependence may additionally be combined with a sequence index as well, as for example in $\eta_k^{(\tau)}$.

Of particular interest here is the fluid domain, which technically is always determined through the deformation of the solid. As this deformation may depend on the scales τ and h, on a sequence index as well as on time, we may indicate this dependence in the notation. For example, we might write $\Omega_l^{(h)}(t) := \Omega \setminus \eta_l^{(h)}(t, Q)$.

Coordinates in the reference domain Q will always be denoted by x and coordinates in the physical domain Ω by y. The gradient operator ∇ will always refer to these spatial coordinates and never to time. Similarly $\operatorname{Lip}(v(t)) = \operatorname{Lip}_y(v(t))$ is the Lipschitz constant of a function $v(t)$ only with respect to spatial coordinates, except for one specific occasion when we will explicitly write $\operatorname{Lip}_{t,y}(v)$ for the Lipschitz constant in space-time.

In order to avoid confusion with changing domains, we will almost always write out all time integrations and apart from a few exceptions try to only use spatial norms and inner products. We will always give the relevant domain if there is any chance of doubt. In particular, a subscript like A will always denote the L^2 norm or inner product with respect to this domain:

$$\|f\|_A^2 := \int_A |f|^2 \, dx \quad \text{and} \quad \langle f, g \rangle_A = \int_A f \cdot g \, dx.$$

For other norms we will always specify the domain (but omit the codomain), e.g. we will write $\|v(t)\|_{W^{1,2}(\Omega(t))}$. The only exceptions are the linear operators $DE(\eta)$ and $D_2R(\eta, b)$ which invariably have their domain of definition associated to them. We will thus simply write $\langle DE(\eta), \phi \rangle$ and $\langle D_2R(\eta, b), \phi \rangle$ to denote the underlying $W^{2,q}(Q; \mathbb{R}^n) \times W^{-2,q}(Q; \mathbb{R}^n)$ and $W^{1,2}(Q; \mathbb{R}^n) \times W^{-1,2}(Q; \mathbb{R}^n)$ pairings. Since these only ever occur as linear operators and not as functions, there should be no confusion.

2. Minimizing movements for fluid-structure interactions

In this section, we provide existence of weak solutions to the *parabolic* fluid-structure interaction problem, i.e. the inertia-less balances (1.13)–(1.15) together with the coupling conditions (1.8)–(1.9) as well as Dirichlet boundary conditions for the deformation and a Navier boundary condition stemming from the higher gradients in the energy. This has the interesting difficulty that in fluid-structure interaction a naturally Lagrangian solid has to be coupled with a naturally Eulerian fluid on a variable domain. We will do so without fixing a reference fluid domain and instead use a variational approach to deal with the fluid directly on a varying domain.

We shall work with the following weak formulation:

Definition 2.1 (Weak solution to the parabolic problem). We call a pair (η, v) a *weak solution to the parabolic fluid-structure interaction problem* if it satisfies

$$\begin{aligned} &\eta \in L^\infty([0,T]; \mathcal{E}), && \partial_t \eta \in L^2([0,T]; W^{1,2}(Q; \mathbb{R}^n)), \\ &v \in L^2([0,T]; W^{1,2}(\Omega(t); \mathbb{R}^n)), && \operatorname{div} v(t) = 0 \quad \text{for a.a. } t \in [0,T], \end{aligned}$$

as well as $v(t, \eta(t,x)) = \partial_t \eta(t,x)$ for a.a. $t \in [0,T]$ and $x \in \partial Q$ and there exists a $p \in \mathcal{D}'([0,T] \times \Omega)$ with $\operatorname{supp} p \subset [0,T] \times \Omega(t)$, such that

$$\int_0^T [\langle DE(\eta), \phi\rangle + \langle D_2 R(\eta, \partial_t \eta), \phi\rangle + \langle \varepsilon v, \varepsilon \xi\rangle_{\Omega(t)} - \langle p, \operatorname{div} \xi\rangle]\, dt$$
$$= \int_0^T [\rho_f \langle f, \xi\rangle_{\Omega(t)} + \rho_s \langle f \circ \eta, \phi\rangle_Q]\, dt \qquad (2.1)$$

for all $\phi \in L^2([0,T]; W^{2,q}(Q;\mathbb{R}^n))$ with $\phi|_P = 0$ and $\xi \in C_0([0,T]; W_0^{2,q}(\Omega;\mathbb{R}^n))$ such that $\phi = \xi \circ \eta$ on Q where, as before, we set $\Omega(t) := \Omega \setminus \eta(t, Q)$. Moreover, the initial condition for η is satisfied in the sense that

$$\lim_{t \to 0} \eta(t) = \eta_0 \quad \text{in } L^2(Q;\mathbb{R}^n).$$

The main goal of this section is to prove existence of weak solutions to the parabolic fluid-structure interaction problem. In particular, we show the following theorem:

Theorem 2.2 (Existence of a parabolic fluid-structure interaction). *Assume that the energy and dissipation pair (E, R) fulfill Assumptions 1.7 and 1.10. Further, let $\eta_0 \in \mathcal{E}$ and $f \in L^\infty(\Omega;\mathbb{R}^n)$. Then there exists a maximal time $T_{\max} > 0$ such that on the interval $[0, T_{\max})$ there exists a weak solution to the parabolic fluid-structure interaction problem in the sense of Definition 2.1.*

We have either $T_{\max} = \infty$, or $\liminf_{t \to T_{\max}} E(\eta(t)) = \infty$, or $T_{\max}$ is the time of the first collision of the solid with either itself or the container, i.e. the continuation $\eta(T_{\max})$ exists and $\eta(T_{\max}) \in \partial\mathcal{E}$. Furthermore, $p \in L^2([0,T]; L^\infty(\Omega(t))) + L^\infty([0,T]; L^2(\Omega(t)))$ for all $T < T_{\max}$.

In order to prove Theorem 2.2, we shall exploit the natural gradient flow structure of the parabolic fluid-structure evolution. Indeed, at the heart of the proof is the construction of time-discrete approximations via variational problems inspired by De Giorgi's *minimizing movements* method [31] given in (2.3). We refer to Section 2.2 for a detailed proof of Theorem 2.2 and to Section 2.1 for the preliminary material.

Remark 2.3 (Maximal existence time). The maximal existence time in Theorem 2.2 is due not only to possible collisions but also to a possible blow-up of the energy due to the acting forces. It is notable that by Lemma 4.11 such a situation is absent in the full model including inertia, because acting forces can then be estimated using the inertial term.

2.1. Preliminary analysis

We start this section by discussing the relevant geometry of the fluid-solid coupling and derive some necessary properties for the coupled system that will also be of use for the full Navier–Stokes system in Section 4.

Lemma 2.4 (Closedness of $\mathcal{E}$). *Let $(\eta_l)_{l \in \mathbb{N}} \subset \mathcal{E}$ be a sequence such that $\eta_l \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^n)$ and $\sup_{l \in \mathbb{N}} E(\eta_l) < \infty$. Then $\eta \in \mathcal{E}$.*

Proof. The boundary condition holds as $W^{2,q}(Q;\mathbb{R}^n)$ has a continuous trace operator. Similarly the lower semicontinuity of E guarantees $E(\eta) < \infty$. For the Ciarlet–Nečas condition we refer to [26], but note that, due to the higher regularity we employ, a more direct proof would be feasible as well. ∎

Injectivity and boundary regularity of the solid. From the Ciarlet–Nečas condition we know that any $\eta \in \mathcal{E}$ is injective on Q but not necessarily on $\bar{Q}$, so collisions are in principle possible. Nonetheless, we can exclude them for short times as shown via the following two lemmas as well as Corollary 2.19.

Lemma 2.5 (Local injectivity of the boundary). *For any $E_0 < \infty$ there exists a $\delta_0 > 0$ such that all $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ are locally injective with radius δ_0 on ∂Q, i.e.*

$$\eta(x_0) \neq \eta(x_1) \quad \textit{for all } x_0, x_1 \in \partial Q \textit{ with } |x_0 - x_1| < \delta_0.$$

Proof. Assume that there are $x_0, x_1 \in \partial Q$ such that $\eta(x_0) = \eta(x_1)$. Now using embedding theorems and Assumptions 1.7 (S2, S4), $E(\eta) < E_0$ implies that $\nabla\eta$ is uniformly continuous and there exists a uniform lower bound on $\det \nabla\eta$. This also results in uniform continuity of $(\nabla\eta)^{-1} = \frac{\operatorname{cof} \nabla\eta}{\det \nabla\eta}$. Now on the one hand, by the injectivity in the interior, the tangent planes to ∂Q at x_0 and x_1 need to be mapped to the same image plane by $\nabla\eta(x_0)$ and $\nabla\eta(x_1)$ respectively, but with opposite orientations. On the other hand, if x_0 and x_1 are close, then the continuity of $(\nabla\eta)^{-1}$ implies that the tangent planes to ∂Q at x_0 and x_1 are almost oppositely oriented as well, which contradicts the regularity of ∂Q. ∎

Remark 2.6. The preceding proof is much easier to formulate in the case $n = 2$ as one can deal with tangent vectors directly: Consider the positively oriented unit tangent vectors τ_x at $x \in \partial Q$. Then $\nabla\eta(x_0)\tau_{x_0}$ and $\nabla\eta(x_1)\tau_{x_1}$ point in opposite directions and their length is bounded from below. But if x_0 and x_1 are close, then so are the τ_{x_i} and $\nabla\eta(x_i)$, which leads to a contradiction.

Lemma 2.7 (Short time global injectivity preservation). *Fix $E_0 < \infty$ and $\varepsilon_0 > 0$ and let δ_0 be given by the previous lemma. Then there exists a $\gamma_0 > 0$ such that for all $\eta_0 \in \mathcal{E}$ with $E(\eta_0) < E_0$ and*

$$|\eta_0(x_0) - \eta_0(x_1)| > \varepsilon_0 \quad \textit{for all } x_0, x_1 \in \partial Q \textit{ with } |x_0 - x_1| \geq \delta_0 \tag{2.2}$$

we have, for all $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ and $\|\eta_0 - \eta\|_{L^2(Q)} < \gamma_0$,

$$|\eta(x_0) - \eta(x_1)| > \varepsilon_0/2 \quad \textit{for all } x_0, x_1 \in \partial Q \textit{ with } |x_0 - x_1| \geq \delta_0.$$

Proof. Let η_0 be as prescribed and pick $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ and $|\eta(x_0) - \eta(x_1)| \leq \varepsilon_0/2$ for some x_0, x_1 with $|x_0 - x_1| \geq \delta_0$. Then

$$|\eta_0(x_0) - \eta(x_0)| + |\eta_0(x_1) - \eta(x_1)| \geq |\eta_0(x_0) - \eta_0(x_1)| - |\eta(x_0) - \eta(x_1)| > \varepsilon_0/2.$$

So, without loss of generality, we can assume that $|\eta_0(x_0) - \eta(x_0)| \geq \varepsilon_0/4$. But then since η_0 and η are uniformly continuous with the modulus of continuity depending just on E_0, there exists an $r > 0$ such that $|\eta_0(x) - \eta(x)| \geq \varepsilon_0/8$ for all $x \in B_r(x_0) \cap Q$. Thus

$$\|\eta_0 - \eta\|_{L^2(Q)} \geq \sqrt{(\varepsilon_0/8)^2 |B_r(x_0) \cap Q|} =: \gamma_0 > 0. \qquad \blacksquare$$

Since we are concerned with variable-in-time domains for the fluid flow, we recall here the quantification of uniform regular domains. Later we will encounter several analytical results which will be used uniformly with respect to these quantifications.

Definition 2.8. For $k \in \mathbb{N}$ and $\alpha \in [0, 1]$. We call $\Omega \subset \mathbb{R}^n$ a $C^{k,\alpha}$ *domain with characteristics* L, r if for all $x \in \partial\Omega$ there is a $C^{k,\alpha}$ diffeomorphism $\phi_x : B_1(0) \to B_r(x)$ such that $\phi_x : B_1^-(0) \to B_r(x) \cap \Omega$, $\phi_x : B_1^+(0) \to B_r(x) \cap \Omega^c$ and $\phi_x(0) = x$. We require that ϕ_x can be written as a graph over a direction $e_x \in \mathcal{S}^{n-1}$. This means that for $(z', z_n) \in B_1(0)$ we have $\phi_x(z) = \phi_x((z', 0)) + r e_x z_n$. Moreover, we assume that $\|\phi_x\|_{C^{\alpha,k}(B_1(0))} + \|\phi_x^{-1}\|_{C^{\alpha,k}(B_r(x))} \leq L$.

Collecting the regularity that comes from the energy bounds leads to an important (locally) uniform estimate on the $C^{1,\alpha}$ regularity of the fluid domains.

Corollary 2.9 (Uniform $C^{1,\alpha}$ domains). *Fix $E_0 < \infty$ and $\eta_0 \in \mathcal{E}$ with $E(\eta_0) < E_0$ and satisfying* (2.2) *for some $\varepsilon_0 > 0$. Then for all $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ and $\|\eta_0 - \eta\|_{L^2(Q)} < \gamma_0$ for γ_0 from Proposition* 2.7 *the set $\Omega_\eta := \Omega \setminus \eta(Q)$ is a $C^{1,\alpha}$domain with characteristics L, r depending only on E_0, η_0 and ε_0.*

Global velocity and a global Korn inequality. A useful tool when dealing with fluid-structure interaction in the bulk is the global Eulerian velocity field, which is defined on the unchanging domain Ω. In particular, this will allow us to circumvent the problem of talking about convergence on a changing domain.

Definition 2.10 (The global velocity field). Let $\eta \in \mathcal{E}$ be a given deformation. Let $v \in W^{1,2}(\Omega; \mathbb{R}^n)$ be a divergence free fluid velocity and $b \in W^{1,2}(Q; \mathbb{R}^n)$ a solid velocity satisfying the coupling condition $v \circ \eta = b$ on $\partial Q \setminus P$. Then the corresponding global velocity $u \in W_0^{1,2}(\Omega; \mathbb{R}^n)$ is defined by

$$u(y) := \begin{cases} v(y) & \text{if } y \in \Omega_\eta := \Omega \setminus \eta(Q), \\ b \circ \eta^{-1}(y) & \text{if } y \in \eta(Q). \end{cases}$$

Note that this definition does not involve a reference time-scale directly. The solid velocity b is allowed to be a time derivative $b := \partial_t \eta$ or a discrete derivative $b := \frac{\eta_{k+1} - \eta_k}{\tau}$. Furthermore, this definition is invertible. Given u and knowing η, both v and b can be reconstructed, and those reconstructed velocities will satisfy the coupling condition as above.

When deriving a priori estimates, the only bounds on the velocities that will be available are in the form of the dissipation. As this is given in terms of a symmetrized derivative, we will need to use Korn-type inequalities, whose constants are generally domain-dependent. However, another benefit of the global velocity and its constant domain is that the inequalities for the solid and the fluid can be merged into one global Korn inequality.

Lemma 2.11 (Global Korn inequality). *Fix $E_0 > 0$. Then there exists $c_{gK} = c_{gK}(E_0) > 0$ such that for any $\eta \in \mathcal{E}$ with $E(\eta) < E_0$ and any $b \in W^{1,2}(Q;\mathbb{R}^n)$ and $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ with $u|_{\partial\Omega} = 0$ and $b|_P = 0$ and satisfying the coupling condition $u \circ \eta = b$ in Q we have*

$$c_{gK}\|u\|_{W^{1,2}(\Omega)} \leq \frac{\nu}{2}\|\varepsilon u\|_{\Omega_\eta} + R(\eta, b),$$

where we define $\Omega_\eta = \Omega \setminus \eta(Q)$.

Proof. On the reference domain Q we have by the chain rule $\nabla b = \nabla(u \circ \eta) = (\nabla u) \circ \eta \cdot \nabla\eta$. Using this we can estimate, in analogy to Proposition A.4, as η is a diffeomorphism,

$$\begin{aligned}\int_{\Omega\setminus\Omega_\eta} |\nabla u|^2\, dy &= \int_Q |(\nabla u)\circ\eta|^2 \det\nabla\eta\, dx = \int_Q |(\nabla b)\cdot(\nabla\eta)^{-1}|^2 \det\nabla\eta\, dx \\ &\leq \int_Q |\nabla b|^2 \frac{|\operatorname{cof}\nabla\eta|^2}{\det\nabla\eta}\, dx \leq \frac{\|\eta\|_{C^1}^{2n-2}}{\epsilon_0}\int_Q |\nabla b|^2\, dx \leq \frac{\|\eta\|_{C^1}^{2n-2}}{\epsilon_0} c_K R(\eta, b),\end{aligned}$$

where $\epsilon_0 > 0$ is the uniform lower bound on $\det\nabla\eta$ as given in assumption S2, $\|\eta\|_{C^1}$ is uniformly bounded by embeddings and we use the Korn-type inequality from assumption R3.

But now we can apply Korn's inequality to the fixed domain Ω to get a constant c_Ω for which

$$c_\Omega\|u\|_{W^{1,2}(\Omega)}^2 \leq \|\varepsilon u\|_\Omega^2 = \|\varepsilon u\|_{\Omega_\eta}^2 + \|\varepsilon u\|_{\Omega\setminus\Omega_\eta}^2 \leq \|\varepsilon u\|_{\Omega_\eta}^2 + \frac{\|\eta\|_{C^1}^{2n-2}}{\epsilon_0} c_K R(\eta, b).$$

Collecting all the constants then proves the lemma. ∎

2.2. *Proof of Theorem 2.2*

As mentioned before, we will show Theorem 2.2 in several steps using a time discretization in the form of a minimizing movements iteration.

Step 1: Existence of the discrete approximation. For this we will fix a time-step size τ. Setting $\eta_0^{(\tau)} := \eta_0$ and assuming $\eta_k^{(\tau)} \in \mathcal{E}$ given we define $(\eta_{k+1}^{(\tau)}, v_{k+1}^{(\tau)})$ as the minimizer of

$$E(\eta) + \tau R\Big(\eta_k^{(\tau)}, \tfrac{\eta - \eta_k^{(\tau)}}{\tau}\Big) + \tau\frac{\nu}{2}\|\varepsilon v\|_{\Omega_k^{(\tau)}}^2 - \rho_s\tau\Big\langle f\circ\eta_k^{(\tau)}, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\Big\rangle_Q - \rho_f\tau\langle f, v\rangle_{\Omega_k^{(\tau)}} \tag{2.3}$$

over $\eta \in \mathcal{E}$, $v \in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$ with $\operatorname{div} v = 0$, $v|_{\partial\Omega} = 0$ and $\frac{\eta - \eta_k^{(\tau)}}{\tau} = v \circ \eta_k^{(\tau)}$ in M.

We then repeat this process until we reach $k\tau > T$. Notice that in the coupling condition in (2.3), we implicitly assumed that the solid is free of collisions, i.e. $\eta_k \notin \partial\mathcal{E}$. We will show in Corollary 2.19 that for small enough T this will always be the case.

We will now show that this problem has a (not necessarily unique) solution which satisfies a discrete approximation of our problem in the form of an Euler–Lagrange equation.

Remark 2.12. In (2.3) we minimize the sum of the energy and the dissipation needed to reach the current step from the previous one. In this context, we view the Stokes potential as dissipative damping on the solid. Now, as far as the deformation is concerned, the scheme is implicit in the energy and *implicit-explicit* in the dissipation. In particular, in the Stokes potential, the dependence on the deformation manifests itself explicitly through the domain and implicitly through the coupled boundary values. Explicit-implicit schemes are commonly used in fluid-structure interactions (see e.g. [71]). Moreover, it is a common way to produce solutions in solid mechanics if the dissipation depends on the state variables [59]. Equality of tractions need not then be imposed but follows automatically from the variational approach.

Proposition 2.13 (Existence of solutions to (2.3)). *Assume that $\eta_k^{(\tau)} \in \mathcal{E}$. Then the iterative problem (2.3) has a minimizer, i.e. $\eta_{k+1}^{(\tau)}$ and $v_{k+1}^{(\tau)}$ are defined. Furthermore, if $\eta_{k+1}^{(\tau)} \notin \partial\mathcal{E}$ (i.e. $\eta_{k+1}^{(\tau)}$ is injective on $\bar{Q}$) the minimizers obey the Euler–Lagrange equation*

$$\begin{aligned}\langle DE(\eta_{k+1}^{(\tau)}), \phi\rangle + \big\langle D_2 R\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big), \phi\big\rangle + \nu\langle \varepsilon v_{k+1}^{(\tau)}, \nabla\xi\rangle_{\Omega_k^{(\tau)}}\\ = \rho_f\langle f, \xi\rangle_{\Omega_k^{(\tau)}} + \rho_s\langle f\circ\eta_k^{(\tau)}, \phi\rangle_Q\end{aligned}$$

for any $\phi \in W^{2,q}(Q;\mathbb{R}^n)$ with $\phi|_P = 0$ and $\xi \in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$ with $\operatorname{div}\xi = 0$, $\xi|_{\partial\Omega} = 0$ and $\xi\circ\eta = \phi$ on ∂M.

Proof. First we investigate existence using the direct method. The class of admissible functions is nonempty, since $(\eta_k^{(\tau)}, 0)$ is a possible competitor with finite energy. Next we show that the functional is bounded from below. As the energy and dissipation have lower bounds by assumption, the only problematic terms are those involving the force f. For those we note that by the weighted Young inequality and using assumption R3 we have

$$\begin{aligned}\big|\langle f\circ\eta_k^{(\tau)}, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\rangle_Q\big| &\le \frac{\delta}{2}\big\|\tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big\|_Q^2 + \frac{1}{2\delta}\|f\circ\eta_k^{(\tau)}\|_Q^2\\ &\le \frac{\delta}{2c_K}R\big(\eta_k^{(\tau)}, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big) + \frac{1}{2\delta}\|f\circ\eta_k^{(\tau)}\|_Q^2,\end{aligned}$$

and also, using Lemma 2.11,

$$\begin{aligned}|\langle f, v\rangle_{\Omega_k^{(\tau)}}| &\le \frac{\delta}{2}\|v\|_{\Omega_k^{(\tau)}}^2 + \frac{1}{2\delta}\|f\|_{\Omega_k^{(\tau)}}^2\\ &\le \frac{\delta}{2c_{gK}}\Big(\|\varepsilon v\|_{\Omega_k^{(\tau)}}^2 + R\big(\eta_k^{(\tau)}, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big)\Big) + \frac{1}{2\delta}\|f\|_{\Omega_k^{(\tau)}}^2.\end{aligned}$$

Now if we choose δ small enough, e.g. $\delta := \min(c_K, c_{gK})/2$, all v- and η-dependent terms can be absorbed to get the lower bound

$$E(\eta) + \tau R\big(\eta_k, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big) + \tau\frac{\nu}{2}\|\varepsilon v\|^2_{\Omega_k^{(\tau)}} - \rho_f\tau\langle f, v\rangle_{\Omega_k^{(\tau)}} - \rho_s\tau\big\langle f\circ\eta_k^{(\tau)}, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big\rangle_Q$$
$$\geq E_{\min} - \tau\frac{\max(\rho_f,\rho_s)}{2\delta}\big(\|f\circ\eta_k^{(\tau)}\|_Q^2 + \|f\|^2_{\Omega_k^{(\tau)}}\big). \tag{2.4}$$

Thus a minimizing sequence $\tilde\eta_l, \tilde v_l$ exists and along that sequence, the energy and dissipation are bounded. So by coercivity of the energy we know that $\tilde\eta_l$ is bounded in $W^{2,q}(Q;\Omega)$ and using the Banach–Alaoglu theorem along with compact embeddings we may extract a subsequence (not relabeled) and a limit $\eta_{\min}$ for which

$$\begin{aligned}
\tilde\eta_l &\rightharpoonup \eta_{\min} && \text{in } W^{2,q}(Q;\Omega),\\
\tilde\eta_l &\to \eta_{\min} && \text{in } C^{1,\alpha^-}(Q;\Omega) \text{ for } 0<\alpha^-<\alpha := 1-n/q.
\end{aligned}$$

By Lemma 2.4 we know that $\eta_{\min}\in\mathcal{E}$. We also know that E and R are lower semicontinuous with respect to the above convergence by assumptions S3 and R1 respectively.

Next we pass to the limit with the fluid velocity. With no loss of generality, we may assume $\tilde v_l$ is a minimizer of the functional in (2.3) holding the deformation $\tilde\eta_l$ fixed. As the functional in (2.3) is convex with respect to the velocity, minimizing is *equivalent* to solving the appropriate Euler–Lagrange equation, in other words, it is equivalent to finding a weak solution to the following classical Stokes boundary value problem:

$$\begin{cases}
-\nu\Delta\tilde v_l + \nabla p = \rho_f f & \text{in } \Omega_k^{(\tau)},\\
\operatorname{div}\tilde v_l = 0 & \text{in } \Omega_k^{(\tau)},\\
\tilde v_l = g_l := \frac{(\tilde\eta_l - \eta_k^{(\tau)})\circ(\eta_k^{(\tau)})^{-1}}{\tau} & \text{in } \partial\Omega_k^{(\tau)}\cap\partial\eta_k^{(\tau)}(Q),\\
\tilde v_l = 0 & \text{in } \partial\Omega.
\end{cases}$$

Now since $\eta_k^{(\tau)}$ is a fixed diffeomorphism, and $\tilde\eta_l$ converges uniformly, the boundary data g_l in this problem converges uniformly as well. Furthermore, the solution operator $L^2(\partial\Omega_k^{(\tau)};\mathbb{R}^n)\to W^{1,2}(\Omega;\mathbb{R}^n)$ associated with this boundary value problem is continuous, which implies the existence of a limit $v_{\min}\in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$ with $\tilde v_l\to v_{\min}$ in $W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$. Then by construction $(\eta_{\min}, v_{\min})$ satisfy the compatibility condition and since $\|\varepsilon v\|_{\Omega_k^{(\tau)}}$ is lower semicontinuous and all terms involving f are continuous, the pair (η, v) is indeed a minimizer to the problem.

Next let us derive the Euler–Lagrange equation. Let $(\eta_{k+1}^{(\tau)}, v_{k+1}^{(\tau)})$ be a minimizer and let $\phi\in C^\infty(Q;\mathbb{R}^n)$ and $\xi\in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n)$. We require the perturbation $(\eta_{k+1}^{(\tau)} + \varepsilon\phi, v_{k+1}^{(\tau)} + \varepsilon\xi/\tau)$ to also be admissible[10] for all small enough ε. From this we derive the conditions $\operatorname{div}\xi = 0$, $\xi|_{\partial\Omega} = 0$, $\phi_P = 0$ and for the coupling we get $\xi\circ\eta_k^{(\tau)} = \phi$ on M.

[10] The different scaling of ϕ and ξ/τ with respect to τ allows us to remove most occurrences of τ in the Euler–Lagrange equation. This does not matter as long as τ is fixed, but is the correct scaling in the limit $\tau\to 0$.

Now since we assume $\eta_{k+1}^{(\tau)} \notin \partial\mathcal{E}$, for small enough ε, we have $\eta_{k+1}^{(\tau)} + \varepsilon\phi \in \mathcal{E}$. Thus we are allowed to take the first variation with respect to $(\phi, \xi/\tau)$, which immediately results in the weak formulation. ∎

Now let us give some a priori estimates on the solutions (2.3). Here, it will be crucial that the approximants are constructed as minimizers of an appropriate functional.

Lemma 2.14 (Parabolic a priori estimates). *We have*

$$E(\eta_{k+1}^{(\tau)}) + \tau R\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\big) + \tau\nu\|\varepsilon v_{k+1}^{(\tau)}\|^2_{\Omega_k^{(\tau)}} \\ \leq E(\eta_k^{(\tau)}) + \tau\rho_f\langle f, v_{k+1}^{(\tau)}\rangle_{\Omega_k} + \tau\rho_s\big\langle f\circ\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\big\rangle_Q.$$

Furthermore, let E_0 be such that $E_0 > E(\eta_0)$. Then there exists a time $T_{E_0} > 0$, depending only on E_0, the difference $E_0 - E(\eta_0)$ and $\|f\|_{L^\infty(\Omega)}$, such that for all $\tau > 0$ and all $N \in \mathbb{N}$ with $N\tau \leq T_{E_0}$ we have

$$E(\eta_N^{(\tau)}) + \frac{\tau}{2}\sum_{k=1}^{N}\Big[\frac{\nu}{2}\|\varepsilon v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + R\big(\eta_{k-1}^{(\tau)}, \tfrac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\big)\Big] \leq E_0.$$

Proof. As before, for fixed k, we may compare the value of the cost functional in (2.3) at the minimizer with its value for the pair $(\eta_k^{(\tau)}, 0)$. As $R(\eta_k^{(\tau)}, 0) = 0$ and the terms involving v vanish for $v = 0$, the comparison yields the first statement.

Now we proceed by induction on N. Assume that $E(\eta_{N-1}^{(\tau)}) \leq E_0$ and let c_{gK} be the Korn constant corresponding to E_0 from Lemma 2.11. Using (2.4) again, we end up with

$$E(\eta_{k+1}^{(\tau)}) + \frac{\tau}{2}R\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\big) + \tau\frac{\nu}{4}\|\varepsilon v_k^{(\tau)}\|^2_{\Omega_k^{(\tau)}} \\ \leq E(\eta_k^{(\tau)}) + \tau\frac{\max(\rho_f, \rho_s)}{2\delta}\|f\|^2_{L^\infty(\Omega)}, \tag{2.5}$$

where for all $k \in \{1, \ldots, N\}$ the δ only depends on c_{gK} and c_K, and thus only on E_0.

Hence we may sum this estimate over k, yielding

$$E(\eta_N^{(\tau)}) + \frac{\tau}{2}\sum_{l=1}^{N}\Big[\frac{\nu}{2}\|\varepsilon v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + R\big(\eta_{k-1}^{(\tau)}, \tfrac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\big)\Big] \\ \leq E(\eta_0) + N\tau\frac{\max(\rho_f, \rho_s)}{2\delta}\|f\|^2_{L^\infty(\Omega)} \leq E_0$$

assuming that $N\tau \leq T_{E_0}$ for $T_{E_0} > 0$ given by $\frac{T\max(\rho_f,\rho_s)}{2\delta}\|f\|^2_{L^\infty(\Omega)} = E_0 - E(\eta_0)$. But then $E(\eta_N^{(\tau)}) \leq E_0$ and we can continue the induction until $N\tau$ reaches T_{E_0}. ∎

Remark 2.15. Clearly, the maximal length of the time interval on which the a priori estimates are true depends on E_0 and could be optimized. However, later we prolong the solution to the maximal existence time, independently of this argument.

Step 2: Time-continuous approximations and their properties. Now we use these iterative solutions to construct approximations of the continuous problem. At this point, we will completely switch over to the global velocity u. We will also approximate the deformation η in two different ways: a piecewise constant approximation, which we will need to keep track of the fluid domain, and a piecewise affine approximation, which will give us the correct time derivative $\partial_t \eta$. To be more precise, we define:

Definition 2.16 (Discrete parabolic approximation). For some $E_0 > E(\eta_0)$ fix $T_{E_0} > 0$ as given by Lemma 2.14. We define the *piecewise constant τ-approximation* as

$$\begin{aligned}
\eta^{(\tau)}(t,x) &:= \eta^{(\tau)}_{k+1}(x) && \text{for } t \in [\tau k, \tau(k+1)),\ x \in Q,\\
\underline{\eta}^{(\tau)}(t,x) &:= \eta^{(\tau)}_{k}(x) && \text{for } t \in [\tau k, \tau(k+1)),\ x \in Q,\\
u^{(\tau)}(t,y) &:= \begin{cases} v^{(\tau)}_k(y) \\ \frac{(\eta^{(\tau)}_{k+1}-\eta^{(\tau)}_k)\circ(\eta^{(\tau)}_k)^{-1}(y)}{\tau} \end{cases} && \begin{array}{l}\text{for } t \in [\tau k, \tau(k+1)),\ y \in \Omega^{(\tau)}_k,\\ \text{for } t \in [\tau k, \tau(k+1)),\ y \in \eta_k(\bar{Q}),\end{array}\\
\Omega^{(\tau)}(t) &:= \Omega^{(\tau)}_k && \text{for } t \in [\tau k, \tau(k+1)),
\end{aligned}$$

where $(\eta^{(\tau)}_k, v^{(\tau)}_k)$ is the iterative solution for time step τ. We also define the *piecewise affine approximation* for η as

$$\tilde{\eta}^{(\tau)}(t,\cdot) := ((k+1) - t/\tau)\eta^{(\tau)}_k - (t/\tau - k)\eta^{(\tau)}_{k+1} \quad \text{for } t \in [\tau k, \tau(k+1)), x \in Q.$$

Note that $\tilde{\eta}^{(\tau)}$ is Lipschitz continuous in time, $\tilde{\eta}^{(\tau)}(k\tau) = \eta^{(\tau)}(k\tau)$ for all $k \in \{0,\dots,N\}$ and

$$\partial_t \tilde{\eta}^{(\tau)}(t) = \frac{\eta^{(\tau)}_{k+1}-\eta^{(\tau)}_k}{\tau} = u^{(\tau)}(t) \circ \underline{\eta}^{(\tau)}(t) \quad \text{for all } t \in (\tau k, \tau(k+1)).$$

Lemma 2.17 (Basic a priori estimates). *For any fixed E_0 and the resulting time T_{E_0} from Lemma 2.14 there exists a constant C independent of τ such that*

$$E(\eta^{(\tau)}(t)) + \int_0^t \left[R(\underline{\eta}^{(\tau)}(t), \partial_t \tilde{\eta}^{(\tau)}(t)) + \frac{\nu}{2}\|\varepsilon u^{(\tau)}\|^2_{\Omega^{(\tau)}(t)} \right] dt \le E_0$$

for all $t \in [0, T_{E_0}]$, and

$$\sup_{t\in[0,T_{E_0}]} \|\eta^{(\tau)}(t)\|_{W^{2,q}(Q)} \le C, \qquad \int_0^{T_{E_0}} \|\partial_t \tilde{\eta}^{(\tau)}\|^2_{W^{1,2}(Q)}\, dt \le C,$$
$$\int_0^T \|u^{(\tau)}\|^2_{W^{1,2}(\Omega)}\, dt \le C.$$

Proof. The first statement is a direct translation of Lemma 2.14 while the last three inequalities follow from this. In particular, since $E(\eta^{(\tau)}(t)) < E_0$ on any of its constancy intervals and thus on all of $[0, T_{E_0}]$, its supremum is bounded. Similarly the two integral inequalities follow from the boundedness of the dissipation combined with the Korn inequalities R3 and Lemma 2.11. ■

Lemma 2.18 (Energy and Hölder estimates). *For any E_0 and the resulting time T_{E_0} from Lemma 2.14, there exists a constant C independent of $\tau < 1$ such that we have the following estimates:*

(1) *For all $t \in [0, T_{E_0}]$,*

$$\|\eta^{(\tau)}(t) - \tilde{\eta}^{(\tau)}(t)\|_{W^{1,2}(Q)} \leq C\sqrt{\tau}.$$

(2) *$E(\eta^{(\tau)}(t))$ is nearly monotone, i.e. for any $t, t_0 \in [0, T_{E_0}]$ with $t - t_0 \geq \tau$ we have*

$$E(\eta^{(\tau)})(t) - E(\eta^{(\tau)})(t_0) \leq C(t - t_0).$$

(3) *$\eta^{(\tau)}(t)$ is nearly Hölder continuous in $W^{1,2}(Q)$, i.e. for any $t, t_0 \in [0, T_{E_0}]$ with $t - t_0 > \tau$ we have*

$$\|\eta^{(\tau)}(t) - \eta^{(\tau)}(t_0)\|_{W^{1,2}(Q)} \leq C\sqrt{t - t_0}.$$

Proof. Consider the lower bound on a single step given in (2.5). Singling out the dissipation of the solid material, and dropping some terms with compatible sign, we get by using the Korn inequality R3,

$$\begin{aligned} c_K \frac{1}{\tau}\|\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}\|_{W^{1,2}(Q)}^2 &\leq \tau R\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}}{\tau}\big) \\ &\leq 2\left(E(\eta_k^{(\tau)}) - E(\eta_{k+1}^{(\tau)}) + \tau \frac{\max(\rho_f, \rho_s)}{2\delta}\|f\|_\infty^2\right) \end{aligned}$$

Now as the energy is bounded uniformly from above and from below, we can derive

$$\|\eta_{k+1}^{(\tau)} - \eta_k^{(\tau)}\|_{W^{1,2}(Q)} \leq C\sqrt{\tau}$$

for some constant C depending only on E_0 and f. In particular, due to the definitions of $\tilde{\eta}^{(\tau)}(t)$ and $\eta^{(\tau)}(t)$, this implies (1).

Further, reordering the terms in a different way, we get

$$E(\eta_{k+1}^{(\tau)}) - E(\eta_k^{(\tau)}) \leq \tau \frac{\max(\rho_f, \rho_s)}{2\delta}\|f\|_\infty^2.$$

Now fix $T \geq t > t_0 \geq 0$ and let $M := \lfloor t/\tau \rfloor$ and $N := \lfloor t_0/\tau \rfloor$. Adding up the inequalities yields

$$E(\eta^{(\tau)}(t)) - E(\eta^{(\tau)}(t_0)) \leq \tau(M - N)\frac{\max(\rho_f, \rho_s)}{2\delta}\|f\|_\infty^2.$$

Now either $\tau \leq t - t_0 < 2\tau$, in which case $\tau(M - N) < 2\tau < 2(t - t_0)$, or $t - t_0 \geq 2\tau$ and thus $\tau(M - N) < (t - t_0) + \tau < \frac{3}{2}(t - t_0)$, so this estimate proves (2).[11]

[11] The lower bound on $t_0 - t$ is somewhat arbitrary and is only due to the jumps in the piecewise constant approximation. As we are generally interested in $\tau \to 0$ for fixed t, t_0, this will not be an issue.

Finally, we again use the first estimate and Hölder's inequality to add up the distances:

$$\begin{aligned}\|\eta^{(\tau)}(t)-\eta^{(\tau)}(t_0)\|_{W^{1,2}(Q)} &\le \sum_{k=N}^{M-1}\|\eta^{(\tau)}_{k+1}-\eta^{(\tau)}_k\|_{W^{1,2}(Q)}\\ &\le \sqrt{\sum_{k=N}^{M-1}\tau}\sqrt{\sum_{k=N}^{M-1}\frac{1}{\tau}\|\eta^{(\tau)}_{k+1}-\eta^{(\tau)}_k\|^2_{W^{1,2}(Q)}}\\ &\le c\sqrt{t-t_0}\sqrt{E(\eta^{(\tau)}(t_0))-E(\eta^{(\tau)}(t))+(t-t_0)\frac{\max(\rho_f,\rho_s)}{2\delta}\|f\|^2_\infty},\end{aligned}$$

which proves (3). ■

A direct consequence of the last estimate is that the solid cannot move much in a short time. In particular, this implies the following result on injectivity:

Corollary 2.19 (Short-time collision exclusion). *If $\eta_0 \in \mathcal{E}$ is injective (i.e. $\eta_0 \notin \partial\mathcal{E}$) then there exists $T_{\text{inj}} > 0$ such that for all τ small enough and all $t \in [0, T_{\text{inj}}]$, the deformations $\eta^{(\tau)}(t)$ and $\tilde{\eta}^{(\tau)}(t)$ are injective (i.e. not in $\partial\mathcal{E}$).*

Proof. If we choose T_{inj} small enough, then the near Hölder continuity from Lemma 2.18 implies that $\|\eta_0 - \eta^{(\tau)}_k\|_Q$ is uniformly small. In particular, we can choose it to be smaller than the constant γ_0 from Proposition 2.7, which then results in injectivity. ■

In the following, we take, for η_0, E_0 fixed, $T \le \min\{T_{\text{inj}}, T_{E_0}\}$. In this way, both the a priori estimates of Lemma 2.14 hold and we may assume injectivity.

Step 3: Existence and regularity of limits. As a next step, we will derive limiting objects as $\tau \to 0$ for the deformation and the global velocity, as well as their mode of convergence.

Proposition 2.20 (Convergence of the time-discrete scheme). *There exists a (not relabeled) subsequence $\tau \to 0$ and a limit*

$$\eta \in C^{1/2}([0,T]; W^{1,2}(Q;\mathbb{R}^n)) \cap C_w([0,T]; W^{2,q}(Q;\mathbb{R}^n)) \cap C^0([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n))$$

for $\alpha = 1 - n/q$ and $u \in L^2([0,T]; W^{1,2}(\Omega;\mathbb{R}^n))$, such that

$$\begin{aligned}&\tilde{\eta}^{(\tau)} \to \eta && \text{in } C^{(1/2)^-}([0,T]; W^{1,2}(Q;\mathbb{R}^n)),\\ &\eta^{(\tau)}, \underline{\eta}^{(\tau)}, \tilde{\eta}^{(\tau)} \rightharpoonup^* \eta && \text{in } L^\infty([0,T]; W^{2,q}(Q;\mathbb{R}^n)),\\ &u^{(\tau)} \rightharpoonup u && \text{in } L^2([0,T]; W^{1,2}(\Omega;\mathbb{R}^n)),\\ &\partial_t\tilde{\eta}^{(\tau)} \rightharpoonup \partial_t\eta && \text{in } L^2([0,T]; W^{1,2}(Q;\mathbb{R}^n)).\end{aligned}$$

Furthermore,

$$\partial_t\eta = u \circ \eta \quad \text{in } [0,T] \times Q$$

and $\eta^{(\tau)}$ converges uniformly to η in the following sense: For all $r > 0$, there exists a $\delta_r > 0$ such that for all $\tau < \delta_r$ and all $|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{\alpha+n}} \le r^{\alpha^-}$ we have

$$|\nabla(\eta^{(\tau)}(t_1,x_1) - \eta(t_2,x_2))| + |\eta^{(\tau)}(t_1,x_1) - \eta(t_2,x_2)| \le Cr^{\alpha^-},$$

for all $0 < \alpha^- < 1 - n/q$. *Finally,*

$$\begin{aligned} \tilde{\eta}^{(\tau)} &\to \eta && \text{in } C^0([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n)), \\ \eta^{(\tau)} &\to \eta,\ \underline{\eta}^{(\tau)} \to \eta && \text{in } L^\infty([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n)). \end{aligned} \tag{2.6}$$

Proof. We apply a weak version of the Arzelà–Ascoli theorem. Let $\{t_i\}_{i\in\mathbb{N}} \subset [0,T]$ be a countable dense set. By the upper bound on the energy and its coercivity, we have a uniform bound on $\|\eta^{(\tau)}(t)\|_{W^{2,q}(Q)}$; thus, by a diagonalization argument we can pick a subsequence of τ's (not relabeled) and limits $\eta(t_i)$ such that $\eta^{(\tau)}(t_i) \rightharpoonup \eta(t_i)$ in $W^{2,q}(Q;\mathbb{R}^n)$ and uniformly strongly in $W^{1,2}(Q;\mathbb{R}^n)$ for all $i \in \mathbb{N}$. Then by the convergence of norms, the Hölder continuity from Lemma 2.18 (3) carries over to

$$\|\eta(t_i) - \eta(t_j)\|_{W^{1,2}(Q)} \le C\sqrt{|t_i - t_j|} \quad \forall i, j \in \mathbb{N}.$$

This means that η has a unique extension onto $[0,T]$ in $C^{1/2}([0,T]; W^{1,2}(Q;\mathbb{R}^n))$.

By compactness arguments one gets

$$\tilde{\eta}^\tau \to \eta \in C^{1/2^-}([0,T]; W^{1,2}(Q;\mathbb{R}^n)).$$

Now pick $t \in [0,T]$ and a new sequence $(t_i)_{i\in\mathbb{N}} \subset [0,T]$ with $t_i \to t$. Due to the uniform $W^{2,q}(Q;\mathbb{R}^n)$ bounds resulting from the bounded energy, the sequence $(\eta(t_i))_{i\in\mathbb{N}}$ has a weakly converging subsequence in $W^{2,q}(Q;\mathbb{R}^n)$, which must converge to $\eta(t)$. As the original sequence $(t_i)_{i\in\mathbb{N}}$ was arbitrary, this means that η is weakly continuous in $W^{2,q}(Q;\mathbb{R}^n)$. By the same argument, $\eta^{(\tau)} \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^n)$ pointwise. By Lemma 2.18 (1), we know that $\tilde{\eta}^{(\tau)}(t)$ converges to the same limit as $\eta^{(\tau)}(t)$ in $W^{1,2}(Q;\mathbb{R}^n)$. Since $\tilde{\eta}^{(\tau)}(t)$ satisfies the same $W^{2,q}(Q;\mathbb{R}^n)$ bounds, we can then also prove weak $W^{2,q}(Q;\mathbb{R}^n)$ convergence by the same argument.

Next we interpolate in order to prove that $\nabla\eta$ is Hölder continuous in space-time, which implies $\eta \in C^0([0,T]; C^{1,\alpha}(Q;\mathbb{R}^n))$.[12] For that we take $(s_1, x_1), (s_2, x_2) \in [0,T] \times Q$ with $B_r \ni x_1, x_2$ (i.e. $|x_1 - x_2| \le 2r$) and $|s_1 - s_2| \le r^{2\alpha+n}$. We have

$$\begin{aligned} |\nabla\eta(s_1,x_1) - \nabla\eta(s_2,x_2)| &\le \left|\nabla\eta(s_1,x_1) - \fint_{B_r} \nabla\eta(s_1)\,dx\right| + \left|\fint_{B_r} \nabla(\eta(s_1) - \eta(s_2))\,dx\right| \\ &\quad + \left|\nabla\eta(s_2,x_2) - \fint_{B_r} \nabla\eta(s_2)\,dx\right| \\ &\le Cr^\alpha + |s_2 - s_2|\left|\fint_{s_1}^{s_2} \fint_{B_r} \partial_t\nabla\eta\,dx\,ds\right| \\ &\le Cr^\alpha + |s_2 - s_2|\left(\fint_{s_1}^{s_2} \fint_{B_r} |\partial_t\nabla\eta|^2\,dx\,ds\right)^{1/2} \\ &\le Cr^\alpha + C\left|\frac{s_1 - s_2}{r^n}\right|^{1/2} \le Cr^\alpha. \end{aligned} \tag{2.7}$$

[12] Note that due to the zero boundary values on P the continuity estimates for η follow directly from the gradient estimates.

By similar arguments, we can also prove that $\eta^{(\tau)} \to \eta$ and $\underline{\eta}^{(\tau)} \to \eta$. To this end, recall that $0 < \alpha^- < \alpha = 1 - n/q$. For $r > 0$ we may choose a finite subset $\{t_i\}_{i=1}^{m_r}$ such that for every $t \in [0, T]$ there exists a t_i such that $|t_i - t| \le r^{2\alpha^- + n}$. Using the Arzelà–Ascoli theorem we may choose a subsequence of τ's and a $\delta_r > 0$ such that for all $\tau \le \delta_r$,

$$\max_{i \in \{1,\dots,m_r\}} \|\eta^{(\tau)}(t_i) - \eta(t_i)\|_{C^{1,\alpha^-}(Q)} \le 1;$$

without loss of generality, we may assume that $\delta_\tau < r^{2\alpha^- + n}$.

Now for all $(s_1, x_1), (s_2, x_2) \in [0, T] \times Q$ with $|x_1 - x_2| \le r$, a ball B_r of radius r such that $x_1, x_2 \in B_r$ and $|s_1 - s_2| \le r^{2\alpha^- + n}$ there is a $t_i \in [s_1, s_2]$ and by an analogous calculation to (2.7) we obtain

$$|\nabla \eta^{(\tau)}(s_1, x_1) - \nabla \eta(s_2, x_2)| \le C r^\alpha,$$

by using the already obtained Hölder continuity of η.

Having the uniform convergence of $\eta^{(\tau)}$ at hand, we finally deduce the convergence of the global velocity field $u^{(\tau)}$. To do so, we use the uniform $L^2([0, T]; W^{1,2}(\Omega; \mathbb{R}^n))$ bound on $u^{(\tau)}$ derived through Lemmas 2.14 and 2.11 to extract another subsequence of τ's such that $u^{(\tau)}$ converges weakly in $L^2([0, T]; W^{1,2}(\Omega; \mathbb{R}^n))$ to some limit u. Further, the uniform $L^2([0, T]; W^{1,2}(Q; \mathbb{R}^n))$ bound on $\partial_t \tilde{\eta}^\tau$ implies that up to a subsequence, $\partial_t \tilde{\eta}^\tau$ converges weakly to $\partial_t \eta$ in $L^2([0, T]; W^{1,2}(Q; \mathbb{R}^n))$.

Directly from the definition, we see that $\partial_t \tilde{\eta}^\tau(t) = u^\tau(t) \circ \underline{\eta}^\tau(t)$ for almost all t. So in particular for all $\phi \in C_0^\infty([0, T] \times Q; \mathbb{R}^n)$,

$$\begin{aligned} \int_0^T \langle \partial_t \tilde{\eta}, \phi \rangle_Q \, dt \leftarrow \int_0^T \langle \partial_t \tilde{\eta}^{(\tau)}, \phi \rangle_Q \, dt &= \int_0^T \langle u^{(\tau)} \circ \underline{\eta}^{(\tau)}, \phi \rangle_Q \, dt \\ &= \int_0^T \langle u^{(\tau)} \circ \eta, \phi \rangle_Q \, dt + \int_0^T \langle u^{(\tau)} \circ \underline{\eta}^{(\tau)} - u^{(\tau)} \circ \eta, \phi \rangle_Q \, dt. \end{aligned}$$

Now the first integral on the last line converges to $\int_0^T \langle u \circ \eta, \phi \rangle_Q \, dt$ as η is a diffeomorphism, while the second vanishes in the limit by the following argument: Let $\pi_s(t, x) := s\underline{\eta}^{(\tau)}(t, x) + (1 - s)\eta(t, x))$. Then

$$\begin{aligned} |u^{(\tau)}(t, \underline{\eta}^{(\tau)}(t, x)) - u^{(\tau)}(t, \eta(t, x))|^2 &= \left| \int_0^1 \frac{\partial}{\partial s} u^{(\tau)}(t, \pi_s(t, x)) \, ds \right|^2 \\ &\le \int_0^1 |\nabla u^{(\tau)}(t, \pi_s(t, x))|^2 \, ds \sup_{t \in [0,T], x \in Q} |\underline{\eta}^{(\tau)}(t, x) - \eta(t, x)|^2. \end{aligned}$$

Now, as $\underline{\eta}^{(\tau)}(t)$ and $\eta(t)$ are both diffeomorphisms with lower bound on the determinant and uniformly close gradients, the linear interpolation π_s also has to be a diffeomorphism. So integrating the equation yields

$$\begin{aligned} &\int_0^T \int_Q |u^{(\tau)}(t, \underline{\eta}^{(\tau)}(t, x)) - u^{(\tau)}(t, \eta(t, x))|^2 \, dx \, dt \\ &\qquad \le c \int_0^T \int_\Omega |\nabla u^{(\tau)}|^2 \, dx \, dt \sup_{t \in [0,T], x \in Q} |\underline{\eta}^{(\tau)}(t, x) - \eta(t, x)|^2. \end{aligned}$$

Here the first term is uniformly bounded and the second converges to 0, by the uniform convergence of $\eta^{(\tau)}$ outlined above.

Thus we have $\partial_t \eta = u \circ \eta$ almost everywhere in Q. ■

Step 4: Convergence of the equation. Using the convergences we derived in Proposition 2.20, we proceed by showing that the discrete Euler–Lagrange equations from Proposition 2.13 converge to the equation satisfied by the weak solution. This is not a straightforward task, as we have to deal with coupled pairs of test functions with the coupling nonlinearly dependent on the deformation. We will deal with this issue by focusing on a global test function ξ on Ω from which we derive the test functions on the discrete level. In order to do so, we need to be able to approximate the test functions smoothly while also maintaining the coupling condition. This is done in Proposition 2.22.

For the approximation of test functions we make use of a Bogovskiĭ-type theorem.

Theorem 2.21 (Bogovskiĭ operator [14, Theorem 2.4]). *Let Ω be a bounded Lipschitz domain. Then there is a linear operator $\mathcal{B} : \{g \in C_0^\infty(\Omega) \mid \int_\Omega g\, dy = 0\} \to C_0^\infty(\Omega)$ such that*

$$\operatorname{div} \mathcal{B}(g) = g.$$

Moreover, for $k \in \{0, 1, 2, \dots\}$ and $a \in (1, \infty)$ the operator extends to Sobolev spaces as an operator $\mathcal{B} : \{g \in W_0^{k-1,a}(\Omega) \mid \int_\Omega g\, dy = 0\} \to W_0^{k,a}(\Omega)$ such that

$$\|\mathcal{B}(g)\|_{W_0^{k,a}(\Omega)} \le c\|g\|_{W_0^{k-1,a}(\Omega)},$$

where the constant just depends on k, a, n and Ω.

Next we state the approximation result. It is introduced in order to approximate test functions and later in Section 4 to extend the Aubin–Lions lemma to the variable domain setup. The proof is quite involved and for that reason put in the appendix (see Appendix A.2).

Proposition 2.22 (Approximation of test functions). *Fix a function*

$$\eta \in L^\infty([0,T]; \mathcal{E}) \cap W^{1,2}([0,T]; W^{1,2}(Q; \mathbb{R}^n)) \quad \text{with} \quad \sup_{t \in T} E(\eta(t)) < \infty$$

such that $\eta(t) \notin \partial\mathcal{E}$ for all $t \in [0,T]$. As before, set $\Omega(t) = \Omega \setminus \eta(t, Q)$. Let $\mathcal{T}_\eta$ be the set of admissible test functions, defined as

$$\mathcal{T}_\eta := \{(\phi, \xi) \in W^{1,2}([0,T]; W^{1,2}(Q; \mathbb{R}^n)) \times L^2([0,T]; W_0^{1,2}(\Omega; \mathbb{R}^n)) \mid \\ \phi = \xi \circ \eta \text{ on } [0,T] \times Q \text{ and } \operatorname{div} \xi(t) = 0 \text{ in } \Omega(t)\}.$$

Then the set

$$\tilde{\mathcal{T}}_\eta := \{(\phi, \xi) \in \mathcal{T}_\eta \mid \xi \in C^\infty([0,T]; C_0^\infty(\Omega; \mathbb{R}^n)), \operatorname{div} \xi(t, y) = 0 \\ \text{for all } t \in [0,T] \text{ and all } y \text{ with } \operatorname{dist}(y, \Omega(t)) < \varepsilon \text{ for some } \varepsilon > 0\}$$

is dense in $\mathcal{T}_\eta$ in the following sense:

For every ε sufficiently small there exists a linear map $(\phi,\xi)\mapsto(\phi_\varepsilon,\xi_\varepsilon)\in\tilde{\mathcal{T}}_\eta$ such that

$$\operatorname{div}(\xi_\varepsilon(t,y))=0 \quad \textit{for all } y\in\Omega \textit{ with } \operatorname{dist}(y,\Omega(t))\le\varepsilon.$$

Moreover, if $\xi\in L^b([0,T];W^{k,a}(\Omega))$, then for $k\in\mathbb{N}$, $a\in(1,\infty)$ and $b\in[1,\infty]$,

$$\xi_\varepsilon\to\xi \quad \textit{in } L^b([0,T];W^{k,a}(\Omega)) \textit{ as } \varepsilon\to 0.$$

If additionally $\eta\in L^b([0,T];W^{k,a}(Q;\mathbb{R}^n))$, with $a=2$, if $k\ge 3$, then

$$\phi_\varepsilon\to\phi \quad \textit{in } L^b([0,T];W^{k,a}(Q;\mathbb{R}^n))\cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n)).$$

Further, in case $\partial_t\xi\in L^2([0,T];W^{1,2}(\Omega))$, we have $\partial_t\xi_\varepsilon\to\partial_t\xi$ in $L^2([0,T];W^{1,2}(Q))$. If additionally $\xi\in L^\infty([0,T];W^{3,a}(\Omega))$ with $a>n$ and $\partial_t\xi\in L^2([0,T];W^{1,2}(\Omega))$, then $\partial_t\phi_\varepsilon\to\partial_t\phi$ in $L^2([0,T];W^{1,2}(Q))$.

Moreover, the following bounds are satisfied at every time instant where the right hand side is finite:

$$\begin{aligned}
\|\xi_\varepsilon(t)\|_{W^{1,2}(\Omega)} &\le c\|\xi(t)\|_{W^{1,2}(\Omega;\mathbb{R}^n))},\\
\|\xi_\varepsilon(t)-\xi(t)\|_{L^2(\Omega)} &\le c\varepsilon^{\frac{2}{n+2}}\|\xi(t)\|_{W^{1,2}(\Omega)},\\
\|\xi_\varepsilon(t)\|_{W^{k,a}(\Omega)} &\le c(\varepsilon)\|\xi(t)\|_{L^2(\Omega;\mathbb{R}^n)},\\
\|\phi_\varepsilon(t)\|_{W^{k,a}(Q)} &\le c\|\xi(t)\|_{C^k(\Omega)}\|\eta(t)\|_{W^{k,a}(Q)}\le c(\varepsilon)\|\xi(t)\|_{L^2(\Omega)}\|\eta(t)\|_{W^{k,a}(Q)},
\end{aligned}$$

where the constant c depends on the bounds of $\eta\in L^\infty([0,T];\mathcal{E})\cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n))$ and the lower bound on the Jacobian of η only. The constant $c(\varepsilon)$ depends additionally on ε.

Having Lemma 2.22 at hand, we now pass to the limit in the Euler–Lagrange equation.

Proposition 2.23 (Limit equation). *The limit pair (η,v) obtained in Proposition 2.20 satisfies*

$$\begin{aligned}
0=\int_0^T [&\langle DE(\eta(t)),\phi\rangle_Q+\langle D_2R(\eta(t),\partial_t\eta(t)),\phi\rangle_Q+\nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)}\\
&-\rho_f\langle f,\xi\rangle_{\Omega(t)}-\rho_s\langle f\circ\eta,\phi\rangle_Q]\,dt
\end{aligned}\tag{2.8}$$

for all pairs $\phi\in L^2([0,T];W^{2,q}(Q;\mathbb{R}^n))$, $\xi\in L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))$ which satisfy $\phi(t,\cdot)=\xi\circ\eta(t)$ on Q and $\operatorname{div}\xi(t)=0$ on $\Omega(t)$.

Proof. First, we use the Minty method to show $\langle DE(\eta^\tau(t)),\phi^\tau\rangle_Q\to\langle DE(\eta(t)),\phi\rangle_Q$. Fix $t\in[0,T]$ and pick $\psi\in C_0^\infty(Q;[0,1])$. Then the pair $((\eta^{(\tau)}-\eta)\psi,0)$ fulfills the coupling condition for the discrete Euler–Lagrange equation and we have

$$\begin{aligned}
&\langle DE(\eta^{(\tau)})-DE(\eta),(\eta^{(\tau)}-\eta)\phi\rangle\\
&\quad=-\langle DE(\eta),(\eta^{(\tau)}-\eta)\phi\rangle-\langle D_2R(\underline{\eta}^{(\tau)},\partial_t\tilde{\eta}^{(\tau)}),(\eta^{(\tau)}-\eta)\phi\rangle+\langle f\circ\underline{\eta}^\tau,(\eta^{(\tau)}-\eta)\phi\rangle.
\end{aligned}$$

As $\eta^{(\tau)}(t) \to \eta(t)$ weakly in $W^{2,q}(Q;\mathbb{R}^n)$ and strongly in $C^{1,\alpha^-}(Q;\mathbb{R}^n)$ and moreover $\partial_t \tilde{\eta}^{(\tau)}(t)$ is uniformly bounded in $L^2([0,T]\times Q;\mathbb{R}^n)$, all three terms on the right hand side converge to 0 when integrated in time and thus for almost all $t \in [0,T]$ by Proposition 2.20. Hence Assumption 1.7 (S6) implies the strong convergence of $\eta^{(\tau)}(t) \to \eta(t)$ in $W^{2,q}(Q;\mathbb{R}^n)$ for almost all $t \in [0,T]$.

By Lemma 2.22, it is enough to show the limit equation for $\xi \in C_0^\infty([0,T]\times\Omega;\mathbb{R}^n)$ which is divergence free on a slightly larger set than the fluid domain. Fix such a ξ. Then since $\eta^{(\tau)}$ converges uniformly to η, $\operatorname{div}\xi = 0$ on $\Omega^{(\tau)}(t)$ for all τ small enough.

Now we construct the matching $\phi^{(\tau)}(t,x) := \xi(t,\eta^{(\tau)}(t,x))$ and $\phi(t,x) := \xi(t,\eta(t,x))$. Then by Lemma A.2 and Proposition A.4, $\phi^{(\tau)} \in L^\infty([0,T];W^{2,q}(Q;\mathbb{R}^n))$ with uniform bounds. Thus by compactness and uniqueness of limits we get $\phi^{(\tau)}(t) \rightharpoonup \phi(t)$ in $W^{2,q}(\mathbb{R}^n)$.

As constructed, the pairs $(\phi^{(\tau)}(t), \xi(t))$ are admissible in the respective Euler–Lagrange equations from Proposition 2.13 and we have

$$
\begin{aligned}
0 = {}& \langle DE(\eta^{(\tau)}(t)), \phi^{(\tau)}(t)\rangle + \langle D_2R(\underline{\eta}^{(\tau)}(t), \partial_t\tilde{\eta}^{(\tau)}(t)), \phi^{(\tau)}(t)\rangle \\
&+ \nu\langle \varepsilon u^{(\tau)}(t), \nabla\xi(t)\rangle_{\Omega^{(\tau)}(t)} - \rho_f\langle f, \xi(t)\rangle_{\Omega^{(\tau)}(t)} - \rho_s\langle f\circ\underline{\eta}^{(\tau)}(t), \phi^{(\tau)}(t)\rangle_Q
\end{aligned}
$$

for all $t \in [0,T]$ and τ small enough.

Now we integrate this equation in time and check each of the terms for convergence. For the first term we note that by the strong convergence of $\eta^{(\tau)}$ in $W^{2,q}(Q;\mathbb{R}^n)$ and Assumption 1.7 (S5), $DE(\eta^{(\tau)}(t))$ converges strongly in $W^{-2,q}(Q;\mathbb{R}^n)$ for every fixed t. Since $\phi^{(\tau)}(t)$ converges weakly and both terms are uniformly bounded in their respective spaces, we get

$$
\int_0^T \langle DE(\eta^{(\tau)}(t)), \phi^{(\tau)}(t)\rangle\, dt \to \int_0^T \langle DE(\eta(t)), \phi(t)\rangle\, dt.
$$

For the next term we find by Proposition 2.20 and the continuity of R in Assumption 1.10 (R1) that $D_2R(\underline{\eta}^{(\tau)}, \partial_t\tilde{\eta}^{(\tau)})$ converges weakly in $L^2([0,T];W^{-1,2}(Q;\mathbb{R}^n))$ and $\phi^{(\tau)}$ converges strongly in $L^2([0,T];W^{1,2}(Q;\mathbb{R}^n))$, which implies that

$$
\int_0^T \langle D_2R(\underline{\eta}^{(\tau)}(t), \partial_t\tilde{\eta}^{(\tau)}(t)), \phi^{(\tau)}(t)\rangle\, dt \to \int_0^T \langle D_2R(\eta(t), \partial_t\eta(t)), \phi(t)\rangle\, dt.
$$

For the next terms, let us first deal with the variable domain by rewriting the terms using characteristic functions. By the uniform convergence of the boundary we have $\chi_{\Omega^{(\tau)}(t)} \to \chi_{\Omega(t)}$ in $L^2([0,T]\times\Omega)$ and we can thus conclude

$$
\begin{aligned}
\int_0^T \langle \nabla u^{(\tau)}(t), \nabla\xi(t)\rangle_{\Omega^{(\tau)}(t)}\, dt &= \int_0^T\int_\Omega \chi_{\Omega^{(\tau)}(t)} \nabla u^{(\tau)}(t) : \nabla\xi(t)\, dy\, dt \\
&\to \int_0^T\int_\Omega \chi_{\Omega(t)}\nabla u(t) : \nabla\xi(t)\, dy\, dt = \int_0^T \langle \nabla u^{(\tau)}(t), \nabla\xi(t)\rangle_{\Omega(\cdot)}\, dt
\end{aligned}
$$

as u converges weakly in $L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))$.

The same approach also works for the forces on the fluid, where the domain is the only variable depending on τ and thus

$$\int_0^T \rho_f \langle f, \xi(t) \rangle_{\Omega^{(\tau)}(t)} \, dt \to \int_0^T \rho_f \langle f, \xi(t) \rangle_{\Omega(t)} \, dt.$$

Finally, we have the forces acting on the solid. Here both sides converge uniformly:

$$\int_0^T \rho_s \langle f \circ \underline{\eta}^{(\tau)}(t), \phi^{(\tau)}(t) \rangle_Q \, dt \to \int_0^T \rho_s \langle f \circ \eta(t), \phi(t) \rangle_Q \, dt.$$

Collecting all the terms then concludes the proof. ■

Step 5: Construction of the pressure. Take an arbitrary $s \in (0,T)$. Since we have excluded collisions on $(0,T)$, we know that $\Omega(t)$ is a uniform Lipschitz domain with bounds in the sense of Corollary 2.9 for all $t \leq s$. Taking $\psi \in C_0^\infty(\Omega(t))$ such that $\int_{\Omega(t)} \psi \, dy = 0$, we can use the Bogovskiĭ operator $\mathcal{B}_t$ defined on $\Omega(t)$ via Theorem 2.21 to define

$$\tilde{P}(t)(\psi) = \nu \langle \varepsilon u, \varepsilon \mathcal{B}_t \psi \rangle_{\Omega(t)} - \rho_f \langle f, \mathcal{B}_t \psi \rangle_{\Omega(t)}.$$

This then gives the estimate

$$|\tilde{P}(t)(\psi)| \leq C \|\mathcal{B}_t\| \, \|\psi\|_{L^2(\Omega(t))},$$

where $\|\mathcal{B}_t\|$ is the operator norm of $\mathcal{B}_t : \{\psi \in L^2(\Omega(t)) \mid \int_{\Omega(t)} \psi \, dy = 0\} \to W^{1,2}(\Omega(t))$ which is bounded by the Lipschitz constant of $\Omega(t)$ by Theorem 2.21. Now since $\{\psi \in L^2(\Omega(t)) \mid \int_{\Omega(t)} \psi \, dy = 0\}$ is a Hilbert space we find a $\tilde{p}(t)$ in that space such that $\tilde{p}(t) \equiv \tilde{P}(t)$.

We can extend the operator to $L^2(\Omega(t))$ in the following way: Take $\varphi(t) \in C_0^\infty(\Omega(t))$ and $\tilde{\varphi}(t) \in C_0^\infty(\Omega \setminus \Omega(t))$ fixed such that $\int_\Omega \varphi(t) \, dy = \int_\Omega \tilde{\varphi}(t) \, dy = 1$ for all $t \in [0,s]$. Since the change of domain in time is uniformly continuous, we may assume further that $\varphi, \tilde{\varphi}$ are C^1 smooth in time. Next we define $\mathcal{B}$ to be the operator of Theorem 2.21 with respect to the full domain Ω.

By taking the fixed pair of test functions

$$\xi_0(t) := \mathcal{B}(\varphi(t) - \tilde{\varphi}(t)), \quad \phi_0(t,x) := \xi_0(t, \eta(t,x)),$$

we may define

$$\begin{aligned} \hat{p}(t,y) = \big(\langle DE(\eta(t)), \phi_0(t) \rangle_Q + \langle D_2 R(\eta(t), \partial_t \eta(t)), \phi_0(t) \rangle_Q + \nu \langle \varepsilon u(t), \varepsilon \xi_0(t) \rangle_{\Omega(t)} \\ - \rho_f \langle f(t), \xi_0(t) \rangle_{\Omega(t)} - \rho_s \langle f(t) \circ \eta(t), \phi_0(t) \rangle_Q \big) \varphi(t,y), \end{aligned}$$

which satisfies $\|\hat{p}\|_{L^2([0,s];L^\infty(\Omega(t)))} \leq C$ with C depending on the energy estimates only. But this allows us to introduce the pressure. For $\psi \in L^1(\Omega(t))$, we define $c_\psi(t) = \int_{\Omega(t)} \psi(t) \, dy$. Now, if $\psi \in L^2([0,T]; L^1(\Omega(t)))$ we find that $c_\psi \in L^2([0,s])$. Hence we may define

$$P(\psi) = \int_0^T \langle \tilde{p}, \psi - c_\psi \varphi \rangle \, dt + \int_0^T \int_{\Omega(t)} \hat{p} \, dy \, c_\psi \, dt.$$

Thus $p \in L^\infty(0,s;L^2(\Omega(t)) + L^2(0,s;L^\infty(\Omega(t))$ is well defined via that operator:

$$\int_0^T \langle \nabla p, \xi \rangle \, dt := P(\operatorname{div} \xi),$$

and satisfies the proposed regularity.

One can now check that it fulfills the right equations. For that it suffices to see that

$$\xi - \mathcal{B}_t(\operatorname{div}(\xi) - c_{\operatorname{div}(\xi)}\varphi) - c_{\operatorname{div}(\xi)}\mathcal{B}(\varphi - \tilde{\varphi}) = \xi - \mathcal{B}_t(\operatorname{div}(\xi) - c_{\operatorname{div}(\xi)}\varphi) - c_{\operatorname{div}(\xi)}\xi_0$$

is divergence free over $\Omega(t)$. Hence (2.1) is satisfied by (2.8) using the test function

$$\big(\phi - c_{\operatorname{div}(\xi)}\phi_0, \xi - \mathcal{B}_t(\operatorname{div}(\xi) - c_{\operatorname{div}(\xi)}\varphi) - c_{\operatorname{div}(\xi)}\xi_0\big).$$

This finally allows us to conclude the theorem:

Proof of Theorem 2.2. For any injective η_0 there is a short interval $[0,T]$ such that for all τ small enough all $\eta_k^{(\tau)}$ are injective according to Corollary 2.19. Passing to the limit in the sequence of the accordingly constructed $(\eta^{(\tau)}, v^{(\tau)})$'s we find, by Proposition 2.13, (η, v) that is a weak solution to the parabolic fluid-structure interaction problem.

Now let $[0, T_{\max})$ be a maximal interval on which a solution (η, v) constructed in this way exists. If $T_{\max} = \infty$ there is nothing to show. The same holds if $T_{\max} < \infty$ and $\liminf_{t\to T_{\max}} E(\eta(t)) = \infty$ or if a self-intersection is approached. Now assume that none of that is the case. Then there exists a sequence of times $t_i \nearrow T_{\max}$ such that $E(\eta(t_i))$ is bounded and there exists a limit, which we will denote $\eta(T_{\max})$.

Now take $E_0 := \liminf_{t\to T_{\max}} E(\eta(t)) \geq E(\eta(T_{\max}))$ due to lower semicontinuity. Following Lemmas 2.14 and 2.18, there exists a minimal time T such that any solution starting with energy below $2E_0$ stays below energy $3E_0$ and is Hölder continuous in time in $[0,T]$. Due to the convergence, we can pick t_i with $T_{\max} - t_i \leq T$ and $E(\eta(t_i)) \leq 2E_0$, which makes the solution Hölder continuous right until $T_{\max}$ and thus $\lim_{t\nearrow T_{\max}} \eta(t) = \eta(T_{\max})$. But then we can use the short-term existence to construct a solution starting from $\eta(T_{\max})$ and appending this to the previous solution yields a contradiction as $T_{\max}$ cannot then be maximal. ■

2.3. The example energy-dissipation pair

Let us now consider the prototypical example we stated in the introduction in the form of (1.6) and (1.7). In particular, we will prove that this energy-dissipation pair fulfills Assumptions 1.7 and 1.10. While doing so, we comment in more detail on the meaning of those assumptions and on how they come into play in the course of the construction. Effectively we will prove the following proposition.

Proposition 2.24. *The example energy and dissipation given in* (1.6) *and* (1.7) *fulfill Assumptions* 1.7 *and* 1.10 *respectively.*

Instead of proving the assumptions in ascending order or order of convenience, we will try to tackle them in the order as they appear in the proof of Theorem 2.2. Furthermore, we will roughly group them by some relevant subtopics.

The minimization problem (S1, S3–S4, R1–R2). We start with the definition of $\eta_{k+1}^{(\tau)}$ in the minimizing movements scheme in (2.3). In order to prove existence of minimizers, we need to invoke the direct method of the calculus of variations. In other words, we need to show compactness and lower semicontinuity, as well as a lower bound for the functional.

The last one seems to be directly stated in S1 together with the quadratic homogeneity in R2. Of course for our example energy, S1 immediately holds, as all terms are non-negative, and R2 is similarly obvious, as $\partial_t \eta$ occurs as a quadratic factor. There is however some hidden difficulty in finding a lower bound for the whole functional, which includes not only energy and dissipation, but also the force terms, which can indeed be negative. To counteract these, we actually use the proper quadratic growth of the dissipation.

Once a lower bound for our minimizing sequence is established, we need to consider compactness. Here the relevant topology for η is the weak $W^{2,q}(Q;\Omega)$ topology and the relevant assumption for compactness is coercivity, in the form of S4. As we have bounded the other terms in the functional from below without involving the energy E, the coercivity is obtained in the simplest way, as $\|\nabla^2\eta\|_{L^q(Q)}^q$ is part of the energy.

As for (weak) lower semicontinuity, we need to verify assumptions S3 and R1 for the example case. First, note that the highest order term $\|\nabla^2\eta\|_{L^q(Q)}^q$ in the energy is weakly lower semicontinuous as it is a convex function of the norm. Second, we find that $q > n$ allows us to pick another subsequence converging in $C^{1,\alpha}$ for some $\alpha < \frac{q-n}{q}$. This allows us to pass to the limit in the other terms.

Converting between Lagrangian and Eulerian setting (S2). Note that as long as we were only discussing the minimization over the solid, the specific choice of $W^{2,q}(Q;\mathbb{R}^n)$ as a space was unimportant and choosing different terms in the integrand might as well have led us to a different space. It however becomes important when adding in the fluid, since it is prescribed with respect to the Eulerian setting which is again determined by the solid deformation η. The key here is assumption S2. Not only does this result in physically reasonable injectivity (in conjunction with the Ciarlet–Nečas condition), but it also allows us to convert between Eulerian and Lagrangian quantities.

To prove this property we follow the ideas of [52] where a similar energy was studied. Define $f(x) := \det \nabla\eta$. If $E(\eta)$ is bounded, then f is bounded in $W^{1,q}(Q)$ and $C^\alpha(Q)$. Now for a fixed ϵ_0 assume that there is $x_0 \in Q$ with $f(x_0) = 2\epsilon_0$. Then

$$
\begin{aligned}
E(\eta) &\geq \int_{B_\delta(x_0)\cap Q} \frac{1}{f(x)^a}\,dx \geq \int_{B_\delta(x_0)\cap Q} \frac{1}{(f(x_0) + |f(x) - f(x_0)|)^a}\,dx \\
&\geq c\,\frac{\delta^n}{(2\epsilon_0 + C\delta^\alpha)^a}.
\end{aligned}
$$

However, if $a\alpha > n$, the right hand side can be arbitrarily large if ϵ_0 and δ are chosen small enough, which is a contradiction.

Uniform bounds (R3). It has long been known that there is a certain mismatch between physically reasonable and mathematically expedient dissipation functionals (see e.g. [3]).

Mathematicians would prefer the dissipation potential to be of the form $\|\partial_t \eta\|^2_{W^{1,2}(Q)}$ and $\|u\|^2_{W^{1,2}(\Omega)}$. This would then lead directly to $L^2([0,T]; W^{1,2}(Q;\mathbb{R}^n))$ and $L^2([0,T]; W^{1,2}(\Omega;\mathbb{R}^n))$ bounds respectively for $\partial_t \eta$ and v as well as their approximations. Instead, for physical reasons we have to consider $R(\eta, \partial_t \eta)$ and $\|\varepsilon v\|^2_{\Omega(t)}$, which are independent of the observer. Thus Korn-type inequalities are required to convert the bounds for the latter into bounds for the former.

As the Korn inequality for the fluid is the classic one and the added difficulties due to the changing domain are overcome by Lemma 2.11, we only need to focus on the solid. For our example, this inequality and thus R3 follows from the main theorems in [78, 81]. See also the discussion in [68], where these results are coupled with an energy similar to ours in the context of a thermoviscoelastic solid (but without a fluid).

Observe that these inequalities require a certain regularity of the deformation $\nabla \eta$ itself. In fact, we need the same properties that allow us to switch between Lagrangian and Eulerian settings, i.e. a uniform lower bound on $\det \nabla \eta$ and continuity of $\nabla \eta$, as otherwise there are known counterexamples for which the inequality fails.

Weak equations (S5, R4). Combined, the assumptions so far are enough to construct iterative minimizers and even to have a subsequence converge to a limit object (η, v) in space-time by weak compactness. It remains to show that these functions do satisfy a weak coupled PDE. This is where assumptions S5 and R4 come in. Both are two-part in nature, requiring both the existence of a derivative and some form of continuity. For the example both follow by direct calculation. Let us start with the dissipation, namely

$$\langle D_2 R(\eta, b), \phi\rangle = \int_Q 2(\nabla b^T \nabla \eta + \nabla \eta^T \nabla b) \cdot (\nabla \phi^T \nabla \eta + \nabla \eta^T \nabla \phi)\, dx.$$

Since we have $C^{1,\alpha}(Q;\mathbb{R}^n)$ bounds on $\nabla \eta$, the $L^2(Q)$ regularity of ∇b $(= \nabla \partial_t \eta)$ is enough to make sense of $D_2 R(\cdot, b)$ as an operator in $W^{-1,2}(Q;\mathbb{R}^n)$. Similarly, the uniform convergence in some Hölder space for $\nabla \eta$ is enough to give this derivative the required continuity with respect to both b and η.

The calculation for the energy is a bit more involved. Restricting ourselves to deformations η of finite energy and thus positive determinant, by a short calculation we get

$$\begin{aligned}\langle DE(\eta), \phi\rangle = \int_Q \Bigg[&\frac{1}{4}\mathcal{C}(\nabla \eta^T \nabla \eta - I) \cdot (\nabla \phi^T \nabla \eta + \nabla \eta^T \nabla \phi) \\ &- a\frac{\operatorname{cof} \nabla \eta}{(\det \nabla \eta)^{a+1}} \cdot \nabla \phi + |\nabla^2 \eta|^{q-2} \nabla^2 \eta \cdot \nabla^2 \phi \Bigg] dx,\end{aligned}$$

where the scalar products are to be understood over all tensorial dimensions.

Again in order to pass to the limit with the energy, we need to make use of the uniform Hölder continuity of $\nabla \eta$ to see that the first two terms in $DE(\eta)$ are well defined and continuous with respect to the corresponding convergence. Finally, the last term is well defined since $\eta \in W^{2,q}(Q;\mathbb{R}^n)$ uniformly, but to show that it is also continuous we need to show strong convergence using the convexity of the quantity.

Improved convergence (S6). As the usual compactness methods will only result in weak compactness, and S5 requires strong convergence, we need a way to improve upon this. For this we rely on an idea that is most commonly attributed to Minty. While it is certainly not true that our energy is convex, the critical, second order term in its derivative $DE(\eta)$ is monotone and this allows us to improve convergence as desired. Assume that as stated $\eta_l \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^n)$. Then after possibly extracting another subsequence with $\eta_l \to \eta$ in $C^{1,\alpha}(Q;\mathbb{R}^n)$, the first two terms of $DE(\eta_l)$ already converge to their respective limits (using the lower bound on $\det \nabla\eta$ given through S2). As a result, the stated conditions on convergence of $\langle DE(\eta_l) - DE(\eta), (\eta_l - \eta)\psi\rangle \to 0$ for all cutoffs $\psi \in C_0^\infty(Q;[0,1])$ are equivalent to those for

$$\langle |\nabla^2\eta_l|^{q-2}\nabla^2\eta_l - |\nabla^2\eta|^{q-2}\nabla^2\eta, \nabla^2((\eta_l - \eta)\psi)\rangle. \tag{2.9}$$

Here the cutoff complicates things slightly, but expanding the right hand side yields terms of lower order ($(\eta_l - \eta) \otimes \nabla^2\psi$ and $\nabla(\eta_l - \eta) \otimes \nabla\psi$) which already converge strongly to 0 and one term of second order, which leaves us with (2.9) where we can now let $\psi \to 1$ by approximation. Now $\eta \mapsto |\nabla^2\eta|^{q-2}\nabla^2\eta$ is a classical example of a monotone operator. Thus the term is bounded from below by 0 and its convergence to 0 implies strong convergence $\eta_l \to \eta$ in $W^{2,q}(Q;\mathbb{R}^n)$, because $(|a|^{q-2}a - |b|^{q-2}b)\cdot(a-b) \geq c|a-b|^q$ for $q \geq 2$ and $a, b \in \mathbb{R}^{n^3}$.

3. Minimizing movements for hyperbolic evolutions

In this section, we will introduce a general method for adding inertial effects to continuum mechanical problems. As can be seen below, the scheme we introduce is able to approximate hyperbolic PDEs via a (forced) gradient flow. We will demonstrate it for the solid evolution under study *including the inertia and dissipation* even though the dissipation is not needed for the construction of the approximations. Moreover, as we will see in the next section, the method turns out to be flexible enough to even apply to problems which are of a mixed Lagrangian/Eulerian type such as fluid-structure interaction. Also note that while this section can be read independently of the previous one, at some places we will use a similar reasoning, which will thus be abridged slightly.

In particular, we keep the notation from the previous section. Thus $\eta : Q \to \mathbb{R}^n$ with $\eta \in \mathcal{E}$ is the deformation of the solid specimen, and E and R are its elastic energy and dissipation. For simplicity we will also use the same set of assumptions (i.e. Assumptions 1.7 and 1.10, respectively), though many of them could be relaxed, as they are intended for interaction with the fluid. Furthermore, as the only relevant domain Q is kept fixed, we will suppress the dependence of the inner products and the resulting L^2 norms on Q.

The problem we thus want to solve is to find the deformation of the viscoelastic solid specimen moving inertially in space subject to an action of forces. In other words, we need to solve the balance of momentum (Newton's second law) that reads

$$\rho\partial_t^2\eta + D_2R(\eta, \partial_t\eta) + DE(\eta) = f \circ \eta \quad \text{in } [0,T] \times Q, \tag{3.1}$$

where $\rho = \rho_s$ is a constant density and f some external force, not necessarily conservative. In addition, we will require that $\eta \in \mathcal{E}$, which implies that it satisfies given Dirichlet boundary conditions on P. On the other parts of the boundary $\partial Q \setminus P$ we assume natural Neumann-type (free) boundary conditions that will result from minimization. Finally, we will add appropriate initial conditions to (3.1),

$$\eta(0) = \eta_0 \quad \text{and} \quad \partial_t \eta(0) = \eta^* \text{ in } Q. \tag{3.2}$$

As usual, we translate this into a notion of weak solution.

Definition 3.1 (Weak solution to the inertial problem for solids). We call a function $\eta \in L^\infty([0,T];\mathcal{E}) \cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n))$ such that $\partial_t \eta \in C_w^0([0,T];L^2((Q;\mathbb{R}^n))$ and $\eta(0) = \eta_0$ a *weak solution to the inertial problem of the viscoelastic solid* (3.1) *with initial conditions* (3.2) if

$$\int_0^T [\langle DE(\eta), \phi\rangle + \langle D_2 R(\eta, \partial_t \eta), \phi\rangle - \langle f \circ \eta, \phi\rangle - \rho\langle \partial_t \eta, \partial_t \phi\rangle]\, dt + \rho\langle \eta_*, \phi(0)\rangle_Q = 0$$

for all $\phi \in C^\infty([0,T];C^\infty(Q;\mathbb{R}^n))$ with $\phi|_{[0,T]\times P} = 0$ such that $\phi(T) = 0$.

Observe that we restrict the solution to the closed set $\mathcal{E}$ and thus will only work with injective deformations on Q. This will be of particular interest to us as this property is relevant for modeling fluid-structure interactions.

The main goal of this section will be to prove the following theorem.

Theorem 3.2 (Existence of solutions for solids). *Assume that Assumption* 1.7 (*with* $\Omega = \mathbb{R}^n$) *and Assumption* 1.10 *hold. Assume that* $\eta_0 \in \mathcal{E} \setminus \partial\mathcal{E}$, $E(\eta) < \infty$, $\eta_* \in L^2(Q;\mathbb{R}^n)$ *and* $f \in C^0([0,\infty) \times \mathbb{R}^n;\mathbb{R}^n)$. *Then there exists a weak solution to* (3.1) *according to Definition* 3.1 *on* $[0,T]$. *Furthermore,* $T > 0$ *can be chosen in such a way that* $T = \infty$ *or* $\eta(T) \in \partial\mathcal{E}$.

As described in the introduction, we will first solve what we will call the *time-delayed problem*:

$$\rho \tfrac{\partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h)}{h} = -DR_2(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)) - DE(\eta^{(h)}(t)) + f \circ \eta^{(h)}(t). \tag{3.3}$$

For any fixed h, (3.3) has the structure of a gradient flow, yet one with a nonlocality in time in the form of a time-delayed[13] term $\partial_t \eta^{(h)}(t-h)$. Now the important observation is that on the interval $[0,h]$, $\partial_t \eta^{(h)}(t-h)$ is not part of the solution but actually given through the initial data. Thus, on this interval, the problem can be solved using parabolic methods.

[13]Problems with a time delay have long been studied in continuum mechanics, usually in the form of a convolution in time with an integral kernel backwards in time. This is done to model memory-type effects of the material. At this point we would like to emphasize that our time-delayed equation is not modeling any physical behaviour but is simply used as an approximation of the actual problem.

But then, once we know the solution on $[0, h]$, we can use this as data for the problem on $[h, 2h]$ and iterate. To allow for an iteration process, we in particular need to know that the solution obtained from the previous step is admissible to play the role of data in the next step. This is guaranteed by proving a suitable energy inequality, *a key element of the proof*. In our case, for the time-delayed problem on $[0, h]$, the energy inequality will have the form

$$E(\eta^{(h)}(h)) + \frac{\rho}{2h}\int_0^h \|\partial_t \eta^{(h)}(t)\|^2\,dt + \int_0^h R(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\,dt \\ \leq E(\eta^{(h)}(0)) + \frac{\rho}{2h}\int_0^h \|\partial_t\eta^{(h)}(t-h)\|^2\,dt + \int_0^h \langle f\circ\eta^{(h)}, \partial_t\eta^{(h)}\rangle\,dt.$$

Let us elaborate on the terms in this inequality: On the right hand side, we have the potential energy E of the initial data, as well as the averaged kinetic energy $\frac{\rho}{2}\fint_0^h \|\partial_t\eta^{(h)}\|^2\,dt$ of the “previous step”. On the left hand side, we have the potential energy at the end of the step, as well as the averaged kinetic energy of the current step.

So not only have we bounded the initial data for the next step in terms of the initial data of the previous step, which allows for an iterative process, but also we have an estimate suitable to employ a telescope argument. The resulting uniform bounds for $\eta^{(h)}$ are independent of h, thus they allow us to deduce a priori estimates and, in turn, to pass to the limit $h \to 0$ in order to obtain a solution including inertia.

We will show the existence of weak solutions for the time-delayed problem in detail in Section 3.1 before proving Theorem 3.2 in Section 3.2.

3.1. *The time-delayed problem*

Throughout this subsection we will assume $h > 0$ is fixed. In order to solve the time-delayed problem, we first need to give a precise definition of its weak formulation.

Definition 3.3 (Weak solutions to the time-delayed equation for solids). Let $w \in L^2([0, h] \times Q; \mathbb{R}^n)$. We call $\eta \in L^\infty([0, h]; \mathcal{E}) \cap W^{1,2}([0, h]; W^{k_0,2}(Q; \mathbb{R}^n))$ a *weak solution to the time-delayed equation* (3.3) if $\eta(0) = \eta_0$ and

$$0 = \int_0^h \left[\langle DE_h(\eta), \phi\rangle + \langle D_2 R_h(\eta, \partial_t\eta), \phi\rangle - \langle f\circ\eta, \phi\rangle + \frac{\rho}{h}\langle \partial_t\eta - w, \phi\rangle\right] dt \tag{3.4}$$

for all $\phi \in C^\infty([0, h] \times Q; \mathbb{R}^n)$ with $\phi|_{[0,h]\times P} = 0$.

In this definition, w will play the role of the given data $\partial_t\eta(t-h)$. In addition, as we assume $h > 0$ is fixed, we will not highlight the h-dependence for any of the given quantities. Note that in Definition 3.3 we used the regularized forms of the energy and dissipation potentials that read

$$E_h(\eta) = E(\eta) + h^{a_0}\|\nabla^{k_0}\eta\|^2, \quad R_h(\eta, b) := R(\eta, b) + h\|\nabla^{k_0} b\|^2, \tag{3.5}$$

where we choose k_0 such that $k_0 - n/2 \geq 2 - n/q$, which implies that $W^{k_0,2}(Q;\mathbb{R}^n) \subset W^{2,q}(Q;\mathbb{R}^n)$ compactly. This actually has no direct impact on the existence of time-delayed solutions. Instead, it is a mollifying strategy which will allow us to test the Euler–Lagrange equation with $\partial_t \eta$ in order to obtain the previously mentioned energy inequality. A similar term will also help us with some regularity issues in the fluid-structure interaction problem later in Lemma 4.11. See Remark 3.10 for more discussion of the need of a regularizer.

Remark 3.4 (Properties of the regularizing energy and dissipation). For all $h > 0$, we find that E_h fulfills Assumption 1.7 with $W^{2,q}(Q;\mathbb{R}^n)$ replaced by $W^{k_0,2}(Q;\mathbb{R}^n)$, and R_h fulfills Assumption 1.10 with $W^{1,2}(Q;\mathbb{R}^n)$ replaced by $W^{k_0,2}(Q;\mathbb{R}^n)$, where we may replace R3 by

$$c(\|\nabla\lambda\|^2 + h\|\nabla^{k_0}\lambda\|^2) \leq R_h(\eta,\lambda) \leq C(\|\nabla\lambda\|^2 + h\|\nabla^{k_0}\lambda\|^2).$$

Now the bulk of this subsection will be devoted to proving the following theorem:

Theorem 3.5 (Existence of time-delayed solutions for solids). *Let $\eta_0 \in \mathcal{E} \cap W^{k_0,2}(Q;\mathbb{R}^n) \setminus \partial\mathcal{E}$, $w \in L^2([0,h]\times Q;\mathbb{R}^n)$ and $f \in C^0([0,h]\times Q;\mathbb{R}^n)$. Then there exists a weak solution to the time-delayed equation* (3.3) *in the sense of Definition* 3.3 *or there exists a solution on a shorter interval $[0,h_{\max}]$ such that $\eta(h_{\max}) \in \partial\mathcal{E}$.*[14]

Before we start, let us discuss how the time-delayed problem can still be seen as a type of parabolic gradient flow. In particular, let us compare it to the classical parabolic gradient flow problem at its root, which reads

$$DE_h(\eta(t)) = -D_2R_h(\eta(t),\partial_t\eta(t)) + f\circ\eta(t).$$

This problem consists of three components: energy, dissipation and forces. Our goal is to identify each of the two additional terms in the time-delayed problem with one of those three in order to show that we are still solving a similar problem.

Let us start with the delayed time derivative $\frac{\rho}{h}w(t) = \frac{\rho}{h}\partial_t\eta(t-h)$. As we work in the interval $[0,h]$, this is just a given function, not depending on $\eta|_{[0,h]}$. But then any such function plays the role of a force. In fact, in contrast to the actual forces we consider in the problem, it is a force given in the reference configuration and thus even easier to handle.

The other term, $\frac{\rho}{h}\partial_t\eta(t)$, can be seen as stemming from a quadratic dissipation potential $\hat{R}(\eta,b) := \hat{R}(b) := \frac{\rho}{2h}\|b\|^2$, so that $D_2\hat{R}(\eta(t),\partial_t\eta(t)) = \frac{\rho}{h}\partial_t\eta(t)$. By this reasoning, we claim that in general, if there is a method of solving the gradient flow problem, then the same method can solve the corresponding time-delayed problem.

Proof of Theorem 3.5. The construction is performed by what we call the *hyperbolic minimizing movements*. This is done by a time discretization of the interval $[0,h]$ by some

[14]Note that a posteriori (see Corollary 2.19) it will be shown that (in dependence on η_0) there is always a minimal time-length $h_{\min}$ for which it can be guaranteed that $\eta(t) \notin \partial\mathcal{E}$ for $t \in [0,h_{\min}]$.

fixed time-step size τ. Given $\eta_k^{(\tau)}$, we then recursively solve the following minimization problem to obtain $\eta_{k+1}^{(\tau)}$:

$$\min_{\eta\in\mathcal{E}} E_h(\eta) + \tau R_h\big(\eta_k^{(\tau)}, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big) - \tau\big\langle f_k^{(\tau)}\circ\eta_k, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big\rangle + \tau\frac{\rho}{2h}\Big\|\tfrac{\eta-\eta_k^{(\tau)}}{\tau} - w_k^{(\tau)}\Big\|^2 \tag{3.6}$$

where $w_k^{(\tau)} = \fint_{k\tau}^{(k+1)\tau} w\,dt \in L^2(Q;\mathbb{R}^n)$ and $f_k^{(\tau)} = \fint_{k\tau}^{(k+1)\tau} f\,dt \in L^2(Q;\mathbb{R}^n)$ are time averages.

Note that (3.6) is not quite in the form suggested by the previous discussion. Instead we deliberately wrote the last term as a quadratic difference, to give the problem a bit more structure. Note that when the last term is expanded, these two approaches only differ by a constant, which has no effect on the minimization.

Now using the coercivity of E in a way similar to, but easier than in the proof of Proposition 2.13, we find that a (possibly nonunique) minimizer exists and a short calculation shows that it satisfies (assuming that $\eta_{k+1}^{(\tau)} \notin \partial\mathcal{E}$) the Euler–Lagrange equation

$$\begin{aligned} 0 = \langle DE_h(\eta_{k+1}^{(\tau)}),\phi\rangle + \big\langle D_2R_h\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big),\phi\big\rangle - \langle f_k^{(\tau)}\circ\eta_k^{(\tau)},\phi\rangle \\ + \frac{\rho}{2h}\big\langle \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau} - w_k^{(\tau)},\phi\big\rangle \end{aligned} \tag{3.7}$$

for all $\phi \in W^{2,q}(Q;\mathbb{R}^n)$ with $\phi|_P = 0$.

Next we follow in the steps of Lemma 2.14 (see Remark 1.5 for a discussion of some interesting differences) and derive a simple initial energy estimate by comparing the value of the functional in (3.6) at the minimizer $\eta_{k+1}^{(\tau)}$ with its value at $\eta_k^{(\tau)}$:

$$\begin{aligned} E_h(\eta_{k+1}^{(\tau)}) + \tau R_h\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big) - \tau\big\langle f_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big\rangle \\ + \tau\frac{\rho}{2h}\Big\|\tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau} - w_k^{(\tau)}\Big\|^2 \le E_h(\eta_k^{(\tau)}) + \tau\frac{\rho}{2h}\|w_k^{(\tau)}\|^2. \end{aligned} \tag{3.8}$$

This estimate can be summed so that, using the triangle and the weighted Young inequality, we can derive, for any N such that $\tau N \le h$,

$$\begin{aligned} E_h(\eta_N) + \sum_{k=0}^{N-1} \tau\Big[R_h\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big) + c\Big\|\tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\Big\|^2\Big] \\ \le E_h(\eta_0) + C\int_0^h [\|w\|^2 + \|f\|^2]\,dt \end{aligned}$$

for some $C, c > 0$ depending on h but independent of τ. Here we have used Jensen's inequality to show

$$\tau\sum_{k=0}^{N-1}\|w_k^{(\tau)}\|^2 = \tau\sum_{k=0}^{N-1}\Big\|\fint_{k\tau}^{(k+1)\tau} w\,dt\Big\|^2 \le \tau\sum_{k=0}^{N-1}\fint_{k\tau}^{(k+1)\tau}\|w\|^2\,dt = \int_0^{N\tau}\|w\|^2\,dt$$

and a similar estimate for f. In particular, this also allows us to apply Proposition 2.7 to conclude that $\eta_k^{(\tau)}$ is always injective and the Euler–Lagrange equation is well defined.

If we now define the piecewise constant and piecewise affine approximations

$$\begin{aligned}
\eta^{(\tau)}(t) &= \eta^{(\tau)}_{k+1} && \text{for } k\tau \le t < (k+1)\tau,\\
\underline{\eta}^{(\tau)}(t) &= \eta^{(\tau)}_{k} && \text{for } k\tau \le t < (k+1)\tau,\\
\tilde{\eta}^{(\tau)}(t) &= \big(\tfrac{t}{\tau} - k\big)\eta^{(\tau)}_{k+1} + \big(k+1-\tfrac{t}{\tau}\big)\eta^{(\tau)}_{k} && \text{for } k\tau \le t < (k+1)\tau,
\end{aligned}$$

where in particular

$$\partial_t \tilde{\eta}^{(\tau)}(t) = \frac{\eta^{(\tau)}_{k+1} - \eta^{(\tau)}_{k}}{\tau} \quad \text{for } k\tau < t < (k+1)\tau,$$

our energy estimate turns into a uniform (in τ and t) bound on $E_h(\eta^{(\tau)}(t))$, as well as a uniform (in τ) bound on $\int_0^h [R_h(\underline{\eta}^{(\tau)}, \partial_t \tilde{\eta}^{(\tau)}) + c\|\partial_t \tilde{\eta}^{(\tau)}\|^2]\,dt$. Now using the properties of energy and dissipation from our assumptions, this gives a uniform bound in $L^\infty([0,h]; W^{k_0,2}(Q;\mathbb{R}^n))$ on $\eta^{(\tau)}$, $\underline{\eta}^{(\tau)}$ and $\tilde{\eta}^{(\tau)}$ as well as a uniform $L^2([0,h]; W^{k_0,2}(Q;\mathbb{R}^n))$ bound on $\partial_t \tilde{\eta}^{(\tau)}$.

Analogously to Proposition 2.20, we may extract a converging subsequence and a single limit $\eta \in W^{1,2}([0,T]; W^{k_0,2}(Q)) \cap C^0([0,T]; C^{1,\alpha}(Q))$. In particular, we get

$$\begin{aligned}
\tilde{\eta}^{(\tau)} &\rightharpoonup \eta && \text{in } W^{1,2}([0,T]; W^{k_0,2}(Q;\mathbb{R}^n)),\\
\underline{\eta}^{(\tau)}, \eta^{(\tau)} &\rightharpoonup^* \eta && \text{in } L^\infty([0,T]; W^{k_0,2}(Q;\mathbb{R}^n)),\\
\tilde{\eta}^{(\tau)} &\to \eta && \text{in } L^\infty([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n)),\\
\underline{\eta}^{(\tau)}, \eta^{(\tau)} &\to \eta && \text{in } L^\infty([0,T]; C^{1,\alpha^-}(Q;\mathbb{R}^n)),
\end{aligned}$$

for all $0 < \alpha^- < \alpha := 1 - n/q$.

This is already enough to pass to the limit in all the terms in the Euler–Lagrange equation (3.7); note that due to the added regularizing terms we use the strong convergence and the linearity in the highest gradient to pass to the limit in $DE(\eta^{(\tau)})$. ■

Time-delayed energy inequality. In the proof of Theorem 3.5 we already gave an initial, somewhat crude energy estimate on the discrete level. Now that we have a solution of the time-delayed equation, we can give a much stronger, "physical" energy inequality, which will turn out to be crucial in what follows.

Lemma 3.6 (Time-delayed energy inequality for the solid). *Let the deformation η in $L^\infty([0,h]; \mathcal{E}) \cap W^{1,2}([0,h] \times Q; \mathbb{R}^n)$ be a weak solution to the time-delayed equation in the sense of Definition 3.3. Then for all $t \in [0,h]$, we have*

$$\begin{aligned}
E_h(\eta(t)) + \frac{\rho}{2h}\int_0^t \|\partial_t \eta\|^2\,dt + \int_0^t 2R_h(\eta, \partial_t\eta)\,dt&\\
\le E_h(\eta_0) + \frac{\rho}{2h}\int_0^t \|w\|^2\,dt + \int_0^t \langle f\circ\eta, \partial_t\eta\rangle\,dt&.
\end{aligned}$$

Proof. We use $\chi_{[0,t]}\partial_t\eta$ as a test function in the weak equation.[15] From this we get

$$\begin{aligned}0 &= \int_0^t \left[\langle DE_h(\eta), \partial_t\eta\rangle + \langle D_2R_h(\eta,\partial_t\eta), \partial_t\eta\rangle - \langle f\circ\eta, \partial_t\eta\rangle + \frac{\rho}{h}\langle \partial_t\eta - w, \partial_t\eta\rangle \right] dt \\ &= E_h(\eta(t)) - E_h(\eta(0)) + \int_0^t \left[2R_h(\eta,\partial_t\eta) - \langle f\circ\eta, \partial_t\eta\rangle + \frac{\rho}{h}\langle \partial_t\eta - w, \partial_t\eta\rangle \right] dt,\end{aligned}$$

where in particular we have used the fact that $\langle D_2R_h(\eta,\partial_t\eta), \partial_t\eta\rangle = 2R_h(\eta,\partial_t\eta)$ by the quadratic nature of R_h. Finally, we use Young's inequality on the last term in the form of

$$\begin{aligned}\langle \partial_t\eta - w, \partial_t\eta\rangle &= \|\partial_t\eta\|^2 - \langle w, \partial_t\eta\rangle \geq \|\partial_t\eta\|^2 - \frac{\|\partial_t\eta\|^2}{2} - \frac{\|w\|^2}{2} \\ &= \frac{\|\partial_t\eta\|^2}{2} - \frac{\|w\|^2}{2}.\end{aligned}$$

Reordering the terms then closes the proof. ■

3.2. *Proof of Theorem 3.2*

We will start the proof by directly using its two key ingredients, the two results from the previous section. First we iteratively use the existence of time-delayed solutions on short intervals $[0,h]$ to construct a time-delayed solution on the longer interval $[0,T]$.

Step 1: Iterated time-delayed solutions and energy estimates. For fixed h we start with given initial deformation $\eta_0 \in \mathcal{E}$ and we use the initial velocity as a constant right hand side $w_0(t) = \eta_*$ for $t \in [0,h]$. This allows us to find $\tilde\eta_1$ as a solution of the time-delayed problem. Then we iterate the constructions, i.e. given $\eta_l \in \mathcal{E}$ and $w_l \in L^2([0,h]\times Q;\mathbb{R}^n)$, we find a solution $\tilde\eta_{l+1} \in L^\infty([0,h];\mathcal{E}) \cap W^{1,2}([0,h];W^{k_0,2}(Q;\mathbb{R}^n))$ to the time-delayed equation using Theorem 3.5. We then set $\eta_{l+1} = \tilde\eta_{l+1}(h)$ and $w_{l+1} = \tilde\eta_{l+1}$ as data for the next step for which they are admissible by Lemma 3.6.

From these ingredients we construct $\eta^{(h)} : [0,T]\times Q \to \mathbb{R}^n$ using

$$\eta^{(h)}(t,x) := \tilde\eta_{l+1}(t-hl) \quad \text{for } hl \leq t \leq h(l+1).$$

Directly from the definition we see that $\eta^{(h)}$ fulfills

$$\begin{aligned}0 = \int_0^T \Big[&\langle DE_h(\eta^{(h)}(t)), \phi\rangle + \langle D_2R_h(\eta^{(h)}(t), \partial_t\eta^{(h)}(t)), \phi\rangle \\ &- \langle f\circ\eta^{(h)}(t), \phi\rangle + \frac{\rho}{h}\langle \partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h), \phi\rangle \Big] dt \end{aligned} \tag{3.9}$$

[15] Note that this is the point where we rely on R_h, since to test $DE(\eta)$, we need $\phi \in L^2([0,T];W^{2,q}(Q;\mathbb{R}^n))$, but bounding $R(\eta,\partial_t\eta)$ only gives us an $L^2([0,T];W^{1,2}(Q;\mathbb{R}^n))$ bound. See also Remark 3.10.

for all $\phi \in C^\infty([0,h] \times Q; \mathbb{R}^n)$ with $\phi|_{[0,h]\times P} = 0$. Furthermore, exploiting the energy inequality (Lemma 3.6) yields

$$
\begin{aligned}
E_h(\eta^{(h)}((l+1)h)) &+ \frac{\rho}{2} \fint_{lh}^{(l+1)h} \|\partial_t \eta^{(h)}\|^2 \, dt + \int_{lh}^{(l+1)h} 2R_h(\eta^{(h)}, \partial_t \eta^{(h)}) \, dt \\
&\leq E_h(\eta^{(h)}(lh)) + \frac{\rho}{2} \fint_{(l-1)h}^{lh} \|\partial_t \eta^{(h)}\|^2 \, dt + \int_{lh}^{(l+1)h} \langle f \circ \eta, \partial_t \eta^{(h)} \rangle \, dt.
\end{aligned}
$$

Taking $t \in [lh, (l+1)h]$, after summing the above over $1, \ldots, l$ and adding the energy inequality for $\tilde{\eta}^{l+1}$ from Lemma 3.6 we find the following crucial estimate:

$$
\begin{aligned}
(E) := E_h(\eta^{(h)}(t)) &+ \frac{\rho}{2} \fint_{t-h}^{t} \|\partial_t \eta^{(h)}\|^2 \, ds + \int_0^t 2R(\eta^{(h)}, \partial_t \eta^{(h)}) \, ds \\
&\leq E_h(\eta_0) + \frac{\rho}{2} \|\eta_*\|^2 + \int_0^t \langle f \circ \eta, \partial_t \eta^{(h)} \rangle \, ds
\end{aligned}
\tag{3.10}
$$

for all $t \in [0, T]$. Now, as before, we need to estimate the force term using Young's inequality. This gives

$$
(E) \leq C_0 + C_1 \frac{T}{\delta} + \frac{\delta}{2} \int_0^T \|\partial_t \eta^{(h)}\|^2 \, ds
$$

for some constants C_0, C_1 resulting from the given data and independent of h. Dropping the positive terms involving E and R_h on the left hand side, multiplying by h and adding up implies

$$
\begin{aligned}
\frac{\rho}{2} \int_0^T \|\partial_t \eta^{(h)}\|^2 \, ds &= \sum_{l=0}^{N} \frac{\rho}{2} \int_{lh}^{(l+1)h} \|\partial_t \eta^{(h)}\|^2 \, ds \\
&\leq hN \left(C_0 + C_1 \frac{T}{\delta} + \frac{\delta}{2} \int_0^T \|\partial_t \eta^{(h)}\|^2 \, ds \right)
\end{aligned}
$$

for $hN = T$.[16] Now choosing $\delta := \frac{\rho}{2T}$ allows us to absorb the integral on the right hand side in the left and we end up with a uniform estimate of the form

$$
\frac{\rho}{4} \int_0^T \|\partial_t \eta^{(h)}\|^2 \, ds \leq TC_0 + C_2' T^2,
$$

which also implies that $(E) \leq TC_0 + C_2' T^2$. Note that in contrast to the parabolic setup from the last section, up to this point there was no need to apply Korn's inequality. In particular, as we used the inertial term to estimate the force term, we obtain a uniform bound on the energy without exploiting the dissipative terms, i.e. we already know that $\sup_{t \in [0,T]} E(\eta^h(t)) \leq TC_0 + C_2' T^2$. Now, using this estimate, we may apply Lemma 2.11

[16] There is no need to assume that T is a multiple of h, but we will do so for the sake of simplification.

without restrictions on the final time T to find that

$$\sup_{t\in[h,T]}\left(\fint_{t-h}^{h}\|\partial_t\eta^{(h)}\|^2\,ds + E(\eta^{(h)}(t)) + h^{a_0}\|\nabla^{k_0}\eta^{(h)}(t)\|^2\right)\le C,$$
$$\int_0^T [\|\partial_t\eta^{(h)}\|^2_{W^{1,2}(Q)} + h\|\partial_t\eta^{(h)}\|^2_{W^{k_0,2}(Q)}]\,ds\le C, \tag{3.11}$$

with constant $C = C(T)$ independent of h. Moreover, it allows us to conclude that $\eta^{(h)}(t)$ is always injective by Proposition 2.7.

By the identical arguments to the ones used in the proof of Lemma 2.20, we can now choose a subsequence which converges to a limit function $\eta\in C_w([0,T];W^{2,q}(Q;\mathbb{R}^n))\cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n))\cap C^0([0,T];C^{1,\alpha}(Q;\mathbb{R}^n))$. In particular, we obtain

$$\begin{aligned}
\eta^{(h)} &\rightharpoonup \eta && \text{in } W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n)),\\
\eta^{(h)} &\rightharpoonup^* \eta && \text{in } L^\infty([0,T];W^{2,q}(Q;\mathbb{R}^n)),\\
\eta^{(h)} &\to \eta && \text{in } C^0([0,T];C^{1,\alpha^-}(Q;\mathbb{R}^n)),
\end{aligned}$$

for all $0<\alpha^-<\alpha := 1-n/q$. Moreover, the weak lower semicontinuity implies that

$$\sup_{t\in[0,T]}\left(\|\partial_t\eta(t)\|^2 + E(\eta(t))\right)\le C \quad\text{and}\quad \int_0^T\|\partial_t\eta\|^2_{W^{1,2}(Q)}\,ds\le C \tag{3.12}$$

with the same constant as before.

Step 2: Improving convergence. Our final goal is to prove convergence of the weak equation (3.9), which is satisfied by the time-delayed approximation $\eta^{(h)}$, to the weak inertial equation (Definition 3.1), as this then implies that the limit η is a weak solution. The crucial term here is $DE(\eta^{(h)})$ which requires strong convergence of $\eta^{(h)}$ in $W^{2,q}(Q;\mathbb{R}^n)$. For this we want to use the Minty-type property of the energy, which requires convergence of the other terms in the equation. We achieve this convergence by the Aubin–Lions lemma, for which in turn we need another estimate on the discrete difference quotient.

Lemma 3.7 (Length h bounds (solid)). *Fix $T>0$. Then there exists a constant C, depending only on the initial data and T, such that for $k_0 > 2+\frac{(q-2)n}{2q}$ the following holds:*

$$\int_0^T\left\|\tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h}\right\|^2_{W^{-k_0,2}(Q)}dt\le C,$$

where $\partial_t\eta$ is extended by η_* for negative times.

Proof. Pick $\phi\in C_0^\infty(Q;\mathbb{R}^n)$. Then, using the time-delayed equation, we have

$$\begin{aligned}
\rho_s\left|\left\langle\tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h},\phi\right\rangle_Q\right| &\le |\langle DE(\eta^{(h)}(t)),\phi\rangle| + h^{a_0}|\langle\nabla^{k_0}\eta^{(h)},\nabla^{k_0}\phi\rangle|\\
&\quad + |\langle D_2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t)),\phi\rangle| + h|\langle\nabla^{k_0}\partial_t\eta^{(h)},\nabla^{k_0}\phi\rangle| + |\langle f(t),\phi\rangle_Q|\\
&\le (\|DE(\eta^{(h)}(t))\|_{W^{-2,q}(Q)} + h^{a_0}\|\nabla^{k_0}\eta^{(h)}(t)\|_Q + \|f\|_\infty)\|\phi\|_{W^{k_0,2}(Q)}\\
&\quad + (\|D_2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t))\|_{W^{-1,2}(Q)} + h\|\nabla^{k_0}\partial_t\eta^{(h)}(t)\|_Q)\|\phi\|_{W^{k_0,2}(Q)}.
\end{aligned}$$

Now for the first set of terms, we note that they are uniformly bounded by Assumption 1.7 (S5) and (3.11). For the second set, we note that the quadratic growth of $R(\eta,\cdot)$ in $W^{1,2}(Q;\mathbb{R}^n)$ implies the linear growth of $D_2 R$, thus equally (3.11) implies boundedness when integrated in time. ∎

Note that in the previous lemma the h by which time is shifted is the same h as the sequence index. Thus even though $\partial_t \eta^{(h)}$ is already continuous, we can only ever compare at fixed distances in the form of multiples of h. This is an unavoidable consequence of the way the estimate is obtained, using the equation. In particular, we cannot use the Aubin–Lions lemma directly to conclude that $\partial_t \eta^{(h)}$ converges strongly in $C([0,T];L^2(Q;\mathbb{R}^n))$. Instead we will prove the strong convergence for averages $\partial_t \eta^{(h)}$ over time intervals of length h, which turn out to be much more natural in this context and are in fact the same averages that also occur in the energy inequality.

Lemma 3.8 (Aubin–Lions (solid)). *Let $b^{(h)}(t) := \fint_t^{t+h} \partial_t \eta^{(h)} ds$. We have (for a subsequence $h \to 0$)*

$$b^{(h)} \to \partial_t \eta \quad \text{in } C^0([0,T];L^2(Q;\mathbb{R}^n)).$$

Proof. By the fundamental theorem of calculus we have

$$\partial_t b^{(h)} = \tfrac{\partial_t \eta^{(h)}(t+h) - \partial_t \eta^{(h)}(t)}{h}.$$

Now $b^{(h)}$ is uniformly bounded in $L^\infty([0,T];W^{1,2}(Q;\mathbb{R}^n))$ by the energy estimate and $\partial_t b^{(h)}$ is uniformly bounded in $L^2([0,T];W_0^{-k_0,2}(Q;\mathbb{R}^n))$ by the previous lemma. Thus we can apply the classical Aubin–Lions lemma [83], yielding the existence of a subsequence converging in $C^0([0,T];L^2(Q;\mathbb{R}^n))$. It remains to associate the limit function with $\partial_t \eta$. For that taking $h_0 > 0$ and $\phi \in C_0^\infty([h_0, T-h_0] \times Q)$, for all $h \in (0,h_0)$ we find, by the weak convergence $\partial_t \eta^{(h)} \rightharpoonup \partial_t \eta$ (and the Lebesgue point theorem), that

$$\begin{aligned}
\int_0^T \langle b^{(h)}, \phi \rangle_Q \, dt &= \fint_0^h \int_0^T \langle \partial_t \eta^{(h)}(t+s), \phi(t) \rangle_Q \, dt \, ds \\
&= \fint_0^h \int_0^T \langle \partial_t \eta^{(h)}(\tau), \phi(\tau - s) \rangle_Q \, d\tau \, ds \to \int_0^T \langle \partial_t \eta(\tau), \phi(\tau) \rangle_Q \, d\tau. \quad ∎
\end{aligned}$$

Finally, we will use a Minty-type argument to improve convergence.

Lemma 3.9 (Minty trick). *$\eta^{(h)}(t) \to \eta(t)$ strongly in $W^{2,q}(Q;\mathbb{R}^n)$ for a.a. $t \in [0,T]$.*

Proof. As in the last section we will rely on Assumption 1.7 (S6). Let $h_0 > 0$ and $h \in (0,h_0)$. Further take $\psi \in C_0^\infty((h_0, T-h_0) \times Q; \mathbb{R}^+)$ with $\operatorname{dist}(\operatorname{supp}(\psi), \partial Q) > h_0$. Accordingly we define the approximation $\eta_{\delta_h} := (\eta \chi_{[0,T]\times Q}) * \gamma_{\delta_h}$ for $\delta_h = h^{a_1} < h_0$, where γ_δ is the standard convolution kernel in space-time. This implies that $(\eta^{(h)} - \eta_{\delta_h})\psi$ is a valid test function for (3.9). Moreover, by the standard convolution estimates we find

$$\begin{aligned}
\|\eta_{\delta_h} \psi\|_{W^{k_0,2}(Q)} &\le c h^{a_1(2-q/n-k_0+2/n)} \|\eta\|_{W^{2,q}(Q)}, \\
\|\partial_t \eta_{\delta_h} \psi\|_{W^{k_0,2}(Q)} &\le c h^{(1-k_0)a_1} \|\partial_t \eta\|_{W^{1,2}(Q)}.
\end{aligned}$$

Also $\eta_{\delta_h} \to \eta$ strongly as $h \to 0$ in all norms in which η is bounded. Now we calculate

$$\begin{aligned}
0 &\leq \limsup_{h\to 0} \int_0^T \langle DE(\eta^{(h)}(t)) - DE(\eta(t)), (\eta^{(h)} - \eta)\psi\rangle \, dt \\
&= \limsup_{h\to 0} \int_0^T [\langle DE(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle \\
&\quad + \langle \underbrace{DE(\eta^{(h)}(t))}_{\text{bd. in } W^{-2,q}(Q)}, \underbrace{(\eta - \eta_{\delta_h})\psi}_{\to 0 \text{ in } W^{2,q}(Q)} \rangle] \, dt \\
&= \limsup_{h\to 0} \int_0^T [\langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle \\
&\quad - 2h^{a_0/2}\langle h^{a_0/2}\nabla^{k_0}(\eta^{(h)}(t)), \nabla^{k_0}(\eta^{(h)} - \eta_{\delta_h})\psi\rangle] \, dt \\
&\leq \limsup_{h\to 0} \int_0^T [\langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle \\
&\quad + 4h^{a_0}\|\nabla^{k_0}(\eta^{(h)}(t))\| \, \|\eta_{\delta_h}\psi\|_{W^{k_0,2}}] \, dt \\
&\leq \limsup_{h\to 0} \int_0^T [\langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle + c h^{a_0/2-(2-q/n-k_0+2/n)a_1}] \, dt \\
&= \limsup_{h\to 0} \int_0^T \langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle \, dt
\end{aligned}$$

by (3.13) and by choosing a_1 small enough. The final term can be rewritten using (3.9) as

$$\begin{aligned}
\int_0^T \langle DE_h(\eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle \, dt &= \int_0^T \Big[-\langle D_2 R_h(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle \\
&+ \langle f \circ \eta^{(h)}(t), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle + \frac{\rho_s}{h}\langle \partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle \Big] \, dt.
\end{aligned}$$

On the right hand side we may pass to the limit $h \to 0$. In particular, observe that

$$\begin{aligned}
\langle D_2 R_h(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle &= \langle D_2 R(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)), (\eta^{(h)} - \eta_{\delta_h})\psi\rangle \\
&+ 2h\langle \nabla^{k_0}\partial_t \eta^{(h)}, \nabla^{k_0}((\eta^{(h)} - \eta_{\delta_h})\psi)\rangle \to \langle D_2 R(\eta(t), \partial_t \eta(t)), (\eta - \eta_{\delta_h})\psi\rangle
\end{aligned}$$

by the strong convergence of $\eta^{(h)}$ in $W^{1,2}(Q;\mathbb{R}^n)$, the weak convergence of $\partial_t \eta^{(h)}$ in $W^{1,2}(Q;\mathbb{R}^n)$ and since

$$\begin{aligned}
&h|\langle \nabla^{k_0}\partial_t \eta^{(h)}, \nabla^{k_0}((\eta^{(h)} - \eta_{\delta_h})\psi)\rangle| \\
&\qquad\leq h^{1/2-a_0/2}\|\sqrt{h}\,\nabla^{k_0}\partial_t \eta^{(h)}\| \, \|h^{a_0/2}\nabla^{k_0}((\eta^{(h)} - \eta_{\delta_h})\psi)\|,
\end{aligned}$$

which converges to zero a.e. using the energy estimates and (3.13) by choosing $a_0 < 1$ and $a_1 < 1$ suitably. The force-term converges, since all terms involved converge strongly. For the last term, we use the discrete integration by parts in time (i.e. shift the term involving $t-h$) to get

$$\begin{aligned}\int_0^T \frac{\rho_s}{h}\langle \partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h), (\eta^{(h)}(t) - \eta_{\delta_h}(t))\psi(t)\rangle \, dt \\ = -\rho_s \int_0^T \big\langle \partial_t \eta^{(h)}(t), \big(\tfrac{\eta^{(h)}(t+h)-\eta^{(h)}(t)}{h} - \tfrac{\eta_{\delta_h}(t+h)-\eta_{\delta_h}(t)}{h}\big)\psi(t+h)\big\rangle \, dt \\ - \rho_s \int_0^T \big\langle \partial_t \eta^{(h)}(t), (\eta^{(h)}(t) - \eta_{\delta_h}(t)) \tfrac{\psi(t+h)-\psi(t)}{h}\big\rangle \, dt.\end{aligned}$$

Now note that the first difference quotient is equal to $w^{(h)}$ as it was defined in Lemma 3.8 and thus converges strongly to $\partial_t \eta$ in $L^2([0,T]\times Q;\mathbb{R}^n)$, while the other difference quotients only involve constant functions and their mollifications and thus also converge in the same space. As a result, the entire right hand side converges strongly to 0 in $L^2([0,T]\times Q;\mathbb{R}^n)$ and the left hand sides is bounded. Thus the total limit is 0 and via Assumption 1.7 (S6), we have $\eta^{(h)}(t) \to \eta(t)$ in $W^{2,q}(Q;\mathbb{R}^n)$ for almost all $t \in [0,T]$. ■

Step 3: Limit equation. With all the necessary ingredients at hand, we can finally consider the weak equation (3.9) for arbitrary test functions. For the first three terms we have, as before,

$$\begin{aligned}\int_0^T [\langle DE_h(\eta^{(h)}(t)), \phi\rangle + \langle D_2 R_h(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)), \phi\rangle + \langle f \circ \eta^{(h)}(t), \phi\rangle] \, dt \\ \to \int_0^T [\langle DE(\eta(t)), \phi\rangle + \langle D_2 R(\eta(t), \partial_t \eta(t)), \phi\rangle + \langle f \circ \eta(t), \phi\rangle] \, dt,\end{aligned}$$

where the regularizing terms vanish by the same estimates as in the proof of Lemma 3.9.

This leaves us with the last term, where we shift the discrete derivative to the test function again and get

$$\begin{aligned}\int_0^T \frac{\rho}{h}\langle \partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h), \phi\rangle \, dt = -\rho \int_0^T \big\langle \partial_t \eta^{(h)}(t), \tfrac{\phi(t+h)-\phi(t)}{h}\big\rangle \, dt \\ \to -\rho \int_0^T \langle \partial_t \eta(t), \partial_t \phi\rangle \, dt.\end{aligned}$$

From this, we get solutions on the interval $[0,T]$.

Step 4: Continuation until collision. Using the short term existence, we can now employ a continuation argument: Assume that $\eta : [0, T_{\max}) \to \mathcal{E}$ is a solution on a maximal interval. Then either $T_{\max} = \infty$ or we can use the energy inequality to show existence of a unique limit $\eta(T_{\max})$ similar to what we did at the end of the proof of Theorem 2.2. Then $\eta(T_{\max}) \notin \partial\mathcal{E}$ would allow us to reapply the short time existence, which would be a contradiction. This finishes the proof of Theorem 3.2.

Remark 3.10 (On the proof of the energy inequality). In the proof of the energy inequality of Lemma 3.6 we used a regularization term in the dissipation to simplify the proof. Since we will need that term later on in the fluid-structure interaction, this only seemed natural, but it should be noted that strictly speaking, it was not necessary. The same result is still true if we only use R. To show this directly, one can use some techniques from the theory of minimizing movements, specifically the so-called Moreau–Yosida approximation.

4. The unsteady fluid-structure interaction problem

We will now combine the methods developed in the last two sections to show existence of weak solutions (in the sense of Definition 1.1) for a general fluid-structure interaction problem. In contrast to previous works (see [21, 50, 65, 71–74] as well as the discussion in the introduction) we work in arbitrary dimension and consider a bulk solid that can undergo large elastic deformations. But most importantly, we consider the full nonlinear equation, both for the fluid in the form of the incompressible Navier–Stokes equation with its transport term and full nonlinear elasticity of the solid.

Before embarking on the technical discussion let us highlight some aspects regarding the convective term in the Navier–Stokes equation. Indeed, the very presence of this term necessitates the use of techniques beyond those presented in the previous two sections. At this point, it is instructive to recall some of the arguments behind the derivation of the Navier–Stokes equation. The natural way to deal with inertia in a moving fluid is to transport it along the flow of the fluid, usually by employing the well known concept of a flow map.

A flow map is the fluid counterpart of the deformation of the solid. Indeed, let, as before, $\Omega(t)$ denote the fluid domain at a given time t. Now for a fixed t_0, a *flow map* is a family $\Phi_s : \Omega(t_0) \to \Omega$ for $s \in [0, T - t_0]$, which we say is *generated* by v if $\Phi_0(y) = y$ and $\partial_s \Phi_s(y) = v(t_0 + s, \Phi_s(y))$.

If it exists and has some regularity, it has to be a volume preserving diffeomorphism, which allows us to compare $v(t_0 + h, \Phi_h(y))$ and $v(t_0, y)$ for any $y \in \Omega(t_0)$. From this we are able to obtain the material derivative via the chain rule:

$$\lim_{h \searrow 0} \tfrac{v(t_0, \Phi_h(y)) - v(t_0, y)}{h} = \partial_t v(t_0, y) + \nabla v(t_0, y) \cdot v(t_0, y).$$

This kind of difference quotient will be the Eulerian counterpart to the ordinary difference quotient for $\partial_t \eta$ in the previous section.

Having explained the idea, we immediately have to note that the existence of such a flow map is not guaranteed, even in the case of the Navier–Stokes equation without additional interaction. We will thus additionally use the fact that we no longer need such a flow map in the limit $h \to 0$. This allows us to add an h-dependent regularization term for the fluid flow, similar to those already introduced for the solid.

Additionally we note that in turn to obtain the proper weak equation, we already need to construct a discretized version of Φ along with our minimization procedure. As an added benefit, as we let $\tau \to 0$, we are able to prove convergence of this discretization, directly giving us a flow map for any $h > 0$, without having to resort to additional ODE arguments.

4.1. An intermediate, time-delayed problem

As in the previous section, let us start by deriving a time-delayed equation, similar to Section 3.1.

Definition 4.1 (Time-delayed solution). Let $\Omega_0 = \Omega \setminus \eta_0(Q)$, $f \in C^0([0,h] \times \Omega; \mathbb{R}^n)$ and $w \in L^2([0,h] \times \Omega; \mathbb{R}^n)$. We call the pair $\eta : [0,h] \times Q \to \Omega, u : [0,h] \times \Omega \to \mathbb{R}^n$ a *weak solution to the time-delayed inertial equation* if it satisfies

$$\begin{aligned} 0 = \langle DE_h(\eta), \phi\rangle_Q + \langle D_2 R_h(\eta, \partial_t \eta), \phi\rangle_Q + \left\langle \rho_s \tfrac{\partial_t \eta - w \circ \eta_0^{-1}}{h}, \phi \right\rangle_Q - \rho_s \langle f \circ \eta, \phi\rangle_Q \\ + \nu \langle \varepsilon u, \varepsilon \xi\rangle_{\Omega(t)} + h \langle \nabla^{k_0} u, \nabla^{k_0} \xi\rangle_{\Omega(t)} + \left\langle \rho_f \tfrac{u \circ \Phi - w}{h}, \xi \circ \Phi \right\rangle_{\Omega_0} - \rho_f \langle f, \xi\rangle_{\Omega(t)} \end{aligned} \tag{4.1}$$

for almost all $t \in [0, h]$ and all $\phi \in C^0([0, h]; W^{k_0,2}(Q; \mathbb{R}^n))$, $\xi \in C^0([0, h]; W^{k_0,2}(\Omega; \mathbb{R}^n))$ satisfying $\operatorname{div} \xi|_{\Omega(t)} = 0$, $\xi|_{\partial\Omega} = 0$, $\phi|_P = 0$ and the coupling conditions

$$\xi \circ \eta = \phi \quad \text{and} \quad u \circ \eta = \partial_t \eta \quad \text{in } Q.$$

Here we define $\Omega(t) = \Omega \setminus \eta(t, Q)$ and $\Phi : [0, h] \times \Omega_0 \to \Omega$ solves $\partial_t \Phi = u \circ \Phi$ and $\Phi_0(y) = y$.

The construction of the time-delayed solution shares many similarities to that of the weak solution defined in Definition 2.1 combined with ideas from the construction of the time-delayed solutions for solids in Definition 3.3. However, an important addition here is the *flow map* Φ. Note that in this subsection, the map will always start at $t = 0$. This allows us to take a temporary Lagrangian point of view, as Ω_0 will play the role of a reference configuration for the fluid.

As in Section 3, we will construct the time-delayed solutions by time discretization. Notice that, due to the way that Φ is linked with the equation, we already need to begin its construction in the discrete setting. Here, we make use of the additional regularizing dissipation terms for v, as they will allow us to construct Φ in the limit.

In this subsection we will prove the following existence theorem:

Theorem 4.2 (Existence of time-delayed solutions). *Let $\eta_0 \in \mathcal{E} \cap W^{k_0,2}(Q; \mathbb{R}^n) \setminus \partial\mathcal{E}$, $w \in L^2([0,h] \times Q; \mathbb{R}^n)$ and $f \in C^0([0,h] \times Q; \mathbb{R}^n)$. Then there exists a solution (η, v) to the time-delayed equation as given in Definition 4.1 on the interval $[0, h]$, or there exists a solution on a shorter interval $[0, h_{\max}]$ such that $\eta(h_{\max}) \in \partial\mathcal{E}$.*[17] *Furthermore, $\Phi(t, \cdot)$ is a volume preserving diffeomorphism between Ω_0 and $\Omega(t)$.*

[17] Note that a posteriori (see Corollary 4.12) it will be shown that (in dependence on η_0) there is always a minimal time length $h_{\min}$ for which it can be guaranteed that $\eta(t) \notin \partial\mathcal{E}$ for $t \in [0, h_{\min}]$.

Let us now begin with the proof of this theorem. Parts that are identical to one of the previous proofs will only be sketched.

Proof of Theorem 4.2, Step 1: Constructing an iterative approximation. Fix a step size $\tau > 0$. We again proceed iteratively, this time constructing both the pair (η, v) and Φ. We start with the given $\eta_0^{(\tau)} := \eta_0$ and $\Phi_0^{(\tau)} := \mathrm{id}$. Assuming $\eta_k^{(\tau)} \in \mathcal{E}$ and $\Phi_k^{(\tau)} : \Omega_0 \to \Omega_k^{(\tau)}$ are given, we define $(\eta_{k+1}^{(\tau)}, v_{k+1}^{(\tau)})$ as a solution to the following problem:

$$\begin{aligned}
\text{Minimize}\quad & E_h(\eta) + \tau R_h\big(\eta_k^{(\tau)}, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big) + \frac{\tau\rho_s}{2h}\Big\| \tfrac{\eta-\eta_k^{(\tau)}}{\tau} - w_k^{(\tau)} \circ \eta_0 \Big\|^2 \\
& + \tau\frac{\nu}{2}\|\varepsilon v\|_{\Omega_k}^2 + \frac{\tau h}{2}\|\nabla^{k_0} v\|_{\Omega_k^{(\tau)}}^2 + \frac{\tau\rho_f}{2h}\|v\circ\Phi_k^{(\tau)} - w_k^{(\tau)}\|_{\Omega_0}^2 \\
& - \tau\big\langle f\circ\eta, \tfrac{\eta-\eta_k^{(\tau)}}{\tau}\big\rangle - \tau\langle f\circ\Phi_k^{(\tau)}, v\circ\Phi_k^{(\tau)}\rangle_{\Omega_0} \\
\text{subject to}\quad & \eta\in\mathcal{E}, v\in W^{1,2}(\Omega_k^{(\tau)};\mathbb{R}^n) \text{ with } \operatorname{div} v = 0, v|_{\partial\Omega} = 0 \\
& \text{and } \tfrac{\eta-\eta_k^{(\tau)}}{\tau} = v\circ\eta_k^{(\tau)} \text{ in } P.
\end{aligned}\tag{4.2}$$

Here, as before, $\Omega_k^{(\tau)} = \Omega \setminus \eta_k^{(\tau)}(Q)$ and we define $w_k^{(\tau)}(y) = ⨍_{k\tau}^{(k+1)\tau} w(t,y)\,dt$ for all $y \in \Omega$. Finally, we update Φ_k to Φ_{k+1} using

$$\Phi_{k+1}^{(\tau)} := (\mathrm{id} + \tau v_{k+1}^{(\tau)}) \circ \Phi_k^{(\tau)}.$$

Note that at this point, using the coupling condition, we can immediately derive $\Phi_{k+1}^{(\tau)}(\partial\Omega_0) = \partial\Omega_{k+1}^{(\tau)}$ but we still need to show that a similar property holds in the interior. This will be done in Step 2a of the proof. For now we can simply assume $v_{k+1}^{(\tau)}$ is extended by 0 in the definition of $\Phi_{k+1}^{(\tau)}$.

Proposition 4.3 (Existence of iterative solutions). *The iterative problem (4.2) has a solution, i.e. $\eta_{k+1}^{(\tau)}$ and $v_{k+1}^{(\tau)}$ are defined. Furthermore, the minimizers obey the following equation:*

$$\begin{aligned}
&\langle DE_h(\eta_{k+1}^{(\tau)}), \phi\rangle + \big\langle D_2R_h\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big), \phi\big\rangle + \frac{\rho_s}{h}\big\langle \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau} - w_k^{(\tau)}\circ\eta_0, \phi\big\rangle_Q \\
&\quad + \nu\langle\varepsilon v_{k+1}^{(\tau)}, \varepsilon\xi\rangle_{\Omega_k^\tau} + h\langle\nabla^{k_0} v_{k+1}^{(\tau)}, \nabla^{k_0}\xi\rangle_{\Omega_k^{(\tau)}} + \frac{\rho_f}{h}\langle v_{k+1}^{(\tau)}\circ\Phi_k^{(\tau)} - w_k^{(\tau)}, \xi\circ\Phi_k^{(\tau)}\rangle_{\Omega_0} \\
&\qquad = \rho_f\langle f\circ\Phi_k^{(\tau)}, \xi\circ\Phi_k^{(\tau)}\rangle_{\Omega_0} + \rho_s\big\langle f\circ\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big\rangle_Q,
\end{aligned}$$

where $\phi \in W^{2,q}(Q;\mathbb{R}^n)$, $\phi|_P = 0$ and $\xi \in W_0^{1,2}(\Omega;\mathbb{R}^n)$ are such that

$$\phi = \xi\circ\eta_k \quad \text{on } Q \quad \text{and} \quad \operatorname{div}\xi|_{\Omega_k} = 0.$$

Proof. The proof differs from the quasistatic case in Proposition 2.13 only in the occurrence of the additional terms for the effects of inertia. As both are nonnegative, we still have a minimizing sequence $(\tilde{\eta}_l, \tilde{v}_l)$ bounded in the same spaces as in the proof of Proposition 2.13. In particular, due to compact embeddings, we can assume that for a subsequence both converge strongly in $L^2(Q;\mathbb{R}^n)$ and $L^2(\Omega_k^{(\tau)};\mathbb{R}^n)$ respectively. As the inertial terms are continuous with respect to this convergence, this minimizing sequence will again converge to a minimizer. In fact, establishing the lower bound on the sequence is easier in this case, as the two force terms can now be estimated against the inertial terms directly, without having to resort to a potentially energy-dependent Korn inequality. (See the corresponding calculations in the proof of Theorem 3.5 and Remark 1.5 for more details.)

Further, with regard to the Euler–Lagrange equation, we can treat the additional terms individually. Since both are quadratic functionals of η and v respectively, and neither involves any derivatives, this is straightforward. Note that again we are able to remove a factor of τ from the final term by scaling ϕ and ξ differently than η and v. ∎

Now as before, our minimization yields a discrete energy inequality by comparing minimizers.

Lemma 4.4 (Discrete energy inequality and estimates). *We have*

$$\begin{aligned}
E_h(\eta_{k+1}^{(\tau)}) &+ \tau R_h\big(\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big) + \tau\frac{\rho_s}{2h}\big\|\tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau} - w_k^{(\tau)}\circ\eta_0\big\|_Q^2 \\
&+ \tau\frac{\nu}{2}\|\varepsilon v_{k+1}^{(\tau)}\|^2_{\Omega_k^{(\tau)}} + \frac{\tau h}{2}\|\nabla^{k_0} v_{k+1}^{(\tau)}\|^2_{\Omega_k^{(\tau)}} + \tau\frac{\rho_f}{2h}\|v_{k+1}^{(\tau)}\circ\Phi_k^{(\tau)} - w_k^{(\tau)}\|^2_{\Omega_0} \\
&\le E_h(\eta_k^{(\tau)}) + \tau\frac{\rho_s}{2h}\|w_k^{(\tau)}\circ\eta_0\|_Q^2 + \tau\frac{\rho_f}{2h}\|w_k^{(\tau)}\|^2_{\Omega_0} + \tau\rho_f\langle f\circ\Phi_k^{(\tau)}, v\circ\Phi_k^{(\tau)}\rangle_{\Omega_0} \\
&\quad + \tau\rho_s\big\langle f\circ\eta_k^{(\tau)}, \tfrac{\eta_{k+1}^{(\tau)}-\eta_k^{(\tau)}}{\tau}\big\rangle_Q
\end{aligned}$$

and there exist $c, C > 0$ independent of τ and N (with $N \in \mathbb{N}$ satisfying $N\tau \le h$) such that

$$\begin{aligned}
E_h(\eta_N^{(\tau)}) &+ \sum_{k=1}^N \tau\Big[R_h\big(\eta_{k-1}^{(\tau)}, \tfrac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\big) + c\big\|\tfrac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau} - w_k^{(\tau)}\circ\eta_0\big\|_Q^2 \\
&+ \nu\|\varepsilon v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + \frac{\tau h}{2}\|\nabla^{k_0} v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + c\|v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)} - w_k^{(\tau)}\|^2_{\Omega_0}\Big] \\
&\le E_h(\eta_0) + C\Big(\int_0^h \|w\circ\eta_0\|_Q^2\,dt + \int_0^h \|w\|^2_{\Omega_0}\,dt + \|f\|^2_\infty\Big).
\end{aligned}$$

Proof. As in Lemma 2.14, we compare the minimizer $(\eta_{k+1}^{(\tau)}, v_{k+1}^{(\tau)})$ in (4.2) with the pair $(\eta_k^{(\tau)}, 0)$ to get the first inequality. For the second we add up all those inequalites for $k \le N-1$ and absorb the force terms as in the proof of Theorem 3.5. ∎

As before, this immediately implies that for h small enough all $\eta_k^{(\tau)}$ will be in $\mathcal{E}\setminus\partial\mathcal{E}$.

Proof of Theorem 4.2, Step 2: Constructing interpolations. Now we unfix τ and define the following interpolants:

$$\begin{aligned}
\eta^{(\tau)}(t,x) &= \eta^{(\tau)}_{k+1}(x) && \text{for } \tau k \le t < \tau(k+1),\\
\underline{\eta}^{(\tau)}(t,x) &= \eta^{(\tau)}_{k}(x) && \text{for } \tau k \le t < \tau(k+1),\\
\tilde{\eta}^{(\tau)}(t,x) &= \tfrac{\tau(k+1)-t}{\tau}\eta^{(\tau)}_{k}(x) + \tfrac{t-\tau k}{\tau}\eta^{(\tau)}_{k+1}(x) && \text{for } \tau k \le t < \tau(k+1),\\
u^{(\tau)}(t,y) &= \begin{cases} v^{(\tau)}_k(y) & \text{for } \tau k \le t < \tau(k+1),\ y \in \Omega^{(\tau)}_k,\\ \frac{(\eta^{(\tau)}_{k+1}-\eta^{(\tau)}_k)\circ(\eta^{(\tau)}_k)^{-1}}{\tau} & \text{for } \tau k \le t < \tau(k+1),\ y \in \Omega\setminus\Omega^{(\tau)}_k,\end{cases}\\
\Phi^{(\tau)}(t,y) &= \Phi^{(\tau)}_{k-1}(y) && \text{for } \tau k \le t < \tau(k+1),\\
\tilde{\Phi}^{(\tau)}(t,y) &= \tfrac{\tau(k+1)-t}{\tau}\Phi^{(\tau)}_{k-1}(x) + \tfrac{t-\tau k}{\tau}\Phi^{(\tau)}_{k}(x) && \text{for } \tau k \le t < \tau(k+1),
\end{aligned}$$

as well as $\Omega^{(\tau)}(t) = \Omega^{(\tau)}_k$ for $\tau k \le t < \tau(k+1)$.

Now using the a priori estimate of Lemma 4.4, we derive some uniform bounds on those functions.

Lemma 4.5 (Uniform bounds in τ). *The following quantities are bounded independently of τ:*

$$\begin{aligned}
&\sup_{t\in[0,h]} (E_h(\underline{\eta}^{(\tau)}(t)) + E_h(\eta^{(\tau)}(t))),\\
&\sup_{t\in[0,h]} (\|\underline{\eta}^{(\tau)}\|_{W^{k_0,2}(Q)} + \|\eta^{(\tau)}\|_{W^{k_0,2}(Q)} + \|\tilde{\eta}^{(\tau)}\|_{W^{k_0,2}(Q)}),\\
&\int_0^h [\|\partial_t\tilde{\eta}^{(\tau)}\|^2_{W^{k_0,2}(Q)} + \|u^{(\tau)}\|^2_{W^{k_0,2}(Q)} + \|u^{(\tau)}\circ\Phi^{(\tau)}\|^2_{\Omega_0}]\,dt.
\end{aligned}$$

Furthermore, by definition $\partial_t\tilde{\Phi}^{(\tau)} = u^{(\tau)}\circ\Phi^{(\tau)}$ whenever $\Phi^{(\tau)}(t,y)\in\Omega^{(\tau)}(t)$ and $t\notin\tau\mathbb{N}$.

Proof. First we note that the right hand side of the second estimate in Lemma 4.4 only depends on the initial data η_0 and w as well as the force f. This gives us uniform bounds on $E_h(\eta_k)$ and thus an L^∞ bound on $E_h(\eta^{(\tau)}(t,\cdot))$. By the properties of the energy, Assumption 1.7 and its regularized version, this also results in a uniform bound on $\|\eta_k\|_{W^{k_0,2}(Q)}$ and thus in $L^\infty([0,h];W^{k_0,2}(Q;\mathbb{R}^n))$ bounds on $\eta^{(\tau)}$ and $\tilde{\eta}^{(\tau)}$. By the properties of the dissipation in Assumption 1.10 and using the bound on the energy, we get

$$\begin{aligned}
c_K &\int_0^h [\|\partial_t\nabla\tilde{\eta}^{(\tau)}\|^2_Q + h\|\nabla^{k_0}\partial_t\tilde{\eta}^{(\tau)}\|^2_Q]\,dt\\
&\le c\int_0^h R_h(\underline{\eta}^{(\tau)},\partial_t\tilde{\eta}^{(h)})\,dt \le c\sum_{k=0}^N \tau R_h\Big(\underline{\eta}^{(\tau)},\tfrac{\eta^{(\tau)}_k-\eta^{(\tau)}_{k-1}}{\tau}\Big)
\end{aligned}$$

where we know the right hand side to be bounded. Using Poincaré's inequality, as $\partial_t\tilde{\eta}^{(\tau)}|_P = 0$, this extends to a uniform $L^2([0,T];W^{k_0,2}(Q;\mathbb{R}^n))$ bound on $\partial_t\tilde{\eta}^{(\tau)}$. For the fluid, we use Proposition A.4, as well as the global Korn inequality of Lemma 2.11, to get a $c > 0$ such that

$$\int_0^h [C_{gK} \|u^{(\tau)}\|^2_{W^{1,2}(\Omega)} + ch \|\nabla^{k_0} u^{(\tau)}\|^2_\Omega]\, dt$$
$$\leq \int_0^h \left[R(\underline{\eta}^{(\tau)}, \partial_t \tilde{\eta}^{(\tau)}) + \frac{\nu}{2} \|\varepsilon u^{(\tau)}(t)\|^2_{\Omega^{(\tau)}(t)} \right] dt$$
$$+ h \int_0^h [\|\partial_t \nabla^{k_0} \tilde{\eta}^{(\tau)}\|^2_Q + \|\nabla^{k_0} u^{(\tau)}\|^2_{\Omega^{(\tau)}(t)}]\, dt,$$

which is uniformly bounded using the energy estimate again. The $L^2([0, h]; W^{k_0,2}(\Omega; \mathbb{R}^n))$ estimate then follows by interpolating the missing intermediate derivatives.

For the last estimate, we have

$$\int_0^h \|u^{(\tau)} \circ \Phi^{(\tau)}\|^2_{\Omega_0}\, dt = \sum_{k=0}^N \tau \|u_k^{(\tau)} \circ \Phi_k^{(\tau)}\|^2$$
$$\leq \sum_{k=0}^N \tau \frac{3}{2} (\|u_k^{(\tau)} \circ \Phi_k^{(\tau)} - w_k\|^2 + \|w_k\|^2),$$

which again consists of two bounded sums. ∎

Proof of Theorem 4.2, Step 2a: Bounds on $\Phi^{(\tau)}$. We now arrive at a delicate point in the existence proof for the time-delayed problem: establishing the properties of and suitable bounds on $\Phi^{(\tau)}$. The challenge here is that $\Phi^{(\tau)}$ is defined via concatenation of an unbounded (for $\tau \to 0$) number of functions and thus is highly nonlinear. As any linearizing would break the coupling properties needed, we will instead rely on using high enough regularity of the functions involved.

We start by proving the following:

Proposition 4.6 (Higher regularity for the velocity). *There are $\tau_0 > 0$ and $\alpha > 0$ such that for all $\tau \in (0, \tau_0)$, $\Phi_k^{(\tau)} : \Omega_0 \to \Omega_k$ is a diffeomorphism with $1/2 \leq \det \nabla \Phi_k^{(\tau)} \leq 2$ for all $k < h/\tau$, and*

$$\sum_{k=1}^N \tau \|v_k^{(\tau)}\|^2_{C^{1,\alpha}(\Omega_{k-1}^{(\tau)})} \leq \mathcal{K}$$

for any $N < h/\tau$, where $\mathcal{K}$ and τ_0 only depend on $w, h, E(\eta_0)$ and f.

Proof. As k_0 is such that $k_0 - n/2 \geq 2 - n/q$, we know that $W_0^{k_0,2}(\Omega; \mathbb{R}^n)$ embeds into $C^{1,\alpha}(\Omega; \mathbb{R}^n)$ for some $\alpha > 0$. Thus

$$\sum_{k=1}^N \tau \|v_k^{(\tau)}\|^2_{C^{1,\alpha}(\Omega_{k-1}^{(\tau)})} \leq \int_0^h \|u^{(\tau)}\|^2_{C^{1,\alpha}(\Omega)}\, dt \leq c \int_0^h \|u^{(\tau)}\|^2_{W^{k_0,2}(\Omega)}\, dt,$$

which is uniformly bounded by Lemma 4.5.

Now we need to show the properties of Φ_N. By the chain rule, the multiplicative nature of the determinant and its expansion (Lemma A.1) we have

$$\begin{aligned}\det \nabla \Phi_N^{(\tau)} &= \prod_{k=1}^N [\det(I + \tau \nabla v_k^{(\tau)})] \circ \Phi_{k-1}^{(\tau)} \\ &= \prod_{k=1}^N \Big[1 + \tau \underbrace{\operatorname{tr}(\nabla v_k^{(\tau)})}_{=\operatorname{div} v_k^{(\tau)} = 0} + \sum_{l=2}^n \tau^l M_l(\nabla v_k^{(\tau)}) \Big] \circ \Phi_{k-1}^{(\tau)},\end{aligned}$$

where M_l are homogeneous polynomials of degree l. By the inequality between the arithmetic and geometric means, we then have

$$\begin{aligned}\det \nabla \Phi_N^{(\tau)} &\le \left(\sum_{k=1}^N \frac{1}{N} \Big(1 + \sum_{l=2}^n \tau^l M_l(\nabla v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)}) \Big) \right)^N \\ &\le \left(1 + \frac{1}{N} \sum_{k=1}^N \sum_{l=2}^n \tau^l c_l \operatorname{Lip}(v_k^{(\tau)})^l \right)^N,\end{aligned}$$

where $\operatorname{Lip}(v_k^{(\tau)})$ denotes the Lipschitz constant of $v_k^{(\tau)}$ with respect to its domain $\Omega_{k-1}^{(\tau)}$. Now as $(1 + a/N)^N$ is increasing for $a > 0$, with limit $\exp(a)$, we can further estimate

$$\begin{aligned}&\le \exp\Big(\sum_{k=1}^N \sum_{l=2}^n \tau^l c_l \operatorname{Lip}(v_k^{(\tau)})^l \Big) \\ &= \exp\Big(\sum_{l=2}^n c_l \tau^{l/2} \sum_{k=1}^N (\tau \operatorname{Lip}(v_k^{(\tau)})^2)^{l/2} \Big) \le \exp\Big(\sum_{l=2}^n c_l \tau^{l/2} \mathcal{K}^{l/2} \Big),\end{aligned}$$

where we have used the fact that $l \ge 2$ and

$$\tau \operatorname{Lip}(v_{k_0}^{(\tau)})^2 \le \sum_{k=1}^N \tau \operatorname{Lip}(v_k^{(\tau)})^2 \le \sum_{k=1}^N \tau \|v_k^{(\tau)}\|^2_{C^{1,\alpha}(\Omega_{k-1}^{(\tau)})} \le \mathcal{K}.$$

In a similar fashion, we can get a lower estimate

$$(\det \nabla \Phi_N^{(\tau)})^{-1} \le \left(\sum_{k=1}^N \frac{1}{N} \Big(1 + \sum_{l=2}^n \tau^l M_l(\nabla v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)}) \Big)^{-1} \right)^N \le \exp\Big(2 \sum_{l=2}^n c_l \tau^{l/2} \mathcal{K}^{l/2} \Big)$$

using $\frac{1}{1+a} \le \frac{1}{1-|a|} \le 1 + 2|a|$ for $|a|$ small enough. Thus for τ_0 small enough,

$$1/2 \le \det \nabla \Phi_N^{(\tau)} \le 2.$$

Now we know from the boundary condition that $\Phi_N^{(\tau)}$ is an orientation preserving diffeomorphism on $\partial \Omega_0$ as it is given by $\eta_N^{(\tau)} \circ \eta_0^{-1}$ and id on the respective parts of the boundary. We also know that Ω_0 and $\Omega_N^{(\tau)}$ are domains with the same topology as there were no collisions. But then $\Phi_N^{(\tau)}$ has to be a diffeomorphism by a degree argument. ∎

An immediate consequence of the last proof is the following:

Corollary 4.7 (Regularity of $\Phi^{(\tau)}$). *The maps $\Phi^{(\tau)}(t,\cdot)$ are uniformly Lipschitz continuous, i.e. Lipschitz continuous with respect to x such that the constants are bounded independently of τ and t. Furthermore,*

$$\lim_{\tau\to 0} \det \nabla\Phi^{(\tau)} = 1.$$

Proof. By the estimates in the previous proof, we find that $\lim_{\tau\to 0} \det \nabla\Phi^{(\tau)} = 1$. It remains to prove Lipschitz regularity.

Here we proceed in the same fashion as in the preceding proof:

$$\begin{aligned}
\operatorname{Lip}(\Phi_N^{(\tau)}) &\le \prod_{l=1}^{N}(1+\tau \operatorname{Lip}(v_l^{(\tau)})) \le \left(\frac{1}{N}\sum_{l=1}^{N}(1+\tau \operatorname{Lip}(v_l^{(\tau)}))\right)^N \\
&= \left(1+\frac{1}{N}\sum_{l=1}^{N}\tau \operatorname{Lip}(v_l^{(\tau)})\right)^N \le \exp\Big(\sum_{l=1}^{N}\tau \operatorname{Lip}(v_l^{(\tau)})\Big) \\
&\le \exp\left(\sqrt{\sum_{l=1}^{N}\tau}\sqrt{\sum_{l=1}^{N}\tau \operatorname{Lip}(v_l^{(\tau)})^2}\right) \le \exp\big(\sqrt{h}\sqrt{\mathcal{K}}\big). \qquad \blacksquare
\end{aligned}$$

Proof of Theorem 4.2*, Step 3: Convergence of the equation.* Relying on the Banach–Alaoglu theorem as well as the classical Aubin–Lions lemma, we pick up a subsequence of τ's and find functions $\eta \in W^{1,2}([0,h];W^{k_0,2}(Q;\mathbb{R}^n))$, $u \in L^2([0,h];W^{k_0,2}(\Omega;\mathbb{R}^n))$, $\Phi \in C^0([0,h];W^{1,\infty}(\Omega_0;\mathbb{R}^n))$ such that

$$\begin{aligned}
&\eta^{(\tau)}, \underline{\eta}^{(\tau)}, \tilde{\eta}^{(\tau)} \rightharpoonup^* \eta && \text{in } L^\infty([0,h];W^{k_0,2}(Q;\mathbb{R}^n)),\\
&\partial_t\tilde{\eta}^{(\tau)} \rightharpoonup \partial_t\eta && \text{in } L^2([0,h];W^{k_0,2}(Q;\mathbb{R}^n)),\\
&u^{(\tau)} \rightharpoonup u && \text{in } L^2([0,h];W^{k_0,2}(\Omega;\mathbb{R}^n)),\\
&\Phi^{(\tau)} \to \Phi && \text{in } C^0([0,h];C^\alpha(\Omega_0;\mathbb{R}^n)),
\end{aligned}$$

and we define $\Omega(t) = \Omega \setminus \eta(t,Q)$. Moreover, due to Lemma 4.7 we know that Φ is Lipschitz with constant $\exp(\sqrt{Lh})$ and $\det\nabla\Phi = 1$ almost everywhere. We also remark that $\Phi(t,\cdot)|_{\partial\Omega_0}$ is injective as long as there is no collision in the solid (which we have already excluded), and that again $\Phi(t,\cdot):\Omega_0 \to \Omega(t)$ is a volume preserving diffeomorphism.

Finally, we can conclude that

$$\partial_t\Phi = \lim_{\tau\to 0}\partial_t\tilde{\Phi}^{(\tau)} = \lim_{\tau\to 0} u^{(\tau)}\circ\Phi^{(\tau)} = u\circ\Phi$$

almost everywhere.

Then Φ fulfills the requirements in Definition 4.1 and v and η are coupled in the right way, as before. What is left is to show that these functions fulfill the weak equation (4.1). This is indeed very similar to the proofs of Proposition 2.20 and Theorem 3.5.

As before, we use Lemma 2.22 and pick a test function $\xi \in C_0^\infty([0,h] \times \Omega; \mathbb{R}^n)$ such that $\operatorname{div} \xi = 0$ in a neighborhood of the fluid domain. From this we can construct $\phi^{(\tau)} := \xi \circ \underline{\eta}^{(\tau)}$ and use those to test the discrete Euler–Lagrange equation from Proposition 4.3.

Most of the terms, including all those related to the solid, have already been dealt with in Proposition 2.20 and Theorem 3.5. It remains to handle the additional regularization term, the inertial effects of the fluid and the force term for the fluid which has been slightly modified here.

We start with the latter, where we simply note that $\Phi^{(\tau)}$ converges uniformly and thus any concatenation with a uniformly continuous function such as given by $f \circ \Phi^{(\tau)}$ converges uniformly as well. Therefore

$$\int_0^h \langle f \circ \Phi^{(\tau)}, \xi \circ \Phi^{(\tau)} \rangle_{\Omega_0} \, dt \to \int_0^T \langle f \circ \Phi, \xi \circ \Phi \rangle_{\Omega_0} \, dt = \int_0^h \langle f, \xi \rangle_{\Omega(t)} \, dt,$$

where the last equality is true as Φ is volume preserving.

Of greater interest is the inertial term of the fluid, where we have

$$\int_0^h \langle u^{(\tau)} \circ \Phi^{(\tau)} - w^{(\tau)}, \xi \circ \Phi^{(\tau)} \rangle_{\Omega_0} \, dt \to \int_0^h \langle u \circ \Phi - w, \xi \circ \Phi \rangle_{\Omega_0} \, dt$$

as the right side of the inner product converges uniformly and the left side at least weakly in $L^2([0,h] \times \Omega_0; \mathbb{R}^n)$. Here, we have introduced the notation

$$w^{(\tau)}(t) := w_k^{(\tau)} \quad \text{if } \tau k \le t < \tau(k+1).$$

Then by the Lebesgue differentiation theorem, $w^{(\tau)} \to w$ in $L^2([0,h] \times \Omega; \mathbb{R}^n)$ and $w^{(\tau)} \circ \eta_0^{-1} \to w \circ \eta_0^{-1}$ in $L^2([0,h] \times Q; \mathbb{R}^n)$.

Finally, since as before $\chi_{\Omega^{(\tau)}(t)} \nabla^{k_0} \xi \to \chi_{\Omega(t)} \nabla^{k_0} \xi$ in $L^2([0,h]; L^2(\Omega; \mathbb{R}^n))$, we have

$$\int_0^h \langle \nabla^{k_0} u^{(\tau)}, \nabla^{k_0} \xi \rangle_{\Omega^{(\tau)}(t)} \, dt \to \int_0^h \langle \nabla^{k_0} u, \nabla^{k_0} \xi \rangle_{\Omega(t)} \, dt$$

by the corresponding weak convergence of $u^{(\tau)}$. This finishes the proof. ∎

A posteriori energy inequality. We close this section with an energy inequality analogous to Lemma 3.6. As before, this will be the central estimate that allows us to let $h \to 0$ and pass to the limit with the equation.

Lemma 4.8 (Energy inequality for time-delayed solutions). *Assume that (η, v) is a weak solution to the time-delayed equation (4.1), as constructed in Theorem 4.2. Then we have the following energy inequality:*

$$\begin{aligned} E_h(\eta(h)) &+ \int_0^h [2R_h(\eta, \partial_t \eta) + \nu \|\varepsilon v\|_{\Omega(t)}^2 + h \|\nabla^{k_0} v\|_{\Omega(t)}^2] \, dt \\ &\qquad + \int_0^h \left[\frac{\rho_f}{2} \|v\|_{\Omega(t)}^2 + \frac{\rho_f}{2} \|\partial_t \eta\|_Q^2 \right] dt \\ &\le E_h(\eta(0)) + \int_0^h [\rho_f \langle f, v \rangle_{\Omega(t)} + \rho_s \langle f \circ \eta, \partial_t \eta \rangle_Q] \, dt \\ &\qquad + \int_0^h \left[\frac{\rho_f}{2h} \|w\|_{\Omega_0}^2 + \frac{\rho_f}{2h} \|w \circ \eta_0^{-1}\|_Q^2 \right] dt. \end{aligned}$$

Proof. We insert $(\partial_t \eta, v)$ as test functions in (4.1). These have the correct coupling and boundary conditions. We need to be careful with regularity here and thus have to rely on the added regularizing terms. From these we know that $\partial_t \eta \in L^2([0,h]; W^{k_0,2}(Q;\mathbb{R}^n))$ and it can thus be used in the duality pairing with $DE_h(\eta)$. We hence obtain

$$0 = \int_0^h \Big[\langle DE_h(\eta), \partial_t \eta\rangle + \langle D_2 R_h(\eta, \partial_t \eta), \partial_t \eta\rangle + \big\langle \rho_s \tfrac{\partial_t \eta - w \circ \eta_0^{-1}}{h}, \partial_t \eta\big\rangle_Q \\ + \nu\langle \varepsilon v, \varepsilon v\rangle_{\Omega(t)} + \big\langle \rho_f \tfrac{v\circ\Phi - w}{h}, v \circ \Phi\big\rangle_{\Omega_0} - \rho_f \langle f, v\rangle_{\Omega_0} - \rho_s \langle f \circ \eta, \partial_t \eta\rangle_Q\Big]\, dt.$$

Now the first term is just the time derivative of the energy and thus its integral is $E_h(\eta(h)) - E_h(\eta(0))$, while for the second term we recall that due to the 2-homogeneity of the dissipation, $\langle D_2 R_h(\eta, \partial_t \eta), \partial_t \eta\rangle_Q = 2R_h(\eta, \partial_t \eta)$. Finally, we estimate the inertial terms using Young's inequality in the form $\langle a - b, a\rangle = |a|^2 - \langle b, a\rangle \geq \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2$. Reordering terms according to their sign proves the estimate. ∎

4.2. *Proof of Theorem 1.2*

Similarly to the proof of Theorem 3.2, we will use time-delayed solutions constructed in the previous subsection to approximate weak solutions to the fluid-structure interaction problem (1.1)–(1.11). The main added difficulty, when compared to Section 3, is in dealing with the inertial effects of the fluid. A particular problem here is that the flow map itself will not persist in the limit $h \to 0$. However, since it is only needed for a flow of length h, the goal is simply to find the right reformulation such that the limit quantities still exist. In particular, the material derivative $\partial_t v + v \cdot \nabla v$ will only be obtained in a weak sense. Furthermore, we note that due to the changing domain, we generally use convergence of u instead of v. With all this in mind, let us begin the proof.

Proof of Theorem 1.2, Step 1: Constructing another iterative approximation. We now iteratively construct an approximate solution to the to the fluid-structure interaction problem (1.1)–(1.11) using time-delayed solutions.

For some fixed h assume that η_0 with finite energy $E_h(\eta_0)$, $v_0 : \Omega_0 := \Omega \setminus \eta_0(Q) \to \mathbb{R}^n$ satisfying $\operatorname{div} v_0 = 0$, and $\eta_* : Q \to \mathbb{R}^n$ are given. Set $w_0(t,y) = v_0(y)$ for $y \in \Omega_0$ and $w_0 = \eta_* \circ \eta_0^{-1}$ otherwise.[18]

For $\eta_l : Q \to \Omega$, $w_l : [0,h] \times \Omega \to \mathbb{R}^n$ and $\Omega_l := \Omega \setminus \eta_l(Q)$ given, we rely on Theorem 4.2 to construct time-delayed solutions to (1.1)–(1.11) according to Definition 4.1 on $[0,h]$ with the given data, which we will denote using $\tilde{\eta}_{l+1}, v_{l+1}, \Phi_{l+1}$. Observe in particular that

$$\Phi_{l+1}(s)(\Omega_l) = \Omega \setminus \tilde{\eta}_{l+1}(s, Q).$$

[18] Note that for this first step, v_0 and η_* do not need to fulfill a coupling condition $\eta_* = v_0 \circ \eta_0$ on $\partial Q \setminus P$ yet. This is completely reasonable from the mathematical point of view, as initial values will only be taken in the L^2 sense, so there is no trace theorem to make sense of this condition.

We then set $\eta_{l+1} := \tilde{\eta}_{l+1}(h,\cdot)$, $\Omega_{l+1} := \Omega \setminus \eta_{l+1}(Q)$, and define $w_{l+1} : [0,h] \times \Omega \to \mathbb{R}^n$ by

$$w_{l+1}(t,\cdot) = \begin{cases} v_{l+1}(t,\cdot) \circ \Phi_{l+1}(t,\cdot) \circ \Phi_{l+1}(h,\cdot)^{-1} & \text{on } \Omega_{l+1}, \\ \partial_t \eta(t,\cdot) \circ \eta(t,\cdot)^{-1} & \text{on } \Omega \setminus \Omega_{l+1}, \end{cases}$$

which will again allow us to find time-delayed solutions according to Definition 4.1. Indeed, $E_h(\eta_{l+1}) < \infty$ by the energy inequality of Lemma 4.8, and since Φ_{l+1} is volume preserving, we have $\int_0^h \|w_{l+1}\|^2_{\Omega_{l+1}}\, dt = \int_0^h \|v_{l+1}\|^2_{\Omega_k(t)}\, dt < \infty$, and a similar estimate for the solid. Hence we can iterate until we reach a collision or until $E(\eta_l)$ or w_l diverges (as we will see in Lemma 4.10, neither of the last two can happen in finite time).

Now we construct the h-approximation.

Definition 4.9 (h-approximation). For $h > 0$ and all $l \in \mathbb{N}_0$ such that $lh < T$, let $\tilde{\eta}_l$, v_l and Φ_l be time-delayed solutions as constructed above. Then we define the approximations $\eta^{(h)} : [0,T] \times Q \to \Omega$, $u^{(h)} : [0,T] \times \Omega \to \mathbb{R}^n$ and $\Phi_s^{(h)} : [0,T] \times \Omega \to \Omega$ for $s \in [-h,h]$ by

$$\begin{aligned}
\eta^{(h)}(t,x) &:= \tilde{\eta}_l(t-lh,x) && \text{for } t \in [lh,(l+1)h), \\
\Omega^{(h)}(t) &:= \Omega_l(t-hl) && \text{for } t \in [lh,(l+1)h), \\
v^{(h)}(t,y) &:= v_l(t-lh,y) && \text{for } t \in [lh,(l+1)h),\ y \in \Omega^{(h)}(t), \\
u^{(h)}(t,y) &:= \begin{cases} v^{(h)}(t,y) \\ \partial_t \eta^{(h)}(t,(\eta^{(h)}(t))^{-1}(y)) \end{cases} && \begin{matrix} \text{for } t \in [0,T),\ y \in \Omega^{(h)}(t), \\ \text{for } t \in [0,T),\ y \in \eta^{(h)}(t,Q), \end{matrix} \\
\rho^{(h)}(t,y) &:= \begin{cases} \rho_f \\ \frac{\rho_s}{\det(\nabla \eta^{(h)}(t,(\eta^{(h)}(t))^{-1}(y)))} \end{cases} && \begin{matrix} \text{for } t \in [0,T),\ y \in \Omega^{(h)}(t), \\ \text{for } t \in [lh,(l+1)h),\ y \notin \Omega^{(h)}(t). \end{matrix}
\end{aligned}$$

Moreover, for $y \in \Omega^{(h)}(t)$ and $s \in [-h,h]$ we define, for $t \in [lh,(l+1)h)$,

$$\Phi_s^{(h)}(t,\cdot) :=$$
$$\begin{cases} \Phi_l(t+s-lh) \circ (\Phi_l(t-lh))^{-1} & \text{if } t+s \in [lh,(l+1)h), \\ \Phi_{l+1}(t+s-(l+1)h) \circ \Phi_l(h) \circ (\Phi_l(t-lh))^{-1} & \text{if } (l+1)h \le t+s < (l+2)h, \\ \Phi_{l-1}(t+s-(l-1)h) \circ (\Phi_{l-1}(h))^{-1} \circ (\Phi_l(t-lh))^{-1} & \text{if } (l-1)h \le t+s < lh. \end{cases}$$

For $y \in \eta^{(h)}(t,Q)$ and $s \in [-h,h]$ we define

$$\Phi_s^{(h)}(t) := \eta^{(h)}(t+s) \circ (\eta^{(h)}(t))^{-1},$$

Note that in contrast to the usage in the proof of Theorem 4.2, where the $\Phi(t,\cdot)$ always corresponded to the flow starting from the initial configuration of the fluid, we now use a full flow map $\Phi_s^{(h)}(t,\cdot)$ which corresponds to the flow from time t to time $t+s$. In particular, $\Phi_l(r)$ maps the fluid at time lh to the fluid at time $lh+r$ for $r \in [0,h]$, so we always need to use the previous multiples of h as intermediate steps in defining $\Phi_s^{(h)}$.

This being said, what we will use in the coming proofs is not the definition but the fact that $\Phi_s^{(h)}$ is the flow map of $u^{(h)}$. In particular, we will rely on the resulting properties that are shown in the following lemma.

Lemma 4.10 (The global flow map). *For all $h > 0$, the flow map defined above is continuous in space-time and satisfies*

$$\partial_s \Phi_s^{(h)}(t,y) = u^{(h)}(t+s, \Phi_s^{(h)}(t,y)). \tag{4.3}$$

Moreover, $\Phi_s^{(h)}(t,\cdot)$ is density preserving, i.e.

$$\det(\nabla \Phi_s^{(h)}(t,y)) = \begin{cases} 1 & \text{for } y \in \Omega^{(h)}(t), \\ \frac{\rho^{(h)}(t+s,\Phi_s^{(h)}(t,y))}{\rho^{(h)}(t,y)} & \text{for } y \in \eta^{(h)}(t,Q). \end{cases}$$

The inverse of the flow map is given by $(\Phi_s^{(h)}(t))^{-1} = \Phi_{-s}^{(h)}(t+s)$.

Proof. For all $y \in \Omega^{(h)}(t) \cup \eta^{(h)}(t,Q)$ we find (by the chain rule and Theorem 4.2) that

$$\partial_s \Phi_s^{(h)}(t,y) = u^{(h)}(t+s, \Phi_s^{(h)}(t,y)).$$

For $s = 0$ the function $\Phi_0^{(h)}(t) = \mathrm{id}$ is trivially continuous over Ω, and by the a priori estimates also u is uniformly Lipschitz continuous (in dependence on h). Hence by a standard argument for ordinary differential equations, $\Phi_s^{(h)}(t,y)$ is continuous over Ω.

The identity for the determinant follows by Theorem 4.2 for the fluid part and by the chain rule and the definition of $\rho^{(h)}$ for the solid part. Furthermore, the inverse of the flow map is given as the respective flow in the opposite direction, which directly follows from the definition. ∎

While it would also be possible to define $\Phi_s^{(h)}(t)$ for larger s, for the remainder of the proof we only need $s \in [-h,h]$.

With the h-approximation defined, (4.1) translates to

$$\begin{aligned}\int_0^T \Big[&\langle DE_h(\eta^{(h)}),\phi\rangle + \langle DR_h(\eta^{(h)},\partial_t\eta^{(h)}),\phi\rangle + \rho_s\big\langle \tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h},\phi\big\rangle_Q \\ &+ \langle \varepsilon v^{(h)},\varepsilon\xi\rangle_{\Omega^{(h)}(t)} + \rho_f\big\langle \tfrac{v^{(h)}(t)\circ\Phi_h^{(h)}(t-h)-v^{(h)}(t-h)}{h},\xi(t)\circ\Phi_h^{(h)}(t-h)\big\rangle_{\Omega^{(h)}(t-h)}\Big]\,dt \\ &= \int_0^T [\rho_s\langle f\circ\eta^{(h)},\phi\rangle_Q + \rho_f\langle f,\xi\rangle_{\Omega^{(h)}(t)}]\,dt \end{aligned} \tag{4.4}$$

for all $\phi \in C^0([0,T];W^{k_0,2}(Q;\mathbb{R}^n))$, $\xi \in C^0([0,T];W_0^{k_0,2}(\Omega;\mathbb{R}^n))$ satisfying $\operatorname{div}\xi|_{\Omega(t)} = 0$, $\xi|_{\partial\Omega} = 0$, $\phi|_P = 0$ and the coupling conditions $\xi\circ\eta = \phi$ and $u\circ\eta = \partial_t\eta$ in Q.

Observe that by the definition of $\rho^{(h)}$ above, we find by a change of variables the following identity for the global momentum:

$$\begin{aligned}\big\langle &\tfrac{\rho^{(h)}u^{(h)}(t)\circ\Phi_h^{(h)}(t-h)-\rho^{(h)}u^{(h)}(t-h)}{h},\xi(t)\circ\Phi_h^{(h)}(t-h)\big\rangle_\Omega \\ &= \rho_f\big\langle \tfrac{v^{(h)}(t)\circ\Phi_h^{(h)}(t-h)-v^{(h)}(t-h)}{h},\xi(t)\circ\Phi_h^{(h)}(t-h)\big\rangle_{\Omega^{(h)}(t-h)} \\ &\quad + \rho_s\big\langle \tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h},\phi\big\rangle_Q, \end{aligned} \tag{4.5}$$

which holds for the same set of test functions as (4.4).

From Lemma 4.8 we deduce the following a priori estimate:

Lemma 4.11 (A priori estimate (full problem)). *For any $t \in [0, T]$ we have*

$$E_h(\eta^{(h)}(t)) + \fint_{t-h}^{t} \left[\frac{\rho_f}{2} \|u^{(h)}\|^2_{\Omega^{(h)}(s)} + \frac{\rho_s}{2} \|\partial_t \eta^{(h)}\|^2_Q \right] ds$$
$$+ \int_0^t [R_h(\nabla \eta^{(h)}, \partial_t \eta^{(h)}) + \nu \|\varepsilon u^{(h)}\|^2_{\Omega^{(h)}(s)} + h\|\nabla^{k_0} u^{(h)}\|^2_{\Omega^{(h)}(s)}]\, ds$$
$$\leq E_h(\eta_0) + \frac{1}{2}\|v_0\|^2_{\Omega_0} + \int_0^t [\rho_f \langle f, u^{(h)}\rangle_{\Omega^{(h)}(s)} + \rho_s \langle f \circ \eta^{(h)}, \partial_t \eta^{(h)}\rangle_Q]\, ds,$$

and moreover there exist $C, c > 0$ independent of h such that

$$E_h(\eta^{(h)}(t)) + c \fint_{t-h}^{t} [\|u^{(h)}\|^2_{\Omega^{(h)}(s)} + \|\partial_t \eta^{(h)}\|^2_Q]\, ds$$
$$+ \int_0^t [R_h(\nabla \eta^{(h)}, \partial_t \eta^{(h)}) + \nu \|\varepsilon u^{(h)}\|^2_{\Omega^{(h)}(s)} + h\|\nabla^{k_0} u^{(h)}\|^2_{\Omega^{(h)}(s)}]\, ds \leq C + Ct^2.$$

In both these estimates, $u^{(h)}$ and $\partial_t \eta^{(h)}$ are continued by their initial values to $t < 0$.

Proof. Lemma 4.8 translates for any $l \in \mathbb{N}_0$ such that $lh < T$ to

$$E_h(\eta_{l+1}) + \fint_0^s \left[\frac{\rho_f}{2} \|v_{l+1}\|^2_{\tilde{\Omega}_l(t)} + \frac{\rho_s}{2} \|\partial_t \tilde{\eta}_{l+1}\|^2_Q \right] dt$$
$$+ \int_0^s [R_h(\nabla \tilde{\eta}_{l+1}, \partial_t \nabla \tilde{\eta}_{l+1}) + \nu \|\varepsilon v_{l+1}\|^2_{\tilde{\Omega}_l(t)} + h\|\nabla^{k_0} v_{l+1}\|^2_{\Omega^{(h)}(t)}]\, dt$$
$$\leq E_h(\eta_l) + \fint_0^s \left[\frac{\rho_f}{2} \|w_l\|^2_{\Omega_l} + \frac{\rho_s}{2} \|w_l \circ \eta_l\|_Q \right] dt$$
$$+ \int_0^s [\langle f, v_{l+1}\rangle_{\Omega_l} + \rho_s \langle f \circ \tilde{\eta}_l, \partial_t \tilde{\eta}_l\rangle_Q]\, dt$$

for $s \in [0, h]$. Now by construction $\|w_l(t, \cdot)\|_{\Omega_l} = \|v_l(t, \cdot)\|_{\tilde{\Omega}_{l-1}(t)}$ and $\|w_l \circ \eta_l\|_Q = \|\partial_t \tilde{\eta}_l\|_Q$, and thus we can use a telescope argument to get the first energy inequality as we did in Lemma 3.6.

Next we use Young's inequality for the two force terms, and proceeding as for Lemma 2.14 and the corresponding estimate in Theorem 3.2 gives

$$\int_0^T \left[\frac{\rho_f}{2} \|v^{(h)}\|^2_{\Omega^{(h)}(t)} + \frac{\rho_s}{2} \|\partial_t \eta^{(h)}\|^2_Q \right] dt$$
$$\leq \sum_{l=1}^{N} h \fint_{(l-1)h}^{lh} \left[\frac{\rho_f}{2} \|v^{(h)}\|^2_{\Omega^{(h)}(t)} + \frac{\rho_s}{2} \|\partial_t \eta^{(h)}\|^2_Q \right] dt$$
$$\leq hN\left(C + \frac{C}{\delta} T \|f\|^2_\infty + \int_0^T \left[\frac{\delta \rho_f}{2} \|v^{(h)}\|^2_{\Omega^{(h)}(t)} + \frac{\delta \rho_s}{2} \|\partial_t \eta^{(h)}\|^2_Q \right] dt \right),$$

from which as before in Theorem 3.2 choosing $\delta = \frac{1}{2T}$ yields the desired estimate. ■

Corollary 4.12 (Minimal no-collision time). *Assume that $\eta_0 \notin \partial\mathcal{E}$. Then there is a $T > 0$ depending only on η_0, v_0 and f such that $\eta^{(h)}(s)$ is injective on $\bar{Q}$ for all $t \in [0,T]$ and h small enough, i.e. $\eta^{(h)}(t) \notin \partial\mathcal{E}$ and thus there is no collision.*

Proof. From the final estimate in the proof of Lemma 4.11 we get

$$\|\eta^{(h)} - \eta_0\|_Q^2 = \left\| \int_0^T \partial_t \eta^{(h)} \, dt \right\|_Q^2 \leq \int_0^T \|\partial_t \eta^{(h)}\|_Q^2 \, dt \leq TC(1+T^2).$$

Using this bound for small enough T then allows us to apply the short-distance injectivity result of Proposition 2.7. ∎

As a direct consequence of the uniform bounds of $\det(\nabla\eta^{(h)})$, the definitions of E_h and R_h as well as Lemma 2.11, we obtain

Corollary 4.13 (Korn-type estimate). *There is a constant C, just depending on the energy estimate in Lemma 4.11, such that*

$$\sup_{t\in[0,T-h]} \fint_t^{t+h} \|u^{(h)}(s)\|_\Omega^2 \, ds + \int_0^T \|\partial_t \eta^{(h)}\|_{W^{1,2}(Q)}^2 dt + \int_0^T \|u^{(h)}\|_{W^{1,2}(\Omega)}^2 \, dt \leq C,$$

$$\sup_{t\in[0,T]} h^{a_0} \|\eta^{(h)}\|_{W^{k_0,2}(Q)}^2 + h \int_0^T \|\partial_t \eta^{(h)}\|_{W^{k_0,2}(Q)}^2 \, dt + \int_0^T \|u^{(h)}\|_{W^{k_0,2}(\Omega)}^2 \, dt \leq C.$$

Proof of Theorem 1.2, Step 2: The weak time derivative. In the following, we may understand $\partial_t \eta^{(h)}$ and $u^{(h)}$ to be extended by their initial values for $t \in [-h, 0]$.

Lemma 4.14 (Length h bounds (fluid)). *Fix $T > 0$. Then there exists a constant C, depending only on the initial data, such that the following holds:*

(1) $\int_0^T \left\| \frac{\partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h)}{h} \right\|_{W^{-k_0,2}(Q)}^2 dt \leq C$,

(2) $\|\xi(t) - \xi(t-s_0) \circ \Phi_{-s_0}^{(h)}(t)\|_\Omega \leq Ch \operatorname{Lip}_{t,y}(\xi)$ *for all* $\xi \in C_0^\infty([0,T]\times\Omega)$ *and* $s_0 \in [-h,h]$,

(3) $\|\xi - \xi \circ \Phi_{s_0}^{(h)}(t)\|_\Omega \leq Ch \operatorname{Lip}_y(\xi)$ *for all* $\xi \in C_0^\infty(\Omega)$, $s_0 \in [-h,h]$ *and* $t \in [0,T]$.

Here we use Lip_y and $\operatorname{Lip}_{t,y}$ to distinguish the Lipschitz constants with respect to space and space-time respectively.

Proof. The first estimate is shown in almost the same way as Lemma 3.7. Indeed, as we only test by functions that vanish on the boundary, we can afford to set ξ to 0 on the fluid domain.

For the second estimate, let $\xi \in C_0^\infty([0,T]\times\Omega;\mathbb{R}^n)$ and calculate

$$\begin{aligned}\int_\Omega |\xi(t) - \xi(t-s_0) \circ \Phi_{-s_0}^{(h)}(t)|^2 \, dy &= \int_\Omega \left| \int_{-s_0}^0 \partial_s (\xi(t+s) \circ \Phi_s^{(h)}(t)) \, ds \right|^2 dy \\ &= \int_\Omega \left| \int_{-s_0}^0 [\nabla\xi(t+s) \cdot u^{(h)}(t+s) + \partial_t \xi(t+s)] \circ \Phi_s^{(h)}(t) \, ds \right|^2 dy\end{aligned}$$

$$
\begin{aligned}
&\leq s_0 \int_{-s_0}^{0} \int_\Omega |[\nabla\xi(t+s)\cdot u^{(h)}(t+s) + \partial_t \xi(t+s)] \circ \Phi_s^{(h)}(t)|^2 \,dy\,ds \\
&\leq h^2 \operatorname{Lip}_{t,y}(\xi)^2 \fint_{t-h}^{t} \int_\Omega \det(\nabla\Phi_{-s}^{(h)}(t+s))(|u^{(h)}(s)|^2+1)\,ds \\
&\leq Ch^2 \operatorname{Lip}_{t,y}(\xi)^2 \fint_{t-h}^{t} (\|u^{(h)}(s)\|_\Omega + 1)^2\,ds
\end{aligned}
$$

using the uniform bounds of $\det(\nabla\Phi_{-s}^{(h)}(t+s))$ (Lemma 4.10) and Corollary 4.13. This implies (2). The third assertion follows by the same arguments. ■

The next proposition estimates the weak time derivative of the global momentum.

Proposition 4.15. *There is an $m \geq k_0$ and a constant C independent of h such that for all $\xi \in C^0([0,T]; W_0^{m,2}(\Omega;\mathbb{R}^n))$ with* $\operatorname{div}\xi = 0$ *on $\Omega(t)$,*

$$
\int_0^T \left|\left\langle \tfrac{(\rho^{(h)}u^{(h)})(t)-(\rho^{(h)}u^{(h)})(t-h)}{h}, \xi(t)\right\rangle_\Omega\right| dt \leq C\|\xi\|_{L^2([0,T];W^{m,2}(\Omega))}.
$$

Proof. Let $\xi \in C^0([0,T]\times\Omega;\mathbb{R}^n)$ with $\operatorname{div}\xi(t) = 0$ on $\Omega^{(h)}(t)$ for all $t \in [0,T]$ and define $\phi := \xi \circ \eta^{(h)}$. Let us first split the integrand in two along the flow map:

$$
\begin{aligned}
\left\langle \tfrac{(\rho^{(h)}u^{(h)})(t)-(\rho^{(h)}u^{(h)})(t-h)}{h}, \xi(t)\right\rangle_\Omega &= \left\langle \tfrac{(\rho^{(h)}u^{(h)})(t)-(\rho^{(h)}u^{(h)})(t-h)\circ\Phi_{-h}^{(h)}(t)}{h}, \xi(t)\right\rangle_\Omega \\
&\quad - \left\langle \tfrac{(\rho^{(h)}u^{(h)})(t-h)-(\rho^{(h)}u^{(h)})(t-h)\circ\Phi_{-h}^{(h)}(t)}{h}, \xi(t)\right\rangle_\Omega =: J_1(t) - J_2(t).
\end{aligned}
$$

Now we estimate $J_1(t)$ by changing variables on the fluid domain and using (4.4):

$$
\begin{aligned}
\int_0^T |J_1(t)|\,dt &= \int_0^T \Big|\rho_s \left\langle \tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h}, \phi\right\rangle_Q \\
&\quad + \rho_f \left\langle \tfrac{u^{(h)}(t)\circ\Phi_h^{(h)}(t-h)-u^{(h)}(t-h)}{h}, \xi(t)\circ\Phi_h^{(h)}(t-h)\right\rangle_{\Omega^{(h)}(t-h)}\Big|\,dt \\
&\leq \int_0^T [|\langle DE(\eta^{(h)}),\phi\rangle| + h^{a_0}|\langle\nabla^{k_0}\eta^{(h)},\nabla^{k_0}\phi\rangle_Q| + |\langle D_2R_h(\eta^{(h)},\partial_t\eta^{(h)}),\phi\rangle| \\
&\quad + h|\langle\nabla^{k_0}\partial_t\eta^{(h)},\nabla^{k_0}\phi\rangle_Q| + \nu|\langle\varepsilon u^{(h)},\varepsilon\xi\rangle_{\Omega(t)}| + h|\langle\nabla^{k_0}u^{(h)},\nabla^{k_0}\xi\rangle_{\Omega(t)}| \\
&\quad + \rho_f|\langle f,\xi\rangle_{\Omega(t)}| + \rho_s|\langle f\circ\eta,\phi\rangle_Q|]\,dt \\
&\leq c\int_0^T [\|DE(\eta^{(h)})\|_{W^{-2,q}(Q)}\|\xi(t)\|_{W^{2,q}(\Omega)}\|\eta^{(h)}(t)\|_{W^{2,q}(Q)} \\
&\quad + \|D_2R(\eta^{(h)},\partial_t\eta^{(h)})\|_{W^{-1,2}(Q)}\|\xi(t)\|_{W^{2,q}(\Omega)}\|\eta^{(h)}(t)\|_{W^{2,q}(Q)} \\
&+ \|\eta^{(h)}(t)\|_{W^{k_0,2}(Q)}(h^{a_0}\|\nabla^{k_0}\eta^{(h)}\|\,\|\xi(t)\|_{C^{k_0}(\Omega)} + h\|\partial_t\eta^{(h)}(t)\|_{W^{k_0,2}(Q)}\|\xi(t)\|_{C^{k_0}(\Omega)}) \\
&\quad + h\|u^{(h)}(t)\|_{W^{k_0,2}(Q)}\|\xi\|_{W^{k_0,2}(Q)} + \|\varepsilon u^{(h)}\|_{\Omega(t)}\|\varepsilon\xi\|_{\Omega(t)} + \|f\|_\infty\|\xi\|_{\Omega(t)}]\,dt,
\end{aligned}
$$

where we have used the fact that by Proposition A.4 we have the estimates

$$
\begin{aligned}
\|\phi(t)\|_{W^{2,q}(Q)} &\leq c\|\xi(t)\|_{W^{2,q}(\Omega)}\|\eta^{(h)}(t)\|_{W^{2,q}(Q)}, \\
\|\phi(t)\|_{W^{k_0,2}(Q)} &\leq c\|\xi(t)\|_{C^{k_0}(\Omega)}\|\eta^{(h)}(t)\|_{W^{k_0,2}(Q)}.
\end{aligned}
$$

From the energy estimate in Lemma 4.11 we additionally know that $\|\eta^{(h)}(t)\|_{W^{2,q}(Q)}$ and $h^{a_0/2}\|\eta^{(h)}(t)\|_{W^{k_0,2}(Q)}$ are uniformly bounded in h and t. Thus every term is a product of a quantity which has (at least) a uniform $L^2([0,T])$ bound using the energy estimate and a term which can be estimated against $\|\xi(t)\|_{C^{k_0}(\Omega)}$. Choosing m such that $W^{m,2}(\Omega;\mathbb{R}^n)$ embeds into $C^{k_0}(\Omega;\mathbb{R}^n)$ then gives us

$$\int_0^T |J_1(t)|\,dt \le C\|\xi\|_{L^2([0,T];W^{m,2}(\Omega))}.$$

For $J_2(t)$ we first note that by the density preserving nature of Φ (see Lemma 4.10) we can obtain, by a change of variables,

$$\langle(\rho^{(h)}u^{(h)})(t-h)\circ\Phi^{(h)}_{-h}(t),\xi(t)\rangle_\Omega = \langle(\rho^{(h)}u^{(h)})(t-h),\xi(t)\circ\Phi^{(h)}_h(t-h)\rangle_\Omega$$

and thus

$$J_2(t) = \left\langle(\rho^{(h)}u^{(h)})(t-h),\tfrac{\xi(t)-\xi(t)\circ\Phi^{(h)}_h(t-h)}{h}\right\rangle_\Omega \le \|(\rho^{(h)}u^{(h)})(t-h)\|_\Omega C\,\mathrm{Lip}_y(\xi(t))$$

using Lemma 4.14 (3) as well as the uniform L^∞ bounds of $\rho^{(h)}$ and Corollary 4.13. ∎

Proof of Theorem 1.2, Step 3: Convergence to the limit. Now we again use our uniform bounds in h to find a converging subsequence and suitable limit functions for the approximating sequences:

Lemma 4.16 (Weak compactness). *There exists a subsequence of $h\to 0$ (not relabeled) and limit functions $\eta\in C_w([0,T];W^{2,q}(Q;\mathbb{R}^n))\cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^n))$ and $u\in L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n))$ such that*

$$\begin{aligned} \eta^{(h)} &\rightharpoonup \eta && \text{in } C_w([0,T];W^{2,q}(Q;\mathbb{R}^n)),\\ \partial_t\eta^{(h)} &\rightharpoonup \partial_t\eta && \text{in } L^2([0,T];W^{1,2}(Q;\mathbb{R}^n)),\\ u^{(h)} &\rightharpoonup u && \text{in } L^2([0,T];W^{1,2}(\Omega;\mathbb{R}^n)). \end{aligned}$$

Proof. Using the estimates from Lemma 4.11, we know that $E(\eta^{(h)}(t))$ is bounded independently of t and h and that $\int_0^T R(\eta^{(h)},\partial_t\eta^{(h)})\,dt$ is also uniformly bounded in h. Thus by Assumptions 1.7 and 1.10 we can pick a subsequence and a limit η such that the first two assertions hold.

Finally, we use the global Korn inequality of Lemma 2.11 to show that $\int_0^T \|u^{(h)}\|^2_{W^{1,2}(\Omega)}\,dt$ is uniformly bounded and find a limit u (after possibly choosing another subsequence) such that the last assertion is true. ∎

Exactly by the same argument as for Lemmas 3.8 and 3.9 we get:

Corollary 4.17 (Aubin–Lions & Minty (coupled solid)). *Let $b^{(h)}: t\mapsto \fint_{t-h}^t \partial_t\eta^{(h)}ds$. Then for a subsequence of h's (not relabeled) we have*

$$b^{(h)}\to\partial_t\eta \quad \text{in } L^2([0,T];L^2(Q;\mathbb{R}^n)),\qquad \eta^{(h)}\to\eta \quad \text{in } L^q([0,T];W^{2,q}(Q;\mathbb{R}^n)).$$

In particular, for almost all $t \in [0,T]$ we have

$$DE(\eta^{(h)}(t)) \to DE(\eta(t)) \quad \textit{in } W^{-2,q}(Q;\mathbb{R}^n).$$

We now want to prove a similar result for the Eulerian velocity $u^{(h)}$. While we have an estimate on the time derivative of $⨍_{-h}^{0} \rho^{(h)} u^{(h)}(t+s)\,ds$ in Proposition 4.15, this estimate is in a dual space of functions which are divergence free on the fluid domain and thus in a time- and h-dependent space. As a consequence, we are no longer in the realm of classical Aubin–Lions-type theorems and instead we need to prove a similar result directly.

Lemma 4.18 (Aubin–Lions (fluid)). *For each $t \in [0,T]$ and $h > 0$ define $\widetilde{m}^{(h)}(t) \in L^2(\Omega;\mathbb{R}^n)$ by*

$$\widetilde{m}^{(h)}(t) := ⨍_{-h}^{0} (\rho^{(h)} u^{(h)})(t+s)\,ds.$$

For all (sufficiently small) $\delta > 0$ there exists a subsequence of h's (not relabeled) such that for all $A \in C_0^\infty([0,T]\times\Omega;\mathbb{R}^{n\times n})$,

$$\int_0^T \langle (u^{(h)})_\delta, A\widetilde{m}^h \rangle_\Omega \,dt \to \int_0^T \langle (u)_\delta, A\rho u \rangle_\Omega \,dt,$$

where $(\cdot)_\delta$ is the regularization operator defined in Lemma 2.22, since $u^{(h)}$ plays the role of a test function here.

Proof. We begin with a couple of observations. First, as the operator $(\cdot)_\delta$ introduced in Lemma 2.22 is bounded and linear, we find that (for a nonrelabeled subsequence) $(u^{(h)})_\delta \rightharpoonup (u)_\delta$ as $h \to 0$ in $L^2([0,T]; W^{1,2}(\Omega))$; cf. also Lemma 4.16.

Next, as $\widetilde{m}^{(h)}$ is uniformly bounded in $L^\infty([0,T]; L^2(\Omega))$ (see Corollary 4.13), we find that (after possibly choosing another subsequence) there exists an $\widetilde{m} \in L^\infty([0,T]; L^2(\Omega))$ such that $\widetilde{m}^{(h)} \rightharpoonup^* \widetilde{m}$ in that space. Since for $\xi \in C_0^\infty([0,T]\times\Omega)$ we have

$$\begin{aligned}\int_0^T \langle \widetilde{m}^{(h)}, \xi \rangle_\Omega \,dt &= ⨍_{-h}^{0} \int_0^T \langle (\rho^{(h)} u^{(h)})(t+s), \xi(t) \rangle_\Omega \,dt\,ds \\ &= \int_0^T \left\langle (\rho^{(h)} u^{(h)})(t), ⨍_{-h}^{0} \xi(t-s)\,ds \right\rangle_\Omega dt \to \int_0^T \langle \rho u, \xi \rangle_\Omega \,dt,\end{aligned}$$

we also know that $\widetilde{m} = \rho u$ almost everywhere.

Take a sequence $(h_i)_i$ with $h_i \to 0$ chosen such that all convergences outlined above, including the one in Corollary 4.17, hold true. Next fix $\epsilon > 0$. We aim to show that there is an N_ϵ such that for another (nonrelabeled) subsequence and all $j > i > N_\epsilon$,

$$\left| \int_0^T \langle (u^{(h_i)}(t))_\delta, A(\widetilde{m}^{(h_i)}(t) - \widetilde{m}^{(h_j)}(t)) \rangle_\Omega \,dt \right| \le c\epsilon, \tag{4.6}$$

which implies the result. Our strategy is based on the approach introduced in [75, Theorem 5.1]. Thus, we will split the time interval $[0,T]$ into a finite number of subintervals of length σ, and depending on ϵ we will first choose the regularizing parameter δ and then the length parameter σ which will finally yield the sought number N_ϵ. Due to the changing fluid domain, we first need to ensure that $(u^{(h_i)}(t))_\delta$ are all divergence free on a fixed domain in which, for a given t, all $\Omega^{(h_i)}$ are included. For this, we use the uniform convergence of $\eta^{(h_i)} \to \eta$ that allows us for any given $\delta > 0$ to take h_i small enough (N_ϵ large enough) such that $\hat{\Omega}_\delta(t) = \bigcap_{i\geq N_\epsilon} \Omega^{(h_i)}(t)$ and $\check{\Omega}_\delta(t) = \bigcup_{i\geq N_\epsilon} \Omega^{(h_i)}(t)$ satisfy a small Hausdorff distance condition

$$\sup_{t\in[0,T]} \sup_{i\geq N_\epsilon} \Big(\sup_{y\in\Omega^{(h_i)}(t)} \operatorname{dist}(y, \hat{\Omega}_\delta(t)) + \sup_{y\in\check{\Omega}_\delta(t)} \operatorname{dist}(y, \Omega^{(h_i)}(t)) \Big) \leq \delta. \tag{4.7}$$

Next we may use the approximation introduced in Lemma 2.22 for $u^{(h_i)}$. The regularity of the domain allows us to assume that

$$\operatorname{div}((u^{(h_i)})_\delta(t)) = 0 \quad \text{in } \check{\Omega}_\delta(t).$$

Moreover, Lemma 2.22 implies that for almost every t and every $m \in \mathbb{N}$,

$$\begin{aligned} \|(u^{(h_i)}(t))_\delta\|_{W^{m,2}(\Omega)} &\leq c(\delta, m)\|u^{(h_i)}(t)\|_{W^{1,2}(\Omega)}, \\ \|(u^{(h_i)}(t))_\delta\|_{L^2([0,T];W^{1,2}(\Omega))} &\leq c\|u^{(h_i)}(t)\|_{L^2([0,T];W^{1,2}(\Omega))}, \\ \|(u^{(h_i)}(t))_\delta - u^{(h_i)}(t)\|_{L^2([0,T];L^2(\Omega))} &\leq c\delta^{\frac{2}{2+n}}\|u^{(h_i)}(t)\|_{L^2([0,T];W^{1,2}(\Omega))}. \end{aligned} \tag{4.8}$$

The parameter δ will be chosen later depending on ϵ. Furthermore, we choose (in dependence on δ) $\sigma > 0$ and $N \in \mathbb{N}$ such that $T = N\sigma$.

Now for any $k \in \{0, \dots, N\}$,

$$\|\tilde{m}^{(h_i)}(\sigma k)\|_\Omega^2 \leq \fint_{k\sigma-h}^{k\sigma} \|(\rho^{(h)}u^{(h)})(t)\|_\Omega^2 \, dt \leq C\|\rho^{(h)}\|_\infty \leq C\rho_{\max}$$

by the volume density preserving nature of Φ and by Lemma 4.11. Here, $\rho_{\max}$ is a uniform upper bound on the density in the fluid and the solid; the latter can easily be derived from the energy bounds. As usual we continue v and $\partial_t\eta$ to negative times by their initial data. We can thus use compact embeddings to find a subsequence of $h_i \to 0$ such that $\tilde{m}^{(h_i)}(\sigma k)$ converges strongly in $(W^{1,2}(\Omega;\mathbb{R}^n) \cap \{\operatorname{div} v|_{\check{\Omega}_\delta(t)} = 0\})^*$ for all $k \in \{0, \dots, N-1\}$. In particular, we can choose N_ϵ in such a way that for all $i, j \geq N_\epsilon$,

$$\|\tilde{m}^{(h_i)}(\sigma k) - \tilde{m}^{(h_j)}(\sigma k)\|_{(W^{1,2}(\Omega;\mathbb{R}^n)\cap\{\operatorname{div} v|_{\check{\Omega}_\delta(t)}=0\})^*} \leq \epsilon. \tag{4.9}$$

Now we rewrite, for $t \in [\sigma k, \sigma(k+1))$,

$$\begin{aligned} &\langle (u^{(h_i)}(t))_\delta, A(\tilde{m}^{(h_i)}(t) - \tilde{m}^{(h_j)}(t))\rangle_\Omega \\ &= \langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_i)}(t) - A\tilde{m}^{(h_i)}(\sigma k)\rangle_\Omega + \langle (u^{(h_i)}(t))_\delta, A(\tilde{m}^{(h_i)}(\sigma k) - \tilde{m}^{(h_j)}(\sigma k))\rangle_\Omega \\ &\quad + \langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_j)}(\sigma k) - A\tilde{m}^{(h_j)}(t)\rangle_\Omega \\ &=: \mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t). \end{aligned}$$

For $i, j \geq N_\epsilon$ we find, using (4.9) and (4.8),

$$\int_0^T \mathrm{II}(t)\, dt \leq C\epsilon.$$

The other two terms are estimated using the continuity of $\widetilde{m}^{(h_i)}$ in time. Indeed,

$$\begin{aligned}\partial_\theta \widetilde{m}^{(h_i)}(\theta, y) &= \partial_\theta \left(\fint_{-h_i}^0 (\rho^{(h_i)} u^{(h_i)})(\theta + s)\, ds \right) \\ &= \frac{1}{h_i} \partial_\theta \left(\int_{\theta-h_i}^{\theta} (\rho^{(h_i)} u^{(h_i)})(s)\, ds \right) = \tfrac{(\rho^{(h_i)} u^{(h_i)})(\theta) - (\rho^{(h_i)} u^{(h_i)})(\theta - h_i)}{h_i}\end{aligned}$$

and thus

$$\begin{aligned}\left| \int_0^T \mathrm{I}(t)\, dt \right| &= \left| \int_0^T \left\langle (u^{(h_i)}(t))_\delta, \int_{\sigma k}^t A \partial_\theta \widetilde{m}^{(h_i)}(\theta)\, d\theta \right\rangle_\Omega dt \right| \\ &\leq \sum_k \int_{\sigma k}^{(\sigma+1)k} \int_{\sigma k}^t \left| \langle (u^{(h_i)}(t))_\delta, A \tfrac{(\rho^{(h_i)} u^{(h_i)})(\theta) - (\rho^{(h_i)} u^{(h_i)})(\theta - h_i)}{h_i} \rangle_\Omega \right| d\theta\, dt \\ &= \sum_k \int_{\sigma k}^{\sigma(k+1)} \int_\theta^{\sigma(k+1)} \left| \langle (u^{(h_i)}(t))_\delta, A \tfrac{(\rho^{(h_i)} u^{(h_i)})(\theta) - (\rho^{(h_i)} u^{(h_i)})(\theta - h_i)}{h_i} \rangle_\Omega \right| dt\, d\theta \\ &\leq \sum_k \int_{\sigma k}^{\sigma(k+1)} \int_0^\sigma \left| \langle (u^{(h_i)}(\theta + s))_\delta, A \tfrac{(\rho^{(h_i)} u^{(h_i)})(\theta) - (\rho^{(h_i)} u^{(h_i)})(\theta - h_i)}{h_i} \rangle_\Omega \right| ds\, d\theta \\ &\leq \|A\|_\infty \int_0^\sigma \int_0^T \left| \langle (u^{(h_i)}(\theta + s))_\delta, \tfrac{(\rho^{(h_i)} u^{(h_i)})(\theta) - (\rho^{(h_i)} u^{(h_i)})(\theta - h_i)}{h_i} \rangle_\Omega \right| dt\, ds \\ &\leq \|A\|_\infty \int_0^\sigma \|(u^{(h_i)}(\cdot + s))_\delta\|_{L^2([0,T];W^{m,2}(\Omega))}\, ds \\ &\leq \|A\|_\infty C_\delta \sigma \|u^{(h_i)}\|_{L^2([0,T];W^{1,2}(\Omega))}\end{aligned}$$

using Proposition 4.15.

Using an analogous estimate on $\mathrm{III}(t)$, we find (4.6) by choosing σ small enough. ■

Observe that due to the strong convergence of $\partial_t \eta^{(h)}$ (and consequently of u on the sets $\eta^{(h)}(t, Q)$) we also find

$$\int_0^T \langle (u^{(h)})_\delta, A\widetilde{m}^{(h)} \rangle_{\Omega^{(h)}(t)}\, dt \to \rho_f \int_0^T \langle (u)_\delta, Au \rangle_{\Omega(t)}\, dt \tag{4.10}$$

for all $A \in C_0^\infty(\Omega)$.

Proof of Theorem 1.2, Step 3a: Passing to the limit with the coupled PDEs. In the following we assume that T is small enough such that there is a sequence $(\eta^{(h)}, u^{(h)})$ of approximate solutions on the interval $[0, T + h]$. Later it will be discussed how to prolong the solution up to the point of contact.

As before in the proof of Proposition 2.20 and Theorem 4.2, we use Lemma 2.22 to restrict ourselves to test functions $\xi \in C_0^\infty(\Omega;\mathbb{R}^n)$ with $\operatorname{div}\xi = 0$ in a neighborhood of $\Omega(t)$. We then construct $\phi^{(h)} := \xi \circ \eta^{(h)}$ and pass to the limit $h \to 0$. Most of the terms in the weak equation (4.4) converge by the same arguments as in the previous sections.

What is different is the inertial term of the fluid. Again we transfer the difference quotient to the test function. For that we have to take into account the flow map $\Phi_h^{(h)}$:

$$\begin{aligned}
&\int_0^T \Big\langle \tfrac{u^{(h)}(t)\circ\Phi_h^{(h)}(t-h)-u^{(h)}(t-h)}{h}, \xi(t)\circ\Phi_h^{(h)}(t-h)\Big\rangle_{\Omega^{(h)}(t-h)}\,dt \\
&\qquad = -\int_0^T \Big\langle u^{(h)}(t), \tfrac{\xi(t+h)\circ\Phi_h^{(h)}(t)-\xi(t)}{h}\Big\rangle_{\Omega^{(h)}(t)}\,dt \\
&\qquad = -\int_0^T \Big\langle (u^{(h)}(t))_\delta, \tfrac{\xi(t+h)\circ\Phi_h^{(h)}(t)-\xi(t)}{h}\Big\rangle_{\Omega^{(h)}(t)}\,dt \\
&\qquad\quad + \int_0^T \Big\langle (u^{(h)}(t))_\delta - u^{(h)}(t), \tfrac{\xi(t+h)\circ\Phi_h^{(h)}(t)-\xi(t)}{h}\Big\rangle_{\Omega^{(h)}(t)}\,dt =: -\mathrm{I}^{\delta,h} + \mathrm{II}^{\delta,h},
\end{aligned}$$

where $(u^{(h)}(t))_\delta$ is a regularization in space, as defined in Lemma 2.22. Since the right hand side in the scalar product of $\mathrm{II}^{\delta,h}$ is uniformly bounded in $L^\infty([0,T];L^2(\Omega(t);\mathbb{R}^n))$, using (4.8) we know that $\mathrm{II}^{\delta,h}$ vanishes as $\delta \to 0$ (uniformly in h). For the first term we expand

$$\begin{aligned}
\mathrm{I}^{\delta,h} &= \int_0^T \Big\langle (u^{(h)}(t))_\delta, \fint_0^h \partial_s(\xi(t+s)\circ\Phi_s^{(h)}(t))\,ds\Big\rangle_{\Omega^{(h)}(t)}\,dt \\
&= \int_0^T \Big\langle (u^{(h)}(t))_\delta, \fint_0^h (\partial_t\xi(t+s) - u^{(h)}(t+s)\cdot\nabla\xi(t+s))\circ\Phi_s^{(h)}(t)\,ds\Big\rangle_{\Omega^{(h)}(t)}\,dt \\
&= \int_0^T \fint_0^h \langle (u^{(h)}(t))_\delta\circ\Phi_{-s}^{(h)}(t+s), \partial_t\xi(t+s) - u^{(h)}(t+s)\cdot\nabla\xi(t+s)\rangle_{\Omega^{(h)}(t+s)}\,ds\,dt \\
&= \int_0^T \fint_0^h \langle (u^{(h)}(t))_\delta, \partial_t\xi(t+s) - u^{(h)}(t+s)\cdot\nabla\xi(t)\rangle_{\Omega^{(h)}(t+s)}\,ds\,dt \\
&\quad + \int_0^T \fint_0^h \langle (u^{(h)}(t))_\delta, u^{(h)}(t+s)\cdot\nabla(\xi(t)-\xi(t+s))\rangle_{\Omega^{(h)}(t+s)}\,ds\,dt \\
&\quad + \int_0^T \fint_0^h \langle (u^{(h)}(t))_\delta\circ\Phi_{-s}^{(h)}(t+s) - (u^{(h)})_\delta(t), \\
&\qquad\qquad\qquad \partial_t\xi(t+s) - u^{(h)}(t+s)\cdot\nabla\xi(t+s)\rangle_{\Omega^{(h)}(t+s)}\,ds\,dt.
\end{aligned}$$

Since $\|\nabla(\xi(t)-\xi(t+s))\|_{L^\infty(\Omega)} \le Ch\|\partial_t\nabla\xi\|_{L^\infty([0,T]\times\Omega)}$, the second term in the last sum converges to zero as $h \to 0$. For the third term, we may use Lemma 4.14 to see that the L^2 norm of the left hand side in the scalar product is bounded by $Ch\,\mathrm{Lip}_y((u^{(h)}(t))_\delta)$, which is in turn bounded by $hC_\delta\|u^{(h)}(t)\|_{W^{1,2}(\Omega)}$, so that this term vanishes for $h \to 0$. For the first term we aim to apply Lemma 4.18. To do so, we take $A_\delta \in C^0([0,T];C_0^\infty(\hat\Omega_\delta))$ such that $A_\delta(t) \to \chi_{\Omega(t)}$ almost everywhere in Ω. Hence by

Lemma 4.18 in the form of (4.10) we find that

$$
\begin{aligned}
\lim_{h\to 0} \mathrm{I}^{\delta,h} &= \lim_{h\to 0}\int_0^T \fint_0^h \langle (u^{(h)}(t))_\delta, \partial_t\xi(t+s) - u^{(h)}(t+s)\cdot\nabla\xi(t)\rangle_{\Omega^{(h)}(t+s)}\,ds\,dt\\
&= \int_0^T \langle (u(t))_\delta, \partial_t\xi - u\cdot\nabla\xi A_\delta(t)\rangle_{\Omega(t)}\,dt\\
&\quad + \lim_{h\to 0}\int_0^T \fint_0^h \langle (u^{(h)}(t))_\delta, u^{(h)}(t+s)\cdot\nabla\xi(t)(A_\delta(t) - \chi_{\Omega^{(h)}}(t+s))\rangle_\Omega\,ds\,dt.
\end{aligned}
$$

The last term is estimated by Hölder's inequality and Sobolev embedding. Indeed, for $a < \frac{n}{n-2}$ we find, by (4.8),

$$
\begin{aligned}
&\left|\int_0^T \fint_0^h \langle (u^{(h)}(t))_\delta, u^{(h)}(t+s)\cdot\nabla\xi(t)(A_\delta(t) - \chi_{\Omega^{(h)}}(t+s))\rangle_\Omega\,ds\,dt\right|\\
&\quad\le \int_0^T \|(u^{(h)}(t))_\delta\|_{L^{2a}(\Omega)} \fint_0^h \|u^{(h)}(t+s)\|_{L^a(\Omega)}\|A_\delta(t)-\chi_{\Omega^{(h)}}(t+s)\|_{L^{2a'}(\Omega)}\,ds\,dt\\
&\quad\le c\|(u^{(h)}(t))_\delta\|_{L^2([0,T];W^{1,2}(\Omega))}\sup_{t\in T}\left(\fint_0^h \|u^{(h)}(t+s)\|^2\,ds\right)^{1/2}\\
&\qquad\times\left(\fint_0^h \|A_\delta(t)-\chi_{\Omega^{(h)}}(t+s)\|^2_{L^2([0,T];L^{2a'}(\Omega))}\,ds\right)^{1/2}\\
&\quad\le c\left(\fint_0^h \|A_\delta(\cdot)-\chi_{\Omega^{(h)}}(\cdot+s)\|^2_{L^2([0,T];L^{2a'}(\Omega))}\,ds\right)^{1/2}.
\end{aligned}
$$

By the uniform convergence of $\eta^{(h)}\to\eta$, we find that

$$
\begin{aligned}
&\lim_{h\to 0}\left(\fint_0^h \|A_\delta(\cdot) - \chi_{\Omega^{(h)}}(\cdot+s)\|^2_{L^2([0,T];L^{2a'}(\Omega))}\,ds\right)^{1/2}\\
&\qquad\qquad = \|A_\delta - \chi_\Omega\|_{L^2([0,T];L^{2a'}(\Omega))}.
\end{aligned}
$$

Finally, by passing to the limit $\delta\to 0$ we have

$$
\lim_{\delta\to 0}\lim_{h\to 0}(-\mathrm{I}^{\delta,h} + \mathrm{II}^{\delta,h}) = -\int_0^T \langle u, \partial_t\xi - u\cdot\nabla\xi\rangle_{\Omega(t)}\,dt.
$$

Thus we have shown that we obtain the right equation in the limit:

$$
\begin{aligned}
&\int_0^T [-\rho_s\langle\partial_t\eta, \partial_t\phi\rangle_Q - \rho_s\langle v, \partial_t\xi - v\cdot\nabla\xi\rangle_{\Omega(t)}\\
&\qquad\qquad + \langle DE(\eta),\phi\rangle + \langle D_2R(\eta,\partial_t\eta),\phi\rangle + \nu\langle\varepsilon v, \varepsilon\xi\rangle_{\Omega(t)}]\,dt\\
&= \int_0^T [\rho_s\langle f\circ\eta, \phi\rangle_Q + \rho_f\langle f,\xi\rangle_{\Omega(t)}]\,dt - \rho_s\langle\eta_*, \phi(0)\rangle_Q - \rho_f\langle v_0, \xi(0)\rangle_{\Omega_0}. \qquad (4.11)
\end{aligned}
$$

Proof of Theorem 1.2, Step 3b: Reconstruction of the pressure. As we do not want to consider the time derivatives of the operator $\mathcal{B}_t$, we cannot follow the same lines as in the proof of Theorem 2.2. Instead, we have to proceed in a global manner. We construct the pressure as a distribution.

Let $\psi \in C_0^\infty([0,T] \times \Omega)$. Take $\mathcal{B}$ to be the operator from Theorem 2.21 with respect to the domain Ω. To apply this operator to ψ, we need to normalize its mean by picking a $\tilde{\psi} \in C_0^\infty([0,T] \times \Omega)$ with $\operatorname{supp}(\tilde{\psi}(t)) \cap \Omega(t) = \emptyset$ and $\int_\Omega \tilde{\psi}(t)\, dy = -\int_\Omega \psi(t)\, dy$ for all $t \in [0,T]$.

Now let $\xi(t) := \mathcal{B}(\psi(t) + \tilde{\psi}(t))$, $\phi(t,x) := \xi(t, \eta(t,x))$ and define a linear operator by

$$P(\psi) := \int_0^T [\langle DE(\eta(t)), \phi\rangle + \langle D_2 R(\eta(t), \partial_t \eta(t)), \phi\rangle + \nu \langle \varepsilon u, \varepsilon \xi\rangle_{\Omega(t)} \\ - \rho_f \langle f, \xi\rangle_{\Omega(t)} - \rho_s \langle f \circ \eta, \phi\rangle_Q - \rho_s \langle \partial_t \eta, \partial_t \phi\rangle_Q - \rho_f \langle u, \partial_t \xi - u \cdot \nabla \xi\rangle_{\Omega(t)}]\, dt.$$

Note that $P(\psi)$ is independent of the choice of $\tilde{\psi}$: If $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are two such choices with corresponding ξ_1 and ξ_2, then $\xi_1 - \xi_2 = \mathcal{B}(\tilde{\psi}_1 - \tilde{\psi}_2)$ has divergence 0 on $\Omega(t)$ and thus the above integral is the same because of (4.11). In particular, if $\operatorname{supp}(\psi(t)) \subset \eta(t, Q)$ (for all $t \in [0,T]$), we may choose $\tilde{\psi} \equiv \psi$, which implies (by the linearity of $\mathcal{B}$) that $P(\psi) = 0$. Hence $\operatorname{supp}(P) \subset [0,T] \times \overline{\Omega(t)}$.

Furthermore, it can be estimated that

$$\int_0^T [\langle DE(\eta(t)), \phi\rangle + \langle D_2 R(\eta(t), \partial_t \eta(t)), \phi\rangle + \nu \langle \varepsilon u, \varepsilon \xi\rangle_{\Omega(t)} \\ - \rho_f \langle f, \xi\rangle_{\Omega(t)} - \rho_s \langle f \circ \eta, \phi\rangle_Q]\, dt \leq C \|\phi\|_{L^1([0,T], W^{2,q}(Q))} + \|\xi\|_{L^2([0,T]; W^{1,2}(\Omega))}$$

via the known bounds on the terms in the weak equation. Finally, using Proposition A.4 we know that $\|\phi\|_{W^{2,q}(Q)} \leq C \|\xi\|_{W^{2,q}(\Omega)}$. Consequently, by the properties of the Bogovskiĭ operator we find

$$C \|\phi\|_{L^1([0,T], W^{2,q}(Q))} + \|\xi\|_{L^2([0,T]; W^{1,2}(\Omega))} \\ \leq C \|\psi\|_{L^1([0,T], W^{2,q}(Q))} + C \|\psi\|_{L^2([0,T]; L^2(\Omega))},$$

where we note that $\tilde{\psi}(t)$ can be chosen as a multiple of a fixed C_0^∞-function and thus its norm only needs to depend on $|\int_\Omega \psi(t)\, dy| \leq c \|\psi(t)\|_\Omega$. Additionally, for the other remaining terms we have

$$\left| \int_0^T \langle \partial_t \eta, \partial_t \phi\rangle_Q\, dt \right| \leq \|\partial_t \eta\|_{L^2([0,T] \times Q)} \|\phi\|_{W^{1,2}([0,T]; L^2(Q))},$$

$$\left| \int_0^T \langle u, \partial_t \xi\rangle_{\Omega(t)}\, dt \right| \leq \|u\|_{L^2([0,T] \times \Omega)} \|\xi\|_{W^{1,2}([0,T]; L^2(\Omega))},$$

$$\left| \int_0^T \langle u, u \cdot \nabla \xi\rangle_{\Omega(t)}\, dt \right| \leq \|u^2\|_{L^a([0,T]; L^b(\Omega))} \|\xi\|_{L^{a'}([0,T]; W^{1,b'}(\Omega))},$$

where $a, b \in (1, \infty)$ are chosen in such a way that $|u|^2 \in L^a([0,T]; L^b(\Omega))$, which is possible since $|u|^2 \in L^\infty([0,T]; L^1(\Omega)) \cap L^1([0,T], L^p(\Omega))$ (with $p = \frac{n}{n-2}$ for $n > 2$ or p arbitrarily large for $n = 2$). Now bounding the norms of ξ and ϕ in terms of ψ as before proves that $P \in \mathcal{D}'([0,T] \times \Omega)$. Thus p is well defined via that operator, and expanding

$$\int_0^T \langle \nabla p, \xi \rangle \, dt = P(\operatorname{div} \xi)$$

proves that it fulfills the right equations for $\xi \in C^\infty([0,T] \times \Omega)$. Moreover, it can be decomposed as

$$\begin{aligned} p \in{} & L^\infty([0,T]; W^{-1,q}(\Omega)) + L^2([0,T] \times \Omega) \\ & + W^{-1,2}([0,T]; W^{-1,2}(\Omega)) \cap L^{a'}([0,T]; W^{-1,b'}(\Omega)). \end{aligned} \tag{4.12}$$

Proof of Theorem 1.2, Step 4: Energy inequality & maximal interval of existence. Above, we have shown existence of coupled weak solutions u, η on $[0,T]$ for some $T > 0$. As before, we can now pick a maximal interval $[0, T_{\max})$ and use the energy bounds to conclude that either $T_{\max} = \infty$ or there exists a limit $\eta(T_{\max}) \in \partial \mathcal{E}$.

Finally, we observe that (1.12) follows by Lemma 4.11.

Appendix A

A.1. Some technical lemmas

Here we gather the proofs of some technical lemmas.

Lemma A.1 (Expansion of the determinant). *Let $A \in \mathbb{R}^{n \times n}$. Then*

$$\det(I + \tau A) = 1 + \tau \operatorname{tr} A + \sum_{l=2}^n \tau^l M_l(A),$$

where $M_l(A)$ is a homogeneous polynomial of degree l in the entries of A. Note that this is a finite sum.

Proof. Consider the Leibniz formula

$$\det(I + \tau A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^n (\delta_{i,\pi(i)} + \tau A_{i,\pi(i)}),$$

where S_n is the set of permutations of $\{1, \dots, n\}$. We expand the product and order the terms by the exponent of the factor τ^l and thus by the number of terms $\tau A_{i,\pi(i)}$ that are taken while expanding the product. This will directly yield the homogeneous polynomial $M_l(A)$.

For τ^0 and τ^1, the only nonzero terms occur for $\pi = \mathrm{id}$, otherwise there will be at least one factor $\delta_{i,\pi(i)}$ for $i \neq \pi(i)$. For τ^0 this means we only choose the $\delta_{i,i}$ terms and for τ^1 we can choose any one $\tau A_{i,i}$ term. Thus $M_0(A) = 1$ and $M_1(A) = \operatorname{tr} A$. ∎

Lemma A.2 (Invertible maps). *Let $\eta \in W^{2,q}(Q;\mathbb{R}^n)$ be injective and such that* $\det \nabla\eta > \epsilon_0 > 0$ *for some $\epsilon_0 < 1$ and $\eta|_P = \gamma$. Then $\eta^{-1} \in W^{2,q}(\eta(Q);\mathbb{R}^n)$ and $\|\eta^{-1}\|_{W^{2,q}(\eta(Q))} \leq c\|\eta\|^{2n-1}_{W^{2,q}(Q)}/\epsilon_0^2$, where c depends only on q, Q, γ and n.*

Proof. Due to the condition on the determinant, $\nabla\eta$ is invertible, and furthermore we have the well-known formula

$$\nabla(\eta^{-1}) = (\nabla\eta)^{-1} \circ \eta^{-1} = \tfrac{(\operatorname{cof}\nabla\eta)^T}{\det\nabla\eta} \circ \eta^{-1}.$$

Now we take the derivative of $\nabla(\eta^{-1}) \circ \eta$ to get

$$(\nabla^2(\eta^{-1})) \circ \eta \cdot \nabla\eta = \nabla(\nabla(\eta^{-1}) \circ \eta) = \tfrac{\nabla(\operatorname{cof}\nabla\eta)^T}{\det\nabla\eta} - \tfrac{(\operatorname{cof}\nabla\eta)^T \otimes (\operatorname{cof}\nabla\eta)}{(\det\nabla\eta)^2} \cdot \nabla^2\eta.$$

Integrating then yields

$$\begin{aligned}\int_{\eta(Q)} |(\nabla^2(\eta^{-1}))|^q\, dy &= \int_Q |(\nabla^2(\eta^{-1})) \circ \eta|^q \det\nabla\eta\, dx \\ &= \int_Q \Big| \tfrac{\nabla(\operatorname{cof}\nabla\eta)^T}{\det\nabla\eta} - \tfrac{(\operatorname{cof}\nabla\eta)^T \otimes (\operatorname{cof}\nabla\eta)}{(\det\nabla\eta)^2} \cdot \nabla^2\eta \Big|^q \det\nabla\eta\, dx.\end{aligned}$$

Now the determinants in the denominators can be estimated by ϵ_0, while the numerators all consist of one second derivative multiplied with a number of first derivatives, which we can estimate by their supremum:

$$\leq \int_Q C\big(\tfrac{|\nabla^2\eta|\|\nabla\eta\|_\infty^{n-2}}{\epsilon_0^{1-1/q}} + \tfrac{\|\nabla\eta\|_\infty^{2n-2}|\nabla^2\eta|}{\epsilon_0^{2-1/q}}\big)^q\, dx \leq C\,\frac{\|\nabla^2\eta\|^q_{L^q(\eta(Q))}\|\nabla\eta\|_\infty^{q(2n-2)}}{\varepsilon^{2q}}.$$

Using the Morrey embedding $\|\nabla\eta\|_\infty \leq \|\nabla\eta\|_{C^\alpha} \leq C\|\eta\|_{W^{2,q}(Q)}$ and collecting the terms then shows

$$\|\nabla^2(\eta^{-1})\|_{L^q(\eta(Q))} \leq C\,\frac{\|\eta\|^{2n-1}_{W^{2,q}(Q)}}{\epsilon_0^2}.$$

Finally, as we have partially known boundary values because $\eta^{-1}|_{\gamma(P)} = \gamma^{-1}$, the lower order estimates follow from a Poincaré inequality. ∎

For the next result we need an interpolation. We begin by recalling the following result, which follows for instance from the interpolation estimate in [87, Theorem 2.13] which implies together with the usual Sobolev embeddings [88, Theorem 2.5.1 and Remark 2.5.2] that for all $m \in [0,\infty)$, $\alpha \in [1,\infty)$ and all Lipschitz domains Ω satisfying

$$m \leq l \quad \text{and} \quad \frac{1}{\alpha} - \frac{m}{n} \geq \frac{1}{\gamma} - \frac{l}{n} = \frac{k-l}{ka} + \frac{l}{2k} - \frac{l}{n}$$

we have the estimate

$$\|g\|_{W^{m,\alpha}} \leq C\|g\|^{l/k}_{W^{k,2}}\|g\|^{(k-l)/k}_{L^a}. \tag{A.1}$$

Lemma A.3. *Let $Q \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $q > n$, and let $k \in \mathbb{N}$ be defined as*

$$k = 2 + \frac{n+1}{2} \text{ if } n \text{ is odd}, \quad k = 3 + \frac{n}{2} \text{ if } n \text{ is even}. \tag{A.2}$$

For every $\eta \in W^{2,q}(Q) \cap W^{k,2}(Q)$, there is a constant c depending on Q, n, k and $\|\eta\|_{W^{2,q}(Q)}$ such that

$$\sum_{l=1}^{k} \sum_{a \in \{1,\ldots,n\}^l} \left\| \nabla^{k-l} \prod_{i=1}^{l} \partial_{a_i} \eta \right\| \leq c \|\eta\|_{W^{k,2}(Q)}.$$

Proof. Observe that since $\nabla \eta$ is uniformly bounded by the $W^{2,q}(Q;\mathbb{R}^n)$ norm, we find that

$$\sum_{a \in \{1,\ldots,n\}^l} \left\| \nabla^{k-l} \prod_{i=1}^{l} \partial_{a_i} \eta \right\| \leq c \sum_{\beta \in \mathbb{N}_0^l,\ |\beta| = k-l} \left\| \prod_{i=1}^{l} |\nabla^{\beta_i} \nabla \eta| \right\|.$$

The estimate for $l = 1$ is direct. Next assume that $l \geq 2$ and $\beta \in \mathbb{N}_0^l$ with $|\beta| = k - l$ is such that all $\beta_i \neq 1$. Now by Hölder's and Young's inequalities,

$$\prod_{i=1}^{l} \|\nabla^{\beta_i} \nabla \eta\| \leq c \sum_{\beta_i > 1} \|\nabla^{\beta_i - 1} \nabla^2 \eta\|_{2(k-l)/\beta_i}^{(k-l)/\beta_i}.$$

Next we seek to interpolate $\nabla^2 \eta$ between $W^{2,q}$ and $W^{k-2,2}$. For that we wish to use (A.1). Hence we have to prove that

$$\frac{\beta_i}{2(k-l)} \geq \frac{k-1-\beta_i}{q(k-2)} + \frac{\beta_i - 1}{2(k-2)}. \tag{A.3}$$

Since $l \geq 2$ we find (by multiplying (A.3) with $k-2$) that (A.3) holds true whenever

$$\frac{1}{2} \geq \frac{k - \beta_i - 1}{n} \iff n \geq 2(k - \beta_i - 1),$$

which is satisfied by the definition of k as long as $\beta_i \geq 2$.

Hence we may use (A.3):

$$\|\nabla^{\beta_i - 1} \nabla^2 \eta\|_{\frac{2(k-l)}{\beta_i}}^{\frac{k-l}{\beta_i}} \leq c \|\nabla^2 \eta\|_{L^q(Q)}^{\frac{k-l}{\beta_i} \frac{k-1-\beta_i}{k-2}} \|\nabla^2 \eta\|_{W^{k-2,2}(Q)}^{\frac{k-l}{\beta_i} \frac{\beta_i - 1}{k-2}} \leq c \|\nabla^2 \eta\|_{W^{k-2,2}(Q)},$$

using $\frac{k-l}{\beta_i} \frac{\beta_i - 1}{k-2} \leq 1$.

The last case is proved inductively. First with no loss of generality we take $\beta_1 = 1$. Then $\sum_{i=2}^{l} \beta_i \leq k - l - 1$ and using Hölder's inequality and Sobolev embedding implies

$$\left\| |\nabla^2 \eta| \prod_{i=1}^{l-1} |\nabla^{\beta_i} \eta| \right\| \leq \|\nabla^2 \eta\|_n \left\| \prod_{i=1}^{l-1} |\nabla^{\beta_i} \eta| \right\|_{\frac{2n}{n-2}} \leq c \left\| \prod_{i=1}^{l-1} |\nabla^{\beta_i} \eta| \right\|_{W^{1,2}(Q)}.$$

If now $\beta_i \neq 1$ for all $i > 1$, the estimate follows by the above case for $\nabla^{k-(l-1)} \prod_{i=1}^{l-1} \partial_{a_i} \eta$. If not, we may assume that $\beta_2 = 1$ and repeat the argument again. After at most l steps (in which case $k \geq 2l$), we get the result. ∎

Proposition A.4 (Space isomorphisms). *Let $\eta \in W^{2,q}(Q; \mathbb{R}^n)$ be such that* $\det \nabla \eta > \epsilon_0 > 0$ *and $\eta|_P = \gamma$. Then the map*

$$\eta^\# : W^{2,q}(\eta(Q); \mathbb{R}^n) \to W^{2,q}(Q; \mathbb{R}^n), \quad \xi \mapsto \xi \circ \eta,$$

is a linear isomorphism with operator norm $\|\eta^\#\| \leq C \|\eta\|^2_{W^{2,q}(Q)} / \epsilon_0^{1/q}$, where c only depends on q, Q, γ and n. Moreover, if $q > n$ and additionally $\eta \in W^{k,2}(Q; \mathbb{R}^n)$ and $\xi \in C^k(\eta(Q; \mathbb{R}^n))$, for k defined in (A.2), then

$$\|\xi \circ \eta\|_{W^{k,2}(Q)} \leq c \|\eta\|_{W^{k,2}(Q)} \|\xi\|_{C^k(Q)},$$

where the constant depends on Ω, n, k and $\|\eta\|_{W^{2,q}(Q)}$ only.

Proof. Linearity follows immediately from the definition. Now we calculate

$$\begin{aligned}\|\nabla^2(\xi \circ \eta)\|_{L^q(Q)} &= \|((\nabla^2 \xi) \circ \eta \cdot \nabla \eta) \cdot \nabla \eta + (\nabla \xi) \circ \eta \cdot \nabla^2 \eta\|_{L^q(Q)} \\ &\leq C \big(\|\nabla \eta\|^2_\infty \|(\nabla^2 \xi) \circ \eta\|_{L^q(Q)} + \|\nabla \xi\|_\infty \|\nabla^2 \eta\|_{L^q(Q)} \big),\end{aligned}$$

and use

$$\epsilon_0 \|(\nabla^2 \xi) \circ \eta\|^q_{L^q(Q)} \leq \int_Q |(\nabla^2 \xi) \circ \eta|^q \det \nabla \eta \, dx = \|\nabla^2 \xi\|^q_{L^q(\eta(Q))}$$

to estimate the first term. Then using Poincaré's inequality and the usual Morrey embeddings we get

$$\|\xi \circ \eta\|_{W^{2,q}(Q)} \leq C \|\xi\|_{W^{2,q}(\eta(Q)} \frac{\|\eta\|^2_{W^{2,q}(Q)}}{\varepsilon^{1/q}},$$

which proves that $\eta^\#$ is a linear map with the given operator norm. Now as $(\eta^\#)^{-1} = (\eta^{-1})^\#$ we conclude that it is also an isomorphism by the previous lemma.

For the second estimate we observe that

$$\|\nabla^k(\xi \circ \eta)\| \leq c \sum_{l=1}^{k} \sum_{a \in \{1,\ldots,n\}^l} \|\xi\|_{C^l(\eta(Q))} \left\| \nabla^{k-l} \prod_{i=1}^{l} \partial_{a_i} \eta \right\|,$$

which finishes the proof by Lemma A.3. ∎

A.2. *Proof of Lemma* 2.22

The proof is split into two parts. The first part constructs an extension of solenoidality. The second part shows how this extension can then be convoluted. We will also need the following Poincaré-type lemma:

Lemma A.5 (Poincaré's lemma for thin regions). *Let $S_0 \subset \mathbb{R}^n$ be an $(n-1)$-dimensional rectifiable set and $\Phi : S_0 \times [0, \varepsilon_0] \to \mathbb{R}^n$ an injective L-bi-Lipschitz function such that* $\Phi(\cdot, 0) = \mathrm{id}$. *Define $S_\varepsilon = \Phi(S_0, [0, \varepsilon])$ for $\varepsilon \in [0, \varepsilon_0]$. Then for all $f \in W^{1,a}(S_\varepsilon)$ with $f|_{S_0} = 0$ in the trace sense we have*

$$\|f\|_{L^a(S_\varepsilon)} \leq c\varepsilon \|f\|_{W^{1,a}(S_\varepsilon)} \quad \text{for all } f \in W^{1,a}(S_\varepsilon(t); \mathbb{R}^n) \text{ with } f|_{\Omega(t)} = 0, \tag{A.4}$$

where c is independent of ε.

Proof. By density arguments it is enough to prove the theorem for smooth functions. Now for $z \in S_0$ and $s_0 \in [0, \varepsilon_0]$ we find

$$\begin{aligned}|f(\Phi(z, s_0))| = |f(\Phi(z, s_0)) - f(\Phi(z, 0))| &= \left| \int_0^{s_0} \partial_s f(\Phi(z, s))\, ds \right| \\ &\leq \int_0^{s_0} |(\nabla f)(\Phi(z, s))|\, |\partial_s \Phi(z, s)|\, ds \leq \int_0^{s_0} L|(\nabla f)(\Phi(z, s))|\, ds.\end{aligned}$$

But then integrating over the whole domain yields

$$\begin{aligned}\int_{S_\varepsilon} |f(y)|^a dy &= \int_{S_0} \int_0^\varepsilon |f(\Phi(z, s_0))|^a |J(z, s_0)|\, ds_0\, dz \\ &\leq \int_{S_0} \int_0^\varepsilon \left(\int_0^{s_0} L|\nabla f(\Phi(z, s))|\, ds \right)^a |J(z, s_0)|\, ds_0\, dz \\ &\leq \int_{S_0} \int_0^\varepsilon \fint_0^\varepsilon \varepsilon^a L^a |\nabla f(\Phi(z, s))|^a |J(z, s_0)|\, ds\, ds_0\, dz \\ &= \varepsilon^a L^a \int_{S_0} \int_0^\varepsilon L^a |\nabla f(\Phi(z, s))|^a |J(z, s)|\, \|J\|_\infty \|J^{-1}\|_\infty\, ds\, dz \\ &= L^a \varepsilon^a \|J\|_\infty \|J^{-1}\|_\infty \int_{S_\varepsilon} |\nabla f(y)|^a\, dy,\end{aligned}$$

where $J(z, s)$ is the Jacobian of Φ which is bounded from above as well as away from zero because Φ is bi-Lipschitz. ∎

Lemma A.6 (Extension of the solenoidal region). *Fix a function*

$$\eta \in L^\infty([0, T]; \mathcal{E}) \cap W^{1,2}([0, T]; W^{1,2}(Q; \mathbb{R}^n)) \quad \text{with} \quad \sup_{t \in T} E(\eta(t)) < \infty,$$

such that $\eta(t) \notin \partial\mathcal{E}$ for all $t \in [0, T]$. As before, set $\Omega(t) = \Omega \setminus \eta(t, Q)$.

Let $\xi \in L^2([0, T]; W_0^{1,2}(\Omega; \mathbb{R}^n))$ be such that $\operatorname{div} \xi(t) = 0$ *on $\Omega(t)$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, there exists ξ_ε such that* $\operatorname{div} \xi(t, y) = 0$ *for all $y \in \Omega$ with* $\operatorname{dist}(y, \Omega(t) \cup \partial\Omega) < \varepsilon$ *and there are constants c independent of ξ such that for a.e. $t \in [0, T]$,*

$$\|\xi_\varepsilon\|_{W^{1,2}(\Omega)} \leq c\|\xi\|_{W^{1,2}(\Omega)} \quad \text{and} \quad \|\xi_\varepsilon - \xi\|_{L^2(\Omega)} \leq c\varepsilon^{\frac{2}{n+2}} \|\xi\|_{W^{1,2}(\Omega)}.$$

Additionally, for any $k \in \mathbb{N}$ and $a \in (1,\infty)$ such that $\xi \in L^2([0,T]; W^{k,a}(Q;\mathbb{R}^n))$ we have

$$\|\xi_\varepsilon - \xi\|_{L^2([0,T];W^{k,a}(\Omega))} \to 0 \quad \text{as } \varepsilon \to 0,$$

and similarly if $\xi \in W^{1,2}([0,T]; W^{1,\infty}(\Omega;\mathbb{R}^n))$ then also

$$\|\partial_t(\xi - \xi_\varepsilon)\|_{L^2([0,T];W^{1,2}(\Omega))} \to 0 \quad \text{as } \varepsilon \to 0. \tag{A.5}$$

Proof. We begin by defining

$$S_\varepsilon(t) := \{y \in \Omega \mid \operatorname{dist}(y, \eta(t,\partial Q)) \le \varepsilon\}$$

and introduce the cutoff function $\psi_\varepsilon : [0,T] \times \Omega \to [0,1]$ such that

$$\chi_{S_\varepsilon(t)} \le \psi_\varepsilon(t) \le \chi_{S_{2\varepsilon}(t)} \quad \text{and} \quad \|\psi_\varepsilon(t)\|_{C^l} \le c/\varepsilon^l \quad \text{for } l \in \mathbb{N}.$$

Due to the regularity of η, we may assume that $\partial_t \psi_\varepsilon$ is uniformly bounded and such that

$$\|\partial_t \psi_\varepsilon\|_{L^2([0,T]\times\Omega)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{A.6}$$

We also pick $\tilde\psi \in C_0^\infty([0,T] \times \Omega; \mathbb{R}^n)$ such that $\operatorname{supp}(\tilde\psi(t)) \cap S_{\varepsilon_0}(t) = \emptyset$ for some $\varepsilon_0 > 0$ and $\int_\Omega \tilde\psi(t)\,dy = 1$ for all t. Using this we then define

$$\xi_\varepsilon(t) := \xi(t) - \mathcal{B}\big(\psi_\varepsilon(t) \operatorname{div} \xi(t) - b_\varepsilon(t)\tilde\psi(t)\big),$$

where $\mathcal{B}$ is the Bogovskiĭ operator on Ω and $b_\varepsilon(t) := \int_\Omega \psi_\varepsilon(t) \operatorname{div} \xi(t)\,dy$ is used to keep the mean. Then by definition the function

$$\operatorname{div} \xi_\varepsilon(t) = (1 - \psi_\varepsilon(t)) \operatorname{div} \xi(t) - b_\varepsilon(t)\tilde\psi(t)$$

vanishes on $S_\varepsilon(t)$ as required, and

$$\begin{aligned}\|\xi - \xi_\varepsilon\|_{W^{k,a}(\Omega)} &= \big\|\mathcal{B}\big(\psi_\varepsilon(t) \operatorname{div} \xi(t) - b_\varepsilon(t)\tilde\psi(t)\big)\big\|_{W^{k,a}(\Omega)} \\ &\le c\|\psi_\varepsilon(t) \operatorname{div} \xi(t) - b_\varepsilon(t)\tilde\psi(t)\|_{W^{k-1,a}(\Omega)} \le c\|\psi_\varepsilon(t) \operatorname{div} \xi(t)\|_{W^{k-1,a}(\Omega)} + c|b_\varepsilon(t)|\end{aligned}$$

is the main quantity we need to estimate.

Let us begin with the special case $k = 0, a = 2$. Here we use the embedding $L^{\frac{2n}{2+n}}(\Omega;\mathbb{R}^n) \subset W^{-1,2}(\Omega;\mathbb{R}^n)$ and apply Hölder's inequality to show that

$$\begin{aligned}\|\xi - \tilde\xi_\varepsilon\|_{L^2(\Omega)} &\le c\|\psi_\varepsilon \operatorname{div} \xi\|_{W^{-1,2}(\Omega)} + c|b_\varepsilon| \le c\|\psi_\varepsilon \operatorname{div} \xi\|_{L^{\frac{2n}{n+2}}(\Omega)} \\ &\le c\|\psi_\varepsilon\|_{L^n(S_{2\varepsilon}(t))}\|\operatorname{div} \xi\|_{L^2(\Omega)} \le c|S_{2\varepsilon}|^{1/n}\|\xi\|_{W^{1,2}(\Omega)} \le c\varepsilon^{1/n}\|\xi\|_{W^{1,2}(\Omega)}.\end{aligned}$$

For $k \ge 1$ we first note that $|b_\varepsilon(t)| \le c\|\operatorname{div} \xi\|_{L^2(S_\varepsilon(t))} \to 0$ for each fixed ξ and furthermore

$$\begin{aligned}\|\psi_\varepsilon(t) \operatorname{div} \xi(t)\|_{W^{k-1,a}(\Omega)} &\le c \sum_{l=0}^{k-1} \|\psi_\varepsilon(t)\|_{C^{k-1-l}(\Omega)}\|\nabla^l \operatorname{div} \xi\|_{L^a(S_{2\varepsilon}(t))} \\ &\le c \sum_{l=0}^{k-1} \varepsilon^{-(k-1-l)}\|\nabla^l \operatorname{div} \xi\|_{L^a(S_{2\varepsilon}(t))}.\end{aligned}$$

In particular, for $k = 1, a = 2$ we have $k - 1 - l = 0$, so this immediately proves that $\|\xi_\varepsilon\|_{W^{1,2}(\Omega)} \leq c\|\xi\|_{W^{1,2}(\Omega)}$ independently of ξ. For $k > 1$ we will apply the Poincaré inequality of Lemma A.5. For this we make use of the fact that $S_{\varepsilon_0} \setminus \Omega(t)$ is a small neighborhood of a uniformly Lipschitz boundary and thus can be written in the required way using η itself. Furthermore, for any $l < k$ we have $\nabla^l \operatorname{div} \xi = 0$ on $\Omega(t)$ and thus also on $\partial\Omega(t)$ in the trace sense. This then gives $\|\nabla^l \operatorname{div} \xi\|_{L^a(S_{2\varepsilon}(t))} \leq c\varepsilon^{k-1-l}\|\xi\|_{W^{k,a}(S_{2\varepsilon}(t))}$, which is enough to finish the estimate.

Finally, let us consider the time derivative. As $\mathcal{B}$ is a linear operator, we have

$$\begin{aligned}
&\|\partial_t(\xi_\varepsilon - \xi)\|_{W^{1,2}(\Omega)} = c\left\|\partial_t\big(\psi_\varepsilon(t) \operatorname{div} \xi(t) - b_\varepsilon(t)\tilde{\psi}(t)\big)\right\|_{L^2(\Omega)} \\
&\quad \leq c\|\partial_t(\psi_\varepsilon(t) \operatorname{div} \xi(t))\|_{L^2(S_{2\varepsilon}(t))} + c|\partial_t b_\varepsilon(t)|\,\|\tilde{\psi}(t)\|_{L^2(\Omega)} + c|b_\varepsilon(t)|\,\|\partial_t\tilde{\psi}(t)\|_{L^2(\Omega)}.
\end{aligned}$$

For the last term we have already shown that $|b_\varepsilon(t)| \to 0$ and $\tilde{\psi}$ does not depend on ε. For the second to last term we note that

$$|\partial_t b_\varepsilon(t)| = \left|\int_\Omega \partial_t(\psi_\varepsilon(t) \operatorname{div} \xi(t))\, dy\right| \leq \|\partial_t(\psi_\varepsilon(t) \operatorname{div} \xi(t))\|_{L^2(\Omega)},$$

which is the same as the first term and for which we use the estimate

$$\begin{aligned}
&\|\partial_t(\psi_\varepsilon(t) \operatorname{div} \xi(t))\|_{L^2(\Omega)} \\
&\qquad \leq \|\partial_t\psi_\varepsilon(t)\|_{L^2(\Omega)}\|\xi(t)\|_{W^{1,\infty}(S_{2\varepsilon}(t))} + \|\psi_\varepsilon(t)\|_{L^2(\Omega)}\|\partial_t\xi(t)\|_{W^{1,\infty}(S_{2\varepsilon}(t))},
\end{aligned}$$

which implies (A.5) by (A.6) and Hölder's inequality.

■

Proof of Lemma 2.22. First we apply Lemma A.6 to find a function $\hat{\xi}$ with $\hat{\xi} = 0$ on $\Omega(t)$ and on an ε-neighborhood of $\partial(\Omega \setminus \Omega(t))$. Thus taking convolution with γ_{ε^2} does not affect the zero boundary values (if ε is small enough).

We will now apply Lemma A.6 again to $\hat{\xi}_\varepsilon * \gamma_{\varepsilon^2}$ and call the result ξ_ε, a function which is smooth by Theorem 2.21. Moreover, since all operations are linear we find that $\xi_\varepsilon \in C_0^\infty(\Omega; \mathbb{R}^n)$ is divergence free in $\Omega(t) \cup S_\varepsilon$. By collecting all the properties of the approximation, we find that

$$\|\xi - \xi_\varepsilon\|_{W^{l,a}(\Omega)} \to 0$$

for $l \leq k - 1$. Moreover,

$$\begin{aligned}
\|\xi - \xi_\varepsilon\|_{W^{1,2}(\Omega)} &\leq c\|\xi\|_{W^{1,2}(\Omega)}, \\
\|\xi - \xi_\varepsilon\|_{L^2(\Omega)} &\leq c\varepsilon^{\frac{2}{n+2}}\|\xi\|_{W^{1,2}(\Omega)}.
\end{aligned}$$

Next we turn to the estimates for $\phi_\varepsilon := \xi_\varepsilon \circ \eta$. They follow by Proposition A.4 and standard convolution estimates. First, for $k > 2$ we find

$$\|\phi_\varepsilon\|_{W^{k,2}(Q)} \leq c\|\xi_\varepsilon\|_{C^k(\Omega)}\|\eta\|_{W^{k,2}(Q)} \leq c(\varepsilon)\|\xi\|_{L^2(\Omega)}\|\eta\|_{W^{k,a}(Q)}.$$

Second, in case $\xi \in L^\infty([0,T]; W^{2,a}(\Omega))$, we find

$$\|\phi_\varepsilon - \phi\|_{W^{2,a}(Q)} \le c\|\xi_\varepsilon - \xi\|_{W^{2,a}(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Finally, for the time derivative in case $\partial_t \xi \in L^\infty([0,T]; W^{1,2}(\Omega;\mathbb{R}^n))$ as well as $\xi \in L^\infty([0,T]; W^{3,a}(\Omega;\mathbb{R}^n))$ with $a > n$ we find by Sobolev embedding that

$$\begin{aligned}\|\partial_t(\phi_\varepsilon - \phi)\|_{W^{1,2}(Q)} &\le c\|\partial_t(\xi_\varepsilon - \xi)\|_{W^{1,2}(\Omega)} + c\|\nabla(\xi_\varepsilon - \xi)\|_{W^{1,\infty}(\Omega)}\|\partial_t\eta\|_{W^{1,2}(Q)} \\ &\le c\|\partial_t(\xi_\varepsilon - \xi)\|_{W^{1,2}(\Omega)} + c\|\xi_\varepsilon - \xi\|_{W^{3,a}(\Omega)}\|\partial_t\eta\|_{W^{1,2}(Q)},\end{aligned}$$

which implies the assertions for the time derivatives by (A.5). ■

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