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Jason P. Bell · Keping Huang · Wayne Peng · Thomas J. Tucker

A Tits alternative for endomorphisms of the projective line

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Abstract. We prove an analog of the Tits alternative for endomorphisms of $\mathbb{P}^1$. In particular, we show that if S is a finitely generated semigroup of endomorphisms of $\mathbb{P}^1$ over $\mathbb{C}$, then either S has polynomially bounded growth or S contains a nonabelian free semigroup. We also show that if f and g are polarizable maps over any field of any characteristic and $\mathrm{Prep}(f) \neq \mathrm{Prep}(g)$, then for all sufficiently large j, the semigroup $\langle f^j, g^j \rangle$ is a free semigroup on two generators.

Keywords. Tits alternative, semigroups, preperiodic points, height functions, rational functions, free semigroups

1. Introduction

The Tits alternative [65] is a celebrated result in the theory of linear groups. It says that a finitely generated linear group contains either a solvable subgroup of finite index or a nonabelian free group. In general, a group G is said to satisfy the *Tits alternative* if each of its finitely generated subgroups contains either a solvable subgroup of finite index or a nonabelian free group. Many classes of groups have now been shown to satisfy the Tits alternative [4, 10, 16, 17, 30, 35, 39, 46].

When one instead considers the structure of linear groups as semigroups, an even stronger dichotomy is obtained. A result of Longobardi, Maj, and Rhemtulla [44] (see also Milnor [48] and Wolf [66]) combined with the Tits alternative implies that a finitely

Jason P. Bell: Department of Pure Mathematics, University of Waterloo,
Waterloo, ON, N2L 3G1, Canada; jpbell@uwaterloo.ca

Keping Huang: Institute for Advanced Study in Mathematics, Harbin Institute of Technology,
Harbin, P.R. China; keping.huang@rochester.edu

Wayne Peng: Mathematics Division, National Center for Theoretical Sciences,
Taipei, 106 Taiwan; wayne.peng@ncts.tw

Thomas J. Tucker: Department of Mathematics, University of Rochester,
Rochester, NY 14627, USA; thomas.tucker@rochester.edu

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generated linear group is either virtually nilpotent or contains a nonabelian free semigroup. Okniński and Salwa [57] later showed that if S is a finitely generated cancellative linear semigroup, then either S contains a nonabelian free semigroup or the group generated by S is virtually nilpotent (recall that a semigroup S is said to be cancellative if $ab = ac$ and $ba = ca$ each imply that $b = c$ for all $a, b, c \in S$). Rosenblatt [62] showed that a polycyclic group either has a nilpotent subgroup of finite index or contains a free subsemigroup on two generators. Results from the theory of growth of groups then give that the growth of a finitely generated cancellative linear semigroup is either exponential or polynomially bounded. The cancellativity condition here is crucial as Okniński [56] has also produced finitely generated non-cancellative linear semigroups of intermediate growth (see also [12]).

We note that a semigroup S contains a nonabelian free semigroup if and only if it contains a free semigroup on two generators. As with groups, it is not difficult to see that a free semigroup on two generators must contain a free semigroup on n generators for any positive integer n.

We prove the following variant of the Tits alternative for semigroups of endomorphisms of $\mathbb{P}^1_{\mathbb{C}}$.

Theorem 1.1. *Let $\mathcal{S} \subset \mathbb{C}(x)$ be a finitely generated semigroup of endomorphisms of $\mathbb{P}^1$. Then either $\mathcal{S}$ has polynomially bounded growth or $\mathcal{S}$ contains a nonabelian free semigroup.*

We say that an endomorphism of $\mathbb{P}^1$ of degree greater than 1 is *special* if it is conjugate by a homography to a monomial, a Chebyshev polynomial, or a Lattès map; if it is not conjugate to such a map, then it is *nonspecial*. When $\mathcal{S}$ contains a nonspecial rational function of degree greater than 1, we obtain a stronger dichotomy.

Theorem 1.2. *Let $\mathcal{S}$ be a finitely generated semigroup of endomorphisms of $\mathbb{P}^1_{\mathbb{C}}$ such that some element of $\mathcal{S}$ is nonspecial and of degree greater than* 1. *Then either $\mathcal{S}$ has linear growth or $\mathcal{S}$ contains a nonabelian free semigroup.*

Let $f : X \to X$ be a self-map of a set X. Recall that $x_0 \in X$ is a *preperiodic point* of f if the orbit $\{x_0, f(x_0), f^2(x_0), \dots\}$ is a finite set. Let $\mathrm{Prep}(f)$ denote the set of preperiodic points of f. We derive Theorem 1.1 from a result relating common preperiodic points of endomorphisms of $\mathbb{P}^1$ with free subsemigroups. The techniques used for this result also work in the setting of morphisms of projective varieties that are polarized by the same ample line bundle. For V a projective variety, a morphism $f : V \to V$ is said to be *polarized* by the ample line bundle $\mathcal{L}$ if there is an integer $d > 1$ such that $f^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$. The notion of polarization is due to Zhang [70]. Any endomorphism of degree greater than 1 of the projective space $\mathbb{P}^n$ is polarized by $\mathcal{O}_{\mathbb{P}^n}(1)$; it is also true that any polarized morphism on a variety V comes from restricting an endomorphism of projective space to V for some embedding of V into a projective space (see [5, 18]). Polarized morphisms give rise to canonical height functions with good properties (see [9] and Section 3.2). In the theorem below and throughout this paper, we let $\langle f, g\rangle$ denote the semigroup generated under composition by f and g whenever f and g are two maps from a set to itself.

Theorem 1.3. *Let V be a projective variety over a field K, and let $f, g : V \to V$ be endomorphisms polarized by the same line bundle $\mathcal{L}$. If* $\mathrm{Prep}(f) \neq \mathrm{Prep}(g)$*, then for large enough positive integer j, the semigroup $\langle f^j, g^j \rangle$ is a free semigroup on two generators.*

Remark 1.4. One can use results of Jiang and Zieve [31] to extend Theorem 1.1 to characteristic p when the elements of $\mathcal{S}$ are all polynomials whose degrees are not divisible by p. In this setting, one can use their techniques to give a more precise version of Theorem 1.3.

Ritt [61] studied the semigroup of polynomials under composition. He gave necessary and sufficient conditions for two polynomials to commute under composition and determined relations for the semigroup of polynomials under composition. It is very possible that in this case, some of the results here can be obtained using Ritt's work, although there do appear to be some additional subtleties involved. We also point out that the Tits alternative has been considered for automorphism groups of algebraic varieties, with a complete result for projective varieties in characteristic 0 (see [16] for a survey) as well as some results in characteristic p (see [28]). The Tits alternative has also been proved for the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ in all characteristics (see [10]).

Pakovich [58] has proved that if $\mathcal{S}$ is a semigroup of nonspecial endomorphisms of $\mathbb{P}^1$ over the complex numbers satisfying a strong cancellativity property, then either $\mathcal{S}$ contains a nonabelian free semigroup or $\mathcal{S}$ is left amenable. Hindes [26] has proved that certain conditions on a semigroup of endomorphisms of $\mathbb{P}^1$ over $\overline{\mathbb{Q}}$ guarantee that the semigroup is free; the conditions are much more restrictive than those of Theorem 1.3 but have the advantage of ensuring that certain semigroups are free (and do not merely contain a nonabelian free semigroup).

2. Preliminaries

We give a brief overview of the basics of semigroups. See [27] for the theory of semigroups and [38] for generalities on growth functions. Let $\mathcal{S}$ be a finitely generated semigroup and let S be a finite set of generators for $\mathcal{S}$. Then we can form the *growth function* of $\mathcal{S}$ with respect to the generating set S as follows. We define $d_S(n) = |S^{\leq n}|$, where $S^{\leq n}$ is the set of elements of $\mathcal{S}$ that can be expressed as a product of elements of S of length at most n. The function $d_S(n)$ is weakly increasing as a function of n and while this function depends upon our choice of generating set, we observe that if T is another generating set for $\mathcal{S}$ then there exists a positive integer c such that $T \subseteq S^{\leq c}$ and $S \subseteq T^{\leq c}$, so we have the inequalities

$$d_T(n) \leq d_S(cn) \quad \text{and} \quad d_S(n) \leq d_T(cn).$$

Thus if we declare that two weakly increasing functions $f, g : \mathbb{N} \to \mathbb{N}$ are *asymptotically equivalent* whenever there is a positive integer C such that $f(n) \leq g(Cn)$ and

$g(n) \le f(Cn)$, then the growth function is independent of our choice of generating set up to this asymptotic equivalence. One says that $\mathcal{S}$ has *polynomially bounded growth* if $d_{\mathcal{S}}(n) = \mathrm{O}(n^\kappa)$ for some positive constant κ. More specifically, $\mathcal{S}$ has *polynomial growth of degree* κ if there are positive constants C_1, C_2, κ such that $C_1 n^\kappa \le d_S(n) \le C_2 n^\kappa$ for all n; in particular, $\mathcal{S}$ has *linear growth* if there are positive constants C_1, C_2 such that $C_1 n \le d_S(n) \le C_2 n$ for all n; and $\mathcal{S}$ has *exponential growth* if there is a positive constant $C > 1$ such that $d_S(n) > C^n$ for all n sufficiently large. It is not difficult to check that these growth properties are all preserved under asymptotic equivalence, so we can speak unambiguously of $\mathcal{S}$ having these properties without making reference to a generating set.

A semigroup $\mathcal{S}$ is *left cancellative* if whenever $as = at$ with $a, s, t \in \mathcal{S}$ we have $s = t$; right cancellativity is defined analogously. A *cancellative* semigroup is one that is both left and right cancellative. Note that if $\mathcal{S}$ is a semigroup of surjective maps, then $\mathcal{S}$ is right cancellative since $sa = ta$ implies that $s = t$ whenever a is surjective. Hence, in particular, semigroups of nonconstant endomorphisms of $\mathbb{P}^1$ are right cancellative; on the other hand $x^2 \circ (-x) = x^2 \circ (x)$, so they are not always left cancellative.

3. Proof of Theorem 1.3

3.1. A variant of the ping-pong lemma

In our work here, the functions τ and τ_f will be real-valued height functions (either Weil or Moriwaki). The arguments in this section work in a more general setting, and we state Proposition 3.1 accordingly.

Let $\mathcal{U}$ be a set, let $\tau : \mathcal{U} \to \mathbb{R}$ be any function that is not bounded in absolute value, and let $f : \mathcal{U} \to \mathcal{U}$ be a surjective map. Assume that there are positive real numbers $d_1 > 1$ and $C(f)$ such that

$$|\tau(f(z)) - d_1\tau(z)| < C(f). \tag{3.1}$$

We say that d_1 is the *degree* $\deg_\tau(f)$ of f. Suppose now that $g : \mathcal{U} \to \mathcal{U}$ is a surjective map of degree d_2, so that for some positive real number $C(g)$ we have

$$|\tau(g(z)) - d_2\tau(z)| < C(g). \tag{3.2}$$

We let $C = \max(C(f), C(g))$. It automatically follows that $\deg_\tau(f \circ g) = \deg_\tau(f)\deg_\tau(g)$ whenever these quantities are defined.

Recall that one can use (3.1) and (3.2) to construct canonical functions as follows:

$$\begin{aligned} \tau_f(z) &= \lim_{n\to\infty} \frac{\tau(f^n(z))}{d_1^n} = \tau(z) + \sum_{i=0}^{\infty} \frac{\tau(f^{i+1}(z)) - d_1\tau(f^i(z))}{d_1^{i+1}}, \\ \tau_g(z) &= \lim_{n\to\infty} \frac{\tau(g^n(z))}{d_2^n} = \tau(z) + \sum_{i=0}^{\infty} \frac{\tau(g^{i+1}(z)) - d_2\tau(g^i(z))}{d_2^{i+1}}. \end{aligned} \tag{3.3}$$

Note that

$$\left|\sum_{i=0}^{\infty} \frac{\tau(f^{i+1}(z)) - d_1\tau(f^i(z))}{d_1^{i+1}}\right| < C\sum_{i=1}^{\infty}\frac{1}{d_1^i},$$
$$\left|\sum_{i=0}^{\infty} \frac{\tau(g^{i+1}(z)) - d_2\tau(g^i(z))}{d_2^{i+1}}\right| < C\sum_{i=1}^{\infty}\frac{1}{d_2^i}.$$

The telescoping sum argument above is due to Tate and was used by Call and Silverman [9] in their construction of canonical heights. Kawaguchi [32, 33] has further developed the theory of canonical heights in the context of semigroups. These kinds of arguments also appear in many other contexts.

Let $d = \min(d_1, d_2) > 1$, and let

$$C' = C\sum_{i=1}^{\infty}\frac{1}{d^i}.$$

Using the same telescoping series argument, we see that for any n, we have

$$\begin{aligned}\left|\frac{\tau(f^n(z))}{d_1^n} - \tau_f(z)\right| &< \frac{C'}{d_1^n},\\ \left|\frac{\tau(g^n(z))}{d_2^n} - \tau_g(z)\right| &< \frac{C'}{d_2^n}.\end{aligned} \tag{3.4}$$

We will prove the following variant of the ping-pong lemma of Klein and Fricke [36, 37], which also played an important role in the proof of the Tits alternative.

Proposition 3.1. *Let $\mathcal{U}$ be a set, let $\tau : \mathcal{U} \to \mathbb{R}$ be a function that is unbounded in absolute value and let $f, g : \mathcal{U} \to \mathcal{U}$ be surjective maps that satisfy (3.1) and (3.2). Let τ_f and τ_g be as defined in (3.3). Suppose that there is some $z \in \mathcal{U}$ such that $\tau_f(z) \neq \tau_g(z)$. Then for all sufficiently large j the following hold:*

(i) *$wf^j \neq ug^j$ for all $w, u \in \langle f, g\rangle$;*

(ii) *the semigroup $\langle f^j, g^j\rangle$ is free on two generators.*

We begin with one more definition. Let $w = \varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_1$, where each φ_j equals f or g. The degree of w is then

$$\deg_\tau w = \prod_{j=1}^{m} \deg_\tau \varphi_j. \tag{3.5}$$

Since $|\tau(w(z)) - \deg_\tau w \cdot \tau(z)|$ is bounded for all z (by (3.1) and (3.2)) and τ is unbounded, we see that the definition in (3.5) is independent of the word representing w.

Lemma 3.2. *Let $w = \varphi_m \circ \cdots \circ \varphi_1$ where φ_i is equal to f or g for each i. Let $w_i = \varphi_i \circ \cdots \circ \varphi_1$ (for $i \le m$). With notation as above, we have*

$$\left|\frac{\tau(w(z))}{\deg_\tau w} - \frac{\tau(w_j(z))}{\deg_\tau w_j}\right| < \frac{C'}{d^j} \tag{3.6}$$

for $j = 1, \ldots, m$.

Proof. Let $e_\ell = \deg_\tau \varphi_\ell$ for each $\ell = 1, \dots, m$. Then $\deg_\tau w_j = \prod_{\ell=1}^{j} e_\ell$. Thus, as in (3.1) and (3.2), we have a telescoping series

$$\frac{\tau(w(z))}{\deg_\tau w} - \frac{\tau(w_i(z))}{\deg_\tau w_i} = \sum_{j=i}^{m-1} \frac{\tau(w_{j+1}(z)) - e_{j+1}\tau(w_j(z))}{\prod_{\ell=1}^{j+1} e_\ell}. \tag{3.7}$$

Now

$$|\tau(w_{j+1}(z)) - e_{j+1}\tau(w_j(z))| = |\tau(\varphi_{j+1}(w_j(z)) - (\deg_\tau \varphi_{j+1})\tau(w_j(z))| < C$$

for all j by (3.1) and (3.2). Thus

$$\left|\sum_{j=i}^{m-1} \frac{\tau(w_{j+1}(z)) - e_{j+1}\tau(w_j(z))}{\prod_{\ell=1}^{j+1} e_\ell}\right| \le \frac{1}{d^j}\sum_{i=1}^{\infty}\frac{C}{d^i} \le \frac{C'}{d^j}.$$

This completes the proof, by (3.7). ∎

Proof of Proposition 3.1. (i) We choose x_0 so that $\tau_f(x_0) \neq \tau_g(x_0)$ and define $\varepsilon = |\tau_f(x_0) - \tau_g(x_0)|$. Choose j so that $C'/d^j < \varepsilon/4$, where $d = \min(d_1, d_2)$ as above. Let wf^j and ug^j be words in f and g such that $\deg_\tau wf^j = \deg_\tau ug^j$. Then, by (3.4) and (3.6), we have

$$\left|\tau_f(x_0) - \frac{\tau(wf^j(x_0))}{\deg_\tau wf^j}\right| < \varepsilon/2,$$
$$\left|\tau_g(x_0) - \frac{\tau(ug^j(x_0))}{\deg_\tau ug^j}\right| < \varepsilon/2.$$

Thus,

$$\frac{\tau(wf^j(x_0))}{\deg_\tau wf^j} \neq \frac{\tau(ug^j(x_0))}{\deg_\tau ug^j}.$$

Since $\deg_\tau wf^j = \deg_\tau ug^j$, it implies that

$$wf^j(x_0) \neq ug^j(x_0). \tag{3.8}$$

(ii) Now, let $w' = \varphi_m \circ \cdots \circ \varphi_1$ and $w'' = \theta_n \circ \cdots \circ \theta_1$, where each φ_i and θ_k is equal to f^j or g^j. Suppose that $w' = w''$. We will show by induction on $\max(m, n)$ that $m = n$ and $\theta_i = \varphi_i$ for $i = 1, \dots, m$. If $\max(m, n) = 1$, then $m = n = 1$. Because $\tau_f \neq \tau_g$, we have $f^j \neq g^j$, and hence $\theta_1 = \varphi_1$ as words in f^j and g^j. For the inductive step, consider an identity $\varphi_m \circ \cdots \circ \varphi_1 = \theta_n \circ \cdots \circ \theta_1$. If $\varphi_1 = \theta_1$, we can cancel from the right because f and g are surjective and then apply the inductive hypothesis. If not, then $w \neq w'$ by (i). ∎

3.2. *Height functions*

We will prove Theorem 1.3 in this section. The plan is to let τ be a height function, either a Weil height h or a Moriwaki height $\mathfrak{h}$, use Proposition 3.1, and use the fact that certain canonical heights will be zero at exactly the preperiodic points. There may be other sorts of functions where Proposition 3.1 may be used though. For example, if we let $\tau : \mathbb{C} \to \mathbb{R}$ be defined by $\tau(z) = \log \max(|z|, 0)$, and if $f, g \in \mathbb{C}[x]$ are polynomials of degree greater than 1, we see that τ_f and τ_g vanish precisely on the filled Julia sets of f and g respectively. Since the Julia set is simply the boundary of the filled Julia set, Proposition 3.1 implies that if the Julia sets of f and g are not equal, then for all large enough j, the semigroup $\langle f^j, g^j \rangle$ is free on two generators.

For a more general exposition of the Weil height functions, see [6,40]. The Moriwaki height functions we use were introduced in [52,53].

3.2.1. Basics of height functions. Let K be a finitely generated field. If K is not finite, then there is a set $\mathbb{M}_K$ of nontrivial absolute values $|\cdot|_v$ on K along with positive integers e_v such that the product formula

$$\prod_{v \in \mathbb{M}_K} |z|_v^{e_v} = 1$$

holds for all nonzero $z \in K$.

When K is a number field, these are simply the usual archimedean and p-adic absolute values, suitably normalized. When K is a function field over a field k, we choose the absolute values from prime divisors on a variety V over k, such that the residue fields are

- finite extensions of $\mathbb{Q}$ when we are in characteristic 0,
- finite extensions of $\mathbb{F}_p$ when we are in characteristic p.

In either case, see [6, Section 1.4] for more details and [6, Sections 1.3.6 and 1.3.12] for the issue of normalization. When K is a function field over k, the set of $z \in K$ such that $|z|_v = 1$ for all $v \in \mathbb{M}_K$ is called the *field of constants*.

One can extend the $|\cdot|_v$ to $\bar{K}$. We present here the extension of $|\cdot|_v$ to K^{sep}, following [6]. We consider a finite-dimensional separable extension field L of K and a place w of L with $w \mid v$. For any $z \in L$, we define $\|z\|_w := |N_{L_w/K_v}(z)|_v$, $|z|_w := |N_{L_w/K_v}(z)|^{1/[L:K]}$, and $|z|_v := \prod_{w|v} |z|_w$. One can check the compatibility by the properties of the norm and the local degree. For the general case, see [6, Sections 1.3.6 and 1.3.12]. Thus we obtain a Weil height function on $\mathbb{P}^n(\bar{K})$ by defining

$$h_{\mathbb{P}^n}(z_0 : \cdots : z_n) = \frac{1}{m} \sum_{v \in \mathbb{M}_K} \sum_{i=1}^{m} \log \max(|z_0^{[i]}|_v, \ldots, |z_n^{[i]}|_v),$$

where $(z_0^{[i]} : \cdots : z_n^{[i]})$, $i = 1, \ldots, m$, is the set of Galois conjugates of $(z_0 : \cdots : z_n)$ in $\bar{K}$ over K (note that while this does depend on our choice of coordinates, a change of coordinates will only change the definition by a bounded constant – see, for example, [6, Lemma 1.5.3] for details).

When $\mathcal{L}$ is an ample line bundle on V, we can associate a height function h to $\mathcal{L}$ by letting $\iota : V \to \mathbb{P}^n$ be an embedding such that $\iota^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{L}^{\otimes r}$ (such ι and r exist when $\mathcal{L}$ is ample) and taking $h_{\mathcal{L}}(z) = \frac{1}{r} h_{\mathbb{P}^n}(\iota(z))$.

3.2.2. The Northcott property and the Moriwaki heights.

Theorem 3.3 (Northcott, [55], [1, Section 1.2]). *Let K be a number field or finitely generated function field in characteristic p with a finite field of constants. Let $h_{\mathcal{L}}$ be as above. For any constants A and B there are at most finitely many $z \in V(\bar{K})$ such that $h_{\mathcal{L}}(z) \leq A$ and $[K(z) : K] \leq B$.*

Northcott's theorem does not hold over function fields of characteristic 0 for Weil heights. However, Moriwaki [52,53] has used metrics on line bundles and Arakelov intersection theory to associate a height function $\mathfrak{h}_{\mathcal{L}}$ to an ample line bundle $\mathcal{L}$ such that a form of Northcott's theorem does hold. We have the following (see [52, 53]).

Theorem 3.4 (Moriwaki). *For any constants A and B there are at most finitely many $z \in V(\bar{K})$ such that $\mathfrak{h}_{\mathcal{L}}(z) \leq A$ and $[K(z) : K] \leq B$.*

3.2.3. Canonical heights and polarized dynamical systems. Suppose V is a projective variety defined over K, and $\mathcal{L}$ is an ample line bundle on V with associated height function $h_{\mathcal{L}}$. Suppose f is an endomorphism of V defined over K such that $f^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$, where $d > 1$, we have

$$|h_{\mathcal{L}}(f(z)) - dh_{\mathcal{L}}(z)| < C \tag{3.9}$$

for all $z \in V(\bar{K})$ and some $C = C(f) > 0$. We can attach a canonical height to f as in (3.3) by letting $h_f(z) = \lim_{n\to\infty} h_{\mathcal{L}}(f^n(z))/d^n$. See [9, Theorem 1.1] for more details and the uniqueness of the canonical height.

The same construction works for Moriwaki heights (see [69, Section 2.4]) and provides a canonical height $\mathfrak{h}_f$. Note that $h_f(f(z)) = dh_{\mathcal{L}}(z)$ by construction, so if $z \in \operatorname{Prep}(f)$, then clearly $h_f(z) = 0$.

Corollary 3.5 (Northcott–Moriwaki). *We have $h_f(z) = 0$ (resp. $\mathfrak{h}_f(z) = 0$) if and only if $z \in \operatorname{Prep}(f)$.*

3.3. *Proof of Theorem 1.3*

We use Weil heights to treat the case where the field K is a number field or a function field of characteristic p. For function fields of characteristic 0, we use Moriwaki heights. Note that we could also treat the case of endomorphisms of $\mathbb{P}^1$ over function fields of characteristic 0 using Weil heights rather than Moriwaki heights, since Baker [1] has proved a dynamical form of Northcott's theorem for Weil heights in the case of endomorphisms of $\mathbb{P}^1$, assuming a nonisotriviality condition. This may be possible in higher dimensions, too, as there are more general dynamical Northcott-type results for the nonisotrivial maps due to Chatzidakis and Hrushovski [13, 14] (see also [20]), but the nonisotriviality conditions there are a good deal more complicated.

Proof of Theorem 1.3. If K is a number field or a function field with a finite field of constants, set $\tau = h_{\mathcal{L}}$. For other function fields set $\tau = \mathfrak{h}_{\mathcal{L}}$. Equation (3.9) shows that we are in the setting of Section 3.1: τ_f and τ_g correspond to the canonical heights. By Theorem 3.3 (resp. Theorem 3.4) and Corollary 3.5, $\mathrm{Prep}(f) \neq \mathrm{Prep}(g)$ implies $\tau_f \neq \tau_g$. So, the conclusion follows from Proposition 3.1. ∎

The following corollary answers a conjecture posed by Cabrera and Makienko [8].

Corollary 3.6. *Let $f, g \in \mathbb{C}(x)$ both have degree greater than 1. Let μ_f and μ_g be the measures of maximal entropy for f and g, respectively. If $\mu_f \neq \mu_g$, then there is a j such that $\langle f^j, g^j \rangle$ is a free semigroup on two generators.*

Proof. Theorem 1.5 of [69] (see also [11]) states that if $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is infinite for $f, g \in \mathbb{C}(x)$, then $\mu_f = \mu_g$. ∎

Remark 3.7. In fact, Theorem 1.5 of [69] implies that if $\mathrm{Prep}(f) = \mathrm{Prep}(g)$ for f and g polarizable endomorphisms of a projective variety defined over a finitely generated field K, then the canonical measures associated to f and g are equal at every place of K. However, equality of these measures at a single place is a much weaker condition than equality of the set of preperiodic points. For example, polarized morphisms having good reduction at a nonarchimedean place v will have the same canonical measure at v. Even over $\mathbb{C}$, one can have $\mu_f = \mu_g$ but $\mathrm{Prep}(f) \neq \mathrm{Prep}(g)$: take $f(x) = x^2$ and $g(x) = ax^2$ where $|a| = 1$ but a is not a root of unity, for example. We need $\mu_f = \mu_g$ over all places to conclude that $\mathrm{Prep}(f) = \mathrm{Prep}(g)$. On the other hand, equality of measures of maximal entropy for *nonspecial rational functions* of degree greater than 1 over $\mathbb{C}$ has powerful consequences, due to work of Levin [41] and Levin–Przytycki [42]; the results of Ye [67] that we use in the next section rely on the results of Levin and Levin–Przytycki.

3.4. *Some counterexamples*

Propositions 4.8 and 4.10 provide converses to Theorem 1.3 for endomorphisms of $\mathbb{P}^1$ in characteristic 0. One might ask more generally if it is true that when $f, g : V \to V$ are morphisms polarized by the same line bundle, the equality $\mathrm{Prep}(f) = \mathrm{Prep}(g)$ must imply that $\langle f, g \rangle$ cannot contain a nonabelian free semigroup. It turns out that this is not true for polarized morphisms of varieties of dimension greater than one in characteristic 0, as Example 3.8 shows. In characteristic p, it is not even true for polynomials, as Example 3.9 shows.

Example 3.8. Let A be an abelian variety defined over a number field such that $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{R}$ is the Hamiltonian quaternion algebra $\mathbb{H}$. (That such abelian varieties exist is well-known; see [43, Theorem B.33], for example.) By [21, Theorem 2], $1 + 2i$ and $1 + 2j$ generate a free multiplicative subgroup of $\mathbb{H}$ of rank 2. Let $\dagger$ be the Rosati involution associated to the theta divisor Θ (see [47, Section 17] for more details). We also have $\phi^\dagger\phi = [1 - 2i][1 + 2i] = [5]$ and $\psi^\dagger\psi = [1 - 2j][1 + 2j] = [5]$. Therefore ϕ and ψ are both polarized by Θ on A by [59, Proposition 3.1]. Since ϕ and ψ both commute with $[m]$ for any m, we must have $\mathrm{Prep}(\phi) = \mathrm{Prep}(\psi) = A_{\mathrm{tors}}$.

Example 3.9. Let $K = \mathbb{F}_p$ and let $d, e > 1$ be integers such that $p \nmid de$. If $f(x) = x^d$ and $g \in \mathbb{F}_p[x]$ is any polynomial of degree e that is not a monomial, then $\langle f, g \rangle$ is a free semigroup on two generators by [31, Lemma 3.1]. Note that $\mathrm{Prep}(f) = \mathrm{Prep}(g) = \overline{\mathbb{F}}_p$ since f and g are both defined over $\mathbb{F}_p$.

4. Proofs of Theorems 1.1 and 1.2

Throughout this section, $\mathcal{S}$ will denote a finitely generated semigroup of endomorphisms of $\mathbb{P}^1_{\mathbb{C}}$, and $\mathcal{S}^+$ will denote the set of elements in $\mathcal{S}$ of degree greater than 1.

We will now prove Theorems 1.1 and 1.2. We will do so by proving results on the growth of finitely generated semigroups $\mathcal{S}$ such that $\mathrm{Prep}(f) = \mathrm{Prep}(g)$ for all $f, g \in \mathcal{S}$ with degrees greater than 1; these are Propositions 4.8 and 4.10. We then combine these with Theorem 1.3.

Lemma 4.1. *Suppose that S is a finite set of maps from a set X to itself, and f is a map that sends all of X to a single element of X. Let $S_1 = S \cup \{f\}$. Then $|S^{\le n}| \le |S_1^{\le n}| \le 2|S^{\le n}|$.*

Proof. It will suffice to show that the number of words in $S_1^{\le n}$ containing f is bounded by $|S^{\le n}|$. Let $w \in S_1^{\le n}$ contain f. We write $w = w_1 f w_2$ where w_1 does not contain f (w_1 and w_2 may be empty). Let x_0 be the element of X such that $f(x) = x_0$ for all $x \in X$. Then $w(x) = w_1(x_0)$ for all $x \in X$. Since $w_1 \in S^{\le n}$, there are at most $|S^{\le n}|$ such $w_1(x_0)$, and our proof is done. ∎

Remark 4.2. While Lemma 4.1 allows us to treat semigroups of rational functions containing constant maps, we cannot expect to obtain results in higher dimensions for semigroups containing morphisms that are neither constant nor finite, because of the examples in [56].

Lemma 4.3. *Let $f \in \mathbb{C}(x)$ be a nonspecial rational function of degree greater than* 1. *Then the set of $\sigma \in \mathbb{C}(x)$ of degree 1 such that $\mu_f = \mu_{\sigma f}$ is finite.*

Proof. Set $\mu = \mu_f = \mu_g$. Its support $\mathcal{J}$ is the Julia set of f and g. Then $\sigma(\mathcal{J}) = \mathcal{J}$. Recall that by [41, Definition 1], the rational function $\sigma : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a symmetry of $\mathcal{J}$ when f is nonspecial if the following condition is satisfied: $x \in \mathcal{J}$ if and only if $\sigma(x) \in \mathcal{J}$. Thus, σ is a symmetry of $\mathcal{J}$. Since f is nonspecial the set of such symmetries is finite by [41, Theorem 1]. ∎

We will use the following result due to Ye [67]. Since his argument appears with slightly different notation as an implication in the proof of [67, Theorem 1.5, p. 393], rather than as a lemma or theorem, we provide here a proof using the same argument. Recall that if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is an endomorphism of $\mathbb{P}^1$ and x_0 is a fixed point, one defines the *multiplier* of f at x_0 to be the complex number λ such that the tangent map Df_{x_0} is the multiplication by λ. If $x_0 \neq \infty$, then λ is just $f'(x_0)$.

Lemma 4.4. *Let $f, g \in \mathbb{C}(x)$ be endomorphisms of $\mathbb{P}^1$ over the complex numbers such that $\mu_f = \mu_g$ and $\deg f = \deg g > 1$. Suppose that there is an x_0 in $\mathcal{J}_f = \mathcal{J}_g$ such that $f(x_0) = g(x_0) = x_0$ and the multipliers satisfy $f'(x_0) = g'(x_0)$. Then $f = g$.*

Proof. We may assume $x_0 \neq \infty$ (otherwise, conjugate by $z \mapsto 1/z$). Set $\mu = \mu_f = \mu_g$ and $\mathcal{J} = \mathcal{J}_f = \mathcal{J}_g$. Since x_0 is fixed by f and g and is in the Julia set of f and g, we must have $f'(x_0) = g'(x_0) \neq 0$, so f and g are both one-one in a neighborhood of x_0. Hence, there is a neighborhood W of x_0 and univalent functions $\tilde{g}, \tilde{f}$ on a neighborhood of x_0 containing W such that $\tilde{g} \circ g$, $g \circ \tilde{g}$, $\tilde{f} \circ f$, and $f \circ \tilde{f}$ are all the identity on W. There is then an open disc U around x_0 contained in W such that $f(U)$, $g(U)$, $\tilde{g}(U)$, and $\tilde{f}(U)$ are all contained in W. Let $R_1 = f \circ \tilde{g}$ and let $R_2 = g \circ \tilde{f}$; then $R_1 \circ R_2$ and $R_2 \circ R_1$ are both the identity on U. Then for any open subset V of U, we have $\mu(\tilde{g}(V)) = \mu(V)/\deg g$, since g is one-one on $\tilde{g}(V)$, and $\mu(f(\tilde{g}(V))) = (\deg f)\mu(\tilde{g}(V))$, since f is one-one on $\tilde{g}(V)$. Since $\deg f = \deg g$, we therefore have

$$\mu(R_1(V)) = \mu(V). \tag{4.1}$$

Likewise, we have

$$\mu(R_2(V)) = \mu(V). \tag{4.2}$$

Suppose that R_1 is not the identity on any neighborhood of x_0. Then, since R_1 has multiplier equaling 1 at x_0, it determines attracting and repelling petals near x_0; note that a repelling petal for R_1 is an attracting petal for R_2 (see [50, Definition 10.6]). The union of the repelling and attracting petals for R_1 contains a punctured disc D around x_0 [50, Theorem 10.7]. Furthermore, for any attracting petal $\mathcal{P}$ for R_1, there is a choice of coordinates such that $|R_1(z) - x_0| < |z - x_0|$ for all $z \in \mathcal{P}$, and likewise for any attracting petal of R_2 by [50, Theorem 10.9] (see also [3, Theorems 6.5.4 and 6.5.7]). Hence there is an open subset X of $U \cap D \cup \{x_0\}$ such that $R_i(X \cap \mathcal{P}) \subseteq X \cap \mathcal{P}$ for any attracting petal $\mathcal{P}$ of R_i, for $i = 1, 2$. Let Y be an open subset of $\mathcal{P} \cap X$, where $\mathcal{P}$ is an attracting petal for R_i. We may apply (4.1) and (4.2) to any $R_i^n(Y)$ because $R_i^n(Y) \subseteq U$ for any n. Since R_i^n converges uniformly to $\mathcal{P} \to x_0$ on $\mathcal{P}$ (see [50, p. 108]), we must have $\lim_{n\to\infty} \mu(R_i^n(Y)) = 0$, and thus $\mu(Y) = 0$. Hence $\mu(\mathcal{P} \setminus \{x_0\}) = 0$ at each petal, a contradiction because a Julia set has no isolated point. Therefore, R_1 must be the identity. ∎

Lemma 4.5. *Let $f, g \in \mathbb{C}(x)$ be endomorphisms of $\mathbb{P}^1$ over the complex numbers such that* $\mathrm{Prep}(f) = \mathrm{Prep}(g)$ *and* $\deg f = \deg g > 1$. *Then* $\mathrm{Prep}(w) = \mathrm{Prep}(f) = \mathrm{Prep}(g)$ *for all $w \in \langle f, g\rangle$.*

Proof. Let $\mathrm{Prep} = \mathrm{Prep}(f) = \mathrm{Prep}(g)$. Then $f(\mathrm{Prep}) \subseteq \mathrm{Prep}$ and $g(\mathrm{Prep}) \subseteq \mathrm{Prep}$. Let $w \in \langle f, g\rangle$. Then we have $w(\mathrm{Prep}) \subseteq \mathrm{Prep}$. Since Prep contains at most finitely many points defined over any finitely generated field by Theorems 3.3 and 3.4, it follows that for any $z \in \mathrm{Prep}$, the orbit of z is finite under w, so $\mathrm{Prep} \subseteq \mathrm{Prep}(w)$. Since $\mathrm{Prep}(f)$ and $\mathrm{Prep}(g)$ are infinite, it follows from [2, Theorem 1.2] that $\mathrm{Prep}(w) = \mathrm{Prep}(f) = \mathrm{Prep}(g)$. ∎

Lemma 4.6. *Let $f, g \in \mathbb{C}(x)$ be nonspecial rational functions of degree greater than 1 such that* $\operatorname{Prep}(f) = \operatorname{Prep}(g)$ *and* $\deg f = \deg g$. *Let $\mathcal{J}$ denote $\mathcal{J}_f = \mathcal{J}_g$. Suppose there is a periodic cycle $\{x_1, \dots, x_r\}$ for f and g in $\mathcal{J}$ such that $f(x_i) = g(x_i) = x_{i+1}$ for $i = 1, \dots, r-1$ and $f(x_r) = g(x_r) = x_1$. Then we have the following:*

(1) *$f'(x_1)/g'(x_1)$ is a root of unity.*

(2) *If $f'(x_1) = g'(x_1)$, then $f = g$.*

Proof. Let $\eta_i = f'(x_i)$ and let $\lambda_i = g'(x_i)$ for $i = 1, \dots, r$. Note that none of the η_i and λ_i are zero since the x_i are in the Julia set for f and g. Now let $w_1 = fg^{r-1}$ and let $w_2 = g^r$. Then $w_1(x_2) = w_2(x_2) = x_2$. Furthermore, $\operatorname{Prep}(w_1) = \operatorname{Prep}(w_2)$ by Lemma 4.5. We have

$$w_1'(x_2) = \eta_1 \prod_{i=2}^{r} \lambda_i \tag{4.3}$$

and

$$w_2'(x_2) = \lambda_1 \prod_{i=2}^{r} \lambda_i \tag{4.4}$$

by the chain rule.

Since $\operatorname{Prep}(w_1) = \operatorname{Prep}(w_2) = \operatorname{Prep}(f) = \operatorname{Prep}(g)$ and $\mathcal{J}$ contains a point that is periodic for both w_1 and w_2, it follows from [67, Theorem 1.5] that there is an n such that $w_1^n = w_2^n$ for the iterates w_1^n and w_2^n. By (4.3) and (4.4), $(\eta_1/\lambda_1)^n = 1$, so $f'(x_1)/g'(x_1)$ is a root of unity, as desired. Furthermore, if $f'(x_1) = g'(x_1)$, then (4.3) and (4.4) imply that $w_1'(x_2) = w_2'(x_2)$, which means that $w_1 = w_2$, by Lemma 4.4. Now, if $fg^{r-1} = g^r$, then $f = g$ by right cancellation. ∎

Lemma 4.7. *Let $\mathcal{S}$ be a finitely generated semigroup of endomorphisms of $\mathbb{P}^1_{\mathbb{C}}$. Suppose that $\mathcal{S}^+$ contains a nonspecial endomorphism f and that* $\operatorname{Prep}(g) = \operatorname{Prep}(f)$ *for all $g \in \mathcal{S}^+$. Then there is a constant N such that for all $d \geq 1$, the number of elements of $\mathcal{S}$ of degree d is less than or equal to N.*

Proof. For all $g \in \mathcal{S}$, $\operatorname{Prep}(gf) = \operatorname{Prep}(f)$. Thus, by Lemma 4.3, the set $\{\sigma \in \mathcal{S} : \deg(\sigma) = 1\}$ is finite.

Now, let $\mathcal{J}$ be the Julia set, and let $\operatorname{Prep}(\mathcal{S}^+)$ be the set of preperiodic points of the elements of $\mathcal{S}^+$. Then $g(\operatorname{Prep}(\mathcal{S}^+)) = \operatorname{Prep}(\mathcal{S}^+)$ for all $g \in \mathcal{S}^+$. Choose $y_0 \in \mathbb{C} \cap \operatorname{Prep}(\mathcal{S}^+)$. Let K be a finitely generated field over which y_0 and every element of $\mathcal{S}$ are defined. Since $\operatorname{Prep}(g) \cap K$ is finite for each g of degree greater than 1 (by Theorems 3.3 and 3.4), it follows that the orbit $\mathcal{O}$ of y_0 under $\mathcal{S}^+$ is finite. After change of coordinates, we may assume that $\mathcal{O}$ does not contain the point at infinity.

Let n be the number of roots of unity in K. Let $g \in \mathcal{S}^+$ have degree d. We will show that there are at most n elements $h \in \mathcal{S}^+$ such that $\deg h = d$ and $g|_{\mathcal{O}} = h|_{\mathcal{O}}$. Let $\{x_1, \dots, x_r\}$ be a periodic cycle for g in $\mathcal{O}$ (note that there must be one since $\mathcal{O}$ is finite) such that $g(x_i) = x_{i+1}$ for $i = 1, \dots, r-1$ and $g(x_r) = x_1$. Then for any $h \in \mathcal{S}$ such that $\deg h = d$ and $h|_{\mathcal{O}} = g|_{\mathcal{O}}$, we may apply Lemma 4.6 to conclude that $f'(x_1)/g'(x_1)$ is a root of unity in K. Furthermore, given any $h_1, h_2 \in \mathcal{S}$ of degree d such that $h_1|_{\mathcal{O}} = h_2|_{\mathcal{O}} = g|_{\mathcal{O}}$, we have $h_1 = h_2$ whenever $h_1'(x_1) = h_2'(x_1)$. Thus, the number of $h \in \mathcal{S}$ such that $\deg h = d$ and $h|_{\mathcal{O}} = f|_{\mathcal{O}}$ is bounded by the number of roots of unity in K, which is n. Since $\mathcal{O}$ is finite, there are $|\mathcal{O}|^{|\mathcal{O}|}$ maps from $\mathcal{O}$ to itself, so there are at most $n|\mathcal{O}|^{|\mathcal{O}|}$ elements of $\mathcal{S}$ of degree d for any $d > 1$. ∎

Proposition 4.8. *Suppose that $\mathcal{S}$ is a finitely generated semigroup of endomorphisms of $\mathbb{P}^1_{\mathbb{C}}$ such that every element of $\mathcal{S}^+$ is nonspecial and has the same set of preperiodic points. Then*

(i) *there is some $c \geq 2$ such that every element of $\mathcal{S}$ has degree c^n for some $n \geq 0$;*

(ii) *$\mathcal{S}$ has linear growth.*

Proof. Since $\mu_f = \mu_g$ for any $f, g \in \mathcal{S}^+$ (by [69, Theorem 1.5], as in Corollary 3.6), there are m and n such that $(\deg f)^m = (\deg g)^n$ (by [42, Theorem A]). Since the set $\mathcal{D}$ of possible degrees of elements in $\mathcal{S}^+$ is a finitely generated subsemigroup of the positive natural numbers under multiplication, it follows that $\mathcal{D}$ is contained in a semigroup generated by a single element, and (i) follows.

For (ii), it suffices to treat the case where every element of $\mathcal{S}$ is nonconstant, by Lemma 4.1. Let $f_1, \ldots, f_s$ be generators for $\mathcal{S}$. Then for each i, there is some $c \geq 2$ such that $\deg(f_i) = c^{m_i}$, by the previous paragraph, since all $f_i \in \mathcal{S}^+$ have the same set of preperiodic points. Let $M = \max(m_1, \ldots, m_s)$. Now, consider the set $S^{\leq n}$ of elements in $\mathcal{S}$ formed by taking a words of length at most n in $\{f_1, \ldots, f_s\}$. These elements all have degrees in $\{1, c, c^2, \ldots, c^{Mn}\}$, so there is a natural number N such that $|S^{\leq n}| \leq MNn$ for all n, by Lemma 4.7. Clearly, $|S^{\leq n}| \geq n + 1$ since the elements id, $f, f^2, \ldots, f^n$ are pairwise distinct for $f \in \mathcal{S}^+$, and so we see that $\mathcal{S}$ has linear growth as claimed. ∎

Lemma 4.9. *Let E be an elliptic curve defined over $\mathbb{C}$ and let $\pi : E \to \mathbb{P}^1$ be a nonconstant morphism. Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism of degree greater than 1 such that there is a nonconstant morphism $\phi_f : E \to E$ with $f \circ \pi = \pi \circ \phi_f$ (i.e., f is a Lattès map associated to π and E). Then for any $g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree greater than 1 such that* $\mathrm{Prep}(g) = \mathrm{Prep}(f)$, *there is a nonconstant $\phi_g : E \to E$ such that $\pi \circ \phi_g = g \circ \pi$. Furthermore, ϕ_g can be written as $m_g + t_g$, where $m_g \in \mathrm{End}(E)$ and t_g is translation by a torsion element of E.*

Proof. Since $\mathrm{Prep}(g) = \mathrm{Prep}(f)$ implies that $g(\mathrm{Prep}(f)) \subseteq \mathrm{Prep}(f)$, we may apply [34, Theorem 27]. The last two lines of the proof of this theorem state that there is an automorphism $\sigma : E \to E$ and a nonconstant morphism $\psi : E \to E$ such that $(\pi \circ \sigma) \circ \psi = g \circ (\pi \circ \sigma)$. Letting $\phi_g = \sigma \circ \psi \circ \sigma^{-1}$ gives $\pi \circ \phi_g \circ \sigma = g \circ \pi \circ \sigma$, which implies that $\pi \circ \phi_g = g \circ \pi$, as desired. The fact that ϕ_g can be written as $m_g + t_g$, where $m_g \in \mathrm{End}(E)$ and t_g is translation by a torsion element of E, follows from [34, Theorem 30] (see also [49, Theorem 3.1]). ∎

Proposition 4.10. *Let $\mathcal{S}$ be a finitely generated semigroup of endomorphisms of $\mathbb{P}^1$ over $\mathbb{C}$. Suppose that for any $f, g \in \mathcal{S}^+$, we have* $\mathrm{Prep}(f) = \mathrm{Prep}(g)$. *Then $\mathcal{S}$ has polynomially bounded growth, and if $\mathcal{S}^+$ contains at least one nonspecial rational function, then $\mathcal{S}$ has linear growth.*

Proof. Let K be a finitely generated field such that every element of $\mathcal{S}$ is defined over K. Again, we may assume that every element of $\mathcal{S}$ is nonconstant, by Lemma 4.1. Because of Proposition 4.8, we need only treat the cases where every element of $\mathcal{S}$ is linear or $\mathcal{S}^+$ contains a special rational function. In the cases where $\mathcal{S}^+$ contains a special rational function, we will show both that $\mathcal{S}^+$ has polynomially bounded growth and that every element of $\mathcal{S}^+$ is special. It follows from this that if $\mathcal{S}^+$ contains at least one nonexceptional rational function, then $\mathcal{S}$ has linear growth by Proposition 4.8.

Case I: Every element of $\mathcal{S}$ is linear. In this case, the result is contained in [57, Theorem 1.5].

Case II: Some element of $\mathcal{S}^+$ is conjugate to x^m. There are $f \in \mathcal{S}^+$ and $\sigma \in \mathrm{PGL}_2(\mathbb{C})$ such that $\sigma^{-1} f \sigma = x^m$ for some integer m with $|m| \geq 2$. Then we may assume that $f(x) = x^m$ is in $\mathcal{S}^+$. Then, for any $g \in \mathcal{S}^+$, the Julia set $\mathcal{J}_g$ is the unit circle, so we have $g(x) = ax^d$ for some $d = d(g) \in \mathbb{Z}$ and some $a = a(g)$ of modulus 1 by [3, Theorem 1.3.1]. Since $\mathrm{Prep}(g)$ consists of the roots of unity along with 0 and ∞, we see that a must be a root of unity. All the elements of $\mathcal{S}$ are defined over K, and there are at most N roots in unity in K for some $N \geq 1$. Thus, there are at most N elements in $\mathcal{S}^+$ of degree d for each $d \geq 2$. Every element σ in $\mathcal{S}$ of degree 1 has the form $\sigma(x) = ax^{\pm 1}$ for some root of unity $a = a(\sigma)$, so there are finitely many elements in $\mathcal{S}$ of degree 1. Thus, if S is a finite set that generates $\mathcal{S}$, then $|S^{\leq n}|$ is bounded above by $O(n^s)$ where $s = |S|$.

Case III: Some element of $\mathcal{S}^+$ is conjugate to a Chebyshev polynomial. Let $f \in \mathcal{S}^+$ be conjugate to $\pm T_m$ where T_m is the Chebyshev polynomial of degree d with $m \geq 2$. Thus, after change of coordinates we may assume that $f = \pm T_m$. Let $g \in \mathcal{S}^+$. Then $\mathcal{J}_g = \mathcal{J}_f = [-2, 2]$, so by [3, Theorem 1.4.1], we have $g = \pm T_d$ for some d. Let S be a finite set that generates $\mathcal{S}$. Then the number of possible degrees for elements of $S^{\leq n}$ is bounded by $O(n^s)$ where $s = |S|$, so the number of elements of $S^{\leq n}$ of degree greater than 1 is bounded by $O(n^s)$ since there are at most two elements in $\mathcal{S}^+$ having the same degree by the above. Since every element of degree 1 in $\mathcal{S}$ sends $[-2, 2]$ to itself, there are at most two elements of degree 1 in $\mathcal{S}$. Hence $\mathcal{S}$ has polynomially bounded growth.

Case IV: Some element of $\mathcal{S}^+$ is Lattès. Suppose that $f \in \mathcal{S}^+$ is Lattès; then there are nonconstant $\phi_f : E \to E$ and $\pi : E \to \mathbb{P}^1$ with the property that $f \circ \pi = \pi \circ \phi_f$. Let $\mathrm{End}(E)$ be the ring of endomorphisms of the algebraic curve E that preserve the group law. After passing to a finite extension, we may assume that π is defined over K (which implies that each ϕ_f is defined over K) and that every element of $\mathrm{End}(E)$ is defined over K. Since $\mathrm{Prep}(g) = \mathrm{Prep}(f)$ for every $g \in \mathcal{S}^+$, Lemma 4.9 implies that for each $g \in \mathcal{S}^+$, there is a nonconstant $\phi_g : E \to E$ such that $g \circ \pi = \pi \circ \phi_g$. Each ϕ_g is defined over K and $\phi_g = m_g + t_g$, where $m_g \in \mathrm{End}(E)$ and t_g is a translation by a torsion point. Because E has at most finitely many torsion points defined over K by [54, Chapter II, Theorem 3], only finitely many ϕ_g of any given degree are possible. Let $\mathcal{S}_1$ denote the set of elements of degree 1 in $\mathcal{S}$. Since $\deg \sigma g = \deg g$ for all $\sigma \in \mathcal{S}_1$, the set $\mathcal{S}_1$ is finite. Let T be a finite set of generators for $\mathcal{S}$. Enlarging T to another finite set if necessary, we may assume that T contains all $\sigma q \tau$ where $\sigma, \tau \in \mathcal{S}_1$ and $q \in \mathcal{S}^+ \cap T$. Since the

m_g commute, the number of words in $m_g + t_g$ for $g \in T$ of length n or less is bounded above by $O(n^{|T|})$. Thus the number of words of length n in elements of T is bounded by $O(n^{|T|})$ as well. Now, let $S = T \cup \mathcal{S}_1$. Then $S^{\leq n} \subseteq T^{\leq n} \cup \mathcal{S}_1$, so $|S^{\leq n}|$ is bounded above by $O(n^{|T|})$ as well (since $\mathcal{S}_1$ is finite), and hence $\mathcal{S}$ has polynomially bounded growth. ∎

Remark 4.11. The proof of Proposition 4.10 shows that for a finitely generated field K and any rational function $f \in K(x)$ that is not a Lattès map, there is a constant $N(f, K)$ such that for any d, the set of endomorphisms g of $\mathbb{P}^1$ of degree d such that $\mathrm{Prep}(g) = \mathrm{Prep}(f)$ has at most $N(f, K)$ elements. This is clearly not true if f is a Lattès map associated to an elliptic E with complex multiplication. Indeed, in that case, $A := \mathrm{End}(E)$ is an order in a quadratic number field M. Let $\phi \in A$ be a lift of f. Then $\deg(f) = |N_{M/\mathbb{Q}}(\phi)|$. Therefore, by [64, Proposition 6.37]), we have

$$\#\{g \in K(x) \mid \deg g = \deg f\} \geq \#\{\psi \in A \mid |N_{M/\mathbb{Q}}(\psi)| = \deg f\}/6,$$

which depends also on M, and hence cannot be bounded by a constant of the type $N(f, K)$.

Proof of Theorem 1.2. By Theorem 1.3, the semigroup $\mathcal{S}$ contains a free subsemigroup on two elements unless $\mathrm{Prep}(f) = \mathrm{Prep}(g)$ for all $f, g \in \mathcal{S}^+$, in which case $\mathcal{S}$ has linear growth by Proposition 4.10. ∎

Proof of Theorem 1.1. As above, the semigroup $\mathcal{S}$ contains a free subsemigroup on two elements unless $\mathrm{Prep}(f) = \mathrm{Prep}(g)$ for all $f, g \in \mathcal{S}^+$, by Theorem 1.3, in which case $\mathcal{S}$ has polynomially bounded growth by Proposition 4.10. ∎

Corollary 4.12. *Let $f, g \in \mathbb{C}(x)$ be two endomorphisms of $\mathbb{P}^1_{\mathbb{C}}$, each having degree greater than* 1. *Then the following are equivalent:*

(i) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is infinite;*

(ii) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$*;*

(iii) *for any $w_1, w_2 \in \langle f, g\rangle$, we have* $\mathrm{Prep}(w_1) = \mathrm{Prep}(w_2)$*;*

(iv) *$\langle f, g\rangle$ has polynomial growth;*

(v) *$\langle f, g\rangle$ does not contain a nonabelian free semigroup;*

(vi) *for any $\ell > 0$, the semigroup $\langle f^\ell, g^\ell\rangle$ is not free on two generators.*

Proof. It is clear that (iii) implies (ii), that (iv) implies (v), and that (v) implies (vi). Lemma 4.5 shows that (ii) implies (iii). Theorem 1.2 of [2] (see also [11, 51, 68, 69]) states that (i) and (ii) are equivalent. Proposition 4.10 shows that (iii) implies (iv). By Theorem 1.3, we see that (vi) implies (iii). ∎

The techniques used to prove Theorems 1.1 and 1.2 do not extend in an obvious way to higher dimensions. We note, however, that for an abelian variety A one can use the fact that $A(K)$ is finitely generated when K is finitely generated (see [54]) along with [57, Theorem 1.5] to prove the following without difficulty.

Theorem 4.13. *Let A be an abelian variety. Let $\mathcal{S}$ be a finitely generated semigroup of finite morphisms from A to itself. Then either $\mathcal{S}$ has polynomially bounded growth or $\mathcal{S}$ contains a nonabelian free semigroup.*

Corollary 4.14. *Let C be an irreducible curve over $\mathbb{C}$ and let $\mathcal{S}$ be a finitely generated semigroup of morphisms from C to itself. Then either $\mathcal{S}$ has polynomially bounded growth or $\mathcal{S}$ contains a nonabelian free semigroup.*

Proof. By Lemma 4.1, we may assume that every element of $\mathcal{S}$ is nonconstant. Any morphism $f : C \to C$ extends to a morphism $\tilde{f} : C' \to C'$ for C' the normalization of the projective closure of C. Hence we may assume that C is projective and nonsingular. If C has genus greater than 1, then $\mathcal{S}$ must be finite (see [25, Example IV.5.2], for instance), so we may assume that C is isomorphic either to $\mathbb{P}^1$ or to an elliptic curve. Applying Theorems 1.1 and 4.13 then gives the desired conclusion, since every nonconstant map on an irreducible curve is finite. ■

5. Further directions

Let K be a field of arbitrary characteristic.

Question 5.1. *Are there endomorphisms $f, g \in K(x)$ of $\mathbb{P}^1$ over K, each of degree greater than 1, such that $\mathrm{Prep}(f) \neq \mathrm{Prep}(g)$ and $\langle f, g\rangle$ is not a free semigroup on two generators?*

One might also ask for something weaker, namely that there is a j depending only on K such that $\langle f^j, g^j\rangle$ must be free whenever $\mathrm{Prep}(f) \neq \mathrm{Prep}(g)$. This might be thought of as analogous to the uniform version of the Tits alternative proved by Breuillard and Gelander [7]. Recent work of DeMarco, Krieger, and Ye [15] suggests it may be possible to do something along these lines using Arakelov–Zhang intersections of adelically-metrized line bundles on $\mathbb{P}^1$ (see also [19,60]).

Question 5.2. *Let V be a projective variety, let $\mathcal{L}$ be an ample line bundle on V, and let $\mathcal{S}$ be a finitely generated semigroup of morphisms f that are polarized by $\mathcal{L}$. Is it true that $\mathcal{S}$ must either have polynomially bounded growth or contain a nonabelian free semigroup?*

It might also be natural to ask for a version of the Tits alternative for semigroups of dominant endomorphisms that says something about the structure of the semigroups rather than the growth. For example, one might ask if it is true that any finitely generated semigroup of dominant endomorphisms must contain either a nilpotent subsemigroup of finite index or a free subsemigroup on two generators (there is a notion of nilpotence for semigroups due to Mal'tsev [45]). The techniques of this paper can be adapted to show that if $\mathcal{S}$ is a finitely generated semigroup of endomorphisms of $\mathbb{P}^1$ such that $\mathcal{S}^+$ is nonempty, then $\mathcal{S}$ contains either an abelian subsemigroup of finite index or a free semigroup on two generators (see Remark 4.11). In higher dimensions, it seems likely that one might also have to allow for the more general possibility of nilpotent subsemigroups of finite index. We note that Grigorchuk [22] has shown that finitely generated cancellative semigroups have polynomially bounded growth if and only if they have a group of left quotients with a nilpotent subgroup of finite index; this extends well-known work of Gromov [23] from the group setting to the cancellative semigroup setting.

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