Port. Math. 81 (2024), 201–246
DOI 10.4171/PM/2124

First-order mean-field games on networks and Wardrop equilibrium

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Abstract. We explore the relationship between Wardrop equilibrium and stationary mean-field games (MFG) on networks with flow-dependent costs. First, we present the notion of Wardrop equilibrium and the first-order MFG model on networks. We then reformulate the MFG problem into a road traffic problem, establishing that the flow distribution of the MFG solution is the corresponding Wardrop equilibrium. Next, we prove that the solution of the MFG model can be recovered using the corresponding Wardrop equilibrium. Next, we examine the cost properties and calibrate MFG with respect to travel cost problems on networks. We propose a novel calibration approach for MFGs. Additionally, we demonstrate that non-monotonic MFGs can be generated by even simple travel costs.

1. Introduction

Models for flows on networks arise in the study of road traffic and pedestrian crowds. We consider stationary models for which the cost of traversing a network edge depends on the agent flow in that edge. These models capture congestion effects and agents' behavior and preferences, including crowd aversion and attempts to minimize travel time. Agents minimize their costs by considering the flow. A well-studied stationary equilibrium is the one following Wardrop's first criterion: "Equal journey times on all used routes, and less than those experienced by a single vehicle on any unused route.", [40, 41]. Multiple authors have studied Wardrop equilibria; see, for example, the survey [23]. In the context of optimal transport, Wardrop equilibrium was addressed in [22] and [21]. Road traffic is modeled on directed networks. Pedestrian network models are inherently undirected, so we introduce the associated directed network. In a road traffic model, an edge is an aggregated entity, hence the model does not describe the microstructure within the edge. In [26], a mean-field game (MFG) model on undirected networks was introduced to address these matters.

Mathematics Subject Classification 2020: 91A16 (primary); 49N80, 76A30 (secondary).
Keywords: Wardrop equilibrium, mean-field games (MFG), networks, flow-dependent costs, road traffic, calibration, non-monotonic MFGs.

The MFG theory was introduced in [29, 33] to describe the dynamics of systems with a large number of rational agents. In these models, each agent optimizes an individual functional, depending on their actions and the distribution of other agents. An MFG model comprises a backward-in-time Hamilton–Jacobi (HJ) equation coupled with either a forward-in-time Fokker–Planck (FP) equation, or a transport equation. The HJ equation describes the optimal behavior of an agent; the FP equation governs the distribution of the agents. MFG models have been used to address pedestrian flows [13], crowds [32], population dynamics [14], building evacuation problems [24], and in the study of traffic flows [11].

Recently, there has been significant interest in studying nonlinear PDEs on networks due to their applications in traffic and pedestrian dynamics. Various notions of viscosity solutions to HJ equations on networks were studied in [1, 30, 38]. Later in [18], the authors proved the equivalence of the viscosity solutions of HJ equations on networks. The existence and uniqueness of viscosity solutions to Eikonal equations on networks are addressed in [20]. More recently, progress on junction problems has been obtained in [35, 36]. Stationary HJ equations on networks were considered in [39] and [31]. Concerning transport phenomena on networks, the survey paper [12] examines numerous results. More recent works in this direction include [17, 25].

Several researchers in the MFG community examined MFGs on networks, particularly second-order problems. Stationary second-order MFGs were studied in [2, 15, 19]. The time-dependent case was studied in [3, 16]. However, these methods for MFGs on networks are not applicable to first-order MFGs, which present a distinct set of phenomena, causing issues such as loss of smoothness in HJ equations and value function discontinuity at vertices. First-order MFGs on networks were considered in [7–10], specifically in the context of optimal visiting problems where agents have multiple targets. The methods in those papers, since they focus on general time-dependent problems, are quite different from ours, where we take full advantage of the stationary nature of the game. Recently, in [6], time-dependent MFGs on networks were analyzed using weak solutions. First-order deterministic MFGs with control on the acceleration were studied in [4, 5]. As far as the authors are aware, there is no systematic approach for solving stationary first-order MFGs on networks, despite their relevance in scenarios that cannot be modeled by second-order MFGs, such as vehicular networks or dense crowds.

The relation between Wardrop equilibrium and MFGs on networks was observed in [26]. However, the exact relationship between these two models was not established there. We address the challenge of converting an MFG on a network into a standard road traffic problem. Moreover, we demonstrate how to utilize the Wardrop equilibrium of the transformed problem to solve the original MFG.

Organization and results

In this paper, we consider a road traffic model, where agents enter through a finite set of vertices with a prescribed flow and leave through a set of exit vertices where they pay an exit cost. Although our MFG model is defined on undirected networks, the road traffic model is closely related to an MFG model on networks. In our MFG model, the entry flows and the exit costs are given, but the flow direction in each edge is not prescribed a priori. We discuss a method to transform one model into the other.

Section 2 outlines the main definitions and results about Wardrop equilibria. We introduce a road traffic model on a directed network, with travel costs that depend the flow in the edges. Furthermore, we prove the uniqueness of the Wardrop equilibrium (refer to Theorem 2.6). Next, we present our MFG model on networks. We start by examining an MFG model on a single edge and recalling the flow technique from [27, 28] (see Section 3). We then define a cost that depends on the flow in the edge for the MFG model (see Definition 5). Then, we establish the connection between the cost and the solution to the MFG system (1.1). In Section 4, we further discuss MFG models on networks. Our model is defined by a one-dimensional, stationary, first-order MFG system for each edge e_k,

$$\begin{cases} H_k(x, u_x(x), m(x)) = 0, \\ (-m(x) D_p H_k(x, u_x(x), m(x)))_x = 0, \end{cases} \tag{1.1}$$

alongside boundary conditions at the vertices. Here, u is the value function for an agent and m is the probability density of agents distribution. The first equation of (1.1) is the Hamilton–Jacobi (HJ) equation and the latter equation is the Fokker–Planck (FP) or transport equation.

The first main contribution of this paper is the MFG problem's reformulation into a road traffic problem and the creation of a method to recover the MFG solution from the corresponding Wardrop equilibrium. In Section 5, we start with the undirected network of the MFG model and construct an associated directed network to define a corresponding road traffic problem. Then we prove that the MFG problem's solution is the Wardrop equilibrium of its corresponding road traffic problem (see Theorem 5.3). Moreover, Proposition 7 proves that the associated road traffic problem has a unique solution. This finding establishes the uniqueness of the solution to the original MFG problem (see Proposition 8). To finalize the connection between MFG and road traffic models, in Section 6, we show the process of retrieving the solution to the MFG problem from the associated Wardrop equilibrium (see Theorem 6.2).

In contrast to the road traffic model where the cost depends on the edges, MFG models enable costs within an edge to vary at different points. Thus, the conversion from MFG and road traffic models averages the MFG's microscopic effects into a macroscopic cost for the road traffic model. Section 7, containing the paper's second main result, examines how the MFG's microscopic properties encoded in the Hamiltonian translate to the road traffic model's macroscopic properties. We also investigate a class of Hamiltonians for which the associated cost function fulfills the requirements for a unique correspondence between MFG and road traffic models.

One can consider the edge's cost as travel time, which can be experimentally measured based on the flow of agents on the edges. Therefore, in principle, these models are simple to calibrate: it is enough to have data on the flow rates and corresponding velocities. This calibration issue is more complex for MFG models. Hence, Section 8 demonstrates the calibration of an MFG model on a single edge using the concepts and results from previous sections. The problem is the following.

Problem 1. *Consider the cost c for an agent to cross an edge. Suppose c is given as a function of the flow and the direction of travel. Determine an MFG model whose cost coincides with c.*

Note that the cost, c, may not be the travel time, and as far as the authors know, before this work, there was no systematic approach to solving Problem 1. We rigorously state the preceding problem in Section 8 and solve it. We demonstrate how non-monotone MFG models arise as solutions to Problem 1 for natural costs and might be necessary for modeling general traffic problems. Non-monotone problems warrant further research due to their surprising modeling implications as agents appear to prefer congested areas.

This work suggests several future research directions, including the exploration of non-monotone MFGs, examining their general properties, and extending our models to dynamic settings. The non-monotonicity of these MFG models introduces the potential for new phenomena, like multiple equilibria or instability, warranting further investigation.

2. Wardrop equilibrium

We examine a steady-state model where agents navigate a network. This network features flow-dependent travel costs on its edges.

After discussing the model's components, we delve into Wardrop's equilibrium concept [41]. In Section 5, we establish a connection between this concept and the MFG model detailed in Section 4. The model comprises the following:

(1) A finite *directed* network (graph), $\widetilde{\Gamma} = (\widetilde{E}, \widetilde{V})$, where $\widetilde{E} = \widetilde{e}_k : k \in 1, 2, \ldots, \widetilde{n}$ represents the set of edges and $\widetilde{V} = \widetilde{v}_i : i \in 1, 2, \ldots, \widetilde{m}$ represents the set of vertices. Each edge $\widetilde{e}_k$ corresponds to a pair of endpoints $(\widetilde{v}_r, \widetilde{v}_i)$, which can be used interchangeably with $\widetilde{e}_k$ when no confusion arises.

(2) The *flow* denotes the number of agents passing a specific point in a given time unit. The flow in edge $\widetilde{e}_k$ is denoted by $\tilde{j}_k$.
The network's flow refers to the $\widetilde{n}$-dimensional vector $\tilde{\boldsymbol{j}} = (\tilde{j}_1, \ldots, \tilde{j}_{\widetilde{n}})$.

(3) A *travel cost* on each edge $\widetilde{e}_k$ defined by a continuous function $\widetilde{c}_k : \widetilde{E} \to \mathbb{R}$. The cost vector for the edges is denoted as $\widetilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}) = (\widetilde{c}_1(\tilde{\boldsymbol{j}}), \ldots, \widetilde{c}_{\widetilde{n}}(\tilde{\boldsymbol{j}}))$. Note that $\langle \widetilde{\boldsymbol{c}}(\boldsymbol{j}), \boldsymbol{j} \rangle := \sum_{k=1}^{\widetilde{n}} \widetilde{c}_k(\boldsymbol{j}) j_k$ indicates the social cost per unit of time corresponding to the distribution of flows $\boldsymbol{j}$.

(4) Agents travel the network entering via $\widetilde{\lambda}$ *entrance vertices* and exiting through $\widetilde{\mu}$ *exit vertices*, which are distinct from the entrance vertices. For simplicity, we assume that the last $\widetilde{\mu}$ vertices in $\widetilde{V}$ are the exit vertices. Additionally, we presume that entrance and exit vertices have an incidence of 1. As mentioned in the remark below, this assumption entails no loss of generality.

Remark 2.1. Should a vertex with an incidence greater than 1 be identified as an entrance, we attach to it an auxiliary entrance edge, designating another vertex as the new entrance vertex. Similarly, for an exit vertex, we relabel it by adding an auxiliary *exit edge*.

(5) A prescribed flow of agents, denoted by the entry flow $\tilde{\boldsymbol{\iota}} = (\tilde{\iota}_1, \ldots, \tilde{\iota}_{\widetilde{\lambda}}) > 0$, is given at entrance vertices, while entry flows for other vertices are set to zero. The information is encoded in a $(\widetilde{m} - \widetilde{\mu})$-dimensional vector $\widetilde{B}$. Each component of $\widetilde{B}$ corresponds to a non-exit vertex.

(6) At the $\widetilde{\mu}$ exit vertices, agents incur an exit cost $\widetilde{\boldsymbol{\phi}}(\widetilde{v}_i)$ for $i = \widetilde{m} - \widetilde{\mu} + 1, \ldots, \widetilde{m}$. This cost is assumed to be zero, resulting in no loss of generality, as explained in Remark 2.2.

Remark 2.2. If the exit cost at an exit vertex is nonzero, we attach an auxiliary exit edge to it and relabel its extra vertex as the new exit vertex. In the auxiliary exit edge, the travel cost is the exit cost, and at the new exit vertex, the exit cost is zero.

Define $\widetilde{K}$ as the $(\widetilde{m} - \widetilde{\mu}) \times \widetilde{n}$ Kirchhoff matrix, obtained by removing $\widetilde{\mu}$ lines corresponding to the exit vertices of $\widetilde{\Gamma}$ from the *incidence matrix* of $\widetilde{\Gamma}$

$$\widetilde{K}_{ik} = \begin{cases} -1 & \text{if } \widetilde{e}_k = (\widetilde{v}_i, \widetilde{v}_r), \\ 1 & \text{if } \widetilde{e}_k = (\widetilde{v}_r, \widetilde{v}_i), \\ 0 & \text{if } \widetilde{v}_i \notin \widetilde{e}_k, \end{cases}$$

where $i \in \{1, 2, \ldots, \widetilde{m} - \widetilde{\mu}\}$ and $k \in \{1, 2, \ldots, \widetilde{n}\}$. Rows of the $\widetilde{K}$ matrix correspond to non-exit vertices, while columns correspond to the edges.

Definition 1. A distribution of flows, $\tilde{j} \geq 0$, is *admissible* if

$$\widetilde{K}\tilde{j} + \widetilde{B} = 0. \tag{2.1}$$

The set of all admissible distributions of flows is denoted by $\mathcal{A}$.

Remark 2.3. Equations in (2.1) correspond to Kirchhoff's law for non-exit vertices.

Lemma 2.4. *The set $\widetilde{\mathcal{A}} := \mathcal{A}/\widetilde{K}$, the quotient of $\mathcal{A}$ by the kernel of $\widetilde{K}$, both defined above, is closed, convex, and bounded.*

Proof. Because we are looking at the quotient, there is a representative of the equivalence class for which each coordinate of $\tilde{j}$ is at most the sum of the entries of $\widetilde{B}$, this is the canonical representative.

Convexity and closedness are due to the linear nature of the admissibility equation (2.1). ∎

Following Smith [40], we define the Wardrop equilibrium.

Definition 2. A distribution of flows $\tilde{j}^* \in \mathcal{A}$ is a *Wardrop equilibrium* if, for all admissible $\tilde{j} \in \mathcal{A}$,

$$\langle \tilde{c}(\tilde{j}^*), \tilde{j}^* - \tilde{j} \rangle \leq 0. \tag{2.2}$$

In Section 6, we prove that Definition 2 implies that in a Wardrop equilibrium, any flow-carrying minimizes the travel cost to an exit.

We use the notion of monotonicity to state and prove a result on the uniqueness of Wardrop equilibria.

Definition 3. A cost $\tilde{c}$ is *monotone* if, for any $\tilde{j}_1, \tilde{j}_2 \in \mathcal{A}$,

$$\langle \tilde{c}(\tilde{j}_1) - \tilde{c}(\tilde{j}_2), \tilde{j}_1 - \tilde{j}_2 \rangle \geq 0. \tag{2.3}$$

If $\tilde{j}_1 \neq \tilde{j}_2$, the inequality in (2.3) is strict; in that case, we say that $\tilde{c}$ is *strictly monotone*.

Example 2.5. If $\tilde{c}_k$ depends only on $\tilde{j}_k$ and is an increasing function, then $\tilde{c}$ is the gradient of a convex function, hence monotone. In fact, in this case, existence of Wardrop equilibria is connected to the existence of a minimizer of such convex function.

Existence of a Wardrop equilibrium is obtained under continuity of the cost function $\tilde{c}$. Strict monotonicity of the cost is sufficient for uniqueness.

Theorem 2.6 (Existence and uniqueness of Wardrop equilibrium). *Suppose the cost $\tilde{c}$ is continuous and strictly monotone on $\mathcal{A}$. Then, there is one, and only one, Wardrop equilibrium.*

Proof. We focus on the issue of existence first. Let $T : \widetilde{\mathcal{A}} \subset \mathbb{R}_{+}^{\tilde{n}} \to \widetilde{\mathcal{A}}$ be defined by

$$T(\boldsymbol{j}) = \mathrm{Proj}_{\widetilde{\mathcal{A}}}(\boldsymbol{j} - \boldsymbol{c}(\boldsymbol{j})),$$

where $\mathrm{Proj}_{\widetilde{\mathcal{A}}}$ is the orthogonal projection from $\mathbb{R}_{+}^{\tilde{n}}$ to $\widetilde{\mathcal{A}}$.

A fixed point of T belongs to the boundary of $\widetilde{\mathcal{A}}$ and is a Wardrop equilibrium. With the result of Lemma 2.4, and Brouwer's Fixed Point Theorem, we obtain the existence.

To prove uniqueness, suppose $\tilde{\boldsymbol{j}}_{\mathbf{1}}$ and $\tilde{\boldsymbol{j}}_{\mathbf{2}}$ are Wardrop equilibria. Then, for any $\tilde{\boldsymbol{j}} \in \mathcal{A}$, we have

$$\langle \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}_{\mathbf{1}}), \tilde{\boldsymbol{j}}_{\mathbf{1}} - \tilde{\boldsymbol{j}} \rangle \leq 0 \quad \text{and} \quad \langle \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}_{\mathbf{2}}), \tilde{\boldsymbol{j}}_{\mathbf{2}} - \tilde{\boldsymbol{j}} \rangle \leq 0.$$

Accordingly,

$$\langle \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}_{\mathbf{1}}) - \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}_{\mathbf{2}}), \tilde{\boldsymbol{j}}_{\mathbf{1}} - \tilde{\boldsymbol{j}}_{\mathbf{2}} \rangle \leq 0,$$

and because $\tilde{\boldsymbol{c}}$ is strictly monotone, $\tilde{\boldsymbol{j}}_{\mathbf{1}} = \tilde{\boldsymbol{j}}_{\mathbf{2}}$. ∎

It is important to add that the unique Wardrop equilibrium is cost efficient, as it does not allow extra flow with positive cost.

Definition 4. A flow distribution $\tilde{\boldsymbol{j}}_{\mathbf{0}} \geq 0$ on $\widetilde{\Gamma}$ is a *flow loop* if it is a nontrivial solution of $\widetilde{K}\tilde{\boldsymbol{j}}_{\mathbf{0}} = 0$.

Proposition 1. *If $\tilde{\boldsymbol{j}}^{*}$ is a Wardrop equilibrium, then*

$$\langle \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}^{*}), \tilde{\boldsymbol{j}}_{\mathbf{0}} \rangle \geq 0,$$

for any flow loop, $\tilde{\boldsymbol{j}}_{\mathbf{0}}$.

Proof. Let $\tilde{\boldsymbol{j}}^{*}$ be a Wardrop equilibrium and let $\tilde{\boldsymbol{j}}_{\mathbf{0}}$ be a flow loop. Set

$$\tilde{\boldsymbol{j}}(\varepsilon) = \tilde{\boldsymbol{j}}^{*} + \varepsilon \tilde{\boldsymbol{j}}_{\mathbf{0}}.$$

Since $\tilde{\boldsymbol{j}}_{\mathbf{0}}$ is a loop, $\tilde{\boldsymbol{j}}(\varepsilon) \in \mathcal{A}$, for $\varepsilon \in \mathbb{R}_{0}^{+}$. The condition for a Wardrop equilibrium (2.2) for the particular $\tilde{\boldsymbol{j}}(\varepsilon)$ implies that

$$0 \geq \langle \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}^{*}), \tilde{\boldsymbol{j}}^{*} - (\tilde{\boldsymbol{j}}^{*} + \varepsilon \tilde{\boldsymbol{j}}_{\mathbf{0}}) \rangle = -\varepsilon \langle \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}^{*}), \tilde{\boldsymbol{j}}_{\mathbf{0}} \rangle, \quad \forall \varepsilon > 0.$$

Thus, $\langle \tilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}^{*}), \tilde{\boldsymbol{j}}_{\mathbf{0}} \rangle \geq 0$. ∎

Now, we examine the structure of the Wardrop equilibrium in a specific case relevant to MFGs on undirected networks. We associate a pair of directed edges with opposite orientations to every undirected edge. Thus, these networks have as a primary building block the loop network in Figure 1.

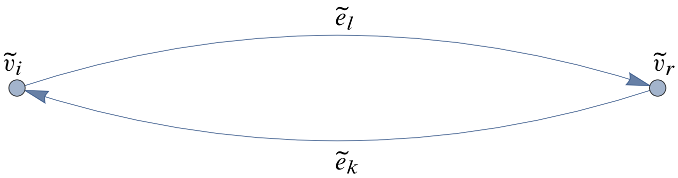

Figure 1. Loop subnetwork.

Proposition 2. *Consider a network containing both edges $\widetilde{e}_k = (\widetilde{v}_r, \widetilde{v}_i)$ and its reverse, $\widetilde{e}_l = (\widetilde{v}_i, \widetilde{v}_r)$. Let $\tilde{j}_k \geq 0$ be the flow in $\widetilde{e}_k$ and $\tilde{j}_l \geq 0$ be the flow in $\widetilde{e}_l$. Let the cost in $\widetilde{e}_k$ and $\widetilde{e}_l$ be $\widetilde{c}_k(\tilde{\boldsymbol{j}})$ and $\widetilde{c}_l(\tilde{\boldsymbol{j}})$, respectively, where $\tilde{\boldsymbol{j}} = (\tilde{j}_k, \tilde{j}_l, \tilde{\boldsymbol{j}}_\$)$ and $\tilde{\boldsymbol{j}}_\$$ is the flow on all the other edges. Furthermore, suppose that the costs $\widetilde{c}_k$ and $\widetilde{c}_l$ satisfy the following condition:*

$$\widetilde{c}_k(\tilde{\boldsymbol{j}}) + \widetilde{c}_l(\tilde{\boldsymbol{j}}) > 0, \quad \forall \tilde{\boldsymbol{j}}. \tag{2.4}$$

Then, any Wardrop equilibrium, $\tilde{\boldsymbol{j}}^* = (\tilde{j}_k^*, \tilde{j}_l^*, \tilde{\boldsymbol{j}}_\$^*)$, satisfies the complementary condition

$$\tilde{j}_k^* \cdot \tilde{j}_l^* = 0. \tag{2.5}$$

Proof. We prove the theorem for a network containing the subnetwork shown in Figure 1. To simplify, we will consider only the edges and flows shown in Figure 1. Assume that $\tilde{\boldsymbol{j}}^* = (\tilde{j}_k^*, \tilde{j}_l^*, \tilde{\boldsymbol{j}}_\$^*)$ is a Wardrop equilibrium and does not satisfy (2.5), i.e., without loss of generality $\tilde{j}_k^* > \tilde{j}_l^* > 0$. Consider the admissible flow $\tilde{\boldsymbol{j}}_{\boldsymbol{0}} = (\tilde{j}_k^* - \tilde{j}_l^*, 0, \tilde{\boldsymbol{j}}_\$^*)$, then by (2.2), we have

$$0 \geq \langle \widetilde{\boldsymbol{c}}(\tilde{\boldsymbol{j}}^*), \tilde{\boldsymbol{j}}^* - \tilde{\boldsymbol{j}}_{\boldsymbol{0}} \rangle = (\widetilde{c}_k(\tilde{\boldsymbol{j}}^*) + \widetilde{c}_l(\tilde{\boldsymbol{j}}^*))\tilde{j}_l^*$$

which contradicts assumption (2.4). Hence, $\tilde{\boldsymbol{j}}^*$ cannot be a Wardrop equilibrium. ∎

With this proposition established, we now analyze cost monotonicity in a network containing the subnetwork in Figure 1.

Example 2.7. The monotonicity condition for the subnetwork in Figure 1 is

$$\langle (\widetilde{c}_1(\tilde{\boldsymbol{j}}), \widetilde{c}_2(\tilde{\boldsymbol{j}})) - (\widetilde{c}_1(\hat{\boldsymbol{j}}), \widetilde{c}_2(\hat{\boldsymbol{j}})), \tilde{\boldsymbol{j}} - \hat{\boldsymbol{j}} \rangle \geq 0,$$

where $\tilde{\boldsymbol{j}} = (\tilde{j}_1, \tilde{j}_2)$ and $\hat{\boldsymbol{j}} = (\hat{j}_1, \hat{j}_2)$. Suppose that $\widetilde{c}_1(\tilde{\boldsymbol{j}}) = \hat{c}_1(\tilde{j}_1 - \tilde{j}_2)$ and $\widetilde{c}_2(\tilde{\boldsymbol{j}}) = \hat{c}_2(\tilde{j}_2 - \tilde{j}_1)$. Then, we have

$$(\hat{c}_1(\tilde{j}_1 - \tilde{j}_2) - \hat{c}_1(\hat{j}_1 - \hat{j}_2))(\tilde{j}_1 - \hat{j}_1) + (\hat{c}_2(\tilde{j}_2 - \tilde{j}_1) - \hat{c}_2(\hat{j}_2 - \hat{j}_1))(\tilde{j}_2 - \hat{j}_2) \geq 0.$$

This example can be generalized to networks constructed from similar subnetworks as in Figure 1. The sum in the monotonicity condition

$$\sum_k (\widetilde{c}_k(\tilde{\boldsymbol{j}}) - \widetilde{c}_k(\hat{\boldsymbol{j}}))(\tilde{j}_k - \hat{j}_k) \geq 0$$

can be organized in related pairs of edges.

3. Mean-field games problem in a single edge

We will focus on a single edge to begin our discussion of MFGs on networks.

3.1. MFG system in an edge

Consider an edge that is identified with the interval [0, 1]. In this edge, the Hamiltonian, $H : [0,1] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$, is smooth and convex in the second variable. The MFG consists of the following system of equations:

$$\begin{cases} H(x, u_x(x), m(x)) = 0, \\ (-m(x) D_p H(x, u_x(x), m(x)))_x = 0. \end{cases} \tag{3.1}$$

The first equation is the Hamilton–Jacobi (HJ) equation, while the second equation is the transport equation. We aim to find the agent's density $m : [0,1] \to \mathbb{R}_0^+$ and the value function $u : [0,1] \to \mathbb{R}$ as the two unknowns.

The constant flow, j, for the second equation in (3.1), leads to an algebraic equation for $m(x, j)$ (denoted by m for simplicity)

$$j = -m D_p H(x, u_x, m), \tag{3.2}$$

see Proposition 5 for details. This procedure is the *current method* from [27]. Here we refer to the currents as flows. It defines, abusing the notation, the distribution of agents $m(\cdot, j)$ and the value function $u(\cdot, j)$ across the edge whose flow is j.

We will demonstrate the application of the flow method using a prototypical family of Hamiltonians for MFGs with congestion. While the Hamiltonian is not defined for $m = 0$, the same method works.

Example 3.1. Let

$$H(x, p, m) = \frac{|p|^2}{2m^\alpha} + V(x) - g(m),$$

where g is an increasing function, V is a smooth function and $0 \leq \alpha \leq 2$ is the congestion strength. Then, (3.1) becomes

$$\begin{cases} \dfrac{|u_x|^2}{2m^\alpha} + V(x) = g(m), \\ (-m^{1-\alpha}u_x)_x = 0. \end{cases} \tag{3.3}$$

Two special values of α exist: $\alpha = 0$ corresponds to the uncongested MFG and $\alpha = 1$ corresponds to the critical congestion model. For the critical congestion model, the system decouples; u becomes a linear function and m can be obtained by solving the first equation in (3.3).

Because the flow, $j = -m^{1-\alpha}u_x$, is constant, we transform (3.3) into the following algebraic system:

$$\begin{cases} \dfrac{j^2}{2m^{2-\alpha}} + V(x) = g(m), \\ j\displaystyle\int_0^1 m^{\alpha-1}\,dx = u(0,j) - u(1,j), \end{cases} \tag{3.4}$$

provided $m > 0$. We refer to the last equation in (3.4) as the *edge equation*. In this equation, either $u(0,j)$ or $u(1,j)$ is given by boundary conditions. The sign of the flow indicates the direction in which the agents travel, from 0 to 1 when $j > 0$ and from 1 to 0 when $j < 0$. The edge equation determines $u(0,j)$ as a function of $u(1,j)$ when $j > 0$ and $u(1,j)$ as a function of $u(0,j)$ when $j < 0$. This result varies based on the specification of boundary conditions, which we will discuss next.

3.2. Travel cost in one edge

Consider the edge $[0,1]$ together with a Hamiltonian $H : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$. Given the flow j in this edge, let $c_{01}(j)$ be the cost of moving from left to right (i.e., from 0 to 1) and $c_{10}(j)$ be the cost of moving from right to left (i.e., from 1 to 0). To retrieve formulas for these costs, we introduce the Lagrangian, L, as follows:

$$L(x,v,m) = \sup_p\{-pv - H(x,p,m)\}.$$

We also define the *value function*, u, given a distribution m, as follows:

$$u(x) = \min_{\substack{\mathbf{x},T,\\ \mathbf{x}(0)=x,\\ \mathbf{x}(T)\in\{0,1\}}} \int_0^T L(\mathbf{x},\dot{\mathbf{x}},m(\mathbf{x}))\,dt + u(\mathbf{x}(T)).$$

The cost of moving from $x \in [0, 1]$ to $y \in [0, 1]$, given the distribution m, is

$$c_{xy}(m) = \min_{\substack{\mathbf{x},T,\\ \mathbf{x}(0)=x,\\ \mathbf{x}(T)=y}} \int_0^T L(\mathbf{x}, \dot{\mathbf{x}}, m(\mathbf{x}))\,dt.$$

We let the cost depend on the flow, j, by substituting $m(\mathbf{x})$ by $m(\mathbf{x}, j)$.

Definition 5. For $j \in \mathbb{R}$, given a distribution of agents $m(\cdot, j)$, the optimal cost of traveling from x to y on the edge is

$$c_{xy}(j) = \min_{\substack{\mathbf{x},T,\\ \mathbf{x}(0)=x,\\ \mathbf{x}(T)=y}} \int_0^T L(\mathbf{x}, \dot{\mathbf{x}}, m(\mathbf{x}, j))\,dt,$$

where the minimum is taken over all Lipschitz trajectories $\mathbf{x}$ and terminal times T.

When $(x, y) = (0, 1)$ or $(x, y) = (1, 0)$, the cost and the boundary conditions are related as follows:

The agent at a vertex of an edge has two options: leaving this edge or crossing it to get to the opposite vertex of the same edge. This is encoded in the following inequalities:

$$\begin{cases} u(1, j) \le c_{10}(j) + u(0, j),\\ u(0, j) \le c_{01}(j) + u(1, j). \end{cases} \tag{3.5}$$

Moreover, the self-consistency in MFGs requires that the direction in which the agent moves to be aligned with the direction of the flow. This is reflected in the following condition:

$$\begin{cases} j > 0 \Rightarrow u(0) = c_{01}(j) + u(1) \quad \Leftrightarrow \quad c_{01}(j) = -\displaystyle\int_0^1 u_x(x, j)\,dx,\\ j < 0 \Rightarrow u(1) = c_{10}(j) + u(0) \quad \Leftrightarrow \quad c_{10}(j) = \displaystyle\int_0^1 u_x(x, j)\,dx. \end{cases} \tag{3.6}$$

Remark 3.2. From (3.6) we note that $-u_x(\cdot, j)$ and $u_x(\cdot, j)$ represent the local costs for $c_{01}(j)$ and $c_{10}(j)$, respectively.

Remark 3.3. The optimality conditions in (3.5) imply the following local compatibility condition:

$$-c_{01}(j) \le c_{10}(j).$$

This can be interpreted as loops having a non-negative cost

$$c_{01}(j) + c_{10}(j) \ge 0.$$

Proposition 3. *If the costs in* (3.5) *are non-negative, then the inequalities in* (3.5) *are redundant.*

Proof. The conditions for optimality (3.5) imply

$$-c_{01}(j) \leq u(1,j) - u(0,j) \leq c_{10}(j).$$

If $j \neq 0$, one of the two inequalities becomes an equality. As the costs are non-negative, the other inequality holds. ∎

Now, we analyze the variational problems that define the costs c_{01} and c_{10}. Starting with c_{01}, we parametrize, in the integral of Definition 5, the velocity v by the space coordinate x, so $dx = v\,dt$ provided that $\dot{x} > 0$, we get

$$c_{01}(j) = \min_{v} \int_0^1 \frac{L(x, v(x,j), m(x,j))}{v(x,j)}\, dx, \tag{3.7}$$

where the minimum is taken within the set $L^\infty([0,1] \times \mathbb{R}, \mathbb{R}^+)$ of positive, essentially bounded functions.

Similarly, for c_{10},

$$c_{10}(j) = \min_{v} \int_0^1 -\frac{L(x, v(x,j), m(x,j))}{v(x,j)}\, dx, \tag{3.8}$$

where the minimum is taken within the set $L^\infty([0,1] \times \mathbb{R}, \mathbb{R}^-)$ of negative, essentially bounded functions.

Proposition 4. *Consider a fixed flow j, if L is superlinear in the second variable, that is,*

$$\lim_{v \to \pm\infty} \frac{L(x,v,m)}{|v|} = +\infty,$$

and $L(\cdot, 0, \cdot) > 0$, then the following statements hold:

(1) *There exist two functions $v_+^* > 0$ and $v_-^* < 0$ that minimize (3.7) and (3.8), respectively. Additionally, these functions satisfy the Euler–Lagrange equation*

$$-\frac{L(x, v(x,j), m(x,j))}{v(x,j)^2} + \frac{D_v L(x, v(x,j), m(x,j))}{v(x,j)} = 0, \quad \forall x \in [0,1]. \tag{3.9}$$

(2) *For any such minimizers, the following holds:*

$$c_{01}(j) = \int_0^1 D_v L(x, v_+^*(x,j), m(x,j))\, dx, \quad \textit{for any } j \in \mathbb{R} \backslash \{0\}. \tag{3.10}$$

Additionally,

$$c_{10}(j) = -\int_0^1 D_v L(x, v_-^*(x,j), m(x,j))\, dx, \quad \textit{for any } j \in \mathbb{R} \backslash \{0\}. \tag{3.11}$$

(3) *If the function $v \mapsto L(x, v, m)$ is strictly convex with respect to v, then the function v_+^* is unique among positive functions, and the function v_-^* is unique among negative functions.*

Proof. Since $L(x, 0, m) > 0$, it follows that

$$\lim_{v \to 0} \frac{L(x, v, m)}{|v|} = +\infty.$$

This, combined with the superlinearity assumption, implies that, for every x and j, there exists a minimizer $v_+^*(x, j) > 0$ of $L(x, v, m(x, j))/v$ which solves (3.9). This pointwise minimizer also minimizes (3.7).

By (3.9), we have

$$D_v L(x, v_+^*(x, j), m(x, j)) = \frac{L(x, v_+^*(x, j), m(x, j))}{v_+^*(x, j)}. \tag{3.12}$$

Thus, we get (3.10).

Now, we prove the uniqueness of the minimizer if L is strictly convex in the velocity variable. Observe that any critical point of $v \mapsto L(x, v, m)/v$ is a minimum. In fact, if v is such a critical point,

$$\begin{aligned} D^2_{vv}\left(\frac{L(x, v, m)}{v}\right) &= D_v\left(\frac{D_v L(x, v, m)}{v} - \frac{L(x, v, m)}{v^2}\right) \\ &= \frac{D^2_{vv} L(x, v, m)}{v} - 2\frac{D_v L(x, v, m)}{v^2} + 2\frac{L(x, v, m)}{v^3} \\ &= \frac{D^2_{vv} L(x, v, m)}{v}, \end{aligned}$$

where we use the first-order condition for critical points to obtain the last equality.

Because $v \mapsto L(x, v, m)/v$ is continuous, for fixed m and x, it cannot have more than one local minimum.

Similarly, we prove results for v_-^*. ∎

Remark 3.4. In (3.10), the cost c_{01} is defined for any $j \in \mathbb{R}$. In particular, when $j < 0$, c_{01} is the cost of moving against the flow.

Proposition 5. *Consider the MFG system (3.1) and the flow equation (3.2). If the Hamiltonian H is strictly convex in p, then*

$$u_x(x, j) = -D_v L\left(x, \frac{j}{m}, m\right), \tag{3.13}$$

where L is the Lagrangian associated to H.

Moreover, m is determined as a function of x and j by

$$H\left(x, -D_v L\left(x, \frac{j}{m}, m\right), m\right) = 0. \tag{3.14}$$

Proof. As H is strictly convex in p, $p \to D_p H(x,p,m)$ is invertible. Moreover, if $D_v L(x,v,m) = -p$ and $v = -D_p H(x,p,m)$, we have

$$p = -D_v L(x, -D_p H(x,p,m), m).$$

Applying the function $v \mapsto -D_v L(x, v/m, m)$ to both sides of (3.2), we obtain the result of the first statement.

Now, by substituting (3.13) in the HJ equation from (3.1), we derive (3.14). ∎

Proposition 6. *Consider the setting of Proposition 5. Let $m(x,j)$ be determined by (3.14). If $j > 0$, then (3.10) becomes*

$$c_{01}(j) = \int_0^1 D_v L\left(x, \frac{j}{m(x,j)}, m(x,j)\right) dx. \tag{3.15}$$

Similarly, if $j < 0$, then (3.11) becomes

$$c_{10}(j) = -\int_0^1 D_v L\left(x, \frac{j}{m(x,j)}, m(x,j)\right) dx. \tag{3.16}$$

Proof. For $j > 0$, we use (3.13) to substitute u_x in (3.6), to get (3.15). The case for $j < 0$ is proven similarly. ∎

Remark 3.5. Note that (3.15) and (3.16) are not valid, in general, for $j < 0$ and $j > 0$, respectively. In these cases, we must use (3.10) and (3.11). Also, one obtains the relations $v_+^*(x,j) = j/m(x,j)$ for $j > 0$ and $v_-^*(x,j) = j/m(x,j)$, for $j < 0$.

4. Mean-field game model on a network

The MFG formulation on networks shares various aspects with the road traffic model, with the key difference being that MFGs are set up in undirected networks.

4.1. The network and the data

In the MFG model, the given information is as follows.

Network. A finite *undirected* network, $\Gamma = (E, V)$, where $E = \{e_k : k \in \{1, 2, \ldots, n\}\}$ is the set of edges and $V = \{v_i : i \in \{1, 2, \ldots, m\}\}$ is the set of vertices. To any edge e_k, we associate the pair (v_r, v_i) of its endpoints.

Entrances and exits. Agents enter the network through λ *entrance vertices* and exit it through μ *exit vertices* (disjoint from the entrance vertices). For convenience, we assume that the last μ vertices in V are the exit vertices. Furthermore, we suppose that entrance and exit vertices have incidence 1. If this is not the case, we proceed as in Remark 2.1.

Entry flows. A flow of agents, the entry flow $\boldsymbol{\iota} = (\iota_1, \dots, \iota_\lambda) > 0$, is prescribed at the entrance vertices. It follows that these are the flows at the *entry edges*, the edges that have an entrance vertex. Entry flows in other vertices are zero.

Exit costs. At the μ exit vertices, agents pay an exit cost denoted by $\boldsymbol{\phi}$. Here, we assume that this exit cost vanishes. If the exit cost is nonzero, an auxiliary edge is added as described in Remark 2.2.

Hamiltonian. On each edge $e_k = (v_r, v_i)$, identified with the [0, 1] interval, there is a smooth Hamiltonian, convex in the momentum variable

$$H_k : [0,1] \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}.$$

Example 4.1. Consider a network with three edges as shown in Figure 2. Assume that v_1 and v_3 be entrance vertices, and let v_2 and v_4 be exit vertices. By attaching an exit edge to v_2 and an entrance edge to v_3, we obtain the network depicted in Figure 3.

4.2. The variables and the costs

On the network Γ, we define the following variables.

Edge flows. Each edge has a *flow* variable, representing the number of agents crossing that edge per unit of time. The flow j_k in the edge $e_k = (v_r, v_i)$ is decomposed into positive and negative parts, with $j_k = j_k^i - j_k^r$, where j_k^i is the flow to the vertex v_i and j_k^r is the flow to the vertex v_r.

Transition flows. The *transition flow* from e_k to e_l through a common vertex v_r is denoted by j_{kl}^r.

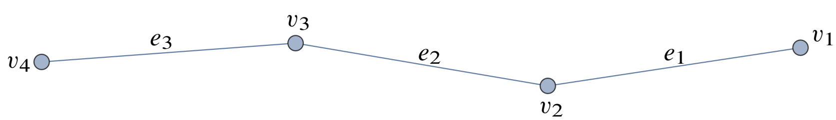

Figure 2. MFG network.

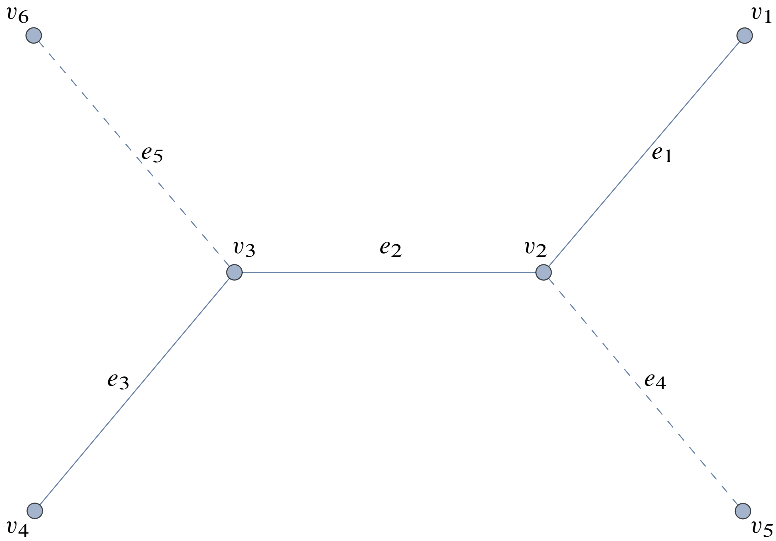

Figure 3. MFG network with entrance and exit edges.

Value function. The *value function* at the vertex v_i of $e_k = (v_r, v_i)$ is denoted by $u_k^i = u^k(1, j_k)$, similarly $u_k^r = u^k(0, j_k)$, see equation (4.2). Note that the notation of the value function depends on an edge-vertex pair rather than solely on a vertex. One of the values u_k^i and u_k^r provides boundary data for the HJ equation on edge e_k. We denote by $\boldsymbol{u}$ the vector of these values.

Travel costs. As per Section 3, edge $e_k = (v_r, v_i)$ has travel cost $c_k^i(j_k)$ from v_r to v_i, and $c_k^r(j_k)$ from v_i to v_r. Note that the cost in the edge depends only on the flow in this edge.

Travel costs on, auxiliary, entrance edges into the network are zero, and the travel costs in the opposite direction are $+\infty$. Travel costs on, auxiliary, exit edges away from the network are the exit costs, and travel costs in the opposite direction are $+\infty$.

Switching costs. Agents pay a *switching cost* ψ_{kl}^i, for moving from e_k to e_l through a common vertex v_i.

Generally, $\psi_{kl}^i \geq 0$. For simplicity, we assume ψ_{kl}^i is independent of the transition flow.

Switching costs are *consistent* if for any vertex v_i with more than two incident edges, a triangle-type inequality in the switching costs holds,

$$\psi_{kl}^i \leq \psi_{kp}^i + \psi_{pl}^i. \tag{4.1}$$

The switching costs are *strictly consistent* if the inequalities above are strict.

Remark 4.2. When adding extra vertices or edges, switching costs are modified as follows: switching cost from an entrance edge to an original edge is 0, and from an original edge to an entry edge is $+\infty$. Similarly, switching cost from an exit edge to an original edge is $+\infty$, and from an original edge to an exit edge is 0. This prevents entry through exits and exit through entrances.

4.3. The equations

Finally, we set up the equations that determine the MFG. These consist of the MFG system in the edges and the optimality conditions at the vertices, as considered in Section 3. Further, we have Kirchhoff's condition, representing the balance of the flow of agents at the different vertices.

MFG. On each edge $e_k = (v_r, v_i)$, identified with the $[0, 1]$ interval, an MFG system is given by

$$\begin{cases} H_k(x, u_x^k, m^k) = 0, \\ -m^k D_p H_k(x, u_x^k, m^k) = j_k, \end{cases} \tag{4.2}$$

where the unknowns are u^k, m^k, and j_k. Note that j_k is given if e_k is an entry edge.

Complementary conditions. The decomposition of flows satisfies

$$j_k^i \cdot j_k^r = 0, \quad j_k^i \geq 0, \quad j_k^r \geq 0, \tag{4.3}$$

where j_k^i is the flow to the vertex v_i and j_k^r is the flow to the vertex v_r.

The same is true for transition flows

$$j_{kl}^r \cdot j_{lk}^r = 0, \quad j_{kl}^r \geq 0, \quad j_{lk}^r \geq 0.$$

Hence, all agents in an edge or transition move in the same direction.

Optimality conditions at the vertices. Agents minimize travel cost by choosing the least expensive path. In particular, they can switch from e_k to e_l through the common vertex v_i by paying a cost ψ_{kl}^i. Hence

$$u_k^r \leq u_l^r + \psi_{kl}^i, \quad \forall\, i, k, l. \tag{4.4}$$

Complementarity conditions at the vertices. A nonzero transition flow j_{kl}^i means that agents move from e_k to e_l through v_i and that $u_k^r = u_l^r + \psi_{kl}^i$. So, we have the following:

$$j_{kl}^i \cdot (u_k^r - u_l^r - \psi_{kl}^i) = 0, \quad \forall i, k, l. \tag{4.5}$$

Optimality conditions in the edges. In the edge $e_k = (v_r, v_i)$, for any j_k we have

$$\begin{cases} u_k^r \leq c_k^i(j_k) + u_k^i, \\ u_k^i \leq c_k^r(j_k) + u_k^r. \end{cases} \tag{4.6}$$

Moreover,

$$\begin{cases} j_k > 0 \implies u_k^r = c_k^i(j_k^i) + u_k^i, \\ j_k < 0 \implies u_k^i = c_k^r(-j_k^r) + u_k^r. \end{cases}$$

The costs c_k^i and c_k^r may not be the same, as discussed in Section 3.

Complementarity conditions in the edges. In the edge $e_k = (v_r, v_i)$, we have the complementary conditions

$$\begin{cases} j_k^i \cdot (u_k^r - u_k^i - c_k^i(j_k)) = 0, \\ j_k^r \cdot (u_k^i - u_k^r - c_k^r(j_k)) = 0. \end{cases} \tag{4.7}$$

Balance equations and Kirchhoff's law. Consider an edge $e_k = (v_r, v_i)$, let $\mathcal{E}$ be the set of indices of the incident edges at v_i, and $\mathcal{E}_k = \mathcal{E} \backslash k$. The flow j_k^i is equal to the sum of the transition flows from e_k to all the other incident edges at v_i

$$\sum_{l \in \mathcal{E}_k} j_{kl}^i = j_k^i; \tag{4.8}$$

this identity models the splitting of the flow at a vertex. We have a similar equation for the gathering of the transition flows; the flow j_k^r is the sum of the transition flows to e_k from all the incident edges at v_i

$$\sum_{l \in \mathcal{E}_k} j_{lk}^i = j_k^r. \tag{4.9}$$

In particular, we have Kirchhoff's law at v_i; that is, the sum of the incoming flows j_k^i is equal to the sum of the outgoing flows $j_l^{r(l)}$,

$$\sum_{k \in \mathcal{E}} j_k^i = \sum_{l \in \mathcal{E}} j_l^{r(l)},$$

where $r(l)$ is the vertex of e_l, different from v_i.

Entry edge equations. Consider an entry edge $e_k = (v_r, v_i)$ with a flow ι_i entering the network through the vertex v_r. For this edge, the following conditions hold:

$$\begin{cases} j_k^i = \iota_i, \\ j_k^r = 0, \end{cases} \tag{4.10}$$

where the first condition states that all flow crosses the edge while the second equation imposes that no agents leave through vertex v_r.

Exit edge equations. In every exit edge $e_l = (v_i, v_s)$, we assume the exit cost vanishes, so we have

$$u_l^s \leq 0,$$

with equality if $j_l^s > 0$. In e_l, we also have

$$j_l^i = 0.$$

5. Reformulation of the MFG as a road traffic model

In this section, we reformulate the MFG model from Section 4 as a road traffic model, as outlined in Section 2. After identifying the network, the flows, and the costs in the road traffic model, we show that the MFG solution corresponds to a Wardrop equilibrium.

While the network in the road traffic model is directed, in the MFG model it is undirected. To establish the correspondence between MFG and road traffic models, we construct a directed network $\overline{\Gamma}$ from the MFG's undirected network Γ.

(1) In the road traffic model, each directed edge corresponds to a flow or transition flow from the MFG model.

(2) Directed edge vertices in the road traffic model are analogous to pairs (e_k, v_i), where e_k is an edge and v_i is one of its vertices in Γ. This pair is built as follows:

- For a flow j_k^i in edge $e_k = (v_r, v_i) \in \Gamma$, the vertices in the new edge correspond to the pairs (e_k, v_r) and (e_k, v_i).
- For a transition flow j_{kl}^i from edge e_k to the edge e_l through the vertex v_i, the new vertices correspond to the pairs (e_k, v_i) and (e_l, v_i).

Remark 5.1. A natural notation for the edges in $\overline{\Gamma}$ is to use the same indices from the corresponding flow in the MFG. For example, $\overline{e}_k^i$ for the edge carrying the flow j_k^i, and $\overline{e}_{kl}^i$ for j_{kl}^i. However, to avoid complex notation and maintain consistency with the preceding notation, we relabel the edges and the vertices and write $\overline{\Gamma} = (\overline{E}, \overline{V})$ for $\overline{E} = \{\overline{e}_\kappa : \kappa \in \{1, 2, \dots, \overline{n}\}\}$ and $\overline{V} = \{\overline{v}_i : i \in \{1, 2, \dots, \overline{m}\}\}$. When the precise correspondence is needed, we use the identification $\overline{e}_\kappa = \overline{e}_k^i$ or $\overline{e}_\kappa = \overline{e}_{kl}^i$.

(3) As the entrance flows at exit vertices and exit flows at entrance vertices are zero, we remove the corresponding edges so that $\overline{\Gamma}$ has entrance and exit vertices with incidence 1.

Example 5.2. Consider the network presented in Example 4.1. We follow the previous steps to transform this network from the MFG to the road traffic setting. Accord-

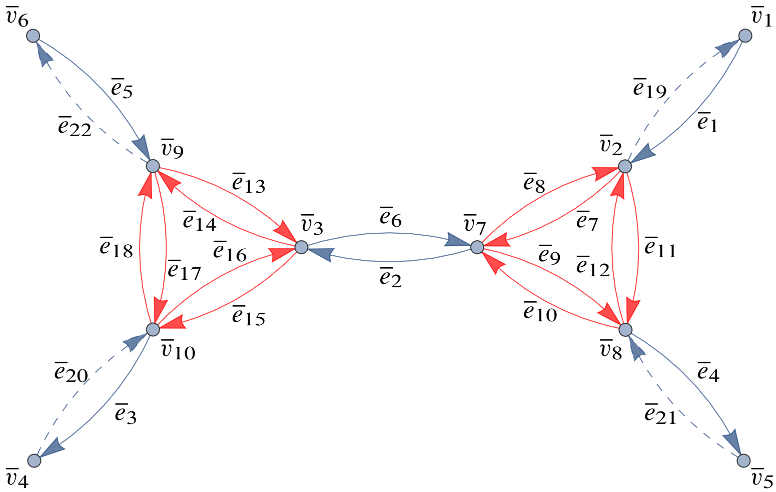

Figure 4. Road traffic network $\overline{\Gamma}$.

ingly, we get the new network $\overline{\Gamma}$ in Figure 4, which consists of 22 edges and 10 vertices. The blue edges correspond to the flows and the red edges correspond to the transition flows. The dashed edges will be removed, as per the third step above, so they are not included in $\overline{\Gamma}$.

The correspondence between the variables in the MFG model and the road traffic model is as follows:

(1) The value function from the MFG is a function on the new vertices because u_k^i is a function on pairs (e_k, v_i), where v_i is a vertex of e_k. While this function is not explicitly present in the road traffic model, it remains relevant in our analysis.

(2) We establish a natural correspondence between flows in the two models, j and $\bar{\jmath}$. If $\overline{e}_\kappa = \overline{e}_k^i$, we set $\bar{\jmath}_\kappa = j_k^i$, and if $\overline{e}_\kappa = \overline{e}_{kl}^i$, we set $\bar{\jmath}_\kappa = j_{kl}^i$. Following the notation from Section 2, the network's flow is denoted by $\bar{\boldsymbol{\jmath}}$.

(3) By combining (4.8) and (4.9) at a pair (e_k, v_i) for which v_i is neither an entrance nor an exit, we derive the Kirchhoff's law at the corresponding vertex of $\overline{\Gamma}$,

$$j_k^r + \sum_{l \in \mathcal{E}_k} j_{kl}^i = j_k^i + \sum_{l \in \mathcal{E}_k} j_{lk}^i, \tag{5.1}$$

which can be expressed in terms of the variable $\bar{\jmath}$.

(4) At entrance vertices with entry flow assignments, we have the first equation in (4.10). This equation, along with (5.1), is encoded as a matrix equation

(similar to (2.1) in Definition 1)

$$K\bar{\boldsymbol{j}} + B = 0.$$

Hence, there is a linear equation for every vertex in $\bar{\Gamma}$ that is not an exit.

(5) Let $\bar{e}_k^i \in \bar{\Gamma}$ and $\bar{e}_k^r \in \bar{\Gamma}$ be the edges corresponding to the undirected edge $e_k = (v_r, v_i) \in \Gamma$, with orientations corresponding to the flows j_k^i and j_k^r, respectively. Based on Section 3, the MFG has two costs c_k^i and c_k^r in the edge e_k. The cost for traveling in $\bar{e}_k^i$ is $\bar{c}_k^i(\boldsymbol{j_k}) = c_k^i(j_k^i - j_k^r)$, where $\boldsymbol{j_k} = (j_k^i, j_k^r)$, and the cost for traveling in $\bar{e}_k^r$ is $\bar{c}_k^r(\boldsymbol{j_k}) = c_k^r(j_k^r - j_k^i)$.

(6) The MFG's switching costs ψ_{kl}^i correspond to constant travel costs in the road traffic model; in the transition edges $\bar{e}_\kappa = \bar{e}_{kl}^i$, the cost is $\bar{c}_\kappa(\boldsymbol{j}) := \bar{c}_{kl}^i(\boldsymbol{j}) = \psi_{kl}^i$.
Applying this to the network in Figure 4, produces the network in Figure 5.

The main result connecting MFGs on networks with road traffic models is the following.

Theorem 5.3. *Let M be an MFG on a network and W be the corresponding road traffic model. Let $\boldsymbol{u}$ be the value of the value function close to the vertices and $\boldsymbol{j}^*$ be the flow on the network. If $(\boldsymbol{u}, \boldsymbol{j}^*)$ solves M, then the corresponding $\bar{\boldsymbol{j}}^*$ is a Wardrop equilibrium for W.*

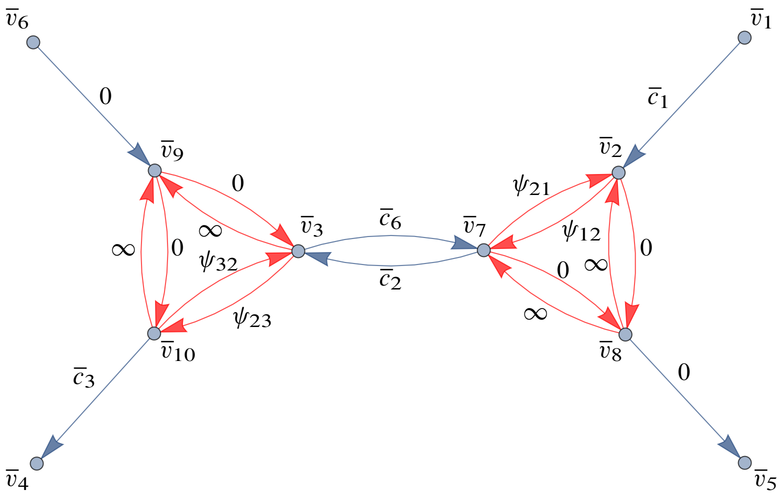

Figure 5. Road traffic network with costs.

Proof. For any admissible $\bar{\boldsymbol{j}}$, we have

$$K\bar{\boldsymbol{j}}^* + B = 0 \quad \text{and} \quad K\bar{\boldsymbol{j}} + B = 0.$$

Subtracting the two equations, we get

$$K(\bar{\boldsymbol{j}}^* - \bar{\boldsymbol{j}}) = 0. \tag{5.2}$$

Let $\boldsymbol{u}$ be the vector of value functions. To retrieve the costs on the edges, multiply (5.2) from the left by $\boldsymbol{u}^{\boldsymbol{T}}$

$$\boldsymbol{u}^{\boldsymbol{T}} K(\bar{\boldsymbol{j}}^* - \bar{\boldsymbol{j}}) = 0.$$

Each flow appears twice, once for each of its vertices and with different signs, except for those flows that point towards an exit vertex. These appear only once. The previous computation can be organized as

$$\sum_{\bar{e}_k^i}(u_k^r - u_k^i)(\bar{j}_k^{i*} - \bar{j}_k^i) + \sum_{\bar{e}_{kl}^i}(u_k^i - u_l^i)(\bar{j}_{kl}^{i*} - \bar{j}_{kl}^i) = 0, \tag{5.3}$$

the first sum is over the edges of W where the vertices of e_k are v_r and v_i; this includes the exit edges since, in that case, $u_k^i \leq 0$. The second sum is over the transition edges of W corresponding to transitions through the vertex v_i from e_k to e_l. Based on the complementarity conditions in the edges, given by (4.7), there are two possible cases for the terms in the first sum in (5.3),

$$\bar{j}_k^{i*} > 0 \Rightarrow u_k^r - u_k^i = \bar{c}_k^i(\bar{\boldsymbol{j}}^*)$$

or, noting (4.6),

$$\bar{j}_k^{i*} = 0 \Rightarrow u_k^r - u_k^i \leq \bar{c}_k^i(\bar{\boldsymbol{j}}^*).$$

Because $\bar{j}_k^i \geq 0$,

$$\bar{j}_k^{i*} = 0 \Rightarrow (u_k^r - u_k^i)(\bar{j}_k^{i*} - \bar{j}_k^i) \geq \bar{c}_k^i(\bar{\boldsymbol{j}}^*)(\bar{j}_k^{i*} - \bar{j}_k^i).$$

Similarly, based on the complementarity conditions at the vertices (4.5), we have two cases for the terms in the second sum in (5.3), either

$$\bar{j}_{kl}^{i*} > 0 \Rightarrow u_k^i - u_l^i = \psi_{kl}$$

or, noting (4.4),

$$\bar{j}_{kl}^{i*} = 0 \Rightarrow u_k^i - u_l^i \leq \psi_{kl}.$$

Because $\bar{j}_{kl}^i \geq 0$,

$$\bar{j}_{kl}^{i*} = 0 \Rightarrow (u_k^i - u_l^i)(\bar{j}_{kl}^{i*} - \bar{j}_{kl}^i) \geq \psi_{kl}(\bar{j}_{kl}^{i*} - \bar{j}_{kl}^i).$$

Using these results in (5.3), we get

$$\sum_{\substack{\bar{e}^i_k,\\ \bar{j}^{i*}_k>0}} \bar{c}^i_k(\bar{\boldsymbol{j}}^*)(\bar{j}^{i*}_k - \bar{j}^i_k) + \sum_{\substack{\bar{e}^i_k,\\ \bar{j}^{i*}_k=0}} \bar{c}^i_k(\bar{\boldsymbol{j}}^*)(\bar{j}^{i*}_k - \bar{j}^i_k) + \sum_{\substack{\bar{e}^i_{kl},\\ \bar{j}^{i*}_{kl}>0}} \psi_{kl}(\bar{j}^{i*}_{kl} - \bar{j}^i_{kl}) + \sum_{\substack{\bar{e}^i_{kl},\\ \bar{j}^{i*}_{kl}=0}} \psi_{kl}(\bar{j}^{i*}_{kl} - \bar{j}^i_{kl}) \leq 0,$$

which implies

$$\langle \bar{\boldsymbol{c}}(\bar{\boldsymbol{j}}^*), \bar{\boldsymbol{j}}^* - \bar{\boldsymbol{j}} \rangle \leq 0.$$

Accordingly, $\bar{\boldsymbol{j}}^*$ is a Wardrop equilibrium. ∎

Next, we prove a uniqueness result for the associated Wardrop equilibrium. For this, we present some definitions.

We say the cost is *reversible* if $c_{01} = c_{10}$, indicating equal travel costs for both directions along an edge. If $c_{01}(-j) = c_{01}(j)$ for every $j \in \mathbb{R}$, then the cost is *even*. We discuss these properties in Section 7.

Proposition 7. *If the costs are reversible, even, increasing for positive flows, and satisfying (2.4) for every edge $e_k = (v_r, v_i)$ of the MFG network, then the equilibrium flow in the road traffic model is unique in the corresponding edge.*

Proof. Recall that $\bar{c}^i_k(\boldsymbol{j_k}) = c^i_k(j^i_k - j^r_k) = c^i_k(j_k)$ is the cost on the edge $e_k = (v_r, v_i)$ in the road traffic model, where c^i_k is the cost from the MFG and $\boldsymbol{j_k} = (j^i_k, j^r_k)$. In the reverse direction, it is $\bar{c}^r_k(\boldsymbol{j_k}) = c^r_k(j^r_k - j^i_k) = c^r_k(-j_k)$. Note that the cost variation

$$\langle \bar{\boldsymbol{c}}(\tilde{\boldsymbol{j}}) - \bar{\boldsymbol{c}}(\hat{\boldsymbol{j}}), \tilde{\boldsymbol{j}} - \hat{\boldsymbol{j}} \rangle,$$

for each edge, can be expressed as

$$\sum_k (c^i_k(\tilde{j}_k) - c^i_k(\hat{j}_k))(\tilde{j}^i_k - \hat{j}^i_k) + (c^r_k(-\tilde{j}_k) - c^r_k(-\hat{j}_k))(\tilde{j}^r_k - \hat{j}^r_k).$$

The costs corresponding to the transition flows are disregarded because they are constant.

Since the cost is reversible, i.e., $c^i_k(j) = c^r_k(j) = c(j)$, we get

$$\sum_k (c(\tilde{j}_k) - c(\hat{j}_k))(\tilde{j}^i_k - \hat{j}^i_k) + (c(-\tilde{j}_k) - c(-\hat{j}_k))(\tilde{j}^r_k - \hat{j}^r_k). \tag{5.4}$$

Apply Proposition 2 with (2.4) to obtain complementarity of the flows, (2.5). It implies four cases for assessing the terms in the expression (5.4).

(1) If $\tilde{j}_k^r = \hat{j}_k^r = 0$, we have

$$(c(\tilde{j}_k^i) - c(\hat{j}_k^i))(\tilde{j}_k^i - \hat{j}_k^i).$$

(2) If $\tilde{j}_k^i = \hat{j}_k^i = 0$, because c is even, we have

$$(c(-\tilde{j}_k^r) - c(-\hat{j}_k^r))(\tilde{j}_k^r - \hat{j}_k^r).$$

(3) If $\tilde{j}_k^i = \hat{j}_k^r = 0$, then, since c is even, we have

$$(c(\tilde{j}_k^r) - c(\hat{j}_k^i))(\tilde{j}_k^r - \hat{j}_k^i).$$

(4) If $\tilde{j}_k^r = \hat{j}_k^i = 0$, we have

$$(c(\tilde{j}_k^i) - c(\hat{j}_k^r))(\tilde{j}_k^i - \hat{j}_k^r).$$

In each case, the sum is non-negative because the cost $c(j)$ is increasing for $j > 0$. Strict inequality occurs if $c(j)$ is strictly increasing and $\tilde{j}_k^i \neq \hat{j}_k^i$. Because the cost is strictly monotone, by Theorem 2.6, the Wardrop equilibrium is unique. ■

Remark 5.4. For general costs, some conditions that implies uniqueness are as follows:

(1) The cost c_k^i is increasing in $\mathbb{R}_0^+$, if $\tilde{j}_k^r = \hat{j}_k^r = 0$.

(2) The cost c_k^r is increasing in $\mathbb{R}_0^+$, if $\tilde{j}_k^i = \hat{j}_k^i = 0$.

(3) The following inequality holds:

$$\tilde{j}_k^r c_k^r(-\hat{j}_k^i) + \hat{j}_k^i c_k^i(-\tilde{j}_k^r) \leq \tilde{j}_k^r c_k^r(\tilde{j}_k^r) + \hat{j}_k^i c_k^i(\hat{j}_k^i)$$

if $\tilde{j}_k^i = \hat{j}_k^r = 0$.

(4) The following inequality holds:

$$\tilde{j}_k^i c_k^i(-\hat{j}_k^r) + \hat{j}_k^r c_k^r(-\tilde{j}_k^i) \leq \tilde{j}_k^i c_k^i(\tilde{j}_k^i) + \hat{j}_k^r c_k^r(\hat{j}_k^r)$$

if $\tilde{j}_k^r = \hat{j}_k^i = 0$.

Remark 5.5. In Proposition 7, we proved the uniqueness of the flows. However, with the same assumptions, we do not have uniqueness of the transition flows. For example, consider 4 intersecting edges at a single vertex, as in Figure 6. Let the flows in edges e_1 and e_3 be $j_1 = j_3 = 10$, and the flows in edges e_2 and e_4 be $j_2 = j_4 = 5$. Possible transition flows are $j_{13} = 10$, $j_{24} = 5$, and $j_{14} = j_{23} = 0$. Another possibility is having $j_{13} = 7.5$ and $j_{24} = j_{14} = j_{23} = 2.5$.

Proposition 8. *In the MFG, with the same assumptions as in Proposition 7, the flow is uniquely defined in each edge.*

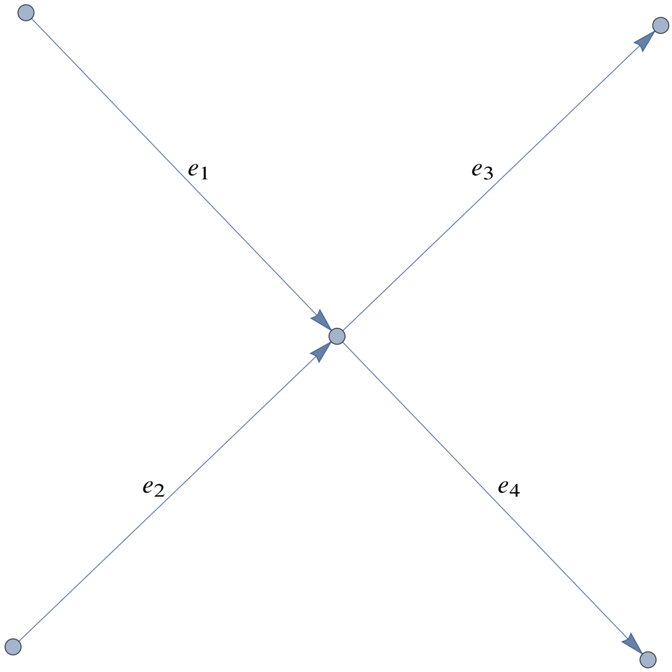

Figure 6. 4 edges intersect in 1 vertex.

Proof. Suppose the MFG has two solutions which the flow may differ in one edge. By Theorem 5.3, both solutions are Wardrop equilibria. Due to the uniqueness of the flows in the road traffic model, they must be the same. ∎

6. Recovering MFG solution from the Wardrop equilibrium

Here we demonstrate how to derive the mean-field game (MFG) solution from the corresponding Wardrop equilibrium. We follow the same procedure as in Section 5 to convert the MFG on a network into a road traffic model. The structure of the network and associated costs are derived from the MFG problem. Subsequently, we derive the MFG solution from the Wardrop equilibrium. This consists in recovering the flows, the transition flows, and the value function. Derive the flow in edge e_k of the MFG by considering the difference between flows in the corresponding directed edges in the Wardrop equilibrium. The correspondence for transition flows is immediate. Kirchhoff's law in $\overline{\Gamma}$, (5.1), implies the splitting and gathering of equations (4.8) and (4.9) for the MFG network Γ.

Proposition 9. *Consider an MFG model with positive costs on the edges* (2.4) *and strictly consistent switching costs* (4.1)*. Let* $\tilde{\boldsymbol{j}}^*$ *be the corresponding Wardrop equilibrium. Then, the splitting and gathering balance conditions,* (4.8) *and* (4.9)*, hold on every non-entrance and non-exit vertex.*

Proof. In the road traffic model, we have Kirchhoff's law (5.1) for every non-entrance and non-exit vertex. Because of (4.3), either $j_k^r = 0$ or $j_k^i = 0$. Without loss of generality, assume that $j_k^i = 0$, then (5.1) becomes

$$j_k^r + \sum_{e_l \in \mathcal{E}} j_{kl}^i = \sum_{e_l \in \mathcal{E}} j_{lk}^i. \tag{6.1}$$

We show that the corresponding transition flows j_{kl}^i are zero. Suppose, by contradiction, that one of the flows j_{kl}^i is nonzero, then by (6.1), we have $j_{\tilde{l}k}^i > 0$, for some $\tilde{l}$. For the Wardrop equilibrium $\tilde{\boldsymbol{j}}^*$, choose $\sigma > 0$ small enough and set $\hat{\boldsymbol{j}}$ by modifying the following coordinates:

$$\hat{j}_{kl}^i = j_{kl}^{i*} - \sigma, \quad \hat{j}_{\tilde{l}k}^i = j_{\tilde{l}k}^{i*} - \sigma, \quad \text{and} \quad \hat{j}_{\tilde{l}l}^i = j_{\tilde{l}l}^{i*} + \sigma.$$

Note that $\hat{\boldsymbol{j}} \in \mathcal{A}$. Because $\tilde{\boldsymbol{j}}^*$ is a Wardrop equilibrium, we get

$$\sigma(-c_{\tilde{l}l}(\tilde{\boldsymbol{j}}^*) + c_{\tilde{l}k}(\tilde{\boldsymbol{j}}^*) + c_{kl}(\tilde{\boldsymbol{j}}^*)) \leq 0,$$

which is

$$\sigma(-\psi_{\tilde{l}l}^i + \psi_{\tilde{l}k}^i + \psi_{kl}^i) \leq 0.$$

Thus,

$$\psi_{\tilde{l}l}^i \geq \psi_{\tilde{l}k}^i + \psi_{kl}^i,$$

which contradicts the strict inequality (4.1). Hence, j_{kl}^i must be zero. Using this in (6.1), we get (4.9). Similarly, we get (4.8). ∎

Next, we present some definitions and a lemma to retrieve the value function.

Definition 6. Consider an admissible flow for a road traffic problem. A *regular vertex* is a vertex that belongs to an edge where the flow is positive.

Definition 7. A *walk* on a network is a sequence $\{v_i, e_i\}_{i=0}^k \cup \{v_{k+1}\}$ of vertices and edges of a graph such that $e_i = (v_i, v_{i+1})$ for $i = 0, \ldots, k$. Consider an admissible flow for a road traffic problem, a *positive walk* is a walk of regular vertices connected by edges where the flows are positive. A *closed positive walk* is a positive walk which is infinite and periodic.

Lemma 6.1. *In a road traffic problem, if all loops have positive costs, then any regular vertex is connected to an exit by a positive walk.*

Proof. Let $\tilde{\boldsymbol{j}}^*$ be a Wardrop equilibrium and let $\tilde{v}$ be a regular vertex. Suppose, by contradiction, that there is a *positive walk* starting at $\tilde{v}$ containing an infinite number of vertices. Because the network is finite, there is a closed positive walk. Let $\mathcal{I}$ be the indices of the edges of this closed positive walk. Next, take $\sigma > 0$ small enough so that one can reduce the flow on the closed positive walk, still keeping it non-negative.[1]

Denoting the new flow $\boldsymbol{j}$, consider the partition $\boldsymbol{j} = (\boldsymbol{j}_0, \boldsymbol{j}_-)$ into the flows that did not change, and the flows that decreased. Consider the same partition for $\tilde{c}(\tilde{\boldsymbol{j}}^*) = (\tilde{c}(\tilde{\boldsymbol{j}}^*)_0, \tilde{c}(\tilde{\boldsymbol{j}}^*)_-)$.

Since the cost on loops is positive, we have

$$\langle \tilde{c}(\tilde{\boldsymbol{j}}^*), \tilde{\boldsymbol{j}}^* - \boldsymbol{j} \rangle = \langle \tilde{c}(\tilde{\boldsymbol{j}}^*)_0, \tilde{\boldsymbol{j}}_0^* - \boldsymbol{j}_0 \rangle + \langle \tilde{c}(\tilde{\boldsymbol{j}}^*)_-, \tilde{\boldsymbol{j}}_-^* - \boldsymbol{j}_- \rangle = \sigma \sum_{k \in \mathcal{I}} \tilde{c}_k(\tilde{\boldsymbol{j}}^*) > 0$$

which contradicts the Wardrop equilibrium condition. ■

Because of Lemma 6.1, we can always connect any regular vertex $\tilde{v}_i \in \tilde{e}_k = (\tilde{v}_r, \tilde{v}_i)$ to an exit vertex. Hence, we define

$$\tilde{u}_k^i = \sum_l \tilde{c}_l(\tilde{j}_l), \tag{6.2}$$

where $\tilde{u}_k^i$ is the candidate for the value function at $\tilde{v}_i$, $\tilde{c}_l(\tilde{j}_l)$ is the cost in each edge of the walk and the sum is taken over the flow-carrying edges.

Proposition 10. *Given the Wardrop equilibrium $\tilde{\boldsymbol{j}}^*$, the value function $\tilde{u}$ in (6.2) is well defined at all regular vertices, i.e., it does not depend on the positive path from the regular vertex to the exit.*

Furthermore, a flow-carrying walk to an exit minimizes the exit cost.

Proof. Let $\tilde{\boldsymbol{j}}^*$ be a Wardrop equilibrium. Consider a regular vertex with outgoing walks, w_1 and w_2, whose costs given $\tilde{\boldsymbol{j}}^*$ are c_1 and c_2. Assume, without loss of generality that $c_2 \geq c_1$. Choose $\sigma > 0$ small enough so that one can divert σ from the flow in w_2 to w_1, forming a new admissible flow distribution $\boldsymbol{j}$. Consider a partition of $\boldsymbol{j} = (\boldsymbol{j}_+, \boldsymbol{j}_0, \boldsymbol{j}_-)$ into the flows that were increased, did not change, and decreased, respectively. Consider the same partition for $\tilde{c}(\tilde{\boldsymbol{j}}^*) = (\tilde{c}(\tilde{\boldsymbol{j}}^*)_+, \tilde{c}(\tilde{\boldsymbol{j}}^*)_0, \tilde{c}(\tilde{\boldsymbol{j}}^*)_-)$. From the definition of Wardrop equilibrium (2.2)

$$\begin{aligned} 0 \geq \langle \tilde{c}(\tilde{\boldsymbol{j}}^*), \tilde{\boldsymbol{j}}^* - \boldsymbol{j} \rangle &= \langle \tilde{c}(\tilde{\boldsymbol{j}}^*)_+, \tilde{\boldsymbol{j}}_+^* - \boldsymbol{j}_+ \rangle + \langle \tilde{c}(\tilde{\boldsymbol{j}}^*)_0, \tilde{\boldsymbol{j}}_0^* - \boldsymbol{j}_0 \rangle \\ &\quad + \langle \tilde{c}(\tilde{\boldsymbol{j}}^*)_-, \tilde{\boldsymbol{j}}_-^* - \boldsymbol{j}_- \rangle \\ &= \sigma(-c_1 + c_2) \\ &> 0, \end{aligned}$$

which contradicts the assumption of $\boldsymbol{j}^*$ being a Wardrop equilibrium. ■

[1]This amounts to subtracting σ from the flow of each nonzero component of a loop.

Proposition 11. *In a road traffic model where all vertices are regular, the value function $\widetilde{u}$ in the road traffic model is the value function u in the MFG model and satisfies* (4.4) *and* (4.6).

Proof. The value function u at vertex v_k is the infimum among the costs of all walks starting at v_k. Hence, by Proposition 10, $\widetilde{u} = u$. By Proposition 10, for $\widetilde{e}_k = (\widetilde{v}_r, \widetilde{v}_i)$, we have

$$\widetilde{u}_k^r \leq \widetilde{c}_k(\tilde{j}_k) + \sum_{l \in w} \widetilde{c}_l(\tilde{j}_l),$$

where w is the sequence of indices of the edges of a positive walk to an exit. By (6.2), we can write the previous inequality as

$$\widetilde{u}_k^r \leq \widetilde{c}_k(\tilde{j}_k) + \widetilde{u}_k^i.$$

If $\tilde{j}_k > 0$, then we have

$$\widetilde{u}_k^r = \widetilde{c}_k(\tilde{j}_k) + \widetilde{u}_k^i.$$

If the edge $\widetilde{e}_k$ is a transition edge, then we can get (4.4) similarly; we just replace the travel costs by the switching costs. ■

Theorem 6.2. *Consider an MFG model with positive costs on the edges* (2.4) *and strictly consistent switching costs* (4.1). *Let $\tilde{\boldsymbol{j}}^*$ be the corresponding Wardrop equilibrium. If all vertices are regular in the road traffic model, then the Wardrop equilibrium $\boldsymbol{j}^*$ satisfies the MFG equations.*

Proof. Proposition 9 assures the balance equations hold for $\boldsymbol{j}^*$ and Proposition 11, assures the value function obeys the optimality conditions. Hence, the MFG equations are satisfied. ■

This shows how to recover the solution of an MFG problem, $(\boldsymbol{u}, \boldsymbol{j}^*)$, from the Wardrop equilibrium of the corresponding road traffic problem, $\tilde{\boldsymbol{j}}^*$.

7. Cost properties

We begin this section by studying cost properties derived from the microstructure of the MFGs in a single edge. We discuss cost reversibility, then we examine monotonicity.

7.1. Cost reversibility

Cost reversibility considers if a single agent's travel direction remains the same irrespective of whether they travel against or with the flow. In a reversible MFG setting, an agent's cost depends solely on the density of other agents and is unaffected by their direction of travel.

Proposition 12. *If the Hamiltonian is an even function in p, i.e., $H(x,-p,m) = H(x,p,m)$, and strictly convex, then we have the following:*

(1) *The cost is reversible; that is, $c_{01}(j) = c_{10}(j)$, $\forall j \in \mathbb{R}\backslash\{0\}$.*

(2) *The cost is even; that is, $c_{01}(-j) = c_{01}(j)$, $\forall j \in \mathbb{R}\backslash\{0\}$.*

Proof. Since H is even, L is also even, i.e.,

$$L(x,-v,m) = L(x,v,m).$$

The density m is determined by solving (3.14). For $j > 0$, traveling from 0 to 1, the optimal velocity is $v_+^* = \frac{j}{m}$, which solves the necessary optimality condition (3.9). For the same j and m, we consider the optimal trajectory connecting 1 to 0. We claim that $v_-^* = -v_+^*$, as it satisfies (3.9). Then, using (3.10) and (3.11), we obtain

$$c_{01}(j) = c_{10}(j).$$

Thus, the cost is reversible.

Next, we prove the second statement. Consider the HJ equation in (3.1) and the definition of the flow in (3.2). If we replace j by $-j$, $p = u_x$ by $-p = -u_x$, and keep m unchanged, the HJ in (3.1) holds because H is even and (3.2) holds because $D_p H$ is odd in p. Hence,

$$m(x,-j) = m(x,j). \tag{7.1}$$

Because H is strictly convex, L is strictly convex. Thus, by Proposition 4, there exists $v_+^* > 0$ unique solution of (3.9) in $\mathbb{R}$. Using (7.1) in (3.9) we get

$$v_+^*(x,-j) = v_+^*(x,j), \quad \forall j.$$

Similarly,

$$v_-^*(x,-j) = v_-^*(x,j), \quad \forall j.$$

Then, (3.10) implies

$$c_{01}(-j) = c_{01}(j).$$

■

Example 7.1 (Even Hamiltonian). Suppose the separable Hamiltonian is

$$H(x,p,m) = \frac{p^2}{2} - m.$$

The corresponding Lagrangian is

$$L(x,v,m) = \frac{v^2}{2} + m.$$

The resulting MFG, with flow j, system is

$$\begin{cases} \dfrac{u_x^2}{2} = m, \\ -mu_x = j. \end{cases}$$

The density m is obtained by substituting $u_x = -\frac{j}{m}$ in the HJ equation, which gives the identity

$$\frac{1}{2}\left(-\frac{j}{m}\right)^2 = m.$$

Accordingly,

$$m = \left(\frac{j^2}{2}\right)^{1/3}.$$

On the other hand, (3.12) implies

$$v = (2m)^{1/2}.$$

Hence, substituting the formula for m in the previous identity, we obtain

$$v = \left(2\left(\frac{j^2}{2}\right)^{1/3}\right)^{1/2}.$$

Using (3.10), we compute the cost as

$$c_{01}(j) = \int_0^1 2^{1/3}|j|^{1/3}\,dx = 2^{1/3}|j|^{1/3}.$$

Note that, for $j > 0$, using (3.15), we have

$$c_{01}(j) = \int_0^1 \frac{j}{m} = \frac{j}{(j^2/2)^{1/3}} = (2j)^{1/3}.$$

Analogously, for $j < 0$, we have

$$c_{10}(j) = -(2j)^{1/3}.$$

7.2. Monotonicity of the costs

From Remark 3.2, the local cost at x for moving from 0 to 1, given the positive flow j, is denoted by $-\frac{\partial u}{\partial x}(x,j)$. Hence, for $j > 0$, $c_{01}(j) = -\int_0^1 \frac{\partial u}{\partial x}(x,j)\,dx$ and $c_{01}'(j) = -\int_0^1 \frac{\partial^2 u}{\partial j \partial x}(x,j)\,dx$.

For general MFGs of the form (3.1), under the monotonicity condition

$$\begin{bmatrix} -\frac{2}{m}D_m H & D_{pm}^2 H \\ D_{pm}^2 H & 2D_{pp}^2 H \end{bmatrix} \text{ is positive definite,} \tag{7.2}$$

where the derivatives of H are calculated at $(x, u_x(x, j), m(x, j))$, the uniqueness of the solution in several cases was proved in [34] (also see [37]). This condition is relevant for the monotonicity of the costs.

Proposition 13. *Let H be smooth, and define*

$$M = mD_mHD_{pp}^2H - (D_pH)^2 - mD_pHD_{pm}^2H.$$

Suppose the monotonicity condition (7.2) holds. Then

$$\begin{bmatrix} \frac{\partial^2 u}{\partial j \partial x} \\ \frac{\partial m}{\partial j} \end{bmatrix} = \frac{1}{M} \begin{bmatrix} -D_mH \\ D_pH \end{bmatrix} \tag{7.3}$$

where the derivatives of H are calculated at $(x, u_x(x, j), m(x, j))$.

Moreover, if (7.2) holds, then

$$\frac{\partial^2 u}{\partial j \partial x}(x, j) < 0.$$

Proof. Differentiate, formally, the HJ equation from (3.1) and in the flow equation (3.2) with respect to j to get the linear system

$$\begin{cases} D_pH(x, u_x, m)\dfrac{\partial^2 u}{\partial j \partial x} + D_mH(x, u_x, m)\dfrac{\partial m}{\partial j} = 0, \\ -mD_{pp}^2H(x, u_x, m)\dfrac{\partial^2 u}{\partial j \partial x} - \left(D_pH(x, u_x, m) + mD_{pm}^2H(x, u_x, m)\right)\dfrac{\partial m}{\partial j} = 1, \end{cases} \tag{7.4}$$

where $\frac{\partial u}{\partial x} = u_x = u_x(x, j)$ and $m = m(x, j)$. The solution of (7.4) is given by (7.3), which proves the first result.

For the second result, notice that (7.2) implies $D_mH < 0$ and the determinant

$$A = -\frac{4}{m}D_mHD_{pp}^2H - (D_{pm}^2H)^2 > 0.$$

Together with Cauchy's inequality, we get that the denominator in (7.3) satisfies

$$\begin{aligned} &mD_mHD_{pp}^2H - (D_pH)^2 - mD_pHD_{pm}^2H \\ &\quad \leq mD_mHD_{pp}^2H - (D_pH)^2 + (D_pH)^2 + \frac{1}{4}m^2(D_{pm}^2H)^2 \\ &\quad = -\frac{Am^2}{4} < 0. \end{aligned}$$

Consequently, (7.2) implies

$$\frac{\partial^2 u}{\partial j \partial x} < 0. \qquad \blacksquare$$

Remark 7.2. If $D_m H(x,u_x,m) < 0$, then m and $c_{01}(j) = -\int_0^1 u_x(x,j)\,dx$ have the same monotonicity with respect to $j > 0$ and opposite otherwise. Because, by the first equation in (7.4) and (3.2),

$$\frac{\partial^2 u}{\partial j\,\partial x} = \frac{m D_m H(x,u_x,m)}{j}\frac{\partial m}{\partial j}.$$

Condition (7.2) is sufficient but not necessary to have monotonicity of the cost, as we show in the following example for separable Hamiltonians.

Example 7.3. Suppose the Hamiltonian is convex in p and separable, that is, it is of the form $H(x,p,m) = \mathcal{H}(x,p) - g(m)$. Then

$$D^2_{pm}H(x,p,m) = 0 \quad \text{and} \quad D_m H(x,p,m) = -g'(m).$$

From (7.3), we get

$$\frac{\partial^2 u}{\partial j\,\partial x} = \frac{g'(m)}{-mg'(m)D^2_{pp}\mathcal{H} - (D_p\mathcal{H})^2}. \tag{7.5}$$

If $g'(m) > 0$, then $\frac{\partial^2 u}{\partial j\,\partial x} < 0$. Now, let $g'(m) < 0$, $\gamma > 1$, and consider a Hamiltonian independent of x

$$H(p,m) = \frac{|p|^\gamma}{\gamma} - g(m).$$

Then we have

$$\begin{aligned} D_p H(p,m) &= p|p|^{\gamma-2}, \\ D_p H(p,m)^2 &= |p|^{2\gamma-2} \implies (D_p H(u_x,m))^2 = (\gamma g(m))^{2-\frac{2}{\gamma}}, \\ D^2_{pp}H(p,m) &= (\gamma-1)|p|^{\gamma-2} \implies D^2_{pp}H(u_x,m) = (\gamma-1)(\gamma g(m))^{1-\frac{2}{\gamma}}. \end{aligned}$$

Replace the above in (7.5) to obtain

$$\frac{\partial^2 u}{\partial j\,\partial x} = \frac{g'(m)}{-mg'(m)(\gamma-1)(\gamma g(m))^{1-\frac{2}{\gamma}} - (\gamma g(m))^{2-\frac{2}{\gamma}}}.$$

Thus, to get $\frac{\partial^2 u}{\partial j\,\partial x} < 0$ when $g'(m) < 0$, we require

$$mg'(m)(\gamma-1) + \gamma g(m) < 0. \tag{7.6}$$

The prior inequality holds when considering

$$g(m) = m^{-\beta}, \quad \beta > 0.$$

In this case, (7.6) becomes

$$\gamma - (\gamma-1)\beta < 0.$$

Hence, for $\beta > \frac{\gamma}{\gamma-1}$, we obtain $\frac{\partial^2 u}{\partial j\,\partial x} < 0$.

Next, we study the monotonicity of c_{01} when $j < 0$. In this case, agents travel against the flow, and Proposition 6 does not apply to compute c_{01}.

Proposition 14. *Let H be smooth and L be its Legendre transform. For $j < 0$, the following holds:*

$$c_{01}'(j) = \int_0^1 \frac{1}{v_+^*} D_m L(x, v_+^*(x, j), m(x, j)) \frac{\partial m(x, j)}{\partial j}\, dx. \tag{7.7}$$

Moreover, if (7.2) holds, then

$$c_{01}'(j) < 0. \tag{7.8}$$

Proof. We use v instead of v_+^*, and omit the arguments for v and m, to simplify notation and write (3.12) as

$$v D_v L(x, v, m) - L(x, v, m) = 0.$$

Formally differentiating with respect to j gives

$$v D_{vv}^2 L(x, v, m) \frac{\partial v}{\partial j} + v D_{vm}^2 L(x, v, m) \frac{\partial m}{\partial j} - D_m L(x, v, m) \frac{\partial m}{\partial j} = 0. \tag{7.9}$$

Additionally, differentiating (3.10) with respect to j yields

$$c_{01}'(j) = \int_0^1 D_{vv}^2 L(x, v, m) \frac{\partial v}{\partial j} + D_{vm}^2 L(x, v, m) \frac{\partial m}{\partial j}\, dx.$$

Noting the relation in (7.9), we get the result in (7.7). To prove the second claim, assume (7.2) holds and notice that in (7.7),

$$\frac{1}{v} D_m L(x, v, m) > 0,$$

and by (7.3), together with (3.2),

$$\frac{\partial m}{\partial j} < 0.$$

Consequently, we obtain (7.8). ■

7.3. Analysis of the velocity field

Lions' monotonicity condition (7.2) is related to crowd aversion; that is, agents try to avoid congested areas. As we have shown before, in this case, both c_{01} and $m(x, \cdot)$ increase with $j > 0$, and both c_{10} and $m(x, \cdot)$ decrease with $j < 0$. It is possible to have a crowd-seeking behavior, where g is decreasing, c_{01} increases with j and $m(x, \cdot)$ decreases with j. This is a phenomenon observed in uncongested highways.

Increased speed results in greater car distances, leading to decreased density. Here, we analyze the dependence of the optimal velocity field

$$v(x,j)=\frac{j}{m(x,j)} \tag{7.10}$$

on the traffic flow and the traffic density. We first examine the flow's impact.

Proposition 15. *Let H and M be as in Proposition 13. For $j>0$,*

$$\frac{\partial v}{\partial j}(x,j)=\frac{vD^2_{pm}H+D_mHD^2_{pp}H}{M}.$$

Moreover, in the separable case, if Lions' condition (7.2) holds, then

$$\frac{\partial v}{\partial j}(x,j)>0.$$

If the reverse Lions' condition holds (that is; the matrix in (7.2) is negative definite), then

$$\frac{\partial v}{\partial j}(x,j)<0.$$

Proof. Formally differentiating (7.10) with respect to j yields

$$\frac{\partial v}{\partial j}(x,j)=\frac{1}{m}-\frac{j}{m^2}\frac{\partial m}{\partial j}=\frac{1}{m}\left(1+D_pH\frac{\partial m}{\partial j}\right).$$

From (7.3) we have

$$\frac{\partial v}{\partial j}(x,j)=\frac{1}{m}\left(1+\frac{(D_pH)^2}{M}\right)=\frac{vD^2_{pm}H+D_mHD^2_{pp}H}{M}.$$

In the separable case, $H(x,p,m)=\mathcal{H}(x,p)-g(m)$, we have

$$\frac{\partial v}{\partial j}(x,j)=\frac{-g'(m)D^2_{pp}\mathcal{H}}{M}. \tag{7.11}$$

If Lions' condition holds, then $g'(m)>0$ and by Proposition 13, $M<0$. Thus,

$$\frac{\partial v}{\partial j}(x,j)>0.$$

If the reverse Lions' condition holds, we get the opposite inequality. ■

Next, we investigate the relationship between traffic velocity and density, denoted as $\frac{\partial m}{\partial v}$. When this is greater than zero, it corresponds to the behavior of the uncongested model.

Proposition 16. *Let M be as in Proposition 13. Let $\widetilde{m}(x,v) := m(x, j(x,v))$, where $j(x,v)$ is obtained from $v(x,j) = j/m(x,j)$. For $j > 0$, in the separable case (as in Example 7.3), we have the following relation:*

$$\frac{\partial \widetilde{m}}{\partial v} = \frac{v}{g'(m) D^2_{pp}\mathcal{H}}. \tag{7.12}$$

Proof. Note first that $M = -mg'(m)D^2_{pp}\mathcal{H} - (D_p\mathcal{H})^2 < 0$. From (7.3) and (7.11) we have

$$\begin{aligned}\frac{\partial \widetilde{m}}{\partial v} &= \frac{\partial m}{\partial j}\frac{\partial j}{\partial v}\\ &= \frac{D_p\mathcal{H}}{M}\frac{-M}{g'(m)D^2_{pp}\mathcal{H}}\\ &= \frac{v}{g'(m)D^2_{pp}\mathcal{H}}.\end{aligned}$$ ■

Remark 7.4. From (7.12), we observe that if g is increasing, then the density $\widetilde{m}$ increases with $v > 0$ while if g is decreasing, then $\widetilde{m}$ decreases with $v > 0$.

A summary of the results of this section. For highway or pedestrian traffic scenarios, we expect that as j increases, m increases and v decreases. But in this model, the road capacity is not limited, so we get counter-intuitive results. For separable H and $j > 0$, if Lions' condition (7.2) holds, then

(1) c_{01} increases as j increases, see Proposition 13;

(2) m decreases as j increases, see the proof of Proposition 14;

(3) v increases as j increases, see Proposition 15.

However, Lions' condition is not needed for the previous points to hold. Example 7.3 gives conditions under which (1) and (3) hold without monotonicity. However, (2) does not hold; see Remark 7.2.

We have presented an analysis of the velocity field based on Lions' monotonicity condition and its implications.

8. Calibration of MFGs

In this section, we address Problem 1; that is, the precise correspondence between MFG and road traffic models. Thus, we consider the following two inverse problems.

Problem 2. *Let W be a road traffic model on a single edge, identified with the interval* $[0, 1]$. *Find L such that, for any flow j,*

$$\int_0^{T(j)} L(x, v, m)\, ds = c(j),$$

where $c : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a reversible flow-dependent travel cost and $T(j) > 0$ is the crossing time for an agent moving at optimal speed.

Problem 3. *Let W be a road traffic model on a single edge, identified with the interval* $[0, 1]$. *Find L such that, for any flow j,*

$$\int_0^{T(j)} L(x, v, m)\, ds = c(j), \quad \text{and} \quad T(j) = c(j),$$

where $c : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a reversible flow-dependent travel cost and $T(j) > 0$ is the crossing time for an agent moving at optimal speed.

While Problem 2 may seem more natural, Problem 3 is more straightforward to calibrate as measuring average speed v as a function of flow is sufficient to obtain the cost $c(j)$. In general, we are not aware of another way to assign a travel cost depending on the flow.

Solutions to these problems are not unique, but each can generate an MFG model associated with the cost c. Non-uniqueness partially stems from the absence of microscopic effects in road traffic models.

8.1. Identification of road traffic cost problems with mean-field games

In this section, we work with a specific class of x-independent Lagrangians and show how to tackle the above problems. There is no uniqueness in this class; see Example 8.1.

In Problem 2, we identify a corresponding MFG model whose travel cost on an edge, for a given flow j, is $c(j)$. We assume that agents travel from 0 to 1. Thus, the flow is positive. We do not expect to recover the MFG microstructure from the road traffic model's macrostructure, so we consider Lagrangians without x-dependency. To have reversible costs, we consider Lagrangians that are even in the velocity. More concretely, we restrict our choice to

$$L(x, v, m) = m^\alpha \mathcal{L}(v) + g(m), \tag{8.1}$$

with $\mathcal{L}$ convex and even, and $0 \le \alpha \le 2$. This class encompasses various examples while being specific enough to yield closed-form formulas for the cost. Because the Lagrangian is even and the cost is reversible, it suffices to solve the problem for $j > 0$.

Moreover, since there are no location preferences in the edge, the incremental cost of the travel is the same everywhere; that is, u_x is constant in x. Therefore, according to the MFG system, the density m is constant with respect to the flow j. The corresponding MFG is equivalent to

$$\begin{cases} m^\alpha \mathcal{H}\left(\dfrac{u_x}{m^\alpha}\right) = g(m), \\ -m\mathcal{H}'\left(\dfrac{u_x}{m^\alpha}\right) = j, \end{cases} \tag{8.2}$$

where $\mathcal{H}$ is the Legendre transform of $\mathcal{L}$. From the second equation in (8.2) and the relation $-\mathcal{L}'(-\mathcal{H}'(p)) = p$, we find

$$u_x = -m^\alpha \mathcal{L}'\left(\frac{j}{m}\right),$$

and transform the MFG system (8.2) into the algebraic system

$$\begin{cases} m^\alpha \mathcal{H}\left(-\mathcal{L}'\left(\dfrac{j}{m}\right)\right) = g(m), \\ c(j) = u(0) - u(1) = m^\alpha \mathcal{L}'\left(\dfrac{j}{m}\right), \end{cases} \tag{8.3}$$

assuming $m > 0$. Suppose that the last equation in (8.3) is solvable for m in terms of j, i.e., there exists an invertible function Ψ_c such that

$$m = \Psi_c(j). \tag{8.4}$$

Now, let $\Phi_j(m) := m^\alpha \mathcal{L}'(\frac{j}{m})$. Then, by (8.4) and the last equation in (8.3), we have

$$\Psi_c(j) = m = \Phi_j^{-1}(c(j)).$$

Hence, we obtain

$$\Psi_c^{-1}(m) = c^{-1}(\Phi_j(m)).$$

Hence, Ψ_c is (locally) well defined when $m \to \Phi_j(m)$ is strictly monotone in m for every j. If, in addition, c is strictly monotone, then Ψ_c is invertible. By inverting (8.4) and substituting in the first equation of (8.3), we get

$$m^\alpha \mathcal{H}\left(-\mathcal{L}'\left(\frac{\Psi_c^{-1}(m)}{m}\right)\right) = g(m).$$

This relation identifies $g(m)$ in terms of c, $\mathcal{L}'$ and $\mathcal{H}$.

Next, we present an example illustrating the application of the previous discussion to solve the inverse problem.

Example 8.1. Let $\alpha \neq 1$ and $\beta > 0$. Consider the Lagrangian, $\mathcal{L}(v) = \beta \frac{v^2}{2}$, with corresponding Hamiltonian, $\mathcal{H}(p) = \frac{p^2}{2\beta}$. The system (8.3) becomes

$$\begin{cases} m^{\alpha-2}\beta j^2 = 2g(m), \\ \quad c(j) = m^{\alpha-1}\beta j, \end{cases}$$

provided $m > 0$. The second equation gives, for $\alpha \neq 1$, m as a function of j,

$$m = \left(\frac{c(j)}{\beta j}\right)^{\frac{1}{\alpha-1}} =: \Psi_c(j). \tag{8.5}$$

If Ψ_c is invertible, we obtain the following:

$$g(m) = \frac{\beta}{2} m^{\alpha-2}(\Psi_c^{-1}(m))^2.$$

Now, consider a linear cost, $c(j) = c_1 + c_2 j$. By (8.5), the density is

$$m = \left(\frac{c_1}{\beta j} + \frac{c_2}{\beta}\right)^{\frac{1}{\alpha-1}}.$$

Assuming $c_1, c_2 \geq 0$, the range of m is

$$\begin{cases} \left[\left(\frac{c_2}{\beta}\right)^{\frac{1}{\alpha-1}}, +\infty\right), & \text{for } \alpha > 1, \\ \left(0, \left(\frac{c_2}{\beta}\right)^{\frac{1}{\alpha-1}}\right), & \text{for } \alpha < 1. \end{cases} \tag{8.6}$$

With the linear cost, Ψ_c is invertible and we have the following expression for $g(m)$:

$$g(m) = \frac{\beta m^{\alpha-2} c_1^2}{2(\beta m^{\alpha-1} - c_2)^2},$$

which can be written as follows:

$$g(m) = \frac{1}{2\beta} \frac{c_1^2}{(m^{\alpha/2} - \frac{c_2}{\beta} m^{1-\alpha/2})^2}.$$

Note that this g is non-monotone. For $\alpha < 1$, it has one singularity at 0 and another one at $m_0 = (\frac{c_2}{\beta})^{\frac{1}{\alpha-1}}$. This last value m_0 is the maximal density. For $\alpha > 1$, there is a singularity at m_0, but it is a minimal density in this case. Figures 7 and 8 illustrate this behavior, where we plot $\frac{1}{g(m)}$ for $\alpha = 0.1$ and $\alpha = 3$, respectively, with $\beta = 1$, $c_1 = 1$, and $c_2 = 0.9$.

The preceding example illustrates an important point: even with the simple linear flow-dependent cost, we obtain a coupling function $g(m)$ that is non-monotone which falls outside standard MFG theory. If $\alpha < 1$, $g(m)$ is neither decreasing nor increasing for the values of m in (8.6). For $\alpha > 1$, $g(m)$ is decreasing for the values of m that satisfy (8.6).

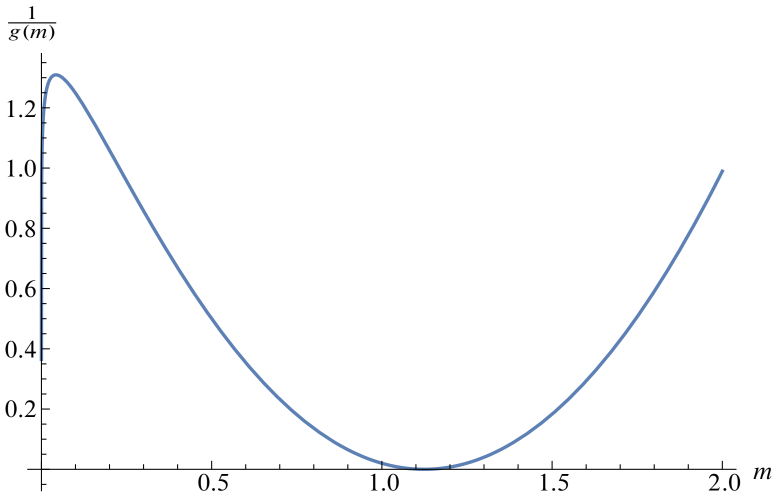

Figure 7. For $\alpha = 0.1$, $\frac{1}{g(m)}$ touches 0 at two points.

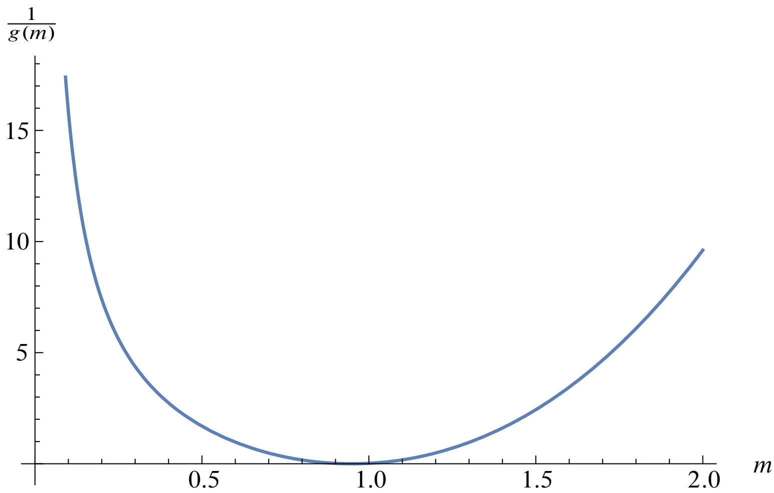

Figure 8. For $\alpha = 3$, $\frac{1}{g(m)}$ touches 0 at one point.

8.2. Identification of road traffic time models with mean-field games

We examine Problem 3. As discussed in Section 7.3, the optimal velocity is $v = \frac{j}{m}$. Given that the edge length is 1 and velocity v is constant, we obtain

$$T = \frac{m}{j}.$$

To address this problem, we refer to L from (8.1) and show how to determine both $\mathcal{L}$ and g. Arguing as in Section 8.1, we obtain

$$\begin{cases} m^\alpha \mathcal{H}\left(-\mathcal{L}'\left(\frac{j}{m}\right)\right) = g(m), \\ m^\alpha \mathcal{L}'\left(\frac{j}{m}\right) = c(j), \\ c(j) = \frac{m}{j}, \end{cases} \tag{8.7}$$

provided $m > 0$. We study the case $j > 0$. From the last equation in (8.7), we get

$$m = jc(j), \quad \forall j > 0. \tag{8.8}$$

Let $\Psi_c(j) = jc(j)$. If c is positive and increasing, then $\Psi_c(j)$ is invertible. Hence,

$$j = \Psi_c^{-1}(m).$$

Furthermore, m can take any value in $\mathbb{R}^+$. Using the identity above, we substitute j into the second equation of (8.7). Next, we use the result in the first equation of (8.7) to get

$$g(m) = m^\alpha \mathcal{H}\left(-\frac{c(\Psi_c^{-1}(m))}{m^\alpha}\right). \tag{8.9}$$

Now, we substitute m from (8.8) in the second equation in (8.7) to get

$$\mathcal{L}'\left(\frac{1}{c(j)}\right) = \frac{c(j)}{(jc(j))^\alpha}.$$

Finally, recalling that $v = \frac{1}{c(j)}$, we have

$$\mathcal{L}'(v) = \frac{v^{\alpha-1}}{(c^{-1}(\frac{1}{v}))^\alpha}. \tag{8.10}$$

If c is increasing, the preceding expression defines $\mathcal{L}'(v)$ for $\lim_{j\to\infty} \frac{1}{c(j)} \leq v \leq \frac{1}{c(0)}$. Thus, with the maximal velocity being $\frac{1}{c(0)}$, we first use (8.10) to determine $\mathcal{L}$, compute its Legendre transform $\mathcal{H}$, and then use (8.9) to obtain g.

In the following example, we show that non-monotone MFGs may arise.

Example 8.2. Consider $c(j) = 1 + j$. Find $\mathcal{L}$ and g for which the cost is $c(j)$.

System (8.7) simplifies to

$$\begin{cases} g(m) = m^{\alpha} \mathcal{H}\left(-\dfrac{2m^{1-\alpha}}{\sqrt{1+4m}-1}\right), \\ \mathcal{L}'(v) = \dfrac{v^{2\alpha-1}}{(1-v)^{\alpha}}, \\ v = \dfrac{1}{1+j}, \\ m = j(1+j). \end{cases}$$

The Lagrangian is convex if $\alpha \geq \frac{1}{2}$. However, the obtained coupling may fail to be monotone, as depicted in Figure 9.

8.3. Example: Braess paradox

We demonstrate the application of MFG calibration in road traffic problems, specifically illustrating how the Braess paradox can arise in MFGs.

Example 8.3 (Braess paradox). Consider the network in Figure 10 with cost $\boldsymbol{c}(\boldsymbol{j}) = (45, |j_2|/100, |j_3|/100, 45)$. Agents traverse the network from the entrance vertex v_1 to the exit vertex v_4, minimizing their travel cost. Assume that the cost at the exit is zero and the flow at the entry is 4000. At v_1, agents will be equally distributed between the edges e_1 and e_2, both having a cost of 65. Adding a new edge $e_5 = (v_2, v_3)$, see Figure 11, with zero travel cost: $\boldsymbol{c}(\boldsymbol{j}) = (45, |j_2|/100, |j_3|/100, 45, 0)$, causes some agents to select the path e_2, e_5, e_3.

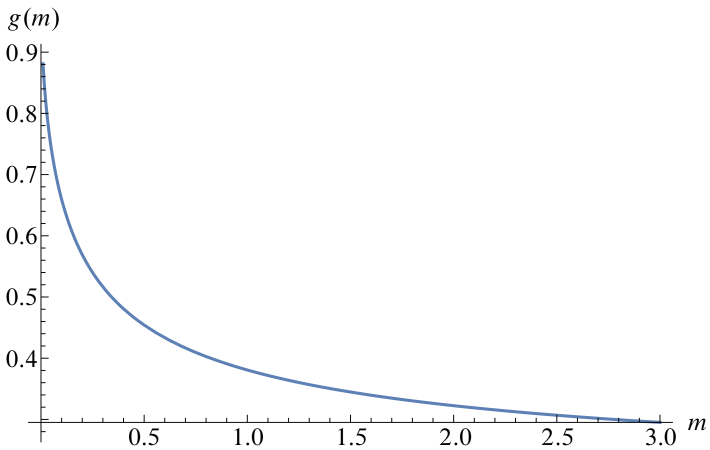

Figure 9. The coupling $g(m)$ decreases with m, for $\alpha = 0.6$.

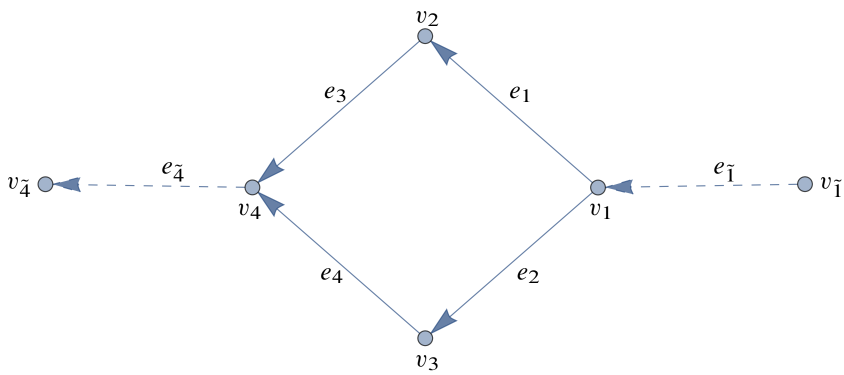

Figure 10. Braess paradox network.

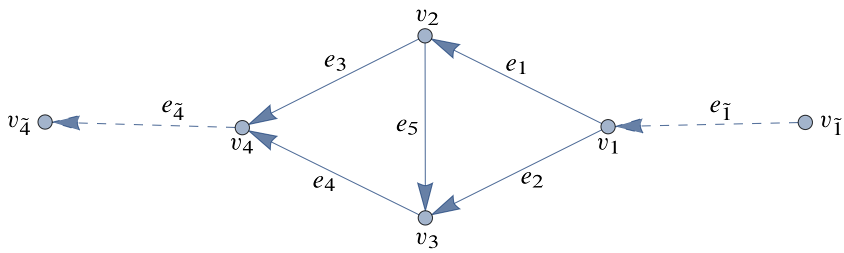

Figure 11. Braess paradox network with edge in the middle.

If all agents are taking this path, this is a Wardrop equilibrium. The addition of e_5 results in increased travel costs for all agents. This is the well-known Braess paradox.

Which Hamiltonians result in the costs associated with the Braess paradox? The computations below establish the MFG for the Braess paradox, addressing this question. For the edges e_3 and e_2, and for $\alpha > \frac{1}{2}$, we have

$$L(v) = \frac{|v|^{2\alpha}}{2\alpha 100^{\alpha}}, \qquad H(p) = 100^{\alpha} \frac{|p|^{\kappa}}{\kappa},$$

with $\kappa = \frac{2\alpha}{2\alpha - 1}$, and

$$g(m) = \frac{100^{(\alpha-1)\kappa}}{\kappa}.$$

The constant costs do not fit the previous calibration. However, in e_1 and e_4, for small ε, we can set the cost to $c(j) = 45 + \varepsilon|j|$. Similarly, in the edge e_5, the cost is set to $c(j) = \varepsilon|j|$.

9. Conclusion and future work

In the first part of this paper, we presented the Wardrop and first-order MFG model on networks. We then demonstrated how to reformulate the MFG problem into a road traffic problem and proved that the MFG solution is the Wardrop equilibrium for the road traffic problem. Furthermore, we proved that the solution of the MFG problem can be recovered from the solution of the corresponding road traffic problem. In the second part of the paper, we studied the cost properties and proposed a novel approach to calibrating MFGs. We demonstrated that even simple travel costs can lead to non-monotone MFGs. This work suggests several future directions, such as investigating non-monotone MFGs and their general properties, as well as extending our models to the dynamic setting. The absence of monotonicity for these MFG models may lead to new phenomena, such as multiple equilibria or instability, which we consider important topics for future research.

Acknowledgments. F. Al Saleh acknowledges the support from King Faisal University.

Funding. The authors were partially supported by King Abdullah University of Science and Technology (KAUST) baseline funds and KAUST OSR-CRG2021-4674.

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Received 3 November 2022; revised 10 January 2024.

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