Explicit moduli of superelliptic curves with level structure

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Abstract. In this article we give an explicit construction of the moduli space of trigonal super-elliptic curves with level 3 structure. The construction is given in terms of point sets on the projective line and leads to a closed formula for the number of connected (and irreducible) components of the moduli space. The results of the article generalise the description of the moduli space of hyperelliptic curves with level 2 structure, due to Dolgachev and Ortland, Runge and Tsuyumine.

1. Introduction

As Mumford describes in [25, §2], 2-torsion divisors on a hyperelliptic curve correspond precisely to degree zero linear combinations of ramification points. Hence, if one takes distinct points $P_1, \ldots, P_{2g+2}$ on $\mathbb{P}^1$ and considers C, the unique hyperelliptic curve ramified over these points, then one can hope to explicitly describe symplectic bases for the 2-torsion Jacobian $\mathrm{Jac}(c)[2]$ in terms of $P_1, \ldots, P_{2g+2}$. Indeed, after choosing an ordering for the branch points, it turns out that there is a natural way to obtain a (full) symplectic level 2 structure on C from combinations of $P_1, \ldots, P_{2g+2}$. One can then obtain any level 2 structure on C via the symplectic group $\mathrm{Sp}(2g, \mathbb{F}_2)$. On top of this, since each choice of ordering will give rise to a different symplectic level 2 structure, this construction naturally defines an embedding of the symmetric group S_{2g+2} into the symplectic group $\mathrm{Sp}(2g, \mathbb{F}_2)$.

Using this construction, Dolgachev and Ortland [13] considered $\mathrm{Hyp}_g[2]$, the moduli space of hyperelliptic curves with level 2 structure. They showed that each irreducible component of $\mathrm{Hyp}_g[2]$ is isomorphic to $M_{0,2g+2}$, the moduli space of smooth rational curves with $2g + 2$ distinct markings, and that the irreducible components of $\mathrm{Hyp}_g[2]$ are indexed by the set of cosets $\mathcal{C} := \mathrm{Sp}(2g, \mathbb{F}_2)/S_{2g+2}$. Dolgachev and Ortland also posed the question: *What are the connected components of* $\mathrm{Hyp}_g[2]$? This question was answered independently by Tsuyumine [37] and Runge [30], who

showed that the irreducible components are also the connected components. Thus, if we let X_c be a variety isomorphic to $M_{0,2g+2}$ and indexed by the coset $c \in \mathcal{C}$, then there is an isomorphism

$$\mathrm{Hyp}_3[2] \cong \bigsqcup_{c \in \mathcal{C}} X_c.$$

It is natural to ask which parts of the above discussion extend to other types of ramified covers. In the present article we proceed in this direction by investigating cyclic covers of $\mathbb{P}^1$ called *superelliptic curves* [23, §1]. We pay special attention to the simplest non-hyperelliptic case, namely trigonal curves. We find (somewhat surprisingly) an almost perfect analogy between the case of cyclic trigonal curves with symplectic level 3 structure and hyperelliptic curves with level 2 structure. Our main result is the following.

Theorem 1.1. *Let g be a positive integer and take the unique integers n and k such that $0 \le k \le 2$ and $g + 2 = 3n - k$. The moduli space $\mathrm{Sup}_g^3[3]$ of trigonal superelliptic curves with level 3 structure has*

$$|\mathrm{Sp}(2g, \mathbb{F}_3)| \cdot \left(\sum_{0 \le i < \lceil \frac{g+2-2k}{6} \rceil} \frac{1}{(3i + k)!(g + 2 - (3i + k))!} + \sum_{\{i \in \mathbb{Z} : i = \frac{g}{2}+1\}} \frac{1}{2(i!)^2} \right)$$

connected components. Each connected component is irreducible and is isomorphic to the moduli space $M_{0,g+2}$ of smooth rational curves with $g + 2$ marked points.

It is plausible that the above theorem can be useful in the context of the conjectures of Bergström–van der Geer [8, Sec. 12] and the present work may be useful in the moduli theory related to 2-spin Hurwitz numbers via [20] and [21]. Other possible directions for future work include cohomological computations, see, e.g., [7] and [10], and investigations of special kinds of superelliptic curves, e.g., Picard curves, Belyi curves or plane curves. Relevant previous work include Accola's characterization of cyclic trigonal curves in terms of vanishing properties of theta functions [1], Kontogeorgis' study of automorphism groups of rational function fields [17], Kopeliovich's computations of Thomae formulae for cyclic covers of the projective line [18], Wangyu's characterization of cyclic covers of the projective line with prime gonality [38], as well as Previato, Shaska and Wijesiri's study of thetanulls for cyclic curves of genus 2 and 3 [29]. For further references, and a survey of the field, see [23] and [33].

The paper is structured as follows. In Sections 2 and 3, we review the necessary background on hyperelliptic and superelliptic curves. We also rephrase some of the classical material into a setup which is mutually compatible and generalisable. In Section 4, we study divisors on superelliptic curves. In particular, we study divisors generated by ramification points and how the Weil pairing behaves on pairs of such

divisors. Finally, in Section 5, we specialize to the case of trigonal curves. Among other results, we obtain an explicit description of the moduli space of superelliptic trigonal curves with level 3 structure and an explicit formula for the number of connected (and irreducible) components of this space (see Theorem 1.1 above).

2. Background on hyperelliptic curves

2.1 (*Overview of section*). In Section 2, we recall facts about hyperelliptic curves and give an overview of an explicit construction of the moduli space of hyperelliptic curves with level 2 structure. This construction, which is based upon finite subsets of the projective line, can be mainly attributed to Mumford [25] and Dolgachev–Ortland [13]. We will present this material in a way that can be readily generalised to superelliptic curves. Throughout Section 2, and the rest of the paper, we work over the complex numbers.

Definition 2.2 (*Hyperelliptic curves*). A *hyperelliptic curve* is a degree 2 morphism (of schemes) $\pi : C \to \mathbb{P}^1$ such that C is a smooth (connected) curve of genus $g > 1$.

Two hyperelliptic curves $\pi : C \to \mathbb{P}^1$ and $\pi' : C' \to \mathbb{P}^1$ are isomorphic if there are isomorphisms $\alpha : C \to C'$ and $\beta : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\pi \circ \alpha = \beta \circ \pi'$.

Remark 2.3 (*Equivalent definition of hyperelliptic curves*). A much more standard definition of a *hyperelliptic curve* is a curve C with genus $g > 1$ such that a degree 2 morphism $\pi : C \to \mathbb{P}^1$ *exists*. However, it is well known that any such degree 2 morphism is unique up to isomorphism, so the two definitions are equivalent (see, for example, [16, Ch. IV, Prop. 5.3]). We use the definition of hyperelliptic curves from Definition 2.2 since it is more in-line with the definition of *superelliptic curves* which will be given in Definition 3.2.

2.4 (*Moduli space of hyperelliptic curves*). Definition 2.2 extends naturally to give a moduli functor. We denote by Hyp_g the (coarse) moduli space parametrising hyperelliptic curves. Moreover, by the uniqueness of the hyperelliptic morphism pointed out in Remark 2.3, we have that there exists a natural inclusion $\mathrm{Hyp}_g \hookrightarrow M_g$. The image of this inclusion is called the *hyperelliptic locus*.

Remark 2.5. In this article, the main focus is on the coarse moduli *scheme* of hyperelliptic curves Hyp_g instead of the moduli *stack* of hyperelliptic curves $\mathcal{Hyp}_g$. A key difference between the two is that $\mathcal{Hyp}_g$ keeps track of the automorphisms of the hyperelliptic curves. The stack theoretic viewpoint for this space and the analogous superelliptic case (appearing in Section 3) was considered in [3].

Remark 2.6. Since we have defined Hyp_g as a moduli space of maps, this moduli space can also naturally be viewed as the *Hurwitz space* $\mathrm{Hur}_g(2)$.

2.7 (*Affine models for hyperelliptic curves*). It is a well-known result that a genus g hyperelliptic curve $\pi : C \to \mathbb{P}^1$ has a non-unique affine model of the form

$$s^2 - f(t) = 0$$

where $f \in \mathbb{C}[t]$ has unique roots and is either of degree $2g + 1$ or $2g + 2$ (see, for example, [25, p. 3.28]). Moreover, if C° is the smooth locus of the above affine variety, then C is the smooth completion of C° and π is the associated projection map arising from the projection to the t coordinate.

The equivalence relation between two isomorphic affine models is described via

$$s^2 - f(t) \sim s^2 - (ct + d)^{2g+2} f\left(\frac{at + b}{ct + d}\right) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

Note that if $s^2 - h(t)$ is the resulting polynomial on the right-hand side, f and h can have degrees differing by 1 depending on whether a root was moved to or away from infinity.

2.8 (*Construction via choosing ramification points*). Consider the above relationship between the affine model and a hyperelliptic curve $\pi : C \to \mathbb{P}^1$, described in Section 2.7. An immediate observation from this relationship is that the ramification points of π occur at the roots of $f(t)$ for $\deg(f) = 2g + 2$ and also at $\pi^{-1}(\infty)$ if $\deg(f) = 2g + 1$.

A result of this observation is that one can construct a unique hyperelliptic curve by choosing a set of $2g + 2$ points in $\mathbb{P}^1$. Taking into account the equivalence described in Section 2.7 we arrive at the following theorem.

Theorem 2.9 ([14,22], *scheme structure of* Hyp_g). *It holds, that there is an isomorphism of schemes*

$$M_{0,2g+2}/S_{2g+2} \to \mathrm{Hyp}_g.$$

Remark 2.10. Versions of the isomorphism from Theorem 2.9 also exist for compactified moduli spaces, e.g., [15, Exm. 6.25] and [5, Cor. 2.5].

Convention 2.11 (*Notation for choice of hyperelliptic curve*). For the remainder of Section 2 we will assume that $\pi : C \to \mathbb{P}^1$ is a hyperelliptic curve of genus g. Moreover, given the equivalence of affine models described in Section 2.7, we will make the assumption that $\infty \in \mathbb{P}^1$ is a branch point of π and the affine model is

$$0 = s^2 - \prod_{i=1}^{2g+1} (t - a_i).$$

We will denote the ramification points as $Q_i := \pi^{-1}(a_i)$ and $Q_\infty := \pi^{-1}(\infty)$.

2.12 (*Natural principal divisors*). Recalling Convention 2.11 we observe that there are natural principal divisors on C given by:

(i) *Horizontal:* $(s) = \left(\frac{u}{v^{g+1}}\right) = \sum_{i=1}^{2g+1}(Q_i - Q_\infty)$.

(ii) *Vertical ramified:* $(t - a_i) = \left(\frac{1-a_i u}{u}\right) = 2Q_i - 2Q_\infty$.

(iii) *Vertical unramified:* $(t - a) = \left(\frac{1-au}{u}\right) = \pi^* P_a - 2Q_\infty$.

Here, $a \in \mathbb{C}^* \setminus \{a_1, \ldots, a_{2g+1}\}$ and $P_a \in \mathbb{P}^1$ is the corresponding divisor. The divisor $\pi^* P_a$ consists of 2 distinct points.

2.13 (*2-torsion in the Jacobian of a hyperelliptic curve*). A key focus of Section 2 is related to the study of 2-torsion within the Jacobian of hyperelliptic curves. In this direction, an immediate observation from Section 2.12 is that for $i \in \{1, \ldots, 2g+1\}$ we have

$$2 \cdot (Q_i - Q_\infty) \sim 0.$$

This shows that each $D_i := [Q_i - Q_\infty]$ is a natural element of $\mathrm{Jac}(C)[2]$. We can also now consider the natural $\mathbb{F}_2$-vector subspace spanned by the classes

$$\Delta := \mathbb{F}_2\text{-Span}\{D_1, \ldots, D_{2g+1}\} \subseteq \mathrm{Jac}(C)[2].$$

On top of this, the horizontal principal divisor from Section 2.12 gives the relationship $0 = \sum_{i=1}^{2g+1} D_i$. This shows that Δ is spanned by any choice of $2g$ classes from $D_1, \ldots, D_{2g+1}$. Indeed, it was shown by Mumford [25, Ch. IIIa, Lem. 2.5] that any of these choices gives a basis for Δ since $\Delta \cong \mathbb{F}_2^{2g}$. Moreover, combining this result with a basis constructed by Dolgachev and Ortland [13, §3, Lem. 2] gives the following key result.

Proposition 2.14 (*A symplectic basis for* $\mathrm{Jac}(C)[2]$). *There are $\mathbb{F}_2$-linear isomorphisms $\mathrm{Jac}(C)[2] \cong \Delta \cong \mathbb{F}_2^{2g}$. The basis $(A_1, \ldots, A_g, B_1, \ldots, B_g)$ of $\mathrm{Jac}(C)[2]$ defined by*

$$A_i := D_{2i-1} + D_{2i} \qquad and \qquad B_i := D_{2i} + \cdots + D_{2g+1},$$

is symplectic with respect to the Weil Pairing. Moreover, the linear map taking this basis to the standard symplectic basis for $\mathbb{F}_2^{2g}$ is an isometry.

Remark 2.15. The methods used in [13, §3, Lem. 2] to prove Proposition 2.14 are more analytic in nature than the methods employed in this article. Indeed, the authors of [13] consider $\pi : C \to \mathbb{P}^1$ as a two-sheeted covering and consider a symplectic basis of cycles in $H_1(C, \mathbb{Z})$ corresponding to the basis elements of Proposition 2.14.

To be more precise, A_i corresponds to a path which goes from Q_{2i-1} along one sheet to Q_{2i} and then returns along the other sheet, while B_i corresponds to a path which goes from Q_{2i} along one sheet to Q_{2g+1} and then returns along the other sheet. The symplectic basis of cycles in $H_1(C, \mathbb{Z})$ is then combined with a normalised basis of $H^0(C, \Omega_C)$ to construct a *branch point period matrix*.

2.16 (*Inclusion of the symmetric group into the symplectic group*). The symmetric group S_{2g+2} permutes the points $Q_1, \ldots, Q_{2g+1}, Q_\infty$ and thus defines an action on the set $\mathsf{D} := \{D_1, \ldots, D_{2g}\}$. Since any permutation of $Q_1, \ldots, Q_{2g+1}, Q_\infty$ will give the same hyperelliptic curve, the action of S_{2g+2} on D extends to give a change of symplectic basis. This defines a group homomorphism $S_{2g+2} \to \mathrm{Sp}(2g, \mathbb{F}_2)$ which is in fact an embedding (see [9, p. 60] for details and further references).

2.17 (*Natural morphism to* $\mathrm{Hyp}_g[2]$). We denote the (coarse) *moduli space of hyperelliptic curves with level 2 structure* by $\mathrm{Hyp}_g[2]$. This is a moduli space parametrising pairs consisting of a hyperelliptic curve $\pi : C \to \mathbb{P}^1$ and an isometry $\eta : \mathbb{F}_2^{2g} \to \mathrm{Jac}(C)[2]$, where $\mathbb{F}_2^{2g}$ uses the standard symplectic form and $\mathrm{Jac}(C)[2]$ uses the Weil pairing. Now, Proposition 2.14 (which extends easily to families) naturally gives a morphism

$$M_{0,2g+2} \to \mathrm{Hyp}_g[2].$$

Moreover, it is shown in [13, §3, Thm. 1] that this morphism is an isomorphism onto an irreducible component of $\mathrm{Hyp}_g[2]$. On top of this, if we consider the morphism that forgets the isometry, $\mathrm{Hyp}_g[2] \to \mathrm{Hyp}_g$, then we obtain the following commutative diagram:

$$
\begin{array}{ccc}
M_{0,2g+2} & \hookrightarrow & \mathrm{Hyp}_g[2] \\
\downarrow & & \downarrow \\
M_{0,2g+2}/S_{2g+2} & \xrightarrow{\ \cong\ } & \mathrm{Hyp}_g.
\end{array}
$$

In fact, the morphism $M_{0,2g+2} \to \mathrm{Hyp}_g[2]$ actually defines an isomorphism to a connected component of $\mathrm{Hyp}_g[2]$. Using this construction, one can describe all of the connected components of $\mathrm{Hyp}_g[2]$ and arrive at the following theorem.

Theorem 2.18 ([37, Thm. 2] & [30, Thm. 4.1], *Decomposition of* $\mathrm{Hyp}_g[2]$). *Consider the inclusion of the symmetric group in the symplectic group described in* 2.16 *and denote the quotient set* $\mathrm{Sp}(2g, \mathbb{F}_2)/S_{2g+2}$ *by* $\mathcal{C}$. *Then, if* X_c *denotes a copy of* $M_{0,2g+2}$ *indexed by* $c \in \mathcal{C}$, *there is an isomorphism of schemes*

$$\mathrm{Hyp}_g[2] \xrightarrow{\cong} \bigsqcup_{c \in \mathcal{C}} X_c,$$

so $\mathrm{Hyp}_g[2]$ *is smooth and the number of connected components of* $\mathrm{Hyp}_g[2]$ *is given by* $|\mathrm{Sp}(2g, \mathbb{F}_2)|/|S_{2g+2}|$.

Remark 2.19. The result of Tsuyumine from [37, Thm. 2] is actually slightly different from the one stated in Theorem 2.18. He shows the more general result that $\mathrm{Hyp}_g[n]$ has $|\mathrm{Sp}(2g, \mathbb{F}_2)|/|S_{2g+2}|$ connected components whenever n is divisible by 2. However, components are only explicitly computed in the case $n = 2$.

Remark 2.20. Modulo minor mistakes (pointed out in [30]), Dolgachev–Ortland [13, §3] presented a weaker version of the above theorem showing that the *irreducible components* of $\mathrm{Hyp}_g[2]$ are indexed by $\mathcal{C}$ and are isomorphic to $M_{0,2g+2}$.

Remark 2.21. Historically, a key purpose of level structures is to rigidify moduli problems. That is, to eliminate the automorphism groups of the objects being parametrised. However, one observation from the constructions in Section 2 is that level 2 structures do not rigidify hyperelliptic curves. To see this, we note that the symplectic basis given in Proposition 2.14 is invariant under the hyperelliptic involution.

3. Background on superelliptic curves

3.1 (*Overview of section*). In Section 3, we will take the results from Section 2 as inspiration and consider a generalisation of hyperelliptic curves. To be precise, we will consider cyclic morphisms of degree p where p is a prime. We note, however, that many of these concepts can also be extended to the non-prime case. We will also assume that $g > 0$.

Definition 3.2 (*Superelliptic curves*). A *superelliptic curve* is a degree p morphism $\pi : C \to \mathbb{P}^1$ such that C is a smooth (connected) curve and the Galois group of π is cyclic. Two superelliptic curves $\pi_1 : C_1 \to \mathbb{P}^1$ and $\pi_2 : C_2 \to \mathbb{P}^1$ are said to be *equivalent* if there are isomorphisms $\psi : C_1 \to C_2$ and $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\pi_2 \circ \psi = \varphi \circ \pi_1$.

Remark 3.3. The term *superelliptic curve* has been used as a generalisation for *hyperelliptic curve* since at least 1978; appearing in the work of Lang [19, Ch. VI §7]. The term is used to describe the zero set of an equation $y^d = f(x)$ where $f(x)$ has no multiple roots. In later work, Baker [6] and Mason [24] used the term *superelliptic equation* to refer to equations $y^d = f(x)$ with general $f(x)$, while others (such as Shorey and Tijdeman [34]) use the term *generalised superelliptic*.

   As will become clear in 3.6 below, our use of the term *superelliptic curve* aligns more closely with that of Baker and Mason, and with the work of Beshaj, Hoxha and Shaska [11], and Beshaj, Shaska and Zhupa [12]. Such objects are also called *cyclic curves* by Sanjeewa and Shaska in [32], and *generalised superelliptic curves of level d* by Malmendier and Shaska in [23, §5.1]. In other work, such as that of Obus and Shaska [27], the term *superelliptic curve* is used to describe curves whose affine model $y^d = f(x)$ has the property of $f(x)$ having distinct roots.

3.4 (*Moduli space of superelliptic curves*). Definition 3.2 extends naturally to give a moduli functor. We denote by Sup_g^p the (coarse) moduli space of superelliptic curves

of degree p and genus g. Here are some well-known relationships with other moduli spaces:

(i)   For $d = 2$ and $g > 1$, Definition 3.2 matches with that of hyperelliptic curves and gives rise to an isomorphism $\mathrm{Hyp}_g \cong \mathrm{Sup}_g^2$. This is because a hyperelliptic curve uniquely determines a morphism $\pi : C \to \mathbb{P}^1$ of degree 2 and because the Galois group of such a morphism is automatically cyclic.

(ii) There is a natural morphism $\mathrm{Sup}_g^p \to M_g$ to the (coarse) moduli space of smooth genus g curves. In general, this morphism is not an immersion such as in the case of $d = 3$ and $g = 3, 4$ (see, for example, [16, Ch. IV, Exm. 5.5.2]). However, it is an immersion in special cases such as for $d = 3$ when $g > 4$ (see, for example, [2, Ch. III, §3, Exer. B-2]).

(iii) For $g > 2$, a curve cannot be both hyperelliptic and degree 3 superelliptic, meaning that the images of Sup_g^2 and Sup_g^3 are disjoint in M_g. (This is a simple application of the base-point-free pencil trick [2].)

3.5 (*Ramification of superelliptic curves*). Since they have a cyclic Galois group, degree p superelliptic curves have the property of being totally ramified at each ramification point (see, for example, [35, Prop. 3.2.10 & Thm. 3.3.7]). In other words, each branch point has a unique preimage. Hence, by the Riemann–Hurwitz formula the number of ramification points is

$$m = \frac{2g}{p-1} + 2.$$

This places a strong condition on the possible choices for g and p. An equivalent condition on g is obtained by specifying m and p to give

$$g = \frac{(p-1)(m-2)}{2}.$$

We note that when $p = 2$ or $p = 3$, superelliptic curves exist for all g by Riemann's existence theorem (see, for example, [16, App. B, Thm. 3.1]). Indeed, for $d = 2$ we recover $m = 2g + 2$ as in Section 2 and when $d = 3$ we obtain $m = g + 2$.

3.6 (*Affine model for superelliptic curves*). It is a well-known result from Kummer theory (see, for example, [36, Tag 09I6]) that a superelliptic curve $\pi : C \to \mathbb{P}^1$ has a (non-unique) affine model given by

$$0 = s^p - f(t)$$

where $f \in \mathbb{C}[t]$ has roots with orders less than p. In particular, if C° is the smooth locus of the above affine variety, then C is the smooth completion of C° and π is the associated projection map arising from the projection to the t coordinate.

We can construct an affine cover of C by considering a second chart. First, define an integer via the ceiling function $n =: \lceil \deg(f)/p \rceil$ and also $\kappa := pn - \deg(f)$. We now define the second chart by

$$0 = u^p - v^\kappa h(v)$$

where $h \in \mathbb{C}[v]$ is the unique polynomial with $h(v) = v^{\deg(f)} f(\frac{1}{v})$. The change of coordinates is then defined by $(t, s) \mapsto (\frac{1}{v}, \frac{u}{v^n})$.

3.7 (*Superelliptic plane curves*). An immediate observation from the two chart cover constructed in 3.6 is that the (singular) affine models for superelliptic curves are naturally embedded in the weighted projective plane $\mathbb{P}(1, 1, n)$.

Furthermore, in the special case where f has no multiple roots and $\kappa = 0$ or $\kappa = 1$, we have that the smooth superelliptic curve itself is a sub-variety of $\mathbb{P}(1, 1, n)$.

Lemma 3.8 (*Equivalent affine models for superelliptic curves*). *The equivalence relation of superelliptic curves from Definition 3.2 is extended to affine models by the relations*

(i) $s^p - f(t) \sim s^p - (ct + d)^{2g+2} f\left(\frac{at+b}{ct+d}\right)$, *for* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{C})$;

(ii) $s^p - \prod_j (t - a_j)^{k_j} \sim s^p - \prod_j (t - a_j)^{\zeta \cdot k_j}$, *for* $\zeta \in \mathbb{F}_p^*$ *and where* $\zeta \cdot k_j$ *should be interpreted as multiplication in* $\mathbb{F}_p$.

In other words, two affine models give rise to equivalent superelliptic curves if and only if they are related by (i) *and* (ii) *above.*

Proof. Property (i) is straightforward and also arises in the hyperelliptic case, hence we consider only property (ii). For this we define $\eta : C \to \mathbb{P}^1$ and $\eta' : C' \to \mathbb{P}^1$ to be the superelliptic curves defined by $s^p - \prod_j (t - a_j)^{k_j}$ and $s^p - \prod_j (t - a_j)^{\zeta \cdot a_j}$ where p divides the sum $\sum_j k_j$. Also, since $k_i \in \mathbb{F}_p^*$ for each j as well as $\zeta \in \mathbb{F}_p^*$, we can assume that $k_1 = 1$.

We begin by considering the monodromies on C arising from the ramification points. Let $x \in \mathbb{P}^1$ be distinct from the branch points and consider m different loops on $\mathbb{P}^1$ starting and finishing at x. The constraint for the loops is that each contains exactly one branch point and different loops contain different branch points.

The monodromy arising from the preimages of the loop around a_i can be described by considering the equation $s^p = r$ while considering the preimage of the unit circle and identifying x with 1. In this case, the preimage of the unit circle is

$$\{(e^{\tau 2\pi i}, e^{\frac{\tau}{p} 2\pi i}) \mid t \in [0, p]\}.$$

The points in the preimage of 1 are the p-th roots of unity and we label them by their power of the primitive root of unity $e^{\frac{2\pi i}{p}}$. Then the associated permutation from the monodromy representation is the p-cycle

$$(1, 2, \dots, (p-1)).$$

Similarly, for the other branch points we consider the equations $s^p = r^{k_j}$. In this case, the associated permutation from the monodromy representation is the p-cycle

$$(k_i, 2 \cdot k_i, \ldots, (p-1) \cdot k_i).$$

Now we apply the same process on C' beginning with $\zeta \cdot k_1 = \zeta$. In this case, the preimage of the unit circle is

$$\{(e^{\tau 2\pi i}, e^{\frac{\tau \zeta}{p} 2\pi i}) \mid t \in [0, p]\}$$

and we again label the points in the preimage of 1 by their power of $e^{\frac{2\pi i}{p}}$. In this way, the ramification monodromy on C' is described by the permutations

$$(\zeta \cdot k_i, 2 \cdot \zeta \cdot k_i, \ldots, (p-1) \cdot \zeta \cdot k_i).$$

Now, define σ to be the inverse of the permutation defined by $n \mapsto \zeta \cdot n$. This gives

$$(\zeta \cdot k_i, 2 \cdot \zeta \cdot k_i, \ldots, (p-1) \cdot \zeta \cdot k_i)\, \sigma = (k_i, 2 \cdot k_i, \ldots, (p-1) \cdot k_i)$$

and shows that the η and η' have the same monodromy data (only a different choice of labelling). The uniqueness part of the Riemann existence theorem now shows that η and η' must be equivalent. ∎

Remark 3.9. Lemma 3.8 can be compared with constructions in [32] (see also [31]).

Remark 3.10 (*Constructing superelliptic curves from points*). A key aspect of the results from Section 2 was the observation (described in Section 2.8) that there is a unique hyperelliptic curve associated to a given choice of branch points. Among other things, this observation leads to the isomorphism $\mathrm{Hyp}_g \cong M_{0,2g+2}/S_{2g+2}$ from Theorem 2.9.

However, in the superelliptic case, things are more complicated. The affine model for superelliptic curves (described in Section 3.6) is of the form $s^p = f(t)$ and suggests that we must also take into consideration the root-orders of $f(t)$. Indeed, this is confirmed by Lemma 3.8.

3.11 (*Notation for indexing components of Sup_g^p*). We will index the possible choices of the affine model $s^p = f(t)$ by indexing the possible combinations for the root-orders of $f(t)$. In this light, it makes sense to group the ramification points with same root orders. If we assume that the representative of the superelliptic curve is *not* branched at $\infty \in \mathbb{P}^1$ then this allows us to express the affine model in the form

$$0 = s^p - \prod_{k=1}^{p-1} \prod_{i=1}^{m_k} (t - \alpha_{k,i})^k.$$

An initial observation from this is that the vector $(m_1, \ldots, m_{p-1})$ has the properties

$$\text{(i)} \quad \sum_{k=1}^{p-1} m_k = m \qquad \text{and} \qquad \text{(ii)} \quad \sum_{k=1}^{p-1} k m_k \equiv 0 \pmod{p}.$$

Moreover, the equivalence relation from Lemma 3.8 shows that for $\zeta \in \mathbb{F}_p^*$, we have that $(m_1, \ldots, m_{p-1})$ and $(m_{\zeta \cdot 1}, \ldots, m_{\zeta \cdot (p-1)})$ give equivalent superelliptic curves. On top of this, there always exists a unique choice of $\zeta \in \mathbb{F}_p^*$ such that the first entry is $m_1 = \max\{m_1, \ldots, m_{p-1}\}$. This leads us to define the following indexing set:

$$\mathsf{M} = \left\{ (m_1, \ldots, m_{p-1}) \in (\mathbb{Z}_{\geq 0})^{p-1} \mid \bullet \; \sum_{k=1}^{p-1} m_k = m, \right.$$
$$\bullet \; \sum_{k=1}^{p-1} k m_k \equiv 0 \pmod{p}, \text{ and}$$
$$\left. \bullet \; m_1 = \max\{m_i\} \right\}.$$

3.12 (*Group describing equivalent affine models*).   As was the case for hyperelliptic curves, we have to account for the inherent labelling of ramification points that an affine model gives. In the hyperelliptic case, all the ramification points were of the same type, so this choice was accounted for by taking the S_{2g+2} quotient.

However, things are more complicated in the superelliptic case. To begin with, for each $\mathsf{m} \in \mathsf{M}$ the group

$$S_{\mathsf{m}} := S_{m_1} \oplus \cdots \oplus S_{m_{p-1}}$$

accounts for permutations of the ramification points within the groupings that correspond to m.

Another complication is that each $\mathsf{m} \in \mathsf{M}$ may define equivalent superelliptic curves in multiple ways. This arises from the equivalence relations described in Lemma 3.8, part (ii). The lemma shows that there is a natural action of $\mathbb{F}_p^*$ on the $(p-1)$-tuples $(m_1, \ldots, m_{p-1})$ which is defined by

$$\zeta \cdot (m_1, \ldots, m_{p-1}) := (m_{\zeta \cdot 1}, \ldots, m_{\zeta \cdot (p-1)}).$$

Moreover, this action preserves properties (i) and (ii) from Section 3.11. Hence we consider the subgroup of $\mathbb{F}_p^*$ which stabilises a given $\mathsf{m} \in \mathsf{M}$ and denote it by

$$\mathrm{Stab}_p(\mathsf{m}) := \left\{ \zeta \in \mathbb{F}_p^* \mid \zeta \cdot \mathsf{m} = \mathsf{m} \right\}.$$

Both S_{m} and $\mathrm{Stab}_p(\mathsf{m})$ are naturally subgroups of S_m. Moreover, both of these inclusions arise from the fact that for any given $\mathsf{m} \in \mathsf{M}$ with $\mathsf{m} = (m_1, \ldots, m_{p-1})$, we

can uniquely describe each $n \in \{1, \ldots, m\}$ by a pair (i, k) where $i \in \{1, \ldots, p-1\}$ and $0 < k \leq m_i$. Explicitly, this relationship is described by

$$n = k + \sum_{0 < j < i} m_j.$$

Using this relationship, we can now write:

(i) The inclusion $\delta : S_\mathsf{m} \hookrightarrow S_m$ is defined by

$$\delta(\sigma_1, \ldots, \sigma_{p-1})\left(k + \sum_{0 < j < i} m_j\right) := \sigma_j(k) + \sum_{0 < j < i} m_j.$$

(ii) The inclusion $\gamma : \mathrm{Stab}_p(\mathsf{m}) \hookrightarrow S_m$ is defined by

$$\gamma(\zeta)\left(k + \sum_{0 < j < i} m_j\right) := k + \sum_{0 < j < \zeta \cdot i} m_j.$$

We observe that as subgroups of S_m, we have that the intersection $S_\mathsf{m} \cap \mathrm{Stab}_p(\mathsf{m})$ is trivial and that $\zeta \cdot \sigma = \sigma \cdot \zeta$ for all $\zeta \in \mathrm{Stab}_p(\mathsf{m})$ and $\sigma \in S_\mathsf{m}$. Hence, we also have

$$\mathsf{A}_\mathsf{m} := S_\mathsf{m} \cdot \mathrm{Stab}_p(\mathsf{m}) \cong S_\mathsf{m} \oplus \mathrm{Stab}_p(\mathsf{m}).$$

The group A_m describes the equivalent ways that $\mathsf{m} \in \mathsf{M}$ can give rise to a superelliptic curve. This results in Proposition 3.13.

Proposition 3.13 (*Decomposition of* Sup_g^p *into connected components*). *The set* M *defined in Section* 3.11 *indexes the connected components of* Sup_g^p *to give*

$$\mathrm{Sup}_g^p = \bigsqcup_{\mathsf{m} \in \mathsf{M}} \mathrm{Sup}_{g,\mathsf{m}}^p.$$

For each $\mathsf{m} = (m_1, \ldots, m_{p-1}) \in \mathsf{M}$ *there is an isomorphism* $M_{0,m}/\mathsf{A}_\mathsf{m} \cong \mathrm{Sup}_{g,\mathsf{m}}^p$ *where* A_m *is the group defined in Section* 3.12. *The* A_m-*quotient map composed with this isomorphism gives a morphism*

$$\Phi_\mathsf{m} : M_{0,m} \to \mathrm{Sup}_{g,\mathsf{m}}^p$$

which is defined by mapping the equivalence class $[[a_1 : b_1], \ldots, [a_m : b_m]] \in M_{0,m}$ *to the equivalence class of superelliptic curves with the representative*

$$0 = s^p - \prod_{k=1}^{p-1} \prod_{i=1}^{m_i} (a_{\varphi(k,i)} t - b_{\varphi(k,i)})^k,$$

where $\varphi(k, i) := k + \sum_{0 < j < i} m_j$.

Remark 3.14 (*Different equivalence conditions*). If we changed the equivalence relation between superelliptic curves from Definition 3.2 to be the stricter condition $\pi : C \to \mathbb{P}^1$ and $\pi' : C' \to \mathbb{P}^1$ are equivalent if there is an isomorphism $\alpha : C \to C'$ such that $\pi \circ \alpha = \pi'$; then much of Section 3 would still apply except with different equivalence conditions.

In this case, the resulting construction in Proposition 3.13 would have been carried out using the configuration space $\mathrm{Conf}^m(\mathbb{P}^1)$ in place of $M_{0,m}$. Moreover, this construction would result in a natural sub-moduli-space of the moduli space of maps $M_g(\mathbb{P}^1, p)$. The resulting sub-moduli-space has links to $(p-1)$-spin Hurwitz theory and was studied by the second-named author in [20, 21].

Remark 3.15 (*Codimension in M_g*). A direct observation from Proposition 3.13 is that superelliptic curves are rare among genus g curves. The image of $\mathrm{Sup}_g^p \to M_g$ has codimension $3g - 3 - (m - 3) = 3g - 1 - \frac{2g}{p-1}$ which we note has a maximum value of $3g - 2$ that occurs when $2g = p - 1$.

4. Divisors on superelliptic curves

4.1 (*Notation for Section 4*). In Section 4, we will consider a fixed superelliptic curve $\pi : C \to \mathbb{P}^1$ with $g > 0$ and a choice of an affine model with the same notation as given in Section 3.6. Namely, π will be given by the equations

$$s^p = f(t) \quad \text{and} \quad u^p = v^\kappa h(v)$$

where $f, h \in \mathbb{C}[t]$ such that for $n =: \lceil \deg(f)/p \rceil$, $\kappa := pn - \deg(f)$ we have that h is the unique polynomial with $v^\kappa h(v) = f(\frac{1}{v})$. The change of coordinates between the two charts is then defined by $(t, s) \mapsto (\frac{1}{v}, \frac{u}{v^n})$.

Moreover, in Section 4, we will assume that these coordinates have been chosen such that π has a branch point at $\infty \in \mathbb{P}^1$. This condition is equivalent to requiring that $p \nmid \deg(f)$ or equivalently requiring that $\kappa \in \mathbb{F}_p^*$.

Lastly, we will assume an ordering for the branch points in $\mathbb{P}^1 \setminus \{\infty\}$. With this ordering assumed we will write $f(t) = \prod_i (t - a_i)^{k_i}$ where we have also invoked the equivalence described in Lemma 3.8 to make this a monic polynomial.

4.2 (*Points at infinity and branch points*). We will follow the standard convention and refer to the preimage $\pi^{-1}(\infty)$ as *the point at infinity* and denote it by Q_∞. In the notation from Section 4.1 this corresponds to the point defined by $v = u = 0$ in the second chart.

4.3 (*Natural principal divisors and the superelliptic divisor class*). Recall from Section 4.1 that $f(t) = \prod_i (t - a_i)^{k_i}$ and observe that $pn = \kappa + \sum_i k_i$. Then we have the natural principal divisors given by:

(i) *Horizontal*: $(s) = \left(\frac{u}{v^n}\right) = \sum_i k_i (Q_i - Q_\infty)$.

(ii) *Vertical ramified*: $(t - a_i) = \left(\frac{1 - a_i u}{u}\right) = pQ_i - pQ_\infty$.

(iii) *Vertical unramified*: $(t - a) = \left(\frac{1 - au}{u}\right) = \pi^* P_a - pQ_\infty$.

Here, $a \in \mathbb{C}^* \setminus \{a_1, \ldots, a_{m-1}\}$ and $P_a \in \mathbb{P}^1$ is the corresponding divisor. The divisor $\pi^* P_a$ consists of p distinct points.

Remark 4.4. In Section 4.3 and in what follows, we will always mean the divisor on the smooth curve C and not the (potentially) singular affine model.

4.5 (*p-torsion in the Jacobian of a superelliptic curve*). A key focus of this article is to study p-torsion within the Jacobian of a superelliptic curve (for related, but somewhat orthogonal, investigations of this topic, see Arul's thesis [4]). An immediate observation from Section 4.3 is that

$$p \cdot (Q_i - Q_\infty) \sim 0$$

for each i. This shows that $[Q_i - Q_\infty]$ is an element of $\mathrm{Jac}(C)[p]$ (cf. the construction in Section 2). We can now consider the $\mathbb{F}_p$-vector subspace of $\mathrm{Jac}(C)[p]$ spanned by the classes $D_i := [Q_i - Q_\infty]$ and denote it by

$$\Delta := \mathbb{F}_p\text{-Span}\{D_1, \ldots, D_{m-1}\} \subseteq \mathrm{Jac}(C)[p].$$

We also have the relationship

$$\sum_i k_i D_i = \sum_i k_i [Q_i - Q_\infty] = [(s)] = 0$$

which shows that $D_1, \ldots, D_{m-2}$ is an $\mathbb{F}_3$-spanning set for Δ. In fact, by the following theorem, this turns out to be a basis for Δ.

Theorem 4.6 ([28, Prop. 6.1] & [39, Thm. 1], *An $\mathbb{F}_p$-basis for Δ*). *Let Δ be the subgroup of $\mathrm{Jac}(C)[p]$ generated by classes of the form $D_i = [Q_i - Q_\infty]$. Then Δ is isomorphic to $\mathbb{F}_p^{m-2}$ and $D_1, \ldots, D_{m-2}$ is an $\mathbb{F}_p$-basis for Δ.*

Remark 4.7. An immediate observation from the construction in Section 4.5 is that for $p > 2$ we will not get all of $\mathrm{Jac}(C)[p] \cong \mathbb{F}_p^{2g}$. Indeed, when $2g = p - 1$, the codimension of Δ in $\mathrm{Jac}(C)[p]$ is $2g - 1$.

4.8 (*Weil pairing on superelliptic curves*). Let $[E]$ and $[E']$ be two elements of $\mathrm{Jac}(C)[p]$ and let $E \in [E]$ and $E' \in [E']$ be divisors with disjoint support. Let f and g be functions on C such that

$$pE = (f) \quad \text{and} \quad pE' = (g).$$

The Weil pairing of $[E]$ and $[E']$ is then defined as the quotient

$$w([E], [E']) := \frac{f(E')}{g(E)} \in \mu_p$$

where $f(E')$ is defined by $f(E') = \prod_{P \in E'} f(P)^{\text{mult}_P(E')}$ and $g(E)$ is analogously defined. The Weil pairing is a symplectic pairing on $\text{Jac}(C)[p]$. For more details and relations to moduli, see, e.g., [2, App. B] and [26, Sec. 7.2].

Proposition 4.9. *In the case where p is odd, the vector space Δ is an isotropic subspace of $\text{Jac}(C)[p]$ with respect to the Weil pairing. In other words, for any two divisor classes D, D' in Δ we have that $w(D, D') = 1$.*

If p is even and D_i is the divisor class defined in Section 4.5 then $w(D_i, D_j) = -1$ for $i \neq j$.

Proof. We will use the notation from Section 4.1. Namely that $\pi : C \to \mathbb{P}^1$ is given by the polynomials $s^p = f(t)$ and $u = v^\kappa g(v)$. Furthermore, using Lemma 3.8, we may assume that $\kappa = 1$ and that $f(t) = \prod_i (t - a_i)^{k_i}$ such that $a_i \in \mathbb{C}^*$ are non-zero and distinct.

From the definition in Section 4.5, we have that Δ is generated by classes with representatives of the form

$$E_i = Q_i - Q_\infty, \quad i = 1, \ldots, m - 1.$$

Moreover, since the Weil pairing is alternating we have that $w([E_i], [E_i]) = 1$, it will suffice to show that $w([E_i], [E_j]) = 1$ for $i \neq j$. Hence, we assume that $i \neq j$.

Since E_i and E_j do not have disjoint support we need to find a divisor F_j in the class E_j such that F_j has support disjoint from E_i. The construction of the divisor F_j is a key aspect of this proof.

To construct F_j we consider the rational function $(t^{n-1} - s)/t^n \in k(C)^*$ and note that it corresponds to $u - v$ in the other chart (where C is defined by the equation $u^p - v\,g(v) = 0$). To determine the divisor $(u - v)$, we consider the isomorphism

$$\mathbb{C}[u, v]/(vg(v) - u^p, u - v) \cong \mathbb{C}[v]/(v(g(v) - v^{p-1}))$$

and recall that $g(v) = \prod_i (1 - a_i v)^{k_i}$. Then we consider the two unique factorisations

$$g(v) - v^{p-1} = \prod_\alpha (1 - b_\alpha v)^{\lambda_\alpha} \quad \text{and} \quad f(t) - t^{p(n-1)} = \prod_\alpha (t - b_\alpha)^{\lambda_\alpha}, \quad (4.1)$$

which are equivalent under the change of variables $u \mapsto \frac{1}{t}$. We then denote by G_α the divisors on C given by the points $(t, s) = (b_\alpha, b_\alpha)$. Then the principal divisor $(u - v)$ is given by

$$(u - v) = Q_\infty + \sum_\alpha \lambda_\alpha G_\alpha - n \cdot \sum_\beta S_\beta$$

where $\sum S_\beta$ is the divisor of the p disjoint points $\pi^{-1}(0)$.

Now we can define F_j by

$$F_j := E_j + (u - v) = Q_j + \sum_\alpha \lambda_\alpha G_\alpha - n \cdot \sum_\beta S_\beta$$

and define the rational function $\psi \in \mathbb{C}(C)^*$ by $\psi = (t - a_j)(t^{n-1} - s)^p / t^{pn}$, while defining the rational function $\varphi \in \mathbb{C}(C)^*$ by $\varphi = (t - a_i)$. We note that by construction we have the properties required to evaluate the Weil pairing. Namely, we have that

(i) $E_i \cap F_j = \emptyset$,

(ii) $(\varphi) = p \cdot E_i$, and

(iii) $(\psi) = p \cdot F_j$.

We will evaluate the numerator and denominator of $w([E_i], [F_j])$ individually beginning with $\psi(E_i)$. First note that in the second chart we have that the rational function ψ is given by $(-1)^p(1 - a_j v)(u - v)^p / v$ and recall that C is defined in this chart by the equation $u^p - vg(v) = 0$. Now the binomial theorem shows that the rational function is given by

$$\psi = (1 - a_j v)\left((-1)^p g(v) + \sum_{\gamma=1}^p \binom{p}{\gamma}(-u)^{p-\gamma} v^{\gamma-1}\right).$$

In this chart we have that Q_i is given by $(v, u) = (\frac{1}{a_i}, 0)$ and Q_∞ is given by $(v, u) = (0, 0)$, hence we can evaluate $\psi(E_i)$ as

$$\psi(E_i) = \frac{\psi(Q_i)}{\psi(Q_\infty)} = \frac{(1 - a_j/a_i)a_i^{1-p}}{(-1)^p} = \frac{a_i - a_j}{(-a_i)^p}. \tag{4.2}$$

Now we will evaluate φ at F_j after recalling that Q_j corresponds to $(t, s) = (a_j, 0)$, G_α corresponds to $(t, s) = (b_\alpha, b_\alpha)$ and S_β corresponds to pairs with $t = 0$. Hence we have

$$\varphi(F_j) = \varphi(Q_j) \cdot \prod_\beta \varphi(G_\alpha)^{\lambda_\alpha} \cdot \prod_\alpha \varphi(S_\beta)^{-n} = \frac{(a_j - a_i) \prod_\alpha (b_\alpha - a_i)^{\lambda_\alpha}}{(-a_i)^{pn}}.$$

Now considering equation (4.1) in the t coordinate and noting that $f(a_i) = 0$ and $\deg(f) = pn - 1$ we have the equations

$$(a_i)^{-p} = \frac{\prod_\alpha (b_\alpha - a_j)^{\lambda_\alpha}}{(-a_i)^{pn}} \quad \text{and} \quad \varphi(F_j) = \frac{-(a_i - a_j)}{(a_i)^p}. \tag{4.3}$$

Finally, combining equations (4.2) and (4.3), we have

$$w([E_i], [F_j]) = \frac{\psi(E_i)}{\varphi(F_j)} = (-1)^{p-1}$$

which is -1 when p is even and 1 when p is odd.   ∎

Corollary 4.10. *The vector space* Δ *is a maximal isotropic subspace of* $\mathrm{Jac}(C)[3]$ *with respect to the Weil pairing.*

Proof. Any maximal isotropic subspace of $\mathrm{Jac}(C)[3]$ has dimension g and, conversely, any isotropic subspace of $\mathrm{Jac}(C)[3]$ of dimension g is a maximal isotropic subspace. Since Δ has dimension g, the result follows immediately from Proposition 4.9. ∎

5. Trigonal superelliptic curves

5.1 (*Overview of section*). The case $p = 3$ is the natural next step from the hyperelliptic case. In this case, superelliptic curves exist for each $g > 0$ and we will see that the key decomposition result of the moduli space of hyperelliptic curves with level 2 structure from Theorem 2.18 has an analogue in the $p = 3$ case.

5.2 (*Properties of the case $p = 3$*). As discussed in Section 3.5, trigonal superelliptic curves exist for every genus $g > 0$ and the number of ramification points is

$$m = g + 2.$$

In this case, we have $\mathbb{F}_3^* \cong S_2$ and the indexing set M from Section 3.11 contains ordered pairs (m_1, m_2). Explicitly, the indexing set can be expressed as

$$\mathsf{M} = \big\{(m - 3i - r, 3i + r) \mid r \in \{0, 1, 2\} \text{ with } m = 3n - r \text{ for some } n \in \mathbb{Z} \text{ and }$$
$$i \in \mathbb{Z} \text{ with } 0 \leq i \leq (m - 2r)/6.\big\}.$$

Some examples of the indexing set are given in below in Table 1. We can also examine the group $\mathsf{A_m}$ describing equivalent affine models from Section 3.12. There are two cases:

(i) $\mathsf{m} = (a, a)$: This case occurs whenever m is even and in this case we have $\mathrm{Stab}_p(\mathsf{m}) = S_2$ and $\mathsf{A_m} \cong S_2 \cdot (S_a \oplus S_a) \cong S_2 \oplus S_a \oplus S_a$. In terms of the explicit description of M given above, this occurs when $i = (m - 2r)/6$.

(ii) $\mathsf{m} = (a, b)$ where $a > b$: In this case we have that $\mathrm{Stab}_p(\mathsf{m})$ is trivial and hence $\mathsf{A_m} \cong S_a \oplus S_b$.

5.3 (*Level 3 structures and moduli description of* $\mathrm{Sup}_g^3[3]$). Here we recall that the definition of a superelliptic curve with a level 3 structure is a pair

$$\big(\pi : C \to \mathbb{P}^1, \eta : \mathbb{F}_3^{2g} \xrightarrow{\cong} \mathrm{Jac}(C)[3]\big)$$

where π is a superelliptic curve and η is an isometry from $\mathbb{F}_3^{2g}$ with the standard symplectic form to $\mathrm{Jac}(C)[3]$ with the Weil pairing. In other words, a level 3 structure is a choice of an ordered symplectic basis for $\mathrm{Jac}(C)[3]$ compatible with the Weil pairing.

g	m	M	g	m	M
1	3	$\{(3,0)\}$	7	9	$\{(9,0),(6,3)\}$
2	4	$\{(\mathbf{2},\mathbf{2})\}$	8	10	$\{(8,2),(\mathbf{5},\mathbf{5})\}$
3	5	$\{(4,1)\}$	9	11	$\{(10,1),(7,4)\}$
4	6	$\{(6,0),(\mathbf{3},\mathbf{3})\}$	10	12	$\{(12,0),(9,3),(\mathbf{6},\mathbf{6})\}$
5	7	$\{(5,2)\}$	11	13	$\{(11,2),(8,5)\}$
6	8	$\{(7,1),(\mathbf{4},\mathbf{4})\}$	12	14	$\{(13,1),(10,4),(\mathbf{7},\mathbf{7})\}$

Table 1. Examples of the indexing set M for $p = 3$. The values of m which have $A_{\mathsf{m}} \neq S_{m_1} \oplus S_{m_2}$ are shown in bold.

The moduli space of $p = 3$ superelliptic curves with level 3 structure is denoted by $\mathrm{Sup}_g^3[3]$ and comes with a natural morphism

$$\mathsf{F} : \mathrm{Sup}_g^3[3] \to \mathrm{Sup}_g^3$$

which forgets the level 3 structure. The group $\mathrm{Sp}(2g, \mathbb{F}_3)$ acts naturally on $\mathrm{Sup}_g^3[3]$ by changing basis and has no fixed points. The morphism F is then equivalent to taking the quotient by $\mathrm{Sp}(2g, \mathbb{F}_3)$ which shows that F is étale of degree $|\mathrm{Sp}(2g, \mathbb{F}_3)|$.

For each $\mathsf{m} \in \mathsf{M}$, we can extend this concept to the sub-moduli space $\mathrm{Sup}_{g,\mathsf{m}}^3$ (defined via the decomposition in Proposition 3.13) by the following diagram where all squares are Cartesian:

$$\begin{array}{ccc} \mathrm{Sup}_{g,\mathsf{m}}^3[3] & \hookrightarrow & \mathrm{Sup}_g^3[3] \\ \downarrow{\scriptstyle\mathsf{F}_\mathsf{m}} & & \downarrow{\scriptstyle\mathsf{F}} \\ \mathrm{Sup}_{g,\mathsf{m}}^3 & \hookrightarrow & \mathrm{Sup}_g^3. \end{array}$$

5.4 (*A natural basis for* $\mathrm{Jac}(C)[3]$). Consider a moduli point $[P_1, \ldots, P_m] \in M_{0,m}$ and let $[\pi : C \to \mathbb{P}^1]$ be the associated point in $\mathrm{Sup}_{g,\mathsf{m}}^3$. Recall from Section 4 that the divisor classes

$$D_j := [Q_j - Q_m]$$

for $j \in \{1, \ldots, g\}$ form a basis for a natural subspace $\Delta \subset \mathrm{Jac}(C)[3]$. It was further shown in Corollary 4.10 that Δ is a Lagrangian (a maximal isotropic subspace) and hence that Δ is a natural choice of polarisation for the pair $(\mathrm{Jac}(C)[3], w)$ where w is the Weil pairing.

Moreover, using standard techniques, we can use the basis $(D_1, \ldots, D_g)$ to define a basis for the complementary isotropic subspace $\Delta^c \subset \mathrm{Jac}(C)[3]$. In particular, for

each $j \in \{1, \dots, g\}$ the conditions

$$w(D_k, E) = 1, \quad k \neq j, \quad E \in \Delta^c$$

define 1-dimensional subspaces $\Lambda_j \subset \Delta^c$. On top of this, the condition

$$w(D_j, E_j) = e^{\frac{2\pi}{3} i}$$

uniquely defines an element $E_j \in \Lambda_j$. So, by construction, we have a natural symplectic basis for $\mathrm{Jac}(C)[3]$ given by $(D_1, \dots, D_g, E_1, \dots, E_g)$.

5.5 (*Natural morphisms to* $\mathrm{Sup}_{g,\mathrm{m}}^3[3]$). Continuing from the situation in Section 5.4, we take a moduli point $[P_1, \dots, P_m] \in M_{0,m}$ and let $[\pi : C \to \mathbb{P}^1]$ be the associated point in $\mathrm{Sup}_{g,\mathrm{m}}^3$. Using the basis from Section 5.4, we can construct an isometry

$$\eta : \mathbb{F}_3^{2g} \xrightarrow{\cong} \mathrm{Jac}(C)[3]$$

by sending the standard basis $(e_1, \dots, e_g, f_1, \dots, f_g)$ for $\mathbb{F}_3^{2g}$ to the natural basis $(D_1, \dots, D_g, E_1, \dots, E_g)$ for $\mathrm{Jac}(C)[3]$. This then gives rise to a natural morphism

$$\psi_{\mathrm{id}} : M_{0,m} \to \mathrm{Sup}_{g,\mathrm{m}}^3[3]$$

defined by mapping the equivalence class $[P_1, \dots, P_m] \in M_{0,m}$ to the equivalence class $[\pi, \eta]$. Here the subscript id refers to the identity in $\mathrm{Sp}(2g, \mathbb{F}_3)$.

We can also extend this concept to any $A \in \mathrm{Sp}(2g, \mathbb{F}_3)$ by first considering the automorphism $\theta_A : \mathrm{Sup}_{g,\mathrm{m}}^3[3] \to \mathrm{Sup}_{g,\mathrm{m}}^3[3]$ defined by $[\pi, \eta] \mapsto [\pi, T_A \circ \eta]$, where $T_A : \mathbb{F}_3 \to \mathbb{F}_3$ is the linear change of basis associated to A. The morphism $\theta_A \circ \psi_{\mathrm{id}}$ is denoted by

$$\psi_A : M_{0,m} \to \mathrm{Sup}_{g,\mathrm{m}}^3[3].$$

Combining this concept with the forgetful morphism and the isomorphism from Proposition 3.13 gives the following commutative diagram:

$$
\begin{array}{ccc}
M_{0,m} & \xrightarrow{\psi_A} & \mathrm{Sup}_{g,\mathrm{m}}^3[3] \\
\downarrow & & \downarrow{\scriptstyle F_\mathrm{m}} \\
M_{0,m}/A_\mathrm{m} & \xrightarrow{\cong} & \mathrm{Sup}_{g,\mathrm{m}}^3.
\end{array}
\tag{5.1}
$$

5.6 (*Natural group homomorphism defined by* ψ_A). For each $\mathrm{m} = (m_1, m_2) \in \mathrm{M}$, $P \in M_{0,m}$ and $A \in \mathrm{Sp}(2g, \mathbb{F}_3)$, the morphism $\psi_A : M_{0,m} \to \mathrm{Sup}_g^3[3]$ from Section 5.4 defines a group homomorphism which we will denote by

$$\Psi_{P,A} : A_\mathrm{m} \to \mathrm{Sp}(2g, \mathbb{F}_3).$$

This homomorphism can be describe explicitly by initially considering the case $A = $ id. To begin, let $[P_1, \ldots, P_m] \in M_{0,m}$ and $(\pi : C \to \mathbb{P}^1, \eta)$ be the superelliptic curve and the level 3 structure which are given by ψ_{id}.

The homomorphism $\Psi_{P,\mathrm{id}}$ is constructed by considering the action of A_m on $M_{0,m}$ and showing that this defines a change of basis for $\mathrm{Jac}(C)[3]$. A straightforward way of accomplishing this is to consider how the generators of A_m define change of basis matrices for $\mathrm{Jac}(C)[3]$.

First consider the generators of $S_m = S_{m_1} \oplus S_{m_2}$ while considering S_m as a subgroup of S_m. Recall that S_m is generated by the $m - 1$ transpositions $(i, i + 1)$ for $i \in \{1, \ldots, m - 1\}$. In the case at hand, S_m is generated by the collection $(S_{m_1} \oplus S_{m_2}) \cap \{(i, i + 1)\}$. We can now consider how these permutations define changes of basis.

(i)   For $i \notin \{m - 1, m - 2\}$: The change of basis $\Psi_{P,\mathrm{id}}(\sigma)$ is defined on each D_j by considering the action of S_m on $\{Q_1, \ldots, Q_m\}$. The result is

$$\Psi_{P,\mathrm{id}}(\sigma) \cdot D_j = \begin{cases} D_j & \text{for } j \notin \{i, i + 1\}, \\ D_{i+1} & \text{for } j = i, \\ D_i & \text{for } j = i + 1, \end{cases}$$

which is the usual "*row-swap*" elementary row-operation.

(ii)   For $\sigma := (g, g + 1) = (m - 2, m - 1)$: The generator $\sigma := (g, g + 1)$ exists in the cases $m_2 = 0$, $m_2 = 1$ and $m_2 > 3$. In these cases we have

$$\Psi_{P,\mathrm{id}}(\sigma) \cdot D_j = \begin{cases} D_j & \text{for } j \neq g, \\ D_{m-1} & \text{for } j = g. \end{cases}$$

To see the associated change of basis, we recall from Section 4.3 that the natural horizontal principal divisor gives:

(a)   For $m_2 = 0$: $D_{m-1} \sim 2 \sum_{i=1}^{m-2} D_i$.

(b)   For $m_2 = 1$: $D_{m-1} \sim \sum_{i=1}^{m-2} D_i$.

(c)   For $m_2 \geq 2$: $D_{m-1} \sim \sum_{i=1}^{m_1} D_i + 2 \sum_{i=m_1+1}^{m-2} D_i$.

(iii)   For $\sigma := (g + 1, g + 2) = (m - 1, m)$: The generator $\sigma := (g, g + 1)$ exists in the cases $m_2 = 0$ and $m_2 > 2$. In these cases we have the same change of basis and relations as those in part (ii) except

$$\Psi_{P,\mathrm{id}}(\sigma) \cdot D_j = D_j + 2D_{m-1}.$$

We now consider the extra generators in the case when $\mathrm{Stab}_p(\mathrm{m}) \neq 0$. This case occurs when $\frac{m}{2} \in \mathbb{Z}_{>0}$ and $\mathrm{m} = (\frac{m}{2}, \frac{m}{2})$ so that $\mathrm{Stab}_p(\mathrm{m}) \cong S_2$. Let ζ be the gener-

ator of $\mathrm{Stab}_p(m)$ then we have

$$\Psi_{P,\mathrm{id}}(\zeta) \cdot D_j = \begin{cases} D_{j+\frac{m}{2}} + 2D_{\frac{m}{2}} & \text{for } 1 \le j < \frac{m}{2} - 1, \\ D_{m-1} + 2D_{\frac{m}{2}} & \text{for } j = \frac{m}{2} - 1, \\ 2D_{\frac{m}{2}} & \text{for } j = \frac{m}{2}, \\ D_{j-\frac{m}{2}} + 2D_{\frac{m}{2}} & \text{for } \frac{m}{2} < j \le m - 2, \end{cases}$$

where D_{m-1} has been calculated above in (ii).

Hence, we have defined a homomorphism $\Psi_{P,\mathrm{id}} : \mathsf{A}_m \to \mathrm{Sp}(2g, \mathbb{F}_3)$. Now, consider the automorphism $\Theta_A : \mathrm{Sp}(2g, \mathbb{F}_3) \to \mathrm{Sp}(2g, \mathbb{F}_3)$ which is defined by $B \mapsto AB$. The group homomorphism $\Psi_{P,A}$ is now defined by $\Psi_{P,A} := \Theta_A \circ \Psi_{P,\mathrm{id}}$.

Lemma 5.7. *Let $A \in \mathrm{Sp}(2g, \mathbb{F}_3)$ and $P \in M_{0,m}$. The morphism ψ_A from Section 5.5 and the group homomorphism $\Psi_{P,A}$ from Section 5.6 are injective.*

Proof. Since for $A \ne \mathrm{id}$, the morphisms ψ_A and Ψ_A are defined by post-composing with automorphisms, we are only required to consider the case when $A = \mathrm{id}$.

To consider whether ψ_{id} is injective take two points in $P, P' \in M_{0,m}$ which have $\psi_{\mathrm{id}}(P) = \psi_{\mathrm{id}}(P')$. We also let $D_1, \ldots, D_g$ and $D'_1, \ldots, D'_g$ be the associated natural bases constructed from divisors in Section 4.5 and corresponding to the images $\psi_{\mathrm{id}}(P)$ and $\psi_{\mathrm{id}}(P')$.

The combination of the condition $\psi_{\mathrm{id}}(P) = \psi_{\mathrm{id}}(P')$ and the fact that the diagram (5.1) commutes shows that P and P' must be in the same A_m-class. Taking $\sigma \in \mathsf{A}_m$ to be such that $P' = \sigma \cdot P$, we observe that this implies $D'_i = \Psi_{P,\mathrm{id}}(\sigma) \cdot D_i$ for all $i \in \{1, \ldots, g\}$. The condition $\psi_{\mathrm{id}}(P) = \psi_{\mathrm{id}}(P')$ is now only true if $D_i = \Psi_{P,\mathrm{id}}(\sigma) \cdot D_i$ for all $i \in \{1, \ldots, g\}$. Hence we have shown that ψ_{id} is injective only if $\Psi_{P,\mathrm{id}}$ is injective for all $P \in M_{0,m}$.

Let $\sigma \in \mathsf{A}_m$ be such that $\Psi_{P,\mathrm{id}}(\sigma) \cdot D_i = D_i$ for all $i \in \{1, \ldots, g\}$. The relations given in Section 4.5 show that $\Psi_{P,\mathrm{id}}(\sigma) \cdot D_{g+1} = D_{g+1}$ as well as $\Psi_{P,\mathrm{id}}(\sigma) \cdot D_{g+2} = D_{g+2} = 0$. Now, let $n \in \{1, \ldots, m\}$ be $n := \sigma^{-1}(m)$. Then the condition $\Psi_{P,\mathrm{id}}(\sigma) \cdot D_n = D_n$ implies

$$Q_n - Q_m \sim Q_m - Q_{\sigma(n)}$$

and hence $D_n = -D_{\sigma(n)}$. This is not one of the relations from Section 4.5, which are shown to be all the relations in Theorem 4.6, so we must have $D_n = 0$ and hence $n = m$. So now, for any $i \in \{1, \ldots, g\}$, the condition $\Psi_{P,\mathrm{id}}(\sigma) \cdot D_i = D_i$ implies

$$Q_i - Q_m \sim Q_{\sigma(i)} - Q_m$$

and $D_i = D_{\sigma(i)}$. Again, the relations from Section 4.5 and Theorem 4.6 show that this implies $\sigma(i) = i$ for any $i \in \{1, \ldots, g\}$. Hence we have that $\sigma = \mathrm{id}$. Thus, we have shown that $\Psi_{P,\mathrm{id}}$ has trivial kernel and so is injective. ∎

Lemma 5.8. *For $A \in \mathrm{Sp}(2g, \mathbb{F}_3)$ the morphism ψ_A is an isomorphism onto an irreducible component of $\mathrm{Sup}^3_{g,\mathrm{m}}[3]$.*

Proof. Denote the quotient map by $\mathsf{q}_\mathrm{m} : M_{0,m} \to M_{0,m}/\mathsf{A}_\mathrm{m}$. We have that both q_m and the morphism F_m forgetting the level 3 structure are finite. Hence we have that ψ_A is finite as well (see, for example, [36, Tag 01WJ & Tag 035D]). Thus, since Lemma 5.7 shows that ψ_A is injective, we have that it is a closed immersion (see, for example, [36, Tag 03BB]).

Since the quotient map q_m and F_m are both finite and surjective, we have that $M_{0,m}$ and $\mathrm{Sup}^3_{g,\mathrm{m}}[3]$ are both the same dimension (see, for example, [36, Tag 01WJ & Tag 0ECG]). Now, let $\mathcal{A} \subseteq \mathrm{Sup}^3_{g,\mathrm{m}}$ be the irreducible component that contains the image of ψ_A, and let $\varepsilon : M_{0,m} \to \mathcal{A}$ be the associated morphism. Since $M_{0,m}$ is irreducible, we must have that ε is surjective.

On top of this, both q_m and F_m are étale, so by the vanishing of cotangent complexes we have that ε is étale as well. In conclusion, ε is a surjective flat closed immersion and hence ε is an isomorphism (see [36, Tag 04PW]). ∎

Corollary 5.9. *Let $x = [\pi] \in \mathrm{Sup}^3_g$ be a geometric point and let $A \in \mathrm{Sp}(2g, \mathbb{F}_3)$. If $[\pi, \eta]$ and $[\pi, \eta']$ are both in the image $\mathrm{Im}\, \psi_A$ then there is a unique $B \in \mathsf{A}_\mathrm{m}$ such that $\eta' = T_B \circ \eta$ (where T_B is defined as in Section 5.5). Moreover, we have that*

$$|\mathrm{Im}\, \psi_A \cap \mathsf{F}_\mathrm{m}^{-1}(x)| = |\mathsf{A}_\mathrm{m}|.$$

Proof. This follows from Lemma 5.8 since ψ_A is an isomorphism on an irreducible component of $\mathrm{Sup}^3_{g,\mathrm{m}}[3]$ and the morphism $M_{0,m} \to \mathrm{Sup}^3_{g,\mathrm{m}}$ described by diagram (5.1) is the fixed-point-free quotient of $M_{0,m}$ by A_m. ∎

Lemma 5.10. *Let $A, A' \in \mathrm{Sp}(2g, \mathbb{F}_3)$. If the images $\mathrm{Im}\, \psi_A$ and $\mathrm{Im}\, \psi_{A'}$ are not disjoint, then $\mathrm{Im}\, \psi_A = \mathrm{Im}\, \psi_{A'}$.*

Proof. Suppose that the images $\mathrm{Im}\, \psi_A$ and $\mathrm{Im}\, \psi_{A'}$ are not disjoint and take a point $[\pi, \eta] \in \mathrm{Sup}^3_g[3]$ in the intersection. Then, by the construction of ψ_A and $\psi_{A'}$, we must have that $\eta = T_A \circ \tau$ and $\eta = T_{A'} \circ \tau'$ for points $[\pi, \tau], [\pi, \tau'] \in \mathrm{Im}\, \psi_{\mathrm{id}}$.

Hence by Corollary 5.9 we have that there is a $B \in \mathsf{A}_\mathrm{m}$ such that $\tau' = T_B \circ \tau$. Combining that with the equality $\eta = T_A \circ \tau = T_{A'} \circ \tau'$ we see that $T_A = T_{A'} \circ T_B$ and hence $A = A'B$.

Now, by the definition of ψ_A we have that $\psi_A = \Theta_{A'} \circ \psi_B$ and $\psi_{A'} = \Theta_{A'} \circ \psi_{\mathrm{id}}$. The result now follows from the observation that $\mathrm{Im}\, \psi_{\mathrm{id}} = \mathrm{Im}\, \psi_B$. ∎

Lemma 5.11. *The scheme $\mathrm{Sup}^3_{g,\mathrm{m}}[3]$ has $|\mathrm{Sp}(2g, \mathbb{F}_3)|/|\mathsf{A}_\mathrm{m}|$ connected components. Each connected component is irreducible and is isomorphic to $M_{0,m}$.*

Proof. We begin by claiming that the morphism

$$\bigsqcup_{A \in \mathrm{Sp}(2g,\mathbb{F}_3)} \psi_A : \bigsqcup_{A \in \mathrm{Sp}(2g,\mathbb{F}_3)} M_{0,m} \to \mathrm{Sup}_g^3[3]$$

is surjective. So see this, take any geometric point $[\pi : C \to \mathbb{P}^1, \eta'] \in \mathrm{Sup}_g^3[3]$ and consider the point $[\pi] \in \mathrm{Sup}_g^3$. We know from diagram (5.1) that the morphism $M_{0,m} \to \mathrm{Sup}_g^3$ is surjective, so let $P \in M_{0,m}$ be any point in the preimage of $[\pi]$ under this morphism. We now have that $\psi_{\mathrm{id}}(P) = [\pi, \eta]$ for some isometry $\eta : \mathbb{F}_3^{2g} \to \mathrm{Jac}(C)[3]$. Considering the composition $\eta' \circ \eta^{-1} \circ \eta = \eta'$, we observe that $\eta' \circ \eta^{-1}$ corresponds to a symplectic change of basis $\mathbb{F}_3^{2g} \to \mathbb{F}_3^{2g}$ and hence $\eta' \circ \eta^{-1} = T_A$ for some $A \in \mathrm{Sp}(2g, \mathbb{F}_3)$. This proves the claimed surjectivity.

Let $\mathcal{B} \subset \mathrm{Sp}(2g, \mathbb{F}_3)$ have the following properties:

(i) $\bigsqcup_{A \in \mathcal{B}} \psi_A$ is surjective.

(ii) If $\mathcal{B}' \subset \mathrm{Sp}(2g, \mathbb{F}_3)$ has the property that $\bigsqcup_{A \in \mathcal{B}'} \psi_A$ is surjective then we have $|\mathcal{B}| \leq |\mathcal{B}'|$.

Such a $\mathcal{B}$ will always exist but may not be unique. Moreover, combining property (ii) with Lemma 5.10 we must have that for each pair $A, A' \in \mathcal{B}$ with $A \neq A'$ that the images $\mathrm{Im}\,\psi_A$ and $\mathrm{Im}\,\psi_{A'}$ are disjoint.

Now, letting $x \in \mathrm{Sup}_g^3$ be a geometric point, we have that

$$\mathsf{F}_m^{-1}(x) = \bigsqcup_{A \in \mathcal{B}} (\mathrm{Im}\,\psi_A \cap \mathsf{F}_m^{-1}(x)).$$

Hence using Corollary 5.9 and the fact that $|\mathsf{F}_m^{-1}(x)| = |\mathrm{Sp}(2g, \mathbb{F}_3)|$ we have

$$|\mathrm{Sp}(2g, \mathbb{F}_3)| = \sum_{A \in \mathcal{B}} |\mathrm{Im}\,\psi_A \cap \mathsf{F}_m^{-1}(x)| = |\mathcal{B}| \cdot |\mathsf{A}_m|.$$

Since $|\mathcal{B}|$ is the number of connected components, the desired result now follows. ∎

Theorem 1.1 now follows immediately from Lemma 5.11 and the description of the indexing set M from Section 5.2. We also summarize what we have obtained in an alternative way to highlight the analogy with Theorem 2.18.

Theorem 5.12 (*Connected components of* $\mathrm{Sup}_g^3[3]$). *For* $m \in \mathsf{M}$, *denote the quotient set* $\mathrm{Sp}(2g, \mathbb{F}_p)/\mathsf{A}_m$ *by* $\mathcal{C}_m$. *Then, if* X_c *denotes a copy of* $M_{0,m}$ *for each* $c \in \mathcal{C}_m$, *there is an isomorphism of schemes*

$$\mathrm{Sup}_g^3[3] \xrightarrow{\cong} \bigsqcup_{m \in \mathsf{M}} \bigsqcup_{c \in \mathcal{C}_m} X_c.$$

In particular, $\mathrm{Sup}_g^3[3]$ *is smooth.*

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