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Asymptotic analysis and optimization of an elastic body surrounded by thin layers

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Abstract. We consider an elastic body surrounded by thin elastic layers along a part of its boundary. We study the asymptotic behavior of the structure as the maximum thickness of the layers tends to zero. We derive an effective boundary integral energy involving a matrix of Borel measures not charging polar sets and having the same support contained in the boundary. We characterize this matrix for three special cases: periodic layers, layers which are determined by a given nonnegative function h, and layers with abrupt changes along self similar fractals. We then consider an optimal control problem, which consists in determining the shape of the best material distribution around the elastic body, under the maximal work of external loads, and characterize the optimal zones on its boundary where possible elastic layers could take place.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_1|, |\Gamma_2| > 0$, where $|\Gamma_i|$; $i = 1, 2$, denotes the Lebesgue measure of Γ_i. Let Σ_ε be an arbitrary layer of maximum thickness $\varepsilon > 0$ extending Ω near Γ_1 (see Figure 1). Without loss of generality, we may suppose that, for small parameter $\varepsilon \in (0, 1)$,

$$\Sigma_\varepsilon = \{s + tn(s);\ s \in \Gamma_1,\ 0 < t < \varepsilon h_\varepsilon(s)\},$$

where $n(s)$ is the outward unit normal on $s \in \Gamma_1$ and h_ε is a positive locally Lipschitz continuous function satisfying

$$\sup_\varepsilon \|h_\varepsilon\|_{L^\infty(\Gamma_1)} \leq 1.$$

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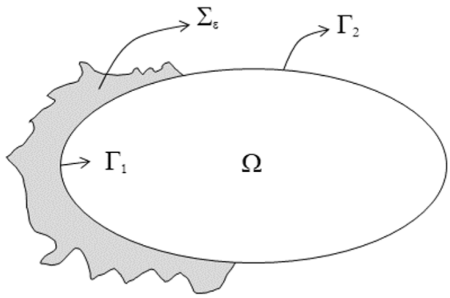

Figure 1. A layer Σ_ε extending the set Ω along the part Γ_1 of its boundary.

We set

$$\Omega_\varepsilon = \Omega \cup \Gamma_1 \cup \Sigma_\varepsilon,$$
$$\Gamma_{1,\varepsilon} = \partial\Sigma_\varepsilon \backslash \partial\Omega.$$

We suppose that Ω is the reference configuration of a linear elastic material. This means that the deformation tensor $e(u) = (e_{ij}(u))_{i,j=1,2,3}$, with

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

for some displacement u, is linked to the stress tensor $\sigma(u) = (\sigma_{ij}(u))_{i,j=1,2,3}$ by Hooke's law,

$$\sigma_{ij}(u) = a_{ijkl} e_{kl}(u); \quad i,j = 1,2,3, \tag{1.1}$$

where a_{ijkl}; $i,j,k,l = 1,2,3$, are material coefficients and where the summation convention with respect to repeated indices has been used and will be used in the sequel. We suppose that

$$a_{ijkl}(x) = a_{jikl}(x) = a_{ljki}(x), \quad \forall i,j,k,l = 1,2,3,\ \forall x \in \Omega, \tag{1.2a}$$
$$c_1\xi_{ij}\xi_{ij} \leq a_{ijkl}(x)\xi_{ij}\xi_{ij} \leq c_2\xi_{ij}\xi_{ij}, \quad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^{3\times3}, \tag{1.2b}$$

where c_1 and c_2 are positive constants. We suppose that Σ_ε is the reference configuration of a linear elastic material with material coefficients εa_{ijkl}; $i,j,k,l = 1,2,3$. We suppose that a perfect adhesion occurs between Ω and Σ_ε along their common interface Γ_1. We suppose that the material in Ω_ε is submitted to volumic forces with density $f \in L^2(\mathbb{R}^3,\mathbb{R}^3)$. We define the sequence of functionals $(\mathcal{F}_\varepsilon)_\varepsilon$ on $L^2(\mathbb{R}^3,\mathbb{R}^3)$ by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int_\Omega \sigma_{ij}(u)e_{ij}(u)dx + \varepsilon\int_{\Sigma_\varepsilon} \sigma_{ij}(u)e_{ij}(u)dx & \text{if } u \in H_0^1(\Omega_\varepsilon,\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3}$$

The equilibrium state of the elastic material in Ω_ε is described by the minimization problem

$$\min_{u\in L^2(\Omega_\varepsilon,\mathbb{R}^3)}\left\{\mathcal{F}_\varepsilon(u)-2\int_\Omega f.u\,dx\right\}. \tag{1.4}$$

Using Γ-convergence methods (see, for instance, [16] and [18]), we prove that the effective potential energy of the material turns out to be of the form

$$\mathcal{F}_0(u)=\begin{cases}\int_\Omega \sigma_{ij}(u)e_{ij}(u)dx+\int_{\Gamma_1}u_iu_j\,d\mu_{ij} & \text{if } u\in H_{\boldsymbol{\mu},\Gamma_2}(\Omega,\mathbb{R}^3),\\ +\infty & \text{otherwise,}\end{cases} \tag{1.5}$$

where $\boldsymbol{\mu}=(\mu_{ij})_{i,j=1,2,3}$ is a symmetric matrix of Borel measures μ_{ij} not charging polar sets (sets of capacity zero), having the same support contained in Γ_1, and satisfying $\mu_{ij}(B)\zeta_i\zeta_j\geq 0$, for every Borel set $B\subset\mathbb{R}^3$ and every $\zeta\in\mathbb{R}^3$, and where

$$H_{\boldsymbol{\mu},\Gamma_2}(\Omega,\mathbb{R}^3)=H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)\cap L^2_{\boldsymbol{\mu}}(\Gamma_1,\mathbb{R}^3), \tag{1.6}$$

with

$$H^1_{\Gamma_2}(\Omega_\varepsilon,\mathbb{R}^3)=\{v\in H^1(\Omega_\varepsilon,\mathbb{R}^3);\ v=0 \text{ on } \Gamma_2\},$$

and

$$L^2_{\boldsymbol{\mu}}(\Gamma_1,\mathbb{R}^3)=\left\{v:\Gamma_1\to\mathbb{R}^3;\ \int_{\Gamma_1}v_iv_j\,d\mu_{ij}<+\infty\right\}.$$

The solution u^0 of the limit problem, stated in Corollary 15, satisfies the following Robin type boundary condition:

$$\sigma_{ij}(u^0)n_j+\mu_{ij}u_i^0=0 \quad \text{on } \Gamma_1,$$

where n is the outward unit normal on $s\in\Gamma_1$. We then consider some special cases. We first consider the case where the thickness of Σ_ε varies periodically along Γ_1. The problem becomes invariant by translation and the measure μ_{ij}; $i,j=1,2,3$, is the Haar measure on Γ_1 with $\mu_{ij}=K_{ij}ds$, where ds is the surface measure on Γ_1 given by the Riemannian metric and K_{ij}; $i,j=1,2,3$, are constants in $\overline{\mathbb{R}}$ satisfying $K_{ij}\zeta_i\zeta_j\geq 0$, $\forall\zeta\in\mathbb{R}^3$. We identify the constants K_{ij}; $i,j=1,2,3$, by constructing appropriate local problems. We secondly suppose that Σ_ε has thickness of the form $\varepsilon h(s)$. We prove in this case that $\mu_{ij}=\kappa_i(s)\frac{ds}{h(s)}\delta_{ij}$, where δ_{ij} denotes Kronecker's symbol and $\kappa_i(s)$; $i=1,2,3$, are material coefficients. We then suppose that Γ_1 is contained in the plane $\{x_3=0\}$ and consider a thin layer Σ_ε with abrupt changes along a self similar fractal Λ with similarity dimension d. We prove that

$$\mu_{ij}=\left(\kappa_i(x_1,x_2)dx_1dx_2|_{\Gamma_1}+\frac{2\pi(c-1/2)\kappa_i(s)}{\mathcal{H}^d(\Lambda)}d\mathcal{H}^d(s)|_\Lambda\right)\delta_{ij},$$

where c is a positive constant given in Theorem 19, $d\mathcal{H}^d$ is the d-dimensional Hausdorff measure, and $\kappa_i(s)$; $i = 1, 2, 3$, are material coefficients.

The asymptotic behavior of the scalar version of Ω surrounded by arbitrary layers Σ_ε of maximum thickness ε was studied in [11]. A general integral on the boundary $\partial\Omega$ written as $\int_{\partial\Omega} u^2 d\mu$ where μ is a nonnegative Borel measure on $\mathbb{R}^3$ not charging polar sets (but possibly $+\infty$ on large subsets of $\mathbb{R}^3$) was obtained at the limit. The characterization of the measure μ was given in terms of suitable asymptotic capacities associated with $\Omega_\varepsilon \backslash \Omega$. The asymptotic behavior for an incompressible viscous flow in Ω surrounded by arbitrary thin layers Σ_ε has been addressed in [19]. A general Navier wall law was obtained, with the proportionality coefficient being a symmetric matrix of Borel measures, having their supports contained in the solid boundary of Ω. Several papers, among which [1,2,5–7,12,14], and [26], have studied the asymptotic behavior of elliptic operators in domains surrounded by thin layers of periodically varying thickness $\varepsilon h(s)$ or with general smooth thickness $h^\varepsilon \leq \varepsilon$.

An important field to which this work is closely related is the asymptotic behavior of a biological body surrounded by thin layers of soft growing tissues resulting from the proliferation of tumor cells. Constitutive models combining the stress-strain relation of linear elasticity with a growth term of avascular tumors have been developed in several papers (see, for instance, [3,4,24,29]). This analysis provides an asymptotic description of stresses in soft tissues growing around a biological body.

This problem has also some implications for modeling the behavior of elastic bodies reinforced with flexible or soft thin elastic layers such as rubber and textile. Some of these materials are indeed known for their nonlinearity in the stress-strain relationship. However, as the reinforced body is modeled as linear elastic with small strains, linear elasticity is assumed here for the material within the layers. Note that materials reinforced with flexible materials as textile are used for the construction of durable and more sustainable elastic structures (see, for instance, [27] and [28]). For two-dimensional plastic layers under longitudinal shear obeying a hardening stress-strain law with a functional energy given by

$$\varepsilon^{p-1} \int_{\Sigma_\varepsilon} |\nabla w(x,y)|^p dx dy,$$

where p is the power hardening parameter and w is the only non-vanishing component of displacement (assumed to be the z-component), we can prove, using the present work and the integral representation theorem of [15], that the effective energy is given by $\int_{\Gamma_1} |w(x,y)|^p d\mu$ where μ is a nonnegative Borel measure on $\mathbb{R}^3$ not charging polar sets.

In Section 6, we consider an optimal control problem which consists in determining the shape of the best material distribution around Ω under the maximal work of external loads. We prove that, for a given quantity $\eta > 0$ of material, there exists an optimal diagonal matrix $\boldsymbol{h} = \mathrm{Diag}(h_i)_{i=1,2,3}$ of Γ_1-measurable functions $h_i : \Gamma_1 \to [0, +\infty)$, such that

$$h_i(u^\eta) = \eta \frac{|u_i^\eta|}{\int_{\Gamma_1} |u_i^\eta| ds},$$

where $u^\eta = (u_1^\eta, u_2^\eta, u_3^\eta)$ is the solution of problem (6.4). We then study the best way to reinforce an elastic body by a flexible elastic layer as η tends to zero. For a biological body, this last study allows to characterize the zones on its boundary where possible soft tissues will grow.

We recall that the scalar version of this problem was investigated as part of the shape optimization of optimal thermal insulators by several authors (see, for instance, [9, 10, 20], and recently [8] and [22]). The problem of optimizing the distribution of material, surrounding a homogeneous elastic plate, which minimizes the energy has been studied in [13].

2. Functional framework

We define the capacity Cap of every compact subset $K \subset \mathbb{R}^3$ as

$$\mathrm{Cap}(K) := \inf\left\{ \int_{\mathbb{R}^N} |\nabla\varphi|^2 dx + \int_{\mathbb{R}^N} |\varphi|^2 dx;\ \varphi \in C_c^\infty(\mathbb{R}^3),\ \varphi \geq 1 \text{ on } K \right\}.$$

For every open subset $U \subset \mathbb{R}^3$, we set

$$\mathrm{Cap}(U) := \sup\left\{ \mathrm{Cap}(K);\ K \subset U, K \text{ compact} \right\}.$$

For every Lebesgue measurable subset $B \subset \mathbb{R}^3$, we define

$$\mathrm{Cap}(B) := \inf\left\{ \mathrm{Cap}(U);\ B \subset U, U \text{ open} \right\}.$$

Let $\mathcal{B}(\mathbb{R}^3)$ be the σ-field of Lebesgue measurable subsets of $\mathbb{R}^3$. A property is said to be true quasi-everywhere (q.e.) on $B \in \mathcal{B}(\mathbb{R}^3)$ if it is true except on a subset of B of capacity Cap equal to 0. A function $u : B \to \overline{\mathbb{R}}$; $B \in \mathcal{B}(\mathbb{R}^3)$, is quasi-continuous on B if, for every $\varepsilon > 0$, there exists an open subset $U_\varepsilon \subset B$ with $\mathrm{Cap}(U_\varepsilon) < \varepsilon$, such that the restriction of u to $B \setminus U_\varepsilon$ is continuous. Every function $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ has a quasi-continuous representant $\tilde{u}$, which is unique quasi-everywhere in Ω, (see, for instance, [30, Theorem 3.1.4]); $\tilde{u}$ is given by

$$\tilde{u}(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) dy \quad \text{for q.e. } x \in \mathbb{R}^3,$$

where $B(x, r)$ is the ball centered at x and of radius $r > 0$. We now define some notions concerning families of subsets of $\mathbb{R}^3$ (see, for instance, [16, Chapter 14]) and a class of functionals of $\mathbb{R}^3$ (see, for instance, [17]).

Definition 1. (1) A subset $\mathcal{D} \subset \mathcal{B}(\mathbb{R}^3)$ is a dense family in $\mathcal{B}(\mathbb{R}^3)$ if, for every $A, B \in \mathcal{B}(\mathbb{R}^3)$ with $\overline{A} \subset \mathring{B}$, there exists $D \in \mathcal{D}$ such that: $\overline{A} \subset \mathring{D} \subset \overline{D} \subset \mathring{B}$, where $\mathring{A}$ (resp. $\overline{A}$) denotes the interior (resp. the closure) of A.

(2) A subset $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$ is a rich family if, for every family $(A_t)_{t\in]0,1[} \subset \mathcal{B}(\mathbb{R}^3)$ such that $\overline{A}_s \subset \mathring{A}_t$, for every $s < t$, the set $\{t \in]0,1[;\ A_t \notin \mathcal{R}\}$ is at most countable.

(3) Let $\mathcal{O}(\mathbb{R}^3)$ be the set of all open subsets of Ω. We consider the class $\mathbb{F}$ of functionals F from $H^1(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{O}(\mathbb{R}^3)$ to $[0, +\infty]$ satisfying:

(a) (*lower semi-continuity*): for every open subset $\omega \in \mathcal{O}(\mathbb{R}^3)$, the functional $u \mapsto F(u, \omega)$ is lower semi-continuous with respect to the strong topology of the space $H^1(\mathbb{R}^3; \mathbb{R}^3)$,

(b) *(measure property)*: for every $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, $\omega \mapsto F(u, \omega)$ is the restriction to $\mathcal{O}(\mathbb{R}^3)$ of a nonnegative Borel measure still denoted $F(u, \omega)$,

(c) (*localization*): for every $\omega \in \mathcal{O}(\mathbb{R}^3)$ and every $u, v \in H^1(\mathbb{R}^3, \mathbb{R}^3)$: $u|_\omega = v|_\omega \Rightarrow F(u, \omega) = F(v, \omega)$,

(d) (C^1-*convexity*): for every $\omega \in \mathcal{O}(\mathbb{R}^3)$, the functional $u \mapsto F(u, \omega)$ is convex on $H^1(\mathbb{R}^3, \mathbb{R}^3)$ and for every $\varphi \in C^1(\mathbb{R}^3)$ with $0 \leq \varphi \leq 1$,

$$F(\varphi u + (1 - \varphi)v, \omega) \leq F(u, \omega) + F(v, \omega).$$

Example 2. We consider the functional F_ε defined on $H^1(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{O}(\mathbb{R}^3)$ by

$$F_\varepsilon(u, \omega) = \begin{cases} 0 & \text{if } \widetilde{u} = 0, \text{ q.e. on } \Gamma_{1,\varepsilon} \cap \omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.1}$$

then $F_\varepsilon \in \mathbb{F}$.

Let us set the following definitions.

Definition 3. (1) A Borel measure λ is absolutely continuous with respect to the capacity Cap if

$$\forall B \in \mathcal{B}(\mathbb{R}^3) : \mathrm{Cap}(B) = 0 \Rightarrow \lambda(B) = 0,$$

(2) $\mathcal{M}_0(\mathbb{R}^3)$ is the set of nonnegative Borel measures which are absolutely continuous with respect to the capacity Cap.

Example 4. For every $E \subset \Omega$ such that $\mathrm{Cap}(E) > 0$, we define the measure ∞_E by

$$\infty_E(B) = \begin{cases} 0 & \text{if } \mathrm{Cap}(B \cap E) = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

for every $B \in \mathcal{B}(\mathbb{R}^3)$. Then ∞_E belongs to $\mathcal{M}_0(\Omega)$.

Note that, for every $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ and every $\omega \in \mathcal{O}(\mathbb{R}^3)$, the functional F_ε defined in (2.1) can be written as

$$F_\varepsilon(u, \omega) = \int_\omega |\tilde{u}|^2 d\infty_{\Gamma_{1,\varepsilon}} = \int_\omega |u|^2 d\infty_{\Gamma_{1,\varepsilon}}. \tag{2.2}$$

We have the following integral representation of functionals of $\mathbb{F}$.

Theorem 5 (See [17, Theorem 7.5]). *For every $F \in \mathbb{F}$, there exist a finite measure $\mu \in \mathcal{M}_0(\mathbb{R}^3)$, a nonnegative Borel measure ν, and a Borel function $g : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, +\infty]$ with $\zeta \mapsto g(x, \zeta)$ convex and lower semi-continuous on $\mathbb{R}^3$, such that, for every $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ and every $\omega \in \mathcal{O}(\mathbb{R}^3)$,*

$$F(u, \omega) = \int_\omega g(x, \tilde{u}(x)) d\mu + \nu(\omega).$$

Moreover, if F is quadratic then the following corollary (see [17, Corollary 8.4]) holds.

Corollary 6. *Let $F \in \mathbb{F}$. Assume that $F(., \omega)$ is quadratic for every $\omega \in \mathcal{O}(\mathbb{R}^3)$. Then, there exist*

(1) *a finite measure $\mu \in \mathcal{M}_0(\mathbb{R}^3)$,*

(2) *a symmetric matrix $(a_{ij})_{i,j=1,2,3}$, of Borel functions from $\mathbb{R}^3$ to $\mathbb{R}$ satisfying $a_{ij}(x)\zeta_i\zeta_j \geq 0$ for every $\zeta \in \mathbb{R}^3$ and for q.e. $x \in \mathbb{R}^3$,*

(3) *for every $x \in \mathbb{R}^3$, a linear subspace $V(x)$ of $\mathbb{R}^3$, such that, for every $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ and every $\omega \in \mathcal{O}(\mathbb{R}^3)$,*

   (a) *if $F(u, \omega) < +\infty$ then $u(x) \in V(x)$ for q.e. $x \in \mathbb{R}^3$,*

   (b) *if $u(x) \in V(x)$, for q.e. $x \in \mathbb{R}^3$, then*

$$F(u, \omega) = \int_\omega a_{ij}(x) u_i(x) u_j(x) d\mu(x).$$

Remark 7. Let $F \in \mathbb{F}$ such that $F(., \omega)$ is quadratic for every $\omega \in \mathcal{O}(\mathbb{R}^3)$, $\mu \in \mathcal{M}_0(\mathbb{R}^3)$ be the associated measure by the above corollary,

$$\Theta = \bigcup_{\omega \in \mathcal{S}(F)} \omega, \tag{2.3}$$

where

$$\mathcal{S}(F) = \{\omega \in \mathcal{O}(\mathbb{R}^N); F(., \omega) < +\infty \text{ q.e. in } \omega\},$$

and $V(x)$ be the linear subspace of $\mathbb{R}^3$ defined by

$$V(x) = \begin{cases} \mathbb{R}^3 & \text{if } x \in \Theta, \\ \{0\} & \text{if } x \in \mathbb{R}^3 \backslash \Theta. \end{cases}$$

We define the 3×3 matrix of measures

$$\begin{aligned}\boldsymbol{\mu} &= (\mu_{ij})_{i,j=1,2,3} \\ &= (a_{ij}\mu)_{i,j=1,2,3} + \infty_{\mathbb{R}^3\setminus\Theta}\mathrm{Id}_3.\end{aligned}$$

Then, for every $u \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ and every $\omega \in \mathcal{O}(\mathbb{R}^3)$,

$$\int_\omega u_i u_j \, d\mu_{ij} = \begin{cases} \int_\omega a_{ij} u_i u_j \, d\mu & \text{if } u(x) \in V(x) \text{ for q.e. } x \in \omega, \\ +\infty & \text{otherwise,} \end{cases}$$

and the functional F can be written as

$$F(u, \omega) = \int_\omega u_i u_j \, d\mu_{ij} = \langle \boldsymbol{\mu} u, u \rangle.$$

3. A priori estimates

We define the sequence of functionals $(\Phi_\varepsilon)_\varepsilon$ on $L^2(\mathbb{R}^3, \mathbb{R}^3)$ by

$$\Phi_\varepsilon(u) = \begin{cases} \int_\Omega \sigma_{ij}(u) e_{ij}(u) dx + \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(u) e_{ij}(u) dx & \text{if } u \in H^1_{\Gamma_2}(\Omega_\varepsilon, \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

We have the following results.

Proposition 8. *Let* $u^\varepsilon \in H^1_{\Gamma_2}(\Omega_\varepsilon, \mathbb{R}^3)$ *such that* $\sup_\varepsilon \Phi_\varepsilon(u^\varepsilon) < +\infty$. *Then*

(1) $\sup_\varepsilon(\int_\Omega |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx) < +\infty$,

(2) $\sup_\varepsilon \int_\Omega |u^\varepsilon|^2 dx < +\infty$ *and* $\sup_\varepsilon \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} |u^\varepsilon|^2 dx < +\infty$.

Proof. (1) Let $s + tn(s) \in \Sigma_\varepsilon$. We have

$$\begin{aligned}(u_i^\varepsilon(s + tn(s)) - u_i^\varepsilon(s))^2 &= \left| \int_0^t \nabla u_i^\varepsilon(s + \zeta n(s)).n(s) d\zeta \right|^2 \\ &\leq \varepsilon \int_0^{\varepsilon h_\varepsilon(s)} |\nabla u_i^\varepsilon(s + \zeta n(s))|^2 d\zeta,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{\Gamma_1} \int_0^{\varepsilon h_\varepsilon(s)} |u_i^\varepsilon(s + \zeta n(s))|^2 d\zeta ds \\ &\quad\leq C\Bigg(\varepsilon \int_{\Gamma_1} (u_i^\varepsilon(s))^2 ds \\ &\qquad + \varepsilon^2 \int_{\Gamma_1} \int_0^{\varepsilon h_\varepsilon(s)} |\nabla u_i^\varepsilon(s + \zeta n(s))|^2 (1 + \zeta \varkappa(s,t)) d\zeta ds\Bigg),\end{aligned} \tag{3.2}$$

where $\varkappa$ is the curvature of Γ_1 and C is a positive constant independent of ε. As $u^\varepsilon = 0$ on Γ_2, we have, using the Korn inequality in Ω and Ω_ε, respectively, that

$$\begin{aligned}\int_\Omega |\nabla u^\varepsilon|^2 dx &\le C \int_\Omega \sigma_{ij}(u^\varepsilon) e_{ij}(u^\varepsilon) dx \\ &\le C \Phi_\varepsilon(u^\varepsilon), \\ \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx &\le C\varepsilon \int_{\Omega_\varepsilon} \sigma_{ij}(u^\varepsilon) e_{ij}(u^\varepsilon) dx \\ &\le C \Phi_\varepsilon(u^\varepsilon),\end{aligned} \tag{3.3}$$

from which we deduce that

$$\sup_\varepsilon \left(\int_\Omega |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx \right) < +\infty. \tag{3.4}$$

(2) Using the Poincaré inequality and the trace theorem, we deduce from (3.3) that

$$\begin{aligned}\int_\Omega |u^\varepsilon|^2 dx &\le C \int_\Omega |\nabla u^\varepsilon|^2 dx \\ &\le C \Phi_\varepsilon(u^\varepsilon), \\ \int_{\Gamma_1} |u^\varepsilon(s)|^2 ds &\le C \int_\Omega |\nabla u^\varepsilon|^2 dx \\ &\le C \Phi_\varepsilon(u^\varepsilon),\end{aligned} \tag{3.5}$$

and, using (3.2)–(3.5), we deduce that

$$\int_{\Sigma_\varepsilon} |u^\varepsilon|^2 dx \le C\varepsilon \Phi_\varepsilon(u^\varepsilon). \tag{3.6}$$

We obtain from (3.5) and (3.6) that

$$\sup_\varepsilon \int_\Omega |u^\varepsilon|^2 dx < +\infty \quad \text{and} \quad \sup_\varepsilon \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} |u^\varepsilon|^2 dx < +\infty. \qquad \blacksquare$$

Remark 9. According to [25, pages 354–355], as $\partial\Omega_\varepsilon$ is locally Lipschitz, it is also uniformly Lipschitz. Then, using [25, Theorem 12.15], we infer that every $v \in H^1_{\Gamma_2}(\Omega_\varepsilon, \mathbb{R}^3)$ has an extension $\mathcal{E}v \in H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ verifying, in particular, $\mathcal{E}v(x) = v(x)$ for almost every (a.e.) $x \in \Omega_\varepsilon$, and

$$\|\mathcal{E}v\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \le (1 + M)\|v\|_{L^2(\Omega_\varepsilon,\mathbb{R}^3)}, \tag{3.7}$$

where M is a positive integer. We deduce from Proposition 8 and inequality (3.7) that, for every sequence $(u^\varepsilon)_\varepsilon$, such that $u^\varepsilon \in H^1_{\Gamma_2}(\Omega_\varepsilon, \mathbb{R}^3)$ and $\sup_\varepsilon \Phi_\varepsilon(u^\varepsilon) < +\infty$, there exists a subsequence, still denoted $(u^\varepsilon)_\varepsilon$, such that $(u^\varepsilon)_\varepsilon$ weakly converges in $H^1(\Omega, \mathbb{R}^3)$, as ε tends to 0, to some $u \in H^1_{\Gamma_2}(\Omega, \mathbb{R}^3)$, and its extension $(\mathcal{E}u^\varepsilon)_\varepsilon$ strongly converges in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ to $\mathcal{E}u$.

4. Convergence

Let $v^\varepsilon \in H^1_{\Gamma_2}(\Omega_\varepsilon, \mathbb{R}^3)$ and $v \in H^1_{\Gamma_2}(\Omega, \mathbb{R}^3)$. We denote in the same way their extensions to $H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$. According to Proposition 8 and Remark 9, we introduce the following topology τ.

Definition 10. A sequence $(u^\varepsilon)_\varepsilon$; $u^\varepsilon \in H^1_{\Gamma_2}(\Omega_\varepsilon, \mathbb{R}^3)$, τ-converges to u, as ε tends to 0, if

(1) $u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u$ $H^1_{\Gamma_2}(\Omega, \mathbb{R}^3)$-weak,
(2) $u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u$ $L^2(\mathbb{R}^3, \mathbb{R}^3)$-strong.

We have the following result.

Lemma 11. *Let $(\Phi_\varepsilon)_\varepsilon$ be the sequence of functionals defined in (3.1). Then $(\Phi_\varepsilon)_\varepsilon$ Γ-converges, with respect to the topology τ, to the functional Φ defined on $L^2(\mathbb{R}^3, \mathbb{R}^3)$ by*

$$\Phi(u) = \begin{cases} \int_\Omega \sigma_{ij}(u) e_{ij}(u)dx & \text{if } u \in W_{\Gamma_2}, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.1}$$

where

$$W_{\Gamma_2} = H^1_{\Gamma_2}(\Omega, \mathbb{R}^3) \cap L^2(\mathbb{R}^3, \mathbb{R}^3).$$

Proof. Let $u \in W_{\Gamma_2}$. We consider the set $\Omega_{0,\varepsilon} = \overline{\Omega} \cup \Sigma_{0,\varepsilon}$, where $\Sigma_{0,\varepsilon}$ is a layer of thickness ε surrounding Ω defined by

$$\Sigma_{0,\varepsilon} = \left\{x \in \mathbb{R}^3;\ 0 < d(x, \partial\Omega) < \varepsilon\right\},$$

where $d(x, \partial\Omega)$ is the Euclidean distance from x to $\partial\Omega$. Let $v^{0,\varepsilon} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\|u - v^{0,\varepsilon}\|_{L^2(\mathbb{R}^3 \setminus \Omega_{0,\varepsilon}, \mathbb{R}^3)} < \varepsilon.$$

We define the function $\widetilde{v^{0,\varepsilon}}$ by

$$\widetilde{v^{0,\varepsilon}} = \begin{cases} v^{0,\varepsilon} & \text{in } \mathbb{R}^3 \setminus \Omega_{0,\varepsilon}, \\ 0 & \text{in } \Omega_{0,\varepsilon}. \end{cases}$$

We consider a mollifier $\rho_\varepsilon \in C^\infty_c(\mathbb{R}^3)$ with support in the ball $B(0, \varepsilon)$ of radius ε centered at the origin such that $\int_{\mathbb{R}^N} \rho_\varepsilon(x)dx = 1$. We then define the sequence $(w^{0,\varepsilon})_\varepsilon$ by $w^{0,\varepsilon} = \rho_\varepsilon * \widetilde{v^{0,\varepsilon}}$ and the sequence $(u^{0,\varepsilon})_\varepsilon$ of test-functions by

$$u^{0,\varepsilon} = \begin{cases} w^{0,\varepsilon} & \text{in } \mathbb{R}^3 \setminus \Omega_{0,\varepsilon}, \\ u \frac{(\varepsilon - d(x,\partial\Omega))}{\varepsilon} & \text{in } \Sigma_{0,\varepsilon}, \\ u & \text{in } \overline{\Omega}. \end{cases}$$

We can easily check that $u^{0,\varepsilon} \in H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$, $(u^{0,\varepsilon})_\varepsilon$ τ-converges to u as ε tends to 0, and

$$\limsup_{\varepsilon\to 0} \Phi_\varepsilon(u^{0,\varepsilon}) \le \int_\Omega \sigma_{ij}(u)e_{ij}(u)dx = \Phi(u). \tag{4.2}$$

Let $(u^\varepsilon)_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ such that $(u^\varepsilon)_\varepsilon$ τ-converges to some u as ε tends to 0. Then, since $\Phi(u^\varepsilon) \le \Phi_\varepsilon(u^\varepsilon)$, we infer that

$$\Phi(u) \le \liminf_{\varepsilon\to 0} \Phi(u^\varepsilon) \le \liminf_{\varepsilon\to 0} \Phi_\varepsilon(u^\varepsilon). \tag{4.3}$$

We deduce from (4.2) and (4.3) that $(\Phi_\varepsilon)_\varepsilon$ Γ-converges to Φ with respect to the topology τ. ■

We introduce the functional G_ε defined on $L^2(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{B}(\mathbb{R}^3)$ by

$$G_\varepsilon(u, B) = \begin{cases} \Phi_\varepsilon(u) + F_\varepsilon(u, B) & \text{if } u \in H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.4}$$

where F_ε is defined in (2.1).

Our main result in this section reads as follows.

Theorem 12. *There exist a rich family $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$ and a symmetric matrix $\boldsymbol{\mu} = (\mu_{ij})_{i,j=1,2,3}$ of Borel measures μ_{ij}, having the same support contained in Γ_1, which are absolutely continuous with respect to the capacity* Cap*, and satisfying*

$$\mu_{ij}(B)\zeta_i\zeta_j \ge 0, \quad \forall \zeta \in \mathbb{R}^3,\ \forall B \in \mathcal{B}(\mathbb{R}^3),$$

such that, for every $u \in W_{\Gamma_2}$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbb{R}^3)$

$$(\Gamma\text{-}\lim_{\varepsilon\to 0} G_\varepsilon)(u, \omega) = \Phi(u) + \int_{\Gamma_1\cap\omega} u_i u_j d\mu_{ij},$$

where the Γ-limit is taken with respect to the topology τ and Φ is the functional defined in (4.1).

Proof. The upper and lower Γ-limits, with respect to the topology τ, exist and are respectively defined on $W_{\Gamma_2} \times \mathcal{B}(\mathbb{R}^3)$ by

$$G^s(u, B) = \inf\{\limsup_{\varepsilon\to 0} G_\varepsilon(u^\varepsilon, B);\ u^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} u\},$$
$$G^i(u, B) = \inf\{\liminf_{\varepsilon\to 0} G_\varepsilon(u^\varepsilon, B);\ u^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} u\}.$$

We see that, for every $B \in \mathcal{B}(\mathbb{R}^3)$, we have $G^s(.,B) \geq \Phi(.)$ and $G^i(.,B) \geq \Phi(.)$. Let us introduce the nonnegative functionals F^s and F^i defined, for every $B \in \mathcal{B}(\mathbb{R}^3)$, by

$$F^a(u,B) = \begin{cases} G^a(u,B) - \Phi(u) & \text{if } u \in H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3), \\ +\infty & \text{otherwise}, \end{cases} \tag{4.5}$$

where $a = s, i$. Let $u \in W_{\Gamma_2}$ and $u^\varepsilon \in H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ be such that $(u^\varepsilon)_\varepsilon$ τ-converges to u. Denoting in the same way the extension of u to the space $H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$, we set $z^\varepsilon = u^\varepsilon - u$. We can easily check that $(z^\varepsilon)_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$, $(z^\varepsilon)_\varepsilon$ τ-converges to 0, and, using (4.5),

$$F^s(u,B) = \inf\{\limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, B));\ z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0\}, \tag{4.6a}$$

$$F^i(u,B) = \inf\{\liminf_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, B));\ z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0\}, \tag{4.6b}$$

where, for any $v \in H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$,

$$\Phi_{0,\varepsilon}(v) = \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(v) e_{ij}(v) dx. \tag{4.7}$$

The functionals F^s and F^i satisfy the following properties.

Lemma 13. (1) *Let $u \in W_{\Gamma_2}$ and $A, B \in \mathcal{O}(\mathbb{R}^3)$. Then*

$$F^s(u, A \cup B) \leq F^s(u,A) + F^s(u,B).$$

(2) *Let $u \in W_{\Gamma_2}$ and $A, B \in \mathcal{O}(\mathbb{R}^3)$ such that $A \cap B = \emptyset$. Let $A', B' \in \mathcal{O}(\mathbb{R}^3)$ such that $\overline{A'} \subset A$ and $\overline{B'} \subset B$. Then*

$$F^i(u, A \cup B) \geq F^i(u,A') + F^i(u,B').$$

(3) *For every open subset $\omega \in \mathcal{O}(\mathbb{R}^3)$, $F^s(.,\omega)$ is lower semi-continuous for the strong topology of $H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$.*

(4) *Let $\omega \in \mathcal{O}(\mathbb{R}^3)$ and $u, v \in W_{\Gamma_2}$ such that $u_{|\omega} = v_{|\omega}$. Then $F^s(u,\omega) = F^s(v,\omega)$.*

(5) *For every $\omega \in \mathcal{O}(\mathbb{R}^3)$, the functional $u \mapsto F^s(u,\omega)$ is convex and C^1-convex on W_{Γ_2}.*

Proof. (1) Let $u \in W_{\Gamma_2}$. Let $\omega \in \mathcal{O}(\mathbb{R}^3)$ such that $\overline{\omega} \subset B \setminus \overline{A}$. Then, owing to (4.6a), there exist two sequences $(z^{1,\varepsilon})_\varepsilon$ and $(z^{2,\varepsilon})_\varepsilon$ in $H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ τ-converging to 0 such that

$$\begin{aligned} F^s(u,A) &= \limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^{1,\varepsilon}) + F_\varepsilon(z^{1,\varepsilon} + u, A)), \\ F^s(u,\omega) &= \limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^{2,\varepsilon}) + F_\varepsilon(z^{2,\varepsilon} + u, \omega)). \end{aligned} \tag{4.8}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that $0 \le \varphi \le 1$ in $\mathbb{R}^3$, $\varphi = 0$ in A and $\varphi = 1$ in ω. We define the sequence $(z^\varepsilon)_\varepsilon$ by

$$z^\varepsilon = (1-\varphi)z^{1,\varepsilon} + \varphi z^{2,\varepsilon}.$$

Then $z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0$ and, according to (4.6a),

$$F^s(u, A\cup\omega) \le \limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, A\cup\omega)). \tag{4.9}$$

Let us denote $\mathbb{R}^9_{\text{sym}}$ the set of 3×3-real symmetric matrices. We define the quadratic form Q by

$$Q(\tau) = a_{ijkl}\tau_{kl}\tau_{ij}, \quad \forall \tau \in \mathbb{R}^9_{\text{sym}}. \tag{4.10}$$

Using (4.10), we have, according to (1.1), that, for every $\eta \in (0,1)$,

$$\begin{aligned}
\Phi_{0,\varepsilon}(z^\varepsilon) &= \varepsilon\int_{\Sigma_\varepsilon} Q(e(z^\varepsilon))dx \\
&= \varepsilon\int_{\Sigma_\varepsilon} Q((1-\varphi)e(z^{1,\varepsilon}) + \varphi e(z^{2,\varepsilon}) + (z^{2,\varepsilon} - z^{1,\varepsilon})\otimes\nabla\varphi)dx \\
&= \varepsilon\int_{\Sigma_\varepsilon} Q\left((1-\eta)\frac{(1-\varphi)e(z^{1,\varepsilon}) + \varphi e(z^{2,\varepsilon})}{1-\eta} + \eta\frac{(z^{2,\varepsilon} - z^{1,\varepsilon})\otimes\nabla\varphi}{\eta}\right).
\end{aligned} \tag{4.11}$$

Using the convexity of Q, we have, for every $\eta \in (0,1)$,

$$\begin{aligned}
\Phi_{0,\varepsilon}(z^\varepsilon) \le{}& \frac{\varepsilon}{1-\eta}\int_{\Sigma_\varepsilon} Q((1-\varphi)e(z^{1,\varepsilon}) + \varphi e(z^{2,\varepsilon}) + (z^{2,\varepsilon} - z^{1,\varepsilon})\otimes\nabla\varphi)dx \\
&+ \frac{\varepsilon}{\eta}\int_{\Sigma_\varepsilon} Q((z^{2,\varepsilon} - z^{1,\varepsilon})\otimes\nabla\varphi_\varepsilon)dx,
\end{aligned}$$

and, since $0 \le \varphi \le 1$, we have, using once again the convexity of Q and the coercivity property (1.2b), that

$$\begin{aligned}
\Phi_{0,\varepsilon}(z^\varepsilon) \le{}& \frac{\varepsilon}{1-\eta}\int_{\Sigma_\varepsilon}(1-\varphi)Q(e(z^{1,\varepsilon}))dx \\
&+ \frac{\varepsilon}{1-\eta}\int_{\Sigma_\varepsilon}\varphi Q(e(z^{2,\varepsilon}))dx \\
&+ \frac{\varepsilon c_2}{\eta}\int_{\Sigma_\varepsilon}|z^{2,\varepsilon} - z^{1,\varepsilon}|^2|\nabla\varphi|^2dx,
\end{aligned}$$

where c_2 is the constant appearing in (1.2b). Then, taking into account the properties of φ, we deduce that

$$\begin{aligned}
\Phi_{0,\varepsilon}(w^\varepsilon) \le{}& \frac{1}{1-\eta}(\Phi_{0,\varepsilon}(z^{1,\varepsilon}) + \Phi_{0,\varepsilon}(z^{2,\varepsilon})) \\
&+ \frac{\varepsilon C}{\eta}\int_{\Sigma_\varepsilon}|z^{2,\varepsilon} - z^{1,\varepsilon}|^2dx,
\end{aligned} \tag{4.12}$$

where C is a positive constant independent of ε and η. On the other hand, as F_ε is C^1-convex and $F_\varepsilon(z^{m,\varepsilon}+u,.)$; $m=1,2$, is the restriction to $\mathcal{O}(\mathbb{R}^3)$ of a nonnegative Borel measure, we have

$$\begin{aligned} & F_\varepsilon(z^\varepsilon+u, A\cup\omega) \\ & \qquad = F_\varepsilon((1-\varphi)(z^{1,\varepsilon}+u)+\varphi(z^{2,\varepsilon}+u), A\cup\omega) \\ & \qquad \le F_\varepsilon(z^{1,\varepsilon}+u, A)+F_\varepsilon(z^{2,\varepsilon}+u,\omega). \end{aligned} \tag{4.13}$$

We deduce from (4.9), using (4.12) and (4.13), that

$$\begin{aligned} F^s(u, A\cup\omega) &\le \limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^\varepsilon)+F_\varepsilon(z^\varepsilon+u, A\cup\omega)) \\ &\le \frac{1}{1-\eta}\limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^{1,\varepsilon})+F_\varepsilon(z^{1,\varepsilon}+u, A)) \\ &\quad + \frac{1}{1-\eta}\limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^{2,\varepsilon})+F_\varepsilon(z^{2,\varepsilon}+u, \omega)) \\ &\quad + \frac{C}{\eta}\limsup_{\varepsilon\to 0}\varepsilon\int_{\Sigma_\varepsilon}|z^{2,\varepsilon}-z^{1,\varepsilon}|^2dx. \end{aligned} \tag{4.14}$$

As $\limsup_{\varepsilon\to 0}\varepsilon\int_{\Sigma_\varepsilon}|z^{2,\varepsilon}-z^{1,\varepsilon}|^2dx=0$, we deduce from (4.14) that

$$\begin{aligned} F^s(u, A\cup\omega) &\le \limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^\varepsilon)+F_\varepsilon(z^\varepsilon+u, A\cup\omega)) \\ &\le \frac{1}{1-\eta}\limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^{1,\varepsilon})+F_\varepsilon(z^{1,\varepsilon}+u, A)) \\ &\quad + \frac{1}{1-\eta}\limsup_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^{2,\varepsilon})+F_\varepsilon(z^{2,\varepsilon}+u, \omega)), \end{aligned}$$

thus, letting η tend to 0, we have, according to (4.8), that

$$F^s(u, A\cup\omega)\le F^s(u,A)+F^s(u,\omega),$$

and, letting ω increase to B, we conclude that

$$F^s(u, A\cup B)\le F^s(u,A)+F^s(u,B).$$

(2) We deduce from (4.6b) that there exists a sequence $(z^\varepsilon)_\varepsilon\subset H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ such that $z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0$ and

$$F^i(u, A\cup B)=\liminf_{\varepsilon\to 0}(\Phi_{0,\varepsilon}(z^\varepsilon)+F_\varepsilon(z^\varepsilon+u, A\cup B)).$$

Let $A', B'\in\mathcal{O}(\mathbb{R}^3)$ such that $\overline{A'}\subset A$ and $\overline{B'}\subset B$. Let $\varphi^1_\varepsilon\in C^\infty_c(\mathbb{R}^3)$ such that $0\le\varphi^1_\varepsilon\le 1$, $\varphi^1_\varepsilon=0$ in $\mathbb{R}^3\setminus A\cap\Sigma_\varepsilon$ and $\varphi^1_\varepsilon=1$ in $\overline{A'\cap\Sigma_\varepsilon}$. Let $\varphi^2_\varepsilon\in C^\infty_c(\mathbb{R}^3)$ such that $0 \le \varphi_\varepsilon^2 \le 1$, $\varphi_\varepsilon^2 = 0$ in $\mathbb{R}^3 \setminus B \cap \Sigma_\varepsilon$ and $\varphi_\varepsilon^2 = 1$ in $\overline{B' \cap \Sigma_\varepsilon}$. Let $z^{m,\varepsilon} = \varphi_\varepsilon^m z^\varepsilon$; $m = 1, 2$. Let us set

$$\Sigma_\varepsilon^1 = A \cap \Sigma_\varepsilon \quad \text{and} \quad \Sigma_\varepsilon^2 = B \cap \Sigma_\varepsilon.$$

Then, using a convexity argument, we deduce that, for every $\eta \in (0, 1)$,

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon}) dx &= \varepsilon \int_{\Sigma_\varepsilon^m} Q(e(\varphi_\varepsilon^m z^\varepsilon)) dx \\ &= \varepsilon \int_{\Sigma_\varepsilon^m} Q(\varphi_\varepsilon^m e(z^\varepsilon) + z^\varepsilon \otimes \nabla \varphi_\varepsilon^m) dx \\ &\le \frac{\varepsilon}{1-\eta} \int_{\Sigma_\varepsilon^m} Q(e(z^\varepsilon)) dx + \frac{\varepsilon c_2}{\eta} \int_{\Sigma_\varepsilon^m} |z^\varepsilon|^2 |\nabla \varphi_\varepsilon^m|^2 dx, \end{aligned} \tag{4.15}$$

where c_2 is the constant appearing in (1.2b). Observing that the diameters of $A \cap \Sigma_\varepsilon \diagdown \overline{A' \cap \Sigma_\varepsilon}$ and $B \cap \Sigma_\varepsilon \diagdown \overline{B' \cap \Sigma_\varepsilon}$ are independent of ε, we infer that $|\nabla \varphi_\varepsilon^1|$ and $|\nabla \varphi_\varepsilon^2|$ on $A \cap \Sigma_\varepsilon \diagdown \overline{A' \cap \Sigma_\varepsilon}$ and on $B \cap \Sigma_\varepsilon \diagdown \overline{B' \cap \Sigma_\varepsilon}$ respectively are uniformly bounded by a positive constant independent of ε. Then, using the fact that $z^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0$ $L^2(\mathbb{R}^3, \mathbb{R}^3)$-strong, we deduce that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon c_2}{\eta} \int_{\Sigma_\varepsilon^m} |z^\varepsilon|^2 |\nabla \varphi_\varepsilon^m|^2 dx = 0,$$

hence, passing to the lower limit in (4.15), we get

$$\liminf_{\varepsilon \to 0} \varepsilon \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon}) dx \le \liminf_{\varepsilon \to 0} \frac{\varepsilon}{1-\eta} \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^\varepsilon) e_{ij}(z^\varepsilon) dx,$$

and, letting η tend to 0,

$$\liminf_{\varepsilon \to 0} \varepsilon \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon}) dx \le \liminf_{\varepsilon \to 0} \varepsilon \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^\varepsilon) e_{ij}(z^\varepsilon) dx. \tag{4.16}$$

Observing that $\int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^{m,\varepsilon}) e_{ij}(z^{m,\varepsilon}) dx = \Phi_{0,\varepsilon}(z^{m,\varepsilon})$; $m = 1, 2$, we deduce, using the fact that $A \cap B = \emptyset$, the inequality (4.16), and the measure property of $F_\varepsilon(z^\varepsilon + u, .)$, that

$$\begin{aligned} F^i(u, A \cup B) &= \liminf_{\varepsilon \to 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, A \cup B)) \\ &\ge \liminf_{\varepsilon \to 0} \left(\varepsilon \sum_{m=1,2} \int_{\Sigma_\varepsilon^m} \sigma_{ij}(z^\varepsilon) e_{ij}(z^\varepsilon) dx + F_\varepsilon(z^\varepsilon + u, A' \cup B') \right) \\ &\ge \liminf_{\varepsilon \to 0} \left(\varepsilon \int_{\Sigma_\varepsilon^1} \sigma_{ij}(z^{1,\varepsilon}) e_{ij}(z^{1,\varepsilon}) dx + F_\varepsilon(z^{1,\varepsilon} + u, A') \right) \\ &\quad + \liminf_{\varepsilon \to 0} \left(\varepsilon \int_{\Sigma_\varepsilon^1} \sigma_{ij}(z^{2,\varepsilon}) e_{ij}(z^{2,\varepsilon}) dx + F_\varepsilon(z^{2,\varepsilon} + u, B') \right) \end{aligned}$$

$$
\begin{aligned}
&= \liminf_{\varepsilon \to 0} (\Phi_{0,\varepsilon}(z^{1,\varepsilon}) + F_\varepsilon(z^{1,\varepsilon} + u, A')) \\
&\quad + \liminf_{\varepsilon \to 0} (\Phi_{0,\varepsilon}(z^{2,\varepsilon}) + F_\varepsilon(z^{2,\varepsilon} + u, A')) \\
&\geq F^i(u, A') + F^i(u, B').
\end{aligned}
$$

(3) Let $(u_k)_k \subset H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ converging to some u in the strong topology of $H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$. Then $(u_k)_k$ τ-converges to u and, since F^s is lower semi-continuous as an upper Γ-limit of a sequence of lower semi-continuous functionals, we have

$$
\liminf_{k \to \infty} F^s(u_k, \omega) \geq F^s(u, \omega),
$$

for every $\omega \in \mathcal{O}(\mathbb{R}^3)$.

(4) Let $\omega \in \mathcal{O}(\mathbb{R}^3)$ and $u, v \in W_{\Gamma_2}$ such that $u_{|\omega} = v_{|\omega}$. Then

$$
F_\varepsilon(z^\varepsilon + u, \omega) = F_\varepsilon(z^\varepsilon + v, \omega),
$$

for every sequence $(z^\varepsilon)_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ τ-converging to 0. This implies that $F^s(u, \omega) = F^s(v, \omega)$.

(5) Let $\varphi \in C^1(\mathbb{R}^3)$ such that $0 \leq \varphi \leq 1$. Let $u, v \in W_{\Gamma_2}$, and $\omega \in \mathcal{O}(\mathbb{R}^3)$. Then, as F_ε is C^1-convex, we have, for every sequence $(z^\varepsilon)_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ τ-converging to 0,

$$
\begin{aligned}
F_\varepsilon(z^\varepsilon + \varphi u + (1-\varphi)v, \omega) &= F_\varepsilon(\varphi(z^\varepsilon + u) + (1-\varphi)(z_\varepsilon + v), \omega) \\
&\leq F_\varepsilon(z^\varepsilon + u, \omega) + F_\varepsilon(z^\varepsilon + v, \omega).
\end{aligned} \tag{4.17}
$$

As $\Phi_{0,\varepsilon}$ is nonnegative, we deduce from (4.17) that, for every $\varepsilon > 0$,

$$
\begin{aligned}
&\limsup_{\varepsilon \to 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + \varphi u + (1-\varphi)v, \omega)) \\
&\quad \leq \limsup_{\varepsilon \to 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, \omega) + \Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + v, \omega)) \\
&\quad \leq \limsup_{\varepsilon \to 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + u, \omega)) + \limsup_{\varepsilon \to 0} (\Phi_{0,\varepsilon}(z^\varepsilon) + F_\varepsilon(z^\varepsilon + v, \omega)).
\end{aligned}
$$

Taking the infimum over all sequences $(z^\varepsilon)_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ τ-converging to 0, we infer that

$$
F^s(\varphi u + (1-\varphi)v, \omega) \leq F^s(u, \omega) + F^s(v, \omega).
$$

We can prove in a similar way that F^s is convex. Thus F^s is C^1-convex. ■

Now, according to the compactness theorem of [18], there exist a subsequence $(\varepsilon_k)_k$ and a countable dense family $\mathcal{D} \subset \mathcal{B}(\mathbb{R}^3)$ such that, for every $u \in W_{\Gamma_2}$ and every $B \in \mathcal{D}$, we have the following Γ-limit taken with respect to the topology τ:

$$
(\Gamma\text{-}\lim_{k \to +\infty} G_{\varepsilon_k})(u, B) = G(u, B), \tag{4.18}
$$

where G_ε is the functional defined in (4.4). Defining for any $B \in \mathcal{D}$ the functional F by

$$F(u,B) = \begin{cases} G(u,B) - \Phi(u) & \text{if } u \in W_{\Gamma_2}, \\ +\infty & \text{otherwise,} \end{cases}$$

we deduce from (4.18) that $F = F^s = F^i$ on $W_{\Gamma_2} \times \mathcal{D}$. We extend F on $W_{\Gamma_2} \times \mathcal{B}(\mathbb{R}^3)$ by setting

$$F(u,B) = \sup_{D \in \mathcal{D}, \overline{D} \subset \mathring{B}} F^s(u,D) = \sup_{D \in \mathcal{D}, \overline{D} \subset \mathring{B}} F^i(u,D). \tag{4.19}$$

We define the family $\mathcal{R}(F)$ by

$$\begin{aligned}\mathcal{R}(F) = \Big\{B \in \mathcal{B}(\mathbb{R}^3);\ \forall u \in W_{\Gamma_2},\ F^s_+(u,B) &= \sup_{D \in \mathcal{D}, \overline{D} \subset \mathring{B}} F^s(u,D) \\ &= \inf_{D \in \mathcal{D}, \overline{B} \subset \mathring{D}} F^s(u,D) = F^s_-(u,B)\Big\}.\end{aligned}$$

Then (see, for instance, [16, Proposition 14.14]) $\mathcal{R}(F)$ is a rich family in $\mathcal{B}(\mathbb{R}^3)$ and $F = F^s = F^s_+ = F^s_- = F^i_+ = F^i_- = F^i$ on $\mathcal{R}(F)$. We deduce that, for every $u \in W_{\Gamma_2}$ and every $B \in \mathcal{R}(F)$,

$$\begin{aligned} F(u,B) &= \inf\Big\{\limsup_{k \to +\infty}(\Phi_{0,\varepsilon_k}(z^{\varepsilon_k}) + F_\varepsilon(z^{\varepsilon_k} + u, B));\ z^{\varepsilon_k} \xrightarrow[k\to\infty]{\tau} 0\Big\} \\ &= \inf\Big\{\liminf_{k \to +\infty}(\Phi_{0,\varepsilon_k}(z^{\varepsilon_k}) + F_\varepsilon(z^{\varepsilon_k} + u, B));\ z^{\varepsilon_k} \xrightarrow[k\to\infty]{\tau} 0\Big\}. \end{aligned} \tag{4.20}$$

Let ε' denote any subsequence of ε. Repeating the above arguments, we deduce that there exist a subsequence $(\varepsilon'_k)_k$, a functional F^*, and a rich family $\mathcal{R}(F^*)$, such that, for every $u \in W_{\Gamma_2}$ and every $B \in \mathcal{R}(F^*)$

$$\begin{aligned} F^*(u,B) &= \inf\Big\{\limsup_{k \to +\infty}(\Phi_{0,\varepsilon'_k}(z^{\varepsilon'_k}) + F_{\varepsilon'_k}(z^{\varepsilon'_k} + u, B));\ z^{\varepsilon'_k} \xrightarrow[k\to+\infty]{\tau} 0\Big\} \\ &= \inf\Big\{\liminf_{k \to +\infty}(\Phi_{0,\varepsilon'_k}(z^{\varepsilon'_k}) + F_{\varepsilon'_k}(z^{\varepsilon'_k} + u, B));\ z^{\varepsilon'_k} \xrightarrow[k\to+\infty]{\tau} 0\Big\}. \end{aligned} \tag{4.21}$$

As $\mathcal{R}(F) \cap \mathcal{R}(F^*)$ is still a rich family, we deduce that, for every $u \in W_{\Gamma_2}$,

$$F(u,.) = F^*(u,.) \text{ on } \mathcal{R}(F) \cap \mathcal{R}(F^*). \tag{4.22}$$

Since a countable intersection of rich families is also a rich family, we can repeat the above process and deduce that there exists a rich family $\mathcal{R}$ in $\mathcal{B}(\mathbb{R}^3)$ on which the above limits coincide. We thus obtain that, with respect to the topology τ, for every $u \in W_{\Gamma_2}$ and every $B \in \mathcal{R}$,

$$(\Gamma\text{-}\lim_{\varepsilon \to 0} G_\varepsilon)(u,B) = G(u,B) = \Phi(u) + F(u,B).$$

We now prove the following.

Lemma 14. *The functional F belongs to the class $\mathbb{F}$.*

Proof. Let ω be any element of $\mathcal{R}(F) \cap \mathcal{O}(\mathbb{R}^3)$. As $F^s(., \omega)$ is lower semi-continuous with respect to the strong topology of $H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ by virtue of Lemma 13 (3), we have, according to (4.19), that the functional $u \mapsto F(u, \omega)$ is lower semi-continuous with respect to the strong topology of $H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$. Owing to Lemma 13 (4), to the fact that $H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3) \subset W_{\Gamma_2}$, and to (4.19), we have that $F(u, \omega) = F(v, \omega)$, for every $\omega \in \mathcal{O}(\mathbb{R}^3)$ and every $u, v \in H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ such that $u_{|\omega} = v_{|\omega}$. According to Lemma 13 (5), the functional $u \mapsto F(u, \omega)$ is convex and C^1-convex on W_{Γ_2} and therefore on $H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$. Let us now prove that, for every $u \in W_{\Gamma_2}$, $\omega \mapsto F(u, \omega)$ is the restriction to $\mathcal{O}(\mathbb{R}^3)$ of a nonnegative Borel measure still denoted $F(u, \omega)$.

Let $u \in W_{\Gamma_2}$. Let $\omega_1, \omega_2 \in \mathcal{O}(\mathbb{R}^3)$ such that $\omega_1 \cap \omega_2 = \emptyset$. Let $B \in \mathcal{B}(\mathbb{R}^3)$ such that $\overline{B} \subset \omega_1 \cup \omega_2$. We have

$$B = (B \cap \omega_1) \cup (B \cap \omega_2) \quad \text{and} \quad \overline{B \cap \omega_i} = \overline{B} \cap \omega_i; \quad i = 1, 2.$$

Then, using Lemma 13 (1), we get

$$\begin{aligned} F^s(u, B) &= F^s(u, (B \cap \omega_1) \cup (B \cap \omega_2)) \\ &\leq F^s(u, B \cap \omega_1) + F^s(u, B \cap \omega_2) \\ &\leq F(u, \omega_1) + F(u, \omega_2), \end{aligned}$$

thus

$$\sup_{\overline{B} \subset \omega_1 \cup \omega_2} F^s(u, B) \leq F(u, \omega_1) + F(u, \omega_2),$$

from which we deduce that

$$F(u, \omega_1 \cup \omega_2) \leq F(u, \omega_1) + F(u, \omega_2). \tag{4.23}$$

Let $u \in W_{\Gamma_2}$. Let $\omega_1, \omega_2 \in \mathcal{O}(\mathbb{R}^3)$ such that $\omega_1 \cap \omega_2 = \emptyset$. Let $B_1 \in \mathcal{B}(\mathbb{R}^3)$, $B_2 \in \mathcal{B}(\mathbb{R}^3)$ and $B \in \mathcal{O}(\mathbb{R}^3)$ such that $\overline{B_1} \cup \overline{B_2} \subset B \subset \overline{B} \subset \omega_1 \cup \omega_2$. We have that

$$\begin{aligned} B &= (B \cap \omega_1) \cup (B \cap \omega_2), \\ (B \cap \omega_1) \cap (B \cap \omega_2) &= \emptyset, \\ \overline{B_i} \subset B \cap \omega_i \subset \overline{B} \cap \omega_i &= \overline{B \cap \omega_i} \subset \omega_i; \quad i = 1, 2. \end{aligned}$$

Then, using Lemma 13 (2), we get

$$\begin{aligned} F(u, \omega_1 \cup \omega_2) &\geq F^i(u, B \cap \omega_1) + F^i(u, B \cap \omega_2) \\ &\geq F^i(u, B_1) + F^i(u, B_2), \end{aligned}$$

thus

$$F(u,\omega_1 \cup \omega_2) \geq \sup_{\overline{B_1}\subset\omega_1} F^i(u,B_1) + \sup_{\overline{B_2}\subset\omega_2} F^i(u,B_2),$$

from which we deduce that

$$F(u,\omega_1 \cup \omega_2) \geq F(u,\omega_1) + F(u,\omega_2). \tag{4.24}$$

We deduce from (4.23) and (4.24) that F is additive on $\mathcal{O}(\mathbb{R}^3)$. Let $(\omega_k)_k$ be any non-decreasing sequence of open sets in $\mathcal{R}(F) \cap \mathcal{O}(\mathbb{R}^3)$ and set $\omega = \bigcup_k \omega_k$. We have, for every k, $F(u,\omega_k) \leq F(u,\omega)$, from which we deduce that

$$\limsup_{k\to\infty} F(u,\omega_k) \leq F(u,\omega). \tag{4.25}$$

On the other hand, using the Borel–Lebesgue theorem, we deduce that, for every $\overline{B} \subset \omega$, there exists k_0 such that $\overline{B} \subset \omega_{k_0}$. Thus

$$F^s(u,B) \leq F(u,\omega_{k_0}) \leq \limsup_{k\to\infty} F(u,\omega_k),$$

from which we deduce that

$$\sup_{\overline{B}\subset\omega} F^s(u,B) \leq \limsup_{k\to\infty} F(u,\omega_k),$$

which yields

$$F(u,\omega) \leq \limsup_{k\to\infty} F(u,\omega_k). \tag{4.26}$$

Therefore, according to (4.25) and (4.26), $F(u,\omega) = \limsup_{k\to\infty} F(u,\omega_k)$. Hence, F is σ-additive on $\mathcal{O}(\mathbb{R}^3)$. Consequently, for every $u \in W_{\Gamma_2}$ (and particularly for every $u \in H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$), $F(u,.)$ is a positive Borel measure, which is outer regular by definition. ■

Since Φ_ε and F_ε are quadratic, we deduce from (4.20) that $F(.,\omega)$ is quadratic for every $\omega \in \mathcal{O}(\mathbb{R}^3) \cap \mathcal{R}$. Then, owing to Lemma 14, we deduce, applying Corollary 6, that there exist a finite measure $\mu \in \mathcal{M}_0(\mathbb{R}^3)$, a symmetric matrix $(a_{ij})_{i,j=1,2,3}$ of Borel functions from $\mathbb{R}^3$ to $\mathbb{R}$ satisfying $a_{ij}(x)\zeta_i\zeta_j \geq 0$ for every $\zeta \in \mathbb{R}^3$ and for q.e. $x \in \mathbb{R}^3$, such that, for every $u \in H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ and every $\omega \in \mathcal{O}(\mathbb{R}^3) \cap \mathcal{R}$,

$$\begin{aligned} F(u,\omega) &= \int_\omega a_{ij}(x)u_i(x)u_j(x)d\mu(x) \\ &= \langle \boldsymbol{\mu} u, u\rangle \\ &= \int_\omega u_i u_j d\mu_{ij}, \end{aligned}$$

where, according to Remark 7, $\boldsymbol{\mu} = (\mu_{ij})_{i,j=1,2,3}$ with $\mu_{ij} = a_{ij}\mu + \infty_{\mathbb{R}^N \setminus \Theta}\delta_{ij}$; $i, j = 1, 2, 3$, Θ being the set defined in (2.3). Let us now define

$$d_\varepsilon = \sup\{d(x,\Omega);\ x \in \operatorname{spt}\infty_{\Gamma_{1,\varepsilon}}\},$$

where $d(x,\Omega)$ is the Euclidean distance from x to Ω and $\operatorname{spt}\infty_{\Gamma_{1,\varepsilon}}$ denotes the support of the measure $\infty_{\Gamma_{1,\varepsilon}}$. Let $u, v \in W_{\Gamma_2}$ with $u = v$ a.e. in Ω. We denote in the same way the extensions of u and v to $H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$. Let $(u^\varepsilon)_\varepsilon \in H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ such that $(u^\varepsilon)_\varepsilon$ τ-converges to u. We suppose that

$$\begin{aligned} G(u,\mathbb{R}^3) &= \Phi(u) + F(u,\mathbb{R}^3) \\ &= \lim_{\varepsilon\to 0} G_\varepsilon(u^\varepsilon,\mathbb{R}^3) \\ &= \lim_{\varepsilon\to 0}(\Phi_\varepsilon(u^\varepsilon) + F_\varepsilon(u^\varepsilon,\mathbb{R}^3)). \end{aligned} \tag{4.27}$$

Let $(v^\varepsilon)_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ such that $(v^\varepsilon)_\varepsilon$ τ-converges to v. We suppose that

$$\lim_{\varepsilon\to 0} \Phi_{0,\varepsilon}(v^\varepsilon) = 0. \tag{4.28}$$

Let us define the sequence $(w^\varepsilon)_\varepsilon$ by $w^\varepsilon = \varphi_\varepsilon u^\varepsilon + (1-\varphi_\varepsilon)v^\varepsilon$, where $(\varphi_\varepsilon)_\varepsilon$ is some sequence in $C^1_0(\mathbb{R}^3)$ such that $0 \le \varphi_\varepsilon \le 1$ in $\mathbb{R}^3$, $\varphi_\varepsilon = 1$ in $\overline{\Omega} \cap \operatorname{spt}\infty_{\Gamma_{1,\varepsilon}}$, $\varepsilon|\nabla\varphi_\varepsilon|^2 \le 2\sqrt{\varepsilon}$, and $\varphi_\varepsilon = 0$ whenever $d(x,\Omega) > d_\varepsilon + \varepsilon^{1/4}$. Then $(w^\varepsilon)_\varepsilon$ τ-converges to v and, using Γ-convergence properties, we deduce that

$$\Phi(v) + F(v,\mathbb{R}^3) \le \liminf_{\varepsilon\to 0}(\Phi_\varepsilon(w^\varepsilon) + F_\varepsilon(w^\varepsilon,\mathbb{R}^3)). \tag{4.29}$$

Since $w^\varepsilon = u^\varepsilon$ q.e. on $\operatorname{spt}\infty_{\Gamma_{1,\varepsilon}}$, we have, according to (2.2),

$$F_\varepsilon(u^\varepsilon,\mathbb{R}^3) = F_\varepsilon(w^\varepsilon,\mathbb{R}^3). \tag{4.30}$$

Using the form (4.11) and the convexity of Q, we have, for every $\eta \in (0,1)$,

$$\begin{aligned} \Phi_\varepsilon(w^\varepsilon) \le{}& \frac{1}{1-\eta}\int_\Omega Q(\varphi_\varepsilon e(u^\varepsilon) + (1-\varphi_\varepsilon)e(v^\varepsilon))dx \\ &+ \frac{\varepsilon}{1-\eta}\int_{\Sigma_\varepsilon} Q(\varphi_\varepsilon e(u^\varepsilon) + (1-\varphi_\varepsilon)e(v^\varepsilon))dx \\ &+ \frac{1}{\eta}\int_\Omega Q((u^\varepsilon - v^\varepsilon)\otimes\nabla\varphi_\varepsilon)dx \\ &+ \frac{\varepsilon}{\eta}\int_{\Sigma_\varepsilon} Q((u^\varepsilon - v^\varepsilon)\otimes\nabla\varphi_\varepsilon)dx, \end{aligned}$$

and, since $0 \le \varphi_\varepsilon \le 1$, we have, using the convexity of Q and the coercivity property (1.2b), that

$$\begin{aligned}\Phi_\varepsilon(w^\varepsilon) \le & \frac{1}{1-\eta}\int_\Omega \varphi_\varepsilon Q(e(u^\varepsilon))dx \\ & + \frac{\varepsilon}{1-\eta}\int_{\Sigma_\varepsilon} \varphi_\varepsilon Q(e(u^\varepsilon))dx \\ & + \frac{1}{1-\eta}\int_\Omega (1-\varphi_\varepsilon) Q(e(v^\varepsilon))dx \\ & + \frac{\varepsilon}{1-\eta}\int_{\Sigma_\varepsilon} (1-\varphi_\varepsilon) Q(e(v^\varepsilon))dx \\ & + \frac{c_2}{\eta}\int_\Omega |u^\varepsilon - v^\varepsilon|^2 |\nabla\varphi_\varepsilon|^2 dx \\ & + \frac{\varepsilon c_2}{\eta}\int_{\Sigma_\varepsilon} |u^\varepsilon - v^\varepsilon|^2 |\nabla\varphi_\varepsilon|^2 dx,\end{aligned}$$

where c_2 is the constant appearing in (1.2b). Then, taking into account the properties of φ_ε, we deduce that

$$\begin{aligned}\Phi_\varepsilon(w^\varepsilon) \le & \frac{1}{1-\eta}\Phi_\varepsilon(u^\varepsilon) \\ & + \frac{\varepsilon}{1-\eta}\int_{\Sigma_\varepsilon} \sigma_{ij}(v^\varepsilon) e_{ij}(v^\varepsilon)dx \\ & + \frac{\varepsilon^{1/2}C}{\eta}\int_{\Sigma_\varepsilon} |u^\varepsilon - v^\varepsilon|^2 dx,\end{aligned} \tag{4.31}$$

where C is a positive constant independent of ε and η. From (4.27)–(4.31) it follows that

$$\Phi(v) + F(v, \mathbb{R}^3) \le \frac{1}{1-\eta}(\Phi(u) + F(u, \mathbb{R}^3)).$$

As $\Phi(u) = \Phi(v)$, we obtain, as $\eta \to 0$, that $F(v, \mathbb{R}^3) \le F(u, \mathbb{R}^3)$ and, changing the role of u and v, that $F(u, \mathbb{R}^3) \le F(v, \mathbb{R}^3)$. Thus

$$F(u, \mathbb{R}^3) = F(v, \mathbb{R}^3) = \int_{\mathbb{R}^3} v_i v_j d\mu_{ij},$$

which implies that $\operatorname{spt} \mu_{ij} \subset \overline{\Omega}$, $\forall i, j = 1, 2, 3$.

Let $u \in H^1_{\Gamma_2}(\Omega, \mathbb{R}^3)$. We have the following inequality:

$$0 \le \Phi(u) + \int_{\mathbb{R}^N} u_i u_j d\mu_{ij} \le \liminf_{\varepsilon\to 0}(\Phi_\varepsilon(u) + F_\varepsilon(u, \mathbb{R}^3)). \tag{4.32}$$

Then, taking $u \in H^1_0(\Omega, \mathbb{R}^3)$, we have $F_\varepsilon(u, \mathbb{R}^3) = 0$, for any $\varepsilon > 0$, and

$$\liminf_{\varepsilon\to 0} \Phi_\varepsilon(u) = \Phi(u).$$

We deduce, using (4.32), that

$$\int_{\Omega\cup\Gamma_2} u_i u_j d\mu_{ij} = 0, \quad \forall i,j = 1,2,3.$$

Therefore, spt $\mu_{ij} \subset \Gamma_1$, $\Theta \subset \Gamma_1$ and $\mu_{ij} = a_{ij}\mu + \infty_{\Gamma_1\setminus\Theta}\delta_{ij}$, $\forall i,j = 1,2,3$. ∎

Let us write the associated limit problem obtained as $\varepsilon \to 0$.

Corollary 15. *Problem (1.4) admits a unique solution u^ε which τ-converges to $u^0 \in H_{\boldsymbol{\mu},\Gamma_2}(\Omega,\mathbb{R}^3)$ which is the unique solution of the minimization problem*

$$\min_{u\in L^2(\mathbb{R}^3,\mathbb{R}^3)} \left\{ \mathcal{F}_0(u) - 2\int_\Omega f.udx \right\},$$

where $H_{\boldsymbol{\mu},\Gamma_2}(\Omega,\mathbb{R}^3)$ is the space of admissible displacements defined in (1.6) and $\mathcal{F}_0$ is the functional defined in (1.5). This solution coincides with the unique solution of the elasticity system

$$\begin{cases} -\sigma_{ij,j}(u) = f_i & \text{in } \Omega;\ i = 1,2,3, \\ \sigma_{ij}(u)n_j + \mu_{ij}u_j = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{4.33}$$

Moreover, we have $\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u^\varepsilon) = \mathcal{F}_0(u^0)$.

Proof. Observe that the functional $\mathcal{F}_\varepsilon$ can be written as $\mathcal{F}_\varepsilon(u) = G_\varepsilon(u,\mathbb{R}^3)$, for every $u \in L^2(\mathbb{R}^3,\mathbb{R}^3)$. As the Dirichlet condition $u = 0$ on $\Gamma_{1,\varepsilon}$ is prescribed in the capacity sense, we can prove (see, for instance, [30]) that, using the classical Poincaré and Korn inequalities, problem (1.4) has a unique solution $u^\varepsilon \in H_0^1(\Omega_\varepsilon,\mathbb{R}^3)$. Using Proposition 8 and Theorem 12, we deduce, according to [16, Theorem 7.8], that the whole sequence $(u^\varepsilon)_\varepsilon$ τ-converges to the unique solution $u^0 \in H_{\boldsymbol{\mu},\Gamma_2}(\Omega,\mathbb{R}^3)$ of the problem

$$\min_{u\in L^2(\mathbb{R}^3,\mathbb{R}^3)} \left\{ \mathcal{F}_0(u) - 2\int_\Omega f.udx \right\} = \lim_{\varepsilon\to 0} \min_{u\in L^2(\Omega_\varepsilon,\mathbb{R}^3)} \left\{ \mathcal{F}_\varepsilon(u) - 2\int_\Omega f.udx \right\},$$

$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u^\varepsilon) = \mathcal{F}_0(u^0)$, and u^0 coincides with the unique solution of problem (4.33). ∎

5. Special cases

5.1. Periodic case

Let $V \subset \mathbb{R}^2$ and $p : V \to \mathbb{R}^3$ be a parameterization of Γ_1 such that p is a one-to-one mapping of class C^2 and the rank of $\nabla p(\vartheta)$ is 2 for each point $\vartheta = (\vartheta_1,\vartheta_2) \in V$. For the sake of simplification, we suppose that $p(V) = \Gamma_1$. Let $Y = (0,1)^2$ and $k \in \mathbb{Z}^2$. We set

$$\begin{aligned} Y_\varepsilon^k &= k\varepsilon + \varepsilon Y, \\ I_\varepsilon &= \{k \in \mathbb{Z}^2;\ Y_\varepsilon^k \subset V\}. \end{aligned} \tag{5.1}$$

According to (5.1), we see that the measure $|V \setminus V_\varepsilon|$ of the set $V \setminus V_\varepsilon$, where $V_\varepsilon = \bigcup_{k\in I_\varepsilon} Y_\varepsilon^k$, tends to 0 as $\varepsilon \to 0$ so that $|\Gamma_1 \setminus \bigcup_{k\in I_\varepsilon} p(Y_\varepsilon^k)|$ tends to 0 as $\varepsilon \to 0$ and $\Gamma_1 \approx \bigcup_{k\in I_\varepsilon} p(Y_\varepsilon^k)$. Let us consider a positive 1-periodic function $h \in C^2(Y)$. We consider here layers Σ_ε of the form

$$\Sigma_\varepsilon = \left\{x = s + tn(s);\ s = p(\vartheta),\ 0 < t < \varepsilon h\Big(\frac{\vartheta}{\varepsilon}\Big),\ \vartheta \in V_\varepsilon\right\}.$$

Then, according to Theorem 12, there exist a rich family $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$ and a symmetric matrix $\boldsymbol{\mu} = (\mu_{ij})_{i,j=1,2,3}$ of Borel measures μ_{ij}, which are absolutely continuous with respect to the capacity Cap, having the same support contained in Γ_1, and satisfying $\mu_{ij}(B)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbb{R}^3$, $\forall B \in \mathcal{B}(\mathbb{R}^3)$, such that, for every $u \in W_{\Gamma_2}$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbb{R}^3)$, we have the following equality:

$$\begin{aligned} &\inf\left\{\liminf_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^\varepsilon) : z^\varepsilon + u = 0 \text{ on } \omega \cap \left\{t = \varepsilon h\Big(\frac{\vartheta}{\varepsilon}\Big)\right\};\ \vartheta \in V_\varepsilon,\ z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0\right\} \\ &\quad = \int_{\Gamma_1\cap\omega} u_i u_j \, d\mu_{ij}, \end{aligned} \tag{5.2}$$

where $\Phi_{0,\varepsilon}$ is defined in (4.7). Since Σ_ε has a periodic structure, the problem

$$\inf\left\{\liminf_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^\varepsilon) : z^\varepsilon + u = 0 \text{ on } \omega \cap \left\{t = \varepsilon h\Big(\frac{\vartheta}{\varepsilon}\Big)\right\};\ \vartheta \in V_\varepsilon,\ z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0\right\},$$

is invariant by translation on V_ε and the measure μ_{ij}; $i,j = 1,2,3$, is the Haar measure on Γ_1. Then

$$\mu_{ij} = K_{ij}\, d\rho, \tag{5.3}$$

on Γ_1, where $d\rho$ is the surface measure on Γ_1 which is given by the Riemannian metric and K_{ij}; $i,j = 1,2,3$, are constants in $\overline{\mathbb{R}}$ satisfying $K_{ij}\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbb{R}^3$. Our purpose is to identify the constants K_{ij}. Let us denote Z_h the set defined by

$$Z_h = \{y = (y_1,y_2,y_3) \in \mathbb{R}^3;\ y' = (y_1,y_2) \in Y,\ y_3 \in (0, h(y'))\}.$$

We consider, for $m = 1,2,3$, the following problem:

$$\begin{cases} \operatorname{div}(w^m) = 0 & \text{in } Z_h, \\ \quad w^m = e^m & \text{on } \{y_3 = h(y')\}, \\ \quad w^m = 0 & \text{on } \{y_3 = 0\}, \\ \quad w^m \text{ is } Y\text{-periodic}, \end{cases}$$

where $e^m = (\delta_{1m}, \delta_{2m}, \delta_{3m})$; $m = 1, 2, 3$. Let $t_h = \max_{y \in Y} h(y)$ and $H > t_h$ be a fixed number. Let us now denote Z_H the set defined by

$$Z_H = \{y \in \mathbb{R}^3;\ y' \in Y, y_3 \in (h(y'), H)\}.$$

We consider, for $m = 1, 2, 3$, the following problem:

$$\begin{cases} \operatorname{div}(\widetilde{w}^m) = 0 & \text{in } Z_H, \\ \quad \widetilde{w}^m = e^m & \text{on } \{y_3 = h(y')\}, \\ \quad \widetilde{w}^m = 0 & \text{on } \{y_3 = H\}, \\ \quad \widetilde{w}^m \text{ is } Y\text{-periodic.} \end{cases} \tag{5.4}$$

Let us define the layer $\Sigma_{H,\varepsilon}$ by

$$\Sigma_{H,\varepsilon} = \left\{x = s + tn(s);\ s = p(\vartheta),\ \varepsilon h\left(\frac{\vartheta}{\varepsilon}\right) < t < \varepsilon H;\ \vartheta \in V_\varepsilon\right\},$$

and the sequence $(z_0^{m,\varepsilon})_\varepsilon$; $m = 1, 2, 3$, of test-functions, by

$$z_0^{m,\varepsilon}(x) = \begin{cases} w^m(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}) & \text{if } x = s + tn(s) \in \Sigma_\varepsilon, \\ \widetilde{w}^m(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}) & \text{if } x = s + tn(s) \in \Sigma_{H,\varepsilon}. \end{cases} \tag{5.5}$$

The properties of the sequence $(z_0^{m,\varepsilon})_\varepsilon$; $m = 1, 2, 3$, are stated in the following.

Lemma 16. *We have*

(1) $(z_0^{m,\varepsilon})_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ *and* $(z_0^{m,\varepsilon})_\varepsilon$ *τ-converges to* 0.

(2) $\lim_{\varepsilon \to 0} \Phi_{0,\varepsilon}(z_0^{m,\varepsilon}) = C_m |\Gamma_1|$, *where*

$$C_m = \int_{Z_h} \sigma_{ij}(w^m) e_{ij}(w^m)\, dy\, dt.$$

Proof. (1) Observing that $z_0^{m,\varepsilon} = e^m$ on $\{t = \varepsilon h(\frac{p^{-1}(s)}{\varepsilon})\}$, and $z_0^{m,\varepsilon} = 0$ on $\{t = 0\} \cup \{t = \varepsilon H\}$, we deduce that $(z_0^{m,\varepsilon})_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$, and, since $z_0^{m,\varepsilon} = 0$ in Ω, $z_0^{m,\varepsilon} \underset{\varepsilon \to 0}{\rightharpoonup} 0$ $H^1(\Omega, \mathbb{R}^3)$-weak. On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^3} |z_0^{m,\varepsilon}|^2 dx &= \int_{\Sigma_\varepsilon \cup \Sigma_{H,\varepsilon}} |z_0^{m,\varepsilon}|^2 dx \\ &= \sum_{k \in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_0^{\varepsilon H} |z_0^{m,\varepsilon}|^2 (1 + t\varkappa(s,t))\, dt\, ds \\ &= \sum_{k \in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_0^{\varepsilon h(\frac{p^{-1}(s)}{\varepsilon})} \left| w^m\left(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}\right)\right|^2 (1 + t\varkappa(s,t))\, dt\, ds \\ &\quad + \sum_{k \in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_{\varepsilon h(\frac{p^{-1}(s)}{\varepsilon})}^{\varepsilon H} \left| \widetilde{w}^m\left(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}\right)\right|^2 (1 + t\varkappa(s,t))\, dt\, ds \end{aligned}$$

$$= \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k} \int_0^{\varepsilon h(\frac{\vartheta}{\varepsilon})} \left| w^m\left(\frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon}\right)\right|^2 (1 + t\varkappa(p(\vartheta), t)) J dt d\vartheta$$
$$+ \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k} \int_{\varepsilon h(\frac{\vartheta}{\varepsilon})}^{\varepsilon H} \left| \widetilde{w}^m\left(\frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon}\right)\right|^2 (1 + t\varkappa(p(\vartheta), t)) J dt d\vartheta, \quad (5.6)$$

where $\varkappa$ is the curvature of Γ_1 and $J = |\det(r_1 r_2 r_3)|$ with $(r_1 r_2) = \nabla p$ and $r_3 = n(p^{-1}(s))$. Observing that J and $\varkappa$ are uniformly bounded by some positive constant independent of ε, we deduce from (5.6) that

$$\int_{\mathbb{R}^3} |z_0^{m,\varepsilon}|^2 dx \le C \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k} \int_0^{\varepsilon h(\frac{\vartheta}{\varepsilon})} \left| w^m\left(\frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon}\right)\right|^2 (1 + t\varkappa(p(\vartheta), t)) dt d\vartheta$$
$$+ C \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k} \int_{\varepsilon h(\frac{\vartheta}{\varepsilon})}^{\varepsilon H} \left| \widetilde{w}^m\left(\frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon}\right)\right|^2 (1 + t\varkappa(p(\vartheta), t)) dt d\vartheta$$
$$\le C \sum_{k\in I_\varepsilon} \varepsilon^3 \int_{Z_h} |w^m(y,t)|^2 (1+\varepsilon t) dy dt$$
$$+ C \sum_{k\in I_\varepsilon} \varepsilon^3 \int_{Z_H} |\widetilde{w}^m(y,t)|^2 (1+\varepsilon t) dy dt,$$

from which we deduce that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} |z_0^{m,\varepsilon}|^2 dx = 0.$$

(2) We have

$$\lim_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z_0^{m,\varepsilon}) = \lim_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_0^{m,\varepsilon}) e_{ij}(z_0^{m,\varepsilon}) dx$$
$$= \lim_{\varepsilon\to 0} \varepsilon \sum_{k\in I_\varepsilon} \int_{p(Y_\varepsilon^k)} \int_0^{\varepsilon h(\frac{p^{-1}(s)}{\varepsilon})} A_{ij}^{m,\varepsilon} B_{ij}^{m,\varepsilon} (1 + t\varkappa(s,t)) dt ds \quad (5.7)$$

where

$$A_{ij}^{m,\varepsilon} = \breve{\sigma}_{ij}\left(w^m\left(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}\right)\right),$$
$$B_{ij}^{m,\varepsilon} = \breve{e}_{ij}\left(w^m\left(\frac{p^{-1}(s)}{\varepsilon}, \frac{t}{\varepsilon}\right)\right),$$

where $\breve{\sigma}_{ij}$ and $\breve{e}_{ij}$; $i, j = 1, 2, 3$, are, respectively, the components of stress and deformation tensors in the local basis r_1, r_2, r_3, with

$$\begin{aligned} \breve{\sigma}_{ij}(u) &= a_{ijkl}\breve{e}_{kl}(u), \\ \breve{e}_{ij}(u) &= \frac{1}{2}(\nabla_j u_i + \nabla_i u_j), \end{aligned} \tag{5.8}$$

where $\nabla_j u_i = \frac{\partial u_i}{\partial s^j} - u_l \Gamma^l_{ij}$; $\Gamma^l_{ij} = \Gamma^l_{ji} = \frac{\partial s^l}{\partial x_k}\frac{\partial^2 x_k}{\partial s^i \partial s^j}$ being the Christoffel symbol of the second kind with

$$x(s) = s + tn = s^1 r_1 + s^2 r_2 + t r_3.$$

Let us set $p^{-1}(s) = \vartheta = \varepsilon y' + \varepsilon k$; $k \in \mathbb{Z}^2$, and $y_3 = \frac{t}{\varepsilon}$. Then, for every $\vartheta \in Y^k_\varepsilon$; $k \in \mathbb{Z}^2$, we have that $J(\vartheta) = \det \nabla p(\varepsilon k) + O(\varepsilon)$ and

$$\begin{aligned} &\sum_{k\in I_\varepsilon} \int_{p(Y^k_\varepsilon)} \int_0^{\varepsilon h(\frac{p^{-1}(s)}{\varepsilon})} A^{m,\varepsilon}_{ij} B^{m,\varepsilon}_{ij}(1 + t\varkappa(s,t))dt\,ds \\ &\quad= \sum_{k\in I_\varepsilon} \int_{Y^k_\varepsilon} \int_0^{\varepsilon h(\frac{\vartheta}{\varepsilon})} C^{m,\varepsilon}_{ij} D^{m,\varepsilon}_{ij}(1 + t\varkappa) J(\vartheta)dt\,d\vartheta \\ &\quad= \sum_{k\in I_\varepsilon} \varepsilon^3 \int_{Z_h} \sigma_{ij}(w^m) e_{ij}(w^m)(1 + \varepsilon y_3 \varkappa) J(\varepsilon y' + \varepsilon k)dy \\ &\quad= \varepsilon |\det \nabla p(\varepsilon k)| C_m + O(\varepsilon)\varepsilon, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} C^{m,\varepsilon}_{ij} &= \sigma_{ij}\left(w^m\left(\frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon}\right)\right), \\ D^{m,\varepsilon}_{ij} &= e_{ij}\left(w^m\left(\frac{\vartheta}{\varepsilon}, \frac{t}{\varepsilon}\right)\right). \end{aligned}$$

Thus, combining (5.7) and (5.9), we deduce that

$$\lim_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^{m,\varepsilon}_0) = \lim_{\varepsilon\to 0} \sum_{k\in I_\varepsilon} \varepsilon^2 |\det \nabla p(\varepsilon k)| C_m = C_m |\Gamma_1|. \qquad \blacksquare$$

The constants K_{ij}; $i, j = 1, 2, 3$, in equality (5.3) are identified in the following.

Theorem 17. *We have*

$$K_{lm} = \left(\int_{Z_h} \sigma_{ij}(w^l) e_{ij}(w^m) dy\,dt\right) \delta_{lm}.$$

Proof. According to (5.2) and (5.3), we have

$$\inf\Big\{\liminf_{\varepsilon\to 0}\Phi_{0,\varepsilon}(z^\varepsilon);\ z^\varepsilon+u=0 \text{ on } \Big\{t=\varepsilon h\Big(\frac{\vartheta}{\varepsilon}\Big)\Big\},\ \forall\vartheta\in V_\varepsilon,\ z^\varepsilon\xrightarrow[\varepsilon\to 0]{\tau}0\Big\}$$
$$=K_{ij}\int_{\Gamma_1}u_iu_j\,d\rho. \tag{5.10}$$

Let $u=-e^m$ on $\overline{\Sigma}_\varepsilon$. We deduce from (5.10) that

$$K_{mm}|\Gamma_1|=\inf\Big\{\liminf_{\varepsilon\to 0}\Phi_{0,\varepsilon}(z^\varepsilon);\ z^\varepsilon=e^m \text{ on } \Big\{t=\varepsilon h\Big(\frac{\vartheta}{\varepsilon}\Big)\Big\},\ \forall\vartheta\in V_\varepsilon,\ z^\varepsilon\xrightarrow[\varepsilon\to 0]{\tau}0\Big\},$$

from which we deduce, using Lemma 16, that

$$K_{mm}|\Gamma_1|\le C_m|\Gamma_1|,$$

hence

$$K_{mm}\le C_m. \tag{5.11}$$

Let $(z^\varepsilon)_\varepsilon\subset H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ such that $z^\varepsilon=e^m$ on $\{t=\varepsilon h(\frac{\vartheta}{\varepsilon})\}$ and $(z^\varepsilon)_\varepsilon$ τ-converges to 0. We have from the definition of the subdifferentiability of convex functionals that

$$\Phi_{0,\varepsilon}(z^\varepsilon)\ge\Phi_{0,\varepsilon}(z_0^\varepsilon)+2\varepsilon\int_{\Sigma_\varepsilon}\sigma_{ij}(z_0^{m,\varepsilon})e_{ij}(z^\varepsilon-z_0^{m,\varepsilon})dx. \tag{5.12}$$

Using Green's formula, we deduce that

$$\varepsilon\int_{\Sigma_\varepsilon}\sigma_{ij}(z_0^{m,\varepsilon})e_{ij}(z^\varepsilon-z_0^{m,\varepsilon})dx=-\varepsilon\int_{\Sigma_\varepsilon}\sigma_{ij,j}(z_0^{m,\varepsilon})(z_i^\varepsilon-z_{0,i}^{m,\varepsilon})dx$$
$$+\varepsilon\int_{\Gamma_1\cup\Gamma_{1,\varepsilon}}\sigma_{ij}(z_0^{m,\varepsilon})n_j(z_i^\varepsilon-z_{0,i}^{m,\varepsilon})ds. \tag{5.13}$$

As $z^\varepsilon-z_0^{m,\varepsilon}=0$ on $\Gamma_{1,\varepsilon}$ and $z_0^{m,\varepsilon}=0$ on Γ_1, we have that

$$\varepsilon\int_{\Gamma_1\cup\Gamma_{1,\varepsilon}}\sigma_{ij}(z_0^{m,\varepsilon})n_j(z_i^\varepsilon-z_{0,i}^{m,\varepsilon})ds=\varepsilon\int_{\Gamma_1}\sigma_{ij}(z_0^{m,\varepsilon})n_jz_i^\varepsilon ds. \tag{5.14}$$

Then, using the expression (5.5) of $z_0^{m,\varepsilon}$ and the trace theorem, we infer that

$$\Big|\varepsilon\int_{\Gamma_1}\sigma_{ij}(z_0^{m,\varepsilon})n_jz_i^\varepsilon dx\Big|\le\Big(\varepsilon\int_{\Gamma_1}(\sigma_{ij}(z_0^{m,\varepsilon})n_j)^2ds\Big)^{1/2}\Big(\varepsilon\int_{\Gamma_1}(z_i^{m,\varepsilon})^2ds\Big)^{1/2}$$
$$\le C\Big(\int_Y(\sigma_{ij}(w^m)n_j)^2dy\Big)^{1/2}\Big(\varepsilon\int_\Omega|\nabla z_i^{m,\varepsilon}|^2dx\Big)^{1/2},$$

from which we deduce that

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\Gamma_1\cup\Gamma_{1,\varepsilon}}\sigma_{ij}(z_0^{m,\varepsilon})n_j(z_i^\varepsilon-z_{0,i}^{m,\varepsilon})ds=0. \tag{5.15}$$

Besides, an easy computation implies that

$$\begin{aligned}\varepsilon\sigma_{ij,j}(z_0^{m,\varepsilon}) &\underset{\varepsilon\to 0}{\longrightarrow} \mathbf{1}_{\Gamma_1}\int_{Z_h}\sigma_{ij,j}(w^m)dydt\\ &= 0,\end{aligned}\tag{5.16}$$

in $L^2(\mathbb{R}^3)$-weak, where $\mathbf{1}_{\Gamma_1}$ is the characteristic function of Γ_1. Thus, combining (5.13), (5.15), and (5.16), we infer that

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\Sigma_\varepsilon}\sigma_{ij}(z_0^{m,\varepsilon})e_{ij}(z^\varepsilon - z_0^{m,\varepsilon})dx = 0,$$

and, passing to the lower limit in (5.12), we deduce that

$$\liminf_{\varepsilon\to 0}\Phi_{0,\varepsilon}(z^\varepsilon) \geq \liminf_{\varepsilon\to 0}\Phi_{0,\varepsilon}(z_0^\varepsilon) = C_m|\Gamma_1|.\tag{5.17}$$

Now, taking in (5.17) the infimum over all sequences $(z^\varepsilon)_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ such that $z^\varepsilon = e^m$ on $\left\{t = \varepsilon h(\frac{\vartheta}{\varepsilon})\right\}$ and $z^\varepsilon \underset{\varepsilon\to 0}{\overset{\tau}{\longrightarrow}} 0$, we deduce that

$$K_{mm}|\Gamma_1| \geq C_m|\Gamma_1|,$$

hence

$$K_{mm} \geq C_m.\tag{5.18}$$

We conclude from (5.11) and (5.18) that $K_{mm} = C_m$. Taking $u = -(e^1 + e^2)$ on $\overline{\Sigma}_\varepsilon$, we deduce from (5.10) that

$$\begin{aligned}(K_{11} + 2K_{12} + K_{22})|\Gamma_1| = \inf\Big\{&\liminf_{\varepsilon\to 0}\Phi_{0,\varepsilon}(z^\varepsilon);\ z^\varepsilon = e^1 + e^2 \text{ on } \left\{t = \varepsilon h\left(\frac{\vartheta}{\varepsilon}\right)\right\},\\ &\forall\vartheta\in V_\varepsilon,\ \text{and } z^\varepsilon \underset{\varepsilon\to 0}{\overset{\tau}{\longrightarrow}} 0\Big\}\\ \leq\ &\liminf_{\varepsilon\to 0}\Phi_{0,\varepsilon}(z_0^{1,\varepsilon} + z_0^{2,\varepsilon}).\end{aligned}\tag{5.19}$$

As

$$\int_{Z_h}\sigma_{ij}(w^m)e_{ij}(w^l)dydt = 0 \quad \text{for } m \neq l,$$

we have

$$\liminf_{\varepsilon\to 0}\Phi_{0,\varepsilon}(z_0^{1,\varepsilon} + z_0^{2,\varepsilon}) = |\Gamma_1|(C_1 + C_2),$$

hence, according to (5.19),

$$K_{12} \leq 0.\tag{5.20}$$

Let $(z^\varepsilon)_\varepsilon \subset H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ such that $z^\varepsilon = e^1 + e^2$ on $\{t = \varepsilon h(\frac{\vartheta}{\varepsilon})\}$ and $z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0$. We have from the definition of the subdifferentiability of convex functionals that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^\varepsilon) \geq \liminf_{\varepsilon\to 0}\Big(&\Phi_{0,\varepsilon}(z_0^{1,\varepsilon} + z_0^{2,\varepsilon}) \\ &+ 2\varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z_0^{1,\varepsilon} + z_0^{2,\varepsilon}) e_{ij}(z^\varepsilon - (z_0^{1,\varepsilon} + z_0^{2,\varepsilon}))dx\Big) \\ \geq{}& |\Gamma_1|(C_1 + C_2),\end{aligned}$$

from which we deduce that

$$K_{12} \geq 0. \tag{5.21}$$

Thus, combining (5.20) and (5.21), we deduce that $K_{12} = 0$. Doing the same for K_{23}, we deduce that $K_{lm} = 0$ for $l \neq m$. ■

5.2. Case of thickness $\varepsilon h(s)$

We suppose here that

$$\Sigma_\varepsilon = \big\{s + tn(s);\ s \in \Gamma_1,\ 0 < t < \varepsilon h(s)\big\},$$

where h is a positive continuous function on Γ_1. Then, according to Theorem 12, there exist a rich family $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$ and a symmetric matrix $\boldsymbol{\mu} = (\mu_{ij})_{i,j=1,2,3}$ of Borel measures μ_{ij}, which are absolutely continuous with respect to the capacity Cap, having the same support contained in Γ_1, and satisfying $\mu_{ij}(B)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbb{R}^3$, $\forall B \in \mathcal{B}(\mathbb{R}^3)$, such that, for every $u \in W_{\Gamma_2}$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$, we have

$$\inf\Big\{\liminf_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^\varepsilon);\ z^\varepsilon + u = 0 \text{ on } \omega \times \{t = \varepsilon h(s)\},\ z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0\Big\} = \int_\omega u_i u_j \, d\mu_{ij}. \tag{5.22}$$

The main result in this subsection reads as follows.

Theorem 18. *The matrix of measures $(\mu_{ij})_{i,j=1,2,3}$ is given by*

$$\mu_{ij} = \kappa_i(s)\frac{ds}{h(s)}\delta_{ij}; \quad i, j = 1, 2, 3.$$

For homogeneous and isotropic materials

$$\kappa_i = \begin{cases} \frac{E}{(1+\nu)} & \text{for } i = 1, 2, \\ \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \text{for } i = 3. \end{cases}$$

Proof. We consider the sequence $(w^{m,\varepsilon})_\varepsilon$; $m = 1, 2, 3$, defined, for $x = s + tn(s) \in \Sigma_\varepsilon$, by $w^{m,\varepsilon}(x) = e^m \frac{t}{\varepsilon h(s)}$, and the sequence $(\widetilde{w}^{m,\varepsilon})_\varepsilon$; $m = 1, 2, 3$, defined, for $x =$

$s + tn(s) \in \Sigma_{\varepsilon,H}$, by $\widetilde{w}^{m,\varepsilon}(x) = e^m \frac{t-\varepsilon H}{\varepsilon(h(s)-H)}$, where H is a positive number such that $H > \sup_{s\in\Gamma_1} h(s)$ and

$$\Sigma_{\varepsilon,H} = \{s + tn(s);\ s \in \Gamma_1, \varepsilon h(s) < t < \varepsilon H\}.$$

Let $\omega \in \mathscr{R} \cap \mathscr{O}(\Gamma_1)$. Let $\delta > 0$ be a small parameter. We define the open set $\omega_\delta \subset \Gamma_1$ by

$$\omega_\delta = \{s \in \Gamma_1;\ d(s,\omega) < \delta\},$$

where $d(s,\omega)$ is the distance between s and ω in curvilinear coordinates on Γ_1. We define the auxiliary layers

$$\begin{aligned}
\Sigma_{\varepsilon,\omega} &= \{s + tn(s);\ s \in \omega,\ 0 < t < \varepsilon h(s)\},\\
\Sigma_{\varepsilon,\omega,H} &= \{s + tn(s);\ s \in \omega,\ \varepsilon h(s) < t < \varepsilon H\},\\
\Sigma_{\varepsilon,\omega_\delta} &= \{s + tn(s);\ s \in \omega_\delta,\ 0 < t < \varepsilon h(s)\},\\
\Sigma_{\varepsilon,\omega_\delta,H} &= \{s + tn(s);\ s \in \omega_\delta,\ \varepsilon h(s) < t < \varepsilon H\}.
\end{aligned}$$

Let $\varphi_{\delta,\varepsilon}$ be a smooth function such that $0 \le \varphi_{\delta,\varepsilon} \le 1$ in $\mathbb{R}^3$ and

$$\varphi_{\delta,\varepsilon} = \begin{cases} 1 & \text{in } \Sigma_{\varepsilon,\omega} \cup \Gamma_{1,\varepsilon,\omega} \cup \Sigma_{\varepsilon,\omega,H},\\ 0 & \text{in } \Sigma_\varepsilon \setminus \overline{\Sigma_{\varepsilon,\omega_\delta} \cup \Sigma_{\varepsilon,\omega_\delta,H}}, \end{cases}$$

where $\Gamma_{1,\varepsilon,\omega} = \partial\Sigma_{\varepsilon,\omega} \cap \partial\Sigma_{\varepsilon,\omega,H}$. Let us define the sequence $(z^{m,\varepsilon}_{0,\omega_\delta})_\varepsilon$; $m = 1, 2, 3$, by

$$z^{m,\varepsilon}_{0,\omega_\delta} = \begin{cases} \varphi_{\delta,\varepsilon} w^{m,\varepsilon} & \text{in } \Sigma_{\varepsilon,\omega_\delta},\\ \varphi_{\delta,\varepsilon} \widetilde{w}^{m,\varepsilon} & \text{in } \Sigma_{\varepsilon,\omega_\delta,H}, \end{cases}$$

so that $z^{m,\varepsilon}_{0,\omega_\delta} \in H^1_{\Gamma_2}(\mathbb{R}^3,\mathbb{R}^3)$ and $z^{m,\varepsilon}_{0,\omega} \xrightarrow[\varepsilon\to 0]{\tau} 0$. We then compute

$$\lim_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^{m,\varepsilon}_{0,\omega_\delta}) = \lim_{\varepsilon\to 0} \varepsilon \int_{\Sigma_{\varepsilon,\omega_\delta}} \sigma_{ij}(z^{m,\varepsilon}_{0,\omega_\delta}) e_{ij}(z^{m,\varepsilon}_{0,\omega_\delta}) dx \tag{5.23a}$$

$$= \lim_{\varepsilon\to 0} \varepsilon \int_{\Sigma_{\varepsilon,\omega}} \sigma_{ij}(w^{m,\varepsilon}) e_{ij}(w^{m,\varepsilon}) dx \tag{5.23b}$$

$$+ \lim_{\varepsilon\to 0} \varepsilon \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} \sigma_{ij}(\varphi_{\delta,\varepsilon} w^{m,\varepsilon}) e_{ij}(\varphi_{\delta,\varepsilon} w^{m,\varepsilon}) dx. \tag{5.23c}$$

We have that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon \int_{\Sigma_{\varepsilon,\omega}} \sigma_{ij}(w^{m,\varepsilon}) e_{ij}(w^{m,\varepsilon}) dx\\
&= \lim_{\varepsilon\to 0} \varepsilon \int_\omega \int_0^{\varepsilon h(s)} \breve{\sigma}_{ij}(w^{m,\varepsilon}) \breve{e}_{ij}(w^{m,\varepsilon})(1 + t\varkappa) dt ds\\
&= \lim_{\varepsilon\to 0} \varepsilon \int_\omega \kappa_m(s) \int_0^{\varepsilon h(s)} \left|\frac{\partial w^{m,\varepsilon}_m}{\partial t}\right|^2 (1 + t\varkappa) dt ds
\end{aligned}$$

$$= \lim_{\varepsilon\to 0} \varepsilon^2 \int_\omega \kappa_m(s) \int_0^{h(s)} \frac{1}{\varepsilon^2 h^2(s)}(1+\varepsilon t\varkappa)dtds$$
$$= \int_\omega \frac{\kappa_m(s)}{h(s)} ds, \tag{5.24}$$

where $\breve{\sigma}_{ij}$ and $\breve{e}_{ij}$ are defined in (5.8), $\varkappa$ is the curvature of Γ_1, and $\kappa_m(s)$; $m = 1, 2, 3$, are material coefficients. On the other hand, using a convexity argument and the coercivity property (1.2b), we infer that, for every $\eta \in (0, 1)$,

$$\begin{aligned}
&\varepsilon \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} \sigma_{ij}(\varphi_{\delta,\varepsilon} w^{m,\varepsilon}) e_{ij}(\varphi_{\delta,\varepsilon} w^{m,\varepsilon}) dx \\
&\quad \le \frac{\varepsilon}{1-\eta} \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} \sigma_{ij}(w^{m,\varepsilon}) e_{ij}(w^{m,\varepsilon}) dx \\
&\qquad + \frac{\varepsilon C}{\eta} \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} |w^{m,\varepsilon}|^2 |\nabla \varphi_{\delta,\varepsilon}|^2 dx \\
&\quad \le \frac{\varepsilon}{1-\eta} \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} \sigma_{ij}(w^{m,\varepsilon}) e_{ij}(w^{m,\varepsilon}) dx \\
&\qquad + \frac{\varepsilon C}{\eta\delta^2} \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} |w^{m,\varepsilon}|^2 dx,
\end{aligned}$$

where C is a positive constant independent of ε. Then, taking $\delta = \sqrt{\varepsilon}$, we deduce, using (5.24) and the fact that $|\omega_\delta \setminus \omega| \to 0$ as $\delta \to 0$, that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{\varepsilon}{1-\eta} \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} \sigma_{ij}(w^{m,\varepsilon}) e_{ij}(w^{m,\varepsilon}) dx \\
&\quad = \lim_{\varepsilon\to 0} \frac{\varepsilon}{1-\eta} \int_{\omega_\delta\setminus\omega} \int_0^{\varepsilon h(s)} \breve{\sigma}_{ij}(w^{m,\varepsilon}) \breve{e}_{ij}(w^{m,\varepsilon})(1+t\varkappa) dtds \\
&\quad = \lim_{\varepsilon\to 0} \frac{\varepsilon}{1-\eta} \int_{\omega_\delta\setminus\omega} \kappa_m(s) \int_0^{\varepsilon h(s)} \left|\frac{\partial w_m^{m,\varepsilon}}{\partial t}\right|^2 (1+t\varkappa) dtds \\
&\quad = \lim_{\varepsilon\to 0} \frac{\varepsilon^2}{1-\eta} \int_{\omega_\delta\setminus\omega} \kappa_m(s) \int_0^{h(s)} \frac{1}{\varepsilon^2 h^2(s)} (1+\varepsilon t\varkappa) dtds \\
&\quad = 0,
\end{aligned}$$

and, since $w^{m,\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0$ $L^2(\mathbb{R}^3, \mathbb{R}^3)$-strong,

$$\lim_{\varepsilon\to 0} \frac{\varepsilon C}{\eta\delta^2} \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} |w^{m,\varepsilon}|^2 dx = \lim_{\varepsilon\to 0} \frac{C}{\eta} \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} |w^{m,\varepsilon}|^2 dx = 0,$$

hence

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Sigma_{\varepsilon,\omega_\delta}\setminus\Sigma_{\varepsilon,\omega}} \sigma_{ij}(\varphi_{\delta,\varepsilon} w^{m,\varepsilon}) e_{ij}(\varphi_{\delta,\varepsilon} w^{m,\varepsilon}) dx = 0. \tag{5.25}$$

Thus, replacing in (5.23b) and (5.23c) by the limits obtained in (5.24) and (5.25) respectively, we get

$$\lim_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^{m,\varepsilon}_{0,\omega_\delta}) = \int_\omega \frac{\kappa_m(s)}{h(s)} ds.$$

According to (5.22), we have, using to the above equality, that, for every $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$,

$$\mu_{mm}(\omega) \leq \lim_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^{m,\varepsilon}_{0,\omega_\delta}) = \int_\omega \frac{\kappa_m(s)}{h(s)} ds. \tag{5.26}$$

Let $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$. Let $(z^{m,\varepsilon})_\varepsilon$; $m = 1, 2, 3$, be any sequence in $H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ such that $z^{m,\varepsilon} = e^m$ on $\omega \times \{t = \varepsilon h(s)\}$ and $z^\varepsilon \xrightarrow[\varepsilon\to 0]{\tau} 0$. Let us consider the subdifferential inequality

$$\Phi_{0,\varepsilon}(z^{m,\varepsilon}) \geq \Phi_{0,\varepsilon}(z^{m,\varepsilon}_{0,\omega_\delta}) + 2\varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z^{m,\varepsilon}_{0,\omega_\delta}) e_{ij}(z^{m,\varepsilon} - z^{m,\varepsilon}_{0,\omega_\delta}) dx. \tag{5.27}$$

By calculations similar to those carried out in (5.13)–(5.16) (in the previous subsection) we deduce that

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \sigma_{ij}(z^{m,\varepsilon}_{0,\omega_\delta}) e_{ij}(z^{m,\varepsilon} - z^{m,\varepsilon}_{0,\omega_\delta}) dx = 0,$$

which implies, passing to the lower limit in (5.27), that

$$\liminf_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^{m,\varepsilon}) \geq \liminf_{\varepsilon\to 0} \Phi_{0,\varepsilon}(z^{m,\varepsilon}_{0,\omega_\delta}),$$

thus, taking the infimum over all sequences $(z^{m,\varepsilon})_\varepsilon$ satisfying the above properties, we get

$$\mu_{mm}(\omega) \geq \int_\omega \frac{\kappa_m(s)}{h(s)} ds. \tag{5.28}$$

Therefore, according to (5.26) and (5.28), we have that

$$\mu_{mm}(\omega) = \int_\omega \frac{\kappa_m(s)}{h(s)} ds.$$

Moreover, replacing $z^{m,\varepsilon}_{0,\omega_\delta}$ by $z^{m,\varepsilon}_{0,\omega_\delta} + z^{l,\varepsilon}_{0,\omega_\delta}$; $l \neq m$, we prove as in the previous subsection, that $\mu_{ml}(\omega) = \mu_{lm}(\omega) = 0$. Thus

$$\mu_{ij} = \kappa_i(s) \frac{ds}{h(s)} \delta_{ij}, \quad \forall i, j = 1, 2, 3.$$

If the material is homogeneous and isotropic with

$$a_{ijkl} = \frac{E}{2(1+\nu)} \left\{ \delta_{ik}\delta_{jl} + \delta_{il}\delta_k + \frac{2\nu}{(1-2\nu)} \delta_{ij}\delta_{kl} \right\},$$

where $E > 0$ is the Young modulus and $\nu \in (0, 1/2)$ is the Poisson ratio, then we obtain, after some computations, that $\kappa_m = \frac{E}{(1+\nu)}$ for $m = 1, 2$, and $\kappa_m = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$ for $m = 3$. ∎

5.3. Fractal layers

Let $\mathbb{R}^{3-}$ be the halfspace defined by

$$\mathbb{R}^{3-} = \{x = (x_1, x_3, x_3) \in \mathbb{R}^3; x_3 < 0\}.$$

Let $\Omega \subset \mathbb{R}^{3-}$ be a bounded open subset with Lipschitz continuous boundary $\partial\Omega$. We set

$$\Gamma_1 = \partial\Omega \cap \{x_3 = 0\},$$
$$\Gamma_2 = \partial\Omega \backslash \Gamma_1.$$

We suppose that $\Gamma_1 \neq \emptyset$. Let N be a positive integer and $\psi_1, \ldots, \psi_N$ be a finite family of contractive similitudes on $\mathbb{R}^2$ with ratio $0 < \rho < 1$. There exists a unique compact subset $\Lambda \subset \mathbb{R}^2$, such that

$$\Lambda = \bigcup_{i=1}^{N} \psi_i(\Lambda).$$

We suppose that the family $(\psi_i)_{i=1,\ldots,N}$ satisfies the following open set condition: there exists a bounded open subset $U \subset \mathbb{R}^2$ such that

$$\psi_i(U) \subset U \quad \forall i = 1, \ldots, N,$$
$$\psi_i(U) \cap \psi_j(U) = \emptyset \quad \text{if } i \neq j.$$

This condition prevents distinct copies $\psi_i(\Lambda)$ from having overlapping interiors. The real number $d = -\ln N/\ln\rho$ is the similarity dimension of Λ. Moreover, there exists a unique Borel regular measure ϖ with unit mass which is invariant for $\{\psi_1, \ldots, \psi_N\}$, that is

$$\int_\Lambda \varphi d\varpi = \sum_{i=1}^{N} \rho^{-d} \int_\Lambda \varphi \circ \psi_i d\varpi, \tag{5.29}$$

for every integrable $\varphi : \Lambda \to \mathbb{R}$, and ϖ is supported on Λ. Indeed, the measure ϖ is given by

$$\varpi = \frac{\mathcal{H}^d \lfloor \Lambda}{\mathcal{H}^d(\Lambda)},$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure. For the definitions of the self-similar fractals, their dimensions and their Hausdorff measures, we refer to [23]. We suppose here that $d > 1$ and $\Lambda \subset \Gamma_1$.

Let us denote $x' = (x_1, x_2)$ and $D(x', R)$ the disk of radius $R > 0$ centered at x'. Let $x'_0 \in \Lambda$ be a fixed point, we define, for every positive integer k and every indices $i_1, \dots, i_k \in \{1, 2, \dots, N\}$,

$$\begin{aligned} \psi_{i_1,\dots,i_k} &= \psi_{i_1} \circ \cdots \circ \psi_{i_k}, \\ x'_{i_1,\dots,i_k} &= \psi_{i_1,\dots,i_k}(x'_0), \\ D_{i_1,\dots,i_k} &= D(x'_{i_1,\dots,i_k}, \rho^{dh}), \end{aligned} \tag{5.30}$$

and set

$$D_k = \bigcup_{i_1,\dots,i_k \in \{1,2,\dots,N\}} D_{i_1,\dots,i_k}.$$

Let $\zeta \in C^1([0, 1])$ such that

$$\zeta > 0, \quad \zeta(1) = 1, \quad \text{and} \quad \int_0^1 \frac{t}{\zeta(t)} dt \geq 1/2.$$

For every $k \in \mathbb{N}$, we define the function h_k on Γ_1 by

$$h_k(x') = \begin{cases} \sum_{i_1,\dots,i_k \in \{1,\dots,N\}} \zeta\Big(\frac{|x'-x'_{i_1,\dots,i_k}|}{\sqrt{\rho^{dk}}}\Big) \mathbf{1}_{D_{i_1,\dots,i_k}} & \text{if } x' \in D_k, \\ 1 & \text{if } x' \in \Gamma_1 \backslash D_k. \end{cases}$$

We define the layer

$$\Sigma_k = \big\{x \in \mathbb{R}^3;\, x' \in \Gamma_1,\, 0 < x_3 < \varepsilon_k h_k(x')\big\},$$

where $\varepsilon_k = \rho^k$. Then, according to Theorem 12, there exist a rich family $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^3)$ and a symmetric matrix $\boldsymbol{\mu} = (\mu_{ij})_{i,j=1,2,3}$ of Borel measures μ_{ij}, which are absolutely continuous with respect to the capacity Cap, having the same support contained in Γ_1, and satisfying $\mu_{ij}(B)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbb{R}^3$, $\forall B \in \mathcal{B}(\mathbb{R}^3)$, such that, for every $u \in W_{\Gamma_2}$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$, we have

$$\begin{aligned} \inf\Big\{ \liminf_{k\to\infty} \Phi_{0,\varepsilon_k}(z^k);\, z^k + u = 0 \text{ on } \omega \times \{x_3 = \varepsilon_k h_k(x')\},\, z^k \xrightarrow[k\to\infty]{\tau} 0 \Big\} \\ = \int_\omega u_i u_j d\mu_{ij}, \end{aligned} \tag{5.31}$$

where

$$\Phi_{0,\varepsilon_k}(z^k) = \varepsilon_k \int_{\Sigma_k} \sigma_{ij}(z^k) e_{ij}(z^k) dx.$$

The main result in this subsection is stated in the following.

Theorem 19. *The matrix of measures* $(\mu_{ij})_{i,j=1,2,3}$ *is given by*

$$\mu_{ij} = \left(\kappa_i(x')dx'|_{\Gamma_1} + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)}\kappa_i(s)d\mathcal{H}^d(s)|_\Lambda\right)\delta_{ij}; \quad i,j=1,2,3,$$

where $c = \int_0^1 \frac{r}{\zeta(r)}dr$ *and* $\kappa_i(s)$*;* $i = 1,2,3$*, are material coefficients. For homogeneous and isotropic materials,*

$$\kappa_i = \begin{cases} \frac{E}{(1+\nu)} & \text{for } i = 1,2, \\ \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \text{for } i = 3. \end{cases}$$

Proof. We consider the sequence $(w^{m,k})_k$; $m = 1,2,3$, defined, for every $x \in \Sigma_k$, by $w^{m,k}(x) = e^m \frac{x_3}{\varepsilon_k h_k(x')}$, and the sequence $(\widetilde{w}^{m,k})_k$; $m = 1,2,3$, defined, for every $x \in \Sigma_{k,H}$, by $\widetilde{w}^{m,k}(x) = e^m \frac{x_3 - \varepsilon_k H}{\varepsilon_k(h_k(x') - H)}$, where H is a positive number such that $H > \sup_{s\in[0,1]} \zeta(s)$ and

$$\Sigma_{k,H} = \{x \in \mathbb{R}^3;\ x' \in \Gamma_1,\ \varepsilon_k h_k(x') < x_3 < \varepsilon_k H\}.$$

Let $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$. Let $(\delta_k)_k$ be a sequence of small positive numbers such that $\lim_{k\to\infty} \delta_k = 0$. We define, for every $k \in \mathbb{N}$, the open set $\omega_{\delta_k} \subset \Gamma_1$ by

$$\omega_{\delta_k} = \{x' \in \Gamma_1;\ d(x',\omega) < \delta_k\},$$

We define the layers

$$\begin{aligned} \Sigma_{k,\omega} &= \{x;\ x' \in \omega,\ 0 < x_3 < \varepsilon_k h(s)\}, \\ \Sigma_{k,\omega,H} &= \{x;\ x' \in \omega,\ \varepsilon_k h_k(x') < x_3 < \varepsilon_k H\}, \\ \Sigma_{k,\omega_{\delta_k}} &= \{x;\ x' \in \omega_{\delta_k},\ 0 < x_3 < \varepsilon_k h(s)\}, \\ \Sigma_{k,\omega_{\delta_k},H} &= \{x;\ x' \in \omega_{\delta_k},\ \varepsilon_k h_k(x') < x_3 < \varepsilon_k H\}. \end{aligned}$$

Let φ_k be a smooth function such that $0 \le \varphi_k \le 1$ in $\mathbb{R}^3$ and

$$\varphi_k = \begin{cases} 1 & \text{in } \Sigma_{k,\omega} \cup \Gamma_{1,\varepsilon_k,\omega} \cup \Sigma_{k,\omega,H}, \\ 0 & \text{in } \Sigma_\varepsilon \setminus \overline{\Sigma_{k,\omega_{\delta_k}} \cup \Sigma_{k,\omega_{\delta_k},H}}, \end{cases}$$

where $\Gamma_{1,\varepsilon_k,\omega} = \partial\Sigma_{k,\omega} \cap \partial\Sigma_{k,\omega,H}$. Let us define, for $m = 1,2,3$, the sequence $(z_0^{m,k})_k$ by

$$z_0^{m,k} = \begin{cases} \varphi_k w^{m,k} & \text{in } \Sigma_{k,\omega_{\delta_k}}, \\ \varphi_k \widetilde{w}^{m,k} & \text{in } \Sigma_{k,\omega_{\delta_k},H}, \end{cases}$$

so that $z_0^{m,k} \in H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ and $z_0^{m,k} \xrightarrow[k\to\infty]{\tau} 0$. We then compute

$$\begin{aligned}
&\lim_{k\to\infty} \varepsilon_k \int_{\Sigma_k} \sigma_{ij}(z_0^{m,k}) e_{ij}(z_0^{m,k}) dx \\
&\quad = \lim_{k\to\infty} \varepsilon_k \int_{\omega\cap D_k} \int_0^{\varepsilon_k h_k(x')} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k}) dx \\
&\qquad + \lim_{k\to\infty} \varepsilon_k \int_{\omega\setminus D_k} \int_0^{\varepsilon_k} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k}) dx \\
&\qquad + \lim_{k\to\infty} \varepsilon_k \int_{\omega_{\delta_k}\setminus\omega} \int_0^{\varepsilon_k} \sigma_{ij}(\varphi_k w^{m,k}) e_{ij}(\varphi_k w^{m,k}) dx.
\end{aligned}$$

As in the previous subsection, we can easily prove that, for $\delta_k = \sqrt{\varepsilon_k}$,

$$\lim_{k\to\infty} \varepsilon_k \int_{\omega_{\delta_k}\setminus\omega} \int_0^{\varepsilon_k} \sigma_{ij}(\varphi_k w^{m,k}) e_{ij}(\varphi_k w^{m,k}) dx = 0.$$

On the other hand, passing in polar coordinates

$$r = \frac{|x' - x'_{i_1,\dots,i_k}|}{\rho^{dk}} \quad \text{and} \quad \theta \in [0, 2\pi],$$

we obtain, using the fact that $\rho^{dk} = \frac{1}{N^k}$, that

$$\begin{aligned}
&\lim_{k\to\infty} \varepsilon_k \int_{\omega\cap D_k} \int_0^{\varepsilon_k h_k(x')} \sigma_{ij}(z_0^{m,k}) e_{ij}(z_0^{m,k}) dx \\
&\quad = \lim_{k\to\infty} \sum_{i_1,\dots,i_k\in\{1,2,\dots,N\}} \int_{\omega\cap D_{i_1,\dots,i_k}} \frac{\kappa_m(x')}{\zeta\left(\frac{|x'-x'_{i_1,\dots,i_k}|}{\sqrt{\rho^{dk}}}\right)} dx' \\
&\quad = \lim_{k\to\infty} \sum_{\substack{i_1,\dots,i_k\in\{1,2,\dots,N\},\\ D_{i_1,\dots,i_k}\subset\Lambda\cap\omega}} 2\pi\rho^{dk}\kappa_m(x'_{i_1,\dots,i_k}) \int_0^1 \frac{r}{\zeta(r)} dr \\
&\quad = 2\pi \int_0^1 \frac{r}{\zeta(r)} dr \lim_{k\to\infty} \sum_{\substack{i_1,\dots,i_k\in\{1,2,\dots,N\},\\ D_{i_1,\dots,i_k}\subset\Lambda\cap\omega}} \frac{\kappa_m(x'_{i_1,\dots,i_k})}{N^k}.
\end{aligned}$$

Then, thanks to property (5.29) which states that the measure $\frac{\mathcal{H}^d \lfloor \Lambda}{\mathcal{H}^d(\Lambda)}$ is invariant for $\{\psi_1, \dots, \psi_N\}$, we deduce from the ergodic theorem of [21, Theorem 6.1], using notations (5.30), that

$$\begin{aligned} &2\pi \int_0^1 \frac{r}{\zeta(r)} dr \lim_{k\to\infty} \sum_{\substack{i_1,\dots,i_k\in\{1,2,\dots,N\},\\ D_{i_1,\dots,i_k}\subset\Lambda\cap\omega}} \frac{\kappa_m(x'_{i_1,\dots,i_k})}{N^k} \\ &\quad = 2\pi \int_0^1 \frac{r}{\zeta(r)} dr \lim_{k\to\infty} \sum_{\substack{i_1,\dots,i_k\in\{1,2,\dots,N\},\\ D_{i_1,\dots,i_k}\subset\Lambda\cap\omega}} \frac{\kappa_m(\psi_{i_1,\dots,i_k}(x'_0))}{N^k} \\ &\quad = \frac{2\pi}{\mathcal{H}^d(\Lambda)} \int_0^1 \frac{r}{\zeta(r)} dr \int_{\Lambda\cap\omega} \kappa_m(s)\, d\mathcal{H}^d(s). \end{aligned} \tag{5.32}$$

We obtain in similar way that

$$\begin{aligned} &\lim_{k\to\infty} \varepsilon_k \int_{\omega\setminus D_k} \int_0^{\varepsilon_k} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k})\, dx \\ &\quad = \lim_{k\to\infty} \varepsilon_k \int_{\omega} \int_0^{\varepsilon_k} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k})\, dx \\ &\qquad - \lim_{k\to\infty} \varepsilon_k \int_{\omega\cap D_k} \int_0^{\varepsilon_k} \sigma_{ij}(w^{m,k}) e_{ij}(w^{m,k})\, dx \\ &\quad = \int_\omega \kappa_m(x')\, dx' - \frac{2\pi}{\mathcal{H}^d(\Lambda)} \int_{\Lambda\cap\omega} \kappa_m(s)\, d\mathcal{H}^d(s). \end{aligned} \tag{5.33}$$

Now, according to (5.31), (5.32), and (5.33), we have, for every $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$,

$$\begin{aligned} \mu_{mm}(\omega) &\le \lim_{k\to\infty} \Phi_{0,\varepsilon_k}(z_0^{m,k}) \\ &= \int_\omega \kappa_m(x')\, dx' + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)} \kappa_i(s) \int_{\omega\cap\Lambda} \kappa_m(s)\, d\mathcal{H}^d(s). \end{aligned} \tag{5.34}$$

Let $\omega \in \mathcal{R} \cap \mathcal{O}(\Gamma_1)$. Let $(z^{m,k})_k$; $m = 1, 2, 3$, be any sequence in $H^1_{\Gamma_2}(\mathbb{R}^3, \mathbb{R}^3)$ such that $z^{m,k} = e^m$ on $\omega \times \{x_3 = \varepsilon_k h_k(x')\}$ and $z^{m,k} \xrightarrow[k\to\infty]{\tau} 0$. Let us consider the subdifferential inequality

$$\Phi_{0,\varepsilon_k}(z^{m,k}) \ge \Phi_{0,\varepsilon_k}(z_0^{m,k}) + 2\varepsilon_k \int_{\Sigma_k} \sigma_{ij}(z_0^{m,k}) e_{ij}(z^{m,k} - z_0^{m,k})\, dx. \tag{5.35}$$

Using (5.13)–(5.16) in the penultimate subsection, we deduce, passing to the lower limit in (5.35), that

$$\liminf_{k\to\infty} \Phi_{0,\varepsilon_k}(z^{m,k}) \ge \liminf_{k\to\infty} \Phi_{0,\varepsilon_k}(z_0^{m,k}),$$

hence, taking the infimum over all sequences $(z^{m,k})_k$ satisfying the above properties, we obtain

$$\mu_{mm}(\omega) \ge \int_\omega \kappa_m(x')\, dx' + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)} \kappa_i(s) \int_{\omega\cap\Lambda} \kappa_m(s)\, d\mathcal{H}^d(s). \tag{5.36}$$

Thus, according to (5.34) and (5.36), we conclude that

$$\mu_{mm}(\omega) = \int_\omega \kappa_m(x')dx' + \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)}\kappa_i(s)\int_{\omega\cap\Lambda}\kappa_m(s)d\mathcal{H}^d(s).$$

Moreover, replacing $z_0^{m,k}$ by $z_0^{m,k} + z_0^{l,k}$; $l \neq m$, we prove as in the last two subsections, that $\mu_{ml}(\omega) = \mu_{lm}(\omega) = 0$. Therefore

$$\begin{aligned}\mu_{ml} = \Bigg(&\int_\omega \kappa_m(x')dx' \\ &+ \frac{2\pi(c-1/2)}{\mathcal{H}^d(\Lambda)}\kappa_i(s)\int_{\omega\cap\Lambda}\kappa_m(s)d\mathcal{H}^d(s)\Bigg)\delta_{ml}, \quad \forall m,l = 1,2,3. \quad \blacksquare\end{aligned}$$

6. Optimization problems

Let $\eta > 0$. We suppose that $\boldsymbol{\mu} = \boldsymbol{h}d\mathbf{s}$, where $\boldsymbol{h}$ is a diagonal matrix $\mathrm{Diag}(h_i)_{i=1,2,3}$ of Γ_1-measurable functions $h_i : \Gamma_1 \to (0,+\infty)$ such that

$$\int_{\Gamma_1} h_i(s)ds = \eta, \quad \forall i = 1,2,3.$$

Let $\mathcal{D}_\eta$ denote the set of all these matrices. Let us consider the following problem:

$$\begin{cases} -\operatorname{div}\sigma(u^h) = f & \text{in } \Omega, \\ \boldsymbol{h}\sigma(u^h)n + u^h = 0 & \text{on } \Gamma_1, \\ u^h = 0 & \text{on } \Gamma_2, \end{cases} \tag{6.1}$$

which has a unique solution $u^h \in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)$. We define the functional $F(\boldsymbol{h},.)$ by

$$F(\boldsymbol{h},u) = \begin{cases} \frac{1}{2}\int_\Omega \sigma_{ij}(u)e_{ij}(u)dx + \frac{1}{2}\sum_{i=1}^3 \int_{\Gamma_1}\frac{u_i^2}{h_i}ds \\ \qquad -\int_\Omega f.udx & \text{if } u \in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

We can easily check that

$$F(\boldsymbol{h},u^h) = -\frac{1}{2}\int_\Omega f.u^h dx. \tag{6.2}$$

We consider the following optimal control problem:

$$\min_{\boldsymbol{h}\in\mathcal{D}_\eta}\ \min_{u\in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)} F(\boldsymbol{h},u). \tag{6.3}$$

According to (6.2), the minimization of F, with respect to u, is equivalent to the maximization of the work of the external loads on Ω. We have the following result.

Theorem 20. *Problem (6.3) admits a unique solution $u^\eta \in H^1_{\Gamma_2}(\Omega, \mathbb{R}^3)$ which satisfies*

$$\begin{cases} -\operatorname{div}\sigma(u) = f & \text{in } \Omega, \\ \sigma_{ij}(u)n_j + \left(\int_{\Gamma_1} |u_i| ds\right)\dfrac{\operatorname{sign}(u_i)}{\eta} = 0 & \text{on } \Gamma_1, \end{cases} \tag{6.4}$$

where sign *is defined by*

$$\operatorname{sign}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

Proof. Let us consider the following equivalent problem to (6.3),

$$\min_{u \in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)} \min_{\boldsymbol{h} \in \mathcal{D}_\eta} F(\boldsymbol{h}, u). \tag{6.5}$$

Let $u \in H^1_{\Gamma_2}(\Omega, \mathbb{R}^3)$ and $\boldsymbol{h}^\eta(u) \in \mathcal{D}_\eta$ be the unique solution of the following problem:

$$\min_{\boldsymbol{h} \in \mathcal{D}_\eta} F(\boldsymbol{h}, u). \tag{6.6}$$

Using Hölder's inequality, we get

$$\left(\int_{\Gamma_1} |u_i| ds\right)^2 \leq \left(\int_{\Gamma_1} h_i ds\right)\left(\int_{\Gamma_1} \frac{u_i^2}{h_i} ds\right),$$

for every $i = 1, 2, 3$. The minimum of problem (6.6), with respect to h_i, is reached when

$$\left(\int_{\Gamma_1} |u_i| ds\right)^2 = \left(\int_{\Gamma_1} h_i ds\right)\left(\int_{\Gamma_1} \frac{u_i^2}{h_i} ds\right),$$

which occurs if and only if h_i has the form

$$h_i^\eta(u) = \eta \frac{|u_i^\eta|}{\int_{\Gamma_1} |u_i^\eta| ds},$$

for every $i = 1, 2, 3$. Let us define

$$\begin{aligned} G_\eta(u) &= F(\boldsymbol{h}^\eta(u), u) \\ &= \frac{1}{2}\int_\Omega \sigma_{ij}(u) e_{ij}(u) dx + \frac{1}{2\eta}\sum_{i=1}^{3}\left(\int_{\Gamma_1} |u_i| ds\right)^2 - \int_\Omega f.u dx. \end{aligned}$$

Then it is possible to see that problem (6.5) becomes

$$\min_{u \in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)} G_\eta(u). \tag{6.7}$$

Then, observing that G_η is strictly convex, coercive and lower semi-continuous with respect to the weak topology of $H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)$, we deduce that problem (6.7) has a unique solution $u^\eta \in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)$ which is the solution of problem (6.4). ■

Example 21. Let us consider a homogeneous isotropic material in the unit ball $\Omega = D(0,1)$ centered at the origin. We suppose that a uniform pressure of intensity P_0 acts on the sphere $\Gamma_1 = \partial D(0,1)$. Then, for $f \equiv 0$, the radial displacement u^η given by

$$\begin{cases} u^\eta_r(r) = \dfrac{P_0\eta}{4\pi} - P_0\dfrac{(1+\nu)(1-2\nu)}{E}(r-R), \\ u^\eta_\theta(r) = 0, \end{cases}$$

satisfies equations (6.4) with $\boldsymbol{h}^\eta(u^\eta) = \mathrm{Diag}(\frac{\eta}{4\pi},0,0)$.

Let Ω be a smoothly bounded open subset of $\mathbb{R}^3$ (we suppose that at least Ω is of a class $C^{1,\alpha}$ with $0<\alpha<1$) and $f \in C(\overline{\Omega},\mathbb{R}^3)$. There exists a unique solution $u^* \in H^2(\Omega,\mathbb{R}^3)\cap H^1_0(\Omega,\mathbb{R}^3)$ to the following problem:

$$\begin{cases} -\operatorname{div}\sigma(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which corresponds to problem (6.7) for $\eta = 0$. Let I_η be the functional defined on $H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)$ by

$$I_\eta(v) = \frac{\eta}{2}\int_\Omega \sigma_{ij}(v)e_{ij}(v)dx + \frac{1}{2}\sum_{i=1}^{3}\left(\int_{\Gamma_1}|v_i|ds\right)^2 + \int_{\Gamma_1}\sigma_{ij}(u^*)n_j v_i ds.$$

Then, $v^\eta = \frac{u^h-u^*}{\eta}$, where u^h is the solution of problem (6.1), is the unique minimizer of I_η. Let $\psi \in H^{1/2}(\Gamma_1,\mathbb{R}^3)$. There exists a unique function v_ψ such that

$$\int_\Omega \sigma_{ij}(v_\psi)e_{ij}(v_\psi)dx = \inf_{\{w\in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3);\, w|_{\Gamma_1}=\psi\}}\int_\Omega \sigma_{ij}(w)e_{ij}(w)dx. \tag{6.8}$$

Let us denote by $\mathcal{M}(\Gamma_1,\mathbb{R}^3)$ the space of vectorial Radon measures on Γ_1. We consider the functional J_η defined on $\mathcal{M}(\Gamma_1,\mathbb{R}^3)$ by

$$J_\eta(\psi) = \begin{cases} \frac{\eta}{2}\int_\Omega \sigma_{ij}(v_\psi)e_{ij}(v_\psi)dx + \frac{1}{2}\sum_{i=1}^{3}\left(\int_{\Gamma_1}|\psi_i|ds\right)^2 & \\ \qquad + \int_{\Gamma_1}\sigma_{ij}(u^*)n_j\psi_i ds & \text{if } \psi \in H^{1/2}(\Gamma_1,\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, according to (6.8), $v^\eta|_{\Gamma_1} = \frac{u^h-u^*}{\eta}|_{\Gamma_1} = \frac{u^h}{\eta}|_{\Gamma_1}$ is the unique minimizer of J_η. This implies that the problem $\min_{v\in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)} I_\eta(v)$ is equivalent to the minimization problem $\min_{\psi\in H^{1/2}(\Gamma_1,\mathbb{R}^3)} J_\eta(\psi)$.

We have the following compactness and Γ-convergence results for J_η.

Theorem 22. (1) $\sup_\eta \sum_{i=1}^{3} \int_{\Gamma_1} |v_i^\eta| ds < +\infty$.

(2) *The sequence* $(J_\eta)_\eta$ Γ*-converges , as* η *tends to zero, with respect to the weak* topology of* $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$*, to the functional* J *defined on* $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$ *by*

$$J(\lambda) = \frac{1}{2}\sum_{i=1}^{3} |\lambda_i|^2(\Gamma_1) + \int_{\Gamma_1} \sigma_{ij}(u^*) n_j \, d\lambda_i,$$

where $|\lambda_i|$ *is the total variation of the measure* λ_i*;* $i = 1, 2, 3$.

Proof. (1) Using the smoothness of Ω, f, and u^*, we infer that there exists a positive constant C such that, for every $i = 1, 2, 3$,

$$\sup_{\Gamma_1} |\sigma_{ij}(u^*) n_j| \leq C,$$

from which we deduce that

$$J_\eta(v^\eta|_{\Gamma_1}) \geq \frac{1}{2}\sum_{i=1}^{3}\left(\int_{\Gamma_1} |v_i^\eta| ds\right)^2 - C\sum_{i=1}^{3}\int_{\Gamma_1} |v_i^\eta| ds. \tag{6.9}$$

Now, observing that $J_\eta(v^\eta|_{\Gamma_1}) \leq J_\eta(0)$, we deduce that

$$\sup_\eta J_\eta(v^\eta|_{\Gamma_1}) \leq 0,$$

and, using (6.9), we get

$$\sup_\eta \sum_{i=1}^{3} \int_{\Gamma_1} |v_i^\eta| ds \leq C.$$

We deduce from the above uniform boundedness that, up to some subsequence,

$$v^\eta|_{\Gamma_1} \xrightarrow[\eta \to 0]{} \lambda \quad \mathcal{M}(\Gamma_1, \mathbb{R}^3)\text{-weak*}. \tag{6.10}$$

(2) The scalar version of this assertion was proved in [20, Theorem 3.5].

(a) *Lower limit inequality.* Let $(\lambda^\eta)_\eta \subset H^{1/2}(\Gamma_1, \mathbb{R}^3)$ such that $\lambda^\eta \xrightarrow[\eta \to 0]{} \lambda$ $\mathcal{M}(\Gamma_1, \mathbb{R}^3)$-weak*. As the functional $v \mapsto |v|$, where $|v|$ is the total variation of the measure v, is lower semi-continuous on $\mathcal{M}(\Gamma_1)$, we have that, for every $i = 1, 2, 3$,

$$\liminf_{\eta \to 0} \int_{\Gamma_1} |\lambda_i^\eta| ds \geq |\lambda_i|(\Gamma_1),$$

from which we deduce that

$$\liminf_{\eta \to 0} J_\eta(\lambda^\eta) \geq J(\lambda). \tag{6.11}$$

(b) *Upper limit inequality.* Without loss of generality, we suppose that $\Omega \subset \{x_3 > 0\}$ and that $\partial\Omega \cap \{x_3 = 0\} = \Gamma_1$. Let us set $x' = (x_1, x_2)$ and define, for $\varepsilon > 0$, the mollifier ς_ε by

$$\varsigma_\varepsilon(x') = \begin{cases} \frac{C_0}{\varepsilon^2} \exp\bigl(-\frac{\varepsilon^2}{\varepsilon^2 - |x'|^2}\bigr) & \text{if } |x'| < \varepsilon, \\ 0 & \text{if } |x'| \geq \varepsilon, \end{cases} \tag{6.12}$$

where $C_0 = (\int_{B(0,1)} \exp(-\frac{1}{1-|y|^2})dy)^{-1}$; $B(0,1)$ being the unit ball of $\mathbb{R}^2$ centered at the origin. Let $(\omega_{[1/\varepsilon]})_\varepsilon$, where $[1/\varepsilon]$ is the integer part of $1/\varepsilon$, be a sequence of open sets such that

$$\begin{cases} \omega_1 \subset \omega_2 \subset \cdots \subset \omega_{[1/\varepsilon]} \subset \cdots \subset \Gamma_1, \\ \displaystyle\bigcup_{\varepsilon>0} \omega_{[1/\varepsilon]} = \Gamma_1, \\ d(\omega_{[1/\varepsilon]}, \partial\Gamma_1) = \varepsilon. \end{cases} \tag{6.13}$$

Observing that, for $\varepsilon \in (0,1)$, $[1/\varepsilon] - 1 = [1/\varepsilon - 1]$, we define the partition of unity $(\varphi_\varepsilon)_\varepsilon$ by

$$\begin{cases} \varphi_\varepsilon \in C_c^\infty(\omega_{[1/\varepsilon]}), & \\ \varphi_\varepsilon(x') = 1 & \text{in } \omega_{[1/\varepsilon]-1}, \\ 0 \leq \varphi_\varepsilon(x') \leq 1 & \text{in } \Gamma_1. \end{cases} \tag{6.14}$$

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathcal{M}(\Gamma_1, \mathbb{R}^3)$. Using (6.12)–(6.14), we define the sequence $(\lambda^\varepsilon)_\varepsilon$ by $\lambda^\varepsilon = (\lambda * \varsigma_\varepsilon)\varphi_\varepsilon$. Then $\lambda^\varepsilon \in C_c^\infty(\Gamma_1, \mathbb{R}^3)$, $|\nabla\lambda^\varepsilon(x')| \leq C/\varepsilon^3$, for every $x' \in \Gamma_1$, and

$$\lambda^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \lambda \quad \mathcal{M}(\Gamma_1, \mathbb{R}^3)\text{-weak*}. \tag{6.15}$$

Let us define the sequence of function $(w^\varepsilon)_\varepsilon$ from Ω to $\mathbb{R}^3$ by

$$w_i^\varepsilon(x) = \frac{\varepsilon - x_3}{\varepsilon}\lambda_i^\varepsilon(x'), \quad \forall i = 1, 2, 3.$$

Then $w^\varepsilon \in H^1_{\Gamma_2}(\Omega, \mathbb{R}^3)$. Let us now set

$$\begin{aligned} \varepsilon &= \eta^{1/16}, \\ w^\eta &= w^{\eta^{1/16}}, \\ \lambda^\eta &= \lambda^{\eta^{1/16}}. \end{aligned} \tag{6.16}$$

Using (6.15) and (6.16), we deduce that

$$\eta \int_\Omega \sigma_{ij}(w^\eta)e_{ij}(w^\eta)dx \leq C\sqrt{\eta},$$

and

$$\limsup_{\eta\to 0} I_\eta(w^\eta) \le \limsup_{\eta\to 0} \frac{1}{2}\sum_{i=1}^{3}\Big(\int_{\Gamma_1} |w_i^\eta| ds\Big)^2 + \limsup_{\eta\to 0}\int_{\Gamma_1} \sigma_{ij}(u^*)n_j w_i^\eta ds$$
$$\le \frac{1}{2}\sum_{i=1}^{3} |\lambda_i|^2(\Gamma_1) + \int_{\Gamma_1} \sigma_{ij}(u^*)n_j d\lambda_i,$$

from which we deduce, since $w^\eta|_{\Gamma_1} = \lambda^\eta$, that

$$\limsup_{\eta\to 0} J_\eta(\lambda^\eta) \le J(\lambda). \tag{6.17}$$

The two inequalities (6.11) and (6.17) imply the second assertion of the theorem. ∎

Let us set

$$\begin{aligned} M_i &= \sup_{\Gamma_1}|\sigma_{ij}(u^*)n_j|,\\ K_i^{\pm} &= \big\{s \in \Gamma_1;\ \sigma_{ij}(u^*)n_j(s) = \pm M_i\big\}. \end{aligned} \tag{6.18}$$

We can now state our result concerning the optimal location where possible elastic layers could take place.

Theorem 23. *We have*

(1) *The sequence $(\frac{u^\eta}{\eta}|_{\Gamma_1})_\eta$, where $u^\eta \in H^1_{\Gamma_2}(\Omega,\mathbb{R}^3)$ is the solution of problem (6.7), converges in $\mathcal{M}(\Gamma_1,\mathbb{R}^3)$-weak*, as η tends to 0, to a measure $\lambda = (\lambda_i)_{i=1,2,3}$ such that $\operatorname{spt}\lambda_i \subset K_i^+ \cup K_i^-$, with λ_i positive in K_i^- and negative in K_i^+, for every $i = 1,2,3$.*

(2) *For every $i = 1,2,3$, $\int_{\Gamma_1} \sigma_{ij}(u^*)n_j d\lambda_i = -M_i$.*

(3) *For every $i = 1,2,3$,*

$$\lim_{\eta\to 0}\int_{\Gamma_1}\Big|\frac{u_i^\eta}{\eta}\Big| ds = |\lambda_i|(\Gamma_1) = M_i$$

and the sequence $(\frac{\boldsymbol{h}^\eta}{\eta}|_{\Gamma_1})_\eta$ converges in $\mathcal{M}(\Gamma_1,\mathbb{R}^3)$-weak, as η tends to 0, to the measure $\boldsymbol{v} = (\boldsymbol{v}_i)_{i=1,2,3}$ given by $\boldsymbol{v}_i M_i = \lambda_i$.*

Proof. (1) Firstly, we deduce from (6.10) that the sequence $(\frac{u^\eta}{\eta}|_{\Gamma_1})_\eta$ converges in $\mathcal{M}(\Gamma_1,\mathbb{R}^3)$-weak*, as η tends to 0, to a measure $\lambda = (\lambda_i)_{i=1,2,3}$ such that

$$J(\lambda) = \min_{\upsilon\in\mathcal{M}(\Gamma_1,\mathbb{R}^3)} J(\upsilon).$$

(2) Let us set

$$\mathcal{M}_1(\Gamma_1,\mathbb{R}^3) = \big\{\upsilon \in \mathcal{M}(\Gamma_1,\mathbb{R}^3);\ |\upsilon_i|(\Gamma_1) = 1,\ i = 1,2,3\big\},$$

and introduce the functional $\widetilde{J}$ defined from $[0, +\infty)^3 \times \mathcal{M}_1(\Gamma_1, \mathbb{R}^3)$ to $\mathbb{R}$ by

$$\begin{aligned}\widetilde{J}(t_1, t_2, t_3, \upsilon_1, \upsilon_2, \upsilon_3) &= J(t_1\upsilon_1, t_2\upsilon_2, t_3\upsilon_3)\\ &= \frac{1}{2}\sum_{i=1}^{3} t_i^2 + \sum_{i=1}^{3} t_i \int_{\Gamma_1} \sigma_{ij}(u^*)n_j\, d\upsilon_i .\end{aligned}$$

We have

$$\min_{\upsilon\in\mathcal{M}(\Gamma_1,\mathbb{R}^3)} J(\upsilon) = \min_{\upsilon\in\mathcal{M}_1(\Gamma_1,\mathbb{R}^3)} \min_{\substack{t_i>0,\\ i=1,2,3}} \widetilde{J}(t_1, t_2, t_3, \upsilon_1, \upsilon_2, \upsilon_3). \tag{6.19}$$

Let us denote $(\mathbf{t}, \boldsymbol{\upsilon})$; $\mathbf{t} = (\mathbf{t}_i)_{i=1,2,3}$ and $\boldsymbol{\upsilon} = (\boldsymbol{\upsilon}_i)_{i=1,2,3}$, the minimizer of the right-hand side of (6.19). One can easily check that if $\int_{\Gamma_1} \sigma_{ij}(u^*)n_j\, d\mu_i \geq 0$, for every $i = 1, 2, 3$, then $\mathbf{t}_i = 0$ and

$$\min_{\substack{t_i>0,\\ i=1,2,3}} \widetilde{J}(t_1, t_2, t_3, \upsilon_1, \upsilon_2, \upsilon_3) = 0,$$

and if $\int_{\Gamma_1} \sigma_{ij}(u^*)n_j\, d\upsilon_i < 0$ for every $i = 1, 2, 3$, then $\mathbf{t}_i = -\int_{\Gamma_1} \sigma_{ij}(u^*)n_j\, d\boldsymbol{\upsilon}_i$,

$$\min_{\substack{t_i>0,\\ i=1,2,3}} \widetilde{J}(t_1, t_2, t_3, \upsilon_1, \upsilon_2, \upsilon_3) = -\frac{1}{2}\sum_{i=1}^{3}\left(\int_{\Gamma_1} \sigma_{ij}(u^*)n_j\, d\upsilon_i\right)^2,$$

and $\boldsymbol{\upsilon} = (\boldsymbol{\upsilon}_i)_{i=1,2,3}$ minimizes $(\int_{\Gamma_1} \sigma_{ij}(u^*)n_j\, d\upsilon_i)_{i=1,2,3}$.

For every $i = 1, 2, 3$, we have that $\int_{\Gamma_1} \sigma_{ij}(u^*)n_j\, d\boldsymbol{\upsilon}_i \geq -M_i$ and the equality holds if and only if $\operatorname{spt} \boldsymbol{\upsilon}_i \subset K_i^+ \cup K_i^-$, $\boldsymbol{\upsilon}_i$ is positive in K_i^- and negative in K_i^+. We deduce, according to (6.18), that $\lambda_i = M_i \boldsymbol{\upsilon}_i$.

(3) As $(\frac{u^\eta}{\eta}|_{\Gamma_1})_\eta$ converges in $\mathcal{M}(\Gamma_1, \mathbb{R}^N)$-weak* to $\lambda = (M_i\boldsymbol{\upsilon}_i)_{i=1,2,3}$, we have that

$$\lim_{\eta\to 0} \int_{\Gamma_1} \left|\frac{u_i^\eta}{\eta}\right| ds = |\lambda_i|(\Gamma_1) = M_i\boldsymbol{\upsilon}_i(\Gamma_1) = M_i,$$

from which we deduce that, for every $i = 1, 2, 3$, the sequence $(\frac{|u_i^\eta|}{\int_{\Gamma_1} |u_i^\eta| ds})_h$ converges in $\mathcal{M}(\Gamma_1)$-weak* to $\boldsymbol{\upsilon}_i$, which means that the sequence $(\frac{h_i^\eta}{\eta}|_{\Gamma_1})_\eta$ converges in $\mathcal{M}(\Gamma_1)$-weak* to $\boldsymbol{\upsilon}_i$. ■

For a biological body, this last theorem provides a tool allowing a characterization of the zones where soft tissues are likely to grow. For a reinforced material this theorem shows that we can have an optimal reinforcement, as η is infinitesimal, if we introduce a material in the points where the tractions $|\sigma_{ij}(u^*)n_j|$ are maximal.

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