Ann. Inst. H. Poincaré
Anal. Non Linéaire 41 (2024), 1179–1237
DOI 10.4171/AIHPC/101

Traveling fronts for Fisher–KPP lattice equations in almost-periodic media

Xing Liang, Hongze Wang, Qi Zhou, and Tao Zhou

Abstract. This paper investigates the existence of almost-periodic traveling fronts for Fisher–KPP lattice equations in one-dimensional almost-periodic media. Using the Lyapunov exponent of the linearized operator near the unstable steady state, we give sufficient conditions for the existence of a minimal speed of traveling fronts. Furthermore, it is shown that almost-periodic traveling fronts share the same recurrence property as the structure of the media. As applications, we give some typical examples which have minimal speed, and the proof of this depends on a dynamical system approach to the almost-periodic Schrödinger operator.

Contents

1. Introduction

1.1. Background and main results

Since the pioneer works of [1, 20, 32], the traveling fronts of reaction–diffusion equations in an unbounded domain have become an important branch of the theory of equations with diffusion. Specifically, traveling fronts of the Fisher–KPP-type equations in continuous media,

$$u_t - (a(x)u_x)_x = c(x)u(1-u), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1.1}$$

and more general reaction–diffusion equations in heterogeneous media have received intense attention in the last few decades.

2020 Mathematics Subject Classification. Primary 35K57; Secondary 35B15,47A75,35P05.
Keywords. Fisher–KPP equation, traveling front, Schrödinger operator, almost-periodicity, KAM theory.

As a simplest heterogenous case, traveling fronts in spatially periodic media were considered widely. First, the definition of spatially periodic traveling waves was provided by [51, 58] independently, and then [26] proved the existence of spatially periodic traveling waves of Fisher–KPP equations in the distributional sense. Then, in the series of works [7, 9], the authors investigated traveling fronts of Fisher–KPP-type equations in high-dimensional periodic media deeply. The traveling fronts of spatially periodic Fisher–KPP-type equations in a discrete lattice,

$$u_t(t,n) - u(t,n+1) - u(t,n-1) + 2u(t,n) = c(n)u(t,n)(1-u(t,n)), \qquad (1.2)$$

were also studied in [21, 37]. Besides the above works, a more general framework was given by [37,57] to study traveling fronts for Fisher–KPP-type equations and more general diffusion systems.

However, few works on traveling waves of Fisher–KPP equations exist in more complicated media. Matano [39] first gave a definition of spatially almost-periodic traveling waves and provided some sufficient conditions for the existence of spatially almost-periodic traveling fronts of reaction–diffusion equations with bistable nonlinearity. In [36], Liang showed the existence and uniqueness of the spatially almost-periodic traveling front of Fisher–KPP equations in one-dimensional almost-periodic media with free boundary. We also notice that the propagation problems of (temporally) nonautonomous reaction–diffusion equations were studied by Shen in a series of works [17, 49, 50].

In this paper, we are concerned with the almost-periodic traveling fronts of the Fisher–KPP equation (1.2). To introduce the definition of almost-periodic traveling fronts, let us first recall the definition of classical and periodic traveling fronts. In homogeneous media, that is, c is a constant sequence, classical traveling fronts are defined by a solution $u(t,n) = U(n-wt)$ with an invariant profile U and a speed w; in periodic media, that is, c is a periodic sequence with period N, periodic traveling fronts are defined by a solution $u(t,n) = U(n-wt,n)$ with a periodic profile U, $U(\xi,n) = U(\xi,n+N)$, and an average speed w. Then it is natural to consider the recurrence of the profile depending on the structure of the media if one tries to generalize the definition of traveling fronts in heterogeneous media. To be exact, in almost-periodic media, generalized traveling fronts need to inherit the almost-periodicity of the media.

Before introducing the definition of generalized traveling fronts with almost-periodic recurrence, we give some notation and background. A sequence $f\colon \mathbb{Z} \to \mathbb{R}$ is Bochner almost periodic if $\{f(\cdot+k) \mid k \in \mathbb{Z}\}$ has a compact closure in $l^\infty(\mathbb{Z})$. Denote by $\mathcal{H}(f)$ the hull of the almost-periodic sequence f, i.e., $\mathcal{H}(f) = \overline{\{f(\cdot+k) \mid k \in \mathbb{Z}\}}$, the closure in $l^\infty(\mathbb{Z})$. The hull $\mathcal{H}(f)$ is isomorphic to $\mathbb{T}^d$ with $d \in \mathbb{N}_+$ or $d = \infty$, and then $f(n) = F(n\alpha)$ for some continuous function F in $\mathbb{T}^d$ [52]; $\alpha \in \mathbb{R}^d \backslash \mathbb{Q}^d$ is called the frequency of f. We say f is quasi-periodic if $d \in \mathbb{N}_+$.

To define the almost-periodic traveling fronts, we prefer to consider not only (1.2) with $c(n) = V(n\alpha)$, but also the family of equations

$$u_t(t,n) - u(t,n+1) - u(t,n-1) + 2u(t,n) = V(n\alpha+\theta)u(t,n)(1-u(t,n)), \quad (1.3)$$

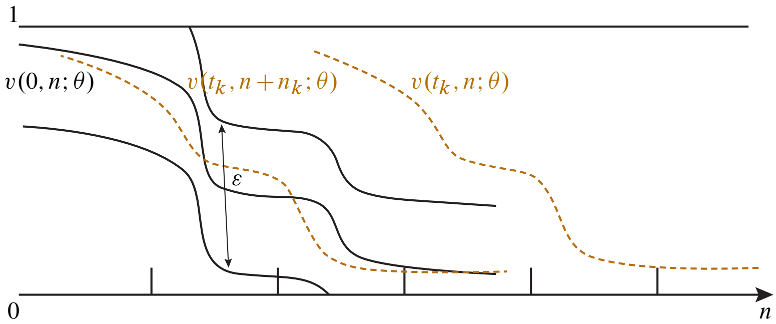

Figure 1

for any $\theta \in \mathbb{T}^d$, $\alpha \in \mathbb{R}^d \backslash \mathbb{Q}^d$, $d \in \mathbb{N}_+$ or $d = \infty$. In this paper, we always assume that $V > 0$ and $d \in \mathbb{N}_+ \cup \{\infty\}$. With these, we have the following precise definition:

Definition 1.1. Let $v(t,n,\theta)\colon \mathbb{R} \times \mathbb{Z} \times \mathbb{T}^d \to \mathbb{R}$. We say $v(t,n,\theta)$ is an *almost-periodic traveling front* (with nonzero average speed) if $v(t,n,\theta)$ is an entire solution of (1.3) for any $\theta \in \mathbb{T}^d$, and if the following properties hold:

(1) $v(t,n,\theta) \to 1$ as $n \to -\infty$, $v(t,n,\theta) \to 0$ as $n \to \infty$, locally uniformly in t and uniformly in θ.

(2) For some $\mathcal{T} \in C^0(\mathbb{T}^d, \mathbb{R})$ and some $U \in C^0(\mathbb{R} \times \mathbb{T}^d, \mathbb{R})$,

$$v(t,n,\theta) = U\left(t - \sum_{i=0}^{n-1} \mathcal{T}(\theta + i\alpha), n\alpha + \theta\right),$$

and U is called the profile of $v(t,n,\theta)$.

(3) We define

$$w := \lim_{|n-k|\to\infty} \frac{n-k}{\sum_{i=k}^{n} \mathcal{T}(\theta + i\alpha)} = \frac{1}{\int_{\mathbb{T}^d} \mathcal{T}\, d\theta} \quad \in \mathbb{R}\backslash\{0\},$$

called the *average wave speed* of $v(t,n,\theta)$.

Remark 1.1. The speed w is nonzero since $\int_{\mathbb{T}^d} \mathcal{T}\, d\theta < \infty$.

Remark 1.2. Actually, the definition of the almost-periodic traveling front implies the time recurrence of such a solution (cf. Lemma 2.2): for any $\varepsilon > 0$, there exist relatively dense sets $\{t_k\}_k \subset \mathbb{R}$ and $\{n_k\} \subset \mathbb{Z}$ such that

$$\sup_{n\in\mathbb{Z}} |v(t_k, n_k + n, \theta) - v(0,n,\theta)| \le \varepsilon,$$

as shown in Figure 1.

It is necessary to point out that a more general extension of traveling fronts, a so-called generalized transition front, was presented by Berestycki and Hamel [8]. For (1.2), the generalized transition front is defined as below.

Definition 1.2. A generalized transition front of (1.2) is an entire solution $u = u(t,n)$ for which there exists a function $N: \mathbb{R} \to \mathbb{Z}$ such that

$$\lim_{n\to-\infty} u(t,n+N(t)) = 1, \quad \lim_{n\to+\infty} u(t,n+N(t)) = 0, \tag{1.4}$$

uniformly in $t \in \mathbb{R}$. We say u has an average speed $w \in \mathbb{R}$, provided

$$\lim_{t-s\to+\infty} \frac{N(t)-N(s)}{t-s} = w.$$

Remark 1.3. In the periodic case, the almost-periodic traveling front in Definition 1.1 is exactly the classical periodic traveling front [21]. However, there may exist a generalized transition front which is not a classical traveling front [8].

A generalized transition front $u = u(t,x)$ of (1.1) can be defined in the same way by assuming $N = N(t): \mathbb{R} \to \mathbb{R}$. Nadin and Rossi [43] investigated the existence of the generalized transition front of (1.1) when a, c are almost periodic. Motivated by the works [9, 21, 37, 41], especially the work of Nadin and Rossi [43], we want to construct almost-periodic traveling fronts via the eigenvalue problem of the linearized operator of (1.3) near the equilibrium state $u \equiv 0$:

$$(\mathcal{L}_\theta u)(n) := u(n+1) + u(n-1) - 2u(n) + V(n\alpha+\theta)u(n) = Eu(n), \quad \theta \in \mathbb{T}^d. \tag{1.5}$$

We will always shorten the notation $\mathcal{L}_0$ to $\mathcal{L}$.

One novelty of the paper is that we will use methods from dynamical systems to study the operator $\mathcal{L}_\theta$, thus to study (1.3). Note that (1.5) can be rewritten as

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A(n) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix},$$

where $A(n) = \begin{pmatrix} E+2-V(n\alpha+\theta) & -1 \\ 1 & 0 \end{pmatrix}$. Let $A_n(\theta) = A((n-1)\alpha+\theta)\cdots A(\theta)$ be the transfer matrix. Then the *Lyapunov exponent* of $\mathcal{L}_\theta$ at energy E is denoted by $L(E)$ and given by

$$L(E) := \lim_{n\to+\infty} \frac{1}{n} \int_{\mathbb{T}^d} \ln \|A_n(\theta)\| \, d\theta \geq 0, \tag{1.6}$$

where $\|\cdot\|$ denotes the matrix norm. The Lyapunov exponent characterizes the decay rate of any solutions of (1.5), and is also a fundamental topic in smooth dynamical systems.

We always use $\Sigma(\mathcal{L})$ to denote the spectrum of $\mathcal{L}$, and denote $\lambda_1 = \max \Sigma(\mathcal{L})$. Once we have this, we can state our main results as follows:

Theorem 1.1. *Denote*

$$w^* := \inf_{E>\lambda_1} \frac{E}{L(E)}, \quad \underline{w} := \lim_{E\searrow\lambda_1} \frac{E}{L(E)}.$$

Then for (1.3)*, the following statements hold:*

(1) *If $w^* < \underline{w}$, then for any $w \in (w^*, \underline{w})$, there exists a time-increasing almost-periodic traveling front with average wave speed w.*

(2) *If $w^* < \underline{w}$, then there exists a time-increasing generalized transition front with average speed w^*.*

(3) *There is no generalized transition front with average speed $w < w^*$.*

Note that the sufficient condition in Theorem 1.1 (1) is fulfilled up to a constant perturbation of V (Lemma 4.1), which was first observed in [43].

It is a remarkable fact that, for any $E > \lambda_1$, there exists a unique positive solution $\phi_E(n)$ of

$$\mathcal{L}\phi_E = E\phi_E, \quad \phi_E(0) = 1, \quad \lim_{n\to\infty} \phi_E(n) = 0,$$

and the limit $\mu(E) := -\lim_{n\to\pm\infty} \frac{1}{n} \ln \phi_E(n) = L(E)$ (see Proposition 3.4). Thus the minimal speed we constructed is the same as that given in [43].

1.2. Applications

Of course, the interesting thing is to give concrete examples in which we can establish almost-periodic traveling fronts for any average wave speed $w > w^*$ (i.e., $\underline{w} = \infty$). Based on [34], Nadin and Rossi [43] showed that if a, c are finitely differentiable quasi-periodic functions with Diophantine frequency α, and c is small enough (the smallness must depend on α), then the operator $Lu = (a(x)u_x)_x + c(x)u$ has a positive almost-periodic function. Consequently, (1.1) has a time-increasing generalized transition front with average speed $w \in (w^*, \infty)$. Here we recall that α is Diophantine (denoted by $\alpha \in \mathrm{DC}_d(\gamma, \tau)$), if there exist $\gamma > 0$, $\tau > d$ such that

$$\inf_{j\in\mathbb{Z}} |\langle k, \alpha\rangle - j| > \frac{\gamma}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^d.$$

Therefore, the natural question is whether one can remove the arithmetic condition of α, or whether one can remove the smallness on c. Based on the former results [2, 59], Corollary 5.5 shows that for any irrational $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, if V is analytic and close to constant, even the closeness is independent of α, then $L(\lambda_1) = 0$, which yields that for any $w \in (w^*, \infty)$, there exists a time-increasing almost-periodic traveling front with average wave speed w. The analyticity of V is necessary, since the Lyapunov exponent might be discontinuous in smooth topology [56].

Corollary 5.4 also states the results for V being just a finitely differentiable quasi-periodic function as in [43]. To be exact, if $\alpha \in \mathrm{DC}_d(\gamma, \tau)$, $V \in C^s(\mathbb{T}^d, \mathbb{R})$ with $s > 6\tau + 2$, and V is small enough, then (1.2) has a time-increasing almost-periodic traveling front with average wave speed $w \in (w^*, \infty)$. However, it is widely believed that if the regularity is worse, then for generic V, the spectrum of (1.5) has no absolutely continuous component [4]. Therefore, most probably $L(\lambda_1) > 0$ by the well-known Kotani theory [33].

As a concrete and typical example of removing the smallness of c, we will take $V(\theta) = 2\kappa\cos\theta + C$, where $C > 2|\kappa|$. Then the corresponding linearized operator can be reduced to the well-known *almost Mathieu operator* (AMO):

$$(\mathcal{L}_{2\kappa\cos-2,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\kappa\cos(\theta + n\alpha)u(n).$$

The AMO was first introduced by Peierls [46], as a model for an electron on a two-dimensional lattice, acted on by a homogeneous magnetic field [24, 48]. Now, if $V(\theta) = 2\kappa\cos\theta + C$, $C > 2|\kappa|$, then we have the following:

Corollary 1.1. *Suppose that* $c(n) = 2\kappa\cos(2\pi n\alpha) + C$*, where* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ *and* $C > 2|\kappa|$*. Then the following statements hold:*

(1) *If* $|\kappa| \leq 1$*, then* (1.3) *has a time-increasing quasi-periodic traveling front with average wave speed* $w \in (w^*, \infty)$.

(2) *If* $|\kappa| > 1$ *and* $w^* < \frac{\lambda_1}{\ln|\kappa|}$*, then* (1.3) *has a time-increasing quasi-periodic traveling front with average speed* $w \in (w^*, \frac{\lambda_1}{\ln|\kappa|})$.

Note that the AMO plays the central role in the Thouless et al. theory of the integer quantum Hall effect [53]. This model has been extensively studied, not only because of its importance in physics [44], but also as a fascinating mathematical object [5, 6, 27, 28]. For us, the example is interesting since Corollary 1.1 (2) provides a rigorous example of $\underline{w} < \infty$ in the discrete setting. However, it is still open whether in this case, (1.2) has a time-increasing almost-periodic traveling front with average wave speed $w \in [\frac{\lambda_1}{\ln|\kappa|}, \infty)$.

In this aspect, we should mention that compared to [43], we are mainly concerned with Fisher–KPP lattice equations, since the main results we mentioned above [2, 3, 14, 15, 28] only work in the discrete case. Whether the corresponding results are valid for the continuous case is widely open.

We also remark that we are mainly concerned with almost-periodic traveling fronts of (1.2) with average wave speed $w \in (w^*, \infty)$, while Nadin and Rossi's example [43] and Corollary 1.1 only give examples where c is quasi-periodic. Therefore, it is interesting and important to give concrete examples of a real almost-periodic sequence $c(n)$, such that (1.2) has an almost-periodic traveling front with average wave speed $w \in (w^*, \infty)$. To state the result clearly, let us show how to construct the desired almost-periodic sequence.

We assume that the frequency $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ belongs to the infinite-dimensional cube $\mathcal{R}_0 := [1,2]^{\mathbb{N}}$, and we endow $\mathcal{R}_0$ with the probability measure $\mathcal{P}$ induced by the product measure of the infinite-dimensional cube $\mathcal{R}_0$. We now define the set of Diophantine frequencies that was first developed by Bourgain [13]:

Definition 1.3 ([40]). Given $\gamma \in (0,1)$, $\tau > 1$, we denote by $\mathrm{DC}_\infty(\gamma,\tau)$ the set of Diophantine frequencies $\alpha \in \mathcal{R}_0$ such that

$$\inf_{n\in\mathbb{Z}} |\langle k,\alpha\rangle - n| \geq \gamma \prod_{j\in\mathbb{N}} \frac{1}{1 + |k_j|^\tau \langle j\rangle^\tau} \quad \forall k \in \mathbb{Z}^\infty,\ 0 < \sum_{j\in\mathbb{N}} |k_j| < \infty,$$

where $\langle j\rangle := \max\{1, j\}$.

As proved in [40], for any $\tau > 1$, Diophantine frequencies $\mathrm{DC}_\infty(\gamma,\tau)$ are typical in the set $\mathcal{R}_0$ in the sense that there exists a positive constant $C(\tau)$ such that

$$\mathcal{P}(\mathcal{R}_0 \backslash \mathrm{DC}_\infty(\gamma,\tau)) \le C(\tau)\gamma.$$

Now we consider the almost-periodic sequence $c(n)$ with Diophantine $\alpha \in \mathrm{DC}_\infty(\gamma,\tau)$. Since α is rationally independent, $\mathcal{H}(c) = \mathbb{T}^\infty = \prod_{i\in\mathbb{N}} \mathbb{T}^1$ with infinite product topology. Denote by $\mathbb{Z}^\infty_* := \{k \in \mathbb{Z}^\infty \mid |k|_1 := \sum_{j\in\mathbb{N}} \langle j\rangle |k_j| < \infty\}$ the set of infinite integer vectors with finite support. For any $f \in C^0(\mathbb{T}^\infty,\mathbb{R})$, and $k \in \mathbb{Z}^\infty_*$, denote $\hat{f}(k) = \int_{\mathbb{T}^\infty} f(\theta)\mathrm{e}^{-2\pi\mathrm{i}\langle k,\theta\rangle}$. Once we have these, we can introduce our precise results as follows:

Corollary 1.2. *Let $\gamma \in (0,1)$, $\tau > 1$, $r > 0$, $\alpha \in \mathrm{DC}_\infty(\gamma,\tau)$. Suppose that $c(n) = V(n\alpha)$ for some $V \in C^0(\mathbb{T}^\infty,\mathbb{R})$. Furthermore, suppose that there exists $\varepsilon = \varepsilon(\gamma,\tau,r)$ such that*

$$\sum_{k\in\mathbb{Z}^\infty_*} |\widehat{V}(k)|\mathrm{e}^{r|k|_1} < \varepsilon(\gamma,\tau,r).$$

Then (1.2) *has a time-increasing almost-periodic traveling front with average wave speed $w \in (w^*,\infty)$.*

Finally, let us outline the novelty of the proofs of these applications. The proofs depend crucially on a dynamical approach to the almost-periodic Schrödinger operator, i.e., in order to study the spectral property of the Schrödinger operator (1.5), one only needs to study the corresponding Schrödinger cocycle (cf. Section 2.6). For analytic quasi-periodic potentials (Corollary 1.1), the result follows from the continuity of the Lyapunov exponent for analytic cocycles. For the almost-periodic case, we need to prove the existence of positive almost-periodic functions. The key observation is Lemma 5.1, which says that if the Schrödinger cocycle is reduced to a constant parabolic cocycle, and the conjugacy is close to the identity, then the corresponding Schrödinger equation has a positive almost-periodic solution. Here, reducibility means the cocycle can be conjugated to a constant cocycle (cf. Section 5). In this aspect, the powerful method is KAM. For almost-periodic Hamiltonian systems, KAM was first developed by Pöschel [47]. One can consult [12, 13, 40] for more studies on similar objects. However, it is well-known that the traditional KAM method only works for *positive measure* parameters, and here we need to *fix* the energy to be the supremum of the spectrum, thus the corresponding cocycle is *fixed*. The method of solving the difficulty is to make good use of the *fibered rotation number*, which was first developed by Herman [25] for quasi-periodic cocycles (not necessarily a quasi-periodic Schrödinger cocycle), and it can be extended to the almost-periodic setting.

1.3. Structure of the paper

In Section 2 we introduce some preliminary knowledge which will be needed in our proof. In Section 3 we introduce and investigate some properties of the generalized principal eigenvalue and the Lyapunov exponent, which turn out to be powerful techniques in constructing the almost-periodic traveling front with any average wave speed $w > w^*$.

Moreover, we also show the reason that the existence of a positive almost-periodic solution of (1.5) implies $\underline{w} = \infty$; cf. Proposition 3.5.

In Section 4 we prove Theorem 1.1 using the following steps: First we establish the almost-periodic traveling front with any average wave speed $w > w^*$ by constructing a super-sub solution, and then get the monotonicity of the fronts in t, thus proving (1) of Theorem 1.1. Next we make use of the properties of spreading speed to deduce that even the generalized transition fronts with average speed $w < w^*$ cannot exist, which yields (2). Last, we construct a generalized transition front with critical wave speed w^* by pulling back the solution of the Cauchy problem associated with the initial datum Heaviside function, and then we finish the proof.

In Section 5 we use the KAM method to get the positive quasi-periodic (almost-periodic) solution of (1.5) with positive infimum when $c = V(\cdot\alpha + \theta)$ is very close to a constant, where V is finitely differentiable. This will help us to prove Corollary 1.2. Finally, we finish all the proofs of the applications.

2. Preliminaries

2.1. Maximum principle, existence and uniqueness for the Cauchy problem

The maximum principle on the whole space can be stated as follows:

Proposition 2.1 (Maximum principle [17]). *Let $v \in \ell^\infty(\mathbb{Z})$. Assume that for any bounded interval $I = [0, t_0] \subset [0, \infty)$, u is bounded in $I \times \mathbb{Z}$. If u satisfies*

$$\begin{cases} u_t(n) - u(n+1) - u(n-1) + (2 - v(n))u(n) \geq 0 & \text{a.e. in } I \times \mathbb{Z}, \\ u(n) \geq 0 & \text{in } \{0\} \times \mathbb{Z}, \end{cases} \tag{2.1}$$

then $u \geq 0$ in $I \times \mathbb{Z}$.

The following Harnack inequality is a very useful technique when we study the properties of the solution of (1.2). We will present it here for the reader's convenience.

Proposition 2.2 (Harnack inequality [38]). *Assume that u is bounded on $(0, \infty) \times \mathbb{Z}$ and solves (2.1). Then for any $(t, n) \in (0, \infty) \times \mathbb{Z}$, $T > 0$, there exists a positive constant $C = C(T, \|v\|_{l^\infty})$ such that*

$$u(t, n) \leq C(T, \|v\|_{l^\infty})u(t + T, m), \quad m \in \{n \pm 1, n\}.$$

Remark 2.1. The proof of the Harnack inequality can be found in [38] with the initial value $u(0, n)$ having finite support. However, we note that the argument can similarly be applied to (2.1), with minor modification.

Combining the Harnack inequality with the maximum principle, we deduce the strong maximum principle as follows:

Corollary 2.1 (Strong maximum principle [38]). *Under the assumption of Proposition 2.1, either* $u \equiv 0$ *or* $u > 0$ *in* $I \times \mathbb{Z}$.

The comparison principle is a consequence of the strong maximum principle, and it is useful for us in constructing the almost-periodic traveling front. To state it, we first give the definition of super-sub solutions:

Let $\bar{u}, \underline{u} \in C(\mathbb{R} \times \mathbb{Z})$ be two bounded functions. We say that $\bar{u}$ is a supersolution of (1.2) if for any given $n \in \mathbb{Z}$, $\bar{u}$ is absolutely continuous in t and satisfies

$$\bar{u}_t - \bar{u}(n+1) - \bar{u}(n-1) + 2\bar{u}(n) - c\bar{u}(1-\bar{u}) \geq 0 \quad \text{for a.e. } t \in (0, \infty),$$

and $\underline{u}$ is a subsolution if for any given $n \in \mathbb{Z}$, $\underline{u}$ is absolutely continuous in t and satisfies

$$\underline{u}_t - \underline{u}(n+1) - \underline{u}(n-1) + 2\underline{u}(n) - c\underline{u}(1-\underline{u}) \leq 0 \quad \text{for a.e. } t \in (0, \infty).$$

The strong comparison principle is given by the following proposition:

Proposition 2.3 (Strong comparison principle). *Let* $\bar{u}$ *and* $\underline{u}$ *be a supersolution and a subsolution of* (1.2) *respectively. If* $\underline{u}(0, n) \leq \bar{u}(0, n)$ *in* $\mathbb{Z}$, *then* $\underline{u} < \bar{u}$ *or* $\underline{u} \equiv \bar{u}$ *in* $(0, \infty) \times \mathbb{Z}$.

Usually, the well-behaved Cauchy problem possesses the property that it admits a unique global solution. Moreover, existence and uniqueness is vital for us when we construct the generalized transition front with minimal speed w^*.

Theorem 2.1 ([45]). *For any initial value* $\varphi \in \ell^\infty(\mathbb{Z})$, *there exists a unique* $u \in C^0(\mathbb{R} \times \mathbb{Z})$ *with* $u(t, \cdot) \in \ell^\infty(\mathbb{Z})$ *for any* $t \in (0, \infty)$ *such that*

$$\begin{cases} u_t(n) - u(n+1) - u(n-1) + 2u(n) = c(n)u(1-u) & \text{in } (0, \infty) \times \mathbb{Z}, \\ u(0, n) = \varphi(n). \end{cases}$$

2.2. Quadratic form and critical operator

Denote by $l_c(\mathbb{Z})$ the space of real-valued functions on $\mathbb{Z}$ with compact support. The associated bilinear form l of $-\mathcal{L}$ is defined on $l_c(\mathbb{Z}) \times l_c(\mathbb{Z})$ as

$$l(\varphi, \psi) := \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{m = \pm 1} (\varphi(n) - \varphi(n+m))(\psi(n) - \psi(n+m)) - c(n)\varphi(n)\psi(n).$$

We denote by $l(\varphi) := l(\varphi, \varphi)$ the induced quadratic form on $l_c(\mathbb{Z})$. Furthermore, we write $l \geq 0$ if $l(\varphi) \geq 0$ for all $\varphi \in l_c(\mathbb{Z})$.

Definition 2.1 ([31]). Let l be a quadratic form associated with the Schrödinger operator $-\mathcal{L}$ such that $l \geq 0$. We say the form is critical if there does not exist a positive $\varpi \in l^\infty_{\text{loc}}(\mathbb{Z})$ (i.e., $\varpi \geq 0$ and $\varpi \not\equiv 0$ on $\mathbb{Z}$) such that $l(\varphi) - \sum_{n \in \mathbb{Z}} \varpi(n)\varphi(n)^2 \geq 0$ for any $\varphi \in l_c(\mathbb{Z})$.

The operator $\mathcal{L}$ is said to be critical if the quadratic form l associated with $-\mathcal{L}$ is critical.

The following well-known formula reveals the connection between the operator $\mathcal{L}$ and the associated quadratic form l.

Lemma 2.1 (Green formula [31]). *For all $\varphi, \psi \in l_c(\mathbb{Z})$, one has*

$$\frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{m=\pm1}(\varphi(n)-\varphi(n+m))(\psi(n)-\psi(n+m))-\sum_{n\in\mathbb{Z}}c(n)\varphi(n)\psi(n)$$
$$=-\sum_{n}(\mathcal{L}\varphi)(n)\psi(n)=-\sum_{n}(\mathcal{L}\psi)(n)\varphi(n).$$

That is, $l(\varphi,\psi)=\langle-\mathcal{L}\varphi,\psi\rangle$.

The critical operator has the following important property, which is useful for us to reveal the connection between the Lyapunov exponent and a positive almost-periodic solution (cf. Proposition 3.5):

Proposition 2.4. [31] *Let $\mathcal{L}$ be a critical operator. Then there exists a unique positive function in $\mathbb{Z}$ such that $\mathcal{L}u \le 0$ (up to scalar multiplication).*

2.3. Properties of the almost-periodic traveling front

An almost-periodic traveling front also implies the almost-periodicity of the time recurrence in the sense of the following lemma.

Lemma 2.2. *Assume $u(t,n,\theta)$ is an almost-periodic traveling front of (1.3). Then for any $\varepsilon > 0$, there exist relatively dense sets $\{t_k\}\subset\mathbb{R}$ and $\{n_k\}\subset\mathbb{Z}$ such that*

$$\sup_{n\in\mathbb{Z}}|u(t_k,n+n_k,\theta)-u(0,n,\theta)|\le\varepsilon.$$

Proof. Denote $\mathcal{T}_n(\theta)=\sum_{i=0}^{n-1}\mathcal{T}(\theta+i\alpha)$. From Definition 1.1 (2), for any $\varepsilon>0$, there exist $\delta>0$ and a relatively dense set $\{n_k\}\subset\mathbb{Z}$ such that if $|n_k\alpha|\le\delta$, then for any $n\in\mathbb{Z}$, $\theta\in\mathbb{T}^d$,

$$|u(0,n,\theta)-u(\mathcal{T}_{n_k},n+n_k,\theta)|$$
$$=|U(-\mathcal{T}_n(\theta),n\alpha+\theta)-U(-\mathcal{T}_n(\theta+n_k\alpha),(n+n_k)\alpha+\theta)|\le\varepsilon.$$

Since the average wave speed $w\in\mathbb{R}\backslash\{0\}$ implies directly that $\mathcal{T}_n/n\to 1/w\neq 0$ uniformly in θ as $|n|\to\infty$, and then it follows from $\sup_{\theta\in\mathbb{T}^d}|\mathcal{T}_n-\mathcal{T}_{n-1}|\le\sup_{\theta\in\mathbb{T}^d}|\mathcal{T}(\theta)|$ that $\{\mathcal{T}_{n_k}(\theta)\}$ is relatively dense. Take $\{t_k\}$ as $\{\mathcal{T}_{n_k}(\theta)\}$, and the result is proved. ∎

An observation is that an almost-periodic traveling front is also a generalized transition front.

Proposition 2.5. *An almost-periodic traveling front of (1.3) with average wave speed w is a generalized transition front of (1.3) with average speed w.*

Conversely, let u be a generalized transition front with average speed $w \neq 0$. Assume $\widetilde{\mathcal{T}} \in C^0(\mathbb{T}^d, \mathbb{R})$ and the almost-periodic sequence $\widetilde{\mathcal{T}}(k\alpha)$, satisfies

(1) *$u(t,n)$ can be represented as $u(t,n) = U(t - \widetilde{\mathcal{T}}_n, n\alpha)$ for some $U \in C^0(\mathbb{T}^d, \mathbb{R})$, where $\widetilde{\mathcal{T}}_n := \sum_{i=0}^{n-1} \widetilde{\mathcal{T}}(i\alpha)$;*

(2) $\sup_t |\widetilde{\mathcal{T}}_{N(t)} - t| < \infty$.

Then $u(t,n)$ is also an almost-periodic traveling front of (1.3) *with $\theta = 0$.*

Proof. We assume without loss of generality that $w > 0$. For any $n \in \mathbb{Z}$, denote $\mathcal{T}_n(\theta) = \sum_{i=0}^{n-1} \mathcal{T}(\theta + i\alpha)$. From Definition 1.1 (3), $w = \lim_{|n|\to\infty} n/\mathcal{T}_n = 1/\int_{\mathbb{T}^d} \mathcal{T}\, d\theta$ exists. For any $\theta \in \mathbb{T}^d$, there exists an absolute constant L such that for any $|n| \geq L$, there holds $\frac{w}{2} \leq \frac{n}{\mathcal{T}_n} \leq \frac{3w}{2}$, and then

$$\begin{cases} \dfrac{2n}{w} \geq \mathcal{T}_n(\theta) \geq \dfrac{2n}{3w} & \text{for } n > L, \\ \dfrac{2n}{w} \leq \mathcal{T}_n(\theta) \leq \dfrac{2n}{3w} & \text{for } n < -L. \end{cases} \tag{2.2}$$

From the above inequality, $\mathcal{T}_n \to \pm\infty$ as $n \to \pm\infty$. Hence

$$N(t) := \min\big\{k \mid \sup_{\theta\in\mathbb{T}^d} |t - \mathcal{T}_k| = \min_n \sup_{\theta\in\mathbb{T}^d} |t - \mathcal{T}_n|\big\}$$

is well defined.

Claim. *For any $t \in \mathbb{R}$, there exists an absolute constant $C > 0$ such that*

$$\sup_{\theta\in\mathbb{T}^d} |t - \mathcal{T}_{N(t)}| \leq C. \tag{2.3}$$

Proof of claim. As we showed in the proof of Lemma 2.2, $\{\mathcal{T}_n\}$ is a relatively dense set, i.e., there exists $R > 0$ such that for any $t \in \mathbb{R}$, $(t - R/2, t + R/2) \cap \{\mathcal{T}_n\} \neq \emptyset$. Now by (2.2) with $\theta = \theta + N(t)\alpha$, for any $|n| \geq \max\{3wR/2, L\}$,

$$\sup_{\theta\in\mathbb{T}^d} |\mathcal{T}_n(\theta + N(t)\alpha)| = \sup_{\theta\in\mathbb{T}^d} |\mathcal{T}_{n+N(t)} - \mathcal{T}_{N(t)}| \geq R.$$

Taking $M = \max\{3wR/2, L\}$, from the definition of $N(t)$, we have

$$\begin{aligned} \sup_{\theta\in\mathbb{T}^d} |t - \mathcal{T}_{N(t)}| \leq R &\leq \sup_k \sup_{|j|\leq M} \sup_{\theta\in\mathbb{T}^d} |\mathcal{T}_{k+j} - \mathcal{T}_k| \\ &\leq \sup_k \sup_{|j|\leq M} \sup_{\theta\in\mathbb{T}^d} |\mathcal{T}_j(\theta + k\alpha)| \leq C, \end{aligned}$$

since $\mathbb{T}^d$ is compact. The claim is proved. ∎

Finally, Definition 1.1 (1) shows that $v(t,n,\theta) \to 1$ as $n \to -\infty$, $v(t,n,\theta) \to \infty$ as $n \to \infty$ uniformly in θ. Thus,

$$v(t, n + N(t), \theta) = v(t - \mathcal{T}_{N(t)}, n, N(t)\alpha + \theta) \to \begin{cases} 1 & \text{as } n \to -\infty, \\ 0 & \text{as } n \to +\infty, \end{cases}$$

uniformly in t. Moreover, by (2.3) and (2.2), one has

$$\begin{aligned}|t-s| &\le |t-\mathcal{T}_{N(t)}| + |s-\mathcal{T}_{N(s)}(\theta)| + |\mathcal{T}_{N(t)}-\mathcal{T}_{N(s)}| \\ &\le 2C + |\mathcal{T}_{N(t)}-\mathcal{T}_{N(s)}| \le 2C + \frac{2}{w}(N(t)-N(s)).\end{aligned}$$

It follows that $|t-s|\to\infty$ implies $N(t)-N(s)\to\infty$. Now applying (2.3) again, we have

$$\begin{aligned}\lim_{t-s\to\infty}\frac{N(t)-N(s)}{t-s} &= \lim_{t-s\to\infty}\frac{N(t)-N(s)}{\mathcal{T}_{N(t)}-\mathcal{T}_{N(s)}}\cdot\frac{\mathcal{T}_{N(t)}-\mathcal{T}_{N(s)}}{t-s} \\ &= \lim_{t-s\to\infty}\frac{N(t)-N(s)}{\mathcal{T}_{N(t)}-\mathcal{T}_{N(s)}}\cdot\frac{(\mathcal{T}_{N(t)}-t)-(\mathcal{T}_{N(s)}-s)+(t-s)}{t-s} \\ &= \lim_{N(t)-N(s)\to\infty}\frac{N(t)-N(s)}{\mathcal{T}_{N(t)}-\mathcal{T}_{N(s)}} = w.\end{aligned}$$

Hence $u(t,n) := v(t,n,\theta)$ is a generalized transition front of (1.3).

Now we prove the converse part. It suffices to verify Definition 1.1 (3). From assumption (2) and $\widetilde{\mathcal{T}}\in C^0(\mathbb{T}^d,\mathbb{R})$, $N(t)\to\infty$ as $t\to\infty$, and then from the definition of the average speed of a generalized transition front, $|N(t)-N(s)|\le\frac{3w}{2}|t-s|$ for $|t-s|\ge L$. Thus, for any $s\in\mathbb{R}$, $\{N(t)\mid t\in\mathbb{R}\}\cap(s-3wL/4, s+3wL/4)\ne\emptyset$. Notice that for any $n\in\mathbb{Z}$, $\inf_{t\in\mathbb{R}}|n-N(t)|$ can be obtained in some $t_n\in\mathbb{R}$ and $\sup_{n\in\mathbb{Z}}\inf_{t\in\mathbb{R}}|n-N(t)| < \frac{3wL}{2}$. Hence

$$\begin{aligned}\lim_{|n-k|\to\infty}\frac{n-k}{\sum_{i=k}^{n}\widetilde{\mathcal{T}}(i\alpha)} &= \lim_{|n-k|\to\infty}\frac{n-N(t_n)-k+N(t_k)+N(t_n)-N(t_k)}{\widetilde{\mathcal{T}}_n-t_n-\widetilde{\mathcal{T}}_k+t_k+t_n-t_k} \\ &= \lim_{|n-k|\to\infty}\frac{N(t_n)-N(t_k)}{t_n-t_k} = w,\end{aligned}$$

which follows from $\sup_{n\in\mathbb{Z}}|\widetilde{\mathcal{T}}_n-t_n|\le\sup_{n\in\mathbb{Z}}(|\widetilde{\mathcal{T}}_n-\widetilde{\mathcal{T}}_{N(t_n)}|+|\widetilde{\mathcal{T}}_{N(t_n)}-t_n|)<\infty$. ∎

2.4. SL(2, ℝ) cocycles, uniformly hyperbolic

Let X be a compact metric, ν be a probability measure on X space, and T be a measure-preserving transformation. Suppose (X,ν,T) is ergodic [54], and a continuous map $A\colon X\to\mathrm{SL}(2,\mathbb{R})$, the group of real 2×2 matrices with determinant 1. An $\mathrm{SL}(2,\mathbb{R})$ cocycle over (T,X) is defined on $X\times\mathbb{R}^2$ such that

$$(T,A)\colon (x,v)\in X\times\mathbb{R}^2\mapsto (Tx, A(x)\cdot v)\in X\times\mathbb{R}^2.$$

For $n\in\mathbb{Z}$, A_n is defined by $(T,A)^n=(T^n,A_n)$, where $A_0(x)=\mathrm{Id}$,

$$A_n(x)=\prod_{j=n-1}^{0}A(T^jx)=A(T^{n-1}x)\cdots A(Tx)A(x),\quad n\ge 1,$$

and $A_{-n}(x) = A_n(T^{-n}x)^{-1}$. The Lyapunov exponent is defined as

$$L(T, A) = \lim_{n\to\infty} \frac{1}{n} \int_X \ln\|A_n(x)\|\, d\nu.$$

We say an $SL(2, \mathbb{R})$ cocycle (T, A) is uniformly hyperbolic if, for every $x \in X$, there exists a continuous splitting $\mathbb{R}^2 = E_s(x) \oplus E_u(x)$ such that for every $n \geq 0$,

$$\begin{aligned} |A_n(x)v(x)| &\leq Ce^{-cn}|v(x)|, \quad v(x) \in E_s(x), \\ |A_{-n}(x)v(x)| &\leq Ce^{-cn}|v(x)|, \quad v(x) \in E_u(x), \end{aligned} \tag{2.4}$$

for some constants $C, c > 0$, and it holds that $A(x)E_s(x) = E_s(Tx)$ and $A(x)E_u(x) = E_u(Tx)$ for every $x \in X$. Here s, u stand for stable and unstable respectively. It is known from [3] that if (T, A) is uniformly hyperbolic, then $L(T, A) > 0$. If $L(T, A) > 0$ and the splitting is not continuous, then we say (T, A) is nonuniformly hyperbolic.

2.5. Fibered rotation number

Assume X is compact and (X, T) is uniquely ergodic with respect to its unique invariant probability measure. For this kind of dynamically defined cocycle (T, A), one can define the rotation number of the cocycle. Let $\mathbb{S}^1$ be the set of unit vectors of $\mathbb{R}^2$. Consider a projective cocycle $F_{T,A}$ on $X \times \mathbb{S}^1$:

$$(x, v) \mapsto \left(Tx, \frac{A(x)v}{\|A(x)v\|}\right).$$

If $A\colon X \to SL(2, \mathbb{R})$ is continuous and homotopic to the identity, then there exists a lift $\widetilde{F}_{(T,A)}$ of $F_{(T,A)}$ to $X \times \mathbb{R}$ such that $\widetilde{F}_{(T,A)}(x, y) = (Tx, y + \tilde{f}_{(T,A)}(x, y))$, where $\tilde{f}_{(T,A)}\colon X \times \mathbb{R} \to \mathbb{R}$ is a continuous lift that satisfies

- $\tilde{f}_{(T,A)}(x, y + 1) = \tilde{f}_{(T,A)}(x, y) + 1$;
- for every $x \in X$, $\tilde{f}_{(T,A)}(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism;
- if π_2 is the projection map $X \times \mathbb{R} \to X \times \mathbb{S}^1\colon (x, y) \mapsto (x, e^{2\pi i y})$, then $F_{(T,A)} \circ \pi_2 = \pi_2 \circ \widetilde{F}_{(T,A)}$.

Meanwhile, define the nth iterate as

$$\widetilde{F}^n_{(T,A)}(x, y) = (T^n x, y + \tilde{f}^n_{(T,A)}(x, y)).$$

Then there exists $\rho \in \mathbb{R}$ such that

$$\frac{\tilde{f}^n_{(T,A)}(x, y)}{n}$$

converges uniformly to ρ in $(x, y) \in X \times \mathbb{R}$, and it is independent of the lift of $F_{T,A}$, up to the addition of an integer [25]. Then ρ is called a fibered rotation number of (T, A), and we denote it by $\mathrm{rot}(T, A)$.

Remark 2.2. In the following, we will always take $X = \mathbb{T}^d$, where $d \in \mathbb{N}_+$ or ∞ (we endow it with the product topology if $d = \infty$), and consider the quasi-periodic (or almost-periodic) cocycle (α, A).

The following two lemmas are useful for us:

Lemma 2.3 ([22]). *For any $A \in \mathrm{SL}(2,\mathbb{R})$, we have*

$$|\mathrm{rot}(\alpha, Ae^F) - \mathrm{rot}(\alpha, A)| < 2\|A\|\,\|F\|_0^{\frac{1}{2}}.$$

Lemma 2.4 ([35]). *The rotation number is invariant under the conjugation map which is homotopic to the identity. More precisely, if $A, B\colon \mathbb{T}^d \to \mathrm{SL}(2,\mathbb{R})$ is continuous and homotopic to the identity, then*

$$\mathrm{rot}(\alpha, B(\cdot+\alpha)^{-1}A(\cdot)B(\cdot)) = \mathrm{rot}(\alpha, A).$$

2.6. Almost-periodic Schrödinger operator

For an almost-periodic sequence, we also have the following:

Proposition 2.6 ([52]). *Let f be a bounded function on $\mathbb{Z}$. The following are equivalent:*

(1) *f is Bochner almost periodic.*

(2) *f is a uniform limit of finite sums of the form*

$$g_N(n) = \sum_{j=1}^{N} a_j e^{2\pi \mathrm{i} \alpha_j^{(N)} n}$$

for $\alpha^1, \ldots, \alpha_N^N \in \mathbb{R}/\mathbb{Z}$.

(3) *There exists a continuous function F on $\mathbb{T}^\infty$ and $\{\theta_j\}_{j=1}^\infty$ in $\mathbb{T}^\infty$ so that*

$$f(n) = F(\theta^n),$$

where $(\theta^n)_j = \theta_j^n$.

As result, we define the almost-periodic Schrödinger operator as a self-adjoint operator on $l^2(\mathbb{Z})$:

$$(\mathcal{L}_{V,\alpha,\theta}u)(n) = u(n+1) + u(n-1) - 2u(n) + V(n\alpha+\theta)u(n) \quad \forall n \in \mathbb{Z},$$

where $V \in C^0(\mathbb{T}^d, \mathbb{R})$, $d \in \mathbb{N}_+$ or $d = \infty$. In particular, if $d \in \mathbb{N}_+$, then $\mathcal{L}_{V,\alpha,\theta}$ is called a quasi-periodic Schrödinger operator. We say θ is the phase, V is the potential, and α is the frequency. It is well known that the spectrum $\Sigma(\mathcal{L}_{V,\alpha,\theta})$ is a compact set of $\mathbb{R}$. Moreover, $\Sigma(\mathcal{L}_{V,\alpha,\theta})$ is independent of θ, and we shorten the notation to $\Sigma(\mathcal{L}_{V,\alpha})$.

For the almost-periodic Schrödinger operator, one can define the *integrated density of states* (abbreviated IDS), denoted by $k(E)$, as

$$k(E) = \lim_{L\to\infty} \frac{\sharp\{\text{eigenvalues (counting multiplicity) of } \mathcal{L}_L \leq E\}}{L-1},$$

where $\mathcal{L}_L$ is the restriction of $\mathcal{L}_{V,\alpha,\theta}$ to the set $I = \{1, \ldots, L-1\}$ with boundary conditions $\frac{u(0)}{u(1)} = \cot\theta$, $u(L) = 0$. The integrated density of states will be crucial for our study of the positive almost-periodic solution. Furthermore, we have the following elementary fact:

Remark 2.3. Suppose E is at the rightmost point of the spectrum. Then by the well-known characterization of the biggest eigenvalue λ^L and E,

$$\lambda^L = \sup_{\substack{u\in l^2(\mathbb{Z}),\, u\not\equiv 0,\\ \operatorname{supp} u \subset I}} \frac{\langle \mathcal{L}u, u\rangle}{\langle u, u\rangle}, \quad E = \sup_{u\in l^2(\mathbb{Z}),\, u\not\equiv 0} \frac{\langle \mathcal{L}u, u\rangle}{\langle u, u\rangle},$$

the rightmost point of the spectrum E is always larger than λ^L. Thus, the number of eigenvalues of $\mathcal{L}_L$ less than or equal to E is always $L-1$. By the definition of the IDS, $k(E) = 1$.

Note that a sequence $(u_n)_{n\in\mathbb{Z}}$ is a formal solution of the eigenvalue equation $\mathcal{L}_{V,\alpha,\theta} u = Eu$ if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = S_E^V(n\alpha + \theta) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix},$$

where

$$S_E^V(\theta) = \begin{pmatrix} E + 2 - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

We call (α, S_E^V) an almost-periodic Schrödinger cocycle if $d = \infty$, and a quasi-periodic Schrödinger cocycle if $d \in \mathbb{N}_+$.

The study of the spectral properties of the almost-periodic Schrödinger operator is closely related to the dynamics of the almost-periodic Schrödinger cocycle. For example, we will need the following two important facts:

Theorem 2.2 ([29]). *The IDS of the Schrödinger operator relates to the fibered rotation number of Schrödinger operator as*

$$k(E) = 1 - 2\operatorname{rot}(\alpha, S_E^V) \pmod{\mathbb{Z}}.$$

Theorem 2.3 ([30]). *Let $\mathcal{L}_{V,\alpha,\omega}$ be an almost-periodic Schrödinger operator. Then we have*

$$\mathbb{R}\backslash\Sigma = \{E \in \mathbb{R} \mid (\alpha, S_E^V) \text{ is uniformly hyperbolic}\}.$$

3. Properties of the linearized problem

3.1. Generalized principal eigenvalue for more-general operators

In this section we will define and study the properties of the generalized principal eigenvalue of a more general operator since it will be needed in our proof. Remarkably, generalized principal eigenvalue theory for elliptic operators is of its own interest, and it turns out to be very useful in studying the maximum principle [11].

Now we consider the operator

$$\mathcal{M}_{a,b,c}\phi(n) := a(n)\phi(n+1) + b(n)\phi(n-1) + c(n)\phi(n),$$

with a, b, c being almost-periodic sequences and $\inf a > 0$, $\inf b > 0$. In particular,

$$\mathcal{M}_{1,1,V(\cdot\alpha+\theta)-2} = \mathcal{L}_\theta$$

defined in (1.5).

For any (maybe unbounded) interval $I \subset \mathbb{Z}$, we define the generalized principal eigenvalue for $\mathcal{M}_{a,b,c}$ as

$$\lambda_1(\mathcal{M}_{a,b,c}, I) = \inf\{\lambda \in \mathbb{R} \mid \exists \phi > 0 \text{ in } \mathbb{Z},\ \mathcal{M}_{a,b,c;I}\phi \leq \lambda\phi \text{ in } I\}, \tag{3.1}$$

where $\mathcal{M}_{a,b,c;I}$ is the restriction of $\mathcal{M}_{a,b,c}$ to the set $I \subset \mathbb{Z}$. If I is bounded, it is exactly the classical principal eigenvalue (the largest eigenvalue). In the case that $I = \mathbb{Z}$, $a = b \equiv 1$, and c is replaced by $c - 2$, we will show below that it coincides with $\lambda_1 = \max \Sigma(\mathcal{L})$. From (3.1), $\lambda_1(\mathcal{M}_{a,b,c}, I)$ is nondecreasing with respect to the inclusion of intervals I.

Proposition 3.1. *There holds*

$$\lambda_1(\mathcal{M}_{a,b,c}, (-N, N)) \nearrow \lambda_1(\mathcal{M}_{a,b,c}, \mathbb{Z}) \quad \text{as } N \to \infty.$$

Remark 3.1. The proof of this proposition is similar to the proofs in [11, Proposition 2.3] and [10, Proposition 4.2], which work for continuous elliptic operators.

Proposition 3.2. *Let $b(n) = a(n-1)$ and I be an interval in $\mathbb{Z}$. Then*

$$\lambda_1(\mathcal{M}_{a,b,c}, I) = \sup_{\substack{v \in l^2(\mathbb{Z}),\ v \not\equiv 0, \\ \operatorname{supp} v \subset I}} \frac{\langle \mathcal{M}_{a,b,c} v, v\rangle}{\langle v, v\rangle}, \tag{3.2}$$

where $\langle\cdot\rangle$ denotes the inner product on $l^2(\mathbb{Z})$. Moreover, $\lambda_1(\mathcal{L}, \mathbb{Z}) = \lambda_1 = \max \Sigma(\mathcal{L})$.

Proof. We shorten the notation: $\mathcal{M}_{a,b,c;N} := \mathcal{M}_{a,b,c;[-N,N]}$, $\lambda_1^N := \lambda_1(\mathcal{M}_{a,b,c}; [-N, N])$.

We only prove the case $I = \mathbb{Z}$. Since $b(n) = a(n-1)$, it is clear that $\mathcal{M}_{a,b,c;N}$ is a bounded self-adjoint operator. Then for any $N \in \mathbb{N}_+$,

$$\begin{aligned} \sup_{\substack{v \in l^2(\mathbb{Z}),\ v \not\equiv 0, \\ \operatorname{supp} v \subset [-N,N]}} \frac{\langle \mathcal{M}_{a,b,c;N} v, v\rangle}{\langle v, v\rangle} &= \|\mathcal{M}_{a,b,c;N}\|, \\ \sup_{v \in l^2(\mathbb{Z}), v \not\equiv 0} \frac{\langle \mathcal{M}_{a,b,c} v, v\rangle}{\langle v, v\rangle} &= \|\mathcal{M}_{a,b,c}\|. \end{aligned} \tag{3.3}$$

Here, $\|\cdot\|$ denotes the norm in the Banach space of linear bounded operators from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$. From (3.3), for any $\varepsilon > 0$, there exists a nonzero v_ε and an $N_\varepsilon \in \mathbb{Z}$ such that $\sum_{|n|\geq N_\varepsilon} |v_\varepsilon|^2 < (\|a\|_{l^\infty} + \|b\|_{l^\infty} + \|c\|_{l^\infty})^{-1}\frac{\varepsilon}{2}$ and

$$\frac{\langle \mathcal{M}_{a,b,c} v_\varepsilon, v_\varepsilon\rangle}{\langle v_\varepsilon, v_\varepsilon\rangle} \geq \|\mathcal{M}_{a,b,c}\| - \frac{\varepsilon}{2}.$$

Define

$$\tilde{v}_\varepsilon = \begin{cases} v_\varepsilon, & n \in [-N_\varepsilon, N_\varepsilon], \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\|\mathcal{M}_{a,b,c}\| \geq \frac{\langle \mathcal{M}_{a,b,c;N_\varepsilon} \tilde{v}_\varepsilon, \tilde{v}_\varepsilon\rangle}{\langle \tilde{v}_\varepsilon, \tilde{v}_\varepsilon\rangle} \geq \|\mathcal{M}_{a,b,c}\| - \varepsilon,$$

and then

$$\|\mathcal{M}_{a,b,c}\| \geq \|\mathcal{M}_{a,b,c;N_\varepsilon}\| \geq \|\mathcal{M}_{a,b,c}\| - \varepsilon.$$

From (3.3) and Proposition 3.1, we have

$$\begin{aligned} \lambda_1(\mathcal{M}_{a,b,c}, \mathbb{Z}) &= \lim_{N\to\infty} \lambda_1(\mathcal{M}_{a,b,c;N}) = \lim_{N\to\infty} \|\mathcal{M}_{a,b,c;N}\| \\ &= \|\mathcal{M}_{a,b,c}\| = \sup_{v\in l^2(\mathbb{Z}), v\not\equiv 0} \frac{\langle M_{a,b,c} v, v\rangle}{\langle v, v\rangle}. \end{aligned}$$

In particular, $\lambda_1(\mathcal{M}_{a,b,c}, \mathbb{Z})$ lies in the rightmost point of the spectrum. ■

From Proposition 3.2, we can deduce that $\lambda_1(\mathcal{L}, \mathbb{Z}) \geq \inf c$. Indeed, taking the test function

$$v_{2N}(n) = \begin{cases} 1, & |n| \leq 2N, \\ 0 & \text{otherwise,} \end{cases}$$

in (3.2), we deduce the generalized principal eigenvalue $\lambda_1(\mathcal{L}, [-N, N]) \geq \inf c$. By Proposition 3.1, we have $\lambda_1(\mathcal{L}, \mathbb{Z}) \geq \inf c$, as desired.

Other generalizations of the principal eigenvalue will be listed below, and they are all indispensable to our proof. Define the following quantities:

$$\begin{aligned} &\underline{\lambda}_1(\mathcal{M}_{a,b,c}) := \sup\{\lambda \mid \exists \phi \in \mathcal{S},\ \mathcal{M}_{a,b,c}\phi \geq \lambda\phi \text{ in } \mathbb{Z}\}, \\ &\bar{\lambda}_1(\mathcal{M}_{a,b,c}) := \inf\{\lambda \mid \exists \phi \in \mathcal{S},\ \mathcal{M}_{a,b,c}\phi \leq \lambda\phi \text{ in } \mathbb{Z}\}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\underline{\lambda}'_1(\mathcal{M}_{a,b,c}) := \sup\{\lambda \mid \exists \phi > 0,\ \phi \in l^\infty(\mathbb{Z}),\ \mathcal{M}_{a,b,c}\phi \geq \lambda\phi \text{ in } \mathbb{Z}\}, \\ &\lambda_1(\mathcal{M}_{a,b,c}) := \lambda_1(\mathcal{M}_{a,b,c}; \mathbb{Z}) = \inf\{\lambda \mid \exists \phi > 0,\ \mathcal{M}_{a,b,c}\phi \leq \lambda\phi \text{ in } \mathbb{Z}\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\underline{\mu}_1(\mathcal{M}_{a,b,c}) := \sup\{\mu \mid \exists \phi \in l^\infty(\mathbb{Z}),\ \inf_{n\in\mathbb{Z}} \phi > 0,\ \mathcal{M}_{a,b,c}\phi \geq \mu\phi \text{ in } \mathbb{Z}\}, \\ &\bar{\mu}_1(\mathcal{M}_{a,b,c}) := \inf\{\mu \mid \exists \phi \in l^\infty(\mathbb{Z}),\ \inf_{n\in\mathbb{Z}} \phi > 0,\ \mathcal{M}_{a,b,c}\phi \leq \mu\phi \text{ in } \mathbb{Z}\}, \end{aligned} \tag{3.6}$$

where

$$\mathcal{S} = \{\phi > 0 \mid \lim_{|n|\to+\infty} \tfrac{\log\phi(n)}{n} = 0,\ \{\tfrac{\phi(n+1)}{\phi(n)}\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})\}.$$

We denote $g \cdot k = g(\cdot + k)$, $g \in \mathcal{H}(f)$, $k \in \mathbb{Z}$, and the hull of triple (a,b,c) by $\mathcal{H}(a,b,c)$:

$$\left\{(a^*,b^*,c^*) \mid a \cdot n_i \to a^*,\ b \cdot n_i \to b^*,\ c \cdot n_i \to c^* \text{ for some } \{n_i\}_{i\in\mathbb{Z}}\right\}.$$

Proposition 3.3. *The following hold:*

(1) $\underline{\lambda}_1(\mathcal{M}_{a,b,c}) = \bar{\lambda}_1(\mathcal{M}_{a,b,c}) = \underline{\mu}_1(\mathcal{M}_{a,b,c}) = \bar{\mu}_1(\mathcal{M}_{a,b,c})$. *Moreover, if* $a(n) = b(n+1)$, *then they all coincide with* $\lambda_1(\mathcal{M}_{a,b,c})$ *and* $\underline{\lambda}_1'(\mathcal{M}_{a,b,c})$.

(2) $\lambda_1(\mathcal{M}_{a',b',c'})$, $\underline{\mu}_1(\mathcal{M}_{a',b',c'})$, $\bar{\mu}_1(\mathcal{M}_{a',b',c'})$, $\bar{\lambda}_1(\mathcal{M}_{a',b',c'})$ *are constant functions with respect to* (a,b,c) *on* $\mathcal{H}(a,b,c)$. *If* $a(n) = b(n+1)$, *then* $\lambda_1(\mathcal{M}_{a,b,c})$, $\underline{\lambda}_1'(\mathcal{M}_{a,b,c})$ *are also constant functions.*

In particular, $\lambda_1(\mathcal{L}_g,\mathbb{Z})$ *is a constant function with respect to* g *on* $\mathcal{H}(c)$.

Proof. (1) It is straightforward to check using (3.4), (3.5), and (3.6) that

$$\underline{\mu}_1(\mathcal{M}_{a,b,c}) \le \underline{\lambda}_1'(\mathcal{M}_{a,b,c}), \quad \lambda_1(\mathcal{M}_{a,b,c}) \le \bar{\mu}_1(\mathcal{M}_{a,b,c}) \tag{3.7}$$

and

$$\underline{\mu}_1(\mathcal{M}_{a,b,c}) \le \underline{\lambda}_1(\mathcal{M}_{a,b,c}), \quad \bar{\lambda}_1(\mathcal{M}_{a,b,c}) \le \bar{\mu}_1(\mathcal{M}_{a,b,c}). \tag{3.8}$$

Moreover, by the standard argument in [11, Theorem 1.7], we can prove that $\underline{\lambda}_1'(\mathcal{M}_{a,b,c}) \le \lambda_1(\mathcal{M}_{a,b,c})$ if $a(n) = b(n+1)$. From [38, Proposition 2.1], we also have $\underline{\lambda}_1(\mathcal{M}_{a,b,c}) \le \bar{\lambda}_1(\mathcal{M}_{a,b,c})$.

For any $\varepsilon > 0$, [38, Lemma 5.2] guarantees the existence of an almost-periodic function u_ε which satisfies

$$\mathcal{M}_{a,b,c}e^{u_\varepsilon} = \varepsilon u_\varepsilon e^{u_\varepsilon}, \quad -\frac{\|c\|_{l^\infty(\mathbb{Z})}}{\varepsilon} < u^\varepsilon < \frac{2 + \|c\|_{l^\infty(\mathbb{Z})}}{\varepsilon}.$$

Now we choose $e^{u_\varepsilon} \in l^\infty(\mathbb{Z})$ with $\inf_{n\in\mathbb{Z}} e^{u_\varepsilon} > 0$ as a test function in (3.6). Meanwhile, one has $\varepsilon u_\varepsilon \to \lambda$ uniformly by [38, Lemma 5.2]. Thus, for any $\delta > 0$, there exists ε_δ such that $|\varepsilon_\delta u_{\varepsilon_\delta} - \lambda| < \delta$ and

$$(\lambda - \delta)e^{u_{\varepsilon_\delta}} \le \mathcal{M}_{a,b,c}e^{u_{\varepsilon_\delta}} = \varepsilon_\delta u_{\varepsilon_\delta} e^{u_{\varepsilon_\delta}} \le (\lambda + \delta)e^{u_{\varepsilon_\delta}}.$$

Then letting $\delta \to 0$, $\underline{\mu}_1(\mathcal{M}_{a,b,c}) \ge \lambda \ge \bar{\mu}_1(\mathcal{M}_{a,b,c})$ follows by the definition. Thus by (3.8), we obtain $\lambda = \underline{\mu}_1(\mathcal{M}_{a,b,c}) = \underline{\lambda}_1(\mathcal{M}_{a,b,c}) = \bar{\lambda}_1(\mathcal{M}_{a,b,c}) = \bar{\mu}_1(\mathcal{M}_{a,b,c})$. Consequently, the desired result is obtained by (3.7).

(2) Analogous to the above argument, for any sequences $\{n_i\}_{i\in\mathbb{Z}}$ such that

$$a \cdot n_i \to a^*, \quad b \cdot n_i \to b^*, \quad c \cdot n_i \to c^*,$$

we have

$$\mathcal{M}_{a\cdot n_i, b\cdot n_i, c\cdot n_i} e^{u_\varepsilon \cdot n_i} = \varepsilon u_\varepsilon \cdot n_i e^{u_\varepsilon \cdot n_i}.$$

Notice when $\varepsilon \to 0$, $\varepsilon u_\varepsilon \to \lambda$ uniformly. Hence for any $\delta > 0$, there exists ε_δ such that $|\varepsilon_\delta u_{\varepsilon_\delta} \cdot n - \lambda| \le \delta$ holds for any n. Thereby,

$$(\lambda + \delta)e^{u_{\varepsilon_\delta} \cdot n_i} \ge \mathcal{M}_{a\cdot n_i, b\cdot n_i, c\cdot n_i} e^{u_{\varepsilon_\delta} \cdot n_i} = \varepsilon_\delta u_{\varepsilon_\delta} \cdot n_i e^{u_{\varepsilon_\delta} \cdot n_i} \ge (\lambda - \delta)e^{u_{\varepsilon_\delta} \cdot n_i}.$$

Note that u_ε is almost periodic for any $\varepsilon > 0$. Passing along a subsequence $i_k \to \infty$, for any $\delta > 0$, it follows from (1) that

$$\lambda - \delta \le \underline{\mu}_1(\mathcal{M}_{a^*,b^*,c^*}) = \underline{\lambda}_1(\mathcal{M}_{a^*,b^*,c^*}) = \bar{\lambda}_1(\mathcal{M}_{a^*,b^*,c^*}) = \bar{\mu}_1(\mathcal{M}_{a^*,b^*,c^*}) \le \lambda + \delta.$$

Finally, letting $\delta \to 0$,

$$\lambda = \underline{\mu}_1(\mathcal{M}_{a^*,b^*,c^*}) = \underline{\lambda}_1(\mathcal{M}_{a^*,b^*,c^*}) = \bar{\lambda}_1(\mathcal{M}_{a^*,b^*,c^*}) = \bar{\mu}_1(\mathcal{M}_{a^*,b^*,c^*}).$$

If $a(n) = b(n+1)$, apply the above argument to $\underline{\lambda}_1'(\mathcal{M}_{a,b,c})$, $\lambda_1(\mathcal{M}_{a,b,c})$; then we are done. ∎

3.2. The Lyapunov exponent of the linearized operator

The Lyapunov exponent is crucial to determine the average wave speed of the almost-periodic traveling front. We will discuss its properties here and illustrate the connection with the existence of positive almost-periodic solution. Recall

$$(\mathcal{L}_\theta u)(n) = u(n+1) + u(n-1) - 2u(n) + V(n\alpha + \theta)u(n),$$

and $\mathcal{L} = \mathcal{L}_0$. Note that Proposition 3.3 tells us $\lambda_1(\mathcal{L}_\theta, \mathbb{Z}) = \lambda_1(\mathcal{L}, \mathbb{Z})$ for any $\theta \in \mathbb{T}^d$. As a consequence of Lemma 3.2, $\lambda_1 = \max \Sigma(\mathcal{L}) = \lambda_1(\mathcal{L}, \mathbb{Z})$. Hence we do not distinguish between them with a slight abuse of notation in the forthcoming paragraphs. First we state the following technical lemma which will be frequently used, and it is an immediate consequence of [38, Lemma 6.2].

Lemma 3.1. *Let $E > \lambda_1$, $n_0 \in \mathbb{Z}$, and ϕ defined in $[n_0, +\infty) \cap \mathbb{Z}$ satisfy*

$$\mathcal{L}\phi \ge E\phi \ \text{ in } (n_0, +\infty) \cap \mathbb{Z}, \qquad \lim_{n\to+\infty} \phi(n) = 0.$$

Then there are positive constants C, δ only depending on E, λ_1, and $\|c(n)\|_{l^\infty}$ such that

$$\phi(n) \le C \max\{\phi(n_0), 0\} e^{-\delta(n-n_0)}.$$

Moreover, $\lim_{E\to\lambda_1} \delta = 0$ *and* $\lim_{E\to+\infty} \delta = +\infty$.

Now we study the Lyapunov exponent $L(E)$ of $\mathcal{L}$ defined in (1.6) and our observation is stated as follows:

Proposition 3.4. *For all $E > \lambda_1$, there exists a unique positive solution $\phi_E \in C^0(\mathbb{Z} \times \mathbb{T}^d, \mathbb{R})$ of*

$$\mathcal{L}_\theta \phi = E\phi \text{ in } \mathbb{Z}, \quad \phi(0) = 1, \quad \lim_{n\to\infty} \phi(n) = 0. \tag{3.9}$$

and the limit

$$L(E) = -\lim_{n\to\pm\infty} \frac{1}{n} \ln \phi_E(n, \theta) > 0 \quad \text{for any } \theta \in \mathbb{T}^d,$$

where the convergence is uniform in $\theta \in \mathbb{T}^d$.

First we need the following lemma:

Lemma 3.2 ([55]). *Let $T: X \to X$ be a uniquely ergodic homeomorphism of the compact metric space X and (T, A) be an* $\mathrm{SL}(2, \mathbb{R})$ *cocycle over the probability space (X, μ). If (T, A) is uniformly hyperbolic, then $\frac{1}{n} \log \|A_n(x)\|$ converges uniformly to a constant.*

Proof of Proposition 3.4. The existence and uniqueness of the positive solution follows from [38, Lemma 6.3]. Now we prove $\phi_E(\cdot, \theta) \in C^0(\mathbb{Z} \times \mathbb{T}^d, \mathbb{R})$, and it is sufficient to prove $\phi_E(\cdot, n_k\alpha + \theta) \to \phi_E(\cdot, \theta^*)$ provided that $\theta + n_k\alpha \to \theta^*$. Indeed, for any $\theta_k \to \theta_*$, since $\alpha \in \mathbb{R}^d/\mathbb{Q}^d$, there exist sequences $\{n_{i,k}\}_{i,k\in\mathbb{Z}}$ such that $n_{i,k}\alpha + \theta \to \theta_k$, and then

$$
\begin{aligned}
|\phi_E(\cdot, \theta_k) - \phi_E(\cdot, \theta^*)| \leq{}& |\phi_E(\cdot, \theta_k) - \phi_E(\cdot, n_{i,k}\alpha + \theta)| \\
&+ |\phi_E(\cdot, n_{i,k}\alpha + \theta) - \phi_E(\cdot, \theta^*)|.
\end{aligned}
$$

Then we have $\phi_E(\cdot, \theta_k) \to \phi_E(\cdot, \theta^*)$, and this implies that $\phi_E(\cdot, \theta) \in C^0(\mathbb{Z} \times \mathbb{T}^d, \mathbb{R})$.

Now we prove $\phi_E(\cdot, n_k\alpha + \theta) \to \phi_E(\cdot, \theta^*)$ provided that $\theta + n_k\alpha \to \theta^*$. For $n_k\alpha + \theta \to \theta^*$, $\phi_E(n, \theta + n_k\alpha)$ converges locally uniformly, up to some subsequence, to some function $\tilde{\phi}_E(n)$. Moreover, we have $\tilde{\phi}_E(n) \leq Ce^{-\delta n}$ for $n > 0$ since $\phi_E(n, n_k\alpha + \theta) \leq Ce^{-\delta n}$ from Lemma 3.1. Therefore $\tilde{\phi}$ satisfies

$$
\mathcal{L}_{\theta^*}\phi = E\phi \text{ in } \mathbb{Z}, \quad \phi(0) = 1, \quad \lim_{n\to\infty} \phi(n) = 0,
$$

which yields that $\tilde{\phi} = \phi_E(n, \theta^*)$ by uniqueness. That is to say, all the convergent subsequences converge to the same limit. Hence $\phi_E(n, n_k\alpha + \theta) \to \phi_E(n, \theta^*)$ locally uniformly in $n \in \mathbb{Z}$.

Now let $X = \mathbb{T}^d$, $d \in \mathbb{N}_+$ or $d = \infty$, $T: \mathbb{T}^d \to \mathbb{T}^d$ with $T\theta = \theta + \alpha$ for any $\theta \in \mathbb{T}^d$, $S^E(\theta) = \begin{pmatrix} E+2-V(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in C^0(\mathbb{T}^d, \mathrm{SL}(2, \mathbb{R}))$. Then (α, S^E) is a cocycle defined on $(\mathbb{T}^d, d\theta)$. Note that $T: \theta \mapsto \theta + \alpha$ is uniquely ergodic on $\mathbb{T}^d$, because $\mathbb{T}^d$ is a compact Abelian group and T is minimal.

Since $E > \lambda_1$, by Theorem 2.3, (T, S^E) is uniformly hyperbolic. Then as a consequence of Lemma 3.2, the limit $L^E := \lim_{n\to\infty} \frac{1}{n} \ln \|S_n^E(\theta)\|$ exists, and it is independent of $\theta \in \mathbb{T}^d$. Moreover, the convergence is uniform in $\theta \in \mathbb{T}^d$. Thus we can deduce that for any $\theta \in \mathbb{T}^d$,

$$
L^E = \lim_{n\to\infty} \frac{1}{n} \ln \|S_n^E(\theta)\| = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}^d} \ln \|S_n^E(\theta)\| \, d\theta = L(E),
$$

where the last equality follows from direct examination of the definition.

Note that in the proof of Lemma 3.2, we deduce that the limit $\lim_{n\to\infty} \frac{1}{n} \ln |S_n^E(\theta) \cdot v|$ exists for any $v \in \mathbb{R}^2 \backslash \{0\}$. Now it follows from (2.4) that uniform hyperbolicity implies that for any $\theta \in \mathbb{T}^d$, and every $n \geq 0$, only $v(\theta) \in E_s(\theta) \subset \mathbb{R}^2$ satisfies $\lim_{n\to\infty} \frac{1}{n} \ln |S_n^E(\theta)v(\theta)| < 0$. Otherwise, all $v \in E_s(\theta)$ violate (2.4).

Denote $v_E(\theta) = (\phi_E(1, \theta), \phi_E(0, \theta))$ for $\theta \in \mathbb{T}^d$. By Lemma 3.1 and (3.9), one has

$$
\lim_{n\to\infty} \frac{1}{n} \ln |S_n^E(\theta) \cdot v_E(\theta)| = \lim_{n\to\infty} \frac{\ln(\phi_E^2(n, \theta) + \phi_E^2(n-1, \theta))}{2n} < 0. \tag{3.10}
$$

Hence $v_E(\theta) \in E_s(g)$, and $\lim_{n\to\infty} \frac{1}{n} \ln |S_n^E(\theta) \cdot v_E(\theta)| = -L(E)$ since $S_n^E(\theta)$ takes values in $\mathrm{SL}(2,\mathbb{R})$.

Meanwhile, since $S_{-n}^E(\theta)v_E(\theta) \in E_s(\theta - n\alpha)$ in (2.4), one has

$$|v_E(\theta)| = |S_n^E(\theta - n\alpha)S_{-n}^E(\theta)v_E(\theta)| \le C\mathrm{e}^{-nL(E)}|S_{-n}^E v_E(\theta)|.$$

Then we can deduce that

$$\lim_{n\to\infty} \frac{1}{n}|S_{-n}^E(\theta)v_E(\theta)| = \lim_{n\to\infty} \frac{\ln(\phi_E^2(-n,\theta) + \phi_E^2(-n+1,\theta))}{n} = L(E). \tag{3.11}$$

From the equation $(\mathcal{L}_g\phi_E(\cdot,\theta))(n) = E\phi_E(n,\theta)$, we have

$$\frac{\phi_E(n+1,\theta)}{\phi_E(n,\theta)} + \frac{\phi_E(n-1,\theta)}{\phi_E(n,\theta)} = 2 + V(n\alpha+\theta) \le M,$$

where M depends on $\|V\|_{L^\infty}$. It follows that

$$\frac{1}{M}\phi_E(n,\theta) \le \phi_E(n+1,\theta) \le M\phi_E(n,\theta).$$

Inserting this inequality into (3.10) and (3.11), $L(E) = \lim_{n\to\pm\infty} -\frac{\ln\phi_E(n,\theta)}{n}$ follows directly. Then the proof is complete. ∎

The concavity and monotonicity of the Lyapunov exponent $L(E)$ will be needed in our construction of the almost-periodic traveling front, and it is given by the following lemma:

Lemma 3.3. *The function $E \mapsto L(E)$ defined on $(\lambda_1, +\infty)$ is concave, nondecreasing and there exists $C > 0$ such that, for $E > \lambda_1$,*

$$\delta \le L(E) \le C\sqrt{E},$$

where δ was given in Lemma 3.1 and $L(E) > \underline{L} := \lim_{E\searrow\lambda_1} L(E)$.

Proof. That the Lyapunov exponent $L(E)$ is nondecreasing and concave follows from [43, Lemma 2.5].

For $E_1 < E_2$, let ϕ_{E_1} and ϕ_{E_2} be obtained in Proposition 3.4. It is straightforward to check that ϕ_{E_2} is a subsolution of the equation satisfied by ϕ_{E_1} in $[0,\infty)$. Applying Lemma 3.1 to $\phi_{E_2} - \phi_{E_1}$, monotonicity follows directly. Also, Lemma 3.1 shows that

$$L(E) = -\lim_{n\to+\infty} \frac{1}{n}\ln\phi_E(n,\theta) \ge -\lim_{n\to+\infty}\frac{C}{n} + \delta = \delta.$$

Let $h_E(n,\theta) = e^{-n\sqrt{E-\inf c}} - \phi_E(n,\theta)$ with $\phi_E(\cdot,\theta)$ satisfying

$$\mathcal{L}_\theta\phi = E\phi \text{ in } \mathbb{Z}, \quad \phi_E(0,\theta) = 1, \lim_{n\to\infty}\phi(n,\theta) = 0.$$

We can check h_E satisfies $\mathcal{L}_\theta h_E \ge Eh_E$ and $\lim_{n\to\infty} h_E(n,\theta) = 0$. By Lemma 3.1, we have $\phi_E(n,\theta) \ge e^{-n\sqrt{E-\inf c}}$, whence $L(E) \le \sqrt{E-\inf c} \le C\sqrt{E}$, as desired. ∎

The existence of a positive almost-periodic solution always implies the Lyapunov exponent $L(E)$ will decay to 0 as $E \searrow \lambda_1$, and it is crucial for us to determine in which case we can establish the almost-periodic traveling front with average wave speed $w \in (w^*, \infty)$ (cf. Corollaries 1.1 and 1.2). First we need a preliminary lemma about the critical operator.

Lemma 3.4. *Suppose that $\mathcal{L}$ admits a positive bounded solution φ of $\mathcal{L}\phi = E\phi$. Then the associated eigenvalue is the generalized principal eigenvalue λ_1 and the following properties hold:*

(1) *$\mathcal{L} - \lambda_1$ is critical.*

(2) *If $\inf \varphi > 0$, then $\mathcal{L}_\theta - \lambda_1$ is critical for any $\theta \in \mathbb{T}^d$.*

(3) *$\inf \varphi > 0$ if and only if φ is almost periodic.*

Proof. Taking φ as the test function in (3.5), it follows from Proposition 3.3 that the associated eigenvalue is exactly the generalized principal eigenvalue λ_1.

(1) Assume by contradiction that $\mathcal{L} - \lambda_1$ is not critical. Denote the quadratic form associated with $\lambda_1 - \mathcal{L}$ by h. Then by Lemma 2.1, there exists a positive function ϖ in $\mathbb{Z}$ such that $h(u) = \langle(\lambda_1 - \mathcal{L})u, u\rangle \geq \langle \varpi u, u\rangle$ for any $u \in l_c$. That is, $\mathcal{L} - \lambda_1 \pm \varpi$ is bounded above by 0 (the supremum of the spectrum of $\mathcal{L} - \lambda_1$: see Proposition 3.2 – i.e., the ground state energy of $\mathcal{L} - \lambda_1$). It follows from [18, Theorem 4] that $\varpi \equiv 0$, which is impossible since ϖ is positive. Hence $\mathcal{L} - \lambda_1$ is critical.

The proofs of (2) and (3) are similar to that of [43, Propsition 1.7]. ■

Although the positive solution of $\mathcal{L}\phi = E\phi$ may not be almost periodic, almost-periodicity can still be revealed in the following way:

Lemma 3.5. *For all $E > \lambda_1$ and any $\theta \in \mathbb{T}^d$, the function $\frac{\phi_E(n+1,\theta)}{\phi_E(n,\theta)}$ is almost periodic in $n \in \mathbb{Z}$.*

Proof. The method is similar to the proof of [43, Lemma 2.4], and Lemma 3.1 is needed in the proof. ■

Once we have this, the next result follows:

Proposition 3.5. *Assume that $\mathcal{L}$ admits a positive almost-periodic solution ϕ of $\mathcal{L}\phi = \lambda_1\phi$. Then we have $\underline{L} := \lim_{E\to\lambda_1} L(E) = 0$.*

Proof. Denote $\phi_E(\cdot) := \phi_E(\cdot, 0)$, which can be obtained in Proposition 3.4. Consider the analogue $\tilde{\phi}_E$ of ϕ_E but with the initial condition

$$\mathcal{L}\tilde{\phi}_E = E\tilde{\phi}_E \ \text{ in } \mathbb{Z}, \quad \tilde{\phi}_E(0) = 1, \quad \lim_{n\to-\infty} \tilde{\phi}_E(n) = 0.$$

The function $\tilde{\phi}_E(-n)$ shares the same properties as ϕ_E. In particular, the limit

$$\tilde{L}(E) := \lim_{n\to\pm\infty} \frac{1}{n} \ln \tilde{\phi}_E(n)$$

exists and is positive. Let $\varphi_E := \sqrt{\tilde{\phi}_E \phi_E}$. Then, by direct computation, we have

$$(\mathcal{L} - E)\varphi_E = -\frac{1}{2}\varphi_E q_E, \tag{3.12}$$

with

$$q_E = -\left(\frac{\phi_E^{\frac{1}{2}}(n+1)}{\phi_E^{\frac{1}{2}}(n)} - \frac{\tilde{\phi}_E^{\frac{1}{2}}(n+1)}{\tilde{\phi}_E^{\frac{1}{2}}(n)}\right)^2 - \left(\frac{\phi_E^{\frac{1}{2}}(n-1)}{\phi_E^{\frac{1}{2}}(n)} - \frac{\tilde{\phi}_E^{\frac{1}{2}}(n-1)}{\tilde{\phi}_E^{\frac{1}{2}}(n)}\right)^2.$$

We claim that q_E converges uniformly to 0 in $\mathbb{R}$ as $E \to \lambda_1$. Otherwise, there exist $\varepsilon > 0$ and two sequences $\{E_i\}_i$ and $\{n_i\}_i$ such that $E_i \to \lambda_1$ and $|q_{E_i}(n_i)| \geq \varepsilon$ for all $i \in \mathbb{N}$. According to Lemma 3.5, $\frac{\phi_E(\cdot+1)}{\phi_E(\cdot)}$ is almost periodic, hence $\{q_{E_i}\}$ is uniformly bounded. Passing along a subsequence n_{i_k} in (3.12), $\frac{\varphi_E(\cdot+n_{i_k})}{\varphi(n_{i_k})}$ converges pointwise to a positive solution φ^* of

$$(\mathcal{L}_{\theta_*} - \lambda_1)\varphi^* = -\frac{1}{2}\varphi^* q,$$

where $\theta_* = \lim_{k\to\infty} n_{i_k}\alpha$ and q is the limit of $\{q_{E_{i_k}}(n + n_{i_k})\}_k$. Since $|q(0)| \geq \varepsilon$, $\mathcal{L}_{\theta_*}$ admits a supersolution which is not a solution. From the assumption, Lemma 3.4 (2), (3), and Proposition 2.4, $\mathcal{L}_{\theta_*}$ admits a unique positive supersolution. However, as will see, φ and ϕ are both positive supersolutions. This is impossible! Therefore we have $q_E \to 0$ uniformly as $E \to \lambda_1$. Finally, we have

$$\tilde{L}(E) + L(E) = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1}\left(-\ln\frac{\phi_E(i+1)}{\phi_E(i)} + \ln\frac{\tilde{\phi}_E(i+1)}{\tilde{\phi}_E(i)}\right)$$

tends to 0 as $E \to \lambda_1$. Then they both tend to 0 as $E \to \lambda_1$ due to the positivity of $L(E)$ and $\tilde{L}(E)$. ■

4. Construction of the fronts

4.1. Construction of almost-periodic fronts

In this section we start to prove Theorem 1.1. First we will consider (1) in Theorem 1.1 in this subsection. The basic idea is to apply the super-sub solution method.

From now on, we set

$$\sigma_E(n,\theta) := -\ln\frac{\phi_E(n+1,\theta)}{\phi_E(n,\theta)} \quad \text{for any } n \in \mathbb{Z}, \theta \in \mathbb{T}^d,$$

where $\phi_E(n,\theta)$ is given by Proposition 3.4. Notice that $\sigma_E(n,\theta)$ is almost periodic by Lemma 3.5.

Before constructing the almost-periodic traveling front, we should first notice the following facts:

Lemma 4.1. *Let w^*, $\underline{w}$ be defined in Theorem 1.1. Then*

(1) *$w^* < \underline{w}$ provided $V(\cdot\alpha + \theta)$ in equation (1.3) is replaced by $V(\cdot\alpha + \theta) + c_0$ with c_0 large enough;*

(2) *w^* is a minimum provided $w^* < \underline{w}$. Moreover, for all $w \in (w^*, \underline{w})$, there exists $E > \lambda_1$ such that $w = \frac{E}{L(E)}$ and $w > \frac{E'}{L(E')}$ for $E' - E > 0$ small enough.*

Proof. For the proofs we refer readers to [43, Lemmas 1.6 and 3.2]. ∎

Now we take $w \in (w^*, \underline{w})$, and let $E > \lambda_1$ be as in Lemma 4.1. For a given almost-periodic sequence σ, we define the operator

$$\begin{aligned}\mathcal{L}_\theta^\sigma \phi &:= \mathrm{e}^{\sum_0^{n-1}\sigma}\mathcal{L}_\theta(\mathrm{e}^{-\sum_0^{n-1}\sigma}\phi)\\ &= \mathrm{e}^{-\sigma(n)}\phi(n+1) + \mathrm{e}^{\sigma(n-1)}\phi(n-1) + V(n\alpha+\theta)\phi(n).\end{aligned}$$

Proposition 4.1. *For any $\theta \in \mathbb{T}^d$, there exist absolute constants $\delta > 0$, $\varepsilon \in (0,1)$, and a function $\eta \in l^\infty(\mathbb{Z})$ satisfying*

$$\inf_{\mathbb{Z}} \eta > 0, \quad -\mathcal{L}_\theta^{(1+\varepsilon)\sigma_E(\cdot,\theta)}\eta \geq (\delta - (1+\varepsilon)E)\eta \ \text{ in } \mathbb{Z}.$$

Proof. As we chose E in Lemma 4.1, there exists $\varepsilon \in (0,1)$ such that

$$\frac{E}{L(E)} > \frac{(1+\varepsilon)E}{L((1+\varepsilon)E)}.$$

Now we define

$$F(\kappa) := \frac{1}{L(E)} - \frac{1+\varepsilon}{L(E+\kappa)}.$$

Then

$$F(E\varepsilon) = \frac{1}{L(E)} - \frac{1+\varepsilon}{L((1+\varepsilon)E)} > 0, \quad F(0) = -\frac{\varepsilon}{L(E)} < 0.$$

As $L(E)$ is concave in (λ_1, ∞), it is continuous. Hence F is continuous. Then there exists $\kappa \in (0, \varepsilon E)$ so that $F(\kappa) = 0$.

Now consider the function

$$\zeta(n,\theta) := \frac{\phi_{E+\kappa}(n,\theta)}{\phi_E^{1+\varepsilon}(n,\theta)} \quad \text{for all } n \in \mathbb{Z}.$$

First, since $e^{-(1+\varepsilon)\sum_0^{n-1}\sigma_E(\cdot,\theta)} = \phi_E^{1+\varepsilon}(n,\theta)$, the positive function ζ satisfies

$$\begin{aligned}\mathcal{L}_\theta^{(1+\varepsilon)\sigma_E(\cdot,\theta)}\zeta &= \mathrm{e}^{(1+\varepsilon)\sum_0^{n-1}\sigma_E(\cdot,\theta)}\mathcal{L}_\theta\big(\mathrm{e}^{-(1+\varepsilon)\sum_0^{n-1}\sigma_E(\cdot,\theta)}\zeta\big)\\ &= \frac{\mathcal{L}_\theta(\zeta\phi_E^{1+\varepsilon}(\cdot,\theta))}{\phi_E^{1+\varepsilon}(n,\theta)} = \frac{\mathcal{L}_\theta(\phi_{E+\kappa}(\cdot,\theta))}{\phi_E^{1+\varepsilon}(n,\theta)}\\ &= (E+\kappa)\zeta.\end{aligned}$$

Then it follows that

$$-(\mathcal{L}_\theta^{(1+\varepsilon)\sigma_E(\cdot,\theta)} - (1+\varepsilon)E)\zeta = (\varepsilon E - \kappa)\zeta \quad \text{in } \mathbb{Z}.$$

Moreover,

$$\begin{aligned}\lim_{n\to\pm\infty} \frac{1}{n}\ln\zeta(n,\theta) &= \lim_{n\to\pm\infty} \frac{1}{n}\ln\phi_{E+\kappa}(n,\theta) - \lim_{n\to\pm\infty} \frac{1+\varepsilon}{n}\ln\phi_E(n,\theta)\\ &= -L(E+\kappa) + (1+\varepsilon)L(E) = 0,\end{aligned}$$

where the last equality follows from the definition of κ.

Next, it follows from Lemma 3.5 that $\frac{\zeta(\cdot+1,\theta)}{\zeta(\cdot,\theta)} \in l^\infty(\mathbb{Z})$, and the coefficients of $\mathcal{M}_{a,b,c} = \mathcal{L}_\theta^{(1+\varepsilon)\sigma_E(\cdot,\theta)} - (1+\varepsilon)E$ are almost periodic. Applying Proposition 3.3 with $\mathcal{M}_{a,b,c}$, then $\lambda := \bar{\lambda}_1(\mathcal{M}_{a,b,c}) = \underline{\lambda}_1(\mathcal{M}_{a,b,c}) = \kappa - \varepsilon E$. Let $\delta' = \varepsilon E - \kappa > 0$. Thus for any $0 < \delta < \delta'$, there exists a function $\eta := \mathrm{e}^{u_{\varepsilon'}(\cdot,\theta)}$ with $|\varepsilon' u_{\varepsilon'} - \lambda| < \delta' - \delta$ which is defined in the proof of Proposition 3.3 and satisfies

$$\mathcal{L}_g^{(1+\varepsilon)\sigma_E(\cdot,\theta)}\eta - (1+\varepsilon)E\eta = \varepsilon' u_{\varepsilon'}\eta \le (\lambda + \delta' - \delta)\eta \le -\delta\eta \quad \text{in } \mathbb{Z}.$$

Moreover, $0 < \inf_{\mathbb{R}} \eta \le \sup_{\mathbb{R}} \eta < +\infty$ since $u_{\varepsilon'}(\cdot,\theta)$ is bounded. ∎

It is clear that the choice of η is not unique. Let us now define $\{\eta(\cdot,\theta)\}_{\theta\in\mathbb{T}^d}$ in the following certain way. Note that for any $0<\delta<\delta'$, the function $\eta = e^{u_{\varepsilon'}}$ with $|\varepsilon' u_{\varepsilon'} - \lambda| < \delta' - \delta$ in Proposition 4.1 is the unique almost-periodic solution (see [38, Lemma 5.1]) of

$$(\mathcal{L}_\theta^{(1+\varepsilon)\sigma_E(n,\theta)} - (1+\varepsilon)E)\eta = \varepsilon' u_{\varepsilon'}\eta \quad \text{in } \mathbb{Z}.$$

Indeed, almost-periodicity follows from the inequality

$$\begin{aligned}&\|u_{\varepsilon'}\cdot n_i - u_{\varepsilon'}\cdot n_j\|_{l^\infty}\\ &\quad\le \frac{C}{\varepsilon'}\max\big\{\|a\cdot n_i - a\cdot n_j\|_{l^\infty}, \|b\cdot n_i - b\cdot n_j\|_{l^\infty}, \|c\cdot n_i - c\cdot n_j\|_{l^\infty}\big\},\end{aligned}$$

for any sequence $\{n_i\}_i$, $\{n_j\}_j$, where $a = \mathrm{e}^{-(1+\varepsilon)\sigma_E(n,\theta)}$, $b = \mathrm{e}^{(1+\varepsilon)\sigma_E(n-1,\theta)}$, $f\cdot k = f(\cdot + k)$ for any almost-periodic function f. The details can be found in the proof of [38, Lemma 5.2]. Since $\varepsilon' u_{\varepsilon'}(n) \to \lambda$ uniformly in n which follows from [38, Lemma 5.2], one can choose a fixed $\varepsilon_0 > 0$ sufficiently small such that $|\varepsilon_0 u_{\varepsilon_0} - \lambda| < \delta' - \delta$, and it is independent of $\theta \in \mathbb{T}^d$. Now by the argument in Proposition 3.4, $\eta(n,\theta)$ is continuous in $\theta \in \mathbb{T}^d$. In this way, we always denote $\eta(n,\theta) = \mathrm{e}^{u_{\varepsilon_0}(n,\theta)}$.

Define for all $(t,n) \in \mathbb{R}\times\mathbb{Z}, \theta \in \mathbb{T}^d$,

$$\begin{aligned}\bar{u}(t,n,\theta) &:= \min\{1, \phi_E(n,\theta)\mathrm{e}^{Et}\},\\ \underline{u}(t,n,\theta) &:= \max\{0, \phi_E(n,\theta)\mathrm{e}^{Et} - A\eta(n,\theta)\phi_E^{1+\varepsilon}(n,\theta)\mathrm{e}^{(1+\varepsilon)Et}\},\end{aligned}$$

where ε and $\eta(\cdot,\theta)$ are given by Propositions 4.1 and A is a positive constant that is to be specified. Notice that ε is independent of θ.

Denote

$$\mathcal{S}_\theta = \big\{\tilde{u} \text{ is an entire solution of (1.3)} \mid \underline{u}(t,n,\theta) \le \tilde{u}(t,n) \le \bar{u}(t,n,\theta) \text{ in } \mathbb{R}\times\mathbb{Z}\big\}.$$

Proposition 4.2. *There exists a solution u of (1.3) satisfying $u \in \mathcal{S}_\theta$. Moreover, $u = u(t,n,\theta)$ is increasing in t.*

Proof. By the calculation, $\phi_E(n,\theta)\mathrm{e}^{Et}$ is a supersolution on $\mathbb{R}\times\mathbb{Z}$ of (1.2). Then $\bar{u}$ is also a supersolution of equation (1.2). Take $(t,n)\in\mathbb{R}\times\mathbb{Z}$ so that $\underline{u}(t,n,\theta)>0$ and set $\zeta := \phi_E(n,\theta)\mathrm{e}^{Et}$. Then we have

$$\begin{aligned}
&\underline{u}_t(t,n,\theta)-\underline{u}(t,n+1,\theta)-\underline{u}(t,n-1,\theta)+2\underline{u}(t,n,\theta)-V(n\alpha+\theta)\underline{u}(t,n,\theta)\\
&\quad=-(1+\varepsilon)AE\eta(\cdot,\theta)\phi_E^{1+\varepsilon}(\cdot,\theta)\mathrm{e}^{(1+\varepsilon)Et}+Ae^{(1+\varepsilon)Et}\phi_E^{1+\varepsilon}(\cdot,\theta)\mathcal{L}_\theta^{(1+\varepsilon)\sigma_E(\cdot,\theta)}\eta(\cdot,\theta)\\
&\quad=A\zeta^{1+\varepsilon}[\mathcal{L}_\theta^{(1+\varepsilon)\sigma_E(\cdot,\theta)}\eta(\cdot,\theta)-(1+\varepsilon)E\eta(\cdot,\theta)]\\
&\quad\le -A\delta\eta\zeta^{1+\varepsilon}.
\end{aligned}$$

Therefore, as 0 obviously solves (1.2), for $\underline{u}$ to be a subsolution it is sufficient to choose A so large that, for all (t,n) such that $\underline{u}(t,n,\theta)>0$, one has

$$A\delta\eta(\cdot,\theta)\zeta^{1+\varepsilon}\ge V(n\alpha+\theta)\zeta^2.$$

Observe that $\underline{u}(t,n,\theta)>0$ if and only if $A\eta(\cdot,\theta)\zeta^{\varepsilon}(t,n,\theta)<1$, that is,

$$\zeta^{\varepsilon-1}(t,n)>(A\eta(n,\theta))^{\frac{1}{\varepsilon}-1}.$$

Therefore, we can choose A so that $A\ge\frac{\sup_{\mathbb{R}}V^\varepsilon}{\delta^\varepsilon\inf_{\mathbb{R}}\eta(\cdot,\theta)}$.

The above argument also shows that $\underline{u}<(A\eta(\cdot,\theta))^{-\frac{1}{\varepsilon}}$; A can be chosen such that $\underline{u}<1$, whence $\underline{u}\le\bar{u}$.

Define the sequence $\{u_i\}$ as follows: u_i is the solution of (1.3) for $t>-i$ with initial condition $u_i(-i,n,\theta)=\bar{u}(-i,n,\theta)$. By the comparison principle Proposition 2.3, u_i satisfies

$$\forall t>-i,\ n\in\mathbb{Z},\quad \underline{u}(t,n,\theta)\le u_i(t,n,\theta)\le\bar{u}(t,n,\theta).$$

Thus, for $i,j\in\mathbb{N}$ with $j<i$ and for any $0<h<1$, using the monotonicity of $\bar{u}$, we will get

$$u_j(-j,n,\theta)=\bar{u}(-j,n,\theta)\ge\bar{u}(-j-h,n,\theta)\ge u_i(-j-h,n,\theta).$$

Note that $u_i(\cdot-h,\cdot,\theta)$ is also a solution of (1.3). The comparison principle Proposition 2.3 gives us

$$\forall j<i,\ 0<h<1, t>-j,\ n\in\mathbb{Z},\ \theta\in\mathbb{T}^d,\quad u_j(t,n,\theta)\ge u_i(t-h,n,\theta).$$

Now, by the arguments above, we can prove $\{u_i\}_i$ converges locally uniformly to a global function $\underline{u}\le u\le\bar{u}$ of (1.3). Then passing to the limit as $i,j\to\infty$, $u(t,n,\theta)\ge u(t-h,n,\theta)$ for all $(t,n)\in\mathbb{R}\times\mathbb{Z}$ and $0<h<1$, $\theta\in\mathbb{T}^d$. This means that u is nondecreasing in t. If the monotonicity were not strict, then the parabolic maximum principle Corollary 2.1 would imply that u is constant in time, which contradicts $\underline{u}\le u\le\bar{u}$. Then we finish the proof. ∎

One should notice the following fact:

Proposition 4.3. *For any $\tilde{u} \in \mathcal{S}_\theta$, either $\tilde{u}(t,n) < u(t,n,\theta)$, or $\tilde{u}(t,n) \equiv u(t,n,\theta)$ in $\mathbb{R} \times \mathbb{Z}$.*

Proof. Since $\tilde{u} \in \mathcal{S}_\theta$, then, as constructed in Proposition 4.2,

$$\forall j \in \mathbb{Z}_+, \quad \tilde{u}(-j,n) \leq \bar{u}(-j,n,\theta) = u_j(-j,n,\theta).$$

Therefore, the comparison principle Proposition 2.3 gives $\tilde{u}(t,n) \leq u_j(t,n,\theta)$ for $t > -j$. Taking $j \to \infty$, we have

$$\tilde{u}(t,n) \leq u(t,n,\theta) \quad \forall (t,n) \in \mathbb{R} \times \mathbb{Z}.$$

Let $\underline{u} = \tilde{u}(t,n)$, $\bar{u} = u(t,n,\theta)$. By Proposition 2.3, one has either $\tilde{u}(t,n) < u(t,n,\theta)$, or $\tilde{u}(t,n) \equiv u(t,n,\theta)$ in $\mathbb{R} \times \mathbb{Z}$. Thus the proof is complete. ∎

In fact, $u(t,n,\theta)$ is an almost-periodic traveling front of (1.3) as we will show afterwards. To prove it, we note some facts which will be represented below.

Lemma 4.2. *For any $\theta \in \mathbb{T}^d$, we have*

$$u\left(t + \sum_{i=0}^{k-1} \mathcal{T}(\theta + i\alpha), n+k, \theta\right) = u(t,n,\theta + k\alpha) \quad \forall t \in \mathbb{R},\ n,k \in \mathbb{Z},$$

where $\mathcal{T}(\theta) = -\frac{1}{E} \ln \frac{\phi_E(1,\theta)}{\phi_E(0,\theta)}$.

Proof. By the uniqueness of $\phi_E(n,\theta)$ and $\eta(n,\theta)$ (see Proposition 3.4 and the paragraph following Proposition 4.1), we can check that $\phi_E(n, k\alpha + \theta) = \frac{\phi_E(n+k,\theta)}{\phi_E(k,\theta)}$ and $\eta(n+k, \theta) = \eta(n, k\alpha + \theta)$. Combining this and the definitions of $\bar{u}(t,n,\theta)$ and $\underline{u}(t,n,\theta)$, we can verify that

$$\bar{u}\left(t + \sum_{i=0}^{k-1} \mathcal{T}(\theta + i\alpha), n+k, \theta\right) = \bar{u}(t,n,\theta + k\alpha),$$

$$\underline{u}\left(t + \sum_{i=0}^{k-1} \mathcal{T}(\theta + i\alpha), n+k, \theta\right) = \underline{u}(t,n,\theta + k\alpha).$$

Combining these two equations, we can check that

$$u\left(t + \sum_{i=0}^{k-1} \mathcal{T}(\theta + i\alpha), n+k, \theta\right) \in \mathcal{S}_{\theta + k\alpha},$$

$$u\left(t - \sum_{i=0}^{k-1} \mathcal{T}(\theta + i\alpha), n-k, \theta + k\alpha\right) \in \mathcal{S}_{\theta}.$$

Therefore, by Proposition 4.3,

$$u\Big(t+\sum_{i=0}^{k-1}\mathcal{T}(\theta+i\alpha),n+k,\theta\Big)\le u(t,n,\theta+k\alpha),$$

$$u\Big(t-\sum_{i=0}^{k-1}\mathcal{T}(\theta+i\alpha),n-k,\theta+k\alpha\Big)\le u(t,n,\theta),$$

which gives $u(t+\sum_{i=0}^{k-1}\mathcal{T}(\theta+i\alpha),n+k,\theta)=u(t,n,\theta+k\alpha)$ for any $(t,n)\in\mathbb{R}\times\mathbb{Z}$. ∎

To prove u satisfying Definition 1.1 (1), we will consider $u(t,0,\theta+n\alpha)$.

Lemma 4.3. *The function* $u(t,0,\theta+n\alpha)$ *satisfies*

$$\lim_{t\to-\infty}u(t,0,\theta+n\alpha)=0,\quad \lim_{t\to+\infty}u(t,0,\theta+n\alpha)=1,\quad \textit{uniformly in } n\in\mathbb{Z}. \tag{4.1}$$

Proof. For any $(t,n)\in\mathbb{R}\times\mathbb{Z}$, we have

$$u(t,0,\theta+n\alpha)\le\bar{u}(t,0,\theta+n\alpha)\le\phi_E(0,\theta+n\alpha)e^{Et}=e^{Et}.$$

Meanwhile,

$$\begin{aligned}u(t,0,\theta+n\alpha)&\ge\underline{u}(t,0,\theta+n\alpha)\\&\ge e^{Et}\phi_E(0,\theta+n\alpha)-A\eta(0,\theta+n\alpha)\phi_E^{1+\varepsilon}(0,\theta+n\alpha)e^{(1+\varepsilon)Et}\\&\ge e^{Et}-A\Big(\sup_n\eta(n,\theta)\Big)e^{(1+\varepsilon)Et},\end{aligned}$$

where the last inequality follows from $\eta(n+k,\theta)=\eta(n,\theta+n\alpha)$. Therefore,

$$\forall(t,n)\in\mathbb{R}\times\mathbb{Z},\quad e^{Et}(1-Me^{\varepsilon Et})\le u(t,0,\theta+n\alpha)\le e^{Et}, \tag{4.2}$$

for some positive constant M. From the second inequality, we deduce that

$$\lim_{t\to-\infty}u(t,0,\theta+n\alpha)=0\quad\text{uniformly in } n\in\mathbb{Z}.$$

From the first inequality of (4.2), we can see that $\inf_{n\in\mathbb{Z}}u(t,0,\theta+n\alpha)>0$ for $t<0$ small enough. Therefore, it follows from the monotonicity of u in t that

$$\forall t\in\mathbb{R},\qquad \inf_{[t,+\infty)\times\mathbb{Z}}u(t,0,\theta+n\alpha)>0, \tag{4.3}$$

and the quantity $\vartheta:=\lim_{t\to+\infty}\inf_{n\in\mathbb{Z}}u(t,0,\theta+n\alpha)>0$ is well defined. To conclude the proof we only need to prove $\vartheta=1$. Let $\{n_k\}_{k\in\mathbb{Z}_+}\subset\mathbb{Z}$ be such that $u(k,0,\theta+n_k\alpha)\to\vartheta$ as $k\to\infty$. Consider the family of functions $\{p^k\}_{k\in\mathbb{Z}_+}$,

$$p^k(t,n):=u(t+k,n,\theta+n_k\alpha)=u\Big(t+k-\sum_{i=0}^{n-1}\mathcal{T}(\theta+(n_k+i)\alpha),0,\theta+(n+n_k)\alpha\Big),$$

where the last equality follows from Lemma 4.2. Then we have $p^k(0,0) = u(k, 0, \theta + n_k\alpha) \to \vartheta$ as $k \to \infty$, and, for any $(t,n) \in \mathbb{R} \times \mathbb{Z}$,

$$\liminf_{k\to\infty} p^k(t,n) = \liminf_{k\to\infty} u\left(t + k - \sum_{i=0}^{n-1} \mathcal{T}(\theta + (n_k + i)\alpha), 0; \theta + (n + n_k)\alpha\right) \geq \vartheta$$

by (4.3). Moreover, the sequence $\{p^k\}_k$ converges, up to sequences, to a function p satisfying

$$\begin{aligned} p_t(t,n) - p(t,n+1) - p(t,n-1) + 2p(t,n) \\ = V(\theta^* + n\alpha)p(t,n)(1 - p(t,n)) \quad \text{in } \mathbb{R} \times \mathbb{Z}, \end{aligned}$$

where $\theta^* \in \mathbb{T}^d$. Note that p reaches its minimum ϑ at $(0,0)$. Then we have $V(\theta^*)\vartheta(1 - \vartheta) \leq 0$. Therefore, $\vartheta = 1$ since $V(\theta^*) \geq \inf V > 0$ and $0 < \vartheta \leq 1$. ∎

Now we need uniform convergence in $\mathbb{R} \times \mathbb{Z}$ to explain why $u(t,n,\theta)$ is an almost-periodic traveling front. Before that, the following lemma is needed.

Lemma 4.4. *Let two bounded uniformly continuous functions u^1 and u^2 be a subsolution and a supersolution of (1.3), respectively, i.e.,*

$$\begin{aligned} u^1_t(t,n) - u^1(t,n+1) - u^1(t,n-1) + 2u^1(t,n) \\ \leq V(n\alpha + \theta)u^1(t,n)(1 - u^1(t,n)), \quad t \in \mathbb{R}, \\ u^2_t(t,n) - u^2(t,n+1) - u^2(t,n-1) + 2u^2(t,n) \\ \geq V(n\alpha + \theta)u^2(t,n)(1 - u^2(t,n)), \quad t \in \mathbb{R}, \end{aligned}$$

and satisfy $0 \leq u^1 \leq u^2$ in $\mathbb{Z} \times \mathbb{R}$.

If $\inf_{n\in\mathbb{Z}}(u^2 - u^1)(t_0 + \sum_{i=0}^{n-1} \mathcal{T}(\theta + i\alpha), n) = 0$ for some $t_0 \in \mathbb{R}$, then

$$\forall t < t_0, \quad \inf_{n\in\mathbb{Z}} (u^2 - u^1)\left(t + \sum_{i=0}^{n-1} \mathcal{T}(\theta + i\alpha), n\right) = 0.$$

Proof. Let $\{n_k\}_k \subset \mathbb{Z}$ be such that

$$u^2\left(t_0 + \sum_{i=0}^{n_k-1} \mathcal{T}(\theta + i\alpha), n_k\right) - u^1\left(t_0 + \sum_{i=0}^{n_k-1} \mathcal{T}(\theta + i\alpha), n_k\right) \to 0 \quad \text{as } k \to \infty.$$

Consider

$$u^i_k(t,n) := u^i\left(t + \sum_{i=0}^{n_k-1} \mathcal{T}(\theta + i\alpha), n + n_k\right), \quad i = 1, 2.$$

Then the nonnegative function $w_k := u^2_k - u^1_k$ satisfies $\lim_{k\to\infty} w_k(t_0, 0) = 0$ and

$$\frac{d}{dt} w_k(t,n) - \mathcal{D} w_k(t,n) \geq V(\theta + (n + n_k)\alpha)(1 - u^2_k(t,n) - u^1_k(t,n))w_k(t,n),$$

where $\mathcal{D}w_k(t,n) = w_k(t,n+1) + w_k(t,n-1) - 2w_k(t,n)$. Now from Proposition 2.2 we have $w_k(t,n) \leq C(t_0 - t, \|V\|_{L^\infty})w_k(t_0,n)$, where $C(t_0 - t, \|V\|_{L^\infty})$ is a constant which is independent of k. This yields that

$$0 \leq \lim_{k\to\infty} w_k(t,0) \leq \lim_{k\to\infty} C(t_0 - t, \|V\|_{L^\infty})w_k(t_0,0) = 0.$$

Therefore,

$$\begin{aligned}0 \leq \inf_{n\in\mathbb{Z}}(u^2 - u^1)\Big(t + \sum_{i=0}^{n-1}\mathcal{T}(\theta + i\alpha), n\Big) &\leq \inf_{k\in\mathbb{Z}}(u^2 - u^1)\Big(t + \sum_{i=0}^{n-1}\mathcal{T}(\theta + i\alpha), n_k\Big)\\ &= \inf_{k\in\mathbb{Z}} w_k(t,0) \leq \lim_{k\to\infty} w_k(t,0) = 0.\end{aligned}$$

Thus the proof is complete. ∎

Using Lemma 4.4, we have the following result about uniform convergence.

Theorem 4.1. *Assume that $\theta^* - \theta = \lim_{k\to\infty} n_k\alpha$ for some sequence $\{n_k\}_{k\in\mathbb{Z}_+}$. Then*

$$u(t + \sum_{i=0}^{n-1}\mathcal{T}(\theta + (n_k + i)\alpha), n, \theta + n_k\alpha) \to u\Big(t + \sum_{i=0}^{n-1}\mathcal{T}(\theta^* + i\alpha), n, \theta^*\Big)$$

uniformly in $\mathbb{R}\times\mathbb{Z}$.

Proof. Denote $\mathcal{T}_k(\theta) = \sum_{i=0}^{k-1}\mathcal{T}(\theta + i\alpha)$. First, considering a sequence $\{n_k\} \subset \mathbb{Z}$ where $n_k\alpha$ converges in $\mathbb{T}^d$, we prove that $u(t + \mathcal{T}_n(\theta + n_k\alpha), n, \theta + n_k\alpha)$ converges uniformly in $\mathbb{R}\times\mathbb{Z}$. Assume by contradiction that it is false. Then there exist $\{t_k\}_k \subset \mathbb{R}$, $\{m_k\}_k \subset \mathbb{Z}$, and two subsequences $\{n_k^1\}_k$, $\{n_k^2\}_k$ of $\{n_k\}_k$ such that

$$\liminf_{k\to\infty}\big(u(t_k + \mathcal{T}_{m_k}(\theta + n_k^1\alpha), m_k, \theta + n_k^1\alpha) - u(t_k + \mathcal{T}_{m_k}(\theta + n_k^2\alpha), m_k, \theta + n_k^2\alpha)\big) > 0,$$

i.e., $\liminf_{k\to\infty}(u(t_k, 0, \theta + (m_k + n_k^1)\alpha) - u(t_k, 0, \theta + (m_k + n_k^2)\alpha)) > 0$. By Lemma 4.3, $\{t_k\}$ is bounded. Let ζ be a limit point of $\{t_k\}_k$. Then by $0 \leq u_t \leq 4 + \|V\|_{L^\infty}$, which follows from (1.3), u is uniformly continuous in t. Thus,

$$\liminf_{k\to\infty}\big(u(\zeta, 0, \theta + (m_k + n_k^1)\alpha) - u(\zeta, 0, \theta + (m_k + n_k^2)\alpha)\big) > 0.$$

Consider for $j = 1, 2$,

$$\begin{aligned}p_k^j(t,n) &:= u(t, n, \theta + (m_k + n_k^j)\alpha)\\ &= u\big(t - \mathcal{T}_n(\theta + (m_k + n_k^j)\alpha), 0, \theta + (n + m_k + n_k^j)\alpha\big),\end{aligned}$$

where the second equality follows from Lemma 4.2. Then, up to subsequences, $p_k^j(t,n)$ and $\theta + (m_k + n_k^j)\alpha$ $(j = 1, 2)$ converge locally uniformly to p^j and θ^{**}, respectively, as $k \to \infty$. Moreover, $p^1(\zeta, 0) - p^2(\zeta, 0) > 0$, and

$$\begin{aligned}p_t^j(t,n) = {}& p^j(t,n+1) + p^j(t,n-1) - 2p^j(t,n)\\ &+ V(\theta^{**} + n\alpha)p^j(t,n)(1 - p^j(t,n)), \quad j = 1, 2.\end{aligned}$$

Recall that $T(\theta) = -\frac{1}{E}\ln\frac{\phi_E(1,\theta)}{\phi_E(0,\theta)}$. We have for $i = 1, 2$,

$$\lim_{k\to\infty} u(t,0,\theta+(n+m_k+n_k^j)\alpha) = \lim_{k\to\infty} p_k^j\big(t+\mathcal{T}_n(\theta+(m_k+n_k^j)\alpha),n\big) = p^j(t+\mathcal{T}_n(\theta^{**}),n).$$

Combining this with (4.1) and (4.2), we have $p^1(t+\mathcal{T}_n(\theta^{**}),n)/p^2(t+\mathcal{T}_n(\theta^{**}),n) \to 1$ as $t\to\pm\infty$ uniformly in $n\in\mathbb{Z}$. Note that by (4.3),

$$\kappa^* := \sup_{\mathbb{R}\times\mathbb{Z}} \frac{p^1(t+\mathcal{T}_n(\theta^{**}),n)}{p^2(t+\mathcal{T}_n(\theta^{**}),n)}$$

is finite and $\kappa^* > 1$ since $p^1(\zeta,0) - p^2(\zeta,0) > 0$. Moreover, by the uniform continuity of p^1 and p^2, we can find some finite $\bar{t}$ such that

$$\sup_{n\in\mathbb{Z}} \frac{p^1(\bar{t}+\mathcal{T}_n(\theta^{**}),n)}{p^2(\bar{t}+\mathcal{T}_n(\theta^{**}),n)} = \sup_{\mathbb{R}\times\mathbb{Z}} \frac{p^1(t+\mathcal{T}_n(\theta^{**}),n)}{p^2(t+\mathcal{T}_n(\theta^{**}),n)} = \kappa^*.$$

By direct computation, we can show that $\kappa^* p^2$ is a supersolution of (1.3) with θ replaced by θ^{**} since $\kappa^* > 1$. Now we can apply Lemma 4.4 to deduce that

$$\forall t < \bar{t}, \quad \inf_{n\in\mathbb{Z}}\big(\kappa^* p^2(t+\mathcal{T}_n(\theta^{**}),n) - p^1(t+\mathcal{T}_n(\theta^{**}),n)\big) = 0,$$

which contradicts (4.2) when t is sufficiently large. Hence $u(t+\mathcal{T}_n(\theta+n_k\alpha),n,\theta+n_k\alpha)$ converges uniformly in $\mathbb{R}\times\mathbb{Z}$.

Let us now show that $\lim_{k\to\infty} u(t+\mathcal{T}_n(\theta+n_k\alpha),n,\theta+n_k\alpha) = u(t+\mathcal{T}_n(\theta^*),n,\theta^*)$. Denote $v(t+\mathcal{T}_n(\theta^*),n) := \lim_{k\to\infty} u(t+\mathcal{T}_n(\theta+n_k\alpha),n,\theta+n_k\alpha)$. Then we have

$$\lim_{k\to\infty} u(t,n,\theta+n_k\alpha) = \lim_{k\to\infty} u(t-\mathcal{T}_n(\theta+n_k\alpha)+\mathcal{T}_n(\theta+n_k\alpha),n,\theta+n_k\alpha) = v(t,n).$$

Hence $v\in\mathcal{S}_{\theta^*}$, and thus $v(t,n) \le u(t,n,\theta^*)$ by Proposition 4.3. We claim that $v(t,n) \equiv u(t,n,\theta^*)$. Otherwise, $v(t,n) < u(t,n,\theta^*)$. Note that by Lemma 4.2,

$$v(t+\mathcal{T}_n(\theta^*),n) = \lim_{k\to\infty} u(t+\mathcal{T}_n(\theta+n_k\alpha),n,\theta+n_k\alpha) = \lim_{k\to\infty} u(t,0,\theta+(n+n_k)\alpha).$$

Then, for similar reasons to before, we deduce that

$$\kappa' := \sup_{\mathbb{R}\times\mathbb{Z}} \frac{u(t+\mathcal{T}_n(\theta^*),n,\theta^*)}{v(t+\mathcal{T}_n(\theta^*),n)}$$

is finite, $\kappa' > 1$ since $v(t,n) < u(t,n,\theta^*)$, and $\kappa' v$ is a supersolution of (1.3) with θ replaced by θ^*. Applying Lemma 4.4 as before, we obtain a contradiction. Hence $v(t,n) \equiv u(t,n;\theta^*)$, that is to say,

$$u(t+\mathcal{T}_n(\theta+n_k\alpha),n,\theta+n_k\alpha) \to u(t+\mathcal{T}_n(\theta^*),n,\theta^*) \quad \text{as } k\to\infty$$

uniformly in $(t,n)\in\mathbb{R}\times\mathbb{Z}$. ■

We are now in a position to prove the existence of an almost-periodic traveling front.

Theorem 4.2. *The solution $u(t,n,\theta)$ constructed in Proposition 4.2 is an almost-periodic traveling front with average wave speed $w \in (w^*, \underline{w})$.*

Proof. We prove that $u(t,n,\theta)$ satisfies Definition 1.1 (1) first. Denote

$$\mathcal{T}_n(\theta) = \sum_{i=0}^{n-1} \mathcal{T}(\theta + i\alpha).$$

Note that $u(t,n,\theta) = u(t - \mathcal{T}_n(\theta), 0, \theta + n\alpha)$ by Lemma 4.2. From the choice of $\mathcal{T}(\theta)$, we have $\mathcal{T}_n(\theta) \to \pm\infty$ as $n \to \pm\infty$. Then by Lemma 4.3, we deduce that $u(t,n,\theta)$ satisfies Definition 1.1 (1).

By Theorem 4.1, for $\theta + n_k\alpha \to \theta^*$, we have

$$\begin{aligned} u(t,n;\theta + n_k\alpha) &= u(t - \mathcal{T}_n(\theta + n_k\alpha) + \mathcal{T}_n(\theta + n_k\alpha), n, \theta + n_k\alpha) \\ &\to u(t - \mathcal{T}_n(\theta^*) + \mathcal{T}_n(\theta^*), n, \theta^*) \\ &= u(t,n,\theta^*) \end{aligned}$$

locally uniformly in $(t,n) \in \mathbb{R} \times \mathbb{Z}$. Define a function U on $\mathbb{R} \times \mathbb{T}^d$ as

$$U(t,\theta) := u(t,0,\theta).$$

By Theorem 4.1, one can prove that U is continuous on $\mathbb{R} \times \mathbb{T}^d$. Recall that $\mathcal{T}(\theta) = -\frac{1}{E}\ln\frac{\phi_E(1,\theta)}{\phi_E(0,\theta)}$. It follows from Lemma 4.2, i.e., $u(t,n,\theta) = u(t - \mathcal{T}_n(\theta), 0, n\alpha + \theta)$, that

$$u(t,n,\theta) = u(t - \mathcal{T}_n(\theta), 0, \theta + n\alpha) = U(t - \mathcal{T}_n(\theta), n\alpha + \theta).$$

This proves (2).

By Proposition 3.4, one has

$$w(\theta) = \lim_{|n|\to\infty} \frac{n}{\sum_{i=k}^{n+k-1} \mathcal{T}(\theta + i\alpha)} = -\lim_{|n|\to\infty} \frac{nE}{\ln\phi_E(n, \theta + k\alpha)} = \frac{E}{L(E)} \in (w^*, \underline{w}).$$

Hence we finish the proof. ■

Now we complete the proof of Theorem 1.1 (1).

4.2. Nonexistence of fronts with speed less than w^*

Next we turn to prove Theorem 1.1 (3), i.e., there is no generalized transition front with average speed $w < w^*$. Compared to the previous section, we only consider the generalized transition front of (1.2). However, similar arguments can be applied to (1.3) with minor modification. From now on, for the sake of simplicity, we denote $\mathcal{D}\phi(t,n) = \phi(t,n+1) + \phi(t,n-1) - 2\phi(t,n)$, $t,n \in \mathbb{R} \times \mathbb{Z}$ and $u(t,n;s,u_0)$, $t \geq s$, $n \in \mathbb{Z}$ a solution of (1.2) with initial value u_0 starting at time s.

Now we begin with a lemma which will provide a lower bound for the average speed of a generalized transition front.

Lemma 4.5. *Let $u(t,n;0,u^{(0)})$ be a solution of* (1.2)*, and $u(0,n;0,u^{(0)})=u^{(0)}$, where*

$$u^{(0)}(n)=0 \ \text{ if } n>0 \quad \text{and} \quad \inf_{n\le 0} u^{(0)}(n)=\alpha\in(0,1].$$

Then $\lim_{t\to\infty}\inf_{n\le wt} u(t,n;0,u^{(0)})=1$ for all $0\le w<w^*$.

To prove this lemma, we need the following proposition:

Proposition 4.4. *Let $\theta\in\mathbb{T}^d$ and $u^\theta(t,n;0,u_0)$ be the solution of*

$$\begin{cases} u_t(t,n)-\mathcal{D}u(t,n)=V(n\alpha+\theta)u(t,n)(1-u(t,n)) & \text{in } \mathbb{R}\times\mathbb{Z},\\ u(0,n)=u_0(n), \ \text{with } u_0(0)=\beta\in(0,1] \text{ and } u_0(n)=0, & \text{if } n\neq 0. \end{cases} \tag{4.4}$$

Then for any $0\le w<w^*$, we have $\lim_{t\to\infty}\inf_{0\le n\le wt} u^\theta(t,n;0,u_0)=1$ exists uniformly in θ.

Proof. We split the proof into three steps.

Step 1. Show that $\lim_{t\to\infty}\inf_{0\le n\le wt} u^g(t,n;0,u_0)=1$ for any $0\le w<w^*$.

Denote $\mathcal{L}_{\theta,p}\phi:=e^{p\cdot}\mathcal{L}_\theta(e^{-p\cdot}\phi)$, where $\mathcal{L}_\theta$ is the linearized operator given by

$$(\mathcal{L}_\theta\phi)(n)=\phi(n+1)+\phi(n-1)-2\phi(n)+V(n\alpha+\theta)\phi(n).$$

Then as proved in [38, Theorem 2.1], we have

$$\lim_{t\to+\infty}\inf_{0\le n\le wt} u^\theta(t,n;0,u_0)=1 \quad \text{for any } 0\le w<\inf_{p>0}\frac{\underline{\lambda}_1(\mathcal{L}_{\theta,p})}{p},$$

where $\underline{\lambda}_1(\mathcal{L}_{\theta,p})$ is defined as (3.4) with $\mathcal{M}_{a,b,c}$ replaced by $\mathcal{L}_{\theta,p}$. Proposition 3.3 yields that $\underline{\lambda}_1(\mathcal{L}_{\theta,p})=\lambda_1(\mathcal{L}_{0,\theta})$ for any $\theta\in\mathbb{T}^d$, where $\lambda_1(\mathcal{L}_{\theta,p})$ is defined as (3.5), and then we denote it by $k(p)$. We still need to show that $\inf_{p>0}\frac{k(p)}{p}=w^*=\inf_{E>\lambda_1}\frac{E}{L(E)}$ with $L(E)$ given by (1.6).

In fact, by similar arguments to the proof of [38, Theorem 6.1], we can deduce that the map $L\colon(\lambda_1,+\infty)\to(\underline{L},+\infty)$, where $\underline{L}=\lim_{E\searrow\lambda_1}L(E)\ge 0$, admits an inverse, which is exactly $k\colon(\underline{L},+\infty)\to(\lambda_1,+\infty)$. If $\underline{L}>0$, then $k(p)\equiv k(0)=\lambda_1$ for any $p\in[0,\underline{L}]$. Hence

$$w^*=\inf_{E>\lambda_1}\frac{E}{L(E)}=\inf_{p>\underline{L}}\frac{k(p)}{p}=\inf_{p>0}\frac{k(p)}{p}.$$

Step 2. Construct an appropriate subsolution to obtain uniform convergence.

Suppose by contradiction that there exist $\delta\in(0,1)$ and a sequence $\{\theta_l,t_l,n_l\}\subset\mathbb{T}^d\times\mathbb{R}_+\times\mathbb{Z}$ with $0\le n_l\le wt_l$ and $t_l\to+\infty$ such that

$$u^{\theta_l}(t_l,n_l;0,u_0)\le 1-\delta.$$

Without loss of generality, we assume that $\theta_l\to\theta^*$ in $\mathbb{T}^d$. Let $u^{\theta_s}(t,n;0,u_0)$ be the solution of

$$\begin{cases} u_t(t,n)-\mathcal{D}u(t,n)=V(\theta^s+n\alpha)u(t,n)(1-u(t,n)) & \text{in } \mathbb{R}\times\mathbb{Z},\\ u(0,\cdot)=u_0. \end{cases} \tag{4.5}$$

Denote $g^s(n) = (1-s)V(n\alpha + \theta^*) + s \inf V/2$, and $u^{g_s}(t, n; 0, u_0)$ the solution of (4.5) with $V(\theta^* + n\alpha)$ replaced by g^s. Then we choose N large enough such that for $l \geq N$ and $s \in (0, 1)$, $|\theta_l - \theta^*| \leq s \inf_{n\in\mathbb{Z}} V/2$. Hence for $l \geq N$,

$$\theta_l(n) \geq \theta^*(n) - |\theta_l - \theta^*| \geq \theta^*(n) - s \inf_{n\in\mathbb{Z}} V/2 \geq g^s(n).$$

From this, for $l \geq N$, we have

$$\begin{aligned} u_t^{\theta^s}(t, n; 0, u_0) - \mathcal{D}u^{\theta^s}(t, n; 0, u_0) &= g^s(n)u^{g^s}(t, n; 0, u_0)(1 - u^{g^s}(t, n; 0, u_0)) \\ &\leq V(\theta_l + n\alpha)u^{g^s}(t, n; 0, u_0)(1 - u^{g^s}(t, n; 0, u_0)). \end{aligned}$$

That is to say, $u^{g^s}(t, n; 0, u_0)$ is a subsolution of (4.4) with θ replaced by θ_l for any $l \geq N$.

Step 3: End of the proof. As proved in [38, Theorem 2.1], we have

$$\lim_{t\to+\infty} \inf_{0\leq n\leq wt} u^{g^s}(t, n; 0, u_0) = 1 \quad \text{for any } 0 \leq w < \inf_{p>0} \frac{\lambda_1(\mathcal{L}_{g^s,p})}{p}.$$

Proposition 3.1 in [38] also tells us that

$$\begin{aligned} \lim_{s\to 0} \inf_{p>0} \frac{\lambda_1(\mathcal{L}_{g^s,p})}{p} &= \inf_{p>0} \frac{\lambda_1(\mathcal{L}_{g,p})}{p} \\ &= \inf_{E>\lambda_1} \frac{E}{L(E)} = w^*, \end{aligned}$$

where $\mathcal{L}_{g^s,p}\phi := e^{p\cdot}\mathcal{L}_{g^s}(e^{-p\cdot}\phi)$ with

$$\mathcal{L}_{g^s}u := u(n+1) + u(n-1) - 2u(n) + g^s(n)u(n).$$

Hence, for any $0 \leq w < w^*$, we can take $s > 0$ small such that $0 \leq w < \inf_{p>0} \frac{\lambda_1(\mathcal{L}_{g^s,p})}{p}$. Moreover, we have $\lim_{t\to\infty} \inf_{0\leq n\leq wt} u^{g^s}(t, n; 0, u_0) = 1$. On the other hand, since $u^{g^s}(t, n; 0, u_0)$ is a subsolution, it follows from Proposition 2.3 that we have $u^{g^s}(t, n; 0, u_0) \leq u^{\theta_l}(t, n; 0, u_0)$ for l large enough. Therefore,

$$\begin{aligned} \lim_{t\to\infty} \inf_{0\leq n\leq wt} u^{g^s}(t, n; 0, u_0) &\leq \limsup_{l\to\infty} u^{g^s}(t_l, n_l; 0, u_0) \\ &\leq \limsup_{l\to\infty} u^{\theta_l}(t_l, n_l; 0, u_0) \leq 1 - \delta. \end{aligned}$$

That is impossible. Hence $\lim_{t\to\infty} \inf_{0\leq n\leq wt} u^\theta(t, n; 0, u_0) = 1$ exists uniformly in $\theta \in \mathbb{T}^d$. ∎

Proof of Lemma 4.5. Consider the solution $u_k(t, n)$ of

$$\begin{cases} u_t(t, n) - \mathcal{D}u(t, n) = c(n+k)u(t, n)(1 - u(t, n)), & (t, n) \in \mathbb{R} \times \mathbb{Z}, \\ u(0, 0) = \alpha \text{ and } u(0, n) = 0 & \text{if } n \neq 0. \end{cases}$$

Then for any $0 \le w < w^*$, we have $\lim_{t\to\infty} \inf_{0\le n\le wt} u_k(t,n) = 1$ exists uniformly in k by Proposition 4.4. Therefore, the solution $u(t,n;0,u_k^{(0)})$ of

$$\begin{cases} u_t(t,n) - \mathcal{D}u(t,n) = c(n)u(t,n)(1-u(t,n)), & (t,n) \in \mathbb{R}\times\mathbb{Z}, \\ u(0,n) = u_k^{(0)}(n), & n \in \mathbb{Z}, \end{cases}$$

where $u_k^{(0)}(-k) = \alpha$ and $u(0,n) = 0$ if $n \ne -k$, satisfies

$$\lim_{t\to\infty} \inf_{-k\le n\le wt-k} u(t,n;0,u_k^{(0)}) = 1 \quad \text{for any } w < w^* \text{ uniformly in } k \in \mathbb{N},$$

since $u(t,n;0,u_k^{(0)}) = u_k(t,n-k)$ by Theorem 2.1. Then it follows from Proposition 2.3 that

$$\lim_{t\to\infty} \inf_{n\le wt} u(t,n;0,u^{(0)}) = 1 \quad \forall 0 \le w < w^*. \qquad \blacksquare$$

Using Lemma 4.5, we finish the proof of Theorem 1.1 (3).

Proposition 4.5. *Let u be a generalized transition front of equation (1.2) and let N be such that (1.4) holds. Then*

$$\liminf_{t-s\to+\infty} \frac{N(t)-N(s)}{t-s} \ge w^*.$$

In particular, there exists no generalized transition front with average speed $w < w^*$.

Proof. First by (1.4) and Proposition 2.3, we can check that

$$\alpha := \inf_{t\in\mathbb{R},n\le 0} u(t,n+N(t)) > 0 \quad \text{and} \quad \beta := \sup_{t\in\mathbb{R},n>0} u(t,n+N(t)) < 1. \tag{4.6}$$

It is clear that $\alpha < 1$, $\beta > 0$. Assume that, by contradiction, there exist t_k and s_k such that

$$\lim_{k\to\infty} t_k - s_k = +\infty \quad \text{and} \quad \lim_{k\to\infty} \frac{N(t_k)-N(s_k)}{t_k-s_k} = w < w^*.$$

Set $v_k(t,n) := u(t+s_k, n+N(s_k))$. It is clear that $v_k(0,n) \ge u^{(0)}(n)$, where $u^{(0)}(n) = 0$ if $n > 0$, and $u^{(0)}(n) = \alpha$ if $n \le 0$. Thus, by Proposition 2.3, we have $v_k(t,n) \ge u(t,n;0,u^{(0)})$ for $t \ge 0$ and $n \in \mathbb{Z}$, which yields

$$\begin{aligned} u(t_k, N(t_k)+1) &= v_k(t_k-s_k, N(t_k)-N(s_k)+1) \\ &\ge u(t_k-s_k, N(t_k)-N(s_k)+1; 0, u^{(0)}). \end{aligned} \tag{4.7}$$

For the left-hand side of (4.7), we have $u(t_k, N(t_k)+1) \le \beta < 1$ by (4.6). But from Lemma 4.5, the right-hand side of (4.7) converges to 1 as $k \to \infty$ since

$$\lim_{k\to\infty} \frac{N(t_k)-N(s_k)+1}{t_k-s_k} = w < w^*,$$

which is a contradiction. $\blacksquare$

In all, we have proved in Theorem 1.1 (3).

4.3. Construction of the critical fronts

Finally, to verify Theorem 1.1 (2), we only need to consider (1.3) with $c(n) = V(n\alpha)$, i.e., (1.2). First we want to construct the critical front with average speed w^*. By critical front we mean the following:

Definition 4.1. We say that an entire solution u of (1.2) with $0 < u < 1$ is a critical traveling front (to the right) if, for all $(t_0, n_0) \in \mathbb{R} \times \mathbb{Z}$, v is an entire solution of (1.2) such that $v(t_0, n_0) = u(t_0, n_0)$ and $0 < v < 1$. Then

$$u(t_0, n) \geq v(t_0, n) \ \text{ if } n \leq n_0 \quad \text{and} \quad u(t_0, n) \leq v(t_0, n) \ \text{ if } n > n_0.$$

Before going any further, we introduce some useful lemmas.

Lemma 4.6. *Let $u(t, n)$ be an entire solution of (1.2) with $0 < u < 1$. There exists $\delta \in (0, 1)$, such that $|u(t, n) - u(t, n+1)| \leq 1 - \delta$ for all $t, n \in \mathbb{R} \times \mathbb{Z}$.*

Proof. Suppose that the conclusion fails. Then for any $k \in \mathbb{Z}_+$, there exist t_k and n_k such that $|u(t_k, n_k) - u(t_k, n_k + 1)| > 1 - 1/k$. After passing to a subsequence, we have $u(s_k, m_k) - u(s_k, m_k + 1) > 1 - \varepsilon_k$ or $u(s_k, m_k + 1) - u(s_k, m_k) > 1 - \varepsilon_k$ for some s_k, m_k, and ε_k with $\varepsilon_k \to 0$. We only prove the former case. Note that $0 < u(t, n) < 1$. Then

$$u(s_k, m_k) > 1 - \varepsilon_k \quad \text{and} \quad u(s_k, m_k + 1) < \varepsilon_k.$$

Note also that $|\frac{d}{dt} u| \leq 4 + \|V\|_{L^\infty(\mathbb{Z})}$. Then there exists S which is independent of k such that $u(s_k - S, m_k) \geq 3/4 - \varepsilon_k > 1/2$ for k large. On the other hand, from Proposition 2.2, there exists a constant $C(S, \|V\|_{L^\infty})$ which only depends on S, $\|V\|_{L^\infty}$ such that

$$\frac{1}{2} < u(s_k - S, m_k) \leq C(S, \|V\|_{L^\infty}) u(s_k, m_k + 1) \leq C(S, \|V\|_{L^\infty}) \varepsilon_k \to 0 \quad \text{as } k \to 0.$$

This yields a contradiction. ∎

With Lemma 4.6 at hand, we have the following equivalent definition of a generalized transition front.

Lemma 4.7. *Let $u(t, n)$ be a solution of (1.2) with $0 < u < 1$ and*

$$u(t, n) \to 0 \ \text{ as } n \to \infty, \quad u(t, n) \to 1 \ \text{ as } n \to -\infty \quad \text{for any } t \in \mathbb{R}.$$

Then u is a generalized transition front if and only if

$$\sup_{t \in \mathbb{R}} \operatorname{diam}\{n \in \mathbb{Z} \mid \varepsilon \leq u(t, n) \leq 1 - \varepsilon\} < \infty \quad \text{for any } \varepsilon \in (0, 1/2). \tag{4.8}$$

Proof. As we see, (4.8) is satisfied if u is a generalized transition front. Next we prove that u is a generalized transition front provided (4.8) holds. Set $N(t) := \sup\{n \mid u(t, n) \geq 1/4\}$.

Now we claim that $1/4 \leq \inf_{t \in \mathbb{R}} u(t, N(t)) \leq \sup_{t \in \mathbb{R}} u(t, N(t)) < 1$. In fact, if $\sup_{t \in \mathbb{R}} u(t, N(t)) = 1$, then there exists a sequence $\{t_k\} \subset \mathbb{Z}$ such that $u(t_k, N(t_k)) \to 1$.

After passing to a subsequence, $u(t + t_k, n + N(t_k))$ converges locally uniformly to a function $v(t, n)$. Then $v(0, 0) = 1$, and thus $v(t, n) \equiv 1$ by Corollary 2.1. On the other hand, the definition of $N(t)$ gives $v(0, 1) \le 1/4$, which is a contradiction.

For any $\varepsilon < \varepsilon_0 := \min\{1 - \sup_{t\in\mathbb{R}} u(t, N(t)), 1/4, \delta/2\}$, where δ was given in Lemma 4.6, we have

$$N(t) \in \big\{n \in \mathbb{Z} \mid \varepsilon_0 \le u(t,n) \le 1 - \varepsilon_0\big\} \subset \big\{n \in \mathbb{Z} \mid \varepsilon \le u(t,n) \le 1 - \varepsilon\big\}.$$

Denote $L_\varepsilon := \sup_t \operatorname{diam}\{n \in \mathbb{Z} \mid \varepsilon \le u(t,n) \le 1 - \varepsilon\}$. Therefore,

$$n + N(t) \notin \big\{n \in \mathbb{Z} \mid \varepsilon \le u(t,n) \le 1 - \varepsilon\big\} \quad \text{for all } t \in \mathbb{R}$$

if $n > L_\varepsilon + 1$ or $n < -L_\varepsilon - 1$. Note that $u(t, n) \to 0$ as $n \to \infty$ and $u(t, n) \to 1$ as $n \to -\infty$ for any $t \in \mathbb{R}$. Combining this with Lemma 4.6, we have $u(t, n + N(t)) \le \varepsilon$ for $n > L_\varepsilon + 1$, and $u(t, n + N(t)) \ge 1 - \varepsilon$ for $n < -L_\varepsilon - 1$. Therefore, (1.4) holds. ∎

Let us now construct a critical front. Fix any $\theta \in (0, 1)$. For any $k \in \mathbb{Z}_+$, we define

$$H_k(n) = \begin{cases} 1 & \text{if } n \le -k, \\ 0 & \text{if } n > -k. \end{cases}$$

Then by Lemma 4.5 and the continuity of $u(t, n; 0, H_k)$ with respect to t, we can define $s_k := \min\{s \mid u(s, 0; 0, H_k) = \theta\} > 0$. In particular, $u(s_k, 0; 0, H_k) = \theta$. Note that by Theorem 2.1, $u(t, n; s, H_k) = u(t - s, n; 0, H_k)$ for any $t \ge s$. Then $u(0, 0; -s_k, H_k) = \theta$ for any $k \in \mathbb{Z}_+$. The idea is to take the limit of some subsequence from $\{u(t, n; -s_k, H_k)\}$ and prove the resulting function is exactly the critical traveling front. Moreover, it is exactly a generalized transition front with average speed w^*. Before that, an observation about some important properties of s_k is given by the following lemma:

Lemma 4.8. *The sequence $\{s_k\}_{k\in\mathbb{N}_+}$ is strictly increasing and converges to $+\infty$.*

Proof. We first show that s_k is strictly increasing. Assume by contradiction that $s_k \ge s_{k+1}$ for some $k \in \mathbb{Z}_+$. Note that

$$u(t + s_k - s_{k+1}, n; -s_{k+1}, H_{k+1})|_{t=-s_k} = H_{k+1} \le H_k = u(t, n; -s_k, H_k)|_{t=-s_k}.$$

Then by Proposition 2.3 we have

$$u(t + s_k - s_{k+1}, n; -s_{k+1}, H_{k+1}) < u(t, n; -s_k, H_k) \quad \text{for } t > -s_k.$$

In particular,

$$\theta = u(0, 0; -s_{k+1}, H_{k+1}) < u(s_{k+1} - s_k, 0; -s_k, H_k).$$

Notice that $u(-s_k, 0; -s_k, H_k) = 0$. Then by the intermediate value theorem there exists $-s_k < \tau < s_{k+1} - s_k \le 0$ such that

$$\theta = u(\tau, 0; -s_k, H_k) = u(\tau + s_k, 0; 0, H_k).$$

Therefore the definition of s_k gives $s_k \le \tau + s_k$. That is impossible since $\tau < 0$. Hence $s_k < s_{k+1}$.

Next we prove that $\lim_{k\to\infty} s_k = +\infty$. Suppose by contradiction that after passing to a subsequence, $\lim_{k\to\infty} s_k = s_\infty < +\infty$. Let $\phi_{E,k}$ be a solution of

$$\phi(n+1) + \phi(n-1) - 2\phi(n) + c(n-k)\phi(n) = E\phi(n), \quad n \in \mathbb{Z},$$

with $\phi(0) = 1$, $\lim_{n\to+\infty} \phi(n) = 0$, where $E > \lambda_1$. Then $\bar{u}_k(t, n+k) := \min\{1, \phi_{E,k}(n+k)e^{Et}\}$ is a supersolution of

$$u_t(n) - u(n+1) - u(n-1) + 2u(n) = c(n)u(n)(1-u(n)), \quad (t,n) \in \mathbb{R}\times\mathbb{Z}.$$

Note that for any $t \le 0$,

$$\bar{u}_k(t,k) = \min\{1, \phi_{E,k}(k)e^{Et}\} \le \phi_{E,k}(k) \le Ce^{-\delta k},$$

for some constant C only depending on E, λ_1, and $\|g\|_{l^\infty}$, and the last inequality follows from Lemma 3.1. Then we can take K large such that $\bar{u}_K(t, K) \le \theta/2$ for any $t < 0$. Moreover, combining Proposition 3.4 with the above result, there exists K_1 such that for any $t \ge -s_\infty$ and $n \in \mathbb{Z}$,

$$H_{K_1}(n) \le \bar{u}_K(-s_\infty, n+K) \le \bar{u}_K(t, n+K).$$

Then $u(-s_{K_1}, n; -s_{K_1}, H_{K_1}) \le \bar{u}_K(-s_\infty, n+K) \le \bar{u}_K(-s_{K_1}, n+K)$. Thus for any $t \ge -s_{K_1}$ and $n \in \mathbb{Z}$, it follows from Proposition 2.3 that

$$u(t, n; -s_{K_1}, H_{K_1}) \le \bar{u}_K(t, n+K).$$

In particular, we have $\theta = u(0,0; -s_{K_1}, H_{K_1}) \le \bar{u}_K(0,K) \le \theta/2$, which is a contradiction. Then we complete the proof. ■

As $s_k \to \infty$, there exists a subsequence of $\{u(t,n; -s_k, H_k)\}$ such that it converges to some entire solution $u(t,n)$. Moreover, $u(t,n)$ is "steeper" than any other entire solution in the following sense (see [19] for the continuous case):

Lemma 4.9. *Let $u(t,n)$ be a limit of some subsequences of $u(t,n;-s_k,H_k)$. Assume that v is an entire solution of (1.2) with $v(t,n) \in (0,1)$ on $\mathbb{R}\times\mathbb{Z}$. Then for any $t \in \mathbb{R}$, there exists $n_t \in \mathbb{Z} \cup \{\pm\infty\}$ such that*

$$u(t,n) \ge v(t,n) \ \text{if}\ n \le n_t \quad \text{and} \quad u(t,n) \le v(t,n) \ \text{if}\ n > n_t.$$

First we need the following proposition whose proof can be found in [23, Lemma 4]:

Proposition 4.6 ([23]). *Consider the solution $u_1(t,n)$ and $u_2(t,n)$ of (1.2). Denote $w(t,n) = u_1(t,n) - u_2(t,n)$. If $w(t_0,n_0) > 0$ for some $(t_0,n_0) \in \mathbb{R}\times\mathbb{Z}$, $w(t_0,n) > 0$ for $n < n_0$, $w(t_0,n) < 0$ for $n > n_0$, then the following hold:*

(1) *For any* $t \geq t_0$*, if* $w(t,n) > 0$ *for some* $n \in \mathbb{Z}$*, then* $w(t,m) > 0$ *for any* $m < n$*.*

(2) *For any* $t \geq t_0$*, if* $w(t,n) < 0$ *for some* $n \in \mathbb{Z}$*, then* $w(t,m) < 0$ *for any* $m > n$*.*

Proof of Lemma 4.9. First we prove that if $u(t_0,n_0) < v(t_0,n_0)$ for some $(t_0,n_0) \in \mathbb{R}\times\mathbb{Z}$, then $u(t_0,m) \leq v(t_0,m)$ for any $m > n_0$.

Suppose by contradiction that $u(t_0,m) > v(t_0,m)$ for some $m > n_0$. Then we have

$$u(t_0,n_0;-s_k,H_k) < v(t_0,n_0), \quad \text{and} \quad u(t_0,m;-s_k,H_k) > v(t_0,m) \tag{4.9}$$

for some k large enough. It is clear that

$$\begin{cases} 1 = u(-s_k,n;-s_k,H_k) > v(-s_k,n), & n \leq -k, \\ 0 = u(-s_k,n;-s_k,H_k) < v(-s_k,n), & n > -k. \end{cases}$$

Applying Proposition 4.6 with $w(t,n) := u(t,n;-s_k,H_k) - v(t,n)$, we can conclude that for $t \geq -s_k$ and $n \in \mathbb{Z}$ where $w(t,n) < 0$, one has $w(t,m) < 0$ for any $m > n$. This contradicts (4.9).

Similarly, if $u(t_0,n_0) > v(t_0,n_0)$ for some $(t_0,n_0) \in \mathbb{R}\times\mathbb{Z}$, then $u(t_0,m) \geq v(t_0,m)$ for any $m < n_0$. Therefore the existence of n_t follows directly. ■

Now we begin to construct an entire solution by taking the limit of $\{u(t,n;-s_k,H_k)\}$:

Lemma 4.10. *The limit* $u(t,n) := \lim_{k\to\infty} u(t,n;-s_k,H_k)$ *exists locally uniformly in* $(t,n) \in \mathbb{R}\times\mathbb{Z}$*. Moreover, u is an entire solution of* (1.2) *with* $u(0,0) = \theta$*.*

Proof. Define $w(t,n) := u(t,n;-s_k,H_k) - u(t,n;-s_{k+1},H_{k+1})$ for $t \geq -s_k$ and $n \in \mathbb{Z}$. Then for $t > -s_k$, $n \in \mathbb{Z}$, w satisfies

$$w_t(t,n) - w(t,n+1) - w(t,n-1) + 2w(t,n) = c(n) f_k(t,n) w(t,n),$$

where $f_k(t,n) = 1 - u(t,n;-s_k,H_k) - u(t,n;-s_{k+1},H_{k+1})$. Now we prove this lemma in the following two steps.

Step 1. Show that $w(0,n) \geq 0$ for $n < 0$, and $w(0,n) \leq 0$ for $n > 0$.

Assume, by contradiction, that there is $n_1 < 0$ such that $w(0,n_1) < 0$ (the proof is similar in the case where there is $n_1 > 0$ such that $w(0,n_1) > 0$). Note that

$$w(-s_k,n) = \begin{cases} 1 - u(-s_k,n;-s_{k+1},H_{k+1}) > 0 & \forall n \leq -k, \\ -u(-s_k,n;-s_{k+1},H_{k+1}) < 0 & \forall n > -k. \end{cases}$$

We can deduce from Proposition 4.6 that if $w(t,n) < 0$ for some $t > -s_k$ and $n \in \mathbb{Z}$, then $w(t,m) < 0$ for any $m > n$ (similarly, if $w(t,n) > 0$ for some $t > -s_k$ and $n \in \mathbb{Z}$, then $w(t,m) > 0$ for $m < n$). Therefore, $w(0,m) < 0$ for $m > n_1$ contradicts $w(0,0) = 0$.

Step 2. Show that $u_1(t,n) = u_2(t,n)$, where $u_i(t,n)$ $(i = 1,2)$ are any two limits of different subsequences of $u(t,n;-s_k,H_k)$.

From Step 1, we have

$$u(0,n;-s_k,H_k) \geq u(0,n;-s_{k+1},H_{k+1}) \quad \text{for } n < 0,$$
$$u(0,n;-s_k,H_k) \leq u(0,n;-s_{k+1},H_{k+1}) \quad \text{for } n > 0.$$

That is to say, the sequence $\{u(0,n;-s_k,H_k)\}_k$ is nonincreasing if $n < 0$ and is nondecreasing if $n > 0$. Thus the limit $u_0(n) := \lim_{k\to\infty} u(0,n;-s_k,H_k)$ is well defined for $n \in \mathbb{Z}$. Hence $u_1(0,n) = u_2(0,n)$, which yields that $u_1(t,n) = u_2(t,n)$ for $t > 0$ and $n \in \mathbb{Z}$ by Proposition 2.3.

Now we prove $u_1(t,n) = u_2(t,n)$ for all $(t,n) \in \mathbb{R}\times\mathbb{Z}$. From Lemma 4.9, there exist $n_t^i \in \mathbb{Z}\cup\{\pm\infty\}$ $(i = 1,2)$ such that

$$u_1(t,n) \geq u_2(t,n) \ \text{ if } n \leq n_t^1 \quad \text{and} \quad u_1(t,n) \leq u_2(t,n) \ \text{ if } n > n_t^1,$$
$$u_2(t,n) \geq u_1(t,n) \ \text{ if } n \leq n_t^2 \quad \text{and} \quad u_2(t,n) \leq u_1(t,n) \ \text{ if } n > n_t^2.$$

It is clear that $u_1(t,n) = u_2(t,n)$ if $n_t^1 = n_t^2$ for any $t < 0$. Now we assume, without loss of generality, that $n_\tau^1 < n_\tau^2$ for some $\tau < 0$. It follows directly that

$$\begin{cases} u_1(\tau,n) = u_2(\tau,n) & \text{if } n \leq n_\tau^1 \text{ or } n > n_\tau^2, \\ u_1(\tau,n) \leq u_2(\tau,n) & \text{if } n_\tau^1 < n \leq n_\tau^2. \end{cases}$$

It is still required to prove $u_1(\tau,n) = u_2(\tau,n)$ for $n_\tau^1 < n \leq n_\tau^2$. If not, then $u_1(t,n) < u_2(t,n)$ for $t > \tau$ and $n \in \mathbb{Z}$ by Proposition 2.3. That is impossible since $u_1(t,n) = u_2(t,n)$ for any $t > 0$ and $n \in \mathbb{Z}$. Hence $u_1(t,n) = u_2(t,n)$ in $\mathbb{R}\times\mathbb{Z}$ with $u(0,0) = \theta$, which follows from $u(0,0;-s_k,H_k) = \theta$ for any $k \in \mathbb{Z}_+$. ∎

Now we prove Theorem 1.1 (2) by verifying that the above solution is exactly a time-increasing generalized transition front with average speed w^*.

Theorem 4.3. *The solution $u(t,n)$ in Lemma 4.10 is a critical traveling front and a time-increasing generalized transition front with average speed w^*.*

Proof. We prove that $u(t,n)$ is a critical traveling front first. Assume that v is an entire solution of (1.2) such that $v(t_0,n_0) = u(t_0,n_0)$ and $0 < v < 1$. Then by Lemma 4.9, we have

$$u(t_0,n) \geq v(t_0,n) \quad \text{if } n \leq n_0 \quad \text{and} \quad u(t_0,n) \leq v(t_0,n) \ \text{ if } n > n_0.$$

That is to say, $u(t,n)$ is a critical traveling front.

Let $u_w(t,n)$ be the almost-periodic traveling front $u(t,n;c)$ obtained in Theorem 4.2 with average wave speed $w > w^*$. Thus by Proposition 2.5, $u_w(t,n)$ is also a generalized transition front with average speed w. Then a similar argument to that of [42, Theorem 3.1] yields that $\sup_{t\in\mathbb{R}} \operatorname{diam}\{n \in \mathbb{Z} \mid \varepsilon \leq u(t,n) \leq 1-\varepsilon\} < \infty$ for any $\varepsilon \in (0,1/2)$. Therefore, it follows from Lemma 4.7 that $u(t,n)$ is a generalized transition front.

Now we show that $u(t,n)$ possesses an average speed w^*. Set $N(t) := \sup\{n \mid u(t,n) \geq 1/4\}$ and $M_w(t) := \sup\{n \mid u_w(t,n) \geq 1/4\}$. Then by similar arguments to those of [42, Theorem 3.6], we can find L such that, for any $s \in \mathbb{R}$, there exists $\tilde{s} \in \mathbb{R}$ satisfying

$$N(s+\tau) - N(s) \leq M_w(\tilde{s}+\tau) - M_w(\tilde{s}) + L \quad \forall \tau > 0.$$

Then

$$\lim_{\tau\to+\infty} \sup_{s} \frac{N(s+\tau)-N(s)}{\tau} \leq \lim_{\tau\to+\infty} \sup_{\tilde{s}} \frac{M_w(\tilde{s}+\tau)-M_w(\tilde{s})+L}{\tau} = w,$$

for all $w^* < w < \underline{w}$, which gives $\limsup_{t-s\to+\infty} \frac{N(t)-N(s)}{t-s} \leq w^*$. On the other hand, since $u(t,n)$ is a generalized transition front and $N(t)$ satisfies (1.4), we have that $\liminf_{t-s\to+\infty} \frac{N(t)-N(s)}{t-s} \geq w^*$ by Proposition 4.5. Hence w^* is the average speed of u.

Finally, we need to prove $u(t,n)$ is a time-increasing traveling front. Otherwise, there exist some t', τ, n' such that $u(t'+\tau, n') < u(t', n')$. Since $\alpha := \inf_{n<0} u(0,n) \in (0,1)$, then by a similar argument to the proof of Proposition 4.5, we can deduce that $u(t,n) \geq u(t,n;0,u^{(0)})$ with $u^{(0)}(n) = 0$ for $n > 0$ and $u^{(0)}(n) = \alpha$ for $n \leq 0$. Thus, we have $\limsup_{t\to\infty} u(t,n) = 1$. Combining this with the intermediate value theorem, there exists $T_0 > \tau$ such that $u(t'+T_0, n') = u(t', n')$.

As $u(t,n)$ is a critical traveling front, so is $v(t,n) := u(t+T_0, n)$. Combining $u(t',n') = v(t',n')$ and the definition of a critical traveling front, one has

$$\begin{aligned} &u(t',n) \geq v(t',n) \ \text{ if } n < n' \quad \text{and} \quad u(t',n) \leq v(t',n) \ \text{ if } n > n', \\ &v(t',n) \geq u(t',n) \ \text{ if } n < n' \quad \text{and} \quad v(t',n) \geq u(t',n) \ \text{ if } n > n'. \end{aligned}$$

Hence $u(t',n) = v(t',n)$ for $n \in \mathbb{Z}$. Now we get the conclusion that $u(t,n) = v(t,n) = u(t+T_0,n)$ for any $(t,n) \in \mathbb{R}\times\mathbb{Z}$ by similar arguments to Step 2 of Lemma 4.10. This means u is a time-periodic transition front. Then the maximum 1 of $u(t,0)$ can be attained since $\limsup_{t\in\mathbb{R}} u(t,0) = 1$. This contradicts Corollary 2.1. Then the proof is complete. ■

Summing up Theorem 4.2, Proposition 4.5, and Theorem 4.3, we have proved Theorem 1.1.

5. Positive almost-periodic solution of the discrete Schrödinger equation

5.1. Criterion of existence of a positive almost-periodic solution

Theorem 1.1 is an incomplete answer since we cannot determine whether (1.2) has an almost-periodic traveling front, or even a generalized transition front with average wave speed $w \geq \underline{w}$. However, in the case $\underline{w} = \infty$, we can get a complete answer using Theorem 1.1. In this section, we will provide some conditions on c in (1.2) to guarantee

$\underline{w} = \infty$. The idea is to apply Proposition 3.5 and to use the KAM method for constructing the positive almost-periodic solution of $(\mathcal{L}_{V,\alpha,\theta}u)(n) = u(n+1) + u(n-1) - 2u(n) + V(n\alpha + \theta)u(n) = \lambda_1 u(n)$, with V, α, θ to be specified and $\lambda_1 = \max \Sigma(\mathcal{L}_{V,\alpha,\theta})$.

We will first provide a simple criterion, which says that if the Schrödinger cocycle can be reduced to constant parabolic cocycle, and the conjugacy is close to the identity, then the corresponding Schrödinger equation has a positive almost-periodic solution. Recall that an almost-periodic cocycle (α, A) is said to be C^s reducible, where $s \in \mathbb{N}_+ \cup \{\omega\}$ (here ω refers to analyticity), if there exist $B(\cdot) \in C^s(\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$ and a constant matrix $\tilde{A}$ such that

$$B(\theta+\alpha)^{-1}A(\theta)B(\theta) = \tilde{A}.$$

Denote the norm in $C^s(\mathbb{T}^d, *)$ as $\|F\|_s := \sup_{|l|\le s, \theta\in\mathbb{T}^d} \|\partial^l F(\theta)\|$, and for $r > 0$, $C^\omega_r(\mathbb{T}^d, *)$ denotes the bounded analytic space which can be extended to $|\Im\theta_i| \le r, i = 1, \ldots, d$:

$$\left\{F\colon \mathbb{T}^d \to * \mid |F|_r := \sup_{|\Im\theta_i|\le r, i=1,\ldots,d} \|F(\theta)\| < \infty\right\},$$

where $* = \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{sl}(2,\mathbb{R})$, the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$, i.e., the matrix whose trace vanishes, $\|\cdot\|$ denotes the matrix norm, and $\Im$ denotes the imaginary part. Then we have the following:

Lemma 5.1. *Suppose that there exists $B \in C^0(\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$ with $\|B - \mathrm{Id}\|_0 \le \frac{1}{100}$ such that*

$$B^{-1}(\theta+\alpha)S_E^V(\theta)B(\theta) = \tilde{A},$$

where $\tilde{A} \in \mathrm{SL}(2,\mathbb{R})$ is a parabolic matrix (i.e., the trace $|\mathrm{tr}(\tilde{A})| = 2$). Then

$$(\mathcal{L}_{V,\alpha,\theta}u)(n) = u(n+1) + u(n-1) - 2u(n) + V(\theta + n\alpha)u(n) = Eu(n) \tag{5.1}$$

has an almost-periodic positive solution.

Proof. Without loss of generality, we assume $\mathrm{tr}(\tilde{A}) = 2$. Then one can find some $R_\eta := \begin{pmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{pmatrix} \in \mathrm{SO}(2,\mathbb{R})$ with $\eta \in [0, 2\pi)$ such that

$$R_\eta \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} R_{-\eta} = \tilde{A}$$

for some $p \in \mathbb{R}$. Denote

$$B(\theta) := \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$
$$C(\theta) := R_\eta B(\theta) = \begin{pmatrix} C_{11}(\theta) & C_{12}(\theta) \\ C_{21}(\theta) & C_{22}(\theta) \end{pmatrix}.$$

It directly follows that

$$C^{-1}(\theta+\alpha)S_E^V(\theta)C(\theta) = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}. \tag{5.2}$$

Now we can write (5.2) as

$$\begin{pmatrix} E+2-V(\theta) & -1 \\ 1 & 0 \end{pmatrix} C(\theta) = C(\theta+\alpha) \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix},$$

which gives us

$$\begin{aligned}(E+2-V(\theta))C_{11}(\theta) - C_{21}(\theta) &= C_{11}(\theta+\alpha),\\ C_{11}(\theta) &= C_{21}(\theta+\alpha),\end{aligned}$$

that is,

$$C_{11}(\theta+\alpha) + C_{11}(\theta-\alpha) - 2C_{11}(\theta) + V(\theta)C_{11}(\theta) = EC_{11}(\theta).$$

Hence $u(n) = C_{11}(n\alpha+\theta) = C_{21}((n+1)\alpha+\theta)$ is an almost-periodic solution. Now we distinguish the following two cases:

Case 1*:* $\cos^2\eta > \frac{1}{2}$, $\sin^2\eta < \frac{1}{2}$. If $\cos\eta > 0$, then by the assumption $\|B - \mathrm{Id}\|_{C^s} \le \frac{1}{100}$,

$$C_{11}(\theta) = \cos\eta B_{11}(\theta) - \sin\eta B_{21}(\theta) > \frac{\sqrt{2}}{4} \quad \text{for any } \theta \in \mathbb{T}^d.$$

This implies $C_{11}(n\alpha+\theta)$ is a positive almost-periodic solution of (5.1). If $\cos\eta < 0$, then $-C_{11}(n\alpha+\theta)$ is a positive almost-periodic solution of (5.1).

Case 2*:* $\cos^2\eta \le \frac{1}{2}$, $\sin^2\eta \ge \frac{1}{2}$. If $\sin\eta > 0$, one can similarly verify that $C_{21}(\theta) = \cos\eta B_{21}(\theta) + \sin\eta B_{11}(\theta) \ge \frac{\sqrt{2}}{4}$ for any $\theta \in \mathbb{T}^d$. Hence $C_{21}(n\alpha+\theta)$ is a positive almost-periodic solution. If $\sin\eta < 0$, $-C_{21}(n\alpha+\theta)$ is a positive almost-periodic solution of (5.1).

In all, the proof is complete. ■

5.2. Analytic quasi-periodic potential

Motivated by Lemma 5.1, to prove the existence of a positive almost-periodic solution of (5.1), we only need to prove that the corresponding Schrödinger cocycle is reducible to a parabolic constant cocycle:

$$\mathrm{e}^{-Y(\theta+\alpha)} S_E^V(\theta) \mathrm{e}^{Y(\theta)} = \tilde{A}.$$

Moreover, the conjugation is close to constant.

First we state the reducibility result for an analytic quasi-periodic potential. To prove this, we first need a nonresonance cancellation lemma. The result will be the basis of our proof, and we will also use this when we deal with analytic almost-periodic potentials.

Let $\mathcal{B}$ be an $\mathrm{sl}(2,\mathbb{R})$-valued Banach algebra. Assume that for any given $\eta > 0, \alpha \in \mathbb{T}^d$, where $d \in \mathbb{N}_+ \cup \{\infty\}$ and $A \in \mathrm{SL}(2,\mathbb{R})$, we have a decomposition of the Banach space $\mathcal{B}$ into nonresonant spaces and resonant spaces, i.e., $\mathcal{B} = \mathcal{B}^{\mathrm{nre}}(\eta) \oplus \mathcal{B}^{\mathrm{re}}(\eta)$. Here $\mathcal{B}^{\mathrm{nre}}(\eta)$ is defined in the following way: for any $Y \in \mathcal{B}^{\mathrm{nre}}(\eta)$, we have

$$A^{-1}Y(\theta+\alpha)A \in \mathcal{B}^{\mathrm{nre}}(\eta), \quad |A^{-1}Y(\theta+\alpha)A - Y(\theta)| \ge \eta|Y(\theta)|,$$

where $|\cdot|$ is the norm of the Banach space $\mathcal{B}$.

Once we have this, we have the following:

Lemma 5.2. *Assume that* $A \in \mathrm{SL}(2,\mathbb{R})$, $\varepsilon \le (4\|A\|)^{-4}$, *and* $\eta \ge 13\|A\|^2\varepsilon^{\frac{1}{2}}$. *For any* $F \in \mathcal{B}$ *with* $|F| \le \varepsilon$, *there exist* $Y \in \mathcal{B}$ *and* $F^{\mathrm{re}} \in \mathcal{B}^{\mathrm{re}}(\eta)$ *such that*

$$e^{-Y(\theta+\alpha)}Ae^{F(\theta)}e^{Y(\theta)} = Ae^{F^{\mathrm{re}}(\theta)}.$$

Moreover, we have the estimates $|Y| \le \varepsilon^{\frac{1}{2}}$ *and* $|F^{\mathrm{re}}| \le 2\varepsilon$.

Remark 5.1. The proof of the lemma for $\mathcal{B} := C_r^\omega(\mathbb{T}^d, \mathrm{su}(1,1))$ with $d \in \mathbb{N}_+$ can be found in [16, Lemma 3.1], and we can easily see that the proof works for any other Banach algebra.

In our application, we will set $\mathcal{B} := C_r^\omega(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$, where $d \in \mathbb{N}_+ \cup \{\infty\}$, $r > 0$ (the definition of $C_r^\omega(\mathbb{T}^\infty, \mathrm{sl}(2,\mathbb{R}))$ will be introduced later). Define the norm in $\mathcal{B}$ as $|F|_r := \sup_{|\Im\theta|\le r} |F(\theta)|$. The nonresonant space $\mathcal{B}^{\mathrm{nre}}$ will take the truncating operator $\mathcal{T}_K$ on $\mathcal{B}$: for any $K > 0$, we define

$$\mathcal{T}_K F(\theta) = \sum_{k\in\mathbb{Z}^d, |k|<K} \hat{F}(k)\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}$$

and

$$\mathcal{R}_K F(\theta) = \sum_{k\in\mathbb{Z}^d, |k|\ge K} \hat{F}(k)\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}.$$

Obviously, $\mathcal{T}_K F + \mathcal{R}_K F = F$. Now, as a direct application of Lemma 5.2, we have the following:

Proposition 5.1. *Let* $\alpha \in \mathrm{DC}_d(\gamma,\tau)$, $r, \delta, \gamma > 0$, $\tau > d \ge 1$. *Suppose that* $A \in \mathrm{SL}(2,\mathbb{R})$, $F \in C_r^\omega(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$. *For any* $r' \in (0, r)$, *there exists* $c = c(\gamma,\tau,d)$ *such that if* $|\mathrm{rot}(\alpha, A)| \le 2\|A\|\varepsilon^{\frac{1}{2}}$, *and*

$$|F|_r \le \varepsilon < \frac{c(r-r')^{6(1+\delta)\tau}}{\|A\|^6}, \tag{5.3}$$

then there exist $Y, F' \in C_{r'}^\omega(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ *and* $A' \in \mathrm{SL}(2,\mathbb{R})$ *such that*

$$\mathrm{e}^{-Y(\theta+\alpha)}A\mathrm{e}^{F(\theta)}\mathrm{e}^{Y(\theta)} = A'\mathrm{e}^{F'(\theta)}.$$

Moreover, we have the following estimates:

$$|Y|_{r'} \le \varepsilon^{\frac{1}{2}}, \quad |F'|_{r'} \le 4\varepsilon^2, \quad \|A - A'\| \le 2\varepsilon\|A\|.$$

Proof. We only need to apply the nonresonance cancellation lemma (Lemma 5.2). In this case, we will define

$$\Lambda_K = \left\{ f \in C_r^\omega(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R})) \mid f(\theta) = \textstyle\sum_{k\in\mathbb{Z}^d,\ 0<|k|<K} \hat{f}(k)e^{i\langle k,\theta\rangle} \right\},$$

where $K = \frac{2}{r-r'}|\ln\varepsilon|$, and prove that for any $Y \in \Lambda_K$, the operator

$$Y \to A^{-1}Y(\theta+\alpha)A - Y(\theta)$$

has a bounded inverse.

Thus we only need to consider the equation

$$A^{-1}Y(\theta+\alpha)A - Y(\theta) = \mathcal{T}_K G(\theta) - \widehat{G}(0).$$

Without loss of generality, we assume that $A = \begin{pmatrix} e^{i\rho} & p \\ 0 & e^{-i\rho} \end{pmatrix}$, where $e^{\pm i\rho}$ are the two eigenvalues of A, $p \in \mathbb{R}$, and write

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & -Y_{11} \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & -G_{11} \end{pmatrix}.$$

Taking the Fourier transformation for the above equation and comparing the Fourier coefficients, we can get

$$\begin{cases} \widehat{Y}_{11}(k) = \dfrac{\widehat{G}_{11}(k) + p e^{i(\rho+\langle k,\alpha\rangle)}\widehat{Y}_{21}(k)}{e^{i\langle k,\alpha\rangle} - 1}, \\ \widehat{Y}_{12}(k) = \dfrac{\widehat{G}_{12}(k) + p^2 e^{i\langle k,\alpha\rangle}\widehat{Y}_{21}(k) - 2p e^{i(\langle k,\alpha\rangle-\rho)}\widehat{Y}_{11}(k)}{e^{i\langle k,\alpha\rangle-2\rho} - 1}, \\ \widehat{Y}_{21}(k) = \dfrac{\widehat{G}_{21}(k)}{e^{i(\langle k,\alpha\rangle+2\rho)} - 1}. \end{cases}$$

Note that, for any $k \in \mathbb{Z}^d$ with $0 < |k| \le K$, if ε satisfies (5.3), we have

$$|\langle k,\alpha\rangle| \ge \frac{\gamma}{|k|^\tau} \ge \frac{\gamma}{|K|^\tau} \ge \frac{\gamma\varepsilon^{\frac{1}{6(1+\delta)}}}{|\ln\varepsilon|^\tau} \ge \varepsilon^{\frac{1}{6(1+\frac{\delta}{2})}},$$
$$|2\,\mathrm{rot}(\alpha,A) - \langle k,\alpha\rangle| \ge \frac{\gamma}{|K|^\tau} - 4\|A\|\varepsilon^{\frac{1}{2}} \ge \varepsilon^{\frac{1}{6(1+\frac{\delta}{2})}},$$

which implies that

$$|Y(\theta)|_r \le \varepsilon^{-\frac{1}{2+\delta}}|\mathcal{T}_K G(\theta)|_r,$$

and then $\Lambda_K \subset \mathcal{B}_r^{\mathrm{nre}}(\varepsilon^{\frac{1}{2+\delta}})$.

Since $\varepsilon^{\frac{1}{2+\delta}} \ge 13\|A\|^2\varepsilon^{\frac{1}{2}}$, by Lemma 5.2, there exist $Y, F^{\mathrm{re}} \in C_r^\omega(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ such that

$$e^{Y(\theta+\alpha)}Ae^{F(\theta)}e^{-Y(\theta)} = Ae^{F^{\mathrm{re}}(\theta)},$$

where $\mathcal{T}_K F^{\mathrm{re}} = \hat{F}^{\mathrm{re}}(0)$, $\mathcal{R}_K F^{\mathrm{re}} = \sum_{|k|>K}\hat{F}^{\mathrm{re}}(k)e^{i\langle k,\theta\rangle}$. Moreover, we have the estimates $|Y|_r \le \varepsilon^{\frac{1}{2}}$, $|F^{\mathrm{re}}|_r \le 2\varepsilon$. Consequently, for any $r' \in (0,r)$, we have

$$|\mathcal{R}_K F^{\mathrm{re}}(\theta)|_{r'} = \left|\sum_{|k|>K}\hat{F}^{\mathrm{re}}(k)e^{i\langle k,\theta\rangle}\right|_{r'} \le 2\varepsilon e^{-K(r-r')}K^d \le 2\varepsilon^2.$$

Furthermore, we can compute that

$$e^{\hat{F}^{\mathrm{re}}(0)+\mathcal{R}_K F^{\mathrm{re}}(\theta)} = \mathrm{e}^{\hat{F}^{\mathrm{re}}(0)}(\mathrm{Id} + \mathrm{e}^{-\hat{F}^{\mathrm{re}}(0)} O(\mathcal{R}_K F^{\mathrm{re}}(\theta))) = e^{\hat{F}^{\mathrm{re}}(0)}\mathrm{e}^{F'(\theta)},$$

with the estimate

$$|F'(\theta)|_{r'} \le 2|\mathcal{R}_K F^{\mathrm{re}}|_{r'} \le 4\varepsilon^2.$$

Finally, if we denote $A' = A\mathrm{e}^{\hat{F}^{\mathrm{re}}(0)}$, then we get

$$\|A' - A\| \le 2\|A\|\,\|\mathrm{Id} - \mathrm{e}^{\hat{F}^{\mathrm{re}}(0)}\| \le 2\|A\|\varepsilon. \quad \blacksquare$$

5.3. Finitely differentiable quasi-periodic potential

Now we want to get the reducibility result for the finitely differentiable case. Note that for any $f \in C^s(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$, by [60, Lemma 2.1], there exist an analytic sequence $\{f_j\}_{j\ge1}$, $f_j \in C^{\omega}_{\frac{1}{j}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ and a universal constant $C' > 1$ such that

$$\begin{aligned} &\|f_j - f\|_s \to 0 \ \text{ if } j \to +\infty,\\ &|f_j|_{\frac{1}{j}} \le C'\|f\|_s,\\ &|f_{j+1} - f_j|_{\frac{1}{j+1}} \le C'\Big(\frac{1}{j}\Big)^s \|f\|_s. \end{aligned} \tag{5.4}$$

The basic idea is that we approximate a finitely differentiable cocycle by an analytic cocycle. If the analytic cocycle is reducible, then the finitely differentiable cocycle is also reducible. In our case, we will set

$$l_1 = M > \max\{(8\|A\|)^2, 4^{\frac{1}{\delta}}\}, \quad l_j = [M^{(1+\delta)^{j-1}}], \quad j \ge 2.$$

If we assume

$$C'\|F\|_s \le \frac{c}{\|A\|^6 l_1^{\frac{s-1}{1+\delta}}} = \frac{c}{\|A\|^6 M^{\frac{s-1}{1+\delta}}},$$

then by (5.4) we have

$$\begin{aligned} |F_{l_{k+1}} - F_{l_k}|_{\frac{1}{l_{k+1}}} &\le \frac{c}{\|A\|^6 l_k^s M^{\frac{s-1}{1+\delta}}},\\ |F_{l_k}|_{\frac{1}{l_k}} &\le \frac{c}{\|A\|^6 M^{\frac{s-1}{1+\delta}}}. \end{aligned} \tag{5.5}$$

Consequently,

$$\|F - F_{l_k}\|_0 \le \sum_{m=k}^{\infty} |F_{l_m} - F_{l_{m+1}}|_{\frac{1}{l_{m+1}}} \le \frac{c}{\|A\|^6 M^{\frac{s-1}{1+\delta}} l_{k+1}^{\frac{s-1}{1+\delta}}}. \tag{5.6}$$

We also define

$$\varepsilon_0(r,r') = \frac{c(r-r')^{6\tau(1+\delta)}}{\|A\|^6}.$$

Then for any $s > 6\tau(1+\delta)^3 + 1$, where $0 < \delta < 1$, we can compute that for any $m \geq 2^{\frac{1}{\delta}}$,

$$\varepsilon_m := \frac{c}{\|A\|^6 m^{\frac{s-1}{1+\delta}}} \leq \varepsilon_0\Big(\frac{1}{m}, \frac{1}{m^{1+\delta}}\Big).$$

With these parameters, we have the following:

Corollary 5.1. *Let* $\alpha \in \mathrm{DC}_d(\gamma,\tau)$, $\gamma > 0$, $\tau > d$, $A \in \mathrm{SL}(2,\mathbb{R})$, $F \in C^s(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ *with* $s \geq 6\tau(1+\delta)^3 + 1$, *where* $0 < \delta < 1$. *If* $\mathrm{rot}(\alpha, A\mathrm{e}^F) = 0$ *and*

$$\|F\|_s \leq \varepsilon \leq \frac{c}{C'\|A\|^6 M^{\frac{s-1}{1+\delta}}}, \tag{5.7}$$

then there exist $\tilde{A} \in \mathrm{SL}(2,\mathbb{R})$, $Y \in C(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ *with* $\|Y\|_0 \leq 2\varepsilon^{\frac{1}{2}}$ *such that*

$$\mathrm{e}^{-Y(\theta+\alpha)} A e^{F(\theta)} \mathrm{e}^{Y(\theta)} = \tilde{A}.$$

Proof. To prove this, we only need to show inductively that there exist

$$Y_{l_k}, F'_{l_k} \in C^{\omega}_{\frac{1}{l_{k+1}}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R})) \quad \text{and} \quad A_{l_k} \in \mathrm{SL}(2,\mathbb{R})$$

such that

$$\mathrm{e}^{-Y_{l_k}(\theta+\alpha)} A \mathrm{e}^{F_{l_k}(\theta)} \mathrm{e}^{Y_{l_k}(\theta)} = A_{l_k} \mathrm{e}^{F'_{l_k}(\theta)}, \tag{5.8}$$

with the estimates

$$|Y_{l_k}|_{\frac{1}{l_{k+1}}} \leq \sum_{i=1}^{k} \varepsilon_{l_i}^{\frac{1}{2}}, \quad |F'_{l_k}|_{\frac{1}{l_{k+1}}} \leq \varepsilon_{l_k}^2, \quad \|A_{l_k} - A\| \leq 2\|A\|\varepsilon_{l_k}.$$

Once this holds, $\|Y_{l_k}\|_0 \leq |Y_{l_k}|_{\frac{1}{l_{k+1}}} \leq 2\varepsilon^{\frac{1}{2}}$, and as a consequence of (5.8),

$$\begin{aligned} \mathrm{e}^{-Y_{l_k}(\theta+\alpha)} A e^{F(\theta)} \mathrm{e}^{Y_{l_k}(\theta)} &= A_{l_k} \mathrm{e}^{F'_{l_k}(\theta)} + \mathrm{e}^{-Y_{l_k}(\theta+\alpha)} (A e^{F(\theta)} - A e^{F_{l_k}(\theta)}) e^{Y_{l_k}(\theta)} \\ &= A_{l_k} \bar{G}_{l_k}. \end{aligned} \tag{5.9}$$

By (5.6), we have

$$\begin{aligned} \|\bar{G}_{l_k} - \mathrm{Id}\|_0 &\leq \|F'_{l_k}\|_0 + \|A_{l_k}^{-1}\| \, \|\mathrm{e}^{-Y_{l_k}(\theta+\alpha)} (A\mathrm{e}^{F(\theta)} - A\mathrm{e}^{F_{l_k}(\theta)}) \mathrm{e}^{Y_{l_k}(\theta)}\|_0 \\ &\leq 4\varepsilon_{l_k}^2 + 2\|A\|^2 \times \frac{c}{\|A\|^6 M^{\frac{s-1}{1+\delta}} l_{k+1}^{\frac{s-1}{1+\delta}}} \leq \varepsilon_{l_{k+1}}. \end{aligned} \tag{5.10}$$

Taking limits of (5.9), we then have the desired results.

Now let us finish the iteration.

First step. First, by our assumption (5.7), and then by (5.5), we have

$$|F_{l_1}|_{\frac{1}{l_1}} \leq \varepsilon_{l_1} \leq \varepsilon_0\Big(\frac{1}{l_1}, \frac{1}{l_2}\Big),$$

and by Lemma 2.3 we have

$$|\mathrm{rot}(\alpha, A)| = |\mathrm{rot}(\alpha, A\mathrm{e}^F) - \mathrm{rot}(\alpha, A)| \leq 2\|A\|\, \|F\|_0^{\frac{1}{2}} \leq 2\|A\|\varepsilon_{l_1}^{\frac{1}{2}}.$$

It follows from Proposition 5.1 that there exist $Y_{l_1}, F'_{l_1} \in C^{\omega}_{\frac{1}{l_2}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R})), A_{l_1} \in \mathrm{SL}(2,\mathbb{R})$ such that

$$\mathrm{e}^{-Y_{l_1}(\theta+\alpha)} A\mathrm{e}^{F_{l_1}(\theta)}\mathrm{e}^{Y_{l_1}(\theta)} = A_{l_1}\mathrm{e}^{F'_{l_1}(\theta)},$$

with the estimates $|Y_{l_1}|_{\frac{1}{l_2}} \leq \varepsilon_{l_1}^{\frac{1}{2}}, |F'_{l_1}|_{\frac{1}{l_2}} \leq 4\varepsilon_{l_1}^2, \|A_{l_1} - A\| \leq 2\|A\|\varepsilon_{l_1}$.

Induction step. Now, at the $(k+1)$th step, first notice that if we write

$$\begin{aligned}\mathrm{e}^{-Y_{l_k}(\theta+\alpha)} A\mathrm{e}^{F_{l_{k+1}}(\theta)}\mathrm{e}^{Y_{l_k}(\theta)} &= A_{l_k}\mathrm{e}^{F'_{l_k}(\theta)} + \mathrm{e}^{-Y_{l_k}(\theta+\alpha)}(A\mathrm{e}^{F_{l_{k+1}}(\theta)} - A\mathrm{e}^{F_{l_k}(\theta)})\mathrm{e}^{Y_{l_k}(\theta)}\\ &= A_{l_k}G_{l_k}(\theta),\end{aligned}$$

then we have

$$\begin{aligned}|G_{l_k} - \mathrm{Id}|_{\frac{1}{l_{k+1}}} &\leq |F'_{l_k}|_{\frac{1}{l_{k+1}}} + \|A_{l_k}^{-1}\|\, \Big|\mathrm{e}^{-Y_{l_k}(\theta+\alpha)}(A\mathrm{e}^{F_{l_{k+1}}(\theta)} - A\mathrm{e}^{F_{l_k}(\theta)})\mathrm{e}^{Y_{l_k}(\theta)}\Big|_{\frac{1}{l_{k+1}}}\\ &\leq 4\varepsilon_{l_k}^2 + 4\|A\|^2 \times \frac{c}{\|A\|^6 l_k^s} \leq \frac{1}{2}\varepsilon_{l_{k+1}}.\end{aligned}$$

Then, by the implicit function theorem, there exists $\widetilde{F}_{l_k} \in C^{\omega}_{\frac{1}{l_{k+1}}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ with

$$|\widetilde{F}_{l_k}|_{\frac{1}{l_{k+1}}} \leq 2|G_{l_k} - \mathrm{Id}|_{\frac{1}{l_{k+1}}} \leq \varepsilon_{l_{k+1}} \leq \varepsilon_0\Big(\frac{1}{l_{k+1}}, \frac{1}{l_{k+2}}\Big),$$

such that $G_{l_k}(\theta) = \mathrm{e}^{\widetilde{F}_{l_k}(\theta)}$.

On the other hand, by Lemma 2.4, we have

$$\mathrm{rot}(\alpha, A_{l_k}\overline{G}_{l_k}(\theta)) = \mathrm{rot}(\alpha, \mathrm{e}^{-Y_{l_k}(\theta+\alpha)}Ae^{F(\theta)}\mathrm{e}^{Y_{l_k}(\theta)}) = \mathrm{rot}(\alpha, Ae^{F(\theta)}) = 0.$$

Then, by Lemma 2.3 and (5.10), we have

$$|\mathrm{rot}(\alpha, A_{l_k})| \leq |\mathrm{rot}(\alpha, A_{l_k}) - \mathrm{rot}(\alpha, A_{l_k}\overline{G}_{l_k}(\theta))| \leq 2\|\overline{G}_{l_k} - \mathrm{Id}\|_0^{\frac{1}{2}} \leq 2\|A\|\varepsilon_{l_{k+1}}^{\frac{1}{2}}.$$

Consequently, we can apply Proposition 5.1 to the cocycle $(\alpha, A_{l_k}\mathrm{e}^{\widetilde{F}_{l_k}})$, and there exist $\widetilde{Y}_{l_{k+1}}, F'_{l_{k+1}} \in C^{\omega}_{\frac{1}{l_{k+2}}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ such that

$$\mathrm{e}^{-\widetilde{Y}_{l_{k+1}}(\theta+\alpha)} A_{l_k}\mathrm{e}^{\widetilde{F}_{l_k}(\theta)}\mathrm{e}^{\widetilde{Y}_{l_{k+1}}(\theta)} = A_{l_{k+1}}\mathrm{e}^{F'_{l_{k+1}}(\theta)}, \tag{5.11}$$

with $|\tilde{Y}_{l_{k+1}}|_{\frac{1}{l_{k+2}}} \leq \varepsilon_{l_{k+1}}^{\frac{1}{2}}$, $|F'_{l_{k+1}}|_{\frac{1}{l_{k+2}}} \leq \varepsilon_{l_{k+1}}^2$, $\|A_{l_{k+1}} - A_{l_k}\| \leq 2\|A_{l_k}\|\varepsilon_{l_{k+1}}$. Also note that if B, D are small $\mathrm{sl}(2,\mathbb{R})$ matrices, then there exists $C \in \mathrm{sl}(2,\mathbb{R})$ such that

$$\mathrm{e}^B\mathrm{e}^D = \mathrm{e}^{B+D+C},$$

where C is a term of at least two orders in B, D. Thus there exists $Y_{l_{k+1}} \in C^\omega_{\frac{1}{l_{k+2}}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ such that $\mathrm{e}^{Y_{l_{k+1}}} = \mathrm{e}^{\tilde{Y}_{l_{k+1}}}\mathrm{e}^{Y_{l_k}}$, with the estimate

$$|Y_{l_{k+1}}|_{\frac{1}{l_{k+2}}} \leq \sum_{i=1}^{k+1} \varepsilon_i^{\frac{1}{2}}.$$

Moreover, by (5.8) and (5.11), we have

$$\mathrm{e}^{-Y_{l_{k+1}}(\theta+\alpha)} A\mathrm{e}^{F_{l_{k+1}}(\theta)}\mathrm{e}^{Y_{l_{k+1}}(\theta)} = A_{l_{k+1}}\mathrm{e}^{F'_{l_{k+1}}(\theta)},$$

and thus we finish the iteration. ∎

Theorem 5.1. *Let $\alpha \in \mathrm{DC}_d(\gamma,\tau)$, $\gamma > 0$, $\tau > d$, $V \in C^s(\mathbb{T}^d,\mathbb{R})$ with $s > 6\tau + 2$. There exists $\varepsilon = \varepsilon(\gamma,\tau,d,s)$ such that if $\|V\|_s \leq \varepsilon$, then*

$$(\mathcal{L}_{V,\alpha,\theta}u)(n) = u(n+1) + u(n-1) - 2u(n) + V(n\alpha+\theta)u(n) = \lambda_1 u(n)$$

has a positive quasi-periodic solution.

Proof. Write

$$A = \begin{pmatrix} E+2 & -1 \\ 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}.$$

Then $S_V^E(\theta) = A\mathrm{e}^{F(\theta)}$, which is close to constant. Now we consider the energy E which lies at the extreme right endpoint of the spectrum. Since the spectrum is compact, and it is included in $[-4 + \inf V, \sup V]$, then $\|A\| \leq 6$.

By the assumption that $s > 6\tau + 2$, there exists $0 < \delta < 1$, such that $s > 6\tau(1+\delta)^3 + 1$. For such selected δ, we can take

$$\varepsilon \leq \frac{c}{6^6 C' M^{\frac{s-1}{1+\delta}}}.$$

Since the energy E lies in the extreme right endpoint of the spectrum, by Remark 2.3, we have $\mathrm{rot}(\alpha, A\mathrm{e}^F) = 0$. Then by Corollary 5.1, there exist $\tilde{A} \in \mathrm{SL}(2,\mathbb{R})$, $Y \in C^0(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ with $\|Y\|_0 \leq 2\varepsilon^{\frac{1}{2}}$, such that

$$\mathrm{e}^{-Y(\theta+\alpha)}Ae^{F(\theta)}\mathrm{e}^{Y(\theta)} = \tilde{A}.$$

By Lemma 2.4,

$$\mathrm{rot}(\alpha,\tilde{A}) = \mathrm{rot}(\alpha, A\mathrm{e}^F) = 0.$$

Then we can distinguish the following two cases:

Case 1. If $\tilde{A}$ is hyperbolic, then $(\alpha, \tilde{A})$ is uniformly hyperbolic, which implies that $(\alpha, Ae^{F(\theta)})$ is uniformly hyperbolic, since uniformly hyperbolic is conjugacy invariant. However, this contradicts Theorem 2.3.

Case 2. If $\tilde{A}$ is parabolic, since $\|\mathrm{e}^{Y} - \mathrm{Id}\|_0 \leq 2\|Y\|_0 \leq 4\varepsilon^{\frac{1}{2}} < \frac{1}{100}$, then the result follows from Lemma 5.1. Thus we finish the proof. ∎

5.4. Analytic almost-periodic potential

Now we consider the reducibility results in the almost-periodic case. First we define almost-periodic functions in the context of analytic functions on a thickened infinite-dimensional torus $\mathbb{T}_r^\infty$, where $\mathbb{T}_r^\infty$ is defined as

$$\theta = (\theta_j)_{j\in\mathbb{N}}, \quad \theta_j \in \mathbb{C}: \Re(\theta_j) \in \mathbb{T}, \quad |\Im(\theta_j)| \leq r\langle j\rangle.$$

Definition 5.1. For any $r > 0$, we define the space of analytic functions $\mathbb{T}_r^\infty \to \mathrm{sl}(2,\mathbb{R})$ as

$$\begin{aligned} &C_r^\omega(\mathbb{T}^\infty, \mathrm{sl}(2,\mathbb{R})) \\ &\quad := \left\{F(\theta) = \textstyle\sum_{k\in\mathbb{Z}_*^\infty} \hat{F}(k)\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle} \in \mathscr{F}: |F|_r := \textstyle\sum_{k\in\mathbb{Z}_*^\infty} \mathrm{e}^{r|k|_1}|\hat{F}(k)| < \infty\right\}, \end{aligned}$$

where $\mathscr{F}$ denotes the space of pointwise absolutely convergent formal Fourier series $\mathbb{T}_r^\infty \to \mathrm{sl}(2,\mathbb{R})$ as

$$F(\theta) = \sum_{k\in\mathbb{Z}_*^\infty} \hat{F}(k)\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}, \quad \hat{F}(k) \in \mathrm{sl}(2,\mathbb{R}),$$

and $\mathbb{Z}_*^\infty := \{k \in \mathbb{Z}^\infty \mid |k|_1 := \sum_{j\in\mathbb{N}}\langle j\rangle|k_j| < \infty\}$ denotes the set of infinite integer vectors with finite support.

Now we state the following proposition:

Proposition 5.2. *Let $r, \gamma > 0$, $\tau > 1$. Suppose that $\alpha \in \mathrm{DC}_\infty(\gamma,\tau)$, $A \in \mathrm{SL}(2,\mathbb{R})$, $F \in C_r^\omega(\mathbb{T}^\infty, \mathrm{sl}(2,\mathbb{R}))$. For any $r' \in (0,r)$, there exists $c = c(\gamma,\tau)$ such that if $|\mathrm{rot}(\alpha, A)| \leq 2\|A\|\varepsilon^{\frac{1}{2}}$, and*

$$|F|_r \leq \varepsilon < \frac{c\mathrm{e}^{-\frac{1}{(r-r')^2}}}{\|A\|^6}, \tag{5.12}$$

then there exist $Y, F' \in C_{r'}^\omega(\mathbb{T}^\infty, \mathrm{sl}(2,\mathbb{R}))$ and $A' \in \mathrm{SL}(2,\mathbb{R})$ such that

$$\mathrm{e}^{-Y(\theta+\alpha)}A\mathrm{e}^{F(\theta)}\mathrm{e}^{Y(\theta)} = A'\mathrm{e}^{F'(\theta)}.$$

Moreover, we have the estimates

$$|Y|_{r'} \leq \varepsilon^{\frac{1}{2}}, \quad |F'|_{r'} \leq \varepsilon^2, \quad \|A - A'\| \leq 2\varepsilon\|A\|.$$

Proof. Similar to Proposition 5.1, what we need is to apply the nonresonance cancellation lemma (Lemma 5.2). In this case, we will take $\mathcal{B} = C_r^{\omega}(\mathbb{T}^{\infty}, \mathrm{sl}(2,\mathbb{R}))$, and define

$$\Lambda_K = \left\{ f \in C_r^{\omega}(\mathbb{T}^{\infty}, \mathrm{sl}(2,\mathbb{R})) \mid f(\theta) = \sum_{k\in\mathbb{Z}_*^{\infty}, |k|_1\le K} \hat{f}(k) e^{i\langle k,\theta\rangle} \right\},$$

where $K = \frac{2}{r-r'}|\ln\varepsilon|$. By [40, Lemma 2.5], $C_r^{\omega}(\mathbb{T}^{\infty}, \mathrm{sl}(2,\mathbb{R}))$ is a Banach algebra. Moreover, since $\alpha \in \mathrm{DC}_{\infty}(\gamma,\tau)$, we have the following estimate:

Lemma 5.3 ([40, Lemma C.2]). *Let $\alpha \in \mathrm{DC}_{\infty}(\gamma,\tau)$. Then there holds the estimate*

$$\sup_{k\in\mathbb{Z}_*^{\infty}, |k|_1\le K} \prod_{i\in\mathbb{Z}} (1 + |k_i|^{\tau}\langle i\rangle^{\tau}) \le (1+K)^{2\tau K^{\frac{1}{2}}}.$$

Note, by (5.12), one has $(r-r')^2 \ge \frac{1}{|\ln\varepsilon|}$. Hence $K = \frac{2|\ln\varepsilon|}{r-r'} \le 2|\ln\varepsilon|^{\frac{3}{2}}$. Then as a consequence of Lemma 5.3, we have

$$\begin{aligned} |\langle k,\alpha\rangle| &\ge \gamma \prod_{j\in\mathbb{N}} \frac{1}{1+|k_j|^{\tau}\langle j\rangle^{\tau}} \ge \frac{\gamma}{(1+K)^{2\tau K^{\frac{1}{2}}}} \\ &\ge \frac{\gamma}{(1+2|\ln\varepsilon|^{\frac{3}{2}})^{2\tau|\ln\varepsilon|^{\frac{3}{4}}}} \ge 2\varepsilon^{\frac{1}{9}}, \end{aligned}$$

which implies that

$$|2\,\mathrm{rot}(\alpha,A) - \langle k,\alpha\rangle| \ge 2\varepsilon^{\frac{1}{9}} - 4\|A\|\varepsilon^{\frac{1}{2}} \ge \varepsilon^{\frac{1}{9}}.$$

By a similar calculation to Proposition 5.1, these facts imply that for any $Y \in \Lambda_K$,

$$|A^{-1}Y(\theta+\alpha)A - Y(\theta)|_r \ge \varepsilon^{\frac{1}{3}}|Y(\theta)|_r,$$

and then $\Lambda_K \subset \mathcal{B}_r^{\mathrm{nre}}(\varepsilon^{\frac{1}{3}})$.

Since $\varepsilon^{\frac{1}{3}} \ge 13\|A\|^2\varepsilon^{\frac{1}{2}}$, by Lemma 5.2, there exist $Y, F^{\mathrm{re}} \in C_r^{\omega}(\mathbb{T}^{\infty}, \mathrm{sl}(2,\mathbb{R}))$, such that

$$e^{-Y(\theta+\alpha)} A e^{F(\theta)} e^{Y(\theta)} = A e^{F^{\mathrm{re}}(\theta)},$$

where $\mathcal{T}_K F^{\mathrm{re}} = \hat{F}^{\mathrm{re}}(0)$, $\mathcal{R}_K F^{\mathrm{re}} = \sum_{|k|_1>K} \hat{F}^{\mathrm{re}}(k) e^{i\langle k,\theta\rangle}$. Moreover, we have the estimates $|Y|_r \le \varepsilon^{\frac{1}{2}}$, $|F^{\mathrm{re}}|_r \le 2\varepsilon$. Meanwhile, by [40, Lemma 2.3], one has $|\mathcal{R}_K F|_{r'} \le e^{-(r-r')K}|F|_r$. Then for any $r' \in (0,r)$, we have

$$|\mathcal{R}_K F^{\mathrm{re}}(\theta)|_{r'} = \left| \sum_{|k|_1>K} \hat{F}^{\mathrm{re}}(k) e^{i\langle k,\theta\rangle} \right|_{r'} \le 2\varepsilon e^{-K(r-r')} \le 2\varepsilon^3.$$

Furthermore, one can compute that

$$e^{\hat{F}^{\mathrm{re}}(0)+\mathcal{R}_K F^{\mathrm{re}}(\theta)} = e^{\hat{F}^{\mathrm{re}}(0)}(\mathrm{id} + e^{-\hat{F}^{\mathrm{re}}(0)} O(\mathcal{R}_K F^{\mathrm{re}}(\theta))) = e^{\hat{F}^{\mathrm{re}}(0)} e^{F'(\theta)},$$

with the estimate

$$|F'(\theta)|_{r'} \le 2|\mathcal{R}_K F^{\mathrm{re}}|_{r'} \le 4\varepsilon^3 \le \varepsilon^2.$$

Finally, if we denote $A' = Ae^{\hat{F}^{\mathrm{re}}(0)}$, then we get

$$\|A' - A\| \le 2\|A\|\,\|\mathrm{Id} - e^{\hat{F}^{\mathrm{re}}(0)}\| \le 2\|A\|\varepsilon. \qquad \blacksquare$$

As a consequence, we have the following:

Corollary 5.2. *Let* $\alpha \in \mathrm{DC}_\infty(\gamma,\tau)$, $A \in \mathrm{SL}(2,\mathbb{R})$, $F \in C_r^\omega(\mathbb{T}^\infty, \mathrm{sl}(2,\mathbb{R}))$. *For any* $0 < \tilde{r} < r$, *there exists* $\varepsilon = \varepsilon(\tau,\gamma,r) > 0$, *such that if* $\mathrm{rot}(\alpha, A\mathrm{e}^F) = 0$, *and*

$$|F|_r \le \varepsilon \le \frac{c\mathrm{e}^{-\frac{1}{(r-\tilde{r})^2}}}{\|A\|^6},$$

then there exist $\tilde{A} \in \mathrm{SL}(2,\mathbb{R})$, $Y \in C_{\tilde{r}}^\omega(\mathbb{T}^\infty, \mathrm{sl}(2,\mathbb{R}))$ *with* $|Y|_{\tilde{r}} \le 2\varepsilon^{\frac{1}{2}}$, *such that*

$$\mathrm{e}^{-Y(\theta+\alpha)} A e^{F(\theta)} \mathrm{e}^{Y(\theta)} = \tilde{A}.$$

Proof. We will prove this by induction. First we define the sequence

$$r_0 = r, \quad \varepsilon_0 = \varepsilon, \quad r_k - r_{k+1} = \frac{r-\tilde{r}}{(k+2)^2}, \quad \varepsilon_k = \varepsilon^{2^k}.$$

Assume we are at the $(k+1)$-step, i.e., we already construct $Y_k, F_k \in C_{r_k}^\omega(\mathbb{T}^\infty, \mathrm{sl}(2,\mathbb{R}))$, such that

$$\mathrm{e}^{-Y_k(\theta+\alpha)} A\mathrm{e}^{F(\theta)} \mathrm{e}^{Y_k(\theta)} = A_k \mathrm{e}^{F_k(\theta)}, \tag{5.13}$$

with the estimates

$$|Y_k|_{r_k} \le \sum_{i=1}^{k-1} \varepsilon_i^{\frac{1}{2}}, \quad |F_k|_{r_k} \le \varepsilon_k, \quad \|A_k - A_{k-1}\| \le 2\|A_{k-1}\|\varepsilon_{k-1}.$$

First by Lemma 2.4, we have

$$\mathrm{rot}(\alpha, A_k e^{F_k}) = \mathrm{rot}(\alpha, Ae^F) = 0,$$

since the conjugacy e^{Y_k} is homotopic to the identity. Then by Lemma 2.3, we have

$$|\mathrm{rot}(\alpha, A_k)| \le 2\|A_k\|\varepsilon_k^{\frac{1}{2}}.$$

By our selection of ε_0 and the sequence r_k, one can check that

$$\varepsilon_k \le \frac{c\mathrm{e}^{-\frac{1}{(r_k-r_{k+1})^2}}}{\|A_k\|^6}.$$

By Proposition 5.2, there exist $A_{k+1} \in \mathrm{SL}(2,\mathbb{R})$, $\widetilde{Y}_{k+1}, F_{k+1} \in C_{r_{k+1}}^\omega(\mathbb{T}^\infty, \mathrm{sl}(2,\mathbb{R}))$ such that

$$\mathrm{e}^{-\widetilde{Y}_{k+1}(\theta+\alpha)} A_k \mathrm{e}^{F_k(\theta)} \mathrm{e}^{\widetilde{Y}_{k+1}(\theta)} = A_{k+1}\mathrm{e}^{F_{k+1}(\theta)} \tag{5.14}$$

with

$$|\widetilde{Y}_{k+1}|_{r_{k+1}} \le \varepsilon_k^{\frac{1}{2}}, \quad |F_{k+1}|_{r_{k+1}} \le \varepsilon_k^2 = \varepsilon_{k+1}, \quad \|A_k - A_{k+1}\| \le 2\|A_k\|\varepsilon_k.$$

Thus there exists $Y_{k+1} \in C^{\omega}_{r_{k+1}}(\mathbb{T}^{\infty}, \mathrm{sl}(2,\mathbb{R}))$ such that $\mathrm{e}^{Y_{k+1}} = \mathrm{e}^{\widetilde{Y}_{k+1}}\mathrm{e}^{Y_k}$, with the estimate

$$|Y_{k+1}|_{r_{k+1}} \leq \sum_{i=1}^{k} \varepsilon_i^{\frac{1}{2}}.$$

Moreover, by (5.13) and (5.14), we have

$$\mathrm{e}^{-Y_{k+1}(\theta+\alpha)} A \mathrm{e}^{F(\theta)} \mathrm{e}^{Y_{k+1}(\theta)} = A_{k+1}\mathrm{e}^{F_{k+1}(\theta)}. \tag{5.15}$$

Taking limits of (5.15), we get the desired results. ∎

Corollary 5.3. *Let $c(n) = V(n\alpha + \theta)$ be an almost-periodic sequence with frequency $\alpha \in \mathrm{DC}_{\infty}(\gamma, \tau)$ and analytic in the strip $r > 0$. There exists $\varepsilon = \varepsilon(\gamma, \tau, r) > 0$, such that if $|V|_r \leq \varepsilon$, then*

$$(\mathcal{L}_{V,\alpha,\theta}u)(n) = u(n+1) + u(n-1) - 2u(n) + V(n\alpha+\theta)u(n) = \lambda_1 u(n)$$

has a positive almost-periodic solution.

Proof. The proof is the same as Theorem 5.1 if we replace Corollary 5.1 by Corollary 5.2. ∎

5.5. Proof of the applications

In the final subsection, we will give the applications of Theorem 1.1 in various settings, including the quasi-periodic case and almost-periodic case. First, we give the proof of Corollary 1.1.

Proof of Corollary 1.1. By Theorem 1.1, (1.2) has a time-increasing almost-periodic traveling front with average wave speed $w \in (w^*, \underline{w})$, where $w^* = \inf_{E>\lambda_1} \frac{E}{L(E)}$ and $\underline{w} = \lim_{E\searrow\lambda_1} \frac{E}{L(E)}$.

Recall that

$$L(E) = \lim_{n\to+\infty} \frac{1}{n} \int_{\mathcal{H}(c)} \log \|A_n(g)\|\, d\mu$$

with $A_n = A(n-1)\cdots A(0)$, where $A(n) = \begin{pmatrix} E+2-g(n) & -1 \\ 1 & 0 \end{pmatrix}$. In the quasi-periodic case, it is well known that any $g \in \mathcal{H}(c)$ has the form

$$g(n) = V(n\alpha + \theta),$$

for some $\theta \in \mathbb{T}^d$, where V is a continuous function on $\mathbb{T}^d$. Then it is straightforward to check that for any $g(\cdot) = V(\theta + \cdot\alpha)$, one has

$$\begin{pmatrix} E+2-g(\cdot) & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} E+2-V(\theta+\cdot\alpha) & -1 \\ 1 & 0 \end{pmatrix}.$$

It follows directly that

$$A_n(g) = A(n-1)\cdots A(0) = S_V^E(\theta + (n-1)\alpha)\cdots S_V^E(\theta+\alpha)S_V^E(\theta).$$

Then $L(E) = L(\alpha, S_V^E)$, and the result follows from Theorem 1.1 and the following well-known result of Bourgain–Jitomirskaya:

Theorem 5.2 ([14, 15]). *For any $d \in \mathbb{N}_+$, let $\alpha \in \mathbb{T}^d$ be rationally independent and V be an analytic function on $\mathbb{T}^d$. Then $L(\alpha, S_V^E)$ is a continuous function of E.*

By [3, Theorem 10] and [15, Corollary 2], it follows that for any E which belongs to the spectrum of the almost Mathieu operator

$$(\mathcal{L}_{2\kappa-2,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\kappa\cos(\theta+n\alpha)u(n),$$

its Lyapunov exponent satisfies

$$L(E) = \max(0, \ln|\kappa|).$$

Then the result follows directly. Thus the proof is complete. ∎

Next we give some typical examples such that $\underline{L} = \lim_{E\to\lambda_1} L(E) = 0$ which implies that $\underline{w} = \infty$. First we consider the quasi-periodic case; if the potential V is finitely differentiable, then we have the following:

Corollary 5.4. *Let $\alpha \in \mathrm{DC}_d(\gamma,\tau)$, $\gamma > 0$, $\tau > d$, $V \in C^s(\mathbb{T}^d, \mathbb{R})$ with $s > 6\tau + 2$. There exists $\varepsilon = \varepsilon(\gamma,\tau,d,s)$ such that if $\|V\|_s \le \varepsilon$, then (1.2) with $c(n) = V(n\alpha)$ has a time-increasing almost-periodic traveling front with average wave speed $w \in (w^*, \infty)$.*

Proof. By the assumption and Corollary 5.1, we know

$$\begin{aligned}(\mathcal{L}_{V,\alpha,0}u)(n) &= u(n+1) + u(n-1) - 2u(n) + V(n\alpha)u(n)\\ &= \lambda_1 u(n)\end{aligned}$$

has a positive almost-periodic solution. It follows that $\underline{L} = \lim_{E\to\lambda_1} L(E) = 0$ by Proposition 3.5. Then the result follows from Theorem 1.1. ∎

Lemma 5.1 and Proposition 3.5 show that if the corresponding cocycle is reducible, and the conjugacy is close to constant, then $\underline{L} = \lim_{E\to\lambda_1} L(E) = 0$. However, in some cases, we can relax the condition, and the right concept is "*almost reducible*".

Definition 5.2. An analytic cocycle (α, A) is C^ω-almost reducible if the closure of its analytic conjugacy class contains a constant.

Corollary 5.5. *Let $r > 0$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. There exists $\varepsilon = \varepsilon(r)$ such that if $V \in C_r^\omega(\mathbb{T}, \mathbb{R})$, and $|V|_r \le \varepsilon$, then (1.2) with $c(n) = V(n\alpha)$ has a time-increasing almost-periodic traveling front with average wave speed $w \in (w^*, \infty)$.*

Proof. By [59, Corollary 1.3] and [2, Corollary 1.2], there exists $\varepsilon = \varepsilon(r)$ such that if $|V|_r \leq \varepsilon$, then one frequency analytic quasi-periodic Schrödinger cocycle (α, S_V^E) is almost reducible. Clearly, by its definition, any almost-reducible cocycle is not non-uniformly hyperbolic. Thus either $L(E) = 0$ or (α, S_V^E) is uniformly hyperbolic. Then $L(\lambda_1) = 0$ follows from Theorem 2.3 since we only consider λ_1, which is the right endpoint of the spectrum. By Theorem 5.2, the result follows directly. ∎

Finally, we give the application for almost-periodic potentials:

Proof of Corollary 1.2. Choose $\varepsilon = \varepsilon(\gamma, \tau, r)$ defined in Corollary 5.3 such that

$$\sum_{k \in \mathbb{Z}_*^\infty} |\hat{c}(k)| e^{r|k|_1} < \varepsilon(\gamma, \tau, r).$$

Then by Corollary 5.3, we know

$$(\mathcal{L}_{V,\alpha,0} u)(n) = u(n+1) + u(n-1) - 2u(n) + V(n\alpha)u(n) = \lambda_1 u(n)$$

has a positive almost-periodic solution. It follows from Proposition 3.5 that

$$\underline{L} = \lim_{E \to \lambda_1} L(E) = 0.$$

Then Corollary 1.2 (1) follows from Theorem 1.1. ∎

Funding. X. Liang was partially supported by NSFC grant (no. 11971454). Q. Zhou was partially supported by the National Key R&D Program of China (no. 2020YFA0713300), NSFC grant (no. 12071232), the Science Fund for Distinguished Young Scholars of Tianjin (no. 19JCJQJC61300) and the Nankai Zhide Foundation. T. Zhou was partially supported by NSFC grant (no. 12001514).

References

[1] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* **30** (1978), no. 1, 33–76 Zbl 0407.92014 MR 511740

[2] A. Avila, Almost reducibility and absolute continuity I. 2010, arXiv:1006.0704

[3] A. Avila, Global theory of one-frequency Schrödinger operators. *Acta Math.* **215** (2015), no. 1, 1–54 Zbl 1360.37072 MR 3413976

[4] A. Avila and D. Damanik, Generic singular spectrum for ergodic Schrödinger operators. *Duke Math. J.* **130** (2005), no. 2, 393–400 Zbl 1102.82012 MR 2181094

[5] A. Avila and S. Jitomirskaya, The ten martini problem. *Ann. of Math. (2)* **170** (2009), no. 1, 303–342 Zbl 1166.47031 MR 2521117

[6] A. Avila, J. You, and Q. Zhou, Sharp phase transitions for the almost Mathieu operator. *Duke Math. J.* **166** (2017), no. 14, 2697–2718 Zbl 1503.47041 MR 3707287

[7] H. Berestycki and F. Hamel, Front propagation in periodic excitable media. *Comm. Pure Appl. Math.* **55** (2002), no. 8, 949–1032 Zbl 1024.37054 MR 1900178

[8] H. Berestycki and F. Hamel, Generalized transition waves and their properties. *Comm. Pure Appl. Math.* **65** (2012), no. 5, 592–648 Zbl 1248.35039 MR 2898886

[9] H. Berestycki, F. Hamel, and L. Roques, Analysis of the periodically fragmented environment model. II. Biological invasions and pulsating travelling fronts. *J. Math. Pures Appl. (9)* **84** (2005), no. 8, 1101–1146 Zbl 1083.92036 MR 2155900

[10] H. Berestycki, F. Hamel, and L. Rossi, Liouville-type results for semilinear elliptic equations in unbounded domains. *Ann. Mat. Pura Appl. (4)* **186** (2007), no. 3, 469–507 Zbl 1223.35022 MR 2317650

[11] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.* **47** (1994), no. 1, 47–92 Zbl 0806.35129 MR 1258192

[12] L. Biasco, J. E. Massetti, and M. Procesi, An abstract Birkhoff normal form theorem and exponential type stability of the 1d NLS. *Comm. Math. Phys.* **375** (2020), no. 3, 2089–2153 Zbl 1441.35217 MR 4091501

[13] J. Bourgain, On invariant tori of full dimension for 1D periodic NLS. *J. Funct. Anal.* **229** (2005), no. 1, 62–94 Zbl 1088.35061 MR 2180074

[14] J. Bourgain, Positivity and continuity of the Lyapounov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential. *J. Anal. Math.* **96** (2005), 313–355 Zbl 1089.81021 MR 2177191

[15] J. Bourgain and S. Jitomirskaya, Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Stat. Phys.* **108** (2002), no. 5, 1203–1218 Zbl 1039.81019 MR 1933451

[16] A. Cai, C. Chavaudret, J. You, and Q. Zhou, Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles. *Math. Z.* **291** (2019), no. 3-4, 931–958 Zbl 1482.37031 MR 3936094

[17] F. Cao and W. Shen, Spreading speeds and transition fronts of lattice KPP equations in time heterogeneous media. *Discrete Contin. Dyn. Syst.* **37** (2017), no. 9, 4697–4727 Zbl 1394.34027 MR 3661816

[18] D. Damanik, R. Killip, and B. Simon, Schrödinger operators with few bound states. *Comm. Math. Phys.* **258** (2005), no. 3, 741–750 Zbl 1082.47029 MR 2172016

[19] A. Ducrot, T. Giletti, and H. Matano, Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations. *Trans. Amer. Math. Soc.* **366** (2014), no. 10, 5541–5566 Zbl 1302.35209 MR 3240934

[20] R. A. Fisher, The wave of advance of advantageous genes. *Ann. Eugenics* **7** (1937), no. 4, 355–369 Zbl 63.1111.04

[21] J.-S. Guo and F. Hamel, Front propagation for discrete periodic monostable equations. *Math. Ann.* **335** (2006), no. 3, 489–525 Zbl 1116.35063 MR 2221123

[22] S. Hadj Amor, Hölder continuity of the rotation number for quasi-periodic co-cycles in $\mathrm{SL}(2,\mathbb{R})$. *Comm. Math. Phys.* **287** (2009), no. 2, 565–588 Zbl 1201.37066 MR 2481750

[23] D. Hankerson and B. Zinner, Wavefronts for a cooperative tridiagonal system of differential equations. *J. Dynam. Differential Equations* **5** (1993), no. 2, 359–373 Zbl 0777.34013 MR 1223452

[24] P. Harper, Single band motion of conduction electrons in a uniform magnetic field. *Proc. Phys. Soc. A* **68** (1955), no. 10, 874–878 Zbl 0065.23708

[25] M.-R. Herman, Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension 2. *Comment. Math. Helv.* **58** (1983), no. 3, 453–502 Zbl 0554.58034 MR 727713

[26] W. Hudson and B. Zinner, Existence of traveling waves for reaction diffusion equations of Fisher type in periodic media. In *Boundary value problems for functional-differential equations*, pp. 187–199, World Scientific, River Edge, NJ, 1995 Zbl 0846.35062 MR 1375475

[27] S. Jitomirskaya and W. Liu, Universal hierarchical structure of quasiperiodic eigenfunctions. *Ann. of Math. (2)* **187** (2018), no. 3, 721–776 Zbl 1470.47025 MR 3779957

[28] S. Y. Jitomirskaya, Metal-insulator transition for the almost Mathieu operator. *Ann. of Math. (2)* **150** (1999), no. 3, 1159–1175 Zbl 0946.47018 MR 1740982

[29] R. Johnson and J. Moser, The rotation number for almost periodic potentials. *Comm. Math. Phys.* **90** (1983), no. 2, 317–318 Zbl 0497.35026 MR 667409

[30] R. A. Johnson, Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients. *J. Differential Equations* **61** (1986), no. 1, 54–78 Zbl 0608.34056 MR 818861

[31] M. Keller, Y. Pinchover, and F. Pogorzelski, Criticality theory for Schrödinger operators on graphs. *J. Spectr. Theory* **10** (2020), no. 1, 73–114 Zbl 1441.31006 MR 4071333

[32] A. N. Kolmogorov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. Moskow, Ser. Internat., Sec. A.* **1** (1937), 1–25 Zbl 0018.32106

[33] S. Kotani, Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators. In *Stochastic analysis (Katata/Kyoto, 1982)*, pp. 225–247, North-Holland Math. Library 32, North-Holland, Amsterdam, 1984 Zbl 0549.60058 MR 780760

[34] S. M. Kozlov, Ground states of quasiperiodic operators. *Dokl. Akad. Nauk SSSR* **271** (1983), no. 3, 532–536 Zbl 0598.35034 MR 719581

[35] R. Krikorian, Reducibility, differentiable rigidity and Lyapunov exponents for quasiperiodic cocycles on $\mathbb{T} \times \mathrm{SL}(2, \mathbb{R})$. 2004, arXiv:0402333

[36] X. Liang, Semi-wave solutions of KPP–Fisher equations with free boundaries in spatially almost periodic media. *J. Math. Pures Appl. (9)* **127** (2019), 299–308 Zbl 1420.35129 MR 3960145

[37] X. Liang and X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems. *J. Funct. Anal.* **259** (2010), no. 4, 857–903 Zbl 1201.35068 MR 2652175

[38] X. Liang and T. Zhou, Spreading speeds of KPP-type lattice systems in heterogeneous media. *Commun. Contemp. Math.* **22** (2020), no. 1, article no. 1850083 Zbl 1437.37103 MR 4064907

[39] H. Matano, Traveling waves in spatially random media (mathematical economics). *Research Notes of RIMS, Kyoto University* **1337** (2003), 1–9

[40] R. Montalto and M. Procesi, Linear Schrödinger equation with an almost periodic potential. *SIAM J. Math. Anal.* **53** (2021), no. 1, 386–434 Zbl 1458.35025 MR 4201442

[41] G. Nadin, Traveling fronts in space-time periodic media. *J. Math. Pures Appl. (9)* **92** (2009), no. 3, 232–262 Zbl 1182.35074 MR 2555178

[42] G. Nadin, Critical travelling waves for general heterogeneous one-dimensional reaction-diffusion equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **32** (2015), no. 4, 841–873 Zbl 1364.35146 MR 3390087

[43] G. Nadin and L. Rossi, Generalized transition fronts for one-dimensional almost periodic Fisher–KPP equations. *Arch. Ration. Mech. Anal.* **223** (2017), no. 3, 1239–1267 Zbl 1362.35313 MR 3595366

[44] D. Osadchy and J. E. Avron, Hofstadter butterfly as quantum phase diagram. *J. Math. Phys.* **42** (2001), no. 12, 5665–5671 Zbl 1019.81071 MR 1866679

[45] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Appl. Math. Sci. 44, Springer, New York, 1983 Zbl 0516.47023 MR 710486

[46] P. Peierls, Zur Theorie des Diamagnetismus von Leitungselektronen. *Z. Phys.* **80** (1933) 763–791 Zbl 0006.19204

[47] J. Pöschel, Small divisors with spatial structure in infinite-dimensional Hamiltonian systems. *Comm. Math. Phys.* **127** (1990), no. 2, 351–393 Zbl 0702.58065 MR 1037110

[48] A. Rau, Degeneracy of Landau levels in crystals. *Phys. Status Solidi B* **65** (1974), 131–135

[49] W. Shen, Variational principle for spreading speeds and generalized propagating speeds in time almost periodic and space periodic KPP models. *Trans. Amer. Math. Soc.* **362** (2010), no. 10, 5125–5168 Zbl 1225.35048 MR 2657675

[50] W. Shen, Existence, uniqueness, and stability of generalized traveling waves in time dependent monostable equations. *J. Dynam. Differential Equations* **23** (2011), no. 1, 1–44 Zbl 1223.35103 MR 2772198

[51] N. Shigesada, K. Kawasaki, and E. Teramoto, Traveling periodic waves in heterogeneous environments. *Theoret. Population Biol.* **30** (1986), no. 1, 143–160 Zbl 0591.92026 MR 850456

[52] B. Simon, *Szegő's theorem and its descendants*. M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011 Zbl 1230.33001 MR 2743058

[53] D. J. Thouless, D. Kohmoto, M. P. Nightingale, and M. D. Nijs, Quantized Hall conductance in a two dimensional periodic potential. *Phys. Rev. Lett.* **49** (1982), no. 6, 405–408

[54] P. Walters, *An introduction to ergodic theory*. Grad. Texts Math. 79, Springer, New York-Berlin, 1982 Zbl 0958.28011 MR 648108

[55] P. Walters, Unique ergodicity and random matrix products. In *Lyapunov exponents (Bremen, 1984)*, pp. 37–55, Lecture Notes in Math. 1186, Springer, Berlin, 1986 Zbl 0604.60011 MR 850069

[56] Y. Wang and J. You, Examples of discontinuity of Lyapunov exponent in smooth quasiperiodic cocycles. *Duke Math. J.* **162** (2013), no. 13, 2363–2412 Zbl 1405.37032 MR 3127804

[57] H. F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat. *J. Math. Biol.* **45** (2002), no. 6, 511–548 Zbl 1058.92036 MR 1943224

[58] J. X. Xin, Existence of planar flame fronts in convective-diffusive periodic media. *Arch. Rational Mech. Anal.* **121** (1992), no. 3, 205–233 Zbl 0764.76074 MR 1188981

[59] J. You and Q. Zhou, Embedding of analytic quasi-periodic cocycles into analytic quasi-periodic linear systems and its applications. *Comm. Math. Phys.* **323** (2013), no. 3, 975–1005 Zbl 1286.37004 MR 3106500

[60] E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems. I. *Comm. Pure Appl. Math.* **28** (1975), 91–140 Zbl 0309.58006 MR 380867

Received 27 September 2022; accepted 21 July 2023.

Xing Liang
School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China; xliang@ustc.edu.cn

Hongze Wang
School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen, Shenzhen Guangdong 518172, China; wanghongze@cuhk.edu.cn

Qi Zhou
Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China;
qizhou@nankai.edu.cn

Tao Zhou
Center for Pure Mathematics, School of Mathematical Sciences, Anhui University, Hefei,
Anhui 230601, China; tzhou910@ustc.edu.cn