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Coupled variational inequalities and application in electroelasticity

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Abstract. This work is devoted to the mathematical and numerical study of a framework handling a system of coupled variational inequalities. We prove both the existence and the uniqueness of a weak solution to the problem. Then, we introduce a convergent iterative scheme. Using this latter, we decouple the problem into further subproblems and derive their corresponding minimization problems. As a practical application of this class of coupled abstract variational inequalities, we consider a class of problems that model an electroelastic body coming into frictional contact with a rigid electrically conductive foundation. Both electrical and mechanical contacts are of Signorini type. In other words, our model prescribes the mechanical response produced by the foundation and the outflow of the free charges across the contact zone. The last part of this paper is mainly reserved for the numerical resolution of the problem at hand. For this purpose, we have developed an alternating direction method of multipliers and convex dualities to compute and illustrate the solutions.

1. Introduction

The theory of variational and quasi-variational inequalities (of the first and second order) has an important role in the study of both qualitative and numerical analyses of nonlinear boundary value problems. The fundamental importance of this theory is essentially due to its various applications in physics, mechanics, and engineering. As a consequence, the literature in this area is very extensive, and the progress made over the past four decades is very impressive. An important part of this progress has been driven by models from contact mechanics. The mathematical study of variational and quasi-variational inequality is raised and addressed in [18,23,35]. The study of variational and quasi-variational inequalities and their application to contact problems can be found in, for example, [9, 10, 34]. A more general setting of variational and quasi-variational and their applications to contact mechanics, which are well known as hemi-variational inequalities, are introduced and studied in, for example, [21, 28–30].

In this paper, we study an abstract nonlinear, non-coercive, and non-symmetric coupled system. We prove the existence and uniqueness of solutions by employing variational

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inequalities theory's tools [35] and the Banach fixed point theorem [2]. We derive a convergent Bensoussan–Lions iterative scheme [7, 10] in order to decouple the unknowns, for example, "u" from "φ". Using this scheme, we state an equivalent constrained minimization problem to compute "u" and another one to compute "φ". This kind of coupled variational can find a large application in elastic and electroelastic contact problems [13, 14], dynamic electro-viscoelastic contact problem [3] and can be developed to thermo-electroelastic contact problems [1, 6]. In this paper, we choose as application a unilateral contact problem arising in electroelasticity [5, 22, 27, 31, 36] with nonlocal Coulomb friction [23, 32]. The electroelastic body may come into contact with an electrically conductive foundation [14, 26] which may lead to a non-linearity in the problem.

On the contact zone, we assume both contact and electric Signorini conditions. The electric Signorini condition (unilateral condition) controls the electric potential outflow of free charges on the surface of the domain between the body and the conductive foundation. This means that either the potential vanishes in the contact zone or there is an outflow of the free charges in the direction of the conductive foundation. When it is combined with the unilateral contact condition, this condition implies that the normal component of the electric displacement field may vanish even under the contact process. The resulting variational formulation of this problem is given by two nonlinearly coupled variational problems of second kind. The numerical analysis of this class of problems is commonly effectuated by the finite element method [4, 13, 20, 26]. The classical Coulomb friction is approximated by solving a sequence of problems with Tresca friction [11]. The Uzawa block decomposition method is employed in [14]. The method in [14, 15, 24] was initiated by Glowinski, Fortin, Le Tallec, and Marocco in [16, 17, 19] who systematically adopted augmented Lagrangian method for solving nonlinear partial differential equations. The augmented Lagrangian method is adapted for contact problems in elasticity in [24].

In the considered application, since the obtained minimization problems are convex and not differentiable, we propose an employment of an ADMM (alternating direction method of multipliers) decomposition method [8, 14, 15]. In fact, the ADMM allows us to decouple the differentiable part from the non-differentiable one. Thereafter, we derive KKT conditions for the differentiable subproblem, and we develop Fenchel duality for the non-differentiable one [12].

This work is briefly structured as follows. In Section 2, we present the framework of two coupled nonlinear variational inequalities. We prove the existence and uniqueness of the solution through the Banach fixed point theorem. By employing the Bensoussan–Lions scheme, we state an equivalent minimization problem. Section 3 is dedicated to the application of this framework. The results of the previous section are applied to a contact problem in electroelasticity with nonlocal Coulomb friction. ADMM is the basic strategy to decouple the differentiable and non-differentiable parts, then KKT conditions and Fenchel duality are used to compute the solutions. The numerical simulations are visualized with Matlab by adopting the piecewise finite element method and the vectorized Matlab codes [25]. We illustrate the deformation and the distribution of the electric potential in the domain and the Lagrange multipliers are visualized to identify, where the "stick" and "slide" modes occur on the contact zone. Finally, the von Mises distribution over the domain is shown and the performance (CPU time and number of iterations) of the method is demonstrated.

2. Framework: Abstract coupled variational inequalities

The coupled system of variational inequalities under consideration is given by the following problem.

Problem (QV-a). Find $(u, \varphi) \in K_m \times K_e$ such that

$$\begin{aligned} &\langle Au, v-u\rangle_{V^*\times V} + \langle \theta^*\varphi, v-u\rangle_{V^*\times V} + j(u,v) - j(u,u) \\ &\quad \geq \langle f, v-u\rangle_{V^*\times V} \quad \forall v \in K_m, \\ &\langle B\varphi, \xi-\varphi\rangle_{W^*\times W} - \langle \theta u, \xi-\varphi\rangle_{W^*\times W} + \ell(u,\varphi,\xi-\varphi) \\ &\quad \geq \langle q, \xi-\varphi\rangle_{W^*\times W} \quad \forall \xi \in K_e, \end{aligned} \tag{2.1}$$

where $f \in V^*$, $q \in W^*$ are given data, $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are two given reflexive Banach spaces, and $(V^*, \|\cdot\|_{V^*})$ and $(W^*, \|\cdot\|_{W^*})$ are their topological dual spaces, respectively. The notations $\langle\cdot,\cdot\rangle_{V^*\times V}$ and $\langle\cdot,\cdot\rangle_{W^*\times W}$ are the duality pairing between the space and its topological dual space. The sets K_m and K_e are non-empty, convex, and closed subsets of V and W, respectively. In specific applications of this framework, we suppose that V and W are Hilbert spaces.

Our goal is to prove the existence and uniqueness of a solution and provide an equivalent optimization problem to (2.1). So, let us suppose that $A : V \to V^*$ and $B : W \to W^*$ are two symmetric, Lipschitz continuous, and strongly monotone operators, i.e.,

$$\exists M > 0 \quad \text{such that } \|Au - Av\|_{V^*} \leq M\|u-v\|_V \quad \forall u, v \in V, \tag{2.2}$$

$$\exists m_A > 0 \quad \text{such that } \langle Au - Av, u-v\rangle_{V^*\times V} \geq m_A\|u-v\|_V^2 \quad \forall u, v \in V, \tag{2.3}$$

$$\exists N > 0 \quad \text{such that } \|B\varphi - B\xi\|_{W^*} \leq N\|\varphi-\xi\|_W \quad \forall \varphi, \xi \in W, \tag{2.4}$$

$$\exists m_B > 0 \quad \text{such that } \langle B\varphi - B\xi, \varphi-\xi\rangle_{W^*\times W} \geq m_B\|\varphi-\xi\|_W^2 \quad \forall \varphi, \xi \in W, \tag{2.5}$$

and the operator $\theta : V \to W^*$ is continuous, i.e.,

$$\exists \Lambda > 0 \quad \text{such that } \langle \theta u, \varphi\rangle_{W^*\times W} \leq \Lambda\|u\|_V\|\varphi\|_W \quad \forall u \in V,\ \varphi \in W. \tag{2.6}$$

In addition, we suppose that the mapping $j(u, \cdot) : V \to (-\infty, +\infty]$ is proper, strictly convex, and lower semi-continuous with respect to the second component. Also, we suppose that $j(\cdot,\cdot)$ satisfies

$$\begin{cases} \exists k > 0 \text{ such that } j(u_1, v_1) + j(u_2, v_2) - j(u_1, v_2) - j(u_2, v_1), \\ \leq k\|u_1 - u_2\|_V\|v_1 - v_2\|_V \ \forall u_1, u_2, v_1, v_2 \in K_m. \end{cases} \tag{2.7}$$

The mapping $\ell : V \times W \times W \to \mathbb{R}$ satisfies the following conditions:

$$\begin{cases} \text{(a)} & \ell(u,\varphi,\cdot) : \xi \mapsto \ell(u,\varphi,\xi) \text{ is linear and bounded, i.e., } \exists c > 0 \text{ such that} \\ & |\ell(u,\varphi,\xi)| \le c\|\xi\|_W \ \forall (u,\varphi) \in V \times W, \\ \text{(b)} & \ell(\cdot,\cdot,\xi) : (u,\varphi) \mapsto \ell(u,\varphi,\xi) \text{ is Lipschitz continuous, i.e., } \exists \kappa > 0 \text{ such that} \\ & \forall \xi \in W |\ell(u_1,\varphi_1,\xi) - \ell(u_2,\varphi_2,\xi)| \le \kappa(\|u_1 - u_2\|_V + \|\varphi_1 - \varphi_2\|_W)\|\xi\|_W. \end{cases} \tag{2.8}$$

In the remainder of this paper, we will assume the following smallness assumption:

$$2\Lambda + k + \kappa < \min(m_A, m_B). \tag{2.9}$$

This condition has a physical meaning in the application of this model. This will be shown in the next section. In the sequel, we present the result providing the existence and uniqueness of solution to the problem (2.1).

2.1. Well-posedness: Existence and uniqueness of the solution

The proof of the existence and uniqueness of the solution of (2.1) is essentially based on the tools of the theory of variational inequalities [23] and the Banach fixed point theorem [2, page 160, Theorem 6]. To claim the existence and uniqueness, we will need the following notations. Let $X = V \times W$ be a product space of elements of the form $x = (u,\varphi)$, endowed with the following norm $\|\cdot\|_X^2 = \|\cdot\|_V^2 + \|\cdot\|_W^2$.

Let us define the following operator $\mathbb{A} : X \to X^*$, the mappings $\tilde{j}, \widetilde{\ell}$ defined in $X \times X$, and $\tilde{f} \in X^*$ by

$$\langle \mathbb{A}x, y\rangle = \langle Au, v\rangle_{V^*\times V} + \langle B\varphi, \xi\rangle_{W^*\times W} + \langle \theta^*\varphi, u\rangle_{V^*\times V} - \langle \theta u, \xi\rangle_{W^*\times W}, \tag{2.10}$$

$$\tilde{j}(x,y) = j(u,v), \tag{2.11}$$

$$\widetilde{\ell}(x,y) = \ell(u,\varphi,\xi), \tag{2.12}$$

$$\tilde{f} = (f,q) \in X^* \tag{2.13}$$

for all $x = (u,\varphi) \in X$ and $y = (v,\xi) \in X$.

With the above notations, the following lemma is straightforward.

Lemma 2.1. *The couple $x = (u,\varphi) \in K_m \times K_e$ is the solution of the problem (QV-a) if and only if*

$$\langle \mathbb{A}x, y - x\rangle + \tilde{j}(x,y) - \tilde{j}(x,x) + \widetilde{\ell}(x, y - x) \ge \langle \tilde{f}, y - x\rangle_{X^*\times X} \tag{2.14}$$

for all $y = (v,\xi) \in K_m \times K_e$.

Proof. Let us prove the direct implication. Let $x = (u,\varphi) \in K_m \times K_e$ be the solution to the variational inequality (2.1) and let $y = (v,\xi)$ be an arbitrary element in $K_m \times K_e$. We add the first and second inequalities in (2.1) and by the definitions (2.10)–(2.13), we get directly (2.14). To prove the converse implication, we take the test function $y = (v, \varphi)$ in (2.14) to get the first inequality in (2.1) and the test function $y = (u, \xi)$ in (2.14) to obtain the second inequality in (2.1). ∎

According to this lemma, we provide the first result of this paper.

Theorem 2.2. *Under the assumptions* (2.2)–(2.8) *and* (2.9)*, the problem (QV-a) has a unique solution.*

Proof. Let us define the mapping $\mathcal{S} : K_m \times K_e \to K_m \times K_e$ which maps each $w \in K_m \times K_e$ to the solution of the following variational inequality.

Problem (QV-af). Find $\mathcal{S}w \in K_m \times K_e$ such that

$$\begin{aligned}&\langle \mathbb{A}(\mathcal{S}w), y - \mathcal{S}w\rangle + \tilde{j}(w, y) - \tilde{j}(w, \mathcal{S}w) + \tilde{\ell}(w, y - \mathcal{S}w)\\ &\quad \geq \langle \tilde{f}, y - \mathcal{S}w\rangle_{X^*\times X} \quad \forall y \in K_m \times K_e.\end{aligned}$$

The operator $\mathbb{A}$ defined by (2.10) is strongly monotone and Lipschitz continuous. Indeed, let $x_1 = (u_1, \varphi_1)$ and $x_2 = (u_2, \varphi_2)$ be two elements of $K_m \times K_e$. We have

$$\begin{aligned}&\langle \mathbb{A}x_1 - \mathbb{A}x_2, x_1 - x_2\rangle\\ &\quad = \langle Au_1 - Au_2, u_1 - u_2\rangle_{V^*\times V} + \langle B\varphi_1 - B\varphi_2, \varphi_1 - \varphi_2\rangle_{W^*\times W}\\ &\qquad + \langle \theta^*\varphi_1 - \theta^*\varphi_2, u_1 - u_2\rangle_{V^*\times V} - \langle \theta u_1 - \theta u_2, \varphi_1 - \varphi_2\rangle_{W^*\times W}\\ &\quad = \langle Au_1 - Au_2, u_1 - u_2\rangle_{V^*\times V} + \langle B\varphi_1 - B\varphi_2, \varphi_1 - \varphi_2\rangle_{W^*\times W},\end{aligned}$$

by (2.3) and (2.5), we have

$$\begin{aligned}\langle \mathbb{A}x_1 - \mathbb{A}x_2, x_1 - x_2\rangle &\geq m_A\|u_1 - u_2\|_V^2 + m_B\|\varphi_1 - \varphi_2\|_W^2\\ &\geq \min(m_A, m_B)(\|u_1 - u_2\|^2 + \|\varphi_1 - \varphi_2\|^2),\end{aligned}$$

thus,

$$\langle \mathbb{A}x_1 - \mathbb{A}x_2, x_1 - x_2\rangle \geq \min(m_A, m_B)\|x_1 - x_2\|_X^2,$$

and hence, $\mathbb{A}$ is strongly monotone. Let us show that $\mathbb{A}$ is Lipschitz continuous. By (2.2), (2.4), and (2.6), there exists some constant $C > 0$ such that, for every $y \in K_m \times K_e$, we have

$$\begin{aligned}\langle \mathbb{A}x_1 - \mathbb{A}x_2, y\rangle &\leq C(\|u_1 - u_2\|_V\|v\|_V + \|\varphi_1 - \varphi_2\|_W\|\xi\|_W\\ &\qquad + \|u_1 - u_2\|_V\|\xi\|_W + \|\varphi_1 - \varphi_2\|_W\|v\|_V)\\ &\leq C\|x_1 - x_2\|_X\|y\|_X\end{aligned}$$

for $y = \mathbb{A}x_1 - \mathbb{A}x_2$, we have

$$\|\mathbb{A}x_1 - \mathbb{A}x_2\|_X \leq C\|x_1 - x_2\|_X,$$

then $\mathbb{A}$ is Lipschitz continuous. In addition, by the assumption (a) in (2.8) and the fact that $f \in V^*$, the mapping $y \mapsto (\tilde{f}, y)_X - \tilde{\ell}(w, y)$ is linear and bounded. Moreover, for each $u \in V$, the map $v \mapsto j(u, v)$ is proper, strictly convex, and lower semi-continuous; then the problem (QV-af) has unique solution.

It remains to show that the map $\mathcal{S}$ is a contraction. To do this, let $w_1, w_2 \in K_m \times K_e$ be arbitrarily chosen and let $\mathcal{S}w_1, \mathcal{S}w_2$ be the corresponding solutions to the problem (QV-af), respectively. Let us take $y = \mathcal{S}w_2$ in the problem (QV-af) for which the solution is $\mathcal{S}w_1$ and $y = \mathcal{S}w_1$ in the problem (QV-af) for which the solution is $\mathcal{S}w_2$, by adding the resulting inequalities, we get

$$\langle \mathbb{A}(\mathcal{S}w_1 - \mathcal{S}w_2), \mathcal{S}w_1 - \mathcal{S}w_2 \rangle \leq \mathcal{I} + \mathcal{J}, \tag{2.15}$$

where

$$\mathcal{I} = \tilde{j}(w_1, \mathcal{S}w_2) + \tilde{j}(w_2, \mathcal{S}w_1) - \tilde{j}(w_1, \mathcal{S}w_1) - \tilde{j}(w_2, \mathcal{S}w_2)$$

and

$$\mathcal{J} = \tilde{\ell}(w_1, \mathcal{S}w_1 - \mathcal{S}w_2) + \tilde{\ell}(w_2, \mathcal{S}w_2 - \mathcal{S}w_1).$$

By (2.7) and (2.8) (b), we have

$$\mathcal{I} + \mathcal{J} \leq (k + \kappa)\|w_1 - w_2\|_X \|\mathcal{S}w_1 - \mathcal{S}w_2\|_X.$$

Employing the inequality (2.15), we obtain

$$\langle \mathbb{A}(\mathcal{S}w_1 - \mathcal{S}w_2), \mathcal{S}w_1 - \mathcal{S}w_2 \rangle \leq (k + \kappa)\|w_1 - w_2\|_X \|\mathcal{S}w_1 - \mathcal{S}w_2\|_X.$$

The strong monotony of the operator $\mathbb{A}$ implies that

$$\min(m_A, m_B)\|\mathcal{S}w_1 - \mathcal{S}w_2\|_X^2 \leq (k + \kappa)\|w_1 - w_2\|_X \|\mathcal{S}w_1 - \mathcal{S}w_2\|_X.$$

Therefore,

$$\|\mathcal{S}w_1 - \mathcal{S}w_2\|_X \leq q\|w_1 - w_2\|_X.$$

Since

$$q = \frac{k + \kappa}{\min(m_A, m_B)} < 1,$$

the mapping $\mathcal{S}$ is a contraction. Finally, $\mathcal{S}$ have a unique fixed point, and hence, the problem (QV-a) have a unique solution. ∎

Remark 2.3. This result holds true for an operator B form (2.1) that is only symmetric and monotone, i.e.,

$$\langle B\varphi - B\xi, \varphi - \xi \rangle_{W^* \times W} \geq 0 \quad \forall \varphi, \xi \in W,$$

which is weaker than (2.5), and in this case the smallness condition for the uniqueness will be simply $2k + \kappa < m_A$ instead of (2.9).

The proof of the above result allows us to apply the algorithm due to Bensoussan–Lions [7] to approximate the solution to the variational problem (QV-a).

Algorithm 1 Bensoussan–Lions scheme.

- Initialization $i = 0$. $x_0 \leftarrow x_i$ is given,
- Compute $x_{i+1} = \mathcal{S} x_i$, solution to

$$\langle \mathbb{A} x_{i+1}, y - x_{i+1}\rangle + \tilde{j}(x_i, y) - \tilde{j}(x_i, x_{i+1}) + \widetilde{\ell}(x_i, y - x_{i+1}) \\ \geq \langle \tilde{f}, y - x_{i+1}\rangle_{X^* \times X} \quad \forall y \in K_m \times K_e.$$

Since $\mathcal{S}$ is a contraction, we have the following convergence result.

Proposition 2.1. *The solution* (x_i) *generated by Algorithm 1 is convergent; that is,*

$$x_i \to x \quad \textit{strongly in } X \textit{ as } i \to +\infty.$$

By Lemma 2.1, this scheme is equivalent to the following system.

Problem (QV-ai). Find $(u_{i+1}, \varphi_{i+1}) \in K_m \times K_e$ such that

$$\langle A u_{i+1}, v - u_{i+1}\rangle + \langle \theta^* \varphi_{i+1}, v - u_{i+1}\rangle + j(u_i, v) - j(u_i, u_{i+1}) \\ \geq \langle f, v - u_{i+1}\rangle \quad \forall v \in K_m,$$
$$\langle B\varphi_{i+1}, \xi - \varphi_{i+1}\rangle - \langle \theta u_{i+1}, \xi - \varphi_{i+1}\rangle + \ell(u_i, \varphi_i, \xi - \varphi_{i+1}) \\ \geq \langle q, \xi - \varphi_{i+1}\rangle \quad \forall \xi \in K_e.$$

2.2. Optimization

This subsection is devoted to the reformulation of the constrained minimization problem equivalent to the problem (QV-ai).

The aim of this paragraph is to fix the nonlinearity in the second variational inequality of (QV-ai) resulting from the map $\ell(\cdot,\cdot,\cdot)$. The first iterative scheme is as follows: starting with an initial data $(u_0, \varphi_0) \in K_m \times K_e$, we compute a sequence $(u_{i+1}, \varphi_{i+1}) \in K_m \times K_e$ by the following problem.

Problem (PV$_i$). Find $(u_{i+1}, \varphi_{i+1}) \in K_m \times K_e$ such that

$$\langle A u_{i+1}, v - u_{i+1}\rangle + \langle \theta^* \varphi_{i+1}, v - u_{i+1}\rangle + j(u_i, v) - j(u_i, u_{i+1}) \\ \geq \langle f, v - u_{i+1}\rangle \quad \forall v \in K_m, \tag{2.16}$$
$$\langle B\varphi_{i+1}, \xi - \varphi_{i+1}\rangle - \langle \theta u_{i+1}, \xi - \varphi_{i+1}\rangle + \langle h(u_i, \varphi_i), \psi - \varphi_{i+1}\rangle \\ \geq \langle q, \xi - \varphi_{i+1}\rangle \quad \forall \xi \in K_e, \tag{2.17}$$

where we have omitted the index of duality for the sake of simplicity. The operator $h(\cdot,\cdot)$ in (2.17) is obtained from $\ell(\cdot,\cdot,\cdot)$ by Riesz's representation theorem.

Now, our aim is to decouple the variational inequalities and to set the optimization problems. Let us look at the following iterative scheme.

Problem (PV$_{ij}$). Starting with initial guess $(u_{i,j-1}, \varphi_{i,j-1})$, compute, for $j \geq 0$, $u_{i+1,j}$ and $\varphi_{i+1,j}$ by solving

$$\begin{aligned}&\langle Au_{i+1,j}, v-u_{i+1,j}\rangle + \langle \theta^*\varphi_{i+1,j-1}, v-u_{i+1,j}\rangle + j(u_{i,j-1}, v) - j(u_{i,j-1}, u_{i+1,j})\\&\quad\geq \langle f, v-u_{i+1,j}\rangle \quad \forall v \in K_m, \qquad (2.18)\end{aligned}$$

$$\begin{aligned}&\langle B\varphi_{i+1,j}, \xi-\varphi_{i+1,j}\rangle - \langle \theta u_{i+1,j}, \xi-\varphi_{i+1,j}\rangle + \langle h(u_{i,j-1}, \varphi_{i,j-1}), \xi-\varphi_{i+1,j}\rangle\\&\quad\geq \langle q, \xi-\varphi_{i+1,j}\rangle \quad \forall \xi \in K_e. \qquad (2.19)\end{aligned}$$

The convergence of the problem (2.18)–(2.19) is obtained in the following theorem.

Theorem 2.4. *Under* (2.9) *and the same assumptions of Theorem* 2.2*, the iterative scheme in problem (PV$_{ij}$) converges; that is,*

$$u_{i,j} \to u_i, \quad \text{strongly in } V, \text{ as } j \to +\infty,$$
$$\varphi_{i,j} \to \varphi_i, \quad \text{strongly in } W, \text{ as } j \to +\infty.$$

Proof. Everywhere below, we denote $\boldsymbol{x} := x_i$ and $\boldsymbol{x}_j := x_{i,j}$ for $x = \boldsymbol{u}, \boldsymbol{\varphi}$. We choose the test functions like $v = \boldsymbol{u}_j$ and $\xi = \boldsymbol{\varphi}_j$ in (2.16) and (2.17), respectively. By adding the resulting inequalities, we obtain

$$\begin{aligned}&\langle A\boldsymbol{u}, \boldsymbol{u}_j-\boldsymbol{u}\rangle + \langle B\boldsymbol{\varphi}, \boldsymbol{\varphi}_j-\boldsymbol{\varphi}\rangle + \langle \theta^*\boldsymbol{\varphi}_{j-1}, \boldsymbol{u}_j-\boldsymbol{u}\rangle\\&\quad - \langle \theta\boldsymbol{u}, \boldsymbol{\varphi}_j-\boldsymbol{\varphi}\rangle + j(\boldsymbol{u}_{j-1}, \boldsymbol{u}_j) - j(\boldsymbol{u}_{j-1}, \boldsymbol{u}) + \langle h(\boldsymbol{u}, \boldsymbol{\varphi}), \boldsymbol{\varphi}_j-\boldsymbol{\varphi}\rangle\\&\geq \langle f, \boldsymbol{u}_j-\boldsymbol{u}\rangle + \langle q, \boldsymbol{\varphi}_j-\boldsymbol{\varphi}\rangle. \qquad (2.20)\end{aligned}$$

By this way, we put $v = \boldsymbol{u}$ and $\xi = \boldsymbol{\varphi}$ in (2.18) and (2.19), and we get

$$\begin{aligned}&\langle A\boldsymbol{u}_j, \boldsymbol{u}-\boldsymbol{u}_j\rangle + \langle B\boldsymbol{\varphi}_j, \boldsymbol{\varphi}-\boldsymbol{\varphi}_j\rangle + \langle \theta^*\boldsymbol{\varphi}_{j-1}, \boldsymbol{u}-\boldsymbol{u}_j\rangle\\&\quad - \langle \theta\boldsymbol{u}_j, \boldsymbol{\varphi}-\boldsymbol{\varphi}_j\rangle + j(\boldsymbol{u}_{j-1}, \boldsymbol{u}) - j(\boldsymbol{u}_{j-1}, \boldsymbol{u}_j) + \langle h(\boldsymbol{u}, \boldsymbol{\varphi}), \boldsymbol{\varphi}-\boldsymbol{\varphi}_j\rangle\\&\geq \langle f, \boldsymbol{u}-\boldsymbol{u}_j\rangle + \langle q, \boldsymbol{\varphi}-\boldsymbol{\varphi}_j\rangle. \qquad (2.21)\end{aligned}$$

Now, we subtract (2.20) from (2.21), this produces

$$\begin{aligned}&\langle A\boldsymbol{u} - A\boldsymbol{u}_j, \boldsymbol{u}-\boldsymbol{u}_j\rangle + \langle B\boldsymbol{\varphi} - B\boldsymbol{\varphi}_j, \boldsymbol{\varphi}-\boldsymbol{\varphi}_j\rangle \leq \langle \theta(\boldsymbol{u}-\boldsymbol{u}_j), \boldsymbol{\varphi}-\boldsymbol{\varphi}_j\rangle\\&\quad - \langle \theta^*\boldsymbol{\varphi} - \theta^*\boldsymbol{\varphi}_{j-1}, \boldsymbol{u}-\boldsymbol{u}_j\rangle + \langle h(\boldsymbol{u}_{j-1}, \boldsymbol{\varphi}_{j-1}), \boldsymbol{\varphi}-\boldsymbol{\varphi}_j\rangle - \langle h(\boldsymbol{u}, \boldsymbol{\varphi}), \boldsymbol{\varphi}_j-\boldsymbol{\varphi}\rangle, \qquad (2.22)\end{aligned}$$

we get

$$\begin{aligned}m_A\|\boldsymbol{u}_j-\boldsymbol{u}\|^2 + m_B\|\boldsymbol{\varphi}_j-\boldsymbol{\varphi}\|^2 &\leq \Lambda\|\boldsymbol{u}_j-\boldsymbol{u}\|\|\boldsymbol{\varphi}_j-\boldsymbol{\varphi}\| + \Lambda\|\boldsymbol{u}_j-\boldsymbol{u}\|\|\boldsymbol{\varphi}_{j-1}-\boldsymbol{\varphi}\|\\&\quad + \kappa\|\boldsymbol{u}_{j-1}-\boldsymbol{u}\|\|\boldsymbol{\varphi}_j-\boldsymbol{\varphi}\| + \kappa\|\boldsymbol{\varphi}_{j-1}-\boldsymbol{\varphi}\|\|\boldsymbol{\varphi}_j-\boldsymbol{\varphi}\|,\end{aligned}$$

and hence,

$$\|\boldsymbol{x}_j - \boldsymbol{x}\| \leq \frac{\Lambda+\kappa}{\min(m_A, m_B) - \Lambda}\|\boldsymbol{x}_{j-1} - \boldsymbol{x}\|. \qquad (2.23)$$

From the smallness condition (2.9), it follows that

$$\frac{\Lambda+\kappa}{\min(m_A, m_B) - \Lambda} < 1,$$

and this achieves the proof. ∎

The mechanical field and the potential electric are now decoupled, we compute the solutions by introducing auxiliary unknowns, e.g., we will use a constrained minimization problem to decouple the linear elasticity from the contact and friction. Once this is carried out, we compute the electric potential by a second constrained minimization problem. This process will be explained in the remainder part of this paper.

We first start by denoting $\langle \boldsymbol{f}^{j-1}, v\rangle = \langle f, v\rangle - \langle \theta^* \boldsymbol{\varphi}_{j-1}, v\rangle$. The operator A is coercive and symmetric, thus, the problem (2.18) is equivalent to the following constrained minimization problem:

$$(Pm): \begin{cases} \text{Find } \boldsymbol{u}_j \in K_m \text{ such that} \\ J(\boldsymbol{u}_j) + j(\boldsymbol{u}_j) \leq J(\boldsymbol{v}) + j(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in K_m, \end{cases}$$

where $J(\boldsymbol{v}) = \frac{1}{2}\langle A\boldsymbol{v}, \boldsymbol{v}\rangle - \langle \boldsymbol{f}^{j-1}, v\rangle$ and $j(\boldsymbol{u}_j) = j(\boldsymbol{u}_{j-1}, \boldsymbol{u}_j)$. The problem (Pm) admits a unique solution and this is due to the fact that $\boldsymbol{v} \mapsto J(\boldsymbol{v}) + j(\boldsymbol{v})$ is coercive and strictly convex, and $j(\cdot)$ is lower semicontinuous [12].

Once $\boldsymbol{u}_j$ is computed, we are able to compute the electric potential by (2.19). Let us define $\xi \mapsto \hbar(\xi) := \langle h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}), \xi\rangle$ and $\langle \boldsymbol{q}, \xi\rangle = \langle q, \xi\rangle + \langle \theta \boldsymbol{u}, \xi\rangle$ $\forall \xi \in W$. Since the operator B is coercive and symmetric, the variational inequality (2.19) is equivalent to the following constrained minimization problem.

Problem (PVp). Find $\boldsymbol{\varphi}_j \in K_e$ such that

$$\mathfrak{J}(\boldsymbol{\varphi}_j) + \hbar(\boldsymbol{\varphi}_j) \leq \mathfrak{J}(\psi) + \hbar(\psi) \quad \forall \psi \in K_e,$$

where $\mathfrak{J}(\xi) = \frac{1}{2}\langle B\xi, \xi\rangle - \langle \boldsymbol{q}, \xi\rangle$.

The second part of this paper is reserved for an application of this framework. We firstly present our linear electroelastic model (under hypothesis of small deformations), where we introduce new boundary condition on the contact zone (see (3.13)). This condition leads to a variational inequality and then we can apply the studied framework.

3. Application: Electroelastic contact problem

We consider an electroelastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$, with smooth (enough) boundary $\partial\Omega = \Gamma$. Let us denote by $\boldsymbol{n}$ the outer normal to Γ. We note that the summation over repeated indices is adopted. The index that follows a comma means the partial derivative with respect to the corresponding component of the variable. The notation $\mathbb{T}^d$ expresses the space of second-order symmetric tensors on $\mathbb{R}^d$ while "$\cdot$" and $|\cdot|$ will be the inner product and the Euclidean norm on $\mathbb{T}^d$ and $\mathbb{R}^d$, i.e.,

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}_i \boldsymbol{y}_i, \quad |\boldsymbol{y}| = \sqrt{\boldsymbol{y} \cdot \boldsymbol{y}} \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma}_{ij} \boldsymbol{\tau}_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{T}^d.$$

The mechanical field u is decomposed into the normal component $u_{\boldsymbol{n}}$ and tangential component $u_{\boldsymbol{t}}$, i.e., $u_{\boldsymbol{n}} = u \cdot \boldsymbol{n}$ and $u_{\boldsymbol{t}} = u - u_{\boldsymbol{n}}\boldsymbol{n}$. In the same manner, we decompose the stress tensor $\boldsymbol{\sigma}$ into $\sigma_{\boldsymbol{n}}$ and $\boldsymbol{\sigma_t}$ the normal and tangential stress tensor, respectively, that are given by $\sigma_{\boldsymbol{n}} = \boldsymbol{\sigma n} \cdot \boldsymbol{n}$ and $\boldsymbol{\sigma_t} = \boldsymbol{\sigma n} - \sigma_{\boldsymbol{n}}\boldsymbol{n}$. Furthermore, we will use the following notations, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(u) = (\boldsymbol{\varepsilon}_{ij}(u))$ to prescribe the strain tensor defined by $\boldsymbol{\varepsilon}_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$ to be the stress tensor.

Finally, we denote by φ, $E(\varphi) = (E_i(\varphi))$ and $D = (D_i)$ the electric potential, the electric field and the electric displacement, respectively, where $E_i(\varphi) = -\varphi_{,i}$.

Next, we introduce the following usual functional spaces: $H = L^2(\Omega)^d$, and $H^1(\Omega)$ the usual Sobolev space,

$$\mathbb{H} = \left\{\tau = (\tau_{ij})_{ij} \in \mathbb{T}^d \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$
$$H_1 = \{u \in H \mid \boldsymbol{\varepsilon}(u) \in \mathbb{H}\} \quad \text{and} \quad \mathbb{H}_1 = \{\tau \in \mathbb{H} \mid \operatorname{Div} \tau \in H\}$$

endowed with norms $\|\cdot\|_H$, $\|\cdot\|_{\mathbb{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathbb{H}_1}$, respectively.

The equilibrium equations are given by

$$-\operatorname{Div}(\boldsymbol{\sigma}) = f_0 \quad \text{in } \Omega, \tag{3.1}$$
$$\operatorname{div}(D) = q_0 \quad \text{in } \Omega, \tag{3.2}$$

where the constitutive relations for the piezoelectric material are

$$\boldsymbol{\sigma} = \mathcal{T}\boldsymbol{\varepsilon}(u) - \mathcal{E}^* E(\varphi) \quad \text{in } \Omega, \tag{3.3}$$
$$D = \mathcal{E}\boldsymbol{\varepsilon}(u) + \beta E(\varphi) \quad \text{in } \Omega, \tag{3.4}$$

where $\mathcal{T} = (\tau_{ijkl})$ is a (fourth-order) elasticity tensor, $\mathcal{E} = (e_{ijk})$ is a (third-order) piezoelectric tensor, $\mathcal{E}^*$ is the transpose of $\mathcal{E}$, and $\beta = (\beta_{ij})$ is the electric permittivity. In (3.1)–(3.2), we note that we have the following definitions: $\operatorname{Div}(\boldsymbol{\sigma}) := \boldsymbol{\sigma}_{ij,j}$ and $\operatorname{div}(D) := D_{i,i}$ (see [37]).

In order to complete the statement of the classical problem, we need to provide the mechanical and electrical boundary conditions. First, we divide Γ into three disjoint measurable parts Γ_1, Γ_2, Γ_3 such that $\text{meas}(\Gamma_1) > 0$. The body is assumed to be clamped on Γ_1 and surface tractions of density f_2 are applied on Γ_2, whereas the part Γ_3 contains the portion of the boundary that can reach a frictional contact with the foundation. A second partition of Γ, that is $\Gamma = \Gamma_3 \cup \Gamma_a \cup \Gamma_b$, will be assumed. In addition, surface electric charges of density q_2 are applied on Γ_b, and the electric potential vanishes on Γ_b. The trace of v on Γ is denoted by v for the simplicity of the presentation. We resume the above boundary conditions as follows:

$$u = 0 \quad \text{on } \Gamma_1, \tag{3.5}$$
$$\boldsymbol{\sigma n} = f_2 \quad \text{on } \Gamma_2, \tag{3.6}$$
$$\varphi = 0 \quad \text{on } \Gamma_a, \tag{3.7}$$
$$D\boldsymbol{n} = q_2 \quad \text{on } \Gamma_b. \tag{3.8}$$

The Signorini contact conditions are

$$u_{\boldsymbol{n}} \leq g, \quad \sigma_{\boldsymbol{n}} \leq 0, \quad (u_{\boldsymbol{n}} - g)\sigma_{\boldsymbol{n}} = 0 \quad \text{on } \Gamma_3, \tag{3.9}$$

where $g \in L^\infty(\Gamma_3)$ represents the insulator gap at the reference configuration between Γ_3 and the foundation, measured along the direction of $\boldsymbol{n}$.

The nonlocal Coulomb friction condition is expressed by

$$\begin{cases} \|\boldsymbol{\sigma_t}\| \leq \mu(\|u_{\boldsymbol{t}}\|)|\sigma_{\boldsymbol{n}}^\natural(u)| \quad \text{on } \Gamma_3, \\ \|\boldsymbol{\sigma_t}\| < \mu(\|u_{\boldsymbol{t}}\|)|\sigma_{\boldsymbol{n}}^\natural(u)| \Rightarrow u_{\boldsymbol{t}} = 0 \quad \text{on } \Gamma_3, \\ \|\boldsymbol{\sigma_t}\| = \mu(\|u_{\boldsymbol{t}}\|)|\sigma_{\boldsymbol{n}}^\natural(u)| \Rightarrow \exists \lambda \geq 0 \quad \text{such that } u_{\boldsymbol{t}} = -\lambda \boldsymbol{\sigma_t} \text{ on } \Gamma_3, \end{cases} \tag{3.10}$$

where $\sigma_{\boldsymbol{n}}^\natural$ is a mollification (regularization by convolution product see [26]) of $\sigma_{\boldsymbol{n}}$; that is, the convolution of the given $\sigma_{\boldsymbol{n}}$ with $\mathscr{C}_0^\infty$ a mollifier ω_ρ (see [23]) is given by

$$(\sigma_{\boldsymbol{n}}^\natural)(x) = \omega_\rho * \sigma_{\boldsymbol{n}}(x) = \int_{\Gamma_3} \omega_\rho(|x - y|)\sigma_{\boldsymbol{n}}(y)dy,$$

where dy denotes the measure over the boundary Γ_3 and

$$\omega_\rho(x) = \begin{cases} c \exp\left(\frac{\rho^2}{x^2 - \rho^2}\right), & |x| < \rho, \\ 0, & |x| \geq \rho, \end{cases} \tag{3.11}$$

and c is a positive constant.

The electric contact condition is

$$D\boldsymbol{n} = \psi(u_{\boldsymbol{n}} - g)\Phi_c(\varphi - \varphi_0) \quad \text{on } \Gamma_3, \tag{3.12}$$

where $\varphi_0 \in L^\infty(\Gamma_3)$ is the potential of the conductive foundation.

The Signorini electric conditions are

$$\varphi \leq \varphi_f, \quad D\boldsymbol{n} \leq 0, \quad (\varphi - \varphi_f)D\boldsymbol{n} = 0 \quad \text{on } \Gamma_3. \tag{3.13}$$

Remarks 3.1. (1) Let us clarify the connection between (3.12) and the above Signorini electric condition. The condition (3.12) is related to the contact condition in this way.

- If there is no contact, i.e., $u_{\boldsymbol{n}} < g$, there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes, that is, $D\boldsymbol{n} = 0$.
- if the contact occurs, $u_{\boldsymbol{n}} = g$, the normal component of the electric displacement field (or the free charges) is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with p_e as the proportionality factor, this means that $D\boldsymbol{n} = p_e(\varphi - \varphi_0)$. In this case, the Signorini electric conditions are equivalent to

$$\varphi \leq \varphi_f, \quad D\boldsymbol{n} \leq 0, \quad p_e(\varphi - \varphi_f)(\varphi - \varphi_0) = 0 \quad \text{on } \Gamma_3.$$

(2) The boundary condition (3.12) represents a regularized electrical contact condition, similar to that used in [26], where we have

$$\Phi_c(s) = \begin{cases} -c & \text{if } s < -c, \\ s & \text{if } -c \le s \le c, \\ c & \text{if } s > c, \end{cases} \qquad \psi(r) = \begin{cases} 0 & \text{if } r < 0, \\ p_e \delta r & \text{if } 0 \le r \le \frac{1}{\delta}, \\ p_e & \text{if } r > \frac{1}{\delta}, \end{cases}$$

where c is a positive constant and δ is a small parameter.

To resume, we consider the following problem.

Problem 1. *Find the displacement field $u : \Omega \to \mathbb{R}^d$ and the electric potential field $\varphi : \Omega \to \mathbb{R}$ such that (3.1)–(3.12) hold.*

Problem 1 is the strong formulation of (3.1)–(3.12); in this paper, we are looking for a weak solution. To do this, let us introduce the following Hilbert spaces:

$$V = \{v \in H_1 / v = 0 \text{ on } \Gamma_1\}, \quad W = \{\varphi \in H^1(\Omega)/\varphi = 0 \text{ on } \Gamma_a\},$$

and the closed convex subsets of V and W, respectively, given by

$$K_m = \{v \in V / v_{\boldsymbol{n}} \le g \text{ on } \Gamma_3\}, \quad K_e = \{\psi \in W/\psi \le \varphi_f \text{ on } \Gamma_3\}.$$

Moreover, by the Sobolev trace theorem, there exists a constant k_1 depending only on Ω, Γ_1, and Γ_3 such that

$$\|\xi\|_{L^2(\Gamma_3)} \le k_1 \|\xi\|_W \quad \forall \xi \in W. \tag{3.14}$$

Since $\text{meas}(\Gamma_1) > 0$, Korn's inequality holds

$$\|\boldsymbol{\varepsilon}(v)\|_{\mathbb{H}} \ge c_k \|v\|_{H_1} \quad \forall v \in V, \tag{3.15}$$

where $c_k > 0$ is a constant which depends only on Ω and Γ_1. The space V is endowed with the inner product given by

$$(u, v)_V = (\boldsymbol{\varepsilon}(u), \boldsymbol{\varepsilon}(v))_{\mathbb{H}}, \quad \|v\|_V = (v, v)_V^{\frac{1}{2}}. \tag{3.16}$$

and W is endowed with the inner product

$$(\varphi, \psi)_W = (\varphi, \psi)_{H_1}.$$

From the Korn's inequality (3.15), it follows that the norms $\|\cdot\|_V$ and $\|\cdot\|_{H_1}$ are equivalent on V; therefore, $(V, \|\cdot\|_V)$ is a Hilbert space. In addition, by the Sobolev trace theorem, (3.15) and (3.16) there exists a constant $k_0 > 0$ which depends only on the domain Ω, Γ_3, and Γ_1 such that

$$\|v\|_{L^2(\Gamma)^d} \le k_0 \|v\|_V \quad \forall v \in V. \tag{3.17}$$

We assume that the elasticity tensor $\mathcal{T}$, the piezoelectric tensor $\mathcal{E}$, the electric permittivity tensor β and the surface electrical conductivity function ψ satisfy the following assumptions (see [14, 19] for example).

(1) The operator $\mathcal{T} : \Omega \times \mathbb{T}^d \to \mathbb{T}^d$, $\mathcal{T} = (\tau_{ijkl})_{ijkl}$ satisfies $\tau_{ijkl} = \tau_{jikl} = \tau_{lkij} \in L^\infty(\Omega)$ and there exists $C_\mathcal{T}$ a positive constant such that

$$\tau_{ijkl}(x)\xi_j\xi_l \geq C_\mathcal{T}\|\xi\|^2 \quad \forall \xi \in \mathbb{T}^d \ \forall x \in \Omega.$$

(2) The piezoelectric tensor $\mathcal{E} : \Omega \times \mathbb{T}^d \to \mathbb{R}^d$, $E = (e_{ijk})_{ikj}$, is bounded in $L^\infty(\Omega)$, i.e., $\exists \Lambda > 0$ such that

$$\|\mathcal{E}\boldsymbol{\sigma} \cdot v\| \leq \Lambda \|\boldsymbol{\sigma}\|\|v\|.$$

(3) The electric permittivity tensor $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $\beta = (\beta_{ij})_{ji}$, satisfies $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$ and there is C_β, a positive constant, such that

$$\beta_{ij}\zeta_i\zeta_j \geq C_\beta\|\zeta\|^2 \quad \forall \zeta \in \mathbb{R}^d.$$

(4) The electrical conductivity of the surface $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ is a bounded function by a positive constant M_ψ, and $x \mapsto \psi(x, u)$ is measurable on Γ_3 for all $u \in \mathbb{R}$ and is zero for all $u \leq 0$.

(5) The function $u \mapsto \psi(x, u)$ is Lipschitz continuous on $\mathbb{R}$ for all $x \in \Gamma_3$: $\exists L_\psi > 0$ such that

$$|\psi(x, u_1) - \psi(x, u_2)| \leq L_\psi |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}.$$

(6) The mechanical forces satisfy $f_0 \in L^2(\Omega)$, $f_2 \in L^2(\Gamma_2)^d$.

(7) The electrical forces satisfy $q_0 \in L^2(\Omega)$, $q_2 \in L^2(\Gamma_b)$.

(8) The potential of the contact surface satisfies $\varphi_f \in L^\infty(\Gamma_3)$.

(9) The frictional map $\mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ is such that
   (a) there is $L_\mu > 0$ such that $|\mu(\cdot, u_1) - \mu(\cdot, u_2)| \leq L_\mu |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}_+$,
   (b) there exists $\mu^* > 0$ such that $\mu(x, u) \leq \mu^*$ for all $u \in \mathbb{R}_+$, a.e. $x \in \Gamma_3$,
   (c) the function $x \mapsto \mu(x, u)$ is Lebesgue measurable on Γ_3 for all $u \in \mathbb{R}_+$.

   For the sake of simplicity, in the remainder of this paper, we will denote $\mu(x, r) = \mu(r)$.

Suppose that the functions u and φ are regular (enough) and satisfy (3.1)–(3.9), then we have

$$\begin{aligned}
&\int_\Omega \mathcal{T}\boldsymbol{\varepsilon}(u)\boldsymbol{\varepsilon}(v) + \mathcal{E}\boldsymbol{\varepsilon}(v)\nabla\varphi \\
&\quad = \int_\Omega f_0 \cdot v + \int_{\Gamma_2} f_2 \cdot v + \int_{\Gamma_3} (\boldsymbol{\sigma}\boldsymbol{n}) \cdot v \quad \forall v \in V, \\
&\int_\Omega \beta\nabla\varphi\nabla\xi - \mathcal{E}\boldsymbol{\varepsilon}(u)\nabla\xi + \int_{\Gamma_3} \psi(u_{\boldsymbol{n}} - g)\Phi_L(\varphi - \varphi_0)\xi \\
&\quad = \int_\Omega q_0 \cdot \xi - \int_{\Gamma_b} q_2 \cdot \xi \quad \forall \xi \in W.
\end{aligned}$$

By employing the Riesz's representation theorem, there exists $f \in V^*$, and $q \in W^*$ such that

$$\langle f, v\rangle_{V^*\times V} = \int_\Omega f_0 \cdot v + \int_{\Gamma_2} f_2 \cdot v \quad \forall v \in V,$$
$$\langle q, \xi\rangle_{W^*\times W} = \int_\Omega q_0 \cdot \xi + \int_{\Gamma_b} q_b \cdot \xi \quad \forall \xi \in W.$$

We define the mappings $j : V \times V \to \mathbb{R}$ by

$$j(u, v) = \int_{\Gamma_3} \mu(\|u_{\boldsymbol{t}}\|)|\sigma^\natural_{\boldsymbol{n}}(u)|\|v_{\boldsymbol{t}}\| \quad \forall v \in V,$$

and $\ell : V \times W \times W :\to \mathbb{R}$ by

$$\ell(u, \varphi, \xi) = \int_{\Gamma_3} \psi(u_{\boldsymbol{n}} - g)\Phi_L(\varphi - \varphi_0)\xi, \quad u \in V,\ \forall\varphi,\ \xi \in W.$$

By the Riesz's representation theorem, there exists $h : V \times W \to W^*$ such that

$$\langle h(u, \varphi), \xi\rangle_{W^*\times W} = \ell(u, \varphi, \xi),$$

and we introduce the operators $A : V \to V^*$, $B : W \to W^*$ and $\theta : V \to W^*$ defined by

$$\langle Au, v\rangle_{V^*\times V} = (\mathcal{T}\boldsymbol{\varepsilon}(u), \boldsymbol{\varepsilon}(v))_{\mathbb{H}},$$
$$\langle B\varphi, \psi\rangle_{W^*\times W} = (\beta\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d},$$
$$\langle \theta v, \xi\rangle_{W^*\times W} = (\mathcal{E}\boldsymbol{\varepsilon}(v), \nabla\xi)_{L^2(\Omega)^d}.$$

From (1) and (3), we deduce that the operators A and B satisfy the conditions (2.3) and (2.5), respectively.

The resulting weak variational formulation is formulated as a system of two coupled quasi-variational inequalities of the second order, where we have omitted the pairing duality index for the simplicity of presentation. The system is as follows.

Problem (PV). Find a displacement field $u \in K_m$ and an electric potential $\varphi \in K_e$ such that

$$\langle Au, v - u\rangle + \langle \theta^*\varphi, v - u\rangle + j(u, v) - j(u, u) \geq \langle f, v - u\rangle \quad \forall v \in K_m, \tag{3.18}$$
$$\langle B\varphi, \xi - \varphi\rangle - \langle \theta u, \xi - \varphi\rangle + \langle h(u, \varphi), \psi - \varphi\rangle \geq \langle q, \xi - \varphi\rangle \quad \forall \xi \in K_e. \tag{3.19}$$

In this application, the smallness condition (2.9) presumes the boundary constants like the measure of the domain's boundary, trace and electric contact constants (M_ψ, c and L_ψ) and the constants corresponding to the material properties. In practice, the physical meaning of the assumption (2.9) is that the mechanical properties of the body are relatively dominating and the properties on the boundary are weaker than in the interior.

To provide the existence and uniqueness result of the problem (PV), we proceed in the same manner as in [19, 33] and in the first part of this paper.

Let us define the product space $X = V \times W$ with elements of the form $x = (u, \varphi)$, endowed with the norm $\|\cdot\|_X^2 = \|\cdot\|_V^2 + \|\cdot\|_W^2$. Let us denote X^* the dual space of X and let us define the operator $\mathbb{A} : X \to X^*$, the function $\mathfrak{J}$, the function $\boldsymbol{\ell}$ defined on $X \times X$, and $\mathfrak{f} \in X^*$ by

$$\left.\begin{aligned} \langle \mathbb{A}x, y\rangle = \langle Au, v\rangle + \langle B\varphi, \xi\rangle + \langle \theta^*\varphi, u\rangle - \langle \theta u, \xi\rangle, \\ \mathfrak{J}(x, y) = j(u, v), \\ \boldsymbol{\ell}(x, y) = \ell(u, \varphi, \xi), \\ \mathfrak{f} = (f, q) \in X^*, \end{aligned}\right\} \tag{3.20}$$

for all $x = (u, \varphi)$ and $y = (v, \xi) \in X$.

We have the following lemma, which is similar to the one in the first part of this paper.

Lemma 3.2. *The couple $x = (u, \varphi) \in K_m \times K_e$ is the solution to the problem (PV) if and only if x is the solution to the following variational problem:*

$$\langle \mathbb{A}x, y - x\rangle + \mathfrak{J}(x, y) - \mathfrak{J}(x, x) + \boldsymbol{\ell}(x, y - x) \geq (\mathfrak{f}, y - x) \quad \forall y = (y, \xi) \in K_m \times K_e. \tag{3.21}$$

Then, the existence and uniqueness of the weak solution to the problem (PV) is established in the following result.

Theorem 3.3. *Under the assumptions* (1)–(9) *and* (2.9)*, the problem (PV) has a unique solution.*

Proof. Let us consider the following problem obtained by fixing the nonlinear term in equation (3.21). Define the following map:

$$\mathcal{F} : z \in K_m \times K_e \mapsto \mathcal{F}z \in K_m \times K_e,$$

and find $\mathcal{F}z \in K_m \times K_e$ such that

$$\begin{aligned} &\langle \mathbb{A}(\mathcal{F}z), y - \mathcal{F}z\rangle + \mathfrak{J}(z, y) - \mathfrak{J}(z, \mathcal{F}z) + \boldsymbol{\ell}(z, y - \mathcal{F}z) \\ &\quad \geq (\mathfrak{f}, y - \mathcal{F}z)_{X^* \times X} \quad \forall y \in K_m \times K_e. \end{aligned} \tag{3.22}$$

Since the operator $\mathbb{A}$ defined in (3.20) is strongly monotone and Lipschitz continuous, and since the mapping $y \mapsto (\mathfrak{f}, y)_{X^* \times X} - \boldsymbol{\ell}(z, y)$, defining the right-hand side in the variational inequality (3.22), is bounded and the map $v \mapsto j(u, v)$ is proper, strictly convex and lower semi-continuous (see [19]) the problem (3.22) has unique solution.

Now, we will prove that the function defined by $z \in K_m \times K_e \mapsto \mathcal{F}z \in K_m \times K_e$ is a contraction. To do so, we select an arbitrary element (z_1, z_2) in $K_m \times K_e$ and let $(\mathcal{F}z_1, \mathcal{F}z_2)$ be its corresponding solution to (3.22). Let us make the choice $y = \mathcal{F}z_2$ in (3.22), for which the solution is $\mathcal{F}z_1$, and $y = \mathcal{F}z_1$ in (3.22) for which the solution is $\mathcal{F}z_2$. The sum of the resulting inequalities yields

$$\langle \mathbb{A}(\mathcal{F}z_1) - \mathbb{A}(\mathcal{F}z_2), (\mathcal{F}z_1) - (\mathcal{F}z_2)\rangle \leq \mathcal{I} + \mathcal{J}, \tag{3.23}$$

where we have put

$$\mathcal{I} = \mathfrak{J}(z_1, \mathcal{F} z_2) + \mathfrak{J}(z_2, \mathcal{F} z_1) - \mathfrak{J}(z_1, \mathcal{F} z_1) - \mathfrak{J}(z_2, \mathcal{F} z_2),$$
$$\mathcal{J} = \boldsymbol{\ell}(z_1, \mathcal{F} z_1 - \mathcal{F} z_2) + \boldsymbol{\ell}(z_2, \mathcal{F} z_2 - \mathcal{F} z_1).$$

The inequality (3.17) implies that there exists a positive constant k^* such that

$$\mathcal{I} \le \mu^* k^* k_0 \|z_1 - z_2\|_X \|\mathcal{F} z_1 - \mathcal{F} z_2\|_X \tag{3.24}$$

and from the definition of $\boldsymbol{\ell}$ and the inequality (3.14), it follows that

$$\|\mathcal{J}\| \le (M_\psi k_1^2 + cL_\psi k_0 k_1)\|z_1 - z_2\|_X \|\mathcal{F} z_1 - \mathcal{F} z_2\|_X, \tag{3.25}$$

where M_ψ is the bound of ψ and L_ψ is the Lipschitz constant of ψ. Let us put $k = \mu^* k_0 k^*$ and $\kappa = M_\psi k_1^2 + cL_\psi k_0 k_1$, by (2.9) we have since $\Lambda > 0$

$$\kappa + k < 2\Lambda + k + \kappa < \min(m_A, m_B).$$

Then, the existence and uniqueness of the solution to (PV) is guaranteed by applying Theorem 2.2. ■

3.1. Iterative schemes and optimization

This section is devoted to the numerical study of the above problem. The nonlinearly coupled problem (3.18)–(3.19) is completely hard to solve numerically in a direct way. Following the same idea as in the above section, we first solve successively the subproblem in u and then, we use the computed u to solve the sub-problem in φ, in block Gauss–Seidel mode. To this end, let us introduce the iterative schemes.

3.1.1. Iterative schemes. The aim of this paragraph is to fix the nonlinearity in the second variational inequality (3.19) resulting from the operator h. The first iterative scheme is as follows. Starting with an initial data $(u_0, \varphi_0) \in K_m \times K_e$, we compute a sequence $(u_{i+1}, \varphi_{i+1}) \in K_m \times K_e$ by the following problem.

Problem (PV$_i$). Find $(u_{i+1}, \varphi_{i+1}) \in K_m \times K_e$ so that

$$\langle Au_{i+1}, v - u_{i+1}\rangle + \langle \theta_* \varphi_{i+1}, v - u_{i+1}\rangle + j(u_i, v) - j(u_n, u_{i+1})$$
$$\ge \langle f, v - u_{i+1}\rangle \quad \forall v \in K_m, \tag{3.26}$$
$$\langle B\varphi_{i+1}, \xi - \varphi_{i+1}\rangle - \langle \theta u_{i+1}, \xi - \varphi_{i+1}\rangle + \langle h(u_i, \varphi_i), \psi - \varphi_{i+1}\rangle$$
$$\ge \langle q, \xi - \varphi_{i+1}\rangle \quad \forall \xi \in K_e, \tag{3.27}$$

where we have omitted the index of duality for the sake of simplicity.

3.1.2. The constrained minimization problems. Now, in this part, we aim to decouple the mechanical field from the electric potential. Afterwards, we decouple the contact and friction from the elasticity and set the optimization problems. Let us consider at the following iterative scheme.

Starting with initial data $(u_{i,j-1}, \varphi_{i,j-1})$, compute for $j \geq 0$, $u_{i+1,j}$ and $\varphi_{i+1,j}$ by solving

$$\langle Au_{i+1,j}, v - u_{i+1,j}\rangle + \langle \theta^* \varphi_{i+1,j-1}, v - u_{i+1,j}\rangle + j(u_{i,j-1}, v) - j(u_{i,j-1}, u_{i+1,j}) \\ \langle f, v - u_{i+1,j}\rangle \quad \forall v \in K_m, \tag{3.28}$$

$$\langle B\varphi_{i+1,j}, \xi - \varphi_{i+1,j}\rangle - \langle \theta u_{i+1,j}, \xi - \varphi_{i+1,j}\rangle + \langle h(u_{i,j-1}, \varphi_{i,j-1}), \xi - \varphi_{i+1,j}\rangle \\ \geq \langle q, \xi - \varphi_{i+1,j}\rangle \quad \forall \xi \in K_e. \tag{3.29}$$

The convergence of the problem (3.28)–(3.29) is proved in Theorem 2.4.

As the mechanical field and the potential electric are decoupled, we compute the solutions by introducing auxiliary unknowns, e.g., we will use a constrained minimization problem to decouple the linear elasticity from the contact and friction. Once this is achieved, we compute the electric potential by a second constrained minimization problem. This process will be explained in the remainder of this article.

We first start with the following notation $\langle \boldsymbol{f}^{j-1}, v\rangle = \langle f, v\rangle - \langle \theta^* \boldsymbol{\varphi}_{i+1,j-1}, v\rangle$. The operator A is coercive and symmetric thus the problem (3.28) is equivalent to the following constrained minimization problem:

$$(Pm) : \begin{cases} \text{Find } \boldsymbol{u}_j \in K_m \text{ such that} \\ J(\boldsymbol{u}_j) + j(\boldsymbol{u}_j) \leq J(\boldsymbol{v}) + j(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in K_m, \end{cases}$$

where $J(\boldsymbol{v}) = \frac{1}{2}\langle A\boldsymbol{v}, \boldsymbol{v}\rangle - \langle \boldsymbol{f}^{j-1}, v\rangle$ and $j(\boldsymbol{u}_j) = j(\boldsymbol{u}_{j-1}, \boldsymbol{u}_j)$. The problem (Pm) admits a unique solution and this is due to the fact that $J(\cdot) + j(\cdot)$ is coercive and strictly convex and $j(\cdot)$ is lower semicontinuous [12].

Since the solution lives in the admissible displacement K_m, to state a constrained minimization problem, equivalent to the problem (Pm), we introduce several auxiliary unknowns. For the sake of clarity, we drop the subscript j and introduce the set defined by

$$\mathcal{A}_d = \{q \in L^2(\Gamma_3), q - g \leq 0 \text{ on } \Gamma_3\}.$$

The above constrained minimization problem is then reformulated over $\mathcal{A}_d$ as follows, find $(\boldsymbol{u}, p_c, p_f)$ such that

$$J(\boldsymbol{u}) + j(p_f) \leq J(\boldsymbol{v}) + j(q_f) \quad \forall (\boldsymbol{v}, q_f) \in V \times L^2(\Gamma_3), \tag{3.30}$$

$$u_{\boldsymbol{n}} - p_c = 0 \quad \text{on } \Gamma_3, \tag{3.31}$$

$$u_{\boldsymbol{t}} - p_f = 0 \quad \text{on } \Gamma_3, \tag{3.32}$$

and the augmented Lagrangian operator (see [16, 17, 19]) is given by

$$\mathcal{L}_r(v, q; \mu) = J(v) + j(q_f) + \langle \mu_c, v_{\boldsymbol{n}} - q_c\rangle_{\Gamma_3} \\ + \langle \mu_f, v_{\boldsymbol{t}} - q_f\rangle_{\Gamma_3} + \frac{r}{2}\|v_{\boldsymbol{n}} - q_c\|^2_{\Gamma_3} + \frac{r}{2}\|v_{\boldsymbol{t}} - q_f\|^2_{\Gamma_3}. \tag{3.33}$$

The constrained minimization problem (3.30)–(3.32) is then equivalent to the following saddle point problem.

Find $(\boldsymbol{u}, p; \gamma) \in V \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2$ such that $\forall(\boldsymbol{v}, q; \mu) \in V \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2$:

$$\mathcal{L}_r(\boldsymbol{u}, p; \mu) \le \mathcal{L}_r(\boldsymbol{u}, p; \gamma) \le \mathcal{L}_r(\boldsymbol{v}, q; \gamma). \tag{3.34}$$

The components of a saddle point of the problem (3.34) are computed separately by the alternating direction method of multipliers (ADMM, see [8,24] for more details on the introduction and the convergence of this scheme) and it is stated as follows. Starting with $(p^{j-1}; \gamma^{j-1})$,

$$\boldsymbol{u}^j \in \operatorname{argmin}_{\boldsymbol{u}} \mathcal{L}_r(\boldsymbol{u}, p^{j-1}; \gamma^{j-1}), \tag{3.35}$$

$$p^j \in \operatorname{argmin}_p \mathcal{L}_r(\boldsymbol{u}^j, p; \gamma^{j-1}), \tag{3.36}$$

$$\gamma_c^j = \gamma_c^{j-1} + r(\boldsymbol{u}_{\boldsymbol{n}}^j - p_c^j), \tag{3.37}$$

$$\gamma_f^j = \gamma_f^{j-1} + r(\boldsymbol{u}_{\boldsymbol{t}}^j - p_f^j). \tag{3.38}$$

Finally, the solutions to the problems are computed employing Lagrange and Fenchel dualities (see [12]) and are summarized in Algorithm 2.

Once $\boldsymbol{u}$ is computed, we are able to compute the electric potential by (3.29). Let us define

$$\xi \in K_e \mapsto \hbar(\xi) := \langle h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}), \xi\rangle$$

and

$$\langle \boldsymbol{q}, \xi\rangle = \langle q, \xi\rangle + \langle \theta \boldsymbol{u}, \xi\rangle$$

for given $(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1})$. Since $(\beta\cdot,\cdot)_{L^2(\Omega)}$ is coercive and symmetric, the variational (3.29) is equivalent to the following constrained minimization problem.

Problem (PVp). Find $\boldsymbol{\varphi} \in K_e$ such that

$$\mathfrak{J}(\boldsymbol{\varphi}) + \hbar(\boldsymbol{\varphi}) \le \mathfrak{J}(\psi) + \hbar(\psi) \quad \forall \psi \in K_e, \tag{3.44}$$

where

$$\mathfrak{J}(\xi) = \frac{1}{2}\langle B\xi, \xi\rangle - \langle \boldsymbol{q}, \xi\rangle.$$

To handle the difficulty induced by the constraint in K_e, we introduce the set of admissible potentials defined by

$$\Xi = \left\{\pi \in L^2(\Gamma_3), \pi \le \varphi_f \text{ on } \Gamma_3\right\}.$$

The constrained minimization problem (PVp) is equivalent to the following problem.

Find $\boldsymbol{\varphi} \in W$ such that

$$\mathfrak{J}(\boldsymbol{\varphi}) + \hbar(\pi) \le \mathfrak{J}(\psi) + \hbar(\upsilon) \quad \forall \upsilon \in W, \tag{3.45}$$

$$\varphi - \pi = 0 \quad \text{on } \Gamma_3. \tag{3.46}$$

Algorithm 2 Solution to the saddle point problem (3.34).

1: **Initialization** $j = 0 : (p^0; \gamma^0)$

2: **Iterations**

3: **for** $j \geq 1$ **do**

4: Compute $\boldsymbol{u}^j$ by the Euler–Lagrange equation, applied to $\boldsymbol{u} \mapsto \mathcal{L}_r(\boldsymbol{u}, p; \gamma)$, given by

$$\begin{aligned} &\langle A\boldsymbol{u}^j, \boldsymbol{v}\rangle + r\langle \boldsymbol{u}_t^j, \boldsymbol{v}_t\rangle_{\Gamma_3} + r\langle \boldsymbol{u}_n^j, \boldsymbol{v}_n\rangle_{\Gamma_3} \\ &\quad = +\langle rp_c^{j-1} - \gamma_c, \boldsymbol{v}_n\rangle_{\Gamma_3} + \langle rp_f^{j-1} - \gamma_f, \boldsymbol{v}_t\rangle_{\Gamma_3} \\ &\quad\quad - \langle \boldsymbol{f}^{j-1}, \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in V \end{aligned} \tag{3.39}$$

5: Compute $p^j = (p_c^j, p_f^j)$ by

$$p_c^{i+1} = \boldsymbol{u}_n^{i+1} + \frac{1}{r}\left[\gamma_c^i - (\gamma_c^i + r(\boldsymbol{u}_n^{i+1} - g))^+\right], \tag{3.40}$$

$$p_f^j = \begin{cases} \dfrac{|\gamma_f^{j-1} + r\boldsymbol{u}_t^j| - S}{r|\gamma_f^{j-1} + r\boldsymbol{u}_t^j|}(\gamma_f^{j-1} + r\boldsymbol{u}_t^j) & \text{if } |\gamma_f^{j-1} + r\boldsymbol{u}_t^j| > S, \\ 0 & \text{if } |\gamma_f^{j-1} + r\boldsymbol{u}_t^j| \leq S \end{cases} \tag{3.41}$$

6: Update γ_c^j and γ_f^j by

$$\gamma_c^j = \gamma_c^{j-1} + r(\boldsymbol{u}_n^j - p_c^j), \tag{3.42}$$

$$\gamma_f^j = \gamma_f^{j-1} + r(\boldsymbol{u}_t^j - p_f^j) \tag{3.43}$$

7: **end for**

The Lagrangian formulation associated to the constrained minimization problem inequality (3.45)–(3.46) is given by

$$\mathcal{L}_r(\boldsymbol{\varphi}, \pi; \mu) = \mathfrak{J}(\varphi) + \hbar(\pi) + \langle \mu, \boldsymbol{\varphi} - \pi\rangle_{\Gamma_3} + \frac{r}{2}\|\boldsymbol{\varphi} - \pi\|^2_{\Gamma_3}, \tag{3.47}$$

and then the constrained minimization (3.45)–(3.46) problem is equivalent to the saddle point problem

$$\mathcal{L}_r(\boldsymbol{\varphi}, \pi; \eta) \leq \mathcal{L}_r(\boldsymbol{\varphi}, \pi; \mu) \leq \mathcal{L}_r(\psi, \upsilon; \mu) \quad \forall(\psi, \upsilon; \eta) \in W \times L^2(\Gamma_3)^2. \tag{3.48}$$

The saddle point can be computed separately by alternating the solutions, employing ADMM, as follows. Starting with initial guess $(\pi^{j-1}; \mu^{j-1})$, compute

$$\boldsymbol{\varphi}^j \in \operatorname{argmin}_\psi \mathcal{L}_r(\psi, \pi^{j-1}; \mu^{j-1}), \tag{3.49}$$

$$\pi^j \in \operatorname{argmin}_\upsilon \mathcal{L}_r(\boldsymbol{\varphi}^j, \upsilon; \mu^{j-1}), \tag{3.50}$$

$$\mu^j = \mu^{j-1} + r(\pi^j - \boldsymbol{\varphi}^j). \tag{3.51}$$

In order to calculate the solution to the subproblem (3.49), let us consider the following map $\xi \mapsto \mathcal{L}_r(\xi, \pi; \mu) = \mathfrak{J}(\xi) + \frac{r}{2}\|\xi\|^2_{\Gamma_3} - \langle \mu + r\kappa, \xi\rangle_{\Gamma_3}$. The subproblem (3.49) is then equivalent to the Euler–Lagrange equation given by the following:

$$\text{Find } \boldsymbol{\varphi} \in W \text{ such that } \langle B\boldsymbol{\varphi}, \xi\rangle + r\langle \boldsymbol{\varphi}, \xi\rangle_{\Gamma_3} - \langle \boldsymbol{q}, \xi\rangle - \langle \mu - r\pi, \xi\rangle_{\Gamma_3} = 0 \quad \forall \xi \in W,$$

which is equivalent to the following system:

$$\langle B\boldsymbol{\varphi}, \xi\rangle + r\langle \boldsymbol{\varphi}, \xi\rangle_{\Gamma_3} = \langle \boldsymbol{q}, \xi\rangle + \langle \mu - r\pi, \xi\rangle_{\Gamma_3}.$$

To get the solution to the subproblem (3.50), we consider the Gâteaux differentiable mapping $\pi \mapsto \mathcal{L}_r(\boldsymbol{\varphi}, \pi; \mu) = \hbar(\pi) + \frac{r}{2}\|\pi\|^2_{\Gamma_3} - \langle \mu + r\boldsymbol{\varphi}, \pi\rangle_{\Gamma_3}$. The KKT conditions (Lagrange duality) lead to the following equations:

$$r\langle \pi, \varepsilon\rangle_{\Gamma_3} + \langle \boldsymbol{\phi}, \varepsilon\rangle_{\Gamma_3} = \langle \mu + r\boldsymbol{\varphi}, \varepsilon\rangle_{\Gamma_3} - \langle h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}), \varepsilon\rangle_{\Gamma_3} \quad \forall \varepsilon \in L^2(\Gamma_3), \tag{3.52}$$

$$\langle \boldsymbol{\phi}, \pi - \varphi_f\rangle_{\Gamma_3} = 0, \quad \boldsymbol{\phi} \geq 0, \tag{3.53}$$

thus,

$$\pi = \frac{1}{r}(\mu + r\varphi - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}) - \boldsymbol{\phi}), \tag{3.54}$$

we substitute this expression in (3.53), we get

$$\langle \boldsymbol{\phi}, \frac{1}{r}(\mu + r\varphi - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}) - \boldsymbol{\phi}) - \boldsymbol{\varphi}_f\rangle_{\Gamma_3} = 0. \tag{3.55}$$

If $\boldsymbol{\phi} > 0$ then

$$\mu + r\varphi - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}) - \boldsymbol{\phi} - r\boldsymbol{\varphi}_f = 0,$$

hence,

$$\boldsymbol{\phi} = \mu + r(\boldsymbol{\varphi} - \boldsymbol{\varphi}_f) - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}),$$

that is,

$$\boldsymbol{\phi} = (\mu + r(\boldsymbol{\varphi} - \boldsymbol{\varphi}_f) - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}))^+, \tag{3.56}$$

where $x^+ = \max(0, x)$. From (3.52) and (3.56), we obtain

$$\pi = \boldsymbol{\varphi} + \frac{1}{r}\big(\mu - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}) - (\mu + r(\boldsymbol{\varphi} - \boldsymbol{\varphi}_f) - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}))^+\big). \tag{3.57}$$

We summarize the solutions to the subproblems (3.49)–(3.51) in Algorithm 3.

Algorithm 3 Solution to the saddle point problem (3.48).

1: **Initialization** $j = 0 : (\pi_0, \mu_0)$
2: **Iterations**
3: **for** $j \geq 1$ **do**
4: Compute $\boldsymbol{\varphi}^j$ by

$$\langle B\boldsymbol{\varphi}^j, \xi\rangle + r\langle \boldsymbol{\varphi}^j, \xi\rangle_{\Gamma_3} = \langle \boldsymbol{q}, \xi\rangle + \langle \mu^{j-1} - r\kappa^{j-1}, \xi\rangle_{\Gamma_3} \quad \forall \xi \in W. \tag{3.58}$$

5: Compute π^j by

$$\pi^j = \boldsymbol{\varphi}^j + \frac{1}{r}\Big(\mu^{j-1} - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}) - (\mu^{j-1} + r(\boldsymbol{\varphi}^j - \boldsymbol{\varphi}_f) - h(\boldsymbol{u}^{j-1}, \boldsymbol{\varphi}^{j-1}))^+\Big) \tag{3.59}$$

6: Update μ^j by

$$\mu^j = \mu^{j-1} + r(\boldsymbol{\varphi}^j - \pi^j) \tag{3.60}$$

7: **end for**

3.2. Numerical Examples

In practice, to compute the solution to the problem with nonlocal Coulomb friction, a fixed point strategy is used. This process gives rise to a sequence of problems with Tresca friction. Once the solution $(\boldsymbol{u}^j, \boldsymbol{\varphi}^j)$ is obtained with Algorithms 2–3, we update the slip bound $S^{j+1} = \mu(\|\boldsymbol{u}_t^j\|)|\sigma_{\boldsymbol{n}}^\natural(\boldsymbol{u}^j)|$ to get the new solution, this corresponds to the following fixed-point Algorithm 4 which is stopped if the relative error on S^j becomes “small”.

Algorithm 4 FP for problem (PVI).

Initialization $j = 0$. S^0 given.

Iteration $j \geq 1$. Compute successively $(\boldsymbol{u}^j, \boldsymbol{\varphi}^j)$ and S^{j+1} as follows:

1. Compute $(\boldsymbol{u}^j, \boldsymbol{\varphi}^j)$ with Algorithms 2–3.
2. Update the friction functional with $S^{j+1} = \mu(\|\boldsymbol{u}_t^j\|)|\sigma_{\boldsymbol{n}}^\natural(\boldsymbol{u}^j)|$.

We have implemented the algorithms described previously in Matlab, using piecewise linear finite element and vectorized codes [25], on a computer equipped with running Windows 10 with 2.4 GHz clock frequency and 6 GB RAM.

The chosen stopping criterion for the algorithms is as follows:

$$\frac{\|\boldsymbol{x}^j - \boldsymbol{x}^{j-1}\|}{\|\boldsymbol{x}^{j-1}\|} < 10^{-6}.$$

3.3. Updating the slip bound and example

Let us recall that each iterative step leads to a contact problem with given Tresca friction law. The iterative Step 2 in Algorithm 4 updates the slip bound using the data from the previous iteration. Since this represents the crucial step in the algorithm, we need further details. In the right-hand side in Step 2 of Algorithm 4, we recall that

$$(\sigma_{\boldsymbol{n}}^{\natural})(x) = \omega_\rho * \sigma_{\boldsymbol{n}}(x) = \int_{\Gamma_3} \omega_\rho(|x-y|)\sigma_{\boldsymbol{n}}(y)dy.$$

Following [23], the constant c in (3.11) is chosen such that

$$c\int_{-\rho}^{\rho} \exp\left(\frac{\rho^2}{x^2-\rho^2}\right)dx = 1, \tag{3.61}$$

making the change of variable $x = \rho z$, we obtain

$$c = \frac{1}{\rho\int_{-1}^{1}\exp(\frac{1}{z^2-1})dz},$$

and $A = \int_{-1}^{1}\exp(\frac{1}{z^2-1})dz$ needs to be evaluated. Using a Gaussian integration with 4 integration points provides the following value $A = 0.4437$.

The friction function $\mu(\cdot)$ is given by

$$\mu(r) = 0.2 + \frac{2}{tr+2}, \tag{3.62}$$

where t is a given parameter.

As a numerical example, we are dealing with two dimensional ($d = 2$) numerical realization of the problem (3.26)–(3.27) with nonlocal Coulomb friction law. We consider $\Omega = (0,2)\times(0,1)$ with $\Gamma_1 = \{0\}\times[0,1]$, $\Gamma_3 = [0,2]\times\{0\}$ and $\Gamma_2 = [0,2]\times\{1\}$. The material constants are the following.

(1) Elasticity: $E = 58.7102\, GPA$, $\boldsymbol{n} = 0.3912$.

(2) Piezoelectric (C/m^2): $e_{22} = -5.4$, $e_{33} = 15.8$ and $e_{24} = 12.3$.

(3) Dielectric (nF/m): $\beta_{22} = 8.11$, $\beta_{33} = 7.35$.

On Γ_2, non-homogeneous Neumann boundary condition is prescribed $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = -2y$. On Γ_b we consider homogeneous Neumann condition, i.e., $\boldsymbol{D}\boldsymbol{n} = 0$. For the normalized gap between the foundation and Γ_3, we take $g(x) = 0.01$. In addition, we consider a uniform mesh with mesh size $h = 1/32$ (i.e., Figure 1). We choose $r = 600$ (the penalty parameter) for Algorithm 2 and $r = 10^{-3}$ for Algorithm 3. Also, we take $t = 3\times10^4$ in (3.62).

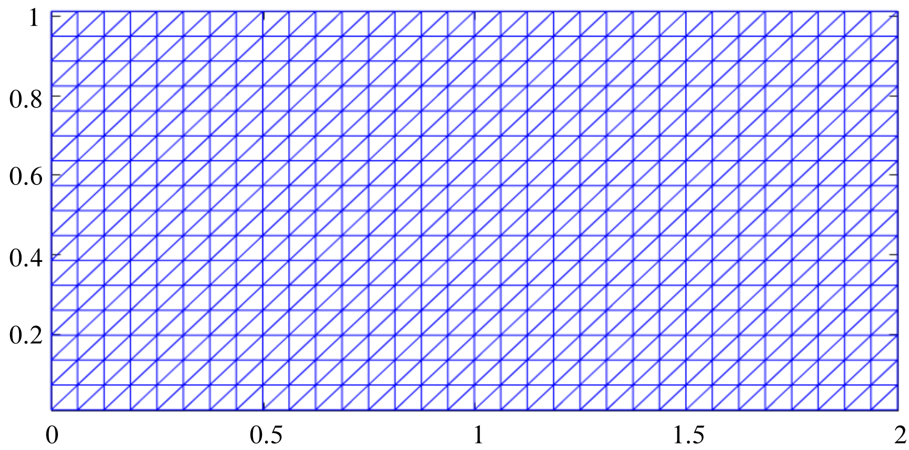


Figure 1. Simple mesh for $\Omega = (0, 2) \times (0, 1)$ with mesh size $h = 1/32$.

When loaded, the deformed configuration is shown in Figure 2. The outside electric potential is $\varphi_f = 1$ and the contour plot of electric potential is visualized in Figure 3. It is clear that there is no outflow of charges, i.e., the potential vanishes on the contact zone. In order to identify the "stick" and "slide" on the contact zone, we visualize the Lagrange multipliers (normal and tangential constraints) and the slip bound of nonlocal Coulomb friction in Figure 4. In the sub-figures of Figure 5, we show the electric potential contour

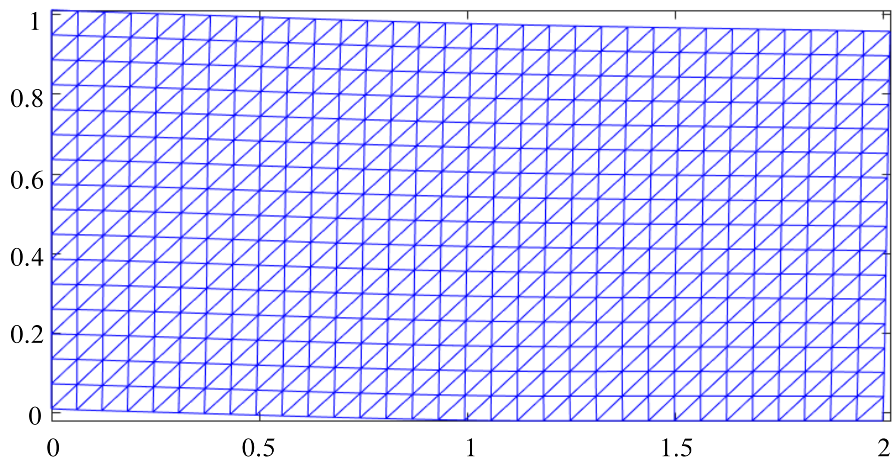


Figure 2. Deformed configuration.

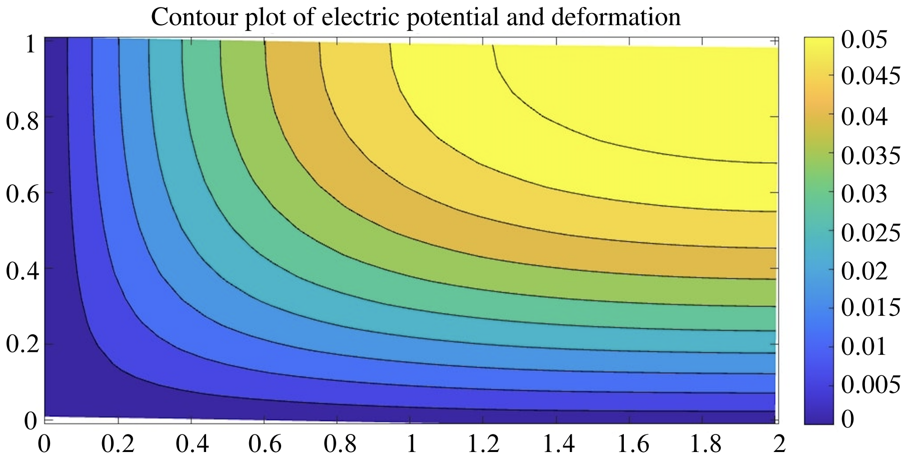


Figure 3. Contour plot of the distribution of electric potential inside the deformed domain.

plot with several choices of φ_f. We can see the value of φ_f for which there may be an outflow between the body and the conductive foundation.

The number of iterations of Algorithms 2, 3 and 4 versus mesh size is reported in Table 1. The different mesh sizes are $1/16$, $1/32$, $1/64$, $1/128$, and $1/256$ which correspond, respectively, to 153, 561, 2145, 8385, and 33153, numbers of nodes. The symbol $\aleph$ designs that the number of iterations is up to 10^3, which means that the smallness condition is not satisfied for mesh of big size. We remark that the number of iterations is

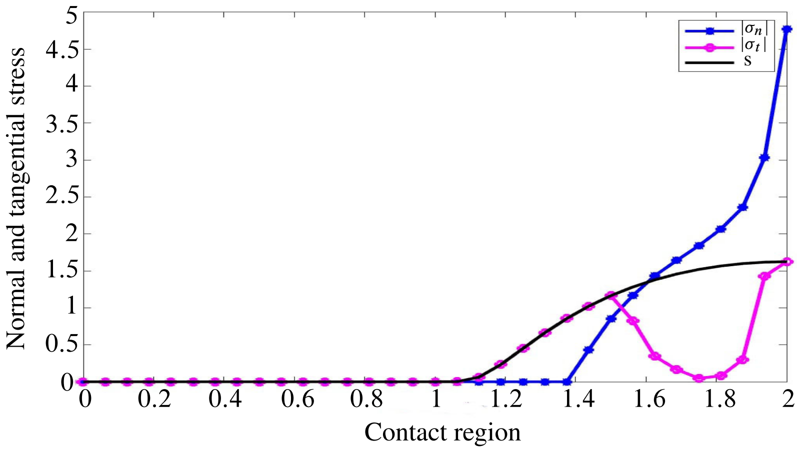
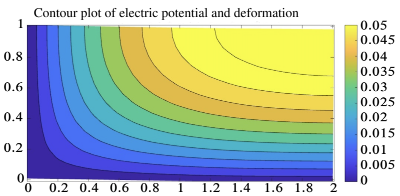
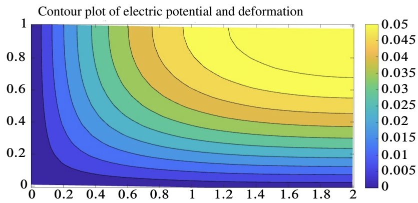


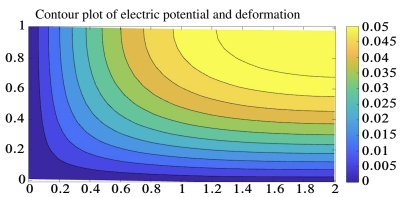
Figure 4. Elastic Lagrange multipliers and slip bound of nonlocal Coulomb friction.



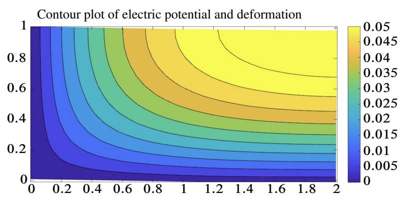
(a) Electric potential contour plot with $\varphi_f = 0.1$.



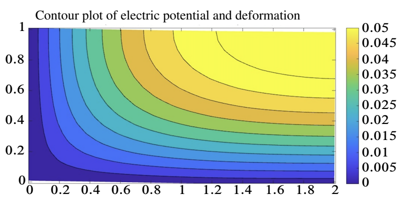
(b) Electric potential contour plot with $\varphi_f = 0.01$.



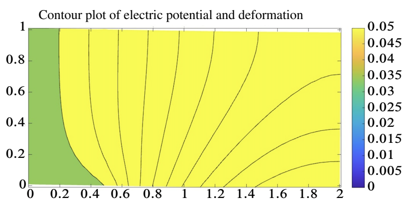
(c) Electric potential contour plot with $\varphi_f = 0.001$.



(d) Electric potential contour plot with $\varphi_f = 0.0005$.



(e) Electric potential contour plot with $\varphi_f = 0.0002$.



(f) Electric potential contour plot with $\varphi_f = 0.0001$.

Figure 5. Visualization of the contour plot of the electric potential with different choices of φ_f.

independent of mesh size and this is because all of the methods and algorithms are constructed in infinite-dimensional spaces.

Conclusion

We proved the existence and uniqueness of the solution to coupled variational inequalities. This kind of variational inequalities can be applied to contact problem in electroelasticity and can be developed to the thermoelctroelasticity. Other future work could be the development of this framework for two coupled nonlinear variational inequalities and hemi-variational inequalities modeling an electroelastic material with a locking effect.

Mesh size	1/16	1/32	1/64	1/128	1/256
Algorithms 2–3 iterations	$\aleph$	36	36	36	36
Algorithm 4 iterations	4	3	3	4	4
CPU (in Sec.)	0.4601	0.0600	0.8620	5.7591	56.4617

Table 1. Total number of iterations in Algorithms 2, 3, and 4.

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Conflicts of Interest

All authors declare that they have no conflicts of interest.

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