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A direct method of moving planes for logarithmic Schrödinger operator

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Abstract. In this paper, we study the radial symmetry and monotonicity of nonnegative solutions to nonlinear equations involving the logarithmic Schrödinger operator $(\mathcal{I}-\Delta)^{\log}$ corresponding to the logarithmic symbol $\log(1+|\xi|^2)$, which is a singular integral operator given by

$$(\mathcal{I}-\Delta)^{\log}u(x) = c_N \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^N} \kappa(|x-y|)dy,$$

where $c_N = \pi^{-\frac{N}{2}}$, $\kappa(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} \mathcal{K}_{\frac{N}{2}}(r)$ and $\mathcal{K}_\nu$ is the modified Bessel function of the second kind with index ν. The proof hinges on a direct method of moving planes for the logarithmic Schrödinger operator.

1. Introduction

The study of Schrödinger equations received a great deal of attention from researchers in the past decades because of its vast applications in several areas of mathematics and mathematical physics. In particular, Schrödinger equations arise in quantum field theory and in the Hartree–Fock theory (see [1,20,21,23]). Recently, there is a surge of interest to investigate integrodifferential operators of order close to zero and associated linear and nonlinear integrodifferential equations (see [5,6,16,18,19,22]). In particular, the logarithmic Laplacian and the logarithmic Schrödinger operator are two interesting examples of such a class of operators. The logarithmic Laplacian was first introduced by Chen and Weth in [5] as a limit of fractional Laplacian (see also [3,4] for the spectral properties of the logarithmic Laplacian). The logarithmic Schrödinger operator $(\mathcal{I}-\Delta)^{\log}$ (see [10]) and the logarithmic Laplacian L_Δ (see [2,5,11,12,24]) have the similar behavior locally concerning to the singularity of kernels but the logarithmic Schrödinger operator eliminates the integrability problem of the logarithmic Laplacian at infinity. To define the logarithmic Schrödinger operator, let us begin with the following observation:

$$\lim_{s\to 0^+} (\mathcal{I}-\Delta)^s u(x) = u(x) \quad \text{for } u \in C^2(\mathbb{R}^N), \tag{1.1}$$

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where, for $s \in (0,1)$, the operator $(\mathcal{I}-\Delta)^s$ stands for the relativistic Schrödinger operator, for sufficiently regular function $u : \mathbb{R}^N \to \mathbb{R}$, which can be represented via hypersingular integral (1.1) (see [8]),

$$(\mathcal{I}-\Delta)^s u(x) = u(x) + c_{N,s} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(0)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \varpi_s(|x-y|)dy, \tag{1.2}$$

where $c_{N,s} = \frac{\pi^{-\frac{N}{2}}4^s}{\Gamma(-s)}$ is a normalization constant and the function ϖ_s is given by

$$\varpi_s(r) = 2^{1-\frac{N+2s}{2}} r^{\frac{N+2s}{2}} \mathcal{K}_{\frac{N+2s}{2}}(r) = \int_0^{+\infty} t^{-1+\frac{N+2s}{2}} e^{-t-\frac{r^2}{4t}} dt. \tag{1.3}$$

Furthermore, if $u \in C^2(\mathbb{R}^N)$, then $(\mathcal{I}-\Delta)^s u(x)$ is well defined by (1.2) for every $x \in \mathbb{R}^N$. Here, the function $\mathcal{K}_\nu$ is the modified Bessel function of the second kind with index $\nu > 0$, and it is given by

$$\mathcal{K}_\nu(r) = \frac{(\frac{\pi}{2})^{\frac{1}{2}} r^\nu e^{-r}}{\Gamma(\frac{2\nu+1}{2})} \int_0^\infty \left(1+\frac{t}{2}\right)^{\nu-\frac{1}{2}} e^{-rt} t^{\nu-\frac{1}{2}} dt$$

for more properties of $\mathcal{K}_\nu$, see, e.g., [7, 9, 10, 14, 15] and references therein.

It is well known that $\mathcal{K}_\nu$ is a real and positive function satisfying

$$\mathcal{K}_\nu'(r) = -\frac{\nu}{r}\mathcal{K}_\nu(r) - \mathcal{K}_{\nu-1}(r) < 0 \tag{1.4}$$

for all $r > 0$, $\mathcal{K}_\nu = \mathcal{K}_{-\nu}$ for $\nu > 0$. Furthermore, for $\nu > 0$ (see [9, 15])

$$\mathcal{K}_\nu(r) \sim \begin{cases} \dfrac{\Gamma(\nu)}{2}\left(\dfrac{r}{2}\right)^\nu, & r \to 0, \\ \dfrac{\sqrt{\pi}}{\sqrt{2}} r^{-\frac{1}{2}} e^{-r}, & r \to \infty. \end{cases} \tag{1.5}$$

It follows from (1.1) that one may expect a Taylor expansion with respect to parameter s of the operator $(\mathcal{I}-\Delta)^s$ near zero for $u \in C^2(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ as follows:

$$(\mathcal{I}-\Delta)^s u(x) = u(x) + s(\mathcal{I}-\Delta)^{\log} u(x) + o(s) \quad \text{as } s \to 0^+. \tag{1.6}$$

The logarithmic Schrödinger operator $(\mathcal{I}-\Delta)^{\log}$ appears as the first-order term in the above expansion.

In this paper, we study the integrodifferential operator $(\mathcal{I}-\Delta)^{\log}$ corresponding to the logarithmic symbol $\log(1+|\xi|^2)$, which is a singular integral operator given by

$$(\mathcal{I}-\Delta)^{\log} u(x) = c_N \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^N} \kappa(|x-y|)dy, \tag{1.7}$$

where $c_N = \pi^{-\frac{N}{2}}\Gamma(\frac{N}{2})$, P.V. stands for the Cauchy principal value of the integral, $\kappa(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} \mathcal{K}_{\frac{N}{2}}(r)$ and $\mathcal{K}_\nu$ is the modified Bessel function of second kind with index ν. One can also easily deduce from (1.4) that $\kappa'(r) < 0$ for $r > 0$. Using the expression (1.7), one can define $(\mathcal{I} - \Delta)^{\log}$ for a quite large class of functions u. To illustrate this, define the space $\mathcal{L}_0(\mathbb{R}^N)$ as the space of locally integrable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\|u\|_{\mathcal{L}_0(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|e^{-|x|}}{(1 + |x|)^{\frac{N+1}{2}}} dx < +\infty.$$

Then, it was proved by [10, Proposition 2.1] that for $u \in \mathcal{L}_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, which is also Dini continuous at some $x \in \mathbb{R}^N$, the quantity $[(\mathcal{I} - \Delta)^{\log} u](x)$ is well defined by the formula (1.7). Let us recall the definition of Dini continuity. Let U be a measurable subset of $\mathbb{R}^N$ and let $u : U \to \mathbb{R}$ be a measurable function. The modulus of continuity $\Psi_{u,x,U} : (0, +\infty) \to [0, +\infty)$ of u at a point $x \in U$ is defined by

$$\Psi_{u,x,U}(r) := \sup_{y \in U, |x-y| \leq r} |u(x) - u(y)|.$$

We call the function u Dini continuous at x if

$$\int_0^1 \frac{\Psi_{u,x,U}(r)}{r} dr < \infty.$$

Using the generalized direct method of moving planes, in this note, we obtain the radial symmetry and monotonicity of nonnegative solutions for the nonlinear equations involving the logarithmic Schrödinger operator (see Theorem 1.1), namely, we consider the nonlinear Schrödinger equation

$$(\mathcal{I} - \Delta)^{\log} u(x) + m u(x) = u^p(x), \quad x \in \mathbb{R}^N, \tag{1.8}$$

with $m > 0$ and $u(x) \geq 0$ for all $x \in \mathbb{R}^N$.

The following results present symmetry and monotonicity properties of Schrödinger equation (1.8).

Theorem 1.1. *Let $u \in \mathcal{L}_0(\mathbb{R}^N)$ be a nonnegative Dini continuous solution of* (1.8) *with $m > 0$ and $1 < p < \infty$. If*

$$\lim_{|x| \to \infty} u(x) = a < \left(\frac{m}{p}\right)^{\frac{1}{p-1}}, \tag{1.9}$$

then u must be radially symmetric and monotone decreasing about some point in $\mathbb{R}^N$.

Remark 1.2. The condition (1.9) in Theorem 1.1 is necessary for applying the method of moving planes using the decay at infinity principle (Theorem 2.3).

The paper is organized as follows: in Section 2, we prove some results for the logarithmic Schrödinger operator. By the direct method of moving planes, we obtain the symmetry and monotonicity of nonnegative solutions for the nonlinear equations involving logarithmic Schrödinger operator in Section 3.

2. Key ingredients for the method of moving planes

This section is devoted to developing basic and key results needed to apply the method of moving planes for establishing the proof of our main result in the next section. We first present some basic notation and nomenclatures which will be beneficial for the rest of the paper.

Choose an arbitrary direction, say, the x_1-direction. For arbitrary $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^N \mid x_1 = \lambda\}$$

be the moving plane, and let

$$\Sigma_\lambda = \{x \in \mathbb{R}^N \mid x_1 < \lambda\}$$

be the region to the left of the plane T_λ, and let

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_N)$$

be the reflection of x about the plane T_λ.

By denoting $u(x^\lambda) := u_\lambda(x)$, we define

$$\omega_\lambda(x) := u_\lambda(x) - u(x), \quad x \in \Sigma_\lambda,$$

to compare the values of $u(x)$ and $u_\lambda(x)$.

The following results on the strong maximum principle for the operator $(\mathcal{I} - \Delta)^{\log}$ can be deduced from [17, Theorem 1.1] (see also [13] and [10, Theorem 6.1]).

Lemma 2.1 (Strong maximum principle). *Let $\Omega \subset \mathbb{R}^N$ be a domain, and let $u \in \mathcal{L}_0(\mathbb{R}^N)$ be a continuous function on $\overline{\Omega}$ satisfying*

$$\begin{cases} (\mathcal{I} - \Delta)^{\log} u(x) \geq 0 & \text{in } \Omega, \\ u(x) \geq 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.1}$$

then $u > 0$ in Ω or $u = 0$ a.e. in $\mathbb{R}^N$.

Now, we will prove the following maximum principles for the logarithmic Schrödinger operator.

Theorem 2.2 (Maximum principle for antisymmetric functions). *Let Ω be a bounded domain in Σ_λ. Assume that $\omega_\lambda \in \mathcal{L}_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is Dini continuous on Ω and is lower semi-continuous on $\overline{\Omega}$. If*

$$\begin{cases} (\mathcal{I} - \Delta)^{\log} \omega_\lambda(x) \geq 0 & \text{in } \Omega, \\ \omega_\lambda(x) \geq 0 & \text{in } \Sigma_\lambda \backslash \Omega, \\ \omega_\lambda(x^\lambda) = -\omega(x) & \text{in } \Sigma_\lambda, \end{cases} \tag{2.2}$$

then

$$\omega_\lambda \geq 0 \quad in\ \Omega. \tag{2.3}$$

Furthermore, if $\omega_\lambda(x) = 0$ *at some point in* Ω*, then we have*

$$\omega_\lambda = 0 \quad a.e.\ in\ \mathbb{R}^N. \tag{2.4}$$

These conclusions hold for unbounded region Ω *if we further assume that*

$$\liminf_{|x|\to\infty} \omega_\lambda(x) \geq 0.$$

Proof. If ω_λ is not nonnegative on Ω, then the lower semi-continuity of ω_λ on $\bar{\Omega}$ implies that there exists a $x^o \in \bar{\Omega}$ such that

$$\omega_\lambda(x^o) := \min_{\bar{\Omega}} \omega_\lambda(x) < 0.$$

One can further deduce from (2.2) that x^o is in the interior of Ω. It follows that

$$\begin{aligned}
(\mathcal{I}-\Delta)^{\log}\omega_\lambda(x^o) &= c_N \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\omega_\lambda(x^o)-\omega_\lambda(y)}{|x^o-y|^N}\kappa(|x^o-y|)dy \\
&= c_N \mathrm{P.V.} \left(\int_{\Sigma_\lambda} \frac{\omega_\lambda(x^o)-\omega_\lambda(y)}{|x^o-y|^N}\kappa(|x^o-y|)dy \right. \\
&\qquad \left. + \int_{\Sigma_\lambda} \frac{\omega_\lambda(x^o)-\omega_\lambda(y^\lambda)}{|x^o-y^\lambda|^N}\kappa(|x^o-y^\lambda|)dy \right).
\end{aligned} \tag{2.5}$$

Since $|x^o-y| \leq |x^o-y^\lambda|$, we have $\frac{1}{|x^o-y|} \geq \frac{1}{|x^o-y^\lambda|}$ and $\kappa(|x^o-y|) \geq \kappa(|x^o-y^\lambda|)$ as κ is a decreasing function, and, therefore,

$$\frac{\omega_\lambda(x^o)-\omega_\lambda(y)}{|x^o-y|^N}\kappa(|x^o-y|) \leq \frac{\omega_\lambda(x^o)-\omega_\lambda(y)}{|x^o-y^\lambda|^N}\kappa(|x^o-y^\lambda|),$$

since $\omega_\lambda(x^o)-\omega_\lambda(y) \leq 0$.

Thus, we obtain from (2.5) that

$$\begin{aligned}
(\mathcal{I}-\Delta)^{\log}\omega_\lambda(x^o) &\leq c_N \mathrm{P.V.} \int_{\Sigma_\lambda} \left(\frac{\omega_\lambda(x^o)-\omega_\lambda(y)}{|x^o-y^\lambda|^N} + \frac{\omega_\lambda(x^o)+\omega_\lambda(y)}{|x^o-y^\lambda|^N} \right)\kappa(|x^o-y^\lambda|)dy \\
&= c_N \mathrm{P.V.} \int_{\Sigma_\lambda} \frac{2\omega_\lambda(x^o)}{|x^o-y^\lambda|^N}\kappa(|x^o-y^\lambda|)dy < 0,
\end{aligned} \tag{2.6}$$

which contradicts (2.2). Therefore, our assumption is wrong, and, consequently, we have $\omega_\lambda(x) \geq 0$ in Ω.

Now, we have proved that $\omega_\lambda(x) \geq 0$ in Ω. If there is some point $\tilde{x} \in \Omega$ such that $\omega_\lambda(\tilde{x}) = 0$, then, from Lemma 2.1, we derive immediately $\omega_\lambda = 0$ a.e. in $\mathbb{R}^N$.

For unbounded domain Ω, the condition

$$\liminf_{|x|\to\infty} \omega_\lambda(x) \geq 0$$

ensures that the negative minimum of ω_λ must be attained at some point x^o, then we can derive the same contradiction as above.

This completes the proof of Theorem 2.2. ■

The following decay at infinity will also be necessary for proving subsequent results.

Theorem 2.3 (Decay at infinity). *Let Ω be an unbounded domain in Σ_λ. Suppose that a Dini continuous $\omega_\lambda \in \mathcal{L}_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a solution to*

$$\begin{cases} (\mathcal{I}-\Delta)^{\log}\omega_\lambda(x) + c(x)\omega_\lambda(x) \geq 0 & \text{in } \Omega, \\ \omega_\lambda(x) \geq 0 & \text{in } \Sigma_\lambda \backslash \Omega, \\ \omega_\lambda(x^\lambda) = -\omega(x) & \text{in } \Sigma_\lambda \end{cases} \tag{2.7}$$

with the measurable function $c(x)$ such that

$$\liminf_{|x|\to\infty} |x|^{\frac{1+N}{2}} c(x) \geq 0. \tag{2.8}$$

Then, there exists a constant $R_o > 0$ such that if

$$\omega_\lambda(x^o) = \min_{\Omega} \omega_\lambda(x) < 0, \tag{2.9}$$

then

$$|x^o| \leq R_o. \tag{2.10}$$

Proof. We prove the assertion by contradiction. Suppose that (2.10) is false, then by (2.7) and (2.9), we have

$$\omega_\lambda(x^o) = \min_{\Sigma_\lambda} \omega_\lambda(x) < 0.$$

After a direct calculation, we obtain

$$\begin{aligned}
&(\mathcal{I}-\Delta)^{\log}\omega_\lambda(x^o) \\
&= c_N \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\omega_\lambda(x^o) - \omega_\lambda(y)}{|x^o - y|^N} \kappa(|x^o - y|) dy \\
&= c_N \mathrm{P.V.} \int_{\Sigma_\lambda} \left(\frac{\omega_\lambda(x^o) - \omega_\lambda(y)}{|x^o - y|^N} \kappa(|x^o - y|) + \frac{\omega_\lambda(x^o) - \omega_\lambda(y^\lambda)}{|x^o - y^\lambda|^N} \kappa(|x^o - y^\lambda|) \right) dy \\
&\leq c_N \mathrm{P.V.} \int_{\Sigma_\lambda} \left(\frac{\omega_\lambda(x^o) - \omega_\lambda(y)}{|x^o - y^\lambda|^N} + \frac{\omega_\lambda(x^o) + \omega_\lambda(y)}{|x^o - y^\lambda|^N} \right) \kappa(|x^o - y^\lambda|) dy \\
&= c_N \mathrm{P.V.} \int_{\Sigma_\lambda} \frac{2\omega_\lambda(x^o)}{|x^o - y^\lambda|^N} \kappa(|x^o - y^\lambda|) dy < 0.
\end{aligned}$$

Now, we fix λ, and when $|x^o| \geq \lambda$, we have $B_{|x^o|}(\breve{x}) \subset \tilde{\Sigma}_\lambda := \mathbb{R}^N \backslash \Sigma_\lambda$ with $\breve{x} = (3|x^o| + x_1^o, (x^o)')$. Then, for $y \in \tilde{\Sigma}_\lambda$, if $|x^o| \geq \frac{R_\infty}{4}$ with sufficiently large R_∞, we can deduce that $|x^o - y| \le |x^o - \check{x}| + |\check{x} - y| \le |x^o| + 3|x^o| = |4x^o|$ which together with the fact that κ is a decreasing function implies that

$$\frac{\kappa(|x^o - y|)}{|x^o - y|} \ge \frac{\kappa(|4x^o|)}{|4x^o|}.$$

Thus, from (1.5) and $\kappa(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} \mathcal{K}_{\frac{N}{2}}(r)$, if R_∞ is sufficiently large, we have

$$\begin{aligned}
\int_{\Sigma_\lambda} \frac{1}{|x^o - y^\lambda|^N} \kappa(|x^o - y^\lambda|) dy &= \int_{\tilde{\Sigma}_\lambda} \frac{\kappa(|x^o - y|)}{|x^o - y|^N} dy \ge \int_{B_{|x^o|}(\check{x})} \frac{\kappa(|4x^o|)}{|4x^o|^N} dy \\
&\ge \int_{B_{|x^o|}(\check{x})} \frac{2^{1-\frac{N}{2}} \mathcal{K}_{\frac{N}{2}}(|4x^o|)}{|4x^o|^{\frac{N}{2}}} dy \\
&\ge \frac{c_\infty \omega_N}{2^{\frac{3N}{2}} |x^o|^{\frac{1+N}{2}} e^{4|x^o|}} := \frac{C}{|x^o|^{\frac{1+N}{2}} e^{4|x^o|}},
\end{aligned} \tag{2.11}$$

where $C = c_\infty \omega_N 2^{-\frac{3N}{2}}$ is a positive constant.

It follows that

$$0 \le (\mathcal{I} - \Delta)^{\log} \omega_\lambda(x^o) + c(x^o)\omega_\lambda(x^o) \le \left(\frac{C}{|x^o|^{\frac{1+N}{2}} e^{4|x^o|}} + c(x^o) \right) \omega_\lambda(x^o),$$

or equivalently,

$$\frac{C}{|x^o|^{\frac{1+N}{2}} e^{4|x^o|}} + c(x^o) \le 0.$$

Now, if $|x^o|$ is sufficiently large, this would contradict (2.8). Therefore, (2.10) holds.

This completes the proof of Theorem 2.3. ∎

3. Proof of the main theorem

Proof of Theorem 1.1. Let $T_\lambda, \Sigma_\lambda, x^\lambda$, and ω_λ be defined as in the previous section. Then, at the points where $\omega_\lambda(x) < 0$, it is easy to verify that, for $\xi_\lambda(x) \in (u_\lambda(x), u(x))$, we have

$$(\mathcal{I} - \Delta)^{\log} \omega_\lambda(x) + m\omega_\lambda(x) = u_\lambda^p(x) - u^p(x) = p\xi_\lambda^{p-1}(x)\omega_\lambda(x) \ge pu^{p-1}(x)\omega_\lambda(x), \tag{3.1}$$

because $\omega_\lambda(x) < 0$ and $\xi_\lambda(x) < u(x)$.

Step 1. We will show that, for sufficiently negative λ,

$$\omega_\lambda(x) \ge 0, \quad x \in \Sigma_\lambda. \tag{3.2}$$

First, from the assumption (1.9), for each fixed λ, $\lim_{|x|\to\infty} \omega_\lambda(x) = 0$. In fact, by (1.9), we have $\lim_{|x|\to\infty} u(x) = a$, and $\lim_{|x|\to\infty} u_\lambda(x) = a$ implying that

$$\lim_{|x|\to\infty} \omega_\lambda(x) = 0.$$

Thus, if (3.2) is false, then the negative minimum of ω_λ can be obtained at some point, say, x^o in Σ_λ, that is,

$$\omega_\lambda(x^o) = \min_{\Sigma_\lambda} \omega_\lambda(x) < 0.$$

Set $c(x) := m - pu^{p-1}(x)$ in (3.1), and then, the assumption (1.9) implies that $c \in L^\infty(\mathbb{R}^N)$ and

$$\lim_{|x|\to\infty} c(x) \geq 0.$$

Consequently, from Theorem 2.3, it follows that there exists $R_o > 0$ (independent of λ), such that

$$|x^o| \leq R_o. \tag{3.3}$$

Therefore, by choosing $\lambda < -R_o$ and, consequently, $|x^\lambda| > R_0$ for $x \in \Sigma_\lambda$, we obtain by (3.3) that

$$\omega_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.4}$$

Step 2. Step 1 provides a starting point, from which we can now move the plane T_λ to the right as long as (3.2) holds to its limiting position. Define

$$\lambda_o := \sup\left\{\lambda \mid \omega_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda\right\}.$$

By (3.3), we know that $\lambda_o < \infty$.

Next, we will show via a contradiction argument that

$$\omega_{\lambda_o}(x) \equiv 0 \quad \forall\, x \in \Sigma_{\lambda_o}. \tag{3.5}$$

Suppose, on the contrary, that

$$\omega_{\lambda_o}(x) \geq 0 \quad \text{and} \quad \omega_{\lambda_o}(x) \not\equiv 0 \quad \text{in } \Sigma_{\lambda_o}, \tag{3.6}$$

then we must have

$$\omega_{\lambda_o}(x) > 0 \quad \forall\, x \in \Sigma_{\lambda_o}. \tag{3.7}$$

In fact, if (3.7) is violated, then there exists a point $\hat{x} \in \Sigma_{\lambda_o}$ such that

$$\omega_{\lambda_o}(\hat{x}) = \min_{\Sigma_{\lambda_o}} \omega_{\lambda_o}(x) = 0.$$

It means that $u_{\lambda_o}(\hat{x}) = u(\hat{x})$. Then, it follows from (3.1) that

$$(\mathcal{I} - \Delta)^{\log}\omega_{\lambda_o}(\hat{x}) = u^p_{\lambda_o}(\hat{x}) - u^p(\hat{x}) = u^p(\hat{x}) - u^p(\hat{x}) = 0.$$

Hence, Theorem 2.2 implies that $\omega_{\lambda_o}(\hat{x}) \equiv 0$ in Σ_{λ_o}, which contradicts (3.6). Thus, (3.7) holds.

Now, we will show that the plane T_λ can be moved further right. More precisely, there exists an $\varepsilon > 0$ such that, for any $\lambda \in [\lambda_o, \lambda_o + \varepsilon)$, we have

$$\omega_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.8}$$

Once it is proved, this will contradict the definition of λ_o. Therefore, (3.5) must be valid.

Let us now prove (3.8). In fact, by (3.7), we have $\omega_{\lambda_o}(x) > 0$, $x \in \Sigma_{\lambda_o}$, which in turn implies that there is a constant $c_o > 0$ and $\delta > 0$ such that

$$\omega_{\lambda_o}(x) \geq c_o > 0, \quad x \in \overline{\Sigma_{\lambda_o - \delta} \cap B_{R_o}(0)}.$$

Since ω_λ is continuous with respect to λ, there exists an $\varepsilon > 0$ such that, for $\lambda \in [\lambda_o, \lambda_o + \varepsilon)$, we have

$$\omega_\lambda(x) \geq 0, \quad x \in \Sigma_{\lambda_o - \delta} \cap B_{R_o}(0). \tag{3.9}$$

Moreover, combining (3.3) with (3.9), we deduce that $w_\lambda(x) \geq 0$ on $\Sigma_{\lambda_0 - \delta}$.

To proceed with the proof, we need the following small volume maximum principle (see [10, Theorem 6.1 (iii)] and [17, Theorem 1.3]).

Lemma 3.1. *Let Ω be an open set of $\mathbb{R}^N$. Consider the following problem on Ω:*

$$\begin{cases} (\mathcal{I} - \Delta)^{\log} u(x) \geq c(x) u & \text{in } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.10}$$

with $c \in L^\infty(\mathbb{R}^N)$.

Then, there exists $\delta > 0$ such that for every open set $\Omega \subset \mathbb{R}^N$ with $|\Omega| \leq \delta$ and any solution $u \in \mathcal{V}_\omega(\Omega)$ of (3.10) in Ω, where the space $\mathcal{V}_\omega(\Omega)$ is given in [10, Section 6], *we have $u \geq 0$ in $\mathbb{R}^N$.*

Consequently, according to Lemma 3.1 (by taking $\Omega = (\Sigma_\lambda \setminus \Sigma_{\lambda_o - \delta}) \cap B_{R_o}(0)$), we obtain that (3.8) holds.

The arbitrariness of the x_1-direction leads to the radial symmetry of $u(x)$ about some point in $\mathbb{R}^N$, and the monotonicity is a consequence of the fact that (3.4) holds.

This completes the proof of Theorem 1.1. ∎

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References

[1] T. Cazenave, *Semilinear Schrödinger equations*. Courant Lect. Notes Math. 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003 Zbl 1055.35003 MR 2002047

[2] H. A. Chang-Lara and A. S. na, Classical solutions to integral equations with zero order kernels. *Math. Ann.* (2023)

[3] H. Chen, D. Hauer, and T. Weth, An extension problem for the logarithmic Laplacian. 2023, arXiv:2312.15689

[4] H. Chen and L. Véron, Bounds for eigenvalues of the Dirichlet problem for the logarithmic Laplacian. *Adv. Calc. Var.* **16** (2023), no. 3, 541–558 Zbl 1518.35515 MR 4609798

[5] H. Chen and T. Weth, The Dirichlet problem for the logarithmic Laplacian. *Comm. Partial Differential Equations* **44** (2019), no. 11, 1100–1139 Zbl 1423.35390 MR 3995092

[6] E. Correa and A. de Pablo, Nonlocal operators of order near zero. *J. Math. Anal. Appl.* **461** (2018), no. 1, 837–867 Zbl 1393.45011 MR 3759570

[7] A. Elgart and B. Schlein, Mean field dynamics of boson stars. *Comm. Pure Appl. Math.* **60** (2007), no. 4, 500–545 Zbl 1113.81032 MR 2290709

[8] M. M. Fall and V. Felli, Sharp essential self-adjointness of relativistic Schrödinger operators with a singular potential. *J. Funct. Anal.* **267** (2014), no. 6, 1851–1877 Zbl 1295.35377 MR 3237776

[9] M. M. Fall and V. Felli, Unique continuation properties for relativistic Schrödinger operators with a singular potential. *Discrete Contin. Dyn. Syst.* **35** (2015), no. 12, 5827–5867 Zbl 1336.35356 MR 3393257

[10] P. A. Feulefack, The logarithmic Schrödinger operator and associated Dirichlet problems. *J. Math. Anal. Appl.* **517** (2023), no. 2, article no. 126656 Zbl 1498.60183 MR 4478365

[11] P. A. Feulefack and S. Jarohs, Nonlocal operators of small order. *Ann. Mat. Pura Appl. (4)* **202** (2023), no. 4, 1501–1529 Zbl 1516.35458 MR 4597591

[12] R. L. Frank, T. König, and H. Tang, Classification of solutions of an equation related to a conformal log Sobolev inequality. *Adv. Math.* **375** (2020), article no. 107395 Zbl 1453.35036 MR 4170219

[13] R. L. Frank and E. Lenzmann, On ground states for the L^2-critical boson star equation. [v1] 2009, [v2] 2010, arXiv:0910.2721v2

[14] J. Fröhlich, B. L. G. Jonsson, and E. Lenzmann, Boson stars as solitary waves. *Comm. Math. Phys.* **274** (2007), no. 1, 1–30 Zbl 1126.35064 MR 2318846

[15] Y. Guo and S. Peng, Symmetry and monotonicity of nonnegative solutions to pseudo-relativistic Choquard equations. *Z. Angew. Math. Phys.* **72** (2021), no. 3, article no. 120 Zbl 1465.35394 MR 4261334

[16] V. Hernández Santamaría and A. Saldaña, Small order asymptotics for nonlinear fractional problems. *Calc. Var. Partial Differential Equations* **61** (2022), no. 3, article no. 92 Zbl 1486.35053 MR 4400612

[17] S. Jarohs and T. Weth, On the strong maximum principle for nonlocal operators. *Math. Z.* **293** (2019), no. 1-2, 81–111 Zbl 1455.35032 MR 4002272

[18] M. Kassmann and A. Mimica, Intrinsic scaling properties for nonlocal operators. *J. Eur. Math. Soc. (JEMS)* **19** (2017), no. 4, 983–1011 Zbl 1371.35316 MR 3626549

[19] A. Laptev and T. Weth, Spectral properties of the logarithmic Laplacian. *Anal. Math. Phys.* **11** (2021), no. 3, article no. 133 Zbl 1476.35153 MR 4279386

[20] G. M. Lieberman, *Second order parabolic differential equations*. World Scientific, River Edge, NJ, 1996 Zbl 0884.35001 MR 1465184

[21] W.-M. Liu and E. Kengne, *Schrödinger equations in nonlinear systems*. Springer, Singapore, 2019 Zbl 1436.81006 MR 3929719

[22] A. Mimica, On harmonic functions of symmetric Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** (2014), no. 1, 214–235 Zbl 1298.60054 MR 3161529

[23] C. Sulem and P. Sulem, *The nonlinear Schrödinger equation: Self-focusing and wave collapse*. Appl. Math. Sci. 139, Springer, New York, 1999 Zbl 0928.35157 MR 1696311

[24] L. Zhang and X. Nie, A direct method of moving planes for the logarithmic Laplacian. *Appl. Math. Lett.* **118** (2021), article no. 107141 Zbl 1475.35093 MR 4226267

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