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Existence and regularity results of parabolic problems with convection term and singular nonlinearity

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Abstract. In this work, we investigate the influence of the convection term and the singular lower order term on the existence and regularity of solutions to the following parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(M(x,t)\nabla u) = -\operatorname{div}(uE(x,t)) + \frac{f}{u^\theta} & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\theta > 0$, $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded smooth domain with $0 \in \Omega$, and $f \in L^m(\Omega \times (0,T))$ with $m \geq 1$ is a non-negative function. The function u_0 is a non-negative function that belongs to the space $L^\infty(\Omega)$ such that

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0, \quad u_0 \geq c_\omega \text{ in } \omega.$$

The main idea of this research explains the combined impact of the convection term and the singular lower order term on the existence and regularity of a solution to the above problem.

1. Introduction

In this paper, we deal with the existence and regularity results of solutions to the following singular parabolic boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(M(x,t)\nabla u) = -\operatorname{div}(uE(x,t)) + \frac{f}{u^\theta} & \text{in } \Omega_T = \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \Gamma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $0 \in \Omega$ is a bounded smooth domain subset of $\mathbb{R}^N$ $(N > 2)$, $\theta > 0$, and $M : \Omega_T \to \mathbb{R}^{N\times N}$ is a bounded measurable matrix, which satisfies the following conditions: there exist two positive constants α_1 and β_1 such that, for a.e. $(x,t) \in \Omega_T$ and $\xi \in \mathbb{R}^N$,

$$\alpha_1|\xi|^2 \leq M(x,t)\xi \cdot \xi, \quad |M(x,t)| \leq \beta_1. \tag{1.2}$$

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Furthermore, the vector field E satisfies

$$|E| \leq \frac{B}{|x|}, \quad B \in \mathbb{R}_+^*. \tag{1.3}$$

$f(x,t)$ is a non-negative measurable function which satisfies

$$f \in L^m(\Omega_T), \quad m \geq 1.$$

Here, $L^m(\Omega_T)$ denotes the Lebesgue space.

There are considerable researches dealing with the problem as (1.1) when $\theta = 0$ or $E \equiv 1$. The problem (1.1) with $\theta = 0$ has been thoroughly investigated in the past by Boccardo et al. in a series of works under different hypotheses on the vector field E. To be more specific, when $\theta = 0$, the stationary case of problem (1.1) becomes

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(uE(x)) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $E \in (L^N(\Omega))^N$, $f \in L^m(\Omega)$, $1 \leq m < N/2$, and M is a bounded measurable matrix. In [3], the author proved the existence and regularity results of solution to problem (1.4) for all $f \in L^m(\Omega)$ with $m \geq 1$. More precisely, they have obtained the following results.

- If $\frac{2N}{N+2} < m < \frac{N}{2}$ and $|B| < \frac{\alpha_1(N-2m)}{m}$, then there exists a weak solution $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$.
- If $1 < m < \frac{2N}{N+2}$ and $|B| < \frac{\alpha_1(N-2m)}{m}$, then there exists a distributional solution $u \in W_0^{1,m^{**}}(\Omega)$.
- If $f \in L^1(\Omega)$ and $E \in (L^2(\Omega))^N$, then there exists an entropy solution such that $\log(1+|u|) \in W_0^{1,2}(\Omega)$.

Some interesting example in [3] showed that the existence and summability results of solution to problem (1.4) obtained in [2] lost with this slightly weaker assumption (1.3). For more details, see [3, Examples 2.1 and 2.2].

Recently, Boccardo and Orsina in [11] studied the existence of distributional solution $u \in W_0^{1,q}(\Omega)$ to problem (1.4) with $q < \frac{N\alpha_1}{B+\alpha_1}$ provided (1.3) holds with $\alpha_1(N-2) \leq B < \alpha_1(N-1)$ and $f \in L^1(\Omega)$. Furthermore, u satisfies

$$\left(\int_\Omega |\nabla u|^q \right)^{\frac{1}{q}} \leq C_E \|f\|_{L^1(\Omega)}.$$

The constant C_E depends on E, α_1, and Ω. For some other related results about elliptic problems with convection term, see the works [4–10, 17, 18] and references therein.

Concerning the evolutive case as problem (1.1) with $\theta = 0$, many authors have investigated this type of problem. Boccardo et al. in [12] have studied problem (1.1) when $\theta = 0$, $f \equiv 0$, $u_0 \in L^1(\Omega)$, and $E \in (L^2(\Omega_T))^N$. In the same kinds, Boccardo, Orsina, and Porzio in [14] have studied problem (1.1) in the case $f \equiv 0$, $\theta = 0$, and E is a non-zero measurable vector field satisfies the following assumption:

$$|E(x,t)| \leq \mu |B(x)|, \quad \mu > 0, \quad B \in L^N(\Omega) \quad \forall (x,t) \in \Omega_T;$$

also, the authors studied problem (1.1) when the vector field is less regular, i.e.,

$$|E(x,t)| \leq \frac{B}{|x|}, \quad B > 0. \tag{1.5}$$

More recently, Farroni–Moscariello [33] and Farroni in [29] studied the following singular parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(M(x,t)\nabla u + A\frac{x}{|x|^2}u\right) = -\operatorname{div} F & \text{in } \Omega_T, \\ u = 0 & \text{on } \Gamma_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $F \in L^2(\Omega_T)$, $u_0 \in L^2(\Omega)$, and M is a measurable, symmetric, matrix field satisfying the uniform bounds

$$\lambda|\xi|^2 \leq \langle M(x,t)\xi, \xi\rangle \leq \kappa|\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \quad (x,t) \in \Omega \times (0,T), \quad 0 < \lambda \leq \kappa.$$

Farroni, Greco, Moscariello et al. in [31] have generalized the problem contained in [29]. For some other results of parabolic equations with convection terms, see [15,32,34,35,39].

If the convection term does not exist (i.e., $E \equiv 0$), problem (1.1) has been extensively studied in the past. De Bonis and De Cave in [19] have studied the existence and regularity of solution to problem (1.1) when the operator is nonlinear with classical Leray–Lions conditions, $\theta > 0$, $0 \leq f \in L^m(\Omega_T)$, $m \geq 1$, and $u_0 \in L^\infty(\Omega)$ such that $u_0 \geq c$ in ω, for all $\omega \subset\subset \Omega$. In the presence of the absorption terms, the existence and regularity of solution to problem (1.1) has been proved in [24, 27]. When the singular term $u^{-\theta}$, $(\theta > 0)$, is replaced by a continuous function h possibly singular at the origin and bounded outside the origin, problem (1.1) has been treated in many works: Oliva and Petitta in [42] have shown the existence of a non-negative distributional solution to problem (1.1), with $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. In the same kinds, Oliva and Magliocca in [38] have proved the existence of non-negative solution to problem (1.1) with a superlinear gradient term which is possibly singular. For more and different aspects concerning singular elliptic and parabolic problems we refer to [20, 22, 23, 25, 26, 28, 30, 40, 41, 43–48] and references therein.

Concerning the case in the presence of the convection and the singular terms (i.e., $\theta > 0$, $E \neq 0$), the literature concerned with this type of problems is more limited. More recently, He and Huang in [36] have studied the following singular elliptic problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(uE(x)) + \frac{f}{u^\theta} & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \leq f \in L^m(\Omega)$ with $m \geq 1$, $\theta > 0$, M is a bounded measurable matrix satisfies some conditions, and the vector field E satisfies condition (1.5). More precisely, the authors have been proved the existence of a solution u to the above problem and satisfy the following summability:

- $u \in H_0^1(\Omega)$ if $\theta = 1$, $A < \frac{\alpha_1(N-2)}{2}$, and $f \in L^1(\Omega)$.
- $u \in H_{\mathrm{loc}}^1(\Omega)$, $u^{\frac{\theta+1}{2}} \in H_0^1(\Omega)$ if $\theta > 1$, $A < \frac{\alpha_1(N-2)}{\theta+1}$, and $f \in L^1(\Omega)$.
- $u \in H_0^1(\Omega)$ if $\theta < 1$, $A < \frac{\alpha_1(N-2)}{2}$, and $f \in L^m(\Omega)$ with $m = \frac{2N}{N+2-\theta(N-2)}$.
- $u \in W_0^q(\Omega)$, $q = \frac{Nm(\theta+1)}{N-m(1-\theta)}$ if $\theta < 1$, $\frac{\alpha_1(N-2)}{2} < A < \frac{\alpha_1(N-2)}{\theta+1}$, and $f \in L^m(\Omega)$ with $1 \leq m < \frac{2N}{N+2+\theta(N-2)}$.

The difficulty of studying problem (1.1) comes from the presence of the convection term $\operatorname{div}(uE(x,t))$, which leads to the noncoercivity of the differential operator $-\operatorname{div}(M(x,t)\nabla u) + \operatorname{div}(E(x,t)u)$ on $L^2(0,T,;H_0^1(\Omega))$ and in the presence of the singular term $u^{-\theta}$, $\theta > 0$. Therefore, in order to overcome the noncoercivity of the operator $-\operatorname{div}(M(x,t)\nabla u) + \operatorname{div}(E(x,t)u)$, we apply truncation method and consider the corresponding approximate Dirichlet problem.

Our main results are in Section 3. More precisely, we start by treating the case $\theta = 1$, the existence of a solution to problem (1.1) given by Theorem 3.3 (see below), and then the regularity of solutions is given by Theorem 3.5. Also, we will state the existence and regularity of solutions to problem (1.1) in the case $\theta > 1$ in Theorems 3.7 and 3.8, respectively. Finally, the existence of solutions to problem (1.1) when $\theta < 1$ is given by Theorems 3.10 and 3.13, and the summability of the solution is given in Theorem 3.11.

Preliminaries and notations. Now, we give the Gagliardo–Nirenberg inequality that we will use afterwards in the proof of main results.

Lemma 1.1 ([21, Theorem 1.2]). *Let v be a function in $W_0^{1,h}(\Omega) \cap L^\rho(\Omega)$ with $h \geq 1$, $\rho \geq 1$. Then, there exists a positive constant C_{GN}, depending on N, h, ρ, and σ, such that*

$$\|v\|_{L^\sigma(\Omega)} \leq C_{GN} \|\nabla v\|^{\eta}_{(L^h(\Omega))^N} \|v\|^{1-\eta}_{L^\rho(\Omega)}$$

for every η and σ satisfying

$$0 \leq \eta \leq 1, \quad 1 \leq \sigma < +\infty, \quad \frac{1}{\sigma} = \eta\left(\frac{1}{h} - \frac{1}{N}\right) + \frac{1-\eta}{\rho}.$$

An immediate consequence of the previous lemma is the following embedding result:

$$\int_{\Omega_T} |v|^\sigma \leq C_{GN} \|v\|^{\frac{\rho h}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_{\Omega_T} |\nabla v|^h,$$

which holds for every function v in $L^h(0,T;W_0^{1,h}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$ with $h \geq 1$, $\rho > 1$, and $\sigma = \frac{h(N+\rho)}{N}$ (see, for instance [21, Proposition 3.1]).

Note that problem (1.1) is related to the following Hardy inequality (see, e.g., [16,50]):

$$H\Big(\int_\Omega \frac{|v|^2}{|x|^2}\Big)^{\frac{1}{2}} \leq \Big(\int_\Omega |\nabla v|^2\Big)^{\frac{1}{2}} \quad \forall v \in W_0^{1,2}(\Omega),$$

where $H = \frac{N-2}{2}$.

For the sake of simplicity, we will often use the simplified notation

$$\int_{\Omega_T} f := \int_0^T \int_\Omega f(x,t)\,dx\,dt \quad \text{and} \quad \int_\Omega f := \int_\Omega f(x)\,dx,$$

when no ambiguity in the integration variables is possible. If not otherwise specified, we will denote by C several constants whose value may change from line to line and, sometimes, in the same line. These values will only depend on the parameters (for instance, C can depend on $N, \alpha_1, \theta, m, T, \Omega, \Omega_T$), but they will never depend on the indexes of the sequences we will often introduce.

Here, we give the definition of a weak solution to problem (1.1).

Definition 1.2. If $\theta \leq 1$, a weak solution to problem (1.1) is a function

$$u \in L^1(0,T; W_0^{1,1}(\Omega))$$

such that

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : u \geq c_\omega \text{ in } \omega \times (0,T), \tag{1.6}$$

$$|M(x,t)\nabla u|, \quad |uE(x,t)| \in L^1(0,T; L^1_{\text{loc}}(\Omega)), \tag{1.7}$$

and

$$\begin{aligned} &-\int_0^T \int_\Omega u\frac{\partial \varphi}{\partial t} + \int_0^T \int_\Omega M(x,t)\nabla u \nabla\varphi \\ &= \int_0^T \int_\Omega uE(x,t)\cdot\nabla\varphi + \int_0^T \int_\Omega \frac{f\varphi}{u^\theta} + \int_\Omega u_0(x)\varphi(x,0), \\ &\qquad \forall \varphi \in C_c^1(\Omega \times (0,T)), \text{ with } \varphi(T) = 0. \end{aligned} \tag{1.8}$$

If $\theta > 1$, a weak solution to problem (1.1) is a function $u \in L^r(0,T; H^r_{\text{loc}}(\Omega))$ with $r > 1$ and $u^{\frac{\theta+1}{2}} \in L^2(0,T; H_0^1(\Omega))$ such that u satisfies (1.6)–(1.8).

2. Approximations problem

First, in order to get the existence and regularity of solutions to problem (1.1), we need to consider the following non-singular approximate problem:

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(M(x,t)\nabla u_n) = -\operatorname{div}\big(\frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x,t)}{1+\frac{1}{n}|E(x,t)|}\big) + \frac{f_n}{(|u_n|+\frac{1}{n})^\theta} & \text{in } \Omega_T, \\ u_n(x,t) = 0 & \text{on } \Gamma_T, \\ u_n(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where

$$f_n = \frac{f}{1 + \frac{1}{n} f} \leq n. \tag{2.2}$$

The following lemma gives the existence of solutions to the approximate problem (2.1).

Lemma 2.1. *Let $B < \frac{\alpha_1(N-2)}{2}$; then problem (2.1) has a non-negative solution*

$$u_n \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(\Omega_T).$$

Proof. For given $n \in \mathbb{N}$ and $v \in L^2(\Omega_T)$, let w be the unique solution to the following problem (see, for instance, [37]):

$$\begin{cases} \frac{\partial w}{\partial t} - \operatorname{div}(M(x,t)w) = -\operatorname{div}\left(\frac{w}{1+\frac{1}{n}|w|} \frac{E(x,t)}{1+\frac{1}{n}|E(x,t)|}\right) + \frac{f_n}{(|v|+\frac{1}{n})^\theta} & \text{in } \Omega_T, \\ w(x,t) = 0 & \text{on } \Gamma_T, \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{2.3}$$

Now, we take w as a test function in (2.3); by (1.2), we have

$$\begin{aligned} \frac{1}{2}\int_\Omega w^2(x,T) + \alpha_1 \int_{\Omega_T} |\nabla w|^2 &\leq \frac{1}{2}\int_\Omega w^2(x,T) + \int_{\Omega_T} M(x,t)\nabla w \cdot \nabla w \\ &\leq \int_{\Omega_T} |wE(x,t)||\nabla w| + \int_{\Omega_T} \frac{|f_n w|}{(|v| + \frac{1}{n})^\theta} + \frac{1}{2}\int_\Omega u_0^2(x) \\ &\leq \int_{\Omega_T} |wE(x,t)||\nabla w| + n^{\theta+1} \int_{\Omega_T} |w| + \frac{1}{2}\int_\Omega u_0^2, \end{aligned} \tag{2.4}$$

where in the last estimate we have used (2.2). Recalling (1.3), and applying Hölder and Hardy inequalities on the second term on the right-hand side of (2.4), we find that

$$\begin{aligned} \int_{\Omega_T} |wE(x,t)||\nabla w| &\leq B \int_{\Omega_T} \frac{|w|}{|x|}|\nabla w| \\ &\leq B\left(\int_{\Omega_T} \frac{|w|^2}{|x|^2}\right)^{\frac{1}{2}} \left(\int_{\Omega_T} |\nabla w|^2\right)^{\frac{1}{2}} \\ &\leq \frac{B}{H}\int_{\Omega_T} |\nabla w|^2, \end{aligned} \tag{2.5}$$

where H is the Hardy constant. Combining (2.4) and (2.5) and using the fact that $u_0 \in L^\infty(\Omega)$, we obtain

$$\frac{1}{2}\int_\Omega w^2(x,T) + \left(\alpha_1 - \frac{B}{H}\right)\int_{\Omega_T} |\nabla w|^2 \leq n^{\theta+1}\int_{\Omega_T} |w| + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{2.6}$$

Since $B < \frac{\alpha_1(N-2)}{2}$, then $\alpha_1 - \frac{B}{H} > 0$. Dropping the first non-negative term and applying the Poincaré inequality, Hölder inequality to the left- and right-hand side of (2.6), respectively, we reach that

$$\int_{\Omega_T} |w|^2 \leq Cn^{\theta+1}\left(\int_{\Omega_T} |w|^2\right)^{\frac{1}{2}} + \frac{C}{2}\|u_0\|^2_{L^2(\Omega)}$$

for some constant $C = C(\alpha_1, N)$. Using this fact and applying Young's inequality, we obtain

$$\|w\|_{L^2(\Omega_T)} \le C_1 := C(\alpha_1, N, B, H, \|u_0\|_{L^2(\Omega)}).$$

Define $w = S(v)$ so that the ball of $L^2(\Omega_T)$ of radius C_1 is invariant for S. It is obvious to verify, applying the embedding, that S is both continuous and compact on $L^2(\Omega_T)$. Therefore, by Schauder's fixed-point theorem, there is $u_n \in L^2(0, T; H_0^1(\Omega))$ satisfying $u_n = S(u_n)$, which implies that u_n satisfies

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(M(x,t)\nabla u_n) = -\operatorname{div}\big(\frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x,t)}{1+\frac{1}{n}|E(x,t)|}\big) + \frac{f_n}{(|u_n|+\frac{1}{n})^\theta} & \text{in } \Omega_T, \\ u_n(x,t) = 0 & \text{on } \Gamma_T, \\ u_n(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Thus, u_n is solution to (2.1). Note that the second term on the right-hand side of (2.1) belongs to $L^\infty(\Omega_T)$ implies that $u_n \in L^\infty(\Omega_T)$; see [1]. Now, taking $u_n^- = \min(0, u_n)$ as a test function in (2.3) and using (1.2), we obtain

$$\int_0^T \int_\Omega \frac{\partial u_n}{\partial t} u_n^- + \int_{\Omega_T} M(x,t)\nabla u_n \cdot \nabla u_n^- = \int_{\Omega_T} u_n E(x,t)\nabla u_n^- + \int_{\Omega_T} \frac{f_n}{(|u_n| + \frac{1}{n})^\theta} u_n^-;$$

therefore,

$$\frac{1}{2}\int_\Omega u_n^-(x,t)^2 - \frac{1}{2}\int_\Omega u_0^-(x)^2 + \alpha_1 \int_{\Omega_T} |\nabla u_n^-|^2 \le \int_{\Omega_T} u_n E(x,t)\nabla u_n^- + \int_{\Omega_T} \frac{f_n}{(|u_n| + \frac{1}{n})^\theta} u_n^-.$$

From (1.3), and applying Hölder and Hardy inequalities, we obtain

$$\begin{aligned} \int_{\Omega_T} u_n E(x,t)\nabla u_n^- &\le \int_{\Omega_T} |u_n^-||E(x,t)||\nabla u_n^-| \\ &\le B \int_{\Omega_T} \frac{u_n^-}{|x|} \\ &\le B\Big(\int_{\Omega_T} \frac{|u_n^-|^2}{|x|^2}\Big)^{\frac{1}{2}} \Big(\int_{\Omega_T} |\nabla u_n^-|^2\Big)^{\frac{1}{2}} \\ &\le \frac{B}{H}\int_{\Omega_T} |\nabla u_n^-|^2. \end{aligned} \tag{2.7}$$

By the last inequality, (2.7) becomes

$$\frac{1}{2}\int_\Omega u_n^-(x,t)^2 + \alpha_1 \int_{\Omega_T} |\nabla u_n^-|^2 \le \frac{B}{H}\int_{\Omega_T} |\nabla u_n^-|^2 + \int_{\Omega_T} \frac{f_n}{(|u_n| + \frac{1}{n})^\theta} u_n^-.$$

Therefore,
$$\frac{1}{2}\int_{\Omega} u_n^-(x,t)^2 + \left(\alpha_1 - \frac{B}{H}\right)\int_{\Omega_T} |\nabla u_n^-|^2 \leq 0.$$
Since $B < \alpha_1 H$, then we deduce that
$$\|u_n^-\|_{L^2(0,T;H_0^1(\Omega))} \leq 0.$$
This implies that $u_n^- = 0$, and so, $u_n \geq 0$ a.e. in Ω_T. ∎

In the following lemma, we prove the strict positivity of the sequence u_n solution to the approximate problem (2.1), which we will apply later in the case $\theta > 1$ in the boundedness of u_n in the space $L^2(0,T;H^1_{\text{loc}}(\Omega))$ as well as in the convergence passages.

Lemma 2.2. *Let u_n be a solution to problem (2.1) given by Lemma 2.1. Then, for every $w \subset\subset \Omega$, there is a positive constant $c_\omega > 0$ (independent of n) such that*
$$u_n \geq c_\omega \quad \text{in } \omega \times (0,T)\ \forall n \in \mathbb{N}.$$

Proof. With some modifications and using the same techniques as in the proof of [36, Lemma 3.2] (see also [13]), we can get the proof of Lemma 2.2. ∎

3. Main results

To show the main results of the present work, we need to obtain a priori estimates on u_n. These estimates will effectively depend on f, θ, and B, so we have three separate cases for evaluation. At this point, we start with $\theta = 1$.

3.1. The case $\theta = 1$

Lemma 3.1. *Assume that $B < \frac{\alpha_1(N-2)}{2}$ and u_n is a solution to (2.1) with $\theta = 1$ and $0 \leq f \in L^1(\Omega_T)$. Then, u_n is uniformly bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.*

Proof. Taking u_n as a test function in (2.1) and using (1.2) and the fact $0 \leq f_n \leq f$, we get
$$\begin{aligned}\frac{1}{2}\int_{\Omega} u_n^2(x,t) + \alpha_1\int_{\Omega_t}|\nabla u_n|^2 &\leq \frac{1}{2}\int_{\Omega} u_n^2(x,t) + \int_{\Omega_t} M(x,t)\nabla u_n\cdot\nabla u_n \\ &\leq \int_{\Omega_T}|u_n E(x,t)||\nabla u_n| + \int_{\Omega_T} f + \frac{1}{2}\int_{\Omega} u_0^2.\end{aligned}$$

By the fact that $f \in L^1(\Omega_T)$ and $u_0 \in L^\infty(\Omega)$, we have
$$\frac{1}{2}\int_{\Omega} u_n^2(x,t) + \alpha_1\int_{\Omega_t}|\nabla u_n|^2 \leq \int_{\Omega_T}|u_n E(x,t)||\nabla u_n| + \|f\|_{L^1(\Omega_T)} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{3.1}$$

Recalling (1.3) and using Hölder's and Hardy inequalities, we estimate the first term on the right-hand side of (3.1) as follows:

$$\begin{aligned}\int_{\Omega_T} |u_n E(x,t)||\nabla u_n| &\leq B \int_{\Omega_T} \frac{|u_n|}{|x|}|\nabla u_n| \\ &\leq B\Big(\int_{\Omega_T} \frac{|u_n|^2}{|x|^2}\Big)^{\frac{1}{2}}\Big(\int_{\Omega_T} |\nabla u_n|^2\Big)^{\frac{1}{2}} \\ &\leq \frac{B}{H}\int_{\Omega_T} |\nabla u_n|^2. \end{aligned} \tag{3.2}$$

Combining (3.1) with (3.2) and passing to the supremum for $t \in [0, T]$, we obtain

$$\frac{1}{2}\|u_n\|^2_{L^\infty(0,T;L^2(\Omega))} + \Big(\alpha_1 - \frac{B}{H}\Big)\int_{\Omega_T} |\nabla u_n|^2 \leq \|f\|_{L^1(\Omega_T)} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} = C.$$

Since $\alpha_1 - \frac{B}{H} > 0$, therefore, we reach that

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C \quad \text{and} \quad \|u_n\|_{L^2(0,T;H^1_0(\Omega))} \leq C.$$

This last affirmation implies the boundedness of the sequence u_n in

$$L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)).$$

■

Remark 3.2. In view of Lemma 3.1, we have $u_n E(x,t)$ bounded uniformly in $L^1(\Omega_T)$. From (1.3), applying Hölder's and Hardy inequalities, we can write

$$\begin{aligned}\int_{\Omega_T} |u_n E(x,t)| &\leq B\int_{\Omega_T} \frac{|u_n|}{|x|} \\ &\leq B|\Omega_T|^{\frac{1}{2}}\Big(\int_{\Omega_T} \frac{|u_n|^2}{|x|^2}\Big)^{\frac{1}{2}} \\ &\leq \frac{B|\Omega_T|^{\frac{1}{2}}}{H}\Big(\int_{\Omega_T} |\nabla u_n|^2\Big)^{\frac{1}{2}} \\ &= \frac{B|\Omega_T|^{\frac{1}{2}}}{H}\|u_n\|_{L^2(0,T;H^1_0(\Omega))} \leq C.\end{aligned}$$

Theorem 3.3. *Let $\theta = 1$, $f \in L^1(\Omega_T)$ with $f \geq 0$, and $B < \frac{\alpha_1(N-2)}{2}$. Then, there is a solution $u \in L^2(0, T; H^1_0(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ in the sense of Definition 1.2.*

Proof. By Lemma 3.1, we have u_n bounded in $L^2(0, T; H^1_0(\Omega))$. Then, there exists a function

$$u \in L^2(0, T; H^1_0(\Omega))$$

such that u_n converges weakly to u in $L^2(0,T;H_0^1(\Omega))$. From Lemma 2.2, we have $\frac{f_n}{u_n+\frac{1}{n}}$ bounded in the space $L^1(0,T;L^1_{\mathrm{loc}}(\Omega))$. On the other hand, by Remark 3.2, we have $u_nE(x,t)$ bounded in $L^1(\Omega_T)$, and $\operatorname{div}(u_nE(x,t))$ is bounded in

$$L^\infty(\Omega_T)\subset L^2(\Omega_T)\subset L^2(0,T;H^{-1}(\Omega));$$

then, we deduce that $\{\frac{\partial u_n}{\partial t}\}_n$ is bounded in the space

$$L^2(0,T;H^{-1}(\Omega))+L^1(0,T;L^1_{\mathrm{loc}}(\Omega));$$

using compactness argument in [49], we obtain

$$u_n\to u\quad \text{strongly in } L^1(\Omega_T) \text{ and a.e. in } \Omega_T. \tag{3.3}$$

In the following lemma, we will prove the convergence a.e. of ∇u_n to ∇u in Ω_T.

Lemma 3.4. *The sequence $\{\nabla u_n\}$ converges to ∇u a.e. in Ω_T.*

Proof. Let $\varphi\in C_c^1(\Omega)$, $\varphi\geq 0$, independent of $t\in(0,T)$, $\varphi=1$ on $w=\operatorname{supp}(\varphi)\subset\subset\Omega$, and we take $T_h(u_n-T_k(u))\varphi$ as a test function in (2.1); we have

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u_n}{\partial t}T_h(u_n-T_k(u))\varphi+\int_{\Omega_T}M(x,t)\nabla u_n\nabla T_h(u_n-T_k(u))\varphi\\
&+\int_{\Omega_T}M(x,t)\nabla u_n\cdot\nabla\varphi T_h(u_n-T_k(u))\\
&\quad\leq\int_{\Omega_T}|u_nE(x,t)||\nabla T_h(u_n-T_k(u))|\varphi+\int_{\Omega_T}|u_nE(x,t)||T_h(u_n-T_k(u))|\nabla\varphi\\
&\qquad+\int_{\Omega_T}\frac{f_n}{(u_n+\frac{1}{n})^\theta}T_h(u_n-T_k(u))\varphi.
\end{aligned}$$

Since $w=\operatorname{supp}(\varphi)\subset\subset\Omega$ and, by Lemma 2.2, we have $u_n\geq c_{\operatorname{supp}(\varphi)}$, then the above inequality becomes

$$\begin{aligned}
&\frac{1}{2}\int_\Omega T_h^2(u_n-T_k(u))\varphi+\alpha_1\int_{\Omega_T}|\nabla T_h(u_n-T_k(u))|^2\\
&\leq\int_{\Omega_T}|u_nE(x,t)|\varphi|\nabla T_h(u_n-T_k(u))|\\
&\quad+\int_{\Omega_T}|u_nE(x,t)||T_h(u_n-T_k(u))|\nabla\varphi\\
&\quad+h\int_{\Omega_T}|u_nE(x,t)||\nabla\varphi|+\frac{h}{c_\omega^\theta}\int_0^T\int_\omega f+\frac{h^2}{2}|\Omega|\\
&\quad-\int_{\Omega_T}M(x,t)\nabla u_n\cdot\nabla\varphi T_h(u_n-T_k(u))\\
&\quad-\int_{\Omega_T}M(x,t)\nabla T_k(u)\nabla T_h(u_n-T_k(u))\varphi.
\end{aligned}$$

By removing the first non-negative term, applying Hardy and Hölder inequalities in the first term on the right-hand side of the above estimate, we get

$$\begin{aligned}\alpha\int_{\Omega_T}|\nabla T_h(u_n-T_k(u))|^2 \le{}& \frac{B}{H}\int_{\Omega_T}|\nabla T_h(u_n-T_k(u))|^2\varphi\\&+\int_{\Omega_T}|u_nE(x,t)||T_h(u_n-T_k(u))|\nabla\varphi\\&+h\int_{\Omega_T}|u_nE(x,t)||\nabla\varphi|+\frac{h}{c_\omega^\theta}\int_0^T\int_\omega f+\frac{h^2}{2}|\Omega|\\&-\int_{\Omega_T}M(x,t)\nabla u_n\cdot\nabla\varphi T_h(u_n-T_k(u))\\&-\int_{\Omega_T}M(x,t)\nabla T_k(u)\nabla T_h(u_n-T_k(u))\varphi.\end{aligned}$$

Since $B<\alpha_1 H$, therefore the above inequality can be written as follows:

$$\begin{aligned}&\left(\alpha-\frac{B}{H}\right)\int_{\Omega_T}|\nabla T_h(u_n-T_k(u))|^2\varphi\\&\le h\int_{\Omega_T}|u_nE(x,t)||\nabla\varphi|+\frac{h}{c_\omega^\theta}\int_0^T\int_\omega f+\frac{h^2}{2}|\Omega|\\&\quad-\int_{\Omega_T}M(x,t)\nabla u_n\cdot\nabla\varphi T_h(u_n-T_k(u))-\int_{\Omega_T}M(x,t)\nabla T_k(u)\nabla T_h(u_n-T_k(u))\varphi.\end{aligned}$$

Since $\nabla T_h(u_n-T_k(u))\neq 0$ (which implies that $u_n\le h+k$), we can easily pass to the limit as n tends to ∞, thanks to (3.3), on the right-hand side of the above inequality, and we use the fact that $\alpha-\frac{B}{H}>0$ so that

$$\left(\alpha-\frac{B}{H}\right)\limsup_{n\to\infty}\int_{\Omega_T}|\nabla T_h(u_n-T_k(u))|^2\varphi\le Ch.$$

To complete the proof of lemma, we can use exactly the same techniques used in the proof of [24, Lemma 7]. Therefore, we find that

$$\nabla u_n\to\nabla u\quad \text{a.e. in } \Omega_T. \tag{3.4}$$

■

Recalling Remark 3.2, (3.3), (3.4) and by Vitali's theorem, we obtain the following convergences:

$$\lim_{n\to+\infty}\int_{\Omega_T}M(x,t)\nabla u_n\cdot\nabla\varphi=\int_{\Omega_T}M(x,t)\nabla u\cdot\nabla\varphi\quad\forall\varphi\in C_c^1(\Omega\times[0,T)) \tag{3.5}$$

and

$$\lim_{n\to+\infty}\int_{\Omega_T}u_nE(x,t)\cdot\nabla\varphi=\int_{\Omega_T}uE(x,t)\cdot\nabla\varphi\quad\forall\varphi\in C_c^1(\Omega\times[0,T)). \tag{3.6}$$

Concerning the passage to the limit on the term on the right of the approximating problem (2.1), since $\operatorname{supp}(\varphi)$ is a compact subset of $\Omega \times [0,T)$, thanks to Lemma 2.2, there exists a constant $c_{\operatorname{supp}(\varphi)} > 0$ such that $u_n \geq c_{\operatorname{supp}(\varphi)}$; then,

$$\left| \frac{f_n}{u_n + \frac{1}{n}} \varphi \right| \leq \frac{\|\varphi\|_{L^\infty(\Omega_T)}}{c_{\operatorname{supp}(\varphi)}} f,$$

for every $(x,t) \in \operatorname{supp}(\varphi)$, since it a.e. converges to $\frac{f}{u}$ for $n \to +\infty$, by Lebesgue theorem, implies that

$$\lim_{n\to+\infty} \int_{\Omega_T} \frac{f_n}{u_n + \frac{1}{n}} \varphi = \int_{\Omega_T} \frac{f}{u} \varphi \quad \forall \varphi \in C_c^1(\Omega \times [0,T)). \tag{3.7}$$

Take now $\varphi \in C_c^1(\Omega \times [0,T))$ as a test function in problem (2.1); by the convergence results (3.3), (3.5), (3.6), (3.7) and letting $n \to +\infty$, we obtain

$$-\int_{\Omega_T} u \frac{\partial \varphi}{\partial t} + \int_{\Omega_T} M(x,t) \nabla u \cdot \nabla \varphi = \int_{\Omega_T} u E(x,t) \cdot \nabla \varphi + \int_{\Omega_T} \frac{f}{u} \varphi + \int_{\Omega} u_0(x) \varphi(x,0). \qquad \blacksquare$$

In the following theorem, we state some summability of u solution to problem (1.1) which depends on B and the summability of f.

Theorem 3.5. *Let $\theta = 1$ and $0 \leq f \in L^m(\Omega_T)$ with $m \geq 1$. Then, solution u to problem (1.1) found in Theorem 3.3 satisfies the following regularity:*

(i) *If $B < \frac{\alpha_1(N-2)}{2}$ and $m > \frac{N}{2} + 1$, then $u \in L^\infty(\Omega_T)$.*

(ii) *If $B < \frac{\alpha_1(N-2)}{2} \frac{Nm}{N-2m+2}$ and $1 \leq m < \frac{N}{2} + 1$, then $u \in L^\sigma(\Omega_T)$ with*

$$\sigma = \frac{2m(N+2)}{N-2m+2}.$$

Proof. Let u_n be a solution of (2.1) given by Lemma 2.1 such that u_n converges to a solution of (1.1). In order to prove (i), we choose $G_k(u_n)$ as a test function in (2.1), where $G_k(s) = (s-k)^+$, $k \geq \max\{1, \|u_0\|_{L^\infty(\Omega)}\}$, we have

$$\int_{\Omega_t} \frac{\partial u_n}{\partial t} G_k(u_n) + \int_{\Omega_t} M(x,t) \nabla u_n \cdot \nabla G_k(u_n) = \int_{\Omega_t} \frac{u_n}{1 + \frac{1}{n}|u_n|} \frac{E(x,t)}{1 + \frac{1}{n}|E(x,t)|} \nabla G_k(u_n) + \int_{\Omega_t} \frac{f_n}{u_n + \frac{1}{n}} G_k(u_n). \tag{3.8}$$

Recalling (1.2), and taking the advantage of the knowledge that the function $G_k(u_n)$ is different from zero only on the set

$$A_{n,k} = \{(x,t) \in \Omega_T : u_n(x,t) \geq k\},$$

and that, on this set, we have $u_n + \frac{1}{n} \geq k \geq 1$, we can get the following estimate:

$$\int_{\Omega_t} M(x,t)\nabla u_n \cdot \nabla G_k(u_n) = \int_0^T \int_{A_{n,k}} M(x,t)\nabla u_n \cdot \nabla u_n$$
$$\geq \alpha_1 \int_0^T \int_{A_{n,k}} |\nabla u_n|^2 = \alpha_1 \int_0^T \int_{\Omega_t} |\nabla G_k(u_n)|^2$$

and

$$\int_{\Omega_t} \frac{\partial u_n}{\partial t} G_k(u_n) = \frac{1}{2}\int_0^T \int_{A_{n,k}} \frac{\partial}{\partial t}(u_n - k)^2 = \frac{1}{2}\int_0^T \int_{A_{n,k}} \frac{\partial}{\partial t}((u_n - k)^+)^2$$
$$= \frac{1}{2}\int_{A_{n,k}} G_k^2(u_n)(t)dx - \frac{1}{2}\int_{A_{n,k}} G_k^2(u_0)(t)dx.$$

From (1.3), applying Hölder's and Hardy's inequalities, we estimate the first term on the right-hand side of (3.8) as follows:

$$\int_{\Omega_t} \frac{u_n}{1+\frac{1}{n}|u_n|} \frac{E(x,t)}{1+\frac{1}{n}|E(x,t)|} \nabla G_k(u_n) = \int_0^T \int_{A_{n,k}} \frac{u_n}{1+\frac{1}{n}|u_n|} \frac{E(x,t)}{1+\frac{1}{n}|E(x,t)|} \nabla u_n$$
$$\int_0^T \int_{A_{n,k}} |u_n(x,t)||\nabla u_n| \leq \int_0^T \int_{A_{n,k}} \frac{|u_n|}{|x|}|\nabla u_n|$$
$$\leq B\left(\int_0^T \int_{A_{n,k}} \frac{|u_n|^2}{|x|^2}\right)^{\frac{1}{2}} \left(\int_0^T \int_{A_{n,k}} |\nabla u_n|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{B}{H}\left(\int_0^T \int_{A_{n,k}} |\nabla u_n|^2\right)^{\frac{1}{2}} \left(\int_0^T \int_{A_{n,k}} |\nabla u_n|^2\right)^{\frac{1}{2}}$$
$$= \frac{B}{H}\int_0^T \int_{A_{n,k}} |\nabla u_n|^2$$
$$= \frac{B}{H}\int_0^T \int_{A_{n,k}} |\nabla G_k(u_n)|^2. \tag{3.9}$$

Since $k \geq \|u_0\|_{L^\infty(\Omega)}$, then $G_k(u_0) = 0$, and from (3.8), (3.9) combined with the above estimates, we obtain

$$\frac{1}{2}\int_{A_{n,k}} |G_k(u_n(x,t))|^2 + \left(\alpha_1 - \frac{B}{H}\right)\int_0^t \int_{A_{k,n}} |\nabla G_k(u_n)|^2 \leq \int_0^t \int_{A_{n,k}} f G_k(u_n).$$

Passing to the supremum for $t \in (0,T)$, we get

$$\frac{1}{2}\|G_k(u_n)\|^2_{L^\infty(0,T;L^2(A_{n,k}))} + \left(\alpha_1 - \frac{B}{H}\right)\int_0^T \int_{A_{n,k}} |\nabla G_k(u_n)|^2 \leq \int_0^T \int_{A_{n,k}} f G_k(u_n).$$

Applying the Hölder inequality on the right-hand side of the above inequality, we find that

$$\|G_k(u_n)\|^2_{L^\infty(0,T;L^2(A_{n,k}))} + 2\Big(\alpha_1 - \frac{B}{H}\Big)\int_0^T\int_{A_{n,k}} |\nabla G_k(u_n)|^2 \leq C\Big(\int_0^T\int_{A_{n,k}} |G_k(u_n)|^{m'}\Big)^{\frac{1}{m'}}. \tag{3.10}$$

Applying Lemma 1.1 (here $\rho = 2$, $h = 2$, $v = G_k(u_n)$), we can write

$$\int_0^T\int_{A_{n,k}} |G_k(u_n)|^{\frac{2(N+2)}{N}} \leq \|u_n\|^{\frac{4}{N}}_{L^\infty(0,T;L^2(A_{n,k}))}\int_0^T\int_{A_{n,k}} |\nabla G_k(u_n)|^2.$$

Invoking (3.10) in the last inequality, we deduce that

$$\int_0^T\int_{A_{n,k}} |G_k(u_n)|^{\frac{2(N+2)}{N}} \leq \Big(\int_0^T\int_{A_{n,k}} |G_k(u_n)|^{m'}\Big)^{\frac{1}{m'}(\frac{2}{N}+1)}. \tag{3.11}$$

By virtue of $m > \frac{N}{2} + 1$, then $\frac{2(N+2)}{Nm'} > 1$. Applying Hölder's inequality with indices $(\frac{2(N+2)}{Nm'}, \frac{2(N+2)}{2(N+2)-Nm'})$ in (3.11), we find that

$$\int_0^T\int_{A_{n,k}} |G_k(u_n)|^{\frac{2(N+2)}{N}} \leq C\Big(\int_0^T\int_{A_{n,k}} |G_k(u_n)|^{\frac{2(N+2)}{N}}\Big)^{\frac{2+N}{2(N+2)}} \times \Big(\int_0^T |A_{n,k}|\Big)^{\frac{1}{m'}(\frac{2}{N}+1)(1-\frac{Nm'}{2(N+2)})}.$$

From now, we can repeat the same techniques used in the proof of [25, Lemma 4] (see also [1]); we deduce that there exists a constant C_∞ independent of n such that

$$\|u_n\|_{L^\infty(\Omega_T)} \leq C_\infty.$$

Therefore, $u_n \in L^\infty(\Omega_T)$, and so, $u \in L^\infty(\Omega_T)$.

Now, we consider $1 < m < \frac{N}{2} + 1$. Choosing $u_n^{2\lambda-1}$, $(\lambda > 1)$, as a test function in (2.1), we have

$$\begin{aligned}
&\frac{1}{2\lambda}\int_\Omega u_n^{2\lambda}(x,t) + (2\lambda-1)\int_{\Omega_t} u_n^{2\lambda-2}M(x,t)\nabla u_n\cdot\nabla u_n \\
&= (2\lambda-1)\int_{\Omega_t} u_n^{2\lambda-1}E(x,t)\nabla u_n + \int_{\Omega_t}\frac{f_n}{u_n+\frac{1}{n}}u_n^{2\lambda-1} + \frac{1}{2\lambda}\int_\Omega u_0^{2\lambda}(x) \\
&\leq (2\lambda-1)\int_{\Omega_t} |u_n|^{2\lambda-1}|E(x,t)||\nabla u_n| + \int_{\Omega_t} f u_n^{2\lambda-2} + \frac{1}{2\lambda}\int_\Omega u_0^{2\lambda}(x).
\end{aligned} \tag{3.12}$$

Condition (1.2) allows us to write

$$\int_{\Omega_t} u_n^{2\lambda-2}M(x,t)\nabla u_n\cdot\nabla u_n \geq \alpha_1\int_{\Omega_t} u_n^{2\lambda-2}|\nabla u_n|^2 = \frac{\alpha_1}{\lambda^2}\int_{\Omega_t}|\nabla u_n^\lambda|^2.$$

From (1.3) and using the Hölder and Hardy inequalities, we can estimate the first term on the right-hand side of (3.12) as follows:

$$\int_{\Omega_t} |u_n|^{2\lambda-1}|E(x,t)||\nabla u_n| \leq B\int_{\Omega_t}\frac{u_n^{2\lambda-1}}{|x|}|\nabla u_n| = \frac{B}{\lambda}\int_{\Omega_t}\frac{u_n^{\lambda}}{|x|}|\nabla u_n^{\lambda}|$$
$$\leq \frac{B}{\lambda H}\int_{\Omega_t}|\nabla u_n^{\lambda}|^2.$$

Using the last two estimates in (3.12) and applying Hölder's inequality, and by the fact that $u_0 \in L^\infty(\Omega)$, we obtain

$$\frac{1}{2\lambda}\int_{\Omega} u_n^{2\lambda}(x,t) + \frac{2\lambda-1}{\lambda}\left(\frac{\alpha_1}{\lambda} - \frac{B}{H}\right)\int_{\Omega_t}|\nabla u_n^{\lambda}|^2$$
$$\leq \|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T} u_n^{(2\lambda-2)m'}\right)^{\frac{1}{m'}} + C(\|u_0\|_{L^{2\lambda}(\Omega)}).$$

Now, passing to the supremum for $t \in [0,T]$, we find that

$$\frac{1}{2\lambda}\|u_n^{\lambda}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{2\lambda-1}{\lambda}\left(\frac{\alpha_1}{\lambda} - \frac{B}{H}\right)\int_{\Omega_T}|\nabla u_n^{\lambda}|^2$$
$$\leq \|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T} u_n^{(2\lambda-2)m'}\right)^{\frac{1}{m'}} + C(\|u_0\|_{L^{2\lambda}(\Omega)})$$
$$= C\left(\int_{\Omega_T} u_n^{(2\lambda-2)m'}\right)^{\frac{1}{m'}} + C. \tag{3.13}$$

Applying Lemma 1.1 (where $h = 2$, $\rho = 2$, $v = u_n^{\lambda}$) and from (3.13), we obtain

$$\int_{\Omega_T}[u_n^{\lambda}]^{\frac{2(N+2)}{N}} \leq \|u_n^{\lambda}\|^{\frac{4}{N}}_{L^\infty(0,T;L^2(\Omega))}\int_{\Omega_T}|\nabla u_n^{\lambda}|^2$$
$$\leq C\left(\int_{\Omega_T} u_n^{(2\lambda-2)m'}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C.$$

By a straightforward simplification, the above estimate becomes

$$\int_{\Omega_T} u_n^{\frac{2\lambda(N+2)}{N}} \leq C\left(\int_{\Omega_T} u_n^{(2\lambda-2)m'}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C. \tag{3.14}$$

Choosing λ such that

$$\sigma = \frac{2\lambda(N+2)}{N} = (2\lambda-2)m' \tag{3.15}$$

yields

$$\lambda = \frac{Nm}{N-2m+2}, \quad \sigma = \frac{2m(N+2)}{N-2m+2}.$$

Note that $\lambda > 1$ is equivalent to $m > 1$ and the condition $B < \frac{\alpha_1(N-2)}{2}\frac{Nm}{N-2m+2}$ ensures that $\frac{\alpha_1}{\lambda} - \frac{B}{H} > 0$. Using (3.15) in (3.14), we reach that

$$\int_{\Omega_T} u_n^\sigma \le C\Big(\int_{\Omega_T} u_n^\sigma\Big)^{(\frac{2}{N}+1)\frac{1}{m'}} + C.$$

Since $m < \frac{N}{2}+1$, then $(\frac{2}{N}+1)\frac{1}{m'} < 1$; we can apply the Young inequality in the above estimate, arriving at

$$\int_{\Omega_T} u_n^\sigma \le C.$$

Hence, the sequence $u_n \in L^\sigma(\Omega_T)$, and so, $u \in L^\sigma(\Omega_T)$. ■

3.2. The case $\theta > 1$

At present, we deal with the case of $\theta > 1$. In this section, we prove the boundedness of some positive power of u_n in $L^2(0,T;H_0^1(\Omega))$; and also, we prove the boundedness of u_n in $L^2(0,T;H^1_{\text{loc}}(\Omega))$.

Lemma 3.6. *Let $\theta > 1$, $B < \frac{\alpha_1(N-2)}{\theta+1}$, u_n be the solution to (2.1) with $0 \le f \in L^1(\Omega_T)$. Then, $u_n^{\frac{\theta+1}{2}}$ is bounded in the space $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, and u_n is bounded in $L^{\frac{N}{N-2}}(0,T;L^{\frac{N(\theta+1)}{N-2}}(\Omega))$. Moreover, $u_n \in L^2(0,T;H^1_{\text{loc}}(\Omega))$.*

Proof. Choosing $u_n^\theta \chi_{(0,t)}$, $t \in (0,T)$ as a test function in (2.1), from (1.2) and using the fact that $u_0 \in L^\infty(\Omega)$, $\frac{u_n^\theta}{(u_n+\frac{1}{n})^\theta} \le 1$, we get

$$\begin{aligned}
&\frac{1}{\theta+1}\int_\Omega u_n^{\theta+1}(x,t) + \alpha_1\theta\int_{\Omega_t} |\nabla u_n|^2 u_n^{\theta-1}\\
&\le \frac{1}{\theta+1}\int_\Omega u_n^{\theta+1}(x,t) + \theta\int_{\Omega_t} M(x,t)\nabla u_n\cdot\nabla u_n u_n^{\theta-1}\\
&\le \theta\int_{\Omega_t} |u_n^\theta E(x,t)||\nabla u_n| + \int_{\Omega_t}\frac{u_n^\theta}{(u_n+\frac{1}{n})^\theta} f + \frac{1}{\theta+1}\int_\Omega u_0^{\theta+1}(x)\\
&\le \theta\int_{\Omega_t} |u_n^\theta E(x,t)||\nabla u_n| + \int_{\Omega_t} f + \frac{1}{\theta+1}\|u_0\|^{\theta+1}_{L^{\theta+1}(\Omega)}.
\end{aligned} \tag{3.16}$$

Applying the Hölder inequality and Hardy inequality to the first term on the right-hand side of (3.16), we find that

$$\begin{aligned}
\theta\int_{\Omega_t} |u_n^\theta E(x,t)||\nabla u_n| &\le \frac{2\theta B}{\theta+1}\int_{\Omega_t}\frac{u_n^{\frac{\theta+1}{2}}}{|x|}|\nabla u_n^{\frac{\theta+1}{2}}|\\
&\le \frac{2\theta B}{(\theta+1)H}\int_{\Omega_t}|\nabla u_n^{\frac{\theta+1}{2}}|^2.
\end{aligned} \tag{3.17}$$

Note that

$$\int_{\Omega_t} |\nabla u_n|^2 u_n^{\theta-1} = \frac{4}{(\theta+1)^2} \int_{\Omega_t} |\nabla u_n^{\frac{\theta+1}{2}}|^2.$$

The last equation combined with (3.17) leads to

$$\begin{aligned} &\frac{1}{\theta+1} \int_\Omega u_n^{\theta+1}(x,t) + \left(\frac{4\alpha_1\theta}{(\theta+1)^2} - \frac{2\theta B}{(\theta+1)H} \right) \int_{\Omega_t} |\nabla u_n^{\frac{\theta+1}{2}}|^2 \\ &\leq \int_{\Omega_t} f + \frac{1}{\theta+1} \|u_0\|^{\theta+1}_{L^{\theta+1}(\Omega)} = C. \end{aligned}$$

Passing now to the supremum for $t \in (0,T)$, and using the fact that $f \in L^1(\Omega_T)$, we obtain

$$\frac{1}{\theta+1} \|u_n\|^{\theta+1}_{L^\infty(0,T;L^{\theta+1}(\Omega))} + \left(\frac{4\alpha_1\theta}{(\theta+1)^2} - \frac{2\theta B}{(\theta+1)H} \right) \int_{\Omega_T} |\nabla u_n^{\frac{\theta+1}{2}}|^2 \leq C.$$

Since $B < \frac{\alpha_1(N-2)}{\theta+1}$, then $\frac{4\alpha_1\theta}{(\theta+1)^2} - \frac{2\theta B}{(\theta+1)H} > 0$. Therefore,

$$\|u_n\|^{\theta+1}_{L^\infty(0,T;L^{\theta+1}(\Omega))} + \int_{\Omega_t} |\nabla u_n^{\frac{\theta+1}{2}}|^2 \leq C. \tag{3.18}$$

By Sobolev embedding theorem and from (3.18), we can write

$$\begin{aligned} \int_0^T \left(\int_\Omega u_n^{\frac{N(\theta+1)}{N-2}} \right)^{\frac{N-2}{N}} &= \int_0^T \left(\int_\Omega u_n^{\frac{\theta+1}{2}2^*} \right)^{\frac{2}{2^*}} \\ &\leq C \int_0^T \int_\Omega |\nabla u_n^{\frac{\theta+1}{2}}|^2 = C \int_{\Omega_T} |\nabla u_n^{\frac{\theta+1}{2}}|^2 \leq C. \end{aligned} \tag{3.19}$$

The estimates (3.18) and (3.19) imply the boundedness of the sequence u_n in the space $L^\infty(0,T;L^{\theta+1}(\Omega)) \cap L^{\frac{N}{N-2}}(0,T;L^{\frac{N(\theta+1)}{N-2}}(\Omega))$ and the boundedness of the sequence $u_n^{\frac{\theta+1}{2}}$ in $L^2(0,T;H_0^1(\Omega))$.

In order to prove $u_n \in L^2(0,T;H^1_{\mathrm{loc}}(\Omega))$, recalling Lemma 2.2 and using (3.18), we have that, for all $\omega \subset\subset \Omega$,

$$c_\omega^{\theta-1} \int_{\omega\times(0.T)} |\nabla u_n|^2 \leq \int_{\Omega_T} u_n^{\theta-1}|\nabla u_n|^2 = \frac{4}{(\theta+1)^2} \int_{\Omega_T} |\nabla u_n^{\frac{\theta+1}{2}}|^2 \leq C. \tag{3.20}$$

This last affirmation implies the boundedness of the sequence $|\nabla u_n|$ in $L^2(\omega \times (0,T))$.

Moreover, u_n is bounded in $L^2(0,T;H^1_{\mathrm{loc}}(\Omega))$; in fact, if $\omega \subset\subset \Omega$ is fixed, using the boundedness of $u_n^{\frac{\theta+1}{2}}$ in $L^2(0,T;H_0^1(\Omega))$, we find that

$$\left(\int_{\omega\times(0.T)} |u_n|^2 \right)^{\frac{1}{2}} \leq C \left(\int_{\omega\times(0.T)} |u_n|^{\theta+1} \right)^{\frac{1}{\theta+1}} \leq C. \tag{3.21}$$

From (3.20) and (3.21), we conclude that u_n is bounded in $L^2(0,T;H^1_{\mathrm{loc}}(\Omega))$. ∎

Now, we state in the following theorem the existence of weak solution to problem (1.1) when $\theta > 1$.

Theorem 3.7. *Let $\theta > 1$, $0 \le f \in L^1(\Omega_T)$, and $B < \frac{\alpha_1(N-2)}{\theta+1}$. Then, there is a solution $u \in L^2(0,T;H^1_{loc}(\Omega))$ and $u^{\frac{\theta+1}{2}} \in L^2(0,T;H^1_0(\Omega))$ in the sense of Definition 1.2. Furthermore,*

$$u \in L^{\frac{N}{N-2}}(0,T;L^{\frac{N(\theta+1)}{N-2}}(\Omega)) \cap L^\infty(0,T;L^{\theta+1}(\Omega)).$$

Proof. The proof of Theorem 3.7 is similar to the proof of Theorem 3.3. ■

In the following theorem, we give the summability of the solution u when $\theta > 1$.

Theorem 3.8. *Let $\theta > 1$, $0 \le f \in L^m(\Omega_T)$ with $m \ge 1$. Then, the solution u of (1.1) given by Theorem 3.7 satisfies the following regularity:*

(i) *If $B < \frac{\alpha_1(N-2)}{2}$ and $m > \frac{N}{2} + 1$, then $u \in L^\infty(\Omega_T)$.*

(ii) *If $B < \frac{\alpha_1(N-2)(N-2m+2)}{Nm(1+\theta)}$ and $1 \le m < \frac{N}{2} + 1$, then $u \in L^\sigma(\Omega_T)$ with $\sigma = \frac{m(N+2)(1+\theta)}{N-2m+2}$.*

Proof. Let u_n be a solution of (2.1) given by Lemma 2.1 such that u_n converges to a solution of (1.1).

The proof of item (i) of Theorem 3.8 is similar to item (i) of Theorem 3.5, so we omit it.

Now, we give the proof of (ii). If $m = 1$, the result comes from the fact that $u^{\frac{\theta+1}{2}} \in L^2(0,T;H^1_0(\Omega))$ and the Sobolev embedding theorem.

If $1 < m < \frac{N}{2} + 1$, taking $u_n^{2\lambda-1}\chi_{(0,t)}$, $t \in (0,T)$ and $\lambda \ge \frac{\theta+1}{2}$, as a test function in (2.1), we have

$$\begin{aligned} &\frac{1}{2\lambda}\int_\Omega u_n^{2\lambda}(x,t) + (2\lambda-1)\int_{\Omega_t} M(x,t)\nabla u_n \cdot \nabla u_n u_n^{2\lambda-2} \\ &\quad = (2\lambda-1)\int_{\Omega_t} u_n^{2\lambda-1}E(x,t)\nabla u_n + \int_{\Omega_t}\frac{f_n}{(u_n+\frac{1}{n})^\theta}u_n^{2\lambda-1} + \frac{1}{2\lambda}\int_\Omega u_0^{2\lambda}(x) \\ &\quad \le (2\lambda-1)\int_{\Omega_t}|u_n|^{2\lambda-1}|E(x,t)||\nabla u_n| + \int_{\Omega_t} f u_n^{2\lambda-1-\theta} + \frac{1}{2\lambda}\int_\Omega u_0^{2\lambda}(x). \end{aligned}$$

Repeating the same argument used in the proof of Theorem 3.5, we have

$$\int_{\Omega_T} u_n^{\frac{2\lambda(N+2)}{N}} \le C\left(\int_{\Omega_T} u_n^{(2\lambda-1-\theta)m'}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C. \tag{3.22}$$

Now, choose λ such that

$$\sigma = \frac{2\lambda(N+2)}{N} = (2\lambda-1-\theta)m'. \tag{3.23}$$

From the last equality, we get the following equalities:

$$\lambda = \frac{Nm(1+\theta)}{2(N-2m+2)}, \quad \sigma = \frac{m(N+2)(1+\theta)}{N-2m+2}.$$

From (3.23), inequality (3.22) becomes

$$\int_{\Omega_T} u_n^\sigma \leq C\Big(\int_{\Omega_T} u_n^\sigma\Big)^{(\frac{2}{N}+1)\frac{1}{m'}} + C.$$

The condition $\lambda > \frac{\theta+1}{2}$ is equivalent to $m > 1$. Since $m < \frac{N}{2}+1$, then $(\frac{2}{N}+1)\frac{1}{m'} < 1$; we can apply the Young inequality to the above estimate, arriving at

$$\int_{\Omega_T} u_n^\sigma \leq C.$$

Hence, the sequence $u_n \in L^\sigma(\Omega_T)$, and so, $u \in L^\sigma(\Omega_T)$. Therefore, the proof of Theorem 3.8 is completed. ■

3.3. The case $\theta < 1$

In this section, we will prove the existence of solution $u \in L^2(0,T;H_0^1(\Omega))$ to problem (1.1) for $m \geq \frac{2(N+2)}{2(N+2)-N(1-\theta)}$ and for some condition assured at B. We will also prove the existence of solution u belonging to some space larger than $L^2(0,T;H_0^1(\Omega))$ if $1 \leq m < \frac{2(N+2)}{2(N+2)-N(1-\theta)}$.

Lemma 3.9. *Let $B < \frac{\alpha_1(N-2)}{2}$, $\theta < 1$. Let u_n be the solution to (2.1) and $0 \leq f \in L^m(\Omega_T)$ with $m = \frac{2(N+2)}{2(N+2)-N(1-\theta)}$. Then, u_n is uniformly bounded in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^{\frac{2(N+2)}{N}}(\Omega_T)$.*

Proof. Let $u_n\chi_{(0,t)}$, $t \in (0,T)$, be a test function in (2.1), and using (1.2) and the fact that $u_0 \in L^\infty(\Omega)$, we obtain

$$\frac{1}{2}\int_\Omega u_n^2(x,t) + \alpha_1\int_{\Omega_t}|\nabla u_n|^2 \leq \int_{\Omega_t}|u_nE(x,t)||\nabla u_n| + \int_{\Omega_t} f u_n^{1-\theta} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{3.24}$$

From (1.3), applying Hölder's and Hardy's inequalities, we estimate the first term on the right-hand side of (3.1) as follows:

$$\begin{aligned}\int_{\Omega_T}|u_nE(x,t)||\nabla u_n| &\leq B\int_{\Omega_T}\frac{|u_n|}{|x|}|\nabla u_n| \\ &\leq B\Big(\int_{\Omega_T}\frac{|u_n|^2}{|x|^2}\Big)^{\frac{1}{2}}\Big(\int_{\Omega_T}|\nabla u_n|^2\Big)^{\frac{1}{2}} \\ &\leq \frac{B}{H}\int_{\Omega_T}|\nabla u_n|^2.\end{aligned} \tag{3.25}$$

Combining (3.25) with (3.24) and applying Hölder's inequality and passing to the supremum for $t \in (0, T)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|u_n\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(\alpha_1 - \frac{B}{H}\right)\int_{\Omega_T} |\nabla u_n|^2 \\
&\le \|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T} u_n^{(1-\theta)m'}\right)^{\frac{1}{m'}} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \\
&= C\left(\int_{\Omega_T} u_n^{(1-\theta)m'}\right)^{\frac{1}{m'}} + C. \qquad (3.26)
\end{aligned}$$

Now, applying Lemma 1.1 (where $h = 2$, $\rho = 2$ and $v = u_n$) and using inequality (3.26), we can write

$$\begin{aligned}
\int_{\Omega_T} u_n^{\frac{2(N+2)}{N}} &\le C\|u_n\|^{\frac{4}{N}}_{L^\infty(0,T;L^2(\Omega))}\int_{\Omega_T} |\nabla u_n|^2 \\
&\le C\left(\int_{\Omega_T} u_n^{(1-\theta)m'}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C. \qquad (3.27)
\end{aligned}$$

Based on the assumption of m, it is easy to check that

$$\frac{2(N+2)}{N} = (1-\theta)m'. \qquad (3.28)$$

Invoking (3.28) in (3.27), we find that

$$\int_{\Omega_T} u_n^{\frac{2(N+2)}{N}} \le C\left(\int_{\Omega_T} u_n^{\frac{2(N+2)}{N}}\right)^{\frac{1-\theta}{2}} + C.$$

Since $\theta < 1$, then $\frac{1-\theta}{2} < 1$; we can apply the Young inequality, obtaining

$$\int_{\Omega_T} u_n^{\frac{2(N+2)}{N}} \le C. \qquad (3.29)$$

This last estimate implies the boundedness of the sequence u_n in $L^{\frac{2(N+2)}{N}}(\Omega_T)$. Since $B < \frac{\alpha_1(N-2)}{2}$, then $\alpha_1 - \frac{B}{H} > 0$, and using (3.29) in (3.26), we obtain

$$\frac{1}{2}\|u_n\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(\alpha_1 - \frac{B}{H}\right)\int_{\Omega_T} |\nabla u_n|^2 \le C\left(\int_{\Omega_T} u_n^{\frac{2(N+2)}{N}}\right)^{\frac{1}{m'}} + C \le C. \qquad (3.30)$$

The estimates (3.29) and (3.30) give the boundedness of the sequence u_n in the space $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L^{\frac{2(N+2)}{N}}(\Omega_T)$. ■

In the following theorem, we establish an existence result for problem (1.1) in the limit case $m = \frac{2(N+2)}{2(N+2)-N(1-\theta)}$.

Theorem 3.10. *Let $\theta < 1$, $B < \frac{\alpha_1(N-2)}{2}$, and $0 \le f \in L^m(\Omega_T)$ with*

$$m = \frac{2(N+2)}{2(N+2) - N(1-\theta)}.$$

Then, there exists a solution $u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^{\frac{2(N+2)}{N}}(\Omega_T)$ in the sense of Definition 1.2.

Proof. To check the proof of Theorem 3.10, we repeat the same proof used in Theorem 3.3. ∎

Theorem 3.11. *Let $\theta < 1$ and $0 \le f \in L^m(\Omega_T)$ with $m \ge \frac{2(N+2)}{2(N+2)-N(1-\theta)}$. Then, the solution u of problem (1.1) found in Theorem 3.10 satisfies the following summability:*

(i) *If $B < \frac{\alpha_1(N-2)}{2}$ and $m > \frac{N}{2} + 1$, then $u \in L^\infty(\Omega_T)$.*

(ii) *If $B < \frac{\alpha_1(N-2)(N-2m+2)}{Nm(1+\theta)}$ and $1 \le m < \frac{N}{2} + 1$, then $u \in L^\sigma(\Omega_T)$ with $\sigma = \frac{m(N+2)(1+\theta)}{N-2m+2}$.*

Proof. Let u_n be a solution of (2.1) given by Lemma 2.1 such that u_n converges to a solution of (1.1).

The proof of item (i) of Theorem 3.11 is similar to item (i) of Theorem 3.5, so we omit it.

(ii) The case $m = \frac{2(N+2)}{2(N+2)-N(1-\theta)}$ is true via the Gagliardo–Nirenberg inequality, since for this value of m one has

$$\sigma = \frac{2(N+2)}{N}.$$

If $\frac{2(N+2)}{2(N+2)-N(1-\theta)} \le m < \frac{N}{2} + 1$, we choose $\varphi(u_n) = u_n^\lambda \chi_{(0,t)}$, $(\lambda \ge 1)$, as a test function in (2.1); we have

$$\begin{aligned} &\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}(x,t) + \lambda \int_{\Omega_t} u_n^{\lambda-1} M(x,t)\nabla u_n \cdot \nabla u_n \\ &\quad \le \lambda \int_{\Omega_t} |u_n^\lambda E(x,t)||\nabla u_n| + \int_{\Omega_t} \frac{u_n^\lambda}{(u_n + \frac{1}{n})^\theta} f + \frac{1}{\lambda+1}\int_\Omega u_0^{\lambda+1}(x). \end{aligned}$$

From the condition (1.2) and the fact that $\frac{1}{(u_n+\frac{1}{n})^\theta} \le \frac{1}{u_n^\theta}$, $u_0 \in L^\infty(\Omega)$, we can write

$$\begin{aligned} &\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}(x,t) + \lambda\alpha_1 \int_{\Omega_t} u_n^{\lambda-1}|\nabla u_n|^2 \\ &\quad \le \lambda \int_{\Omega_t} |u_n^\lambda E(x,t)||\nabla u_n| + \int_{\Omega_t} f u_n^{\lambda-\theta} + C. \end{aligned} \tag{3.31}$$

Observe that

$$\int_{\Omega_t} u_n^{\lambda-1}|\nabla u_n|^2 = \frac{4}{(\lambda+1)^2}\int_{\Omega_t} |\nabla u_n^{\frac{\lambda+1}{2}}|^2. \tag{3.32}$$

Recalling condition (1.3), applying the Hölder and Hardy inequalities, we estimate the first term on the right-hand side of (3.31) as follows:

$$\begin{aligned}\int_{\Omega_t} |u_n^\lambda E(x,t)||\nabla u_n| &\leq B\int_0^t\int_\Omega \frac{u_n^\lambda}{|x|}|\nabla u_n| \\ &= \frac{2B}{\lambda+1}\int_0^t\int_\Omega \frac{u_n^{\frac{\lambda+1}{2}}}{|x|}|\nabla u_n^{\frac{\lambda+1}{2}}| \\ &\leq \frac{2B}{\lambda+1}\int_0^t\left(\int_\Omega \frac{(u_n^{\frac{\lambda+1}{2}})^2}{|x|^2}\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u_n^{\frac{\lambda+1}{2}}|^2\right)^{\frac{1}{2}} \\ &\leq \frac{2B}{\lambda+1}\int_0^t\left(\int_\Omega |\nabla u_n^{\frac{\lambda+1}{2}}|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u_n^{\frac{\lambda+1}{2}}|^2\right)^{\frac{1}{2}} \\ &= \frac{2B}{(\lambda+1)H}\int_{\Omega_t} |\nabla u_n^{\frac{\lambda+1}{2}}|^2. \end{aligned} \tag{3.33}$$

Invoking (3.32), (3.33) in (3.31) and applying Hölder's inequality, we obtain that

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}(x,t) + \frac{2\lambda}{\lambda+1}\left(\frac{2\alpha_1}{\lambda+1} - \frac{B}{H}\right)\int_{\Omega_t}|\nabla u_n^{\frac{\lambda+1}{2}}|^2 \\ &\leq C\left(\int_{\Omega_t} u_n^{(\lambda-\theta)m'}\right)^{\frac{1}{m'}} + C. \end{aligned} \tag{3.34}$$

By some simplification, inequality (3.34) becomes

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega [|u_n(x,t)|^{\frac{\lambda+1}{2}}]^2 + \frac{2\lambda}{\lambda+1}\left(\frac{2\alpha_1}{\lambda+1} - \frac{B}{H}\right)\int_{\Omega_t}|\nabla u_n^{\frac{\lambda+1}{2}}|^2 \\ &\leq C\left(\int_{\Omega_t} u_n^{(\lambda-\theta)m'}\right)^{\frac{1}{m'}} + C. \end{aligned}$$

Now, passing to supremum for $t \in (0,T)$, we get

$$\begin{aligned}&\frac{1}{\lambda+1}\|u_n^{\frac{\lambda+1}{2}}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{2\lambda}{\lambda+1}\left(\frac{2\alpha_1}{\lambda+1} - \frac{B}{H}\right)\int_{\Omega_T}|\nabla u_n^{\frac{\lambda+1}{2}}|^2 \\ &\leq C\left(\int_{\Omega_T} u_n^{(\lambda-\theta)m'}\right)^{\frac{1}{m'}} + C. \end{aligned} \tag{3.35}$$

Recalling Lemma 1.1 (where $v = u_n^{\frac{\lambda+1}{2}}$, $\rho = 2$, $h = 2$) and from (3.35), we have

$$\begin{aligned}\int_{\Omega_T} [u_n^{\frac{\lambda+1}{2}}]^{\frac{2(N+2)}{N}} &\leq \left(\|u_n^{\frac{\lambda+1}{2}}\|^2_{L^\infty(0,T;L^2(\Omega))}\right)^{\frac{2}{N}}\int_{\Omega_T}|\nabla u_n^{\frac{\lambda+1}{2}}|^2 \\ &\leq C\left(\int_{\Omega_T} u_n^{(\lambda-\theta)m'}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C. \end{aligned}$$

Therefore,

$$\int_{\Omega_T} u_n^{\frac{(\lambda+1)(N+2)}{N}} \le C\left(\int_{\Omega_T} u_n^{(\lambda-\theta)m'}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C. \tag{3.36}$$

Now, choosing λ such that

$$\sigma = \frac{(N+2)(\lambda+1)}{N} = (\lambda-\theta)m', \tag{3.37}$$

this implies that

$$\lambda = \frac{(N+2)(m-1)+N\theta m}{N-2m+2} \quad \text{and} \quad \sigma = \frac{m(N+2)(\theta+1)}{N-2m+2}.$$

From (3.37), inequality (3.36) becomes

$$\int_{\Omega_T} u_n^{\sigma} \le C\left(\int_{\Omega_T} u_n^{\sigma}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C. \tag{3.38}$$

The condition $m < N/2+1$ ensures that $(2/N+1)\frac{1}{m'} < 1$; then, applying the Young inequality in (3.38), we find that

$$\int_{\Omega_T} u_n^{\sigma} \le C. \tag{3.39}$$

Note that the condition $m \ge \frac{2(N+2)}{2(N+2)-N(1-\theta)}$ is equivalent to the condition $\lambda \ge 1$. Therefore, inequality (3.39) implies that $u_n \in L^{\sigma}(\Omega_T)$. Thanks to the almost everywhere convergence of u_n, we can use Fatou's lemma, obtaining $u \in L^{\sigma}(\Omega_T)$. Hence, the proof of Theorem 3.11 is completed. ∎

In the following lemma, we will prove some a priori estimate for u_n; the solution of problem (2.1) in the Sobolev space is larger than $L^2(0,T;H_0^1(\Omega))$.

Lemma 3.12. *Let $\theta < 1$, $0 < B < \frac{\alpha_1(N-2)(N-2m+2)}{\sqrt{2}Nm(1+\theta)}$, and $0 \le f \in L^m(\Omega_T)$ with*

$$1 \le m < \frac{2(N+2)}{2(N+2)-N(1-\theta)}.$$

Then, u_n is uniformly bounded in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^{\sigma}(\Omega_T)$, where

$$\sigma = \frac{m(N+2)(\theta+1)}{N-2m+2} \quad \textit{and} \quad q = \frac{m(N+2)(\theta+1)}{N+2-m(1-\theta)}.$$

Proof. We fix $\varepsilon < \frac{1}{n}$, and we take $\varphi(u_n) = ((u_n+\varepsilon)^{\lambda} - \varepsilon^{\lambda})\chi_{(0,t)}$, $(\theta < \lambda \le 1)$ as test function in (2.1); we have

$$\int_{\Omega} \Psi(u_n(x,t)) + \lambda \int_{\Omega_t} (u_n+\varepsilon)^{\lambda-1} M(x,t)\nabla u_n \cdot \nabla u_n$$
$$\le \lambda \int_{\Omega_t} |u_n E(x,t)||\nabla u_n (u_n+\varepsilon)^{\lambda-1}| + \int_{\Omega_t} \frac{(u_n+\varepsilon)^{\lambda}}{(u_n+\frac{1}{n})^{\theta}} f + \int_{\Omega} \Psi(u_0(x)),$$

where $\Psi(s) = \int_0^s \varphi(\ell)d\ell$. From condition (1.2) and the fact that

$$\frac{1}{(u_n + \frac{1}{n})^\theta} \le \frac{1}{(u_n + \varepsilon)^\theta}, \quad u_0 \in L^\infty(\Omega),$$

we can write

$$\begin{aligned}\int_\Omega \Psi(u_n(x,t)) + \lambda\alpha_1 \int_{\Omega_t} (\varepsilon + u_n)^{\lambda-1}|\nabla u_n|^2 \\ \le \lambda \int_{\Omega_t} |u_n E(x,t)(u_n+\varepsilon)^{\lambda-1}\nabla u_n| \\ + \int_{\Omega_t} f(u_n+\varepsilon)^{\lambda-\theta} + C.\end{aligned} \tag{3.40}$$

Observe that

$$\int_{\Omega_t} (u_n+\varepsilon)^{\lambda-1}|\nabla u_n|^2 = \frac{4}{(\lambda+1)^2}\int_{\Omega_t} |\nabla((u_n+\varepsilon)^{\frac{\lambda+1}{2}} - \varepsilon^{\frac{\lambda+1}{2}})|^2. \tag{3.41}$$

From (1.3), applying Hölder's and Hardy's inequalities, we can estimate the first term on the right-hand side of (3.40) as follows:

$$\begin{aligned}&\int_{\Omega_t} |u_n E(x,t)(u_n+\varepsilon)^{\lambda-1}\nabla u_n| \\ &\le B\int_{\Omega_t} \frac{(u_n+\varepsilon)^\lambda|\nabla u_n|}{|x|} \\ &= B\int_{\Omega_t} \frac{(u_n+\varepsilon)^{\frac{\lambda+1}{2}}(u_n+\varepsilon)^{\frac{\lambda-1}{2}}|\nabla u_n|}{|x|} \\ &\le B\Big(\int_{\Omega_t} \frac{((u_n+\varepsilon)^{\frac{\lambda+1}{2}})^2}{|x|^2}\Big)^{\frac{1}{2}}\Big(\int_{\Omega_t}(u_n+\varepsilon)^{\lambda-1}|\nabla u_n|^2\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.42}$$

We use the algebraic inequality

$$(a+b)^2 \le 2a^2 + 2b^2, \quad \forall a \ge 0,\ b \ge 0,$$

and Hardy's inequality; we can write

$$\begin{aligned}\int_{\Omega_t} \frac{((u_n+\varepsilon)^{\frac{\lambda+1}{2}})^2}{|x|^2} &= \int_{\Omega_t} \frac{((u_n+\varepsilon)^{\frac{\lambda+1}{2}} - \varepsilon^{\frac{\lambda+1}{2}} + \varepsilon^{\frac{\lambda+1}{2}})^2}{|x|^2} \\ &\le 2\int_{\Omega_t} \frac{((u_n+\varepsilon)^{\frac{\lambda+1}{2}} - \varepsilon^{\frac{\lambda+1}{2}})^2}{|x|^2} + 2\int_{\Omega_t} \frac{\varepsilon^{\lambda+1}}{|x|^2} \\ &\le \frac{2}{H^2}\int_{\Omega_t} |\nabla((u_n+\varepsilon)^{\frac{\lambda+1}{2}} - \varepsilon^{\frac{\lambda+1}{2}})|^2 + 2\int_{\Omega_t}\frac{\varepsilon^{\lambda+1}}{|x|^2}.\end{aligned} \tag{3.43}$$

Invoking (3.41) and (3.43) in (3.36) and applying Young's inequality, we find that

$$\begin{aligned}
&\int_{\Omega_t} |u_n E(x,t)(u_n+\varepsilon)^{\lambda-1}\nabla u_n| \\
&\le B\left(\frac{2}{H^2}\int_{\Omega_t} |\nabla((u_n+\varepsilon)^{\frac{\lambda+1}{2}}-\varepsilon^{\frac{\lambda+1}{2}})|^2+2\int_{\Omega_t}\frac{\varepsilon^{\lambda+1}}{|x|^2}\right)^{\frac{1}{2}}\left(\int_{\Omega_t}(u_n+\varepsilon)^{\lambda-1}|\nabla u_n|^2\right)^{\frac{1}{2}} \\
&\le \frac{2\sqrt{2}B}{(\lambda+1)H}\left(\int_{\Omega_t} |\nabla((u_n+\varepsilon)^{\frac{\lambda+1}{2}}-\varepsilon^{\frac{\lambda+1}{2}})|^2+H^2\int_{\Omega_t}\frac{\varepsilon^{\lambda+1}}{|x|^2}\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{\Omega_t}\nabla((u_n+\varepsilon)^{\frac{\lambda+1}{2}}+\varepsilon^{\frac{\lambda+1}{2}})|^2\right)^{\frac{1}{2}} \\
&\le \frac{2\sqrt{2}B}{(\lambda+1)H}\int_{\Omega_t} |\nabla((u_n+\varepsilon)^{\frac{\lambda+1}{2}}-\varepsilon^{\frac{\lambda+1}{2}})|^2+\frac{\sqrt{2}BH}{\lambda+1}\int_{\Omega_t}\frac{\varepsilon^{\lambda+1}}{|x|^2}.
\end{aligned}$$

In view of (3.41), (3.42) and applying Hölder's inequality, (3.40) becomes

$$\begin{aligned}
&\int_{\Omega}\Psi(u_n(x,t))+\frac{4\lambda}{\lambda+1}\left(\frac{\alpha_1}{\lambda+1}-\frac{B}{\sqrt{2}H}\right)\int_{\Omega_t}|\nabla((u_n+\varepsilon)^{\frac{\lambda+1}{2}}-\varepsilon^{\frac{\lambda+1}{2}})|^2 \\
&\quad\le C\left(\int_{\Omega_T}(u_n+\varepsilon)^{(\lambda-\theta)m'}\right)^{\frac{1}{m'}}+C+\frac{\sqrt{2}BH}{\lambda+1}\int_{\Omega_t}\frac{\varepsilon^{\lambda+1}}{|x|^2}.
\end{aligned}\tag{3.44}$$

If $\theta\le\lambda<1$, by the definitions of $\varphi(s)$ and $\Psi(s)$, we can get

$$\Psi(s)\ge C_\lambda|s|^{\lambda+1}+\widetilde{C}_\lambda\quad\forall s\ge 0.$$

Since

$$\frac{\sqrt{2}BH}{(\lambda+1)}\int_{\Omega_t}\frac{\varepsilon^{\lambda+1}}{|x|^2}<+\infty,$$

and using the last inequality in (3.44), we find that

$$\begin{aligned}
&C_\lambda\int_{\Omega}|u_n|^{\lambda+1}+\lambda\left(\alpha_1-\frac{B(\lambda+1)}{\sqrt{2}H}\right)\int_{\Omega_t}(u_n+\varepsilon)^{\lambda-1}|\nabla u_n|^2 \\
&\quad\le C\left(\int_{\Omega_T}(u_n+\varepsilon)^{(\lambda-\theta)m'}\right)^{\frac{1}{m'}}+C+\widetilde{C}_\lambda|\Omega|.
\end{aligned}$$

Passing to the supremum for $t\in(0,T)$, we obtain

$$\begin{aligned}
&C_\lambda\|u_n\|^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))}+\lambda\left(\alpha_1-\frac{B(\lambda+1)}{\sqrt{2}H}\right)\int_{\Omega_T}(\varepsilon+u_n)^{\lambda-1}|\nabla u_n|^2 \\
&\quad\le C\left(\int_{\Omega_T}(u_n+\varepsilon)^{(\lambda-\theta)m'}\right)^{\frac{1}{m'}}+C.
\end{aligned}\tag{3.45}$$

Observe that $0 < B < \frac{\alpha_1(N-2)(N-2m+2)}{\sqrt{2}Nm(1+\theta)}$ and $\theta \le \lambda < 1$ lead to $\alpha_1 - \frac{B(\lambda+1)}{\sqrt{2}H} > 0$. Let now $q < 2$; applying Hölder's inequality and using (3.45), we have

$$\begin{aligned}\int_{\Omega_T} |\nabla u_n|^q &= \int_{\Omega_T} \frac{|\nabla u_n|^q}{(u_n+\varepsilon)^{\frac{q(1-\lambda)}{2}}}(u_n+\varepsilon)^{\frac{q(1-\lambda)}{2}} \\ &\le \left(\int_{\Omega_T} \frac{|\nabla u_n|^2}{(u_n+\varepsilon)^{1-\lambda}}\right)^{\frac{q}{2}}\left(\int_{\Omega_T}(u_n+\varepsilon)^{\frac{q(1-\lambda)}{2-q}}\right)^{\frac{2-q}{2}} \\ &\le \left[C\left(\int_{\Omega_T}(u_n+\varepsilon)^{(\lambda-\theta)m'}\right)^{\frac{q}{2m'}} + C\right]\left(\int_{\Omega_T}(u_n+\varepsilon)^{\frac{q(1-\lambda)}{2-q}}\right)^{\frac{2-q}{2}}. \quad (3.46)\end{aligned}$$

Applying Lemma (1.1) (where $\rho = \lambda+1$, $h = q$, $v = |u_n|$) and from (3.45), we get

$$\begin{aligned}&\int_{\Omega_T} |u_n|^{\frac{q(N+\lambda+1)}{N}} \\ &\le \left(\|u_n\|^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))}\right)^{\frac{q}{N}}\int_{\Omega_T}|\nabla u_n|^q \\ &\le \left[C\left(\int_{\Omega_T}(u_n+\varepsilon)^{(\lambda-\theta)m'}\right)^{\frac{q}{2m'}+\frac{q}{Nm'}} + C\right]\left(\int_{\Omega_T}(u_n+\varepsilon)^{\frac{q(1-\lambda)}{2-q}}\right)^{\frac{2-q}{2}}. \quad (3.47)\end{aligned}$$

Let us choose λ such that

$$\sigma = \frac{q(N+\lambda+1)}{N} = (\lambda-\theta)m' = \frac{q(\lambda+1)}{2-q}; \quad (3.48)$$

then, we deduce that

$$\lambda = \frac{(N+2)(m-1)+N\theta m}{N-2m+2}, \quad \sigma = \frac{m(N+2)(\theta+1)}{N-2m+2}, \quad \text{and} \quad q = \frac{m(N+2)(\theta+1)}{N+2-m(1-\theta)}.$$

From (3.48) and letting $\varepsilon \to 0$, inequality (3.47) becomes

$$\int_{\Omega_T} |u_n|^\sigma \le C\left(\int_{\Omega_T}|u_n|^\sigma\right)^{\frac{q}{2m'}+\frac{q}{Nm'}+\frac{2-q}{2}} + C.$$

Since $\lambda < 1$, then we have $m < \frac{2(N+2)}{2(N+2)-N(1-\theta)}$, that is, ensure $\frac{q}{2m'} + \frac{q}{Nm'} + \frac{2-q}{2} < 1$; then applying Young's inequality, we can deduce that

$$\int_{\Omega_T} |u_n|^\sigma \le C. \quad (3.49)$$

Putting (3.48) and (3.49) in (3.46) yields

$$\int_{\Omega_T} |\nabla u_n|^q \le C.$$

The two last estimates prove the boundedness of u_n in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\sigma(\Omega_T)$. ∎

Theorem 3.13. *Let* $\theta < 1$, $0 < B < \frac{\alpha_1(N-2)(N-2m+2)}{\sqrt{2}Nm(1+\theta)}$, *and* $0 \leq f \in L^m(\Omega_T)$ *with* $1 < m < \frac{2(N+2)}{2(N+2)-N(1-\theta)}$. *Then, there exists a solution* $u \in L^q(0,T; W_0^{1,q}(\Omega)) \cap L^\sigma(\Omega_T)$ *to problem* (1.1) *in the sense of Definition* 1.2, *where*

$$\sigma = \frac{m(N+2)(\theta+1)}{N-2m+2} \quad \textit{and} \quad q = \frac{m(N+2)(\theta+1)}{N+2-m(1-\theta)}.$$

Proof. we repeat the same techniques used in the proof of Theorem 3.3, and we obtain the proof of Theorem 3.13. ∎

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