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Energy scattering for a 3D Hartree equation with inverse square potential

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Abstract. This work studies the focusing inhomogeneous nonlinear equation of Hartree type

$$\mathrm{i}\partial_t u - \mathcal{K}_\lambda u + |x|^{-\tau}|u|^{p-2}(\mathcal{J}_\alpha * |\cdot|^{-\tau}|u|^p)u = 0, \quad u(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}.$$

Here, the linear Schrödinger operator reads $\mathcal{K}_\lambda := -\Delta + \frac{\lambda}{|x|^2}$ for some $\lambda > -\frac{1}{4}$. The Riesz potential is $\mathcal{J}_\alpha(x) = C_\alpha |x|^{-(3-\alpha)}$ for certain $0 < \alpha < 3$. The singular decaying term $|x|^{-\tau}$ for some $\tau > 0$ gives an inhomogeneous non-linearity. One considers the inter-critical regime, namely, $1 + \frac{2(1-\tau)+\alpha}{3} < p < 1 + 2(1-\tau) + \alpha$. Moreover, one assumes that $p \geq 2$ in order to avoid a singular term $|u|^{p-2}$. Furthermore, one restricts $\lambda > 0$ because there is no dispersive estimate $L^1 \to L^\infty$ for $\lambda < 0$. Contrarily to the homogeneous case $\tau = 0$, for $\lambda > 0$, there is a ground state which minimizes the associated Gagliardo–Nirenberg-type estimate. The purpose is to investigate the energy scattering of global solutions under the ground state threshold. One uses the method of Dodson–Murphy based on Tao's scattering criteria and Morawetz estimates. The decay of the inhomogeneous term $|x|^{-\tau}$ avoids any spherically symmetric assumption.

1. Introduction

This paper is concerned with the Cauchy problem for a focusing inhomogeneous generalized Hartree equation

$$\begin{cases} \mathrm{i}\partial_t u - \mathcal{K}_\lambda u + |x|^{-\tau}|u|^{p-2}(\mathcal{J}_\alpha * |\cdot|^{-\tau}|u|^p)u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1.1}$$

Here and hereafter, the wave function is $u := u(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$. The linear Schrödinger operator is denoted by $\mathcal{K}_\lambda := -\Delta + \frac{\lambda}{|x|^2}$, where the classical Laplacian operator is $\Delta := \sum_{k=1}^3 \frac{\partial^2}{\partial x_k^2}$. The inhomogeneous singular decaying term is $|\cdot|^{-\tau}$ for some $\tau > 0$. The Riesz potential is defined on $\mathbb{R}^3$ by

$$\mathcal{J}_\alpha := \frac{\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{3}{2}} 2^\alpha} |\cdot|^{\alpha-3}, \quad 0 < \alpha < 3.$$

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In all this note, one assumes that

$$\min\{\tau, \alpha, 3-\alpha, 3-\tau, 2-2\tau+\alpha\} > 0. \tag{1.2}$$

Motivated with the next sharp Hardy inequality [3],

$$\frac{1}{4}\int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|^2}\,dx \le \int_{\mathbb{R}^3} |\nabla f(x)|^2\,dx, \tag{1.3}$$

one assumes that $\lambda > -\frac{1}{4}$, which guarantees that extension of $-\Delta + \frac{\lambda}{|x|^2}$, denoted by $\mathcal{K}_\lambda$, is a positive operator. In the range $-\frac{1}{4} < \lambda < 1 - \frac{1}{4}$, the extension is not unique [12, 29]. In such a case, one picks the Friedrichs extension [12, 23].

Note that by the definition of the operator $\mathcal{K}_\lambda$ and Hardy estimate (1.3), one has

$$\|\sqrt{\mathcal{K}_\lambda}\cdot\| = \left(\|\nabla\cdot\|^2 + \lambda\left\|\frac{\cdot}{|x|}\right\|^2\right)^{\frac{1}{2}} \simeq \|\cdot\|_{\dot{H}^1}.$$

The nonlinear equations of Hartree type, namely, (1.1), model many physical phenomena. Indeed, it is used in nonlinear optical systems with spatially dependent interactions [4]. In particular, when $\lambda = 0$, it can be thought of as modeling inhomogeneities in the medium in which the wave propagates [2, 13]. When $\tau = 0$, it models quantum field equations or black hole solutions to the Einstein's equations [12].

When $\lambda \neq 0$, equation (1.1) is not space-translation invariant, contrarily to the case $\lambda = 0$. It is known that Sobolev norms using $\sqrt{\mathcal{K}_\lambda}$ are not equivalent to the classical ones [15]. This restricts the application of Strichartz estimates to the study of the local well-posedness and scattering of global solutions [21].

The inhomogeneous generalized Hartree equation, namely, (1.1) with $\lambda = 0$, was treated first by the second author [1], where the ground state threshold dichotomy was investigated using a sharp adapted Gagliardo–Nirenberg-type estimate. After that, the second author treated the intermediate case in the sense of the local well-posedness in $\dot{H}^1 \cap \dot{H}^{s_c}$, $0 < s_c < 1$. The scattering under the ground state threshold with spherically symmetric data, was proved by the second author [27] and extended to the non-radial regime in [26, 30]. The well-posedness in the energy-critical regime was investigated recently [16, 17, 25]. The energy critical scattering was treated in [10]. Recently, the inhomogeneous generalized Hartree equation with inverse square potential, namely, (1.1) with $\lambda \neq 0$, was investigated in the inter-critical and the energy-critical regimes. Indeed, in [24], a dichotomy of global existence versus blow-up under the ground state threshold for inter-critical solutions was investigated and a local energy-critical well-posedness theory was developed in [16].

The purpose of this paper is to investigate the scattering of energy solutions to the Schrödinger problem (1.1) in the inter-critical regime and under the ground state threshold. This naturally extends the recent work [24], where the global existence versus finite-time blow-up under the ground state threshold was proved, but the scattering was not treated.

The scattering is obtained by using the new approach of Dodson–Murphy [7] which is based on Tao's scattering criteria [28] and Morawetz estimates.

The rest of this paper is organized as follows. The next section contains the main result and some useful estimates. Sections 3 proves the main result.

2. Background and main result

This section contains the main result and some useful estimates.

2.1. Preliminary

Here and hereafter, one denotes for simplicity some standard Lebesgue and Sobolev spaces and norms as follows:

$$L^r := L^r(\mathbb{R}^3), \quad W^{s,r} := W^{s,r}(\mathbb{R}^3), \quad H^s := W^{s,2};$$
$$\|\cdot\|_r := \|\cdot\|_{L^r}, \quad \|\cdot\| := \|\cdot\|_2.$$

Similarly, one defines Sobolev spaces in terms of the operator $\mathcal{K}_\lambda$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norms

$$\|\cdot\|_{\dot{W}^{1,r}_\lambda} := \|\sqrt{\mathcal{K}_\lambda}\cdot\|_r, \quad \|\cdot\|_{W^{s,r}_\lambda} := \|\langle\sqrt{\mathcal{K}_\lambda}\rangle^s\cdot\|_r,$$

where $\langle\cdot\rangle := (1+|\cdot|^2)^{\frac{1}{2}}$. Take also, for short, the Hilbert space $\dot{H}^1_\lambda := \dot{W}^{1,2}_\lambda$ and $H^1_\lambda := W^{1,2}_\lambda$. Note that, by the definition of the operator $\mathcal{K}_\lambda$ and Hardy estimate, one has

$$\|\cdot\|_{\dot{H}^1_\lambda} := \|\sqrt{\mathcal{K}_\lambda}\cdot\| = \left(\|\nabla\cdot\|^2 + \lambda\left\|\frac{\cdot}{|x|}\right\|^2\right)^{\frac{1}{2}} \simeq \|\cdot\|_{\dot{H}^1}.$$

Let us also define the real numbers

$$\gamma := 3p - 3 - \alpha + 2\tau, \quad \rho := 2p - \gamma,$$

and the source term

$$\mathcal{N}[u] := |x|^{-\tau}(\mathcal{J}_\alpha * |\cdot|^{-\tau}|u|^p)|u|^{p-2}u.$$

If $u \in H^1_\lambda$, one defines the quantities related to energy solutions of (1.1):

$$\mathcal{P}[u] := \int_{\mathbb{R}^3} \bar{u}\mathcal{N}[u]\,dx, \tag{2.1}$$

$$\mathcal{I}[u] := \|\sqrt{\mathcal{K}_\lambda}u\|^2 - \frac{\gamma}{2p}\mathcal{P}[u]; \tag{2.2}$$

$$\mathcal{M}[u] := \int_{\mathbb{R}^3} |u(x)|^2\,dx; \tag{2.3}$$

$$E[u] := \|\sqrt{\mathcal{K}_\lambda}u\|^2 - \frac{1}{p}\mathcal{P}[u]. \tag{2.4}$$

Equation (1.1) enjoys the scaling invariance

$$u_\kappa := \kappa^{\frac{2-2\tau+\alpha}{2(p-1)}} u(\kappa^2\cdot, \kappa\cdot), \quad \kappa > 0.$$

The critical exponent s_c keeps invariant the following homogeneous Sobolev norm:

$$\|u_\kappa(t)\|_{\dot H^\mu} = \kappa^{\mu-(\frac{3}{2}-\frac{2-2\tau+\alpha}{2(p-1)})}\|u(\kappa^2 t)\|_{\dot H^\mu} := \kappa^{\mu-s_c}\|u(\kappa^2 t)\|_{\dot H^\mu}.$$

Two cases are of particular interest in the physical context. The first one $s_c = 0$ corresponds to the mass-critical case which is equivalent to $p = p_c := 1 + \frac{2-2\tau+\alpha}{3}$. This case is related to the conservation of the mass (2.3). The second one is the energy-critical case $s_c = 1$, which corresponds to $p = p^c := 1 + 2 - 2\tau + \alpha$. This case is related to the conservation of the energy (2.4). A particular periodic global solution of (1.1) takes the form $e^{it}\varphi$, where φ satisfies

$$\mathcal{K}_\lambda \varphi + \varphi = |x|^{-\tau}|\varphi|^{p-2}(\mathcal{J}_\alpha * |\cdot|^{-\tau}|\varphi|^p)\varphi, \quad 0 \neq \varphi \in H^1_\lambda. \tag{2.5}$$

The existence of such a ground state is related to the next Gagliardo–Nirenberg-type inequality [24].

Proposition 2.1. *Let* $0 < \alpha < 3$ *and* $1 + \frac{\alpha}{3} < p < p^c$. *Assume that* $\lambda > -\frac{1}{4}$ *and* (1.2) *is satisfied. Thus, the following hold.*

(1) *A sharp constant* $C_{p,\tau,\alpha,\lambda} > 0$ *exists such that, for all* $u \in H^1_\lambda$,

$$\mathcal{P}[u] \le C_{p,\tau,\alpha,\lambda}\|u\|^\rho \|\sqrt{\mathcal{K}_\lambda} u\|^\gamma, \tag{2.6}$$

(2) *Moreover, there exists* φ *as a solution to* (2.5) *satisfying*

$$C_{p,\tau,\alpha,\lambda} = \frac{2p}{\rho}\Big(\frac{\rho}{\gamma}\Big)^{\frac{\gamma}{2}}\|\varphi\|^{-2(p-1)}, \tag{2.7}$$

(3) *Furthermore, one has the following Pohozaev identities:*

$$\mathcal{P}[\varphi] = \frac{2p}{\rho}\mathcal{M}[\varphi] = \frac{2p}{\gamma}\|\sqrt{\mathcal{K}_\lambda}\varphi\|^2. \tag{2.8}$$

In the inter-critical regime $0 < s_c < 1$, one denotes the positive real number $\frac{1}{s_c} - 1 := \alpha_c \in (0,1)$, φ to be a ground state of (2.5) and the scale-invariant quantities

$$\mathcal{ME}[u] := \Big(\frac{\mathcal{M}[u]}{\mathcal{M}[\varphi]}\Big)^{\alpha_c}\Big(\frac{E[u]}{E[\varphi]}\Big),$$
$$\mathcal{MK}[u] := \Big(\frac{\|u\|}{\|\varphi\|}\Big)^{\alpha_c}\Big(\frac{\|\sqrt{\mathcal{K}_\lambda}u\|}{\|\sqrt{\mathcal{K}_\lambda}\varphi\|}\Big),$$
$$\mathcal{MP}[u] := \Big(\frac{\mathcal{M}[u]}{\mathcal{M}[\varphi]}\Big)^{\alpha_c}\Big(\frac{\mathcal{P}[u]}{\mathcal{P}[\varphi]}\Big).$$

Let $e^{-\mathrm{i}\cdot\mathcal{K}_\lambda}$ be the operator associated to the free Schrödinger equation $(\mathrm{i}\partial_t - \mathcal{K}_\lambda) = 0$. Then, by Duhamel integral formula, energy solutions to the problems (1.1) are fixed point of the function

$$f(u) := e^{-\mathrm{i}\cdot\mathcal{K}_\lambda} u_0 + i \int_0^{\cdot} e^{-\mathrm{i}(\cdot - s)\mathcal{K}_\lambda} \big[\mathcal{N}[u(s)]\big]\, ds. \tag{2.9}$$

In the next sub-section, one lists the contribution of this note.

2.2. Main results

The contribution of this note is the next energy scattering under the ground state threshold.

Theorem 2.1. *Let $\lambda > 0$ and τ, α satisfying (1.2). Take $p \in (p_c, p^c)$ such that $p \geq 2$, and let $u \in C_{T^*}(H^1_\lambda)$ be a maximal solution to (1.1). Then, u is global and scatters if one of the following assumptions holds:*

$$\sup_{t\in[0,T^*)} \mathcal{M}\mathcal{P}[u(t)] < 1, \tag{2.10}$$

$$\max\big\{\mathcal{M}\mathcal{E}[u_0], \mathcal{M}\mathcal{K}[u_0]\big\} < 1. \tag{2.11}$$

In view of the results stated in the above theorem, some comments arise and we enumerate them in what follows.

Remarks 2.1. (1) In the first point, the threshold is expressed in terms of the non-conserved potential energy in the spirit of [6].

(2) In the second point, the threshold under the ground state threshold follows the pioneering works of [11, 14].

(3) The assumption $\lambda \geq 0$ exists there is no dispersive estimate $L^1 \to L^\infty$ for $\lambda < 0$; see [22].

(4) Compared with the homogeneous regime $\tau = 0$, the minimizing problem associated to (2.6) is never reached for $\lambda > 0$; see [19].

(5) The above results do not require any radial assumption.

2.3. Useful estimates

In this sub-section, one gathers some standard tools needed in the sequel. Let us start with the Hardy–Littlewood–Sobolev inequality [20].

Lemma 2.1. *Let $0 < \alpha < 3$.*

(1) *Let $s \geq 1$ and $r > 1$ such that $\frac{1}{r} = \frac{1}{s} + \frac{\alpha}{3}$. Then,*

$$\|\mathcal{I}_\alpha * g\|_s \leq C_{s,\alpha}\|g\|_r, \quad \forall g \in L^r.$$

(2) *Let $t \geq 1$ and $1 < s, r < \infty$ be such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{t} + \frac{\alpha}{3}$. Then,*

$$\|f(\mathcal{I}_\alpha * g)\|_t \leq C_{N,s,\alpha}\|f\|_r\|g\|_s, \quad \forall (f,g) \in L^r \times L^s.$$

Now, one gives some estimates related to Schrödinger problem (1.1).

Definition 2.1. A couple of real numbers (q,r) is said to be μ admissible (admissible if $\mu=0$) if

$$3\Big(\frac{1}{2}-\frac{1}{r}\Big)=\frac{2}{q}+\mu,\qquad \frac{6}{3-2\mu}<r<6.$$

For simplicity, one denotes by Γ^μ the set of μ admissible pairs and $\Gamma:=\Gamma^0$. Let also, for any real interval I,

$$\Lambda_\mu(I):=\bigcap_{(q,r)\in\Gamma^\mu}L^q(I,L^r),\qquad \|\cdot\|_{\Lambda_\mu(I)}:=\sup_{(q,r)\in\Gamma^\mu}\|\cdot\|_{L^q(I,L^r)};$$
$$\|\cdot\|_{\Lambda'_{-\mu}(I)}:=\inf_{(q,r)\in\Gamma^{-\mu}}\|\cdot\|_{L^{q'}(I,L^{r'})}.$$

Take also the particular cases

$$\Lambda(I):=\Lambda_0(I),\quad \Lambda'(I):=\Lambda'_0(I),\quad \Lambda_\mu:=\Lambda_\mu((0,\infty)),\quad \Lambda'_{-\mu}:=\Lambda'_{-\mu}((0,\infty)).$$

An essential tool used in this note is Strichartz estimate [5,9,31].

Proposition 2.2. *Let $\lambda>-\frac{1}{4}$, $\mu\in\mathbb{R}$, and $0\in I$ be a real interval. Then, there exists $C>0$ such that*

(1) $\|e^{-i\cdot\mathcal{K}_\lambda}u\|_{\Lambda_\mu(I)}\le C\|u\|_{\dot H^\mu}$;

(2) $\|\int_0^\cdot e^{-i(\cdot-\tau)\mathcal{K}_\lambda}f(\tau)\,d\tau\|_{\Lambda(I)}\le C\|f\|_{\Lambda'(I)}$;

(3) *if* $\lambda\ge 0$, *so* $\|\int_0^\cdot e^{-i(\cdot-\tau)\mathcal{K}_\lambda}f(\tau)\,d\tau\|_{\Lambda_\mu(I)}\le C\|f\|_{\Lambda'_{-\mu}(I)}$.

The above Strichartz estimates are consequence of the next dispersive estimates [8,22].

Proposition 2.3. *There exists $C>0$ such that*

(1) $\|e^{-i\cdot\mathcal{K}_\lambda}u\|_{r'}\le C\frac{\|u\|_r}{|t|^{3(\frac{1}{r}-\frac{1}{2})}}$, *whenever* $\frac{1}{2}\le\frac{1}{r}<\min\{1,1-\frac{\kappa}{3}\}$;

(2) $\|e^{-i\cdot\mathcal{K}_\lambda}u\|_{r'}\le C\frac{\|u\|_r}{|t|^{3(\frac{1}{r}-\frac{1}{2})}}$, *whenever* $r\in[2,\infty]$ *and* $\lambda\ge 0$.

Let $\zeta:\mathbb{R}^3\to\mathbb{R}$ be a convex smooth function. Define the variance potential

$$V_\zeta:=\int_{\mathbb{R}^3}\zeta(x)|u(\cdot,x)|^2\,dx, \tag{2.12}$$

and the Morawetz action

$$M_\zeta=2\Im\int_{\mathbb{R}^3}\bar u(\nabla\zeta\cdot\nabla u)\,dx:=2\Im\int_{\mathbb{R}^3}\bar u(\zeta_j u_j)\,dx, \tag{2.13}$$

where, here and in the sequel, repeated index are summed. Let us give a Morawetz-type estimate for the Schrödinger equation with inverse square potential [18].

Proposition 2.4. *Take $u \in C_T(H^1_\lambda)$ to be a local solution, to (1.1). Let $\zeta : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Then, the following equality holds on $[0, T]$:*

$$\begin{aligned} V_\zeta''[u] &= M_\zeta'[u] \\ &= 4\int_{\mathbb{R}^3} \partial_l\partial_k\zeta\Re(\partial_k u\partial_l\bar{u})\,dx - \int_{\mathbb{R}^3}\Delta^2\zeta|u|^2\,dx + 4\lambda\int_{\mathbb{R}^3}\nabla\zeta\cdot x\frac{|u|^2}{|x|^4}\,dx \\ &\quad + 2\Big(\frac{2}{p}-1\Big)\int_{\mathbb{R}^3}\Delta\zeta\bar{u}\mathcal{N}[u]\,dx + \frac{4}{p}\int_{\mathbb{R}^3}\nabla\zeta\cdot\nabla(|x|^{-\tau})|u|^p(\mathcal{J}_\alpha * |\cdot|^{-\tau}|u|^p)\,dx \\ &\quad + \frac{4}{p}(\alpha - N)\int_{\mathbb{R}^3}|x|^{-\tau}|u|^p\nabla\zeta\Big(\frac{\cdot}{|\cdot|^2}\mathcal{J}_\alpha * |\cdot|^{-\tau}|u|^p\Big)\,dx. \end{aligned}$$

From now on one hides the time variable t for simplicity, spreading it out only when necessary. Moreover, one denotes the centered ball of $\mathbb{R}^3$ with radius $R > 0$ and its complementary, respectively, by $B(R)$ and $B^c(R)$. Furthermore, $C(R, R')$ is the centered annulus of $\mathbb{R}^3$ with small radius R and large radius R'. Finally, the critical Sobolev embedding $H^1 \hookrightarrow L^{2^*}$ gives the index $2^* := 6$. In what follows, one proves the main result of this note.

3. Proof of Theorem 2.1

The proof of the energy scattering is divided into several steps.

3.1. Global existence

The global existence follows by the conservation laws via the next coercivity result.

Lemma 3.1. *Let $u \in H^1_\lambda$ satisfying*

$$\mathcal{M}\mathcal{P}[u] < \nu < 1. \tag{3.1}$$

Then, there is $c(\nu, \varphi) > 0$ such that

$$\|\sqrt{\mathcal{K}_\lambda}u\|^2 < c(\nu, \varphi)E[u], \tag{3.2}$$

$$\mathcal{I}[u] > c(\nu, \varphi)\|\sqrt{\mathcal{K}_\lambda}u\|^2. \tag{3.3}$$

Remark 3.1. It is obvious that one can apply the above result to $\psi_R u$ rather than to u.

Proof. A direct computation gives the useful identities

$$2(p-1)s_c = \gamma - 2, \tag{3.4}$$

$$\alpha_c(\gamma - 2) = \rho. \tag{3.5}$$

Using the Gagliardo–Nirenberg inequality (2.6) via Pohozaev identities (2.8), the explicit expression (2.7), and the equalities (3.4)–(3.5), one writes

$$\begin{aligned}
[\mathcal{P}[u]]^{\frac{\gamma}{2}} &\leq C_{p,\tau,\alpha,\lambda}\big(\|u\|^{2\alpha_c}\mathcal{P}[u]\big)^{\frac{\gamma}{2}-1}\|\sqrt{\mathcal{K}_\lambda}u\|^\gamma \\
&\leq \frac{2p}{\rho}\Big(\frac{\rho}{\gamma}\Big)^{\frac{\gamma}{2}}\|\varphi\|^{-2(p-1)}\big(\mathcal{M}[u]^{\alpha_c}\mathcal{P}[u]\big)^{\frac{\gamma}{2}-1}\|\sqrt{\mathcal{K}_\lambda}u\|^\gamma \\
&\leq \frac{2p}{\rho}\Big(\frac{\rho}{\gamma}\Big)^{\frac{\gamma}{2}}\mathcal{M}[\varphi]^{\frac{\rho-2(p-1)}{2}}[\mathcal{P}[\varphi]]^{\frac{\gamma}{2}-1}\big(\mathcal{M}\mathcal{P}[u]\big)^{\frac{\gamma}{2}-1}\|\sqrt{\mathcal{K}_\lambda}u\|^\gamma \\
&\leq \Big(\frac{\rho}{\gamma}\frac{\mathcal{P}[\varphi]}{\mathcal{M}[\varphi]}\Big)^{\frac{\gamma}{2}}\big(\mathcal{M}\mathcal{P}[u]\big)^{\frac{\gamma}{2}-1}\|\sqrt{\mathcal{K}_\lambda}u\|^\gamma \\
&\leq \big(\mathcal{M}\mathcal{P}[u]\big)^{\frac{\gamma}{2}-1}\Big(\frac{2p}{\gamma}\|\sqrt{\mathcal{K}_\lambda}u\|^2\Big)^{\frac{\gamma}{2}}.
\end{aligned}$$

Thus,

$$\mathcal{P}[u] \leq \frac{2p}{\gamma}\big(\mathcal{M}\mathcal{P}[u]\big)^{\frac{\gamma-2}{\gamma}}\|\sqrt{\mathcal{K}_\lambda}u\|^2. \tag{3.6}$$

This implies that

$$\begin{aligned}
E[u] &= \|\sqrt{\mathcal{K}_\lambda}u\|^2 - \frac{1}{p}\mathcal{P}[u] \\
&\geq \Big(1 - \frac{2}{\gamma}(\mathcal{M}\mathcal{P}[u])^{\frac{\gamma-2}{\gamma}}\Big)\|\sqrt{\mathcal{K}_\lambda}u\|^2.
\end{aligned}$$

The proof of (3.2) follows by (3.1) via the assumption $s_c > 0$, which gives $\gamma > 2$. Moreover, by (3.6) and (3.1), one has

$$\begin{aligned}
\mathcal{I}[u] &= \|\sqrt{\mathcal{K}_\lambda}u\|^2 - \frac{\gamma}{2p}\mathcal{P}[u] \\
&\geq \|\sqrt{\mathcal{K}_\lambda}u\|^2\big(1 - (\mathcal{M}\mathcal{P}[u])^{\frac{\gamma-2}{\gamma}}\big) \\
&\gtrsim \|\sqrt{\mathcal{K}_\lambda}u\|^2.
\end{aligned}$$

This proves (3.3). ∎

3.2. Scattering criteria

Here and hereafter, one denotes a smooth function $\psi \in C_0^\infty(\mathbb{R}^3)$ such that $\psi = 1$ on $B(\frac{1}{2})$, $\psi = 0$ on $B^c(1)$, and $0 \leq \psi \leq 1$. Take also $\psi_R := \psi(\frac{\cdot}{R})$. In this sub-section, one proves the next scattering criteria.

Proposition 3.1. *Take the assumptions of Theorem 2.1. Let $u \in C(\mathbb{R}, H_\lambda^1)$ be a global solution to (1.1). Assume that*

$$0 < \sup_{t\in\mathbb{R}}\|u(t)\|_{H_\lambda^1} := E < \infty.$$

There exist $R, \varepsilon > 0$ depending on E, d, p, τ such that u scatters if

$$\liminf_{t\to\infty} \int_{B(R)} |u(t,x)|^2\,dx < \varepsilon^2. \tag{3.7}$$

Proof. Using an interpolation via the bound in $L^\infty(H^1_\lambda)$, it is sufficient to prove that

$$u \in L^4(L^{2^*}).$$

Moreover, by Sobolev embeddings and Hölder estimate, one writes

$$\|u\|_{L^4_T(L^{2^*})} \le T^{\frac{1}{4}} \|u\|_{L^\infty(H^1_\lambda)}.$$

So, it is sufficient to prove that there is $T > 0$ such that

$$u \in L^4((T,\infty), L^{2^*}).$$

By continuity argument, Strichartz estimate, and Sobolev embedding, the key of the proof of the scattering criterion is the next result.

Proposition 3.2. *Take the assumptions of Proposition 3.1. Then, for any $\varepsilon > 0$, there exist $T, \mu > 0$ satisfying*

$$\|e^{\mathrm{i}(\cdot-T)\mathcal{K}_\lambda} u(T)\|_{L^4((T,\infty),L^{2^*})} \lesssim \varepsilon^\mu.$$

Proof. By the integral formula, one has

$$\begin{aligned} e^{-\mathrm{i}(t-T)\mathcal{K}_\lambda} u(T) &= e^{-\mathrm{i}t\mathcal{K}_\lambda} u_0 + i \int_0^T e^{-\mathrm{i}(t-\tau)\mathcal{K}_\lambda}[\mathcal{N}[u]]\,d\tau \\ &= e^{-\mathrm{i}t\mathcal{K}_\lambda} u_0 + i \left(\int_0^{T-\varepsilon^{-\beta}} + \int_{T-\varepsilon^{-\beta}}^T \right) e^{-\mathrm{i}(t-\tau)\mathcal{K}_\lambda}[\mathcal{N}[u]]\,d\tau \\ &:= e^{-\mathrm{i}t\mathcal{K}_\lambda} u_0 + i \left(\int_{J_1} + \int_{J_2} \right) e^{-\mathrm{i}(t-\tau)\mathcal{K}_\lambda}[\mathcal{N}[u]]\,d\tau \\ &:= e^{-\mathrm{i}t\mathcal{K}_\lambda} u_0 + F_1 + F_2. \end{aligned} \tag{3.8}$$

Now, one estimates the three different parts in (3.8).

The linear term. By Hölder and Strichartz estimates via Sobolev injections, one has

$$\begin{aligned} \|e^{-\mathrm{i}\cdot\mathcal{K}_\lambda} u_0\|_{L^4((T,\infty),L^{2^*})} &\le \|e^{-\mathrm{i}\cdot\mathcal{K}_\lambda} u_0\|^{\frac{1}{2}}_{L^\infty((T,\infty),L^{2^*})} \|e^{-\mathrm{i}\cdot\mathcal{K}_\lambda} u_0\|^{\frac{1}{2}}_{L^2((T,\infty),L^{2^*})} \\ &\le c\|e^{-\mathrm{i}\cdot\mathcal{K}_\lambda} u_0\|^{\frac{1}{2}}_{L^\infty((T,\infty),H^1_\lambda)} \|e^{-\mathrm{i}\cdot\mathcal{K}_\lambda} u_0\|^{\frac{1}{2}}_{L^2((T,\infty),L^{2^*})} \\ &\le c\|e^{-\mathrm{i}\cdot\mathcal{K}_\lambda} u_0\|^{\frac{1}{2}}_{L^2((T,\infty),L^{2^*})}. \end{aligned}$$

Thus, by the dominated convergence theorem via Strichartz estimates and the fact that $(2, 2^*) \in \Gamma$, one may choose $T_0 > \varepsilon^{-\beta} > 0$, where $\beta > 0$ is to pick later, such that

$$\|e^{-\mathrm{i}\cdot\mathcal{K}_\lambda} u_0\|_{L^4((T_0,\infty),L^{2^*})} \le \varepsilon^2. \tag{3.9}$$

The term F_1. First, the integral formula (2.9) gives

$$F_1 = e^{-\mathrm{i}t\mathcal{K}_\lambda}\big(e^{-\mathrm{i}(-T+\varepsilon^{-\beta})\mathcal{K}_\lambda}u(T-\varepsilon^{-\beta}) - u_0\big). \tag{3.10}$$

So, using Strichartz estimate via (3.10), the fact that $(2, 2^*) \in \Gamma$, and an interpolation, one writes

$$\begin{aligned}\|F_1\|_{L^4((T,\infty),L^{2^*})} &\le \|F_1\|^{\frac{1}{2}}_{L^\infty((T,\infty),L^{2^*})}\|F_1\|^{\frac{1}{2}}_{L^2((T,\infty),L^{2^*})}\\ &\le c\|F_1\|^{\frac{1}{2}}_{L^\infty((T,\infty),L^{2^*})}.\end{aligned}$$

Let us prove the next claim:

$$\int_{\mathbb{R}^3}\mathcal{N}[u]\,dx \lesssim \|u\|^{2p-1}_{H^1_\lambda}. \tag{3.11}$$

One decomposes the integral on the unit ball and its complementary as follows:

$$\begin{aligned}&\int_{B(1)}\mathcal{N}[u]\,dx\\ &\quad\le \|u\|_b^{2p-1}\big(\||x|^{-\tau}\|_{L^{a_1}(B(1))}\||x|^{-\tau}\|_{L^c(B(1))} + \||x|^{-\tau}\|_{L^{a_2}(B(1))}\||x|^{-\tau}\|_{L^d(B^c(1))}\big)\\ &\quad\le c\|u\|^{2p-1}_{H^1_\lambda}.\end{aligned}$$

Here, one uses Lemma 2.1 so that

$$\begin{cases} 1+\dfrac{\alpha}{3} = \dfrac{1}{a_1} + \dfrac{2p-1}{b} + \dfrac{1}{c} = \dfrac{1}{a_2} + \dfrac{2p-1}{b} + \dfrac{1}{d},\\ \dfrac{3}{d} < \tau < \min\Big\{\dfrac{3}{c}, \dfrac{3}{a_i}\Big\},\\ \dfrac{1}{6} < \dfrac{1}{b} \le \dfrac{1}{2}.\end{cases}$$

Thus,

$$\begin{cases} 1+\dfrac{\alpha}{3} - \dfrac{2p-1}{b} = \dfrac{1}{a_1} + \dfrac{1}{c} > \dfrac{2\tau}{3},\\ 1+\dfrac{\alpha}{3} - \dfrac{2p-1}{b} = \dfrac{1}{a_2} + \dfrac{1}{d},\\ \dfrac{3}{d} < \tau < \min\Big\{\dfrac{3}{c}, \dfrac{3}{a_i}\Big\},\\ \dfrac{1}{6} < \dfrac{1}{b} \le \dfrac{1}{2}.\end{cases}$$

This reads

$$\frac{2p-1}{6} < \frac{2p-1}{b} < \frac{3+\alpha-2\tau}{3}.$$

This is possible because $p < p_c$. Moreover, by Lemma 2.1, one has

$$\begin{aligned}&\int_{B^c(1)} \mathcal{N}[u]\,dx\\&\le \|u\|_{b_1}^{2p-1}\big(\||x|^{-\tau}\|_{L^{a_1}(B^c(1))}\||x|^{-\tau}\|_{L^{c_1}(B(1))} + \||x|^{-\tau}\|_{L^{a_2}(B^c(1))}\||x|^{-\tau}\|_{L^{d_1}(B^c(1))}\big)\\&\le c\|u\|_{H^1_\lambda}^{2p-1}.\end{aligned}$$

Here, by Lemma 2.1, one has

$$\begin{cases}1+\frac{\alpha}{3}=\frac{1}{a_1}+\frac{2p-1}{b_1}+\frac{1}{c_1}=\frac{1}{a_2}+\frac{2p-1}{b_1}+\frac{1}{d_1},\\ \max\left\{\frac{3}{a_i},d_1\right\}<\tau<\frac{3}{c_1},\\ \frac{1}{6}<\frac{1}{b}\le\frac{1}{2}.\end{cases}$$

Thus,

$$\begin{cases}1+\frac{\alpha}{3}-\frac{2p-1}{b_1}=\frac{1}{a_2}+\frac{1}{d_1}<\frac{2\tau}{3},\\ 1+\frac{\alpha}{3}-\frac{2p-1}{b_1}=\frac{1}{a_1}+\frac{1}{c_1},\\ \max\left\{\frac{3}{a_i},d_1\right\}<\tau<\frac{3}{c_1},\\ \frac{1}{6}<\frac{1}{b_1}\le\frac{1}{2}.\end{cases}$$

This reads

$$\frac{3+\alpha-2\tau}{3}<\frac{2p-1}{b_1}<\frac{2p-1}{2}.$$

This is satisfied because

$$p>p_c>\frac{1}{2}+\frac{3+\alpha-2\tau}{3}.$$

Now, an interpolation via (3.10), (3.11) and Proposition 2.3 implies that

$$\begin{aligned}\|F_1(t)\|_{2^*}&\le\|F_1(t)\|^{\frac13}\|F_1(t)\|_\infty^{\frac23}\\&\le c\|F_1(t)\|_\infty^{\frac23}\\&\le c\left(\int_0^{T-\varepsilon^{-\beta}}|t-s|^{-\frac32}\|\mathcal{N}[u]\|_1\,ds\right)^{\frac23}\\&\le c\big((t-T+\varepsilon^{-\beta})^{1-\frac32}\|u\|_{H^1_\lambda}^{2p-1}\big)^{\frac23}\\&\le c\varepsilon^{\frac{\beta}{3}}.\end{aligned}$$

So, it follows that

$$\|F_1\|_{L^\infty((T,\infty),L^{2^*})}\le c\varepsilon^{\nu},\quad \nu>0. \tag{3.12}$$

Finally, with an interpolation via (3.12), one gets

$$\|F_1\|^2_{L^4((T,\infty),L^{2^*})} \leq \|F_1\|_{L^\infty((T,\infty),L^{2^*})}\|F_1\|_{L^2((T,\infty),L^{2^*})} \leq \varepsilon^\nu, \quad \nu > 0.$$

The term F_2. By the assumption (3.7), one has, for $T > \varepsilon^{-\beta}$ large enough,

$$\int_{\mathbb{R}^3} \psi_R(x)|u(T,x)|^2\,dx < \varepsilon^2.$$

Moreover, a computation with use of (1.1) and Hölder estimate gives

$$\begin{aligned}\left|\frac{d}{dt}\int_{\mathbb{R}^3}\psi_R|u|^2\,dx\right| &= \left|-2\Im\int_{\mathbb{R}^3}\psi_R\bar{u}\Delta u\,dx\right|\\ &= \left|2\Im\int_{\mathbb{R}^3}\bar{u}\nabla\psi_R\cdot\nabla u\,dx\right|\\ &\lesssim \frac{1}{R}.\end{aligned}$$

Then, for any $T-\varepsilon^{-\beta} \leq t \leq T$ and $R > \varepsilon^{-(2+\beta)}$, this yields

$$\|\psi_R u(t)\| \leq \left(\int_{\mathbb{R}^3}\psi_R(x)|u(T,x)|^2\,dx + C\frac{T-t}{R}\right)^{\frac{1}{2}} \leq C\varepsilon.$$

This gives

$$\|\psi_R u\|_{L^\infty([T-\varepsilon^{-\beta},T],L^2)} \leq C\varepsilon. \tag{3.13}$$

Using Strichartz estimate in Proposition 2.2, one writes

$$\begin{aligned}\|F_2\|_{L^4(L^{2^*})} &\leq \|F_2\|_{\Lambda_{\frac{1}{2}}}\\ &\leq \|\mathcal{N}[u]\|_{\Lambda'_{-\frac{1}{2}}(J_2)}\\ &\leq \|\mathcal{N}[u]\|_{L^4(J_2,L^{\frac{6}{5}})}\\ &\leq \|\psi_R\mathcal{N}[u]\|_{L^4(J_2,L^{\frac{6}{5}})} + \|(1-\psi_R)\mathcal{N}[u]\|_{L^4(J_2,L^{\frac{6}{5}})}\\ &:= \|(I)\|_{L^4(J_2)} + \|(II)\|_{L^4(J_2)}.\end{aligned} \tag{3.14}$$

Now, by Hölder estimate via (3.14) and (3.13), one writes, for certain $0 < \theta \leq 1$,

$$\begin{aligned}(I) &\leq \|\psi_R u\|_f\|u\|_f^{2(p-1)}\big(\||x|^{-\tau}\|_{L^{a_1}(B(R))}\||x|^{-\tau}\|_{L^c(B(R))}\\ &\qquad + \||x|^{-\tau}\|_{L^{a_2}(B(R))}\||x|^{-\tau}\|_{L^d(B^c(R))}\big)\\ &\leq c\|\psi_R u\|^\theta\|u\|_{H^1_\lambda}^{2(p-1)+1-\theta}\\ &\leq c\varepsilon^\theta.\end{aligned} \tag{3.15}$$

Here,

$$\begin{cases} \frac{5}{6}+\frac{\alpha}{3}=\frac{1}{a_1}+\frac{2p-1}{f}+\frac{1}{c}=\frac{1}{a_2}+\frac{2p-1}{f}+\frac{1}{d}, \\ \frac{3}{d}<\tau<\min\left\{\frac{3}{a_i},\frac{3}{c}\right\}, \\ \frac{1}{6}<\frac{1}{f}\leq\frac{1}{2}. \end{cases}$$

This gives

$$\begin{cases} \frac{5}{2}+\alpha-\frac{3(2p-1)}{f}=\frac{3}{c}+\frac{3}{a_1}>2\tau, \\ \frac{1}{6}<\frac{1}{f}\leq\frac{1}{2}. \end{cases}$$

So,

$$\frac{2p-1}{6}<\frac{2p-1}{f}<\frac{1}{3}\left(\frac{5}{2}-2\tau+\alpha\right).$$

This is possible because $p<p^c$. Moreover, by Hölder estimate via (3.14) and the properties of ψ, one writes

$$\begin{aligned} (II) & \\ &\leq c\|u\|_e^{2p-1}\left(\||x|^{-\tau}\|_{L^{g_1}(B^c(R))}\||x|^{-\tau}\|_{L^h(B^c(R))}+\||x|^{-\tau}\|_{L^{g_2}(B^c(R))}\||x|^{-\tau}\|_{L^k(B(R))}\right) \\ &\leq cR^{-(g\tau-3)}\|u\|_{H^1_\lambda}^{2p-1} \\ &\leq cR^{-(g\tau-3)}. \end{aligned} \tag{3.16}$$

Here, $g:=\min\{g_1,g_2\}$ and

$$\begin{cases} \frac{5}{6}+\frac{\alpha}{3}=\frac{1}{g_1}+\frac{2p-1}{e}+\frac{1}{h}=\frac{1}{g_2}+\frac{2p-1}{e}+\frac{1}{k}, \\ \max\left\{\frac{3}{g_i},\frac{3}{h}\right\}<\tau<\frac{3}{k}, \\ \frac{1}{6}\leq\frac{1}{e}\leq\frac{1}{2}. \end{cases}$$

This reads

$$\frac{5}{2}+\alpha-2\tau<\frac{3(2p-1)}{e}<\frac{3(2p-1)}{2}.$$

This is possible because $p>p_c$ gives

$$p-1>\frac{1-2\tau+\alpha}{3}.$$

Now, by (3.14), (3.15), and (3.16), one gets for $0 < \beta < \theta$ and $R^{-(g\tau-3)} < \varepsilon^{\beta}$,

$$\begin{aligned}\|F_2\|_{L^4(L^{2^*})} &\leq \|(I)\|\|_{L^4(J_2)} + \|(II)\|_{L^4(J_2)}\\ &\leq c|J_2|^{\frac{1}{4}}\left(R^{-(g\tau-3)} + \varepsilon^{\theta}\right)\\ &\leq c\varepsilon^{-\frac{\beta}{4}}\left(R^{-(g\tau-3)} + \varepsilon^{\theta}\right)\\ &\leq c\varepsilon^{\nu}. \end{aligned} \tag{3.17}$$

The proof is closed via (3.9), (3.12), and (3.17). ■

■

3.3. Virial/Morawetz estimate

The next radial identities will be useful:

$$\begin{aligned}\nabla &= \frac{x}{r}\partial_r,\\ \frac{\partial^2}{\partial x_l \partial x_k} &:= \partial_l \partial_k = \left(\frac{\delta_{lk}}{r} - \frac{x_l x_k}{r^3}\right)\partial_r + \frac{x_l x_k}{r^2}\partial_r^2,\\ \Delta &= \partial_r^2 + \frac{2}{r}\partial_r.\end{aligned}$$

In the rest of this note, one takes a smooth radial function $\zeta(x) := \zeta(|x|)$ such that

$$\zeta : r \to \begin{cases} r^2 & \text{if } 0 \leq r \leq 1,\\ r & \text{if } r > 2.\end{cases}$$

Now, for $R > 0$, take via (2.12) and (2.13),

$$\zeta_R := R^2\zeta\left(\frac{|\cdot|}{R}\right), \quad M_R := M_{\zeta_R}, \quad \text{and} \quad V_R := V_{\zeta_R}.$$

Moreover, one assumes that, in the centered annulus $C(0, R, 2R)$,

$$\partial_r\zeta > 0, \quad \partial_r^2\zeta \geq 0, \quad \text{and} \quad |\partial^{\alpha}\zeta| \leq C_{\alpha}|\cdot|^{1-\alpha}, \quad \forall\, |\alpha| \geq 1. \tag{3.18}$$

Note that, on the centered ball of radius R, one has

$$\partial_{jk}\zeta_R = 2\delta_{jk}, \quad \Delta\zeta_R = 6, \quad \text{and} \quad \Delta^2\zeta_R = 0. \tag{3.19}$$

Moreover, for $|x| > 2R$,

$$\partial_{jk}\zeta_R = \frac{R}{|x|}\left(\delta_{jk} - \frac{x_j x_k}{|x|^2}\right), \quad \Delta\zeta_R = \frac{2R}{|x|}, \quad \text{and} \quad \Delta^2\zeta_R = 0. \tag{3.20}$$

Now, one states a Morawetz-type estimate.

Proposition 3.3. *There is* $0<\varepsilon\ll 1$ *and* $t_n, R_n\to\infty$ *such that*

$$\int_0^T\left(\int_{B(R)}|u(s,x)|^{2^*}\,dx\right)^{\frac{1}{3}}ds\lesssim T^{\frac{1}{1+\varepsilon}}, \tag{3.21}$$

$$\lim_{n\to\infty}\int_{B(R_n)}|u(t_n,x)|^{2^*}\,dx=0. \tag{3.22}$$

Proof. Taking account of Proposition 2.4, one writes

$$\begin{aligned}
V_R''[u]&=4\int_{\mathbb{R}^3}\partial_l\partial_k\zeta_R\Re(\partial_k u\partial_l\bar u)\,dx-\int_{\mathbb{R}^3}\Delta^2\zeta_R|u|^2\,dx\\
&\quad+4\lambda\int_{\mathbb{R}^3}\nabla\zeta_R\cdot x\frac{|u|^2}{|x|^4}\,dx+2\Big(\frac{2}{p}-1\Big)\int_{\mathbb{R}^3}\Delta\zeta_R\bar u\mathcal{N}[u]\,dx\\
&\quad+\frac{4}{p}\int_{\mathbb{R}^3}\nabla\zeta_R\cdot\nabla(|x|^{-\tau})|u|^p(\mathcal{J}_\alpha*|\cdot|^{-\tau}|u|^p)\,dx\\
&\quad+\frac{4}{p}(\alpha-N)\int_{\mathbb{R}^3}|x|^{-\tau}|u|^p\nabla\zeta_R\Big(\frac{\cdot}{|\cdot|^2}\mathcal{J}_\alpha*|\cdot|^{-\tau}|u|^p\Big)\,dx\\
&:=(I)+(I)+(III),
\end{aligned} \tag{3.23}$$

where one decomposes the above integrals as $(\int_{B(R)}+\int_{C(R,2R)}+\int_{B^c(2R)})$. By (3.19) via [27, Section 5], one has

$$\begin{aligned}
(I)&=8\Big(\int_{B(R)}|\nabla u|^2\,dx-\frac{\gamma}{2p}\int_{B(R)}\bar u\mathcal{N}[u]\,dx+\lambda\int_{B(R)}\frac{|u|^2}{|x|^2}\,dx\Big)\\
&\quad+O\Big(\int_{B^c(\frac{R}{2})}\bar u\mathcal{N}[u]\,dx\Big).
\end{aligned} \tag{3.24}$$

Moreover, taking $\not\nabla:=\nabla-\frac{x\cdot\nabla}{|x|^2}x$ to be the angular gradient, by (3.20), it follows that

$$\begin{aligned}
(III)&=4\int_{B^c(2R)}\frac{R}{|x|}\Big(\delta_{jk}-\frac{x_jx_k}{|x|^2}\Big)\Re(\partial_k u\partial_l\bar u)\,dx+4\lambda\int_{B^c(2R)}\frac{R}{|x|}\frac{|u|^2}{|x|^2}\,dx\\
&\quad-2\Big(1-\frac{1}{2p}\Big)\int_{B^c(2R)}\frac{2R}{|x|}\bar u\mathcal{N}[u]\,dx-\frac{4\tau}{p}\int_{B^c(2R)}\frac{R}{|x|}\bar u\mathcal{N}[u]\,dx\\
&\quad+O\Big(\int_{B^c(\frac{R}{2})}\bar u\mathcal{N}[u]\,dx\Big)\\
&=4\int_{B^c(2R)}\frac{R}{|x|}|\not\nabla u|^2\,dx+4\lambda\int_{B^c(2R)}\frac{R}{|x|}\frac{|u|^2}{|x|^2}\,dx\\
&\quad-2\Big(1-\frac{1}{2p}\Big)\int_{B^c(2R)}\frac{2R}{|x|}\bar u\mathcal{N}[u]\,dx-\frac{4\tau}{p}\int_{B^c(2R)}\frac{R}{|x|}\bar u\mathcal{N}[u]\,dx\\
&\quad+O\Big(\int_{B^c(\frac{R}{2})}\bar u\mathcal{N}[u]\,dx\Big)\\
&\gtrsim-R^{-2}\int_{\mathbb{R}^3}|u|^2\,dx-\int_{B^c(\frac{R}{2})}\bar u\mathcal{N}[u]\,dx.
\end{aligned} \tag{3.25}$$

Furthermore, by (3.18), one has

$$\begin{aligned}(II) &:= 4\int_{C(R,2R)} \partial_l\partial_k\zeta_R \Re(\partial_k u \partial_l \bar u)\,dx - \int_{C(R,2R)} \Delta^2\zeta_R |u|^2\,dx \\ &\quad + 4\lambda \int_{C(R,2R)} \nabla\zeta_R\cdot x\frac{|u|^2}{|x|^4}\,dx + 2\Big(\frac{2}{p}-1\Big)\int_{C(R,2R)} \Delta\zeta_R \bar u \mathcal{N}[u]\,dx \\ &\quad + \frac{4}{p}\int_{C(R,2R)} \nabla\zeta_R\cdot\nabla(|x|^{-\tau})|u|^p(\mathcal{J}_\alpha * |\cdot|^{-\tau}|u|^p)\,dx \\ &\quad + \frac{4}{p}(\alpha-N)\int_{C(R,2R)} |x|^{-\tau}|u|^p\nabla\zeta_R\Big(\frac{\cdot}{|\cdot|^2}\mathcal{J}_\alpha * |\cdot|^{-\tau}|u|^p\Big)\,dx \\ &\gtrsim -R^{-3}\int_{\mathbb{R}^3}|u|^2\,dx - \int_{B^c(\frac{R}{2})}\bar u\mathcal{N}[u]\,dx.\end{aligned}$$

Now, by (3.23), (3.24), and (3.25), it follows that, for certain $\varepsilon > 0$,

$$\begin{aligned}V_R''[u] &\gtrsim \int_{B(R)}|\nabla u|^2\,dx - \frac{\gamma}{2p}\int_{B(R)}\bar u\mathcal{N}[u]\,dx + \lambda\int_{B(R)}\frac{|u|^2}{|x|^2}\,dx \\ &\quad - R^{-2}\int_{\mathbb{R}^3}|u|^2\,dx - \int_{B^c(\frac{R}{2})}\bar u\mathcal{N}[u]\,dx \\ &\gtrsim \int_{B(R)}|\nabla u|^2\,dx - \frac{\gamma}{2p}\int_{B(R)}\bar u\mathcal{N}[u]\,dx + \lambda\int_{B(R)}\frac{|u|^2}{|x|^2}\,dx \\ &\quad - R^{-2}\int_{\mathbb{R}^3}|u|^2\,dx - R^{-\varepsilon}\|u\|_{H^1_\lambda}^{2p} \\ &\gtrsim \int_{B(R)}|\nabla u|^2\,dx - \frac{\gamma}{2p}\int_{B(R)}\bar u\mathcal{N}[u]\,dx \\ &\quad + \lambda\int_{B(R)}\frac{|u|^2}{|x|^2}\,dx - cR^{-2} - cR^{-\varepsilon}. \end{aligned} \tag{3.26}$$

Indeed, by Lemma 2.1 and Sobolev embeddings, one gets

$$\begin{aligned}&\int_{B^c(\frac{R}{2})}\bar u\mathcal{N}[u]\,dx \\ &\lesssim \|u\|_e^{2p}\Big(\||x|^{-\tau}\|_{L^{a_1}(B^c(\frac{R}{2}))}\||x|^{-\tau}\|_{L^{b_1}(B(1))} + \||x|^{-\tau}\|_{L^{a_2}(B^c(\frac{R}{2}))}\||x|^{-\tau}\|_{L^{b_2}(B^c(1))}\Big) \\ &\lesssim R^{3-b_2\tau}\|u\|_{H^1_\lambda}^{2p}.\end{aligned}$$

Here,

$$\begin{cases} 1+\frac{\alpha}{3} = \frac{2p}{e}+\frac{1}{a_1}+\frac{1}{b_1} = \frac{2p}{e}+\frac{1}{a_2}+\frac{1}{b_2}, & \\ \max\Big\{\frac{3}{a_i},\frac{3}{b_2}\Big\} < \tau < \frac{3}{b_1}, & 1\le i\le 2, \\ \frac{1}{6}\le\frac{1}{e}\le\frac{1}{2}. & \end{cases}$$

This gives

$$\begin{cases} 3+\alpha-\frac{6p}{e}=\frac{3}{a_2}+\frac{3}{b_2}<2\tau, \\ \frac{1}{6}\leq\frac{1}{e}\leq\frac{1}{2}. \end{cases}$$

Such a choice is possible because $p > p_c$. Moreover, by the identities

$$\int_{\mathbb{R}^3}(\psi_R\bar{u})\mathcal{N}[\psi_R u]\,dx=\int_{\mathbb{R}^3}\bar{u}\mathcal{N}[u]\,dx+O\Big(\int_{B^c(R)}\bar{u}\mathcal{N}[u]\,dx\Big),$$
$$\int_{\mathbb{R}^3}\psi_R^2|\nabla u|^2\,dx=\|\nabla(\psi_R u)\|^2+\int_{\mathbb{R}^3}\psi_R\Delta\psi_R|u|^2\,dx,$$

via (3.26), (3.3), and Sobolev embedding, one writes

$$\begin{aligned} V_R''[u]+cR^{-2}+cR^{-\varepsilon} &\gtrsim \mathcal{I}(\psi_R u) \\ &\gtrsim \|\sqrt{\mathcal{K}_\lambda}(\psi_R u)\|^2 \\ &\gtrsim \|\psi_R u\|_6^2 \\ &\gtrsim \Big(\int_{B(R)}|u|^6\,dx\Big)^{\frac{1}{3}}. \end{aligned} \tag{3.27}$$

Integrating in time the estimate (3.27) and taking $0<\varepsilon\ll 1$, it follows that

$$\begin{aligned} \int_0^T\Big(\int_{B(R)}|u(s,x)|^6\,dx\Big)^{\frac{1}{3}}ds &\lesssim V_R'[u(T)]-V_R'[u_0]+cTR^{-2}+cTR^{-\varepsilon} \\ &\lesssim R+cTR^{-\varepsilon}. \end{aligned}$$

So, (3.21) follows by taking $R=T^{\frac{1}{1+\varepsilon}}$. Moreover, (3.21) gives

$$\frac{2}{T}\int_{\frac{T}{2}}^T\Big(\int_{B(R)}|u(s,x)|^6\,dx\Big)^{\frac{1}{3}}ds\lesssim T^{-\frac{\varepsilon}{1+\varepsilon}}.$$

We conclude the proof of (3.22) by using the mean value theorem. ■

3.4. Proof of the scattering in Theorem 2.1 under (2.10)

Take $R,\varepsilon>0$ given by Proposition 3.1 and $t_n, R_n\to\infty$ given by Proposition 3.3. Letting $n\gg 1$ such that $R_n>R$, one gets by Hölder's inequality that

$$\begin{aligned} \|u(t_n)\|_{L^2(B(R))}^2 &\leq |B(R)|^{\frac{2}{3}}\,\|u(t_n)\|_{L^6(B(R))}^2 \\ &\leq R^2\|u(t_n)\|_{L^6(B(R_n))}^2 \\ &\lesssim \varepsilon^2. \end{aligned}$$

Hence, the scattering of energy global solutions to the focusing problem (1.1) follows from Proposition 3.1.

3.5. Proof of the scattering in Theorem 2.1 under (2.11)

This part follows from the first point in Theorem 2.1 with the next result.

Lemma 3.2. *The assumption* (2.11) *implies* (2.10).

Proof. Taking the real function $g : t \mapsto t^2 - \frac{C_{p,\tau,\alpha,\lambda}}{p} t^\gamma$ and computing using (3.5), one has

$$E[u][M[u]]^{\alpha_c} \geq \|\sqrt{\mathcal{K}_\lambda} u\|^2 \|u\|^{2\alpha_c} - \frac{C_{p,\tau,\alpha,\lambda}}{p} \|u\|^{\rho+2\alpha_c} \|\sqrt{\mathcal{K}_\lambda} u\|^\gamma$$
$$= g(\|\sqrt{\mathcal{K}_\lambda} u\| \|u\|^{\alpha_c}). \tag{3.28}$$

Now, with Pohozaev identities (2.8) via (2.11) and the conservation laws, one has, for some $0 < \varepsilon < 1$,

$$\begin{aligned} g(\|\sqrt{\mathcal{K}_\lambda} u\| \|u\|^{\alpha_c}) &\leq E[u][M[u]]^{\alpha_c} \\ &< (1-\varepsilon) E[\varphi][M[\varphi]]^{\alpha_c} \\ &= (1-\varepsilon) g(\|\sqrt{\mathcal{K}_\lambda} \varphi\| \|\varphi\|^{\alpha_c}). \end{aligned} \tag{3.29}$$

Thus, with time continuity, the assumption (2.11) is invariant under the flow of (1.1) and $T^* = \infty$. Moreover, by Pohozaev identities (2.8), one writes

$$E[\varphi][M[\varphi]]^{\alpha_c} = \frac{\gamma-2}{\gamma} (\|\sqrt{\mathcal{K}_\lambda} \varphi\| \|\varphi\|^{\alpha_c})^2 = \frac{C_{p,\tau,\alpha,\lambda}(\gamma-2)}{2p} (\|\sqrt{\mathcal{K}_\lambda} \varphi\| \|\varphi\|^{\alpha_c})^\gamma.$$

So, with (3.28) and (3.29), one gets

$$1 - \varepsilon \geq \frac{\gamma}{\gamma-2} \left(\frac{\|\sqrt{\mathcal{K}_\lambda} u\| \|u\|^{\alpha_c}}{\|\sqrt{\mathcal{K}_\lambda} \varphi\| \|\varphi\|^{\alpha_c}} \right)^2 - \frac{2}{\gamma-2} \left(\frac{\|\sqrt{\mathcal{K}_\lambda} u\| \|u\|^{\alpha_c}}{\|\sqrt{\mathcal{K}_\lambda} \varphi\| \|\varphi\|^{\alpha_c}} \right)^\gamma.$$

Following the variations of $t \mapsto \frac{\gamma}{\gamma-2} t^2 - \frac{2}{\gamma-2} t^\gamma$ via the assumption (2.11) and a continuity argument, there is a real number denoted also by $0 < \varepsilon < 1$ such that

$$\|\sqrt{\mathcal{K}_\lambda} u(t)\| \|u(t)\|^{\alpha_c} \leq (1-\varepsilon) \|\sqrt{\mathcal{K}_\lambda} \varphi\| \|\varphi\|^{\alpha_c} \quad \text{on } \mathbb{R}. \tag{3.30}$$

Now, by (3.30) and Pohozaev identities (2.8) via (3.5), it follows that, for some real number denoted also by $0 < \varepsilon < 1$,

$$\begin{aligned} \mathcal{P}[u][M[u]]^{\alpha_c} &\leq C_{p,\tau,\alpha,\lambda} \|\sqrt{\mathcal{K}_\lambda} u\|^\gamma \|u\|^{\rho+2\alpha_c} \\ &\leq C_{p,\tau,\alpha,\lambda} (1-\varepsilon) (\|\sqrt{\mathcal{K}_\lambda} \varphi\| \|\varphi\|^{\alpha_c})^\gamma \\ &\leq (1-\varepsilon) \frac{2p}{\gamma} (\|\sqrt{\mathcal{K}_\lambda} \varphi\| \|\varphi\|^{\alpha_c})^2 \\ &\leq (1-\varepsilon) \mathcal{P}[\varphi] M[\varphi]^{\alpha_c}. \end{aligned}$$

This finishes the proof. ∎

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