Van der Corput lemmas for Mittag-Leffler functions. I

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Abstract. In this paper, we study analogues of the van der Corput lemmas involving Mittag-Leffler functions. The generalisation is that we replace the exponential function with the Mittag-Leffler-type function to study oscillatory-type integrals appearing in the analysis of time-fractional partial differential equations. Several generalisations of the first and second van der Corput lemmas are proved. Optimal estimates on decay orders for particular cases of the Mittag-Leffler functions are also obtained. As an application of the above results, the generalised Riemann–Lebesgue lemma and the Cauchy problem for the time-fractional evolution equation are considered.

1. Introduction

In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals. This estimate was first obtained by the Dutch mathematician Johannes Gaultherus van der Corput [28] and named in his honour.

Following Stein [24], let us state the classical van der Corput lemmas as follows.

- **van der Corput first lemma.** Suppose that ϕ is a real-valued and smooth function in $[a, b]$. If ψ is a smooth function, ϕ' is monotonic, and $|\phi'(x)| \geq 1$ for all $x \in (a, b)$, then

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq C\lambda^{-1}, \quad \lambda > 0. \tag{1.1}$$

- **van der Corput second lemma.** Suppose that ϕ is a real-valued and smooth function on $[a, b]$. If $|\phi^{(k)}(x)| \geq 1$, $k \geq 2$ for all $x \in [a, b]$ and ψ is a smooth function, then

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq C\lambda^{-1/k}, \quad \lambda > 0. \tag{1.2}$$

Note that various generalisations of the van der Corput lemmas have been investigated in numerous papers [8,15–17,20,24,25,30]. Multidimensional analogues of the van der Corput lemmas were studied in [3–5,9,10,18,21]. In particular, in [21], the multi-dimensional van der Corput lemma was obtained with constants independent of the phase and amplitude.

A classic natural generalisation of an exponential function is the Mittag-Leffler function defined as (see, e.g., [7, 11])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{R},$$

with the property that

$$E_{1,1}(z) = e^z. \tag{1.3}$$

The main objective of this paper is to obtain the van der Corput lemmas for the integral defined by

$$I_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx, \tag{1.4}$$

where $0 < \alpha \leq 1$, $\beta > 0$, ϕ is a phase and ψ is an amplitude, and $\lambda \geq 0$.

The second part of this paper dealing with integrals of the type

$$\tilde{I}_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}(i^\alpha\lambda\phi(x))\psi(x)dx \tag{1.5}$$

has appeared in [22]. As we see above, the integral (1.4) is different from the integral (1.5), since in (1.4) there is a purely imaginary number i before the phase function, and in (1.5) the fractional power of the imaginary number, i.e., i^α. The main difference between (1.4) and (1.5) is that if to study (1.4) we used the estimate of the Mittag-Leffler functions given by (1.7), then to study the integral (1.5) we used the following estimate of the Mittag-Leffler functions:

$$|E_{\alpha,\beta}(z)| \leq C_1(1 + |z|)^{(1-\beta)/\alpha} \exp(\text{Re}(z^{1/\alpha})) + \frac{C_2}{1 + |z|}, \quad z \in \mathbb{C}, \ |\arg(z)| \leq \mu.$$

Since the asymptotic behaviour of Mittag-Leffler function in these cases is also different, this yields different decay rates. We refer to Section 6 for the detailed comparison between results for $I_{\alpha,\beta}(\lambda)$ in (1.4) and $\tilde{I}_{\alpha,\beta}(\lambda)$ in (1.5).

In view of (1.3)*, we are talking about the generalisation of the van der Corput lemmas replacing $E_{1,1}$ by $E_{\alpha,\beta}$ so that instead of integrals in* (1.1) *and* (1.2) *we have the integral* (1.4)*. As it is clear from the properties of the exponential function, the property whether $\phi \neq 0$ or $\phi(c) = 0$ at some point $c \in [a, b]$ is not important for estimates* (1.1) *and* (1.2)*, as the terms $e^{i\lambda\gamma}$ can be taken out of such integrals for any constant γ without changing the estimates. However, this is no longer the case for more general Mittag-Leffler functions. Therefore, the property whether $\phi \neq 0$ or ϕ vanishes at some point will affect the estimates that we obtain.*

Such integrals as in expression (1.4) arise in the study of solutions of the time-fractional Schrödinger equation and the time-fractional wave equation (see, for example, [12, 13, 23, 26, 27, 29]).

For the convenience of the reader, let us briefly summarise the results of this paper, distinguishing between different sets of assumptions.

van der Corput first lemma.

- Let $-\infty \le a < b \le +\infty$. Let $\phi : [a, b] \to \mathbb{R}$ be a measurable function, and let $\psi \in L^1[a, b]$. If $0 < \alpha < 1$, $\beta > 0$, and $\operatorname{ess\,inf}_{x \in [a,b]} |\phi(x)| > 0$, then we have the estimate

$$|I_{\alpha,\beta}(\lambda)| \lesssim \frac{1}{1 + \lambda}, \quad \lambda \ge 0.$$

  Here, and further, we use $X \lesssim Y$ to denote the estimate $X \le CY$ for some constant C independent on λ.

- Let $-\infty \le a < b \le +\infty$ and $0 < \alpha < 1$, $\beta > 0$. Let $\phi \in L^\infty[a, b]$ be a real-valued differentiable function on $[a, b]$ vanishing at some point $c \in (a, b)$. Let $\psi \in C_0[a, b]$, and let $m = \inf_{x \in \operatorname{supp}(\psi)} |\phi'(x)| > 0$; then

$$|I_{\alpha,\beta}(\lambda)| \lesssim \frac{\log(2 + \lambda)}{1 + \lambda}, \quad \lambda \ge 0.$$

- Let $0 < \alpha < 1$ and $-\infty < a < b < +\infty$. Let $\phi \in C^1[a, b]$ be a real-valued function, ϕ' monotonic, and $\psi' \in L^1[a, b]$.

  (i)   If $\phi'(x) \ne 0$ for all $x \in [a, b]$, then we have that

$$|I_{\alpha,\alpha}(\lambda)| \lesssim \frac{1}{1 + \lambda}, \quad \lambda \ge 0.$$

  (ii)   If $\phi(x) \ne 0$ and $\phi'(x) \ne 0$ for all $x \in [a, b]$, then

$$|I_{\alpha,\alpha}(\lambda)| \lesssim \frac{1}{(1 + \lambda)^2}, \quad \lambda \ge 0.$$

- Let $0 < \alpha < 1$ and $-\infty < a < b < +\infty$. Let $\phi \in C^2[a, b]$ be a real-valued function, and let $\psi \in C^1[a, b]$.

  (i)   If $\phi'(x) \ne 0$ for all $x \in [a, b]$, then we have

$$|I_{\alpha,\alpha}(\lambda)| \lesssim \frac{1}{1 + \lambda}, \quad \lambda \ge 0.$$

  (ii)   If $\phi'(x) \ne 0$ for all $x \in [a, b]$ and $\psi(a) = \psi(b) = 0$, then we have

$$|I_{\alpha,\alpha}(\lambda)| \lesssim \frac{\log(2 + \lambda)}{(1 + \lambda)^2}, \quad \lambda \ge 0.$$

  (iii)   If $\phi(x) \ne 0$ and $\phi'(x) \ne 0$ for all $x \in [a, b]$, then we have

$$|I_{\alpha,\alpha}(\lambda)| \lesssim \frac{1}{(1 + \lambda)^2}, \quad \lambda \ge 0.$$

- Let $-\infty \le a < b \le +\infty$. Let $\phi \in L^\infty[a, b]$ be a real-valued function, and let $\psi \in L^1[a, b]$. If $\beta > 1$ and $\inf_{x \in [a,b]} |\phi(x)| > 0$, then we have the estimate

$$|I_{1,\beta}(\lambda)| \lesssim \frac{1}{(1 + \lambda)^{\beta-1}}, \quad \lambda \ge 0.$$

van der Corput second lemma.

- Let $-\infty < a < b < +\infty$. Let $0 < \alpha < 1$, $\beta > 0$, and ϕ be a real-valued function such that $\phi \in C^k[a, b]$, and let $\psi' \in L^1[a, b]$. If ϕ has finitely many zeros on $[a, b]$ and $|\phi^{(k)}(x)| \geq 1$, $k \geq 2$ for all $x \in [a, b]$, then we have

$$|I_{\alpha,\beta}(\lambda)| \lesssim \frac{\log^{\frac{1}{k}}(2 + \lambda)}{(1 + \lambda)^{\frac{1}{k}}}, \quad \lambda \geq 0.$$

- Let $-\infty < a < b < +\infty$ and $0 < \alpha < 1$. Suppose that ϕ is a real-valued function, and let $\psi' \in L^1[a, b]$. If $\phi \in C^k[a, b]$, $k \geq 2$, and $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a, b]$, then we have the estimate

$$|I_{\alpha,\alpha}(\lambda)| \lesssim \frac{1}{(1 + \lambda)^{\frac{1}{k}}}, \quad \lambda \geq 0.$$

Lower and upper estimates.

- Let $-\infty < a < b < +\infty$ and $0 < \alpha \leq 1/2$, $\beta > 2\alpha$. Let $\phi \in L^\infty[a, b]$ be a real-valued function, and let $\psi \in L^\infty[a, b]$.

  Suppose that $m_1 = \inf_{a \leq x \leq b} |\phi(x)| > 0$ and $m_2 = \inf_{a \leq x \leq b} |\psi(x)| > 0$. Then,

$$|I_{\alpha,\beta}(\lambda)| \leq \frac{K_1 \|\psi\|_{L^\infty}}{(b - a)} \frac{1 + \lambda \|\phi\|_{L^\infty}}{1 + k_1 \lambda^2 m_1^2}, \quad \lambda \geq 0,$$

  where $K_1 = \max\{\frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha+\beta)}\}$ and $k_1 = \min\{\frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}, \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\}$, and

$$|I_{\alpha,\beta}(\lambda)| \geq \frac{m_2}{(b - a)\Gamma(\alpha + \beta)} \frac{\lambda m_1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta)}\lambda^2 \|\phi\|_{L^\infty}^2}, \quad \lambda \geq 0.$$

  If $m_1 = \inf_{a \leq x \leq b} |\phi(x)| = 0$, then

$$|I_{\alpha,\beta}(\lambda)| \geq \frac{m_2}{(b - a)\Gamma(\beta)} \frac{1}{1 + \frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta)}\lambda^2 \|\phi\|_{L^\infty}^2}, \quad \lambda \geq 0.$$

- Let $-\infty < a < b < +\infty$ and $0 < \alpha < 1/2$, $\beta = 2\alpha$. Let $\phi \in L^\infty[a, b]$ be a real-valued function, and let $\psi \in L^\infty[a, b]$. Let $m_1 = \inf_{a \leq x \leq b} |\phi(x)| > 0$ and $m_2 = \inf_{a \leq x \leq b} |\psi(x)| > 0$. Then, we have the following estimates:

$$|I_{\alpha,2\alpha}(\lambda)| \leq \frac{K \|\psi\|_{L^\infty}}{(b - a)} \frac{1 + \lambda \|\phi\|_{L^\infty}\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2 \|\phi\|_{L^\infty}^2\right)}{\left(1 + \min\{\frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}, \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\}\lambda^2 m_1^2\right)^2}, \quad \lambda \geq 0,$$

  and

$$|I_{\alpha,2\alpha}(\lambda)| \geq \frac{m_2}{(b - a)\Gamma(3\alpha)} \frac{\lambda m_1}{1 + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)}\lambda^2 \|\phi\|_{L^\infty}^2}, \quad \lambda \geq 0,$$

  where $K = \max\{\frac{1}{\Gamma(2\alpha)}, \frac{1}{\Gamma(3\alpha)}\}$.
  If $m_1 = \inf_{a \leq x \leq b} |\phi(x)| = 0$ and $m_2 = \inf_{a \leq x \leq b} |\psi(x)| > 0$, then we have

$$|I_{\alpha,2\alpha}(\lambda)| \geq \frac{m_2}{(b - a)\Gamma(2\alpha)} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)}}\lambda^2 \|\phi\|_{L^\infty}^2\right)^2}, \quad \lambda \geq 0.$$

1.1. Preliminaries

In this section, we briefly review some preliminary properties for the sake of the rest of the paper.

1.1.1. Mittag-Leffler function and some of its properties. The Mittag-Leffler-type function is defined as (see, e.g., [7, 11])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{R}. \tag{1.6}$$

The function (1.6) is an entire function of order $\frac{1}{\alpha}$ and type 1 (see, e.g., [6, 7]).

From the definition of function (1.6), we obtain a series of formulas relating the Mittag-Leffler function $E_{\alpha,\beta}(z)$ with elementary and special functions, which we list here for the convenience of the reader.

- In the case $\alpha = 1$ and $\beta = 1$, we have

$$E_{1,1}(z) = \exp(z) = e^z.$$

- If $\beta = 1$, then we have the classical Mittag-Leffler function [14]

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

We will need the following properties.

Proposition 1.1 ([19]). *If $0 < \alpha < 2$, β is an arbitrary real number, and μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, then there is $C > 0$ such that we have*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}, \quad z \in \mathbb{C}, \quad \mu \leq |\arg(z)| \leq \pi. \tag{1.7}$$

Proposition 1.2 ([2]). *The following optimal estimates are valid for the real-valued Mittag-Leffler function:*

$$\frac{1}{1 + \Gamma(1-\alpha)x} \leq E_{\alpha,1}(-x) \leq \frac{1}{1 + \frac{1}{\Gamma(1+\alpha)}x}, \quad x \geq 0, \ 0 < \alpha < 1, \tag{1.8}$$

$$\frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}}x\right)^2} \leq \Gamma(\alpha)E_{\alpha,\alpha}(-x) \leq \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}}x\right)^2}, \quad x \geq 0, \ 0 < \alpha < 1, \tag{1.9}$$

$$\frac{1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}x} \leq \Gamma(\beta)E_{\alpha,\beta}(-x) \leq \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}x}, \quad x \geq 0, \ 0 < \alpha \leq 1, \ \beta > \alpha. \tag{1.10}$$

The following properties of the Mittag-Leffler function with special argument possibly follow from its known properties. But we give simple proofs for completeness.

Proposition 1.3 (Fractional Euler formula). *Let $\alpha, \beta > 0$ and $\phi : [a, b] \to \mathbb{C}$. Then, for all $\lambda \in \mathbb{C}$, we have*

$$E_{\alpha,\beta}(i\lambda\phi(x)) = E_{2\alpha,\beta}(-\lambda^2\phi^2(x)) + i\lambda\phi(x)E_{2\alpha,\beta+\alpha}(-\lambda^2\phi^2(x)). \tag{1.11}$$

Proof. From the representation of the Mittag-Leffler function (1.6), we have

$$
\begin{aligned}
E_{\alpha,\beta}(i\lambda\phi(x)) &= \sum_{j=0}^{\infty} i^j\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} \\
&= \frac{1}{2}\sum_{j=0}^{\infty} i^j\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} + \frac{1}{2}\sum_{j=0}^{\infty} i^j\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} \\
&= \frac{1}{2}\sum_{j=0}^{\infty} i^j\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} + \frac{1}{2}\sum_{j=0}^{\infty} (-i)^j\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} \\
&\quad + \frac{1}{2}\sum_{j=0}^{\infty} i^j\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} - \frac{1}{2}\sum_{j=0}^{\infty} (-i)^j\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} \\
&= \frac{1}{2}\sum_{j=0}^{\infty} (i^j + (-i)^j)\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} \\
&\quad + \frac{1}{2}\sum_{j=0}^{\infty} (i^j - (-i)^j)\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + \beta)} \\
&= \sum_{j=0}^{\infty} i^{2j}\lambda^{2j} \frac{\phi^{2j}(x)}{\Gamma(2\alpha j + \beta)} + \sum_{j=0}^{\infty} i^{2j+1}\lambda^{2j+1} \frac{\phi^{2j+1}(x)}{\Gamma(2\alpha j + \alpha + \beta)} \\
&= E_{2\alpha,\beta}(-\lambda^2\phi^2(x)) + i\lambda\phi(x)E_{2\alpha,\beta+\alpha}(-\lambda^2\phi^2(x)).
\end{aligned}
$$

This completes the proof. ∎

Proposition 1.4. *Let ϕ be a differentiable function such that $\phi'(x) \neq 0$ for all $x \in [a, b]$. Then, for any $\alpha > 0$ and $\lambda \in \mathbb{C}$, we have*

$$E_{\alpha,\alpha}(i\lambda\phi(x)) = \frac{\alpha}{i\lambda\phi'(x)} \frac{d}{dx}(E_{\alpha,1}(i\lambda\phi(x))). \tag{1.12}$$

Proof. Using the property $\Gamma(z + 1) = z\Gamma(z)$ [11, p. 24] of the Euler gamma function, we have

$$
\begin{aligned}
\frac{d}{dx}(E_{\alpha,1}(i\lambda\phi(x))) &= \frac{d}{dx}\sum_{j=0}^{\infty} i^j\lambda^j \frac{\phi^j(x)}{\Gamma(\alpha j + 1)} = \phi'(x)\sum_{j=1}^{\infty} i^j\lambda^j \frac{j\phi^{j-1}(x)}{\Gamma(\alpha j + 1)} \\
&= \frac{\phi'(x)}{\alpha}\sum_{j=1}^{\infty} \frac{i^j\lambda^j\phi^{j-1}(x)}{\Gamma(\alpha j)} = \frac{i\lambda\phi'(x)}{\alpha}\sum_{j=0}^{\infty} \frac{i^j\lambda^j\phi^j(x)}{\Gamma(\alpha j + \alpha)} \\
&= \frac{i\lambda\phi'(x)}{\alpha} E_{\alpha,\alpha}(i\lambda\phi(x)).
\end{aligned}
$$

Hence, we get (1.12). The proof is complete. ∎

2. The generalisations of the van der Corput first lemma

In this section, we consider the integral operator defined by

$$I_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx, \tag{2.1}$$

where $0 < \alpha \leq 1$, $\beta > 0$, ϕ is a phase and ψ is an amplitude, and λ is a positive real number that can vary. We are interested in particular in the behaviour of $I_{\alpha,\beta}(\lambda)$ when λ is large, as for small λ the integral is just bounded.

Theorem 2.1. *Let* $-\infty \leq a < b \leq +\infty$. *Let* $\phi : [a,b] \to \mathbb{R}$ *be a measurable function, and let* $\psi \in L^1[a,b]$. *If* $0 < \alpha < 1$, $\beta > 0$, *and* $m = \operatorname{ess\,inf}_{x \in [a,b]} |\phi(x)| > 0$, *then we have the estimate*

$$|I_{\alpha,\beta}(\lambda)| \leq \frac{M\|\psi\|_{L^1[a,b]}}{1 + m\lambda}, \quad \lambda \geq 0, \tag{2.2}$$

where M *does not depend on* ϕ, ψ, *and* λ.

Proof. As for small λ the integral (2.1) is just bounded, we give the proof for $\lambda \geq 1$. Let $\phi : [a,b] \to \mathbb{R}$ be a measurable function and $\psi \in L^1[a,b]$. Then,

$$|I_{\alpha,\beta}(\lambda)| = \left| \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx \right| \leq \int_a^b |E_{\alpha,\beta}(i\lambda\phi(x))||\psi(x)|dx.$$

Using formula (1.11) and estimate (1.7), we have that

$$|E_{\alpha,\beta}(i\lambda\phi(x))| \leq |E_{2\alpha,\beta}(-\lambda^2\phi^2(x))| + \lambda|\phi(x)||E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x))|$$
$$\leq \frac{C}{1 + \lambda^2\phi^2(x)} + \frac{C\lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)} \leq C\frac{1 + \lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)}. \tag{2.3}$$

As ϕ and ψ do not depend on λ, and $m = \operatorname{ess\,inf}_{x \in [a,b]} |\phi(x)| > 0$, then from (2.3) and using $2(1 + y^2) \geq (1 + y)^2$ we have

$$|I_{\alpha,\beta}(\lambda)| \leq \int_a^b |E_{\alpha,\beta}(i\lambda\phi(x))||\psi(x)|dx$$
$$\leq C \int_a^b \frac{1 + \lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)}|\psi(x)|dx$$
$$\leq 2C \int_a^b \frac{1 + \lambda|\phi(x)|}{(1 + \lambda|\phi(x)|)^2}|\psi(x)|dx$$
$$\leq C_1 \int_a^b \frac{|\psi(x)|}{1 + \lambda|\phi(x)|}dx$$
$$\leq \frac{C_1}{1 + \lambda\operatorname{ess\,inf}_{x \in [a,b]} |\phi(x)|} \int_a^b |\psi(x)|dx$$
$$\leq \frac{M\|\psi\|_{L^1[a,b]}}{1 + m\lambda}.$$

The proof is complete.   ∎

If ϕ can vanish at some point $c \in [a, b]$, we have a weaker decay rate. To handle this case, we require some regularity of ϕ.

Theorem 2.2. *Let $-\infty \le a < b \le +\infty$ and $0 < \alpha < 1$, $\beta > 0$. Let $\phi \in L^\infty[a, b]$ be a real-valued differentiable function on $[a, b]$ vanishing at some point $c \in (a, b)$. Let $\psi \in C_0[a, b]$, and let $m = \inf_{x \in \mathrm{supp}(\psi)} |\phi'(x)| > 0$; then*

$$|I_{\alpha,\beta}(\lambda)| \le M \|\psi\|_{C_0[a,b]} \frac{\log(2 + \|\phi\|_{L^\infty[a,b]}\lambda)}{1 + m\lambda}, \quad \lambda \ge 0, \tag{2.4}$$

where M is a constant independent of ϕ, ψ, and λ.

Proof. Since $I_{\alpha,\beta}(\lambda)$ is bounded for small λ, we can assume that $\lambda \ge 1$.

If $a = -\infty$, we understand $C[a, b]$ as $C(-\infty, b]$; if $b = +\infty$, we understand $C[a, b]$ as $C[a, +\infty)$. Similarly, if $a = -\infty$ and $b = +\infty$, we understand $C[a, b]$ as $C(\mathbb{R})$.

Let

$$\mathrm{supp}(\psi) := [a_1, b_1] \subset (a, b),$$

where $-\infty < a_1 < b_1 < +\infty$. Without loss of generality, suppose that $c \in (a_1, b_1)$. Since $\psi \in C_0[a, b]$, we have

$$|I_{\alpha,\beta}(\lambda)| \le \|\psi\|_{C_0[a,b]} \int_{a_1}^{b_1} |E_{\alpha,\beta}(i\lambda\phi(x))| dx.$$

From (2.3), it follows that

$$|I_{\alpha,\beta}(\lambda)| \le M \|\psi\|_{C_0[a,b]} \int_{a_1}^{b_1} \frac{1 + \lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)} dx.$$

Here, and in what follows, we assume that M is an arbitrary constant independent of λ.

Without loss of generality, we can assume that ϕ is nondecreasing; then $\phi \le 0$, $x \in [a_1, c]$ and $\phi \ge 0$, $x \in [c, b_1]$. Since ϕ is a bounded function, we get

$$|I_{\alpha,\beta}(\lambda)| \le M \|\psi\|_{C_0[a,b]} \int_{a_1}^{b_1} \frac{1 + \lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)} dx$$

$$\le 2M \|\psi\|_{C_0[a,b]} \left[\int_{a_1}^{c} \frac{1 - \lambda\phi(x)}{(1 + \lambda\phi(x))^2} dx + \int_{c}^{b_1} \frac{1 + \lambda\phi(x)}{(1 + \lambda\phi(x))^2} dx \right]$$

$$\le M \|\psi\|_{C_0[a,b]} \left[\underbrace{\int_{a_1}^{c} \frac{1}{1 - \lambda\phi(x)} dx}_{I_1} + \underbrace{\int_{c}^{b_1} \frac{1}{1 + \lambda\phi(x)} dx}_{I_2} \right],$$

thanks to the $2(1 + a^2) \ge (1 + a)^2$, $a \ge 0$. Here, $M > 0$ is the arbitrary constant.

As ϕ is the differentiable function with $m = \inf_{x \in [a,b]} |\phi'(x)| > 0$, we have that

$$I_2 = \int_{c}^{b_1} \frac{1}{1 + \lambda\phi(x)} dx = \frac{1}{\lambda} \int_{c}^{b_1} \frac{1}{\phi'(x)} \frac{\lambda\phi'(x)}{1 + \lambda\phi(x)} dx$$

$$\le \frac{1}{m\lambda} \int_{c}^{b_1} \frac{d(1 + \lambda\phi(x))}{1 + \lambda\phi(x)} = \frac{1}{m\lambda} \log(1 + \lambda\phi(b_1)).$$

Similarly, for I_1, we obtain

$$I_1 = \frac{1}{m\lambda} \log(1 - \lambda\phi(a)).$$

Combining the above estimates, we get (2.4).

The cases $c = a_1$ and $c = b_1$ can be proven in a similar way. The proof is complete. ∎

2.1. The case $E_{\alpha,\alpha}$

In this section, we are interested in a particular case of the integral operator (2.1), when $0 < \alpha < 1$, $\beta = \alpha$, that is, the integral operator $I_{\alpha,\alpha}(\lambda)$. For smoother ϕ and ψ, we get a better estimate than (2.2).

Theorem 2.3. *Let $0 < \alpha < 1$ and $-\infty < a < b < +\infty$. Let $\phi \in C^1[a,b]$ be a real-valued function, ϕ' monotonic, and let $\psi' \in L^1[a,b]$.*

(i) *If $\phi'(x) \geq 1$ for all $x \in [a,b]$, then we have*

$$|I_{\alpha,\alpha}(\lambda)| \leq M_1 \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right] (1+\lambda)^{-1}, \quad \lambda \geq 0, \qquad (2.5)$$

where M_1 does not depend on ϕ, ψ, and λ.

(ii) *If $m = \inf_{x \in [a,b]} |\phi(x)| > 0$ and $\phi'(x) \geq 1$ for all $x \in [a,b]$, then we have*

$$|I_{\alpha,\alpha}(\lambda)| \leq M_2 \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right] (1+\lambda)^{-1}(1+m\lambda)^{-1}, \quad \lambda \geq 0,$$

where M_2 does not depend on ϕ, ψ, and λ.

Proof. For small λ, we have bounded estimate for the integral $I_{\alpha,\alpha}(\lambda)$.

Let $\lambda \geq 1$. First, we consider the integral

$$E(x) = \int_a^x E_{\alpha,\alpha}(i\lambda\phi(s)) ds, \quad x \in [a,b].$$

Then, using property (1.12) and integrating by parts give

$$\begin{aligned}
E(x) &= \int_a^x E_{\alpha,\alpha}(i\lambda\phi(s)) ds \\
&= \frac{\alpha}{i\lambda} \int_a^x \frac{1}{\phi'(s)} \frac{d}{ds}(E_{\alpha,1}(i\lambda\phi(s))) ds \\
&= \frac{\alpha}{i\lambda} E_{\alpha,1}(i\lambda\phi(x)) \frac{1}{\phi'(x)} \\
&\quad - \frac{\alpha}{i\lambda} E_{\alpha,1}(i\lambda\phi(a)) \frac{1}{\phi'(a)} \\
&\quad - \frac{\alpha}{i\lambda} \int_a^x E_{\alpha,1}(i\lambda\phi(s)) \frac{d}{ds}\left(\frac{1}{\phi'(s)}\right) ds.
\end{aligned}$$

Then, for all $x \in [a, b]$, we have

$$|E(x)| \leq \frac{\alpha}{\lambda} \left| \int_a^x E_{\alpha,1}(i\lambda\phi(s)) \frac{d}{ds}\left(\frac{1}{\phi'(s)}\right) ds \right|$$
$$+ \frac{\alpha}{\lambda} \left| E_{\alpha,1}(i\lambda\phi(x)) \frac{1}{\phi'(x)} \right| + \frac{\alpha}{\lambda} \left| E_{\alpha,1}(i\lambda\phi(a)) \frac{1}{\phi'(a)} \right|.$$

We first prove case (ii). Since ϕ' is monotonic and $\phi'(x) \geq 1$ for all $x \in [a, b]$, then $\frac{1}{\phi'}$ is also monotonic, and $\frac{d}{dx}\frac{1}{\phi'(x)}$ has a fixed sign. Therefore, using estimate (2.3), we have

$$|E(x)| \leq \frac{\alpha}{\lambda} \int_a^x |E_{\alpha,1}(i\lambda\phi(s))| \left| \frac{d}{ds}\left(\frac{1}{\phi'(s)}\right) \right| ds$$
$$+ \frac{\alpha}{\lambda} |E_{\alpha,1}(i\lambda\phi(x))| \frac{1}{|\phi'(x)|} + \frac{\alpha}{\lambda} |E_{\alpha,1}(i\lambda\phi(a))| \frac{1}{|\phi'(a)|}$$
$$\leq \frac{C\alpha}{\lambda} \int_a^x \frac{1+\lambda|\phi(s)|}{1+\lambda^2\phi^2(s)} \left| \frac{d}{ds}\left(\frac{1}{\phi'(s)}\right) \right| ds$$
$$+ \frac{C\alpha}{\lambda} \frac{1+\lambda|\phi(x)|}{1+\lambda^2\phi^2(x)} + \frac{C\alpha}{\lambda} \frac{1+\lambda|\phi(a)|}{1+\lambda^2\phi^2(a)}$$
$$\leq \frac{2C\alpha}{\lambda} \int_a^x \frac{1}{1+\lambda|\phi(s)|} \left| \frac{d}{ds}\left(\frac{1}{\phi'(s)}\right) \right| ds$$
$$+ \frac{2C\alpha}{\lambda} \frac{1}{1+\lambda|\phi(x)|} + \frac{2C\alpha}{\lambda} \frac{1}{1+\lambda|\phi(a)|}$$
$$\leq \frac{2C\alpha}{\lambda(1+m\lambda)} \left[2 + \int_a^x \left| \frac{d}{ds}\left(\frac{1}{\phi'(s)}\right) \right| ds \right]$$
$$= \frac{2C\alpha}{\lambda(1+m\lambda)} \left[2 + \left| \int_a^x \frac{d}{ds}\left(\frac{1}{\phi'(s)}\right) ds \right| \right] = \frac{8C\alpha}{\lambda(1+m\lambda)},$$

where $m = \inf_{x\in[a,b]} |\phi(x)|$.

Consequently, thanks to $\frac{1}{\lambda} \leq \frac{2}{1+\lambda}$ for $\lambda \geq 1$, we have

$$|E(x)| \leq \frac{M_1}{(1+\lambda)(1+m\lambda)}, \quad \lambda \geq 1, \tag{2.6}$$

where the constant M_1 does not depend on ϕ and λ.

Now, we write $I_{\alpha,\alpha}(\lambda)$ as

$$I_{\alpha,\alpha}(\lambda) = \int_a^b \frac{d}{dx}(E(x))\psi(x)dx.$$

Integrating by parts and using the estimate (2.6), we obtain

$$|I_{\alpha,\alpha}(\lambda)| \leq \frac{M_1}{(1+\lambda)(1+m\lambda)} \left[|\psi(b)| + \int_a^b |\psi'(x)|dx \right].$$

If $\inf_{x \in [a,b]} |\phi(x)| \geq 0$, then we have

$$|I_{\alpha,\alpha}(\lambda)| \leq \frac{M_1}{1+\lambda} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right].$$

If $\inf_{x \in [a,b]} |\phi(x)| > 0$, then we have

$$|I_{\alpha,\alpha}(\lambda)| \leq \frac{M_2}{(1+\lambda)(1+m\lambda)} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right],$$

where the constant M_2 does not depend on ϕ and λ.   ∎

Theorem 2.4. *Let $0 < \alpha < 1$ and $-\infty < a < b < +\infty$. Let $\phi \in C^2[a,b]$ be a real-valued function and $\psi \in C^1[a,b]$.*

(i)   *If $\phi'(x) \neq 0$ for all $x \in [a,b]$, then we have*

$$|I_{\alpha,\alpha}(\lambda)| \leq M_1 \left[\left\| \left(\frac{\psi}{\phi'} \right)' \right\|_{L^\infty[a,b]} + \left| \frac{\psi(b)}{\phi'(b)} \right| + \left| \frac{\psi(a)}{\phi'(a)} \right| \right] (1+\lambda)^{-1}, \quad \lambda \geq 0,$$

where M_1 does not depend on λ.

(ii)   *If $\phi'(x) \neq 0$ for all $x \in [a,b]$ and $\psi(a) = \psi(b) = 0$, then we have*

$$|I_{\alpha,\alpha}(\lambda)| \leq M_2 \left\| \left(\frac{\psi}{\phi'} \right)' \right\|_{L^\infty[a,b]} \frac{\log(2+\lambda)}{(1+\lambda)^2}, \quad \lambda \geq 0, \qquad (2.7)$$

where M_2 does not depend on λ.

(iii)   *If $\phi(x) \neq 0$ and $\phi'(x) \neq 0$ for all $x \in [a,b]$, then we have*

$$|I_{\alpha,\alpha}(\lambda)| \leq M_3 \left[\left\| \left(\frac{\psi}{\phi'} \right)' \right\|_{L^\infty} + \left| \frac{\psi(b)}{\phi'(b)} \right| + \left| \frac{\psi(a)}{\phi'(a)} \right| \right] (1+\lambda)^{-2}, \quad \lambda \geq 0,$$

where M_3 does not depend on λ.

Proof. For small λ, we have bounded estimate for the integral $I_{\alpha,\alpha}(\lambda)$. Let $\lambda \geq 1$, $\phi \in C^2[a,b]$, and $\psi \in C^1[a,b]$. Then, using property (1.12) and integrating by parts in (2.1) give

$$\begin{aligned}
I_{\alpha,\alpha}(\lambda) &= \int_a^b E_{\alpha,\alpha}(i\lambda\phi(x))\psi(x) dx \\
&= \frac{\alpha}{i\lambda} \int_a^b \frac{1}{\phi'(x)} \frac{d}{dx} (E_{\alpha,1}(i\lambda\phi(x)))\psi(x) dx \\
&= \frac{\alpha}{i\lambda} E_{\alpha,1}(i\lambda\phi(b)) \frac{\psi(b)}{\phi'(b)} - \frac{\alpha}{i\lambda} E_{\alpha,1}(i\lambda\phi(a)) \frac{\psi(a)}{\phi'(a)} \\
&\quad - \frac{\alpha}{i\lambda} \int_a^b E_{\alpha,1}(i\lambda\phi(x)) \frac{d}{dx} \left(\frac{\psi(x)}{\phi'(x)} \right) dx.
\end{aligned}$$

Then,

$$|I_{\alpha,\alpha}(\lambda)| \leq \frac{\alpha}{\lambda}\left|\int_a^b E_{\alpha,1}(i\lambda\phi(x))\frac{d}{dx}\left(\frac{\psi(x)}{\phi'(x)}\right)dx\right|$$
$$+ \frac{\alpha}{\lambda}\left|E_{\alpha,1}(i\lambda\phi(b))\frac{\psi(b)}{\phi'(b)}\right| + \frac{\alpha}{\lambda}\left|E_{\alpha,1}(i\lambda\phi(a))\frac{\psi(a)}{\phi'(a)}\right|.$$

Recall that $\phi \in C^2[a,b]$, $\psi \in C^1[a,b]$ do not depend on λ, and $\phi'(x) \neq 0$ for all $x \in [a,b]$. Let

$$M_\phi = \max_{x\in[a,b]}|\phi(x)| \quad \text{and} \quad m_\phi = \min_{x\in[a,b]}|\phi(x)|.$$

Case (iii). If $\phi(x) \neq 0$ for all $x \in [a,b]$, then using the estimate (2.3) we have

$$|I_{\alpha,\alpha}(\lambda)| \leq \frac{\alpha C}{\lambda}\left[\left\|\left(\frac{\psi}{\phi'}\right)'\right\|_{L^\infty}\int_a^b\frac{1+\lambda|\phi(x)|}{1+\lambda^2\phi^2(x)}dx + \left|\frac{\psi(b)}{\phi'(b)}\right|\frac{1+\lambda|\phi(b)|}{1+\lambda^2\phi^2(b)}\right.$$
$$\left.+ \left|\frac{\psi(a)}{\phi'(a)}\right|\frac{1+\lambda|\phi(a)|}{1+\lambda^2\phi^2(a)}\right]$$
$$\leq \frac{\alpha C}{\lambda}\left[\left\|\left(\frac{\psi}{\phi'}\right)'\right\|_{L^\infty} + \left|\frac{\psi(b)}{\phi'(b)}\right| + \left|\frac{\psi(a)}{\phi'(a)}\right|\right]\frac{1+\lambda M_\phi}{1+\lambda^2 m_\phi^2}.$$

Since $1 + \mu\lambda \geq 1 + \lambda$, when $\mu \geq 1$ and $1 + \nu\lambda \leq 1 + \lambda$, when $\nu \leq 1$, we have

$$|I_{\alpha,\alpha}(\lambda)| \leq \frac{\alpha C}{\lambda}\left[\left\|\left(\frac{\psi}{\phi'}\right)'\right\|_{L^\infty} + \left|\frac{\psi(b)}{\phi'(b)}\right| + \left|\frac{\psi(a)}{\phi'(a)}\right|\right]\frac{1+\lambda M_\phi}{1+\lambda^2 m_\phi^2}$$
$$\leq \frac{\alpha C_1}{\lambda}\left[\left\|\left(\frac{\psi}{\phi'}\right)'\right\|_{L^\infty} + \left|\frac{\psi(b)}{\phi'(b)}\right| + \left|\frac{\psi(a)}{\phi'(a)}\right|\right]\frac{1+\lambda}{1+\lambda^2}$$
$$\leq \frac{2C_1}{\lambda}\left[\left\|\left(\frac{\psi}{\phi'}\right)'\right\|_{L^\infty} + \left|\frac{\psi(b)}{\phi'(b)}\right| + \left|\frac{\psi(a)}{\phi'(a)}\right|\right]\frac{1+\lambda}{(1+\lambda)^2}$$
$$\leq M_3\left[\left\|\left(\frac{\psi}{\phi'}\right)'\right\|_{L^\infty} + \left|\frac{\psi(b)}{\phi'(b)}\right| + \left|\frac{\psi(a)}{\phi'(a)}\right|\right]\frac{1}{(1+\lambda)^2}$$

thanks to $2(1+y^2) \geq (1+y)^2$. Therefore,

$$|I_{\alpha,\alpha}(\lambda)| \leq M_3\left[\left\|\left(\frac{\psi}{\phi'}\right)'\right\|_{L^\infty} + \left|\frac{\psi(b)}{\phi'(b)}\right| + \left|\frac{\psi(a)}{\phi'(a)}\right|\right](1+\lambda)^{-2}, \quad \lambda \geq 0,$$

where the constant M_3 depends on ϕ but does not depend on λ.

Case (i). Let

$$K_{\phi,\psi} = \left[\left\|\left(\frac{\psi}{\phi'}\right)'\right\|_{L^\infty} + \left|\frac{\psi(b)}{\phi'(b)}\right| + \left|\frac{\psi(a)}{\phi'(a)}\right|\right].$$

If $\min_{x \in [a,b]} |\phi(x)| = 0$, then we have

$$
\begin{aligned}
|I_{\alpha,\alpha}(\lambda)| &\leq \frac{\alpha C}{\lambda} K_{\phi,\psi} \left[\int_a^b \frac{1 + \lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)} dx + \frac{1 + \lambda|\phi(b)|}{1 + \lambda^2\phi^2(b)} + \frac{1 + \lambda|\phi(a)|}{1 + \lambda^2\phi^2(a)} \right] \\
&\leq \frac{\alpha C_1}{\lambda} K_{\phi,\psi} \left[\int_a^b \frac{1 + \lambda|\phi(x)|}{(1 + \lambda|\phi(x)|)^2} dx + \frac{1 + \lambda|\phi(b)|}{1 + \lambda^2\phi^2(b)} + \frac{1 + \lambda|\phi(a)|}{1 + \lambda^2\phi^2(a)} \right] \\
&\leq \frac{\alpha C_1}{\lambda} K_{\phi,\psi} \left[\int_a^b \frac{dx}{1 + \lambda|\phi(x)|} + \frac{1 + \lambda|\phi(b)|}{1 + \lambda^2\phi^2(b)} + \frac{1 + \lambda|\phi(a)|}{1 + \lambda^2\phi^2(a)} \right] \\
&\leq \frac{\alpha C_2}{\lambda} K_{\phi,\psi} \left[1 + \frac{1 + \lambda|\phi(b)|}{1 + \lambda^2\phi^2(b)} + \frac{1 + \lambda|\phi(a)|}{1 + \lambda^2\phi^2(a)} \right] \leq K_{\phi,\psi} \frac{M_1}{1 + \lambda},
\end{aligned}
$$

where the constant M_1 depends on ϕ, but not on λ.

Case (ii). If $\phi'(x) \neq 0$ for all $x \in [a,b]$ and $\psi(a) = \psi(b) = 0$, then using the estimate (2.3) we have

$$
|I_{\alpha,\alpha}(\lambda)| \leq \frac{\alpha C}{\lambda} \left\| \left(\frac{\psi}{\phi'} \right)' \right\|_{L^\infty} \int_a^b \frac{1 + \lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)} dx.
$$

Then, repeating the procedure of the proof of Theorem 2.2, we obtain (2.7).   ∎

2.2. The case $E_{1,\beta}$

In this section, we are interested in a particular case of the integral operator (2.1), when $\alpha = 1$, $\beta > 1$, that is, the integral operator $I_{1,\beta}(\lambda)$.

Theorem 2.5. *Let $-\infty \leq a < b \leq +\infty$. Let $\phi \in L^\infty[a,b]$ be a real-valued function, and let $\psi \in L^1[a,b]$. If $\beta > 1$ and $m = \inf_{x \in [a,b]} |\phi(x)| > 0$, then we have the following estimate:*

$$
|I_{1,\beta}(\lambda)| \leq \frac{M \|\psi\|_{L^1}}{(1 + m\lambda)^{\beta - 1}}, \quad \lambda \geq 0,
$$

where M does not depend on λ.

Proof. For small λ, we have bounded estimate for the integral $I_{1,\beta}(\lambda)$. Let $\lambda \geq 1$, $\phi \in L^\infty[a,b]$, and $\psi \in L^1[a,b]$. Then,

$$
|I_{1,\beta}(\lambda)| \leq \int_a^b |E_{1,\beta}(i\lambda\phi(x))| |\psi(x)| dx.
$$

Using formula (1.11) and estimate (1.7), we have that

$$
|E_{1,\beta}(i\lambda\phi(x))| \leq |E_{2,\beta}(-\lambda^2\phi^2(x))| + \lambda|\phi(x)| |E_{2,1+\beta}(-\lambda^2\phi^2(x))|.
$$

The following asymptotic estimate (see [11, p. 43])

$$
\begin{aligned}
E_{2,\beta}(-y) = {}& y^{(1-\beta)/2} \cos\left(\sqrt{y} + \frac{\pi(1-\beta)}{2} \right) \\
& - \sum_{k=1}^N \frac{(-1)^k y^{-k}}{\Gamma(\beta - 2k)} + O\left(\frac{1}{y^{N+1}} \right), \quad y \to \infty,
\end{aligned}
$$

implies that

$$
\begin{aligned}
|I_{1,\beta}(\lambda)| &\leq \int_a^b |E_{1,\beta}(i\lambda\phi(x))||\psi(x)|dx \\
&\leq M\lambda^{(1-\beta)} \int_a^b |\phi(x)|^{(1-\beta)} \left|\cos\left(\lambda\phi(x) + \frac{\pi(1-\beta)}{2}\right)\right| |\psi(x)|dx \\
&\quad + M\lambda^{(1-\beta)} \int_a^b |\phi(x)|^{(1-\beta)} \left|\cos\left(\lambda\phi(x) - \frac{\pi\beta}{2}\right)\right| |\psi(x)|dx \\
&\leq Mm^{1-\beta}\lambda^{(1-\beta)} \int_a^b |\psi(x)|dx \\
&\leq \frac{M\|\psi\|_{L^1}}{(1+m\lambda)^{\beta-1}}.
\end{aligned}
$$

Here, M is a constant that does not depend on λ. ∎

3. The generalisations of the van der Corput second lemma

In this section, we will obtain some generalisations of the van der Corput second lemma for the integral operator (2.1), that is, for

$$
I_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx, \quad -\infty < a < b < +\infty,
$$

where $0 < \alpha \leq 1$, $\beta > 0$.

Theorem 3.1. *Let* $-\infty < a < b < +\infty$. *Let* $0 < \alpha < 1$, $\beta > 0$, *and* ϕ *be a real-valued function such that* $\phi \in C^k[a,b]$. *If* ϕ *has finitely many zeros on* $[a,b]$ *and* $|\phi^{(k)}(x)| \geq 1$, $k \geq 2$ *for all* $x \in [a,b]$, *then we have*

$$
\left|\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))dx\right| \leq M_k \frac{\log^{\frac{1}{k}}(2+\lambda)}{(1+\lambda)^{1/k}}, \quad \lambda \geq 0. \tag{3.1}
$$

Here, M_k *does not depend on* λ.

Proof. For small λ, we have bounded estimate for the integral $I_{\alpha,\beta}(\lambda)$. Let $\lambda \geq 1$ and $k = 2$. Let $c \in [a,b]$ be a point, where $|\phi'(c)| \leq |\phi'(x)|$ for all $x \in [a,b]$. As $\phi''(x)$ is non-vanishing, it cannot be the case that c is the interior local minimum/maximum of $\phi'(x)$. Therefore, either $\phi'(c) = 0$ or c is one of the endpoints a, b. We can assume that $\phi'' \geq 1$.

Let $\phi'(c) = 0$. If $x \in [c + \varepsilon, b]$, then we have

$$
\phi'(x) = \phi'(x) - \phi'(c) = \int_c^x \phi''(s)ds \geq x - c \geq \varepsilon.
$$

We have a similar estimate for $x \in [a, c - \varepsilon]$. Now, we write

$$\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))dx = \left(\int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} + \int_{c+\varepsilon}^b \right) E_{\alpha,\beta}(i\lambda\phi(x))dx.$$

Applying the results of Theorem 2.2 with $m = \varepsilon$ and estimate $\frac{1}{\varepsilon\lambda} \geq \frac{1}{1+\varepsilon\lambda}$ for $\lambda \geq 1$, we have that

$$\left| \int_a^{c-\varepsilon} E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq M(\varepsilon\lambda)^{-1}\log(2+\lambda),$$

and

$$\left| \int_{c+\varepsilon}^b E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq M(\varepsilon\lambda)^{-1}\log(2+\lambda).$$

As

$$\left| \int_{c-\varepsilon}^{c+\varepsilon} E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq 2\varepsilon,$$

we have

$$\left| \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq 2M(\varepsilon\lambda)^{-1}\log(2+\lambda) + 2\varepsilon.$$

Taking $\varepsilon = \frac{\sqrt{\log(2+\lambda)}}{\sqrt{\lambda}}$, we obtain the estimate

$$\left| \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq 2M\frac{\sqrt{\log(2+\lambda)}}{\sqrt{\lambda}} + 2\frac{\sqrt{\log(2+\lambda)}}{\sqrt{\lambda}} \leq M_1\frac{\sqrt{\log(2+\lambda)}}{\sqrt{\lambda}}.$$

This gives inequality (3.1) for $k = 2$. The cases when $c = a$ or $c = b$ can be proved similarly.

Let $k \geq 3$ and $\lambda \geq 1$. Let us prove the estimate (3.1) by induction method on k. We assume that (3.1) is true for $k \geq 3$. And assuming $\phi^{(k+1)}(x) \geq 1$, for all $x \in [a, b]$, we prove the estimate (3.1) for $k + 1$. Let $c \in [a, b]$ be a unique point, where $|\phi^{(k)}(c)| \leq |\phi^{(k)}(x)|$ for all $x \in [a, b]$. If $\phi^{(k)}(c) = 0$, then we obtain $\phi^{(k)}(x) \geq \varepsilon$ on the interval $[a, b]$ outside $(c - \varepsilon, c + \varepsilon)$. Further, we will write $\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))dx$ as

$$\int_a^b E_{\alpha,\beta}(i\lambda\phi(x))dx = \left(\int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} + \int_{c+\varepsilon}^b \right) E_{\alpha,\beta}(i\lambda\phi(x))dx.$$

By inductive hypothesis,

$$\left| \int_a^{c-\varepsilon} E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq M_k\frac{\log^{1/k}(2+\lambda)}{(1+\varepsilon\lambda)^{1/k}}$$

$$\leq M_k\frac{\log^{1/k}(2+\lambda)}{(\varepsilon\lambda)^{1/k}},$$

and

$$\left| \int_{c+\varepsilon}^{b} E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq M_k \frac{\log^{1/k}(2+\lambda)}{(1+\varepsilon\lambda)^{1/k}}$$
$$\leq M_k \frac{\log^{1/k}(2+\lambda)}{(\varepsilon\lambda)^{1/k}}.$$

As

$$\left| \int_{c-\varepsilon}^{c+\varepsilon} E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq 2\varepsilon,$$

we have

$$\left| \int_{a}^{b} E_{\alpha,\beta}(i\lambda\phi(x))dx \right| \leq 2M_k \frac{\log^{1/k}(2+\lambda)}{(\varepsilon\lambda)^{1/k}} + 2\varepsilon.$$

Taking $\varepsilon = \lambda^{-\frac{1}{k+1}} \log^{\frac{1}{k+1}}(2+\lambda)$, we obtain the estimate (3.1) for $k+1$, which proves the result. The cases when $c = a$ or $c = b$ can be proved similarly. ∎

Corollary 3.2. *Let $-\infty < a < b < +\infty$. Let $0 < \alpha < 1$, $\beta > 0$, and ϕ be a real-valued function such that $\phi \in C^k[a,b]$, and let $\psi' \in L^1[a,b]$. If ϕ has finitely many zeros on $[a,b]$ and $|\phi^{(k)}(x)| \geq 1$, $k \geq 2$ for all $x \in [a,b]$, then the estimate*

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \left[|\psi(b)| + \int_{a}^{b} |\psi'(x)|dx \right] \frac{\log^{\frac{1}{k}}(2+\lambda)}{(1+\lambda)^{1/k}}, \quad \lambda \geq 0,$$

holds with some $M_k > 0$ independent of λ.

Proof. We write (2.1) as

$$\int_{a}^{b} E'(x)\psi(x)dx,$$

where

$$E(x) = \int_{a}^{x} E_{\alpha,\beta}(i\lambda\phi(s))ds.$$

Integrating by parts and using the results of Theorem 3.1, we obtain

$$|I_{\alpha,\beta}(\lambda)| \leq M_k \frac{\log^{\frac{1}{k}}(2+\lambda)}{(1+\lambda)^{1/k}} \left[|\psi(b)| + \int_{a}^{b} |\psi'(x)|dx \right],$$

completing the proof. ∎

3.1. The case $E_{\alpha,\alpha}$

In this part, we consider the particular case of integral operator (2.1), when $0 < \alpha < 1$, $\beta = \alpha$. In this case, we can obtain an improvement of the decay order in Theorem 3.1.

Theorem 3.3. *Let* $-\infty < a < b < +\infty$ *and* $0 < \alpha < 1$. *Suppose that* ϕ *is a real-valued function. If* $\phi \in C^k[a, b]$ *and* $|\phi^{(k)}(x)| \geq 1$, $k \geq 2$, *for all* $x \in [a, b]$, *then we have the following estimate:*

$$\left| \int_a^b E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq M_k(1 + \lambda)^{-\frac{1}{k}}, \quad \lambda \geq 0, \tag{3.2}$$

where M_k *does not depend on* λ.

Proof. For small λ, the integral $I_{\alpha,\alpha}(\lambda)$ is bounded. Let $\lambda \geq 1$. First, we prove the case $k = 2$. Let $c \in [a, b]$ be a point, where $|\phi'(c)| \leq |\phi'(x)|$ for all $x \in [a, b]$. As $\phi''(x)$ is non-vanishing, it cannot be the case that c is the interior local minimum/maximum of $\phi'(x)$. Therefore, either $\phi'(c) = 0$ or c is one of the endpoints a, b. Let $\phi'(c) = 0$. If $x \in [c + \varepsilon, b]$, then we have $|\phi'(x)| \geq \varepsilon$. We have a similar estimate for $x \in [a, c - \varepsilon]$.

Now, we write

$$\int_a^b E_{\alpha,\alpha}(i\lambda\phi(x))dx = \left(\int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} + \int_{c+\varepsilon}^b \right) E_{\alpha,\alpha}(i\lambda\phi(x))dx.$$

As ϕ is non-vanishing and ϕ' is monotonic (since $|\phi''(x)| \geq 1$) on $[a, c - \varepsilon]$ and $[c + \varepsilon, b]$, then using (2.5) in Theorem 2.3 we have

$$\left| \int_a^{c-\varepsilon} E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq \frac{M_1}{1 + \varepsilon\lambda} \leq \frac{M_1}{\varepsilon\lambda},$$

and

$$\left| \int_{c+\varepsilon}^b E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq \frac{M_1}{1 + \varepsilon\lambda} \leq \frac{M_1}{\varepsilon\lambda}.$$

As

$$\left| \int_{c-\varepsilon}^{c+\varepsilon} E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq 2\varepsilon,$$

we have

$$\left| \int_a^b E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq 2M_1(\varepsilon\lambda)^{-1} + 2\varepsilon.$$

Taking $\varepsilon = \frac{1}{\sqrt{\lambda}}$, we obtain estimate (3.2) for $k = 2$.

Let us prove the estimate (3.2) by induction method on $k \geq 2$. We assume that (3.2) is true for k. And assuming $\phi^{(k+1)}(x) \geq 1$, for all $x \in [a, b]$, we prove the validity of estimate (3.2) for $k + 1$. Let $c \in [a, b]$ be a unique point, where

$$|\phi^{(k)}(c)| \leq |\phi^{(k)}(x)| \quad \text{for all } x \in [a, b].$$

If $\phi^{(k)}(c) = 0$, then we obtain $\phi^{(k)}(x) \geq \varepsilon$ on the outside $(c - \varepsilon, c + \varepsilon)$. Further, we will write $\int_a^b E_{\alpha,\alpha}(i\lambda\phi(x))dx$ as

$$\int_a^b E_{\alpha,\alpha}(i\lambda\phi(x))dx = \left(\int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} + \int_{c+\varepsilon}^b \right) E_{\alpha,\alpha}(i\lambda\phi(x))dx.$$

By inductive hypothesis, we have

$$\left| \int_a^{c-\varepsilon} E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq M_k(1+\varepsilon\lambda)^{-\frac{1}{k}} \leq M_k(\varepsilon\lambda)^{-\frac{1}{k}},$$

and

$$\left| \int_{c+\varepsilon}^b E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq M_k(1+\varepsilon\lambda)^{-\frac{1}{k}} \leq M_k(\varepsilon\lambda)^{-\frac{1}{k}}.$$

As

$$\left| \int_{c-\varepsilon}^{c+\varepsilon} E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq 2\varepsilon,$$

we have

$$\left| \int_a^b E_{\alpha,\alpha}(i\lambda\phi(x))dx \right| \leq 2M_k(\varepsilon\lambda)^{-\frac{1}{k}} + 2\varepsilon.$$

Taking $\varepsilon = \lambda^{-\frac{1}{k+1}}$, we obtain the estimate (3.2) for $k+1$. The proof is complete. ∎

Corollary 3.4. *Let $-\infty < a < b < +\infty$ and $0 < \alpha < 1$. Let $\phi \in C^k[a,b]$ be a real-valued function, and let $\psi' \in L^1[a,b]$. If $|\phi^{(k)}(x)| \geq 1$, $k \geq 2$ for all $x \in [a,b]$, then we have the following estimate:*

$$|I_{\alpha,\alpha}(\lambda)| \leq M_k\left[|\psi(b)| + \int_a^b |\psi'(x)|dx \right](1+\lambda)^{-\frac{1}{k}}, \quad \lambda \geq 0,$$

where M_k does not depend on λ.

Corollary 3.4 is proved similarly as Corollary 3.2.

4. Optimal bounds of the van der Corput-type estimates

In this section, we find optimal bounds in estimates for (2.1), when $0 < \alpha \leq \frac{1}{2}$ and $\beta \geq 2\alpha$. We assume $-\infty < a < b < +\infty$. By optimality we mean estimates from below showing the sharpness of decay orders in general.

Theorem 4.1. *Let $-\infty < a < b < +\infty$ and $0 < \alpha \leq 1/2$, $\beta > 2\alpha$. Let $\phi \in L^\infty[a,b]$ be a real-valued function, and let $\psi \in L^\infty[a,b]$.*

Suppose that $m_1 = \inf_{a \leq x \leq b} |\phi(x)| > 0$ and $m_2 = \inf_{a \leq x \leq b} |\psi(x)| > 0$, then we have

$$|I_{\alpha,\beta}(\lambda)| \leq \frac{K_1\|\psi\|_{L^\infty}}{(b-a)} \frac{1+\lambda\|\phi\|_{L^\infty}}{1+k_1\lambda^2 m_1^2}, \quad \lambda \geq 0, \tag{4.1}$$

where $K_1 = \max\{\frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha+\beta)}\}$ and $k_1 = \min\{\frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}, \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\}$, and

$$|I_{\alpha,\beta}(\lambda)| \geq \frac{m_2}{(b-a)\Gamma(\alpha+\beta)} \frac{\lambda m_1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta)}\lambda^2\|\phi\|_{L^\infty}^2}, \quad \lambda \geq 0. \tag{4.2}$$

If $m_1 = \inf_{a \leq x \leq b} |\phi(x)| = 0$ *and* $m_2 = \inf_{a \leq x \leq b} |\psi(x)| > 0$. *Then, we have*

$$|I_{\alpha,\beta}(\lambda)| \geq \frac{m_2}{(b-a)\Gamma(\beta)} \frac{1}{1 + \frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta)}\lambda^2 \|\phi\|_{L^\infty}^2}, \qquad \lambda \geq 0. \tag{4.3}$$

Proof. First, we prove estimate (4.1). Let $\phi \in L^\infty[a,b]$ and $\psi \in L^\infty[a,b]$. Then,

$$|I_{\alpha,\beta}(\lambda)| \leq \int_a^b |E_{\alpha,\beta}(i\lambda\phi(x))||\psi(x)|dx.$$

Using formula (1.11) and estimate (1.7), we have that

$$|E_{\alpha,\beta}(i\lambda\phi(x))| \leq E_{2\alpha,\beta}(-\lambda^2\phi^2(x)) + \lambda|\phi(x)|E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x)).$$

The properties of functions ϕ and ψ and the use of estimate (1.10) lead to the result

$$|I_{\alpha,\beta}(\lambda)| \leq \int_a^b |E_{\alpha,\beta}(i\lambda\phi(x))||\psi(x)|dx$$

$$\leq \int_a^b \left(E_{2\alpha,\beta}(-\lambda^2\phi^2(x)) + \lambda|\phi(x)|E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x))\right)|\psi(x)|dx$$

$$\leq \|\psi\|_{L^\infty}\left[\frac{1}{\Gamma(\beta)}\int_a^b \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\lambda^2\phi^2(x)}dx\right.$$

$$\left. + \frac{1}{\Gamma(\alpha+\beta)}\int_a^b \frac{\lambda|\phi(x)|}{1 + \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\lambda^2\phi^2(x)}dx\right]$$

$$\leq \|\psi\|_{L^\infty} \max\left\{\frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha+\beta)}\right\}$$

$$\times \left[\int_a^b \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\lambda^2\phi^2(x)}dx + \int_a^b \frac{\lambda|\phi(x)|}{1 + \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\lambda^2\phi^2(x)}dx\right]$$

$$\leq \frac{\|\psi\|_{L^\infty}}{(b-a)} \max\left\{\frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha+\beta)}\right\}$$

$$\times \left[\frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\lambda^2 m_1^2} + \frac{\lambda\|\phi\|_{L^\infty}}{1 + \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\lambda^2 m_1^2}\right]$$

$$\leq \frac{\|\psi\|_{L^\infty}}{(b-a)} \max\left\{\frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha+\beta)}\right\} \frac{1 + \lambda\|\phi\|_{L^\infty}}{1 + \min\{\frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}, \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\}\lambda^2 m_1^2}.$$

Now, we prove the lower bound estimate (4.3). As $|x + iy| \geq |x|$, we obtain

$$|E_{\alpha,\beta}(i\lambda\phi(x))| \geq E_{2\alpha,\beta}(-\lambda^2\phi^2(x)).$$

Hence, by estimate (1.10), it follows that

$$|I_{\alpha,\beta}(\lambda)| \geq m_2 \int_a^b E_{2\alpha,\beta}(-\lambda^2\phi^2(x))dx$$

$$\geq \frac{m_2}{\Gamma(\beta)} \int_a^b \frac{1}{1 + \frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta)}\lambda^2\phi^2(x)}dx$$

$$\geq \frac{m_2}{(b-a)\Gamma(\beta)} \frac{1}{1 + \frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta)}\lambda^2\|\phi\|_{L^\infty(a,b)}^2}.$$

Now, we prove the lower bound estimate (4.2). As $|x + iy| \geq |y|$, we obtain

$$|E_{\alpha,\beta}(i\lambda\phi(x))| \geq \lambda|\phi(x)|E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x)).$$

Hence, by estimate (1.10), it follows that

$$|I_{\alpha,\beta}(\lambda)| \geq m_2\lambda \int_a^b |\phi(x)|E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x))dx$$

$$\geq \frac{m_2\lambda}{\Gamma(\alpha+\beta)} \int_a^b \frac{|\phi(x)|}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta)}\lambda^2\phi^2(x)}dx$$

$$\geq \frac{m_2}{(b-a)\Gamma(\alpha+\beta)} \frac{\lambda m_1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta)}\lambda^2\|\phi\|_{L^\infty(a,b)}^2}.$$

The proof is complete. ∎

Theorem 4.2. *Let* $-\infty < a < b < +\infty$ *and* $0 < \alpha < 1/2$, $\beta = 2\alpha$. *Let* $\phi \in L^\infty[a,b]$ *be a real-valued function, and let* $\psi \in L^\infty[a,b]$.

Let $m_1 = \inf_{a \leq x \leq b} |\phi(x)| > 0$ *and* $m_2 = \inf_{a \leq x \leq b} |\psi(x)| > 0$; *then we have the following estimates:*

$$|I_{\alpha,2\alpha}(\lambda)| \leq \frac{K\|\psi\|_{L^\infty}}{(b-a)} \frac{1 + \lambda\|\phi\|_{L^\infty}\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2\|\phi\|_{L^\infty}^2\right)}{\left(1 + \min\{\frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}, \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\}\lambda^2 m_1^2\right)^2}, \quad \lambda \geq 0, \quad (4.4)$$

and

$$|I_{\alpha,2\alpha}(\lambda)| \geq \frac{m_2}{(b-a)\Gamma(3\alpha)} \frac{\lambda m_1}{1 + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)}\lambda^2\|\phi\|_{L^\infty}^2}, \quad \lambda \geq 0, \quad (4.5)$$

where $K = \max\{\frac{1}{\Gamma(2\alpha)}, \frac{1}{\Gamma(3\alpha)}\}$.

If $m_1 = \inf_{a \leq x \leq b} |\phi(x)| = 0$ *and* $m_2 = \inf_{a \leq x \leq b} |\psi(x)| > 0$, *then we have*

$$|I_{\alpha,2\alpha}(\lambda)| \geq \frac{m_2}{(b-a)\Gamma(2\alpha)} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)}}\lambda^2\|\phi\|_{L^\infty}^2\right)^2}, \quad \lambda \geq 0. \quad (4.6)$$

Proof. First, we prove the estimate (4.4). Let $\phi \in L^\infty[a, b]$ and $\psi \in L^\infty[a, b]$. Then,

$$|I_{\alpha,2\alpha}(\lambda)| \leq \int_a^b |E_{\alpha,2\alpha}(i\lambda\phi(x))||\psi(x)|dx.$$

Using formula (1.11), we have that

$$|E_{\alpha,2\alpha}(i\lambda\phi(x))| \leq E_{2\alpha,2\alpha}(-\lambda^2\phi^2(x)) + \lambda|\phi(x)|E_{2\alpha,3\alpha}(-\lambda^2\phi^2(x)).$$

The properties of functions ϕ and ψ, using estimates (1.9) and (1.10), imply that

$$\begin{aligned}
|I_{\alpha,2\alpha}(\lambda)| &\leq \int_a^b |E_{\alpha,2\alpha}(i\lambda\phi(x))||\psi(x)|dx \\
&\leq \int_a^b \left(E_{2\alpha,2\alpha}(-\lambda^2\phi^2(x)) + \lambda|\phi(x)|E_{2\alpha,3\alpha}(-\lambda^2\phi^2(x))\right)|\psi(x)|dx \\
&\leq \|\psi\|_{L^\infty}\left[\frac{1}{\Gamma(2\alpha)}\int_a^b \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2\phi^2(x)\right)^2}dx \right.\\
&\qquad\left. + \frac{1}{\Gamma(3\alpha)}\int_a^b \frac{\lambda|\phi(x)|}{1 + \frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}\lambda^2\phi^2(x)}dx\right] \\
&\leq \|\psi\|_{L^\infty}K\left[\int_a^b \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2\phi^2(x)\right)^2}dx \right.\\
&\qquad\left. + \int_a^b \frac{\lambda|\phi(x)|}{1 + \frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}\lambda^2\phi^2(x)}dx\right] \\
&\leq \frac{K\|\psi\|_{L^\infty}}{(b-a)}\left[\frac{1}{\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2 m_1^2\right)^2} \right.\\
&\qquad\left. + \frac{\lambda\|\phi\|_{L^\infty}\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2\|\phi\|_{L^\infty}^2\right)}{\left(1 + \frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}\lambda^2 m_1^2\right)\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2 m_1^2\right)}\right] \\
&\leq \frac{K\|\psi\|_{L^\infty}}{(b-a)}\frac{1 + \lambda\|\phi\|_{L^\infty}\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\lambda^2\|\phi\|_{L^\infty}^2\right)}{\left(1 + \min\{\frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}, \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\}\lambda^2 m_1^2\right)^2},
\end{aligned}$$

where $K = \max\{\frac{1}{\Gamma(2\alpha)}, \frac{1}{\Gamma(3\alpha)}\}$.

Now, we prove estimate (4.6). We have

$$|E_{\alpha,2\alpha}(i\lambda\phi(x))| \geq E_{2\alpha,2\alpha}(-\lambda^2\phi^2(x)).$$

Hence, by estimate (1.9), it follows that

$$|I_{\alpha,2\alpha}(\lambda)| \geq m_2 \int_a^b E_{2\alpha,2\alpha}(-\lambda^2 \phi^2(x)) dx$$

$$\geq \frac{m_2}{\Gamma(2\alpha)} \int_a^b \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)}} \lambda^2 \phi^2(x)\right)^2} dx$$

$$\geq \frac{m_2}{(b-a)\Gamma(2\alpha)} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)}} \lambda^2 \|\phi\|_{L^\infty}^2\right)^2}.$$

The estimate (4.5) can be proved similarly as estimate (4.2) by replacing $\beta = 2\alpha$. In fact,

$$|I_{\alpha,2\alpha}(\lambda)| \geq m_2 \lambda \int_a^b |\phi(x)| E_{2\alpha,3\alpha}(-\lambda^2 \phi^2(x)) dx$$

$$\geq \frac{m_2 \lambda}{\Gamma(3\alpha)} \int_a^b \frac{|\phi(x)|}{1 + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} \lambda^2 \phi^2(x)} dx$$

$$\geq \frac{m_2}{(b-a)\Gamma(3\alpha)} \frac{\lambda m_1}{1 + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} \lambda^2 \|\phi\|_{L^\infty(a,b)}^2}.$$

The proof is complete. ∎

5. Some applications

In this section, we give some applications of the obtained results.

5.1. Generalised Riemann–Lebesgue lemma

The Riemann–Lebesgue lemma is one of the most important results of harmonic and asymptotic analysis. The simplest form of the Riemann–Lebesgue lemma (see, e.g., [1, p. 3]) states that for a function $f \in L^1(a,b)$ we have

$$\lim_{k \to \infty} \int_a^b e^{ikx} f(x) dx = 0.$$

If f is continuously differentiable on $[a,b]$, then it follows that

$$\int_a^b e^{ikx} f(x) dx = \mathcal{O}\left(\frac{1}{k}\right), \quad \text{as } k \to \infty.$$

Next, we consider a generalisation of the Fourier integral in the form

$$\int_a^b E_{\alpha,\beta}(ikx) f(x) dx,$$

where $0 < \alpha \leq 1$, $\beta > 0$ and $-\infty < a < b < +\infty$.

Let $0 < \alpha < 1$, $\beta > 0$ or $\alpha = 1$, $\beta > 1$. If $f \in L^1(a, b)$, then from the results of Theorems 2.1 and 2.5 it follows that

$$\lim_{k \to \infty} \int_a^b E_{\alpha,\beta}(ikx) f(x) dx = 0.$$

If $f \in C^1(a, b)$, then from the results of Theorems 2.1, 2.3, and 2.5 we obtain the following.

- For $0 < \alpha < 1$, $\beta > 0$, $-\infty < a < b < +\infty$, we have

$$\int_a^b E_{\alpha,\beta}(ikx) f(x) dx = \mathcal{O}\left(\frac{1}{k}\right).$$

- For $0 < \alpha < 1$, $\beta = \alpha$, $0 < a < b < +\infty$, we have

$$\int_a^b E_{\alpha,\alpha}(ikx) f(x) dx = \mathcal{O}\left(\frac{1}{k^2}\right).$$

- For $\alpha = 1$, $\beta > 1$, $0 < a < b < +\infty$, we have

$$\int_a^b E_{1,\beta}(ikx) f(x) dx = \mathcal{O}\left(\frac{1}{k^{\beta-1}}\right).$$

5.2. Decay estimates for the time-fractional PDE

Consider the time-fractional evolution equation

$$\mathcal{D}_{0+,t}^{\alpha} \mathcal{L}_x u(t, x) + u_x(t, x) = 0, \quad t > 0, \ x \in \mathbb{R}, \tag{5.1}$$

with Cauchy data

$$u(0, x) = \psi(x), \quad x \in \mathbb{R}, \tag{5.2}$$

where $\mathcal{L}f = \mathcal{F}_{\xi}^{-1}(\sqrt{1 + \xi^2} \mathcal{F}(f))$ and

$$\mathcal{D}_{0+,t}^{\alpha} u(t, x) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} u_s(s, x) ds$$

is the Caputo fractional derivative of order $0 < \alpha < 1$.

By using the direct and inverse Fourier and Laplace transforms, we can obtain a solution to problem (5.1)–(5.2) in the form

$$u(t, x) = \int_{\mathbb{R}} e^{ix\xi} E_{\alpha,1}\left(i \frac{\xi}{\sqrt{1 + \xi^2}} t^{\alpha}\right) \hat{\psi}(\xi) d\xi,$$

where

$$\hat{\psi}(\xi) = \frac{1}{\pi} \int_{\mathbb{R}} e^{-iy\xi} \psi(y) dy.$$

Suppose that $\psi \in L^1(\mathbb{R})$; then classical Riemann–Lebesgue lemma implies that $\hat{\psi} \in C_0(\mathbb{R})$. It is obvious that

$$\frac{\xi}{\sqrt{1+\xi^2}} \quad \text{bounded for all } \xi \in \text{supp}(\hat{\psi})$$

and

$$\left(\frac{\xi}{\sqrt{1+\xi^2}}\right)' > 0, \quad \text{for all } \xi \in \text{supp}(\hat{\psi}).$$

Then, using Theorem 2.2, we obtain the dispersive estimate

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \lesssim \frac{\log(2+t)}{(1+t)^\alpha} \|\hat{\psi}\|_{L^1(\mathbb{R})}, \quad t \geq 0.$$

6. Comparison of results for integrals (1.4) and (1.5)

For the convenience of readers, this section provides comparisons of results for integrals (1.4):

$$I_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx$$

and (1.5):

$$\tilde{I}_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}(i^\alpha\lambda\phi(x))\psi(x)dx.$$

Let us start with the results in $\mathbb{R}$ (i.e., when $a = -\infty$, $b = +\infty$). For both integrals $I_{\alpha,\beta}(\lambda)$ and $\tilde{I}_{\alpha,\beta}(\lambda)$, the following estimate holds:

$$\left.\begin{array}{r} |I_{\alpha,\beta}(\lambda)| \\ |\tilde{I}_{\alpha,\beta}(\lambda)| \end{array}\right\} \lesssim \frac{1}{1+\lambda}, \quad \lambda \geq 0,\ \phi \not\equiv 0.$$

However, if for $I_{\alpha,\beta}(\lambda)$ above the estimate is true in case $0 < \alpha < 1$, $\beta > 0$, then for $\tilde{I}_{\alpha,\beta}(\lambda)$ it is true in case $0 < \alpha < 2$, $\beta \geq \alpha + 1$, and in case $0 < \alpha < 2$, $1 < \beta < \alpha + 1$ integral $\tilde{I}_{\alpha,\beta}(\lambda)$ has the estimate

$$|\tilde{I}_{\alpha,\beta}(\lambda)| \lesssim \frac{1}{(1+\lambda)^{\frac{\beta-1}{\alpha}}}.$$

In addition, if $\psi \in C_0(\mathbb{R})$, ϕ is the real-valued differentiable function vanishing at some point $c \in (\mathbb{R})$ and $\inf_{x \in \text{supp}(\psi)} |\phi'(x)| > 0$, then

$$|I_{\alpha,\beta}(\lambda)| \lesssim \frac{\log(2+\lambda)}{1+\lambda}, \quad \lambda \geq 0.$$

The above estimate was not obtained for $\widetilde{I}_{\alpha,\beta}(\lambda)$ in [22]. In addition, this estimate is also carried out for semi-axes $[a, +\infty)$ and $(-\infty, b]$.

Now, let us move on to the results on a finite interval $[a, b]$, $-\infty < a < b < +\infty$. For both integrals $I_{\alpha,\beta}(\lambda)$ and $\widetilde{I}_{\alpha,\beta}(\lambda)$, the following estimate holds:

$$\lambda^{-\frac{1}{k}} \log^{\frac{1}{k}}(2 + \lambda) \gtrsim \begin{cases} |I_{\alpha,\beta}(\lambda)|, & 0 < \alpha < 1, \ \beta > 0, \\ |\widetilde{I}_{\alpha,\beta}(\lambda)|, & 0 < \alpha < 2, \ \beta \geq \alpha + 1. \end{cases}$$

In the case $\alpha = \beta$, the integrals $I_{\alpha,\beta}(\lambda)$ and $\widetilde{I}_{\alpha,\beta}(\lambda)$ have the same decay rate:

$$\lambda^{-\frac{1}{k}} \gtrsim \begin{cases} |I_{\alpha,\alpha}(\lambda)|, & 0 < \alpha < 1, \\ |\widetilde{I}_{\alpha,\alpha}(\lambda)|, & 0 < \alpha < 2, \end{cases}$$

which corresponds to the classical van der Corput's lemma.

Due to the presence of an Euler-type formula (1.11) and thanks to estimates (1.8), (1.9), (1.10) for $I_{\alpha,\beta}(\lambda)$, $0 < \alpha \leq 1/2$, $\beta \geq 2\alpha$, the following two-sided estimates are valid:

$$c_1 \frac{\lambda}{1 + \lambda^2} \leq |I_{\alpha,\beta}(\lambda)| \leq C_1 \frac{\lambda}{1 + \lambda^2},$$

where C_1 and c_1 are positive constants.

However, the above two-sided estimates do not hold for $\widetilde{I}_{\alpha,\beta}(\lambda)$ and have not been studied in [22].

Conclusion and some open questions

The main object of the paper was to estimate integral operator defined by

$$I_{\alpha,\beta}(\lambda) = \int_a^b E_{\alpha,\beta}(i\lambda\phi(x))\psi(x)dx,$$

with $0 < \alpha \leq 1$, $\beta > 0$.

For $I_{\alpha,\beta}(\lambda)$, the following results have been obtained:

- for different cases of parameters α and β, some analogues of the van der Corput first lemma are obtained;
- for different cases of parameters α and β, some analogues of the van der Corput second lemma are obtained;
- for $0 < \alpha \leq 1/2$ and $\beta \geq \alpha$, bilateral optimal bounds estimates are obtained;
- some applications of the generalised van der Corput lemmas are presented.

In conclusion, we present some open questions related to generalisations of the van der Corput lemmas:

(1) are van der Corput-type estimates valid for $I_{\alpha,\beta}(\lambda)$ with $\alpha = 1$ and $\beta < 1$?

(2) is it possible to obtain bilateral optimal estimates as in Section 4 for $I_{\alpha,\beta}(\lambda)$ with $0 < \alpha \leq 1/2$ and $\beta < 2\alpha$?

(3) how to prove bilateral optimal estimates as in Section 4 for $I_{\alpha,\beta}(\lambda)$ for some particular case $1/2 < \alpha < 1$ and $\beta > 0$?

References

[1] S. Bochner and K. Chandrasekharan, *Fourier Transforms*. Ann. of Math. Stud. 19, Princeton University Press, Princeton, NJ; Oxford University Press, London, 1949 Zbl 0065.34101 MR 0031582

[2] L. Boudabsa, T. Simon, and P. Vallois, Fractional extreme distributions. *Electron. J. Probab.* **25** (2020), article no. 115 Zbl 1472.60021 MR 4161125

[3] J. Bourgain and L. Guth, Bounds on oscillatory integral operators based on multilinear estimates. *Geom. Funct. Anal.* **21** (2011), no. 6, 1239–1295 Zbl 1237.42010 MR 2860188

[4] A. Carbery, M. Christ, and J. Wright, Multidimensional van der Corput and sublevel set estimates. *J. Amer. Math. Soc.* **12** (1999), no. 4, 981–1015 Zbl 0938.42008 MR 1683156

[5] M. Christ, X. Li, T. Tao, and C. Thiele, On multilinear oscillatory integrals, nonsingular and singular. *Duke Math. J.* **130** (2005), no. 2, 321–351 Zbl 1119.42005 MR 2181092

[6] M. A. Evgrafov, *Asymptotic estimates and entire functions*. revised edn., Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1962 MR 0154981

[7] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. Rogosin, *Mittag-Leffler functions, related topics and applications*. 2nd edn., Springer Monogr. Math., Springer, Berlin, 2020 Zbl 1451.33001 MR 4179587

[8] M. Greenblatt, Sharp L^2 estimates for one-dimensional oscillatory integral operators with C^∞ phase. *Amer. J. Math.* **127** (2005), no. 3, 659–695 Zbl 1082.42009 MR 2141648

[9] A. Greenleaf, M. Pramanik, and W. Tang, Oscillatory integral operators with homogeneous polynomial phases in several variables. *J. Funct. Anal.* **244** (2007), no. 2, 444–487 Zbl 1127.35090 MR 2297031

[10] I. Kamotski and M. Ruzhansky, Regularity properties, representation of solutions, and spectral asymptotics of systems with multiplicities. *Comm. Partial Differential Equations* **32** (2007), no. 1-3, 1–35 Zbl 1132.35055 MR 2304141

[11] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. North-Holland Math. Stud. 204, Elsevier Science B.V., Amsterdam, 2006 Zbl 1092.45003 MR 2218073

[12] Y. Luchko, Fractional wave equation and damped waves. *J. Math. Phys.* **54** (2013), no. 3, article no. 031505 Zbl 1302.35408 MR 3059427

[13] Y. Luchko and R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives. *Acta Math. Vietnam.* **24** (1999), no. 2, 207–233 Zbl 0931.44003 MR 1710779

[14] M. G. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$. *C. R. Acad. Sci. Paris* **137** (1903), 554–558 Zbl 34.0435.01

[15] I. R. Parissis, A sharp bound for the Stein–Wainger oscillatory integral. *Proc. Amer. Math. Soc.* **136** (2008), no. 3, 963–972 Zbl 1215.42010 MR 2361870

[16] D. H. Phong and E. M. Stein, Oscillatory integrals with polynomial phases. *Invent. Math.* **110** (1992), no. 1, 39–62 Zbl 0829.42014 MR 1181815

[17] D. H. Phong and E. M. Stein, Models of degenerate Fourier integral operators and Radon transforms. *Ann. of Math. (2)* **140** (1994), no. 3, 703–722 Zbl 0833.43004 MR 1307901

[18] D. H. Phong, E. M. Stein, and J. Sturm, Multilinear level set operators, oscillatory integral operators, and Newton polyhedra. *Math. Ann.* **319** (2001), no. 3, 573–596 Zbl 0993.47034 MR 1819885

[19] I. Podlubny, *Fractional differential equations*. Math. Sci. Eng. 198, Academic Press, San Diego, CA, 1999 Zbl 0924.34008 MR 1658022

[20] K. M. Rogers, Sharp van der Corput estimates and minimal divided differences. *Proc. Amer. Math. Soc.* **133** (2005), no. 12, 3543–3550 Zbl 1083.42002 MR 2163589

[21] M. Ruzhansky, Multidimensional decay in the van der Corput lemma. *Studia Math.* **208** (2012), no. 1, 1–10 Zbl 1248.42019 MR 2891181

[22] M. Ruzhansky and B. T. Torebek, Van der Corput lemmas for Mittag-Leffler functions. II. α-directions. *Bull. Sci. Math.* **171** (2021), article no. 103016 Zbl 1470.42023 MR 4281820

[23] R. K. Saxena, R. Saxena, and S. L. Kalla, Solution of space-time fractional Schrödinger equation occurring in quantum mechanics. *Fract. Calc. Appl. Anal.* **13** (2010), no. 2, 177–190 Zbl 1211.26010 MR 2723852

[24] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*. Princeton Math. Ser. 43, Princeton University Press, Princeton, NJ, 1993 Zbl 0821.42001 MR 1232192

[25] E. M. Stein and S. Wainger, The estimation of an integral arising in multiplier transformations. *Studia Math.* **35** (1970), 101–104 Zbl 0202.12401 MR 0265995

[26] X. Su, S. Zhao, and M. Li, Local well-posedness of semilinear space-time fractional Schrödinger equation. *J. Math. Anal. Appl.* **479** (2019), no. 1, 1244–1265 Zbl 1421.35343 MR 3987083

[27] X. Su, S. Zhao, and M. Li, Dispersive estimates for time and space fractional Schrödinger equations. *Math. Methods Appl. Sci.* **44** (2021), no. 10, 7933–7942 Zbl 1475.35398 MR 4271052

[28] J. G. van der Corput, Zahlentheoretische Abschätzungen. *Math. Ann.* **84** (1921), no. 1-2, 53–79 Zbl 48.0181.02 MR 1512020

[29] S. Wang and M. Xu, Generalized fractional Schrödinger equation with space-time fractional derivatives. *J. Math. Phys.* **48** (2007), no. 4, article no. 043502 Zbl 1137.81328 MR 2319896

[30] L. Xiao, Endpoint estimates for one-dimensional oscillatory integral operators. *Adv. Math.* **316** (2017), 255–291 Zbl 1377.42025 MR 3672907

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