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The phenomenon of revivals on complex potential Schrödinger's equation

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Abstract. The mysterious phenomenon of revivals in linear dispersive periodic equations was discovered first experimentally in optics in the 19th century, then rediscovered several times by theoretical and experimental investigations. While the term has been used systematically and consistently by many authors, there is no consensus on a rigorous definition. In this paper, we describe revivals modulo a regularity condition in a large class of Schrödinger's equations with complex bounded potentials. As we show, at rational times, the solution is given explicitly by finite linear combinations of translations and dilations of the initial datum, plus an additional continuous term.

1. Introduction

Recently, there have been significant developments in the study of revivals in dispersive evolution equations [18]. These phenomena, which are also called dispersive quantisations or Talbot effects, describe a surprising dichotomy in the pointwise behaviour of the solution of time-evolution equations at specific values of the time variable, the so-called rational times, compared to all other generic times. At rational times, the solution revives the shape of the initial datum by finite superpositions, reflections, and rescalings, with a prescribed simple combinatorial rule. See [9] and references therein.

The majority of past investigations about revivals involve equations and boundary conditions with the property that the eigenpairs of the spatial operator can be found explicitly and satisfy precise conditions of modularity and periodicity. The prime example of this is the case of linear dispersive equations with constant coefficients and periodic boundary conditions. In such cases, the techniques for detecting the times at which the revivals appear exploit the specific periodic matching of the eigenvalues, eigenfunctions, and boundary conditions. With the help of summations of Gauss type, the infinite series representation of the solution then reduces to a finite sum, characterising the revivals explicitly. Naturally, the direct applicability of this approach is limited.

The purpose of this paper is to take a different point of view by formulating the revivals phenomena in terms of perturbation theory. Concretely, we ask the question of whether a (large) class of equations exhibit them modulo a regular perturbation. An earlier version of this concept can be found in the works [6–8, 17], about which we give details below.

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We consider the class of linear Schrödinger equations

$$\begin{aligned} &\partial_t u(x,t) = -i(-\partial_x^2 + V(x))u(x,t), \quad x \in (0,\pi),\ t > 0, \\ &u(0,t) = u(\pi,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad x \in (0,\pi), \end{aligned} \tag{1.1}$$

with a complex-valued potential V, subject to Dirichlet boundary conditions, given an initial wavefunction $f \in L^2(0,\pi)$. Our goal is to detect revivals by perturbation from the case $V = 0$. As we will see below, in the large wavenumber asymptotic regime and for small enough V, the simple but non-trivial structure of (1.1) supports the combinatorial argument, involving the Gauss summations, that is valid for the free-space equation.

Our contribution is summarised in the next theorem. It shows that the solution of (1.1) at rational times support revivals modulo a continuous term. This result matches a similar earlier finding, reported in [2]. Indeed, for $V = 0$ and boundary conditions of the type $bu(0,t) = (1-b)\partial_x u(\pi,t)$, where $b \in (0,1)$ is a parameter, an analogous conclusion holds true.

Here, and everywhere below, f° denotes the odd, 2π-periodic extension of the function f and $\langle V\rangle = \frac{1}{\pi}\int_0^\pi V(x)\mathrm{d}x$ the mean of the potential function.

Theorem 1. *Let $V \in H^2(0,\pi)$ with $\|V\|_\infty < \frac{3}{2}$. Then, for $p, q \in \mathbb{N}$ co-prime numbers, the solution $u(x,t)$ to (1.1) at time $t = 2\pi\frac{p}{q}$ is given by*

$$u\Big(x, 2\pi\frac{p}{q}\Big) = w\Big(x, 2\pi\frac{p}{q}\Big) + \frac{1}{q}\, e^{-2\pi i\langle V\rangle\frac{p}{q}} \sum_{k,m=0}^{q-1} e^{2\pi i(m\frac{k}{q} - m^2\frac{p}{q})} f^{\circ}\Big(x - 2\pi\frac{k}{q}\Big),$$

where $w(\cdot,t) \in \mathrm{C}(0,\pi)$ for all fixed $t > 0$.

In case V is real-valued, the same conclusion holds with the weaker assumptions $\|V\|_\infty \in \mathrm{BV}(0,\pi)$ and $\|V\|_\infty < \infty$. In Section 4, we investigate whether the bound on V is necessary in the complex case.

We interpret the conclusion of Theorem 1 by saying that (1.1) supports a weak form of revival. Note that the result does not depend on the orthogonality of the family of eigenfunctions. Our choice of Dirichlet boundary conditions, corresponding to potential barriers at both ends of the finite segment, exhibits all the features of our methodology but is free from unnecessary complications. Indeed, while Neumann and other separated boundary conditions can be treated via a similar approach, in these cases multiplicities and the possibility of eigenvalues which are not semi-simple lead to additional technical distractions.

Phenomena similar to the one described here had already been observed. The investigations conducted in [6, 17] lead to versions of Theorem 1 for periodic V and periodic boundary conditions. The method of proof in these works is different from the one below. It relies on Duhamel's representation and an analysis of the solution in periodic Besov spaces. In a separate development, for the periodic cubic non-linear Schrödinger equation [8] and the Korteweg–de Vries equation [7], it was proved that at all times the difference between the solution and the linear time-evolution is more regular than the initial datum. This directly implies the appearance of weak revivals at rational times also for these two non-linear equations. Numerical evidence of this effect in the non-linear setting was reported in [4, 5]. Our findings complement all these investigations.

The proof of Theorem 1 that we present shows how to apply directly classical perturbation expansions in order to derive existence of revivals. This approach has two main implications worth mentioning. On the one hand, Theorem 1 confirms the conjecture that the revival effect is prevalent in a large class of quantum systems with discrete spectra, when the eigenpairs asymptotics support it [3, Section 6.2, page 116], irrespective of whether the underlying operator is self-adjoint. On the other hand, the present approach might provide a rigorous foundation for tackling the general conjecture formulated in [4, pages 12–13]. The latter states that a linear PDE with a dispersion relation that is asymptotic to a polynomial with integer coefficients, in the large wavenumbers regime, should support a type of revival. Further numerical evidence strengthening the validity of this conjecture can be found in [16] for the case $V = 0$ and various classes of boundary conditions.

The structure of the paper is as follows. In Section 2, we lay down the precise eigenpairs asymptotics, in terms of V, that allow the validity of Theorem 1. All the results that we present in that section are classical, but we include crucial details of their proofs. Section 3 is devoted to the proof of Theorem 1. In the final Section 4, we illustrate our main results by means of examples involving complex Mathieu potentials and discuss the optimality of the different assumptions on V.

2. Spectral asymptotics

Let $V \in H^2(0, \pi)$ be a complex-valued potential function. Denote the Hamiltonian associated to (1.1) by

$$L = -\partial_x^2 + V : H^2(0, \pi) \cap H_0^1(0, \pi) \to L^2(0, \pi).$$

Since

$$\|V\|_\infty = \max_{x \in [0,\pi]} |V(x)| < \infty,$$

then the operator L is closed in the domain above, it has a compact resolvent and its adjoint $L^* = -\partial_x^2 + \overline{V}$ has the same domain. In some of the statements below, we impose the extra condition $\|V\|_\infty < \frac{3}{2}$ of the theorem. Moreover, without loss of generality, we assume in what follows that $\langle V \rangle = 0$.

The boundary value problem (1.1) can be written concisely as follows:

$$\begin{aligned} u_t &= -iLu, \\ u(\cdot, 0) &= f \end{aligned} \tag{2.1}$$

for an initial value $f \in L^2(0,\pi)$. We know from the classical theory of perturbations of one-parameter semigroups that the operator iL is the generator of a C_0 one-parameter semigroup, so the equation has a unique solution in L^2 for any $f \in L^2$. In general, the spectrum of L is not real. However, it is asymptotically close to the real line and the eigenfunctions are asymptotically close to trigonometric functions. Our objective in this section is to determine the leading order of these asymptotics and the precise decay rate of the corrections for the appearance of weak revivals in (2.1). We then give the proof of Theorem 1 in the next section.

The next two lemmas are routine consequences of classical properties of non-self-adjoint Sturm–Liouville operators and their analytic perturbation theory, but we give full details of their validity as they are not standard. They imply that the operator L has an infinite sequence of eigenpairs (eigenfunctions and eigenvalues)

$$\{y_j, w_j^2\}_{j=1}^{\infty} \subset (H^2 \cap H_0^1) \times \mathbb{C}$$

with an asymptotic structure close enough to that of the case $V = 0$.

Lemma 1. *Let w_j^2 be the eigenvalues of L. Then, for $j \to \infty$,*

$$|w_j - j| = \frac{a_3}{j^3} + O(j^{-4}),$$

where $a_3 \in \mathbb{C}$ is a constant that only depends on V. Moreover, if $\|V\|_\infty < \frac{3}{2}$, then each eigenvalue w_j^2 is simple.

Proof. By virtue of the classical Marchenko asymptotic formula [14, Theorem 1.5.1], we know that

$$w_j = j + \frac{a_1}{2j} + \frac{\tilde{a}_3}{8j^3} + O(j^{-4}), \quad k \to \infty,$$

where $a_1 = \langle V \rangle = 0$. This gives the eigenvalue asymptotics.

The family of operators $\alpha \mapsto T_\alpha = -\partial_x^2 + \alpha V$ on the domain $H^2 \cap H_0^1$ is a holomorphic family of type (A) for $\alpha \in \mathbb{C}$; see [13, Example 2.17, page 385]. As the operator T_0 has a compact resolvent, then it follows that T_α have compact resolvent for all $\alpha \in \mathbb{C}$.

Now, assume that $\|V\|_\infty < \frac{3}{2}$. Then, for $|\alpha| \le 1$, all the eigenvalues of the family T_α are simple, as they lie in the $\alpha\|V\|_\infty$-neighbourhood of $\{j^2\}_{j=1}^{\infty}$. In particular, for $\alpha = 1$, all the eigenvalues of L are indeed simple. ∎

We now show that the eigenfunctions of L are close enough to the orthonormal Fourier-sine basis corresponding to $V = 0$:

$$d_j(x) = \sqrt{\frac{2}{\pi}} \sin(j\pi x), \quad j \in \mathbb{N}. \tag{2.2}$$

From the second statement of the next lemma, it follows that the solution to (2.1) is given by

$$u(x,t) = \sum_{j=1}^{\infty} \langle f, y_j^* \rangle e^{-iw_j^2 t} y_j(x)$$

for y_j^* the eigenfunction of L^* scaled to form a bi-orthogonal set paired with y_j, $\langle y_j, y_k^*\rangle = \delta_{jk}$, and the series converges in L^2. This turns out to be crucial for the validity of Theorem 1.

Lemma 2. *In the asymptotic regime $j \to \infty$, the eigenfunctions of L are such that*

$$y_j(x) = c\left[\sin(jx) - \frac{\cos(jx)V_1(x)}{2j} + R_j(x)\right], \tag{2.3}$$

where

$$\|R_j\|_\infty = O(j^{-2})$$

and V_1 is defined by

$$V_1(x) = \int_0^x V(s)\,\mathrm{d}s. \tag{2.4}$$

If $\|V\|_\infty < \frac{3}{2}$, then

$$\left\{\frac{y_j}{\|y_j\|_2}\right\}_{j=1}^\infty$$

is a Riesz basis for $L^2(0,\pi)$.

Proof. According to [14, Lemma 1.4.1], the eigenfunction associated to w_k is

$$y_j(x) = A^+\Phi_j^+(x) + A^-\Phi_j^-(x),$$

where

$$\Phi_j^\pm(x) = e^{\pm i w_j x}\left(1 \pm \frac{V_1(x)}{2iw_j} + \frac{V_2(x)}{(2iw_j)^2} + R_j^\pm(x)\right)$$

for $\|R_j^\pm\|_\infty = O(w_j^{-3})$, V_1 given by (2.4), and

$$V_2(x) = \int_0^x LV_1(s)\,\mathrm{d}s = \int_0^x V(s)V_1(s)\,\mathrm{d}s - V(x) + V(0).$$

Moreover, again from [14, Lemma 1.4.1], we know that $R_j^\pm(0) = 0$. Thus, substituting the boundary conditions $y_j(0) = 0$, we get $-A^- = A^+$. Set the latter equal to 1. Hence,

$$y_j(x) = \sin(w_j x) - \frac{\cos(w_j x)V_1(x)}{2w_j} + \widetilde{R}_j(x),$$

where

$$\|\widetilde{R}_j\|_\infty = O(j^{-2}).$$

Now,

$$\sin(jx + O(j^{-3})x) = \sin(jx) + s_j(x),$$
$$\cos(jx + O(j^{-3})x) = \cos(jx) + c_j(x),$$

where

$$|s_j(x)| + |c_j(x)| \le 2|\cos(O(j^{-3})x) - 1| + 2|\sin(O(j^{-3})x)| \le k_1 j^{-6} + k_2 j^{-3}$$

for all $x \in [0, \pi]$. This gives (2.3), by taking $R_j(x) = \widetilde{R}_j(x) + s_j(x) + c_j(x)$.

Let us now show that if $\|V\|_\infty < \frac{3}{2}$, then the eigenfunctions form a Riesz basis. We aim at applying [13, Theorem 2.20, page 265]. According to [14, Theorem 1.3.1] combined with Lemma 1, the family of eigenfunctions $\{y_j\}$ is complete in $L^2(0, \pi)$. Since it has a dual pair, $\{y_j^*\}$, then it is minimal, and so, therefore, exact [11]. Minimality ensures that $\{y_j\}$ is ω-independent [11]. This gives two of the hypotheses of [13, Theorem 2.20, page 265].

Now, by virtue of (2.3) already proven, there exists a constant $c_4 > 0$ such that

$$\left\| \frac{y_j}{\|y_j\|_2} - d_j \right\|_2 \le \frac{k_3}{j}.$$

Thus,

$$\sum_{j=1}^{\infty} \left\| \frac{y_j}{\|y_j\|_2} - d_j \right\|_2^2 < \infty.$$

This is the other hypothesis required in [13, Theorem 2.20, page 265], and so, indeed[1] $\{\frac{y_j}{\|y_j\|}\}_{j=1}^{\infty}$ is a Riesz basis of $L^2(0, \pi)$. ∎

We discuss the optimality of the condition $\|V\|_\infty < \frac{3}{2}$ and the case where V is real-valued in Section 4. From the above, we get that the eigenvalues of L are

$$\lambda_j = w_j^2 = j^2 + \frac{k_j}{j^2} \quad \text{for suitable } \{k_j\} \in \ell^\infty. \tag{2.5}$$

Let

$$n_j(x) = \sqrt{\frac{2}{\pi}} \cos(j\pi x), \quad j \in \mathbb{N}$$

be the non-constant orthonormal Fourier-cosine basis. Below, we fix the eigenfunctions according to a normalisation of their bi-orthogonal pairs. Concretely, let

$$\phi_j(x) = \gamma_j y_j(x) = \gamma_j d_j(x) - \frac{\gamma_j n_j(x) V_1(x)}{2j} + \gamma_j R_j(x), \tag{2.6}$$

where $\|R_j\|_\infty \le \frac{c}{j^2}$. Without further mention, from now on, the non-zero constants γ_j are chosen such that the associated bi-orthogonal sequence $\{\phi_j^*\}$ is given by

$$\phi_j^*(x) = d_j(x) - \frac{n_j(x)\overline{V}_1(x)}{2j} + R_j^*(x). \tag{2.7}$$

[1]The conclusion of [13, Theorem 2.20, page 265] does not exactly state that the family is a Riesz basis. But the proof implies that the family is equivalent to an orthonormal basis; hence, it is indeed always a Riesz basis.

Then, there exist constants $0 < \widetilde{\gamma}_1 < \widetilde{\gamma}_2 < \infty$ such that

$$\frac{\widetilde{\gamma}_1}{j} < |\gamma_j - 1| < \frac{\widetilde{\gamma}_2}{j} \quad \text{and} \quad \frac{\widetilde{\gamma}_1}{j} < |\|\phi_j^*\|_2 - 1| < \frac{\widetilde{\gamma}_2}{j} \tag{2.8}$$

for all $j \in \mathbb{N}$.

3. Proof of Theorem 1

We now state and prove a crucial lemma, from which Theorem 1 follows as a corollary.

Lemma 3. *If the potential V is such that $\langle V \rangle = 0$ and $\|V\|_\infty < \frac{3}{2}$, then the solution to the time-evolution equation (2.1) is given by*

$$u(x,t) = w(x,t) + \sum_{j=1}^{\infty} \langle f, d_j \rangle e^{-ij^2 t} d_j(x), \tag{3.1}$$

where, for each fixed $t > 0$, $w(\cdot, t) \in C([0,\pi])$.

Proof. We separate the proof into four steps.

Step 1. Consider the L^2 expansion of the initial condition:

$$f(x) = \sum_{j=1}^{\infty} c_j \phi_j(x), \quad c_j = \langle f, \phi_j^* \rangle.$$

According to (2.5),

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{\infty} c_j e^{-i\lambda_j t} \phi_j(x) = \sum_{j=1}^{\infty} c_j e^{-i\left(j^2 + \frac{k_j}{j^2}\right)t} \phi_j(x) \\ &= \sum_{j=1}^{\infty} c_j e^{-ij^2 t} \left(1 - \frac{ik_j}{j^2} \int_0^t e^{-\frac{ik_j}{j^2} s}\, \mathrm{d}s\right) \phi_j(x) = U_1(x,t) - U_2(x,t), \end{aligned}$$

where

$$U_1(x,t) = \sum_{j=1}^{\infty} c_j e^{-ij^2 t} \phi_j(x)$$

and $U_2(x,t)$ has a similar expression but involving the integral above. We treat these two terms separately.

Step 2. Let us show that $U_2 \in C^1([0,\pi])$. Set

$$\zeta_j(x,t) = \frac{ic_j k_j}{j^2} e^{-ij^2 t} \int_0^t e^{-\frac{ik_j}{j^2} s}\, \mathrm{d}s\, \phi_j(x).$$

Then,

$$|\zeta_j(x,t)| \le \frac{\|\{k_j\}\|_\infty \|\phi_j\|_\infty |\langle v, \phi_j^* \rangle|}{j^2} t B_j \le \frac{\|\{k_j\}\|_\infty \|\phi_j\|_\infty \|v\|_2 \|\phi_j^*\|_2}{j^2} t B_j,$$

where

$$B_j = \sup_{s\in[0,t]} \left| e^{-\frac{ik_j s}{j^2}} \right| \le \sup_{s\in[0,t]} e^{\left|\frac{\operatorname{Im} k_j s}{j^2}\right|} \le B < \infty.$$

According to (2.5), the right-hand side constant is independent of j. Here, t is fixed. Moreover, by virtue of (2.6) and (2.8),

$$\max\{\|\phi_j^*\|_2, \|\phi_j\|_\infty\} \le c$$

for all $j \in \mathbb{N}$, where the constant $c > 0$ is independent of j. Hence, by Weierstrass's M-test,

$$U_2(x,t) = \sum_{j=1}^{\infty} \zeta_j(x,t)$$

converges absolutely and uniformly to a C^1 function because each component is C^1.

Step 3. Consider now U_1. According to (2.6),

$$\begin{aligned} U_1(x,t) &= \sum_{j=1}^{\infty} c_j \gamma_j e^{-ij^2 t} d_j(x) - \sum_{j=1}^{\infty} \frac{c_j \gamma_j e^{-ij^2 t}}{2j} n_j(x) V_1(x) + \sum_{j=1}^{\infty} c_j \gamma_j e^{-ij^2 t} R_j(x) \\ &= u_3(x,t) + u_4(x,t) + u_5(x,t). \end{aligned}$$

In this step, we show that u_4 and u_5 are continuous in the variable x. From the asymptotic behaviour of R_j and an identical argument as we used in step 2, we know that u_5 is C^1 in the variable x.

Now, for $u_4(x,t)$, note that

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty$$

because $\{\phi_j^*\}$ is a Riesz basis, e.g., [11, Theorem 7.13]. Then, by the Cauchy–Schwarz inequality,

$$\sum_{j=1}^{\infty} \left| \frac{c_j}{j} \right| < \infty.$$

Thus, since V_1 is a bounded function, for all fixed t, the sequence

$$\left\{ \frac{c_j \gamma_j e^{-ij^2 t} \|V_1\|_\infty}{2j} \right\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}).$$

Hence, for all fixed $t > 0$, the family of sequences (family for $x \in [0,\pi]$)

$$\left\{ \frac{c_j \gamma_j e^{-ij^2 t} n_j(x) V_1(x)}{2j} \right\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}).$$

Therefore, by the dominated convergence theorem (in ℓ^1), we have that

$$\lim_{x \to x_0} u_4(x,t) = u_4(x_0,t)$$

for all $x_0 \in [0,\pi]$. That is, u_4 is a continuous function of the variable x.

Step 4. Finally, we consider $u_3(x,t)$. By (2.7), we know that

$$c_j = \langle f, d_j \rangle - \frac{\langle f V_1, n_j \rangle}{2j} + \langle f, R_j^* \rangle.$$

Then, $u_3(x,t) = u_6(x,t) - u_7(x,t) + u_8(x,t)$, where

$$u_6(x,t) = \sum_{j=1}^{\infty} \langle f, d_j \rangle \gamma_j e^{-ij^2 t} d_j(x), \quad u_7(x,t) = \sum_{j=1}^{\infty} \frac{\langle f V_1, n_j \rangle}{2j} \gamma_j e^{-ij^2 t} d_j(x),$$

$$u_8(x,t) = \sum_{j=1}^{\infty} \langle f, R_j^* \rangle \gamma_j e^{-ij^2 t} d_j(x).$$

We write $\gamma_j = 1 + (\gamma_j - 1)$ and split each term of $u_6(x,t), u_7(x,t), u_8(x,t)$ into two sums.

For $u_6(x,t)$, we have

$$u_6(x,t) = \sum_{j=1}^{\infty} \langle f, d_j \rangle e^{-ij^2 t} d_j(x) + \sum_{j=1}^{\infty} \langle f, d_j \rangle (\gamma_j - 1) e^{-ij^2 t} d_j(x)$$

The first component of u_6 is the second component in the solution representation (3.1). To deal with the second, one we use (2.8). By the Cauchy–Schwarz inequality,

$$\sum_{j=1}^{\infty} \frac{|\langle f, d_j \rangle|}{j} < \infty.$$

Hence, by Weierstrass's M-test again, for each $t > 0$, the series of functions

$$\sum_{j=1}^{\infty} \langle f, d_j \rangle (\gamma_j - 1) e^{-ij^2 t} d_j(x)$$

converges absolutely and uniformly to a C^1 function on $[0,\pi]$.

Now, the function u_7 is written as follows:

$$u_7(x,t) = \sum_{j=1}^{\infty} \frac{\langle f V_1, n_j \rangle}{2j} e^{-ij^2 t} d_j(x) + \sum_{j=1}^{\infty} \frac{\langle f V_1, n_j \rangle}{2j} (\gamma_j - 1) e^{-ij^2 t} d_j(x).$$

The first component of u_7 is continuous as a consequence of an argument similar to that employed for u_4. Indeed, since $\{n_j\}$ is an orthonormal basis and $f V_1 \in L^2(0,\pi)$,

$$\sum_{j=1}^{\infty} \left| \frac{\langle f V_1, n_j \rangle e^{-ij^2 t}}{2j} \right| \leq \|f V_1\|_2 \frac{\pi}{\sqrt{6}},$$

so we can use the sequence

$$\left\{\frac{\langle f V_1, n_j\rangle e^{ij^2 t}}{2j}\right\}_{j=1}^{\infty} \in \ell^1$$

to ensure continuity via the dominated convergence theorem. The second component is a C^1 function since its Fourier-sine coefficients decay like j^{-2} due to (2.8) and the Cauchy–Schwarz inequality. So, in total, $u_7(\cdot, t) \in C([0, \pi])$.

Finally, for the function u_8, we have that

$$u_8(x,t) = \sum_{j=1}^{\infty} \langle f, R_j^* \rangle e^{-ij^2 t} d_j(x) + \sum_{j=1}^{\infty} \langle f, R_j^* \rangle (\gamma_j - 1) e^{-ij^2 t} d_j(x).$$

Here, the first component is C^1 in x. Indeed, since

$$|\langle f, R_j^* \rangle| = \left| \int_0^{\pi} f(x) \overline{R_j^*(x)} \, \mathrm{d}x \right| \leq \frac{\pi \|f\|_2 c}{j^2},$$

the first component is a function whose sine-Fourier coefficients decay like j^{-2}, so it is continuously differentiable in x. The second component is twice differentiable in x, since due to (2.8) it represents a function whose Fourier-sine coefficients decay like j^{-3}. Thus, $u_8(\cdot, t)$ belongs to $C^1([0, \pi])$.

Collecting the statements about u_k from the previous steps, we conclude that the expression of u is as claimed in (3.1), where indeed w is continuous in the variable x. ■

Remark 1. In the proof of this lemma, note that all components of w are C^1, except u_4 and u_7.

The proof of Theorem 1 now follows from Lemma 3 and the combinatorial argument for $V = 0$.

Proof of Theorem 1. We show that if $V = 0$, then the solution to (2.1) at rational times $t_{\mathrm{r}} = 2\pi \frac{p}{q}$ is

$$u(x, t_{\mathrm{r}}) = \sum_{j=1}^{\infty} \langle f, d_j \rangle e^{-ij^2 t_{\mathrm{r}}} d_j(x) = \frac{1}{q} \sum_{k,m=0}^{q-1} e^{2\pi i (-m^2 \frac{p}{q} + m \frac{k}{q})} f^{\mathrm{o}}\left(x - 2\pi \frac{k}{q}\right), \tag{3.2}$$

where f^{o} denotes the odd, 2π-periodic extension of f. Therefore, replacing V with $V - \langle V \rangle$ if needed, and applying Lemma 3, gives Theorem 1.

The proof of (3.2) is as follows. For $t \in \mathbb{R}$, we have that

$$u(x,t) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ij^2 t} \langle f^{\mathrm{o}}, e^{ij(\cdot)} \rangle e^{ijx}.$$

Then, for $t = t_r$, take $j \underset{q}{\equiv} m$ so that $e^{ij^2 t_r} = e^{im^2 t_r}$. Thus,

$$u(x, t_r) = \frac{1}{2\pi} \sum_{m=0}^{q-1} e^{-im^2 t_r} \sum_{\substack{j \in \mathbb{Z} \\ j \underset{q}{\equiv} m}} \langle f^{\circ}, e^{ij(\cdot)} \rangle e^{ijx}.$$

Let the summation on the right-had side be denoted by T. Since

$$\sum_{k=0}^{q-1} e^{2\pi i (m-j) \frac{k}{q}} = \begin{cases} q, & j \underset{q}{\equiv} m, \\ 0, & j \underset{q}{\not\equiv} m, \end{cases}$$

we have

$$\begin{aligned} T &= \frac{1}{q} \sum_{k=0}^{q-1} e^{2\pi i m \frac{k}{q}} \sum_{j \in \mathbb{Z}} e^{-2\pi i \frac{k}{q} j} \big\langle f^{\circ}, e^{ij(\cdot)} \big\rangle e^{ijx} \\ &= \frac{1}{q} \sum_{k=0}^{q-1} e^{2\pi i m \frac{k}{q}} \sum_{j \in \mathbb{Z}} \Big\langle f^{\circ}\Big(\cdot - \frac{2\pi k}{q}\Big), e^{ij(\cdot)} \Big\rangle e^{ijx} \\ &= \frac{1}{q} \sum_{k=0}^{q-1} e^{2\pi i m \frac{k}{q}} f^{\circ}\Big(x - \frac{2\pi k}{q}\Big). \end{aligned}$$

Hence, (3.2) holds true. ∎

Remark 2. For general bounded complex potential V, we know from the Dyson expansion that (3.1) holds true for

$$w(x, t) = \sum_{k=1}^{\infty} w_k(x, t),$$

where $w_k(x, t)$ are explicitly given in terms of integrals of regular functions. These functions are continuous for $x \in (0, \pi)$. By tracking the convergence of the series, it might be possible to establish its continuity, therefore extending the results of this section to all $\|V\|_{\infty} < \infty$. See the ideas described in [1, Section 7].

4. The hypotheses of Theorem 1

In this section, we examine the optimality of the hypotheses of Theorem 1, in terms of the size and the regularity of the potential.

Firstly, we note that, for L self-adjoint, the assumptions on V can be relaxed. According to the asymptotic expansions reported in [10, equations $(4.21)_7$, $(5.4)_2$, and $(5.9)_2$], the identity (2.3) is valid for any $V : [0,\pi] \to \mathbb{R}$ of bounded variation irrespective of the size of $\|V\|_\infty < \infty$. Therefore, by following the same method of proof presented above, it directly follows that the conclusion of Theorem 1 still holds true under these modified hypotheses on V. That is, the solution to (1.1) at rational times is given by

$$u\left(x, 2\pi\frac{p}{q}\right) = w\left(x, 2\pi\frac{p}{q}\right) + \frac{1}{q}e^{-2\pi i\langle V\rangle \frac{p}{q}} \sum_{k,m=0}^{q-1} e^{2\pi i(m\frac{k}{q} - m^2\frac{p}{q})} f^{\circ}\left(x - 2\pi\frac{k}{q}\right) \tag{4.1}$$

for a suitable function $w(\cdot, t)$ continuous in $x \in [0,\pi]$.

In the more general non-self-adjoint setting, for $\|V\|_\infty > \frac{3}{2}$, we only know from the available asymptotic formulas that, for large wavenumbers, all the eigenvalues are simple and the corresponding eigenfunctions form a basis of a subspace $\mathcal{S}$ of finite co-dimension. Despite this, still iL is the generator of a one-parameter semigroup, see Remark 2 above. The solution to (1.1) exists and it is unique for all $f \in L^2$. Moreover, (2.1) has a solution with an L^2-convergent eigenfunction expansion and a version of Theorem 1 can be recovered for all $f \in \mathcal{S}$.

We now present an example of a purely imaginary $V \in C^\infty$ with $\|V\|_\infty = 2$, for which (4.1) appears to still be valid. For this purpose, we choose for V a purely imaginary Mathieu potential.

Let $q \in \mathbb{C}$ and $V(x) = 2q\cos(2x)$. Then, $\langle V\rangle = 0$. The eigenvalue equation associated to the operator L is Mathieu's equation. The eigenvalues of L are

$$\omega_j^2 = b_j(q),$$

the Mathieu characteristic values, which satisfy

$$b_j(q) = j^2 + \frac{1}{2(j^2-1)}q^2 + O(q^4)$$

as $|q| \to 0$. The corresponding eigenfunctions are the Mathieu functions

$$\phi_j(x) = \mathrm{se}_j(x,q)$$

for $j \in \mathbb{N}$. See [12, Section 7.4] and also [15, Section 28.2-7].

In Figure 1, we set

$$f(x) = \chi_{[\frac{3\pi}{8},\frac{5\pi}{8}]}(x)$$

and show a numerical approximation to 100 modes of $u(x,t)$ at time $t = \frac{2\pi}{5}$ for purely imaginary q with increasing modulus. As $|q|$ increases, we show how the correction $w(x,t)$ affects the revivals part of the solution. Note that the conclusions of Theorem 1 only hold for $|q| < \frac{3}{4}$, but the numerical approximation in the figure suggests that this conclusion appears to be valid also when $\|V\|_\infty = 2$ for this potential.

The graphs shown in Figure 2 reinforce the conjecture that the correction term w is continuous beyond the threshold $\|V\|_\infty = \frac{3}{2}$. Indeed, in Figure 2, we show a 100 modes

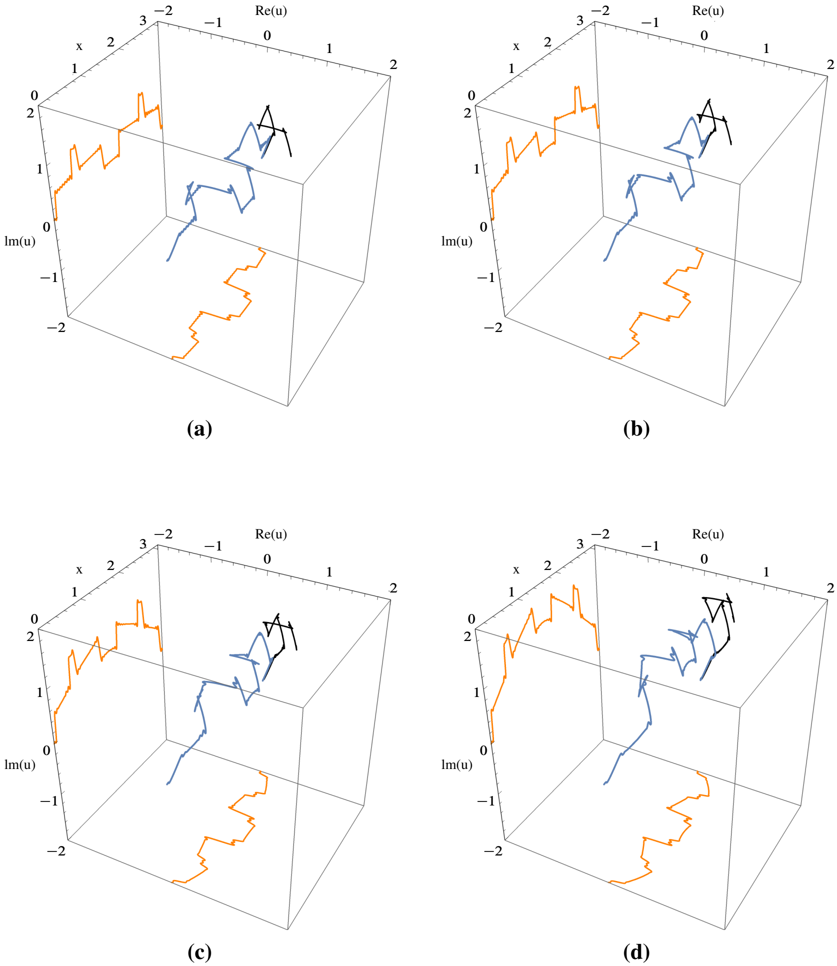

Figure 1. Here, $V(x) = 2q\cos(2x)$ and $t = \frac{2\pi}{5}$. We show an approximation of $u(x,t)$ to 100 modes. The blue curves are the solutions as complex-valued functions of x, the orange curves correspond to projections of the real and imaginary parts of these solutions, and the black curves are the projections corresponding to the curves traced by the solutions in the complex plane for $x \in [0,\pi]$. The figures shown match (a) $q = \frac{i}{4}$, (b) $q = \frac{i}{2}$, (c) $q = \frac{3i}{4}$ and (d) $q = i$.

approximation of

$$u(x,t) - \sum_{j=1}^{\infty} \langle f, d_j \rangle e^{-ij^2 t} d_j(x)$$

for the same data as in Figure 1. For (a)-(b), we confirm the shape of $w(x,t)$. For (c)-(d), note that even when $q = \frac{3}{4}i$ and $q = i$, the difference appears to still be continuous.

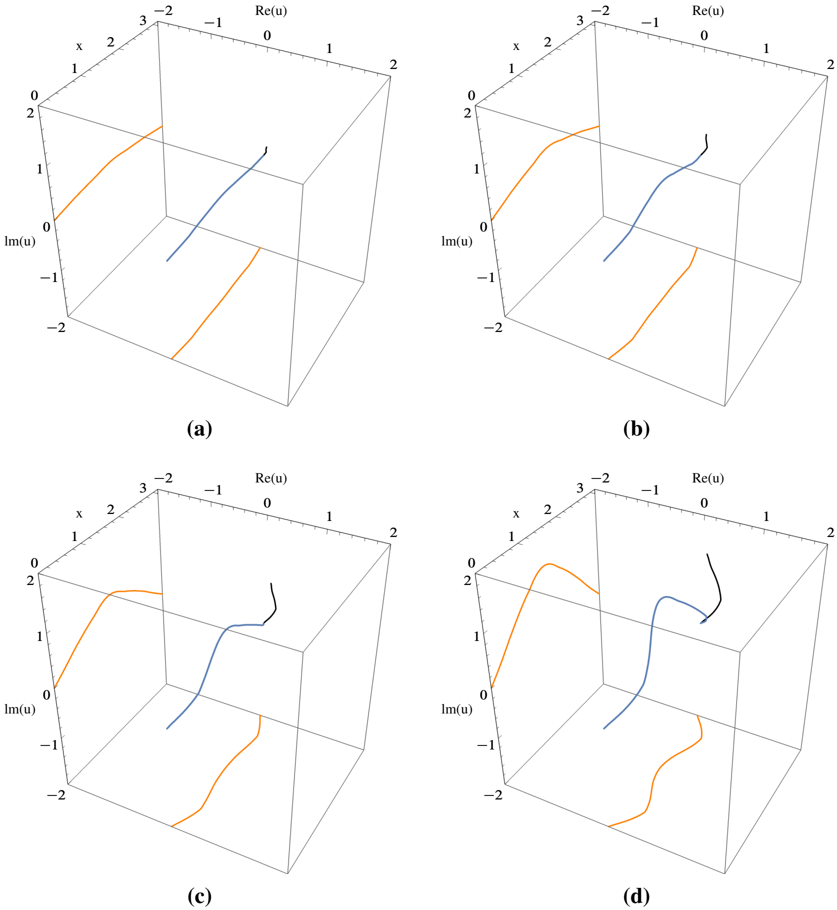

Figure 2. Here, $V(x) = 2q\cos(2x)$ and $t = \frac{2\pi}{5}$. We show an approximation of $w(x,t)$ to 100 modes. The blue curves are the complex-valued functions of x, the orange curves correspond to projections of the real and imaginary parts of these, and the black curves correspond to the graphs traced by $w(x, \frac{2\pi}{5})$ on the complex plane for $x \in [0,\pi]$. The figures shown match (a) $q = \frac{i}{4}$, (b) $q = \frac{i}{2}$, (c) $q = \frac{3i}{4}$, and (d) $q = i$.

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