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A remark on solutions to semilinear equations with Robin boundary conditions

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Abstract. Symmetry properties of solutions to elliptic quasilinear equations have been widely studied in the context of Dirichlet boundary conditions. We show that, in the context of Robin boundary conditions, the symmetry property á la Gidas, Ni, and Nirenberg does not hold in dimension $n \geq 2$, even for superharmonic functions, and we provide an explicit example.

1. Introduction

The task of proving symmetries of solutions to quasilinear or nonlinear PDEs that reflect the symmetries of the domain has interested many authors. In this context, the classical result by Gidas, Ni, and Nirenberg, contained in the celebrated paper [12, Theorem 1], is stated as follows.

Theorem 1.1 (Gidas–Ni–Nirenberg symmetry result). *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f = f_1 + f_2$, where f_1 is a locally Lipschitz function and f_2 is non-decreasing. Then, any positive solution $u \in C^2(\overline{B})$ to the problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

where B is a ball of $\mathbb{R}^n$ of radius R, has to be radial and

$$\frac{\partial u}{\partial r} < 0 \quad \text{for } 0 < r < R.$$

In order to prove this result, the authors make use of the method of moving planes, first introduced by Aleksandrov in [1], and then applied by Serrin in [22] in the context of PDEs.

In the case $n = 2$, Lions in [14] gives an alternative proof of Theorem 1.1, which allows one to consider weaker smoothness assumptions on f and u, under the additional hypothesis $f \geq 0$. The technique used in [14] relies on the Schwartz symmetrization,

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the isoperimetric inequality, and the Pohozaev identity. These techniques were applied and generalized in [13] to the context of the $n-$Laplacian, and, eventually, in [21] to the solutions to the $p-$Laplace equation for any p, in any dimension $n \geq 2$.

The result contained in [12] is a milestone in proving the symmetry of solutions to PDEs: in the linear case, see, for instance, [4, 5, 11, 20] and, for the $p-$Laplace operator, we refer to [3, 7, 9, 10]. In all the aforementioned papers, it is possible to prove the radiality of positive solution to (1) either under the regularity hypothesis on f stated in Theorem 1.1 (using the moving plane method as in [12]) or under the assumption $f \geq 0$ (with symmetrization techniques as in [13, 14, 21]). For a sign-changing f, the Lipschitz continuity property cannot be relaxed to Hölder continuity, as shown in [8]. Indeed, in this case, the author finds a positive solution to (1) that is not radially symmetric.

Recently, the study of symmetrization techniques for PDEs problem with Robin boundary condition has gained attention; see, for instance, [2, 15, 16]. The aim of the present work is to study the behavior of the solution to the Robin problem

$$\begin{cases} -\Delta u = f(u) & \text{in } B, \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B \end{cases} \tag{2}$$

whenever β is a positive parameter. To our knowledge, in the literature, few results deal with symmetry properties of the solutions to differential equations with Robin boundary conditions. For instance, in [6], the authors consider the following problem:

$$\begin{cases} -\varepsilon^2 \Delta u = f(u) - u & \text{in } B, \\ \varepsilon \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial B, \end{cases} \tag{3}$$

where $\varepsilon > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function of the form

$$f(t) = f_1(t) - f_2(t)$$

for $t \geq 0$, with $f_1, f_2 \geq 0$, satisfying some structural growth conditions (for the precise details, see [6, Section 1]). Under these assumptions, it is possible to prove the existence of positive least-energy solutions to (3) by making use of the Palais–Smale condition and the mountain pass lemma. The authors show that there exists a $\beta_* > 1$ such that, for $\beta > \beta_*$, a least-energy solution has the maximum at the center of the ball, while, for $\beta \leq \beta_*$ and $\varepsilon \to 0$, a least-energy solution has the maximum near the boundary, and, consequently, the function cannot be radially symmetric. These results regarding the location of the maximum of a least-energy solution to problem (3) are actually obtained in [6] for the more general case of a bounded and smooth domain $\Omega \subset \mathbb{R}^n$.

For the sake of completeness, we recall that the case $\beta = 0$, i.e., the Neumann problem, and the case $\beta = +\infty$, i.e., the Dirichlet problem, have been studied, for instance, respectively, in [17–19].

In Theorem 2.1, we show that in the one-dimensional case the symmetry result for the solution to (2) holds under the standard hypotheses of Gidas, Ni, and Nirenberg and the additional hypothesis $f \geq 0$. On the other hand, this is not the case for $n \geq 2$, as pointed out in Corollary 3.2:

In dimension $n \geq 2$, there exists a positive superharmonic function φ that is a solution to (2) and that is not radially symmetric.

So, the main novelty of our paper is that we can find a non-radial solution to problem (2) when the nonlinearity f is positive, and, moreover, this solution is explicit (see Theorem 3.1).

2. One-dimensional case: The symmetry holds

We start by analyzing the one-dimensional case.

Theorem 2.1. *Let $R > 0$, and let $I =]-R, R[$ be the open ball of radius R. Let $u \in C^2([-R, R])$ be a solution to*

$$\begin{cases} -u'' = f(u) & \text{in } I, \\ \frac{\partial u}{\partial \nu}(x) + \beta u(x) = 0 & \text{in } x = \pm R, \end{cases}$$

where $\beta > 0$. Let us assume that f satisfies the following assumptions:

(i) *$f \geq 0$ in $\mathbb{R}$, f is not identically zero in $u(I)$,*

(ii) *$f = f_1 + f_2$, where f_1 is locally Lipschitz in $\mathbb{R}$ and f_2 is non-decreasing.*

Then, $u(x) = u(-x)$ for all $x \in [-R, R]$. Moreover,

$$u'(x) < 0, \quad x \in [0, R].$$

Proof. We divide the proof into two steps. In the first step, we prove that the function u is strictly positive, and in the second one, we prove that we can apply the result contained in Theorem 1.1.

Step 1. We start by proving that $u > 0$ in $[-R, R]$. Since $u'' \leq 0$, u' is non-increasing in $]-R, R[$, so the minimum of u on $[-R, R]$ is achieved either in $-R$ or in R. Let us denote by x_m the minimum point of u in $[-R, R]$. From the Robin boundary conditions, we have that

$$-\beta u(x_m) = \frac{\partial u}{\partial \nu}(x_m) \leq 0,$$

and, as a consequence, $u \geq 0$ in $[-R, R]$.

Now, we want to prove that $u > 0$ in $[-R, R]$. By contradiction, we assume that $u(x_m) = 0$. If $x_m = -R$, the Robin boundary conditions imply that

$$0 = \beta u(-R) = -\frac{\partial u}{\partial \nu}(-R) = u'(-R) \geq u'(x), \quad \forall x \in I,$$

where we have observed that u' is non-increasing. This implies that also u is a non-increasing function, so $-R$ should be both a minimum and a maximum. This is not possible, since u should be constant and this contradicts the hypothesis $f \not\equiv 0$ in $u(I)$. Therefore, we have that $x_m = R$ and, arguing as before, we have

$$0 = -\beta u(R) = \frac{\partial u}{\partial \nu}(R) = u'(R) \le u'(x), \quad \forall x \in I,$$

So, u is a non-decreasing function, the point R is both a minimum and a maximum, and we get a contradiction as before.

Step 2. We prove now that $u(R) = u(-R)$. Let us assume by contradiction that $u(R) \neq u(-R)$. Without loss of generality, we can suppose $u(R) < u(-R)$. As a consequence of Step 1, the function u is strictly increasing in a neighborhood of the point $-R$, so, by the continuity of u, there exists $y \in I$ such that $u(y) = u(-R)$. Therefore, the following quantity is well defined:

$$\lambda := \inf\{t \in I : u(t) = u(-R)\},$$

and $\lambda > -R$. Moreover, the continuity of u also implies $u(\lambda) = u(-R)$ and $u(x) > u(-R)$ in $(-R, \lambda)$.

We define now the function $v := u - u(-R)$, which is a positive solution to

$$\begin{cases} -\Delta v = \tilde{f}(v) & \text{in } (-R, \lambda), \\ v = 0 & \text{in } x = -R,\ x = \lambda, \end{cases}$$

where $\tilde{f}(v) = f(v + u(R))$. So, we can use Theorem 1.1 in the interval $(-R, \lambda)$. We have that u is symmetric with respect to the line $x = 2^{-1}(\lambda - R)$, and, as a consequence, we get

$$\frac{du}{d\nu}(-R) = -\frac{du}{dx}(-R) = \frac{du}{dx}(\lambda) < 0. \tag{4}$$

Using (4) and the fact that u' is non-increasing, we obtain

$$\beta u(R) = -\frac{du}{d\nu}(R) = -\frac{du}{dx}(R) \ge -\frac{du}{dx}(\lambda) = -\frac{du}{d\nu}(-R) = \beta u(-R),$$

and, therefore,

$$u(R) \ge u(-R),$$

which is a contradiction.

From Steps 1 and 2, we have that the function $v = u - u(R)$ is a positive solution to

$$\begin{cases} -\Delta v = \tilde{f}(v) & \text{in } I, \\ v = 0 & \text{in } x = \pm R. \end{cases}$$

So, we can conclude by using Theorem 1.1. ∎

We do not know if the hypotheses (i)–(ii) on the function f are the optimal ones to obtain the symmetry result. Nevertheless, we will show in Remark 3.1, in the next session, that the assumption $f \ge 0$ cannot be removed.

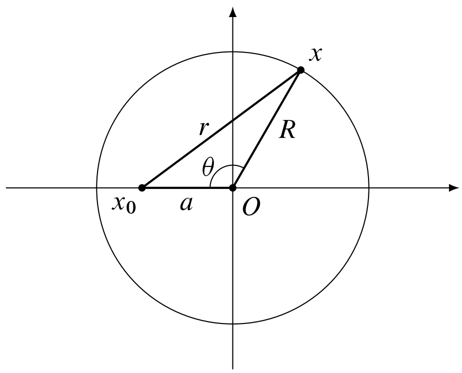

Figure 1. Construction of the function φ.

3. Counterexample: Symmetry breaking in dimension $n \geq 2$

In the following, we will denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$.

Theorem 3.1. *Let $B_R \subset \mathbb{R}^n$, $n \geq 2$, be the ball centered at the origin with radius R, let β be a positive constant, and let $x_0 \neq 0$ in B_R. Then, there exists a positive function $\varphi \in C^\infty(\overline{B_R})$ that is a non-radial function in B_R (i.e., $\varphi(x) \neq \varphi(|x|)$), and it is a solution to*

$$\begin{cases} -\Delta\varphi = f(\varphi) & \text{in } B_R, \\ \frac{\partial\varphi}{\partial\nu} + \beta\varphi = 0 & \text{on } \partial B_R, \end{cases} \tag{5}$$

where

$$f(t) = c_1 t\left[c_2 t^{\frac{1}{\beta R}} + c_3 t^{\frac{2}{\beta R}}\right], \tag{6}$$

with $c_1, c_3 > 0$, $c_2 \in \mathbb{R}$ defined as follows:

$$c_1 = 2\beta R, \quad c_2 = -2(\beta R + 1) + n, \quad c_3 = 2(\beta R + 1)\alpha^2, \quad \alpha^2 = R^2 - |x_0|^2.$$

Proof. We define the following quantities (see Figure 1):

- $a := |x_0|$,
- $\alpha^2 := R^2 - a^2 > 0$.

We show the existence of a positive function $\varphi(x) = \varphi(|x - x_0|) = \varphi(r)$ such that

$$\frac{\partial\varphi}{\partial\nu} + \beta\varphi = 0 \quad \text{on } \partial B_R, \tag{7}$$

where ν is the unit outer normal to ∂B_R. Let us fix $x \in \partial B_R$. Being

$$\nabla(\varphi(x)) = \varphi'(r)\frac{x - x_0}{r}, \quad \nu(x) = \frac{x}{R},$$

the Robin boundary conditions (7) become

$$\varphi'(r)\,\frac{x-x_0}{r}\cdot\frac{x}{R}+\beta\varphi(r)=0. \tag{8}$$

Denoting, now, by θ the angle between the vectors x_0 and x, we have the following relation:

$$\cos(\theta)=\frac{R^2+a^2-r^2}{2aR}. \tag{9}$$

So, from (8) and (9), recalling that $\alpha^2=R^2-a^2$, we have

$$\frac{\varphi'(r)}{rR}(R^2-Ra\cos(\theta))+\beta\varphi(r)=\frac{\varphi'(r)}{2rR}(r^2+\alpha^2)+\beta\varphi(r)=0,$$

and, therefore,

$$\frac{\varphi'(r)}{\varphi(r)}=-(2\beta R)\frac{r}{r^2+\alpha^2}. \tag{10}$$

Integrating (10), we get

$$\varphi(r)=\frac{c}{(r^2+\alpha^2)^{\beta R}}. \tag{11}$$

If we choose $c=1$, we have

$$\begin{aligned}\varphi'(r)&=-\frac{2\beta Rr}{(r^2+\alpha^2)^{\beta R+1}},\\ \varphi''(r)&=-(-\beta R-1)\frac{4\beta Rr^2}{(r^2+\alpha^2)^{\beta R+2}}-\frac{2\beta R}{(r^2+\alpha^2)^{\beta R+1}}.\end{aligned}$$

and, consequently,

$$\begin{aligned}-\Delta\varphi&=-\varphi''(r)-\frac{n-1}{r}\varphi'(r)\\ &=\frac{2\beta R}{(r^2+\alpha^2)^{\beta R+1}}\left[n-2(\beta R+1)+(\beta R+1)\frac{2\alpha^2}{r^2+\alpha^2}\right]\\ &=2\beta R\varphi(r)^{\frac{1}{\beta R}+1}\left[n-2(\beta R+1)+2(\beta R+1)\alpha^2\varphi(r)^{\frac{1}{\beta R}}\right]\\ &=f(\varphi(r)),\end{aligned}$$

where f is the function defined in (6). So, we have proved the desired claim, since we have found a non-radial function of the form $\varphi(x)=\varphi(|x-x_0|)=\varphi(r)$, defined in (11), that satisfies (5). ∎

As a consequence of Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let $n\geq 2$. There exists a positive superharmonic function φ that is a solution to (2) and that is not radially symmetric.*

Proof. In the case $n = 2$, the right-hand side of (6) becomes

$$f(t) = 4\beta R t\big(-\beta R t^{\frac{1}{\beta R}} + \alpha^2(1+\beta R)t^{\frac{2}{\beta R}}\big).$$

We notice that

$$f(t) \geq 0, \quad \text{if } t \geq \left(\frac{\beta R}{\alpha^2(1+\beta R)}\right)^{\beta R}, \tag{12}$$

so the function $f \circ \varphi$ is positive if φ satisfies (12) for all $x \in B_R$, and this follows by imposing the following geometric constraint:

$$\beta \leq \frac{R-a}{R(R+a)}.$$

If $n \geq 3$, we can choose the constant $c_2 \geq 0$; by imposing the condition

$$\beta \leq \frac{n-2}{2R} \tag{13}$$

and, under these assumptions, we have that $f(t) \geq 0$ for $t \geq 0$.

Therefore, we can see that, by imposing the geometrical constraints (12) and (13), respectively, for $n = 2$ and $n \geq 3$, the function φ defined in (11) is an example of positive superharmonic function, which is non-radial and satisfies (5). ∎

We conclude with a remark on the one-dimensional case.

Remark 3.1. The function φ defined in Theorem 3.1, in the case $n = 1$, satisfies the problem

$$\begin{cases} -\varphi'' = f(\varphi) & \text{in } (-R, +R), \\ \frac{\partial \varphi}{\partial \nu} + \beta\varphi = 0 & \text{in } x = \pm R. \end{cases}$$

We note that $\varphi \in C^\infty([-R, R])$ and f is a locally Lipschitz function, but f does not satisfy the hypothesis (i), that is, the positiveness. Indeed, by straightforward computations, we obtain that $f \circ \varphi$ is a sign-changing function in $\varphi([-R, +R])$ for every $\beta > 0$ and $R > a > 0$.

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