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Liouville theorems for a fourth order Hénon equation in the half-space

Abdelbaki Selmi and Cherif Zaidi

Abstract. We investigate here the nonlinear elliptic Hénon-type equation

$$\Delta^2 u = |x|^a |u|^{p-1} u \quad \text{in } \mathbb{R}^n_+, \quad u = \frac{\partial u}{\partial x_n} = 0 \quad \text{in } \partial\mathbb{R}^n_+,$$

where $p > 1$, $a \geq 0$ and $n \geq 5$. Based on the approach of Hu [J. Differential Equations 256 (2014), 1817–1846], we prove Liouville-type theorems for stable solutions and solutions which are stable outside a compact set possibly unbounded and sign-changing. In contrast with the results of Hu (2014), we apply a new method to provide an implicit existence of the fourth-order Joseph–Lundgren exponent. To classify finite Morse index solutions in the supercritical case, we adopt a new method of monotonicity formula together with blowing down sequence. In addition, a difficulty stems from the fact that applying the doubling lemma leads to the singularity. For this reason, we use a more delicate approach to the interval $(n + 4 + 2a, p_{JL2}(n, 0))$. Our analysis uses a combination of some integral estimates, Pohozaev-type identity, and monotonicity formula of solutions.

1. Introduction

We are interested in the Liouville-type theorems, that is, the nonexistence of the solution u which is stable or with finite Morse index of the following problem:

$$\Delta^2 u = |x|^a |u|^{p-1} u \quad \text{in } \mathbb{R}^n_+, \quad u = \frac{\partial u}{\partial x_n} = 0 \quad \text{on } \partial\mathbb{R}^n_+, \tag{1.1}$$

where

$$p > 1,\ a \geq 0,\ n \geq 5, \quad \mathbb{R}^n_+ := \{x = (x', x_n), x' \in \mathbb{R}^{n-1}, x_n > 0\}, \quad \partial\mathbb{R}^n_+ := \{x \in \mathbb{R}^n_+, x_n = 0\}.$$

Liouville-type theorems and properties of the subcritical case have attracted much attention of scientists and many results were obtained. The most remarkable result on this aspect is the first Liouville-type theorem obtained by Gidas and Spruck [14], in which they proved that for $1 < p < \frac{n+2}{n-2}$ the problem

$$-\Delta u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n \tag{1.2}$$

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does not possess positive solutions. Moreover, this result is optimal in the sense that, for any $p \geq \frac{n+2}{n-2}$ and $n \geq 3$, there are infinitely many positive solutions to problem (1.2). Soon afterward, similar results were established in [13] for positive solutions of the subcritical problem in the half-space $\mathbb{R}^n_+$,

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n_+, \quad u = 0 \quad \text{on } \partial\mathbb{R}^n_+. \tag{1.3}$$

Later, Chen, and Li [3] obtained similar nonexistence results for the above two equations by using the moving plane method. These results received wide attention as regards the theory itself and its applications. Particularly, when variational methods cannot be employed, one uses them to establish a prior bound of solutions for general operator, and therefore, existence of solutions may be dealt with via topological methods; see, for instance, [5, 8, 11, 13, 14].

We note that the above-mentioned results only claim that the above equations do not possess positive solutions. In a so important paper [1], Bahri and Lions proved the nonexistence of sign-changing finite Morse index solutions of (1.2) or (1.3), provided that $1 < p < \frac{n+2}{n-2}$. Their proof is based on some integral estimates via Morse index combined with the Pohozaev identity. So, motivated by [13], they used blow-up argument to obtain a relevant L^∞-bound for solutions of semilinear boundary value problems in bounded domain from the boundedness of Morse index (see also [1, 10, 19]). We mention also that when the Palais–Smale; or the Cerami compactness conditions for the energy functional do not seem to follow readily, the proof of existence of solutions is essentially reduced to deriving L^∞-estimate from Liouville-type theorems via Morse index (see, for instance, [9, 26, 27]). After these works, many authors investigated various Liouville-type theorems for solutions with finite Morse indices in subcritical case such as problems with Neumann boundary condition, Dirichlet–Neumann mixed boundary conditions and nonlinear boundary conditions (see [2, 17–20, 29, 32, 33]).

In a famous paper [10], Farina completely classified finite Morse index solutions positive or sign-changing possibly unbounded. In particular, he proved that a smooth nontrivial solution to (1.5) exists if and only if $p \geq p_{JL1}(n)$ and $n \geq 11$, or $p = \frac{n+2}{n-2}$ and $n \geq 3$. Here, $p_{JL1}(n)$ denotes the so-called Joseph–Lundgren exponent (see [10,15]). In addition, similar results were established in [10] for finite Morse index solutions in the upper half-space $\mathbb{R}^n_+$ with homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}^n_+$. His proof makes a delicate application of the classical Moser iteration method. There exist many excellent papers to use the generalization of Moser's iteration technique to discuss the harmonic and fourth-order elliptic equation. (See [4, 6, 16, 28, 30, 31] and the references therein). However, the classical Moser's iterative technique may fail to obtain the similarly complete classification for the biharmonic equation:

$$\Delta^2 u = |u|^{p-1}u. \tag{1.4}$$

Recently, Davila, Dupaigne, Wang, and Wei [7] have derived a monotonicity formula and employed blow down analysis to reduce the nonexistence of nontrivial entire solutions for the problem (1.4) to that of nontrivial homogeneous solutions and gave a complete classification of stable solutions and those of finite Morse index solutions. Furthermore, in a recent paper [6], Dancer, Du, and Guo extended some results in [10] have considered

$$-\Delta u = |x|^a |u|^{p-1} u \quad \text{in } \mathbb{R}^n \tag{1.5}$$

with $a > -2$ they prove that (1.5) has no nontrivial stable solution in $\mathbb{R}^n$ if $1 < p < p_{JL1}(n,a)$ and that for $p \geq p_{JL1}(n,a)$, admits a positive radial stable solution in $\mathbb{R}^n$, where $p_{JL1}(n,a)$ is the Joseph–Lundgren exponent for the Hénon-type equation. In addition, Wang and Ye [28] obtained a Liouville-type result for finite Morse index solutions in $\mathbb{R}^n$, which is a partial extension of results in [6]. However, in case of the biharmonic equation, when $a > 0$, Hu [22] proved Liouville-type theorems for solutions belonging to one of the following classes: stable solutions and finite Morse index solutions positive or sign-changing. His proof is based on a combination of the Pohozaev-type identity, monotonicity formula of solutions and a blowing down sequence.

Relying on Hu's approach [22] and using the technics developed in [7, 10], we give Liouville-type theorems for solutions belonging to one of the following classes: stable solution and finite Morse index solutions of (1.1) possibly unbounded and sign-changing. In contrast with the results of Hu [22], we apply a new method to provide an implicit existence of the fourth-order Joseph–Lundgren exponent. Let us note that, to classify finite Morse index solutions in the supercritical case, we adopt a new method of monotonicity formula together with blowing down sequence. In addition, a difficulty stems from the fact that applying the doubling lemma leads to the singularity. For this reason, we use a more delicate approach to the interval $(n+4+2a, p_{JL2}(n,0))$. Before stating our results, we need to recall some definitions.

Definition 1.1. We say that a solution u of (1.1) belonging to $C^4(\overline{\mathbb{R}^n_+})$ has the following cases.

- It is stable if
$$Q_u(\psi) := \int_{\mathbb{R}^n_+} (\Delta\psi)^2 dx - p \int_{\mathbb{R}^n_+} |x|^a |u|^{p-1} \psi^2 dx \geq 0 \quad \forall \psi \in C_c^2(\overline{\mathbb{R}^n_+}). \tag{1.6}$$
- It is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n_+$ if $Q_u(\psi) \geq 0$ for any $\psi \in C_c^2(\overline{\mathbb{R}^n_+} \backslash \mathcal{K})$.
- It has a Morse index equal to $K \geq 1$ if K is the maximal dimension of a subspace X_K of $C_c^2(\overline{\mathbb{R}^n_+})$ such that $Q_u(\psi) < 0$ for any $\psi \in X_K \backslash \{0\}$.

Remark 1.1. (i) Clearly, a solution is stable if and only if its Morse index is equal to zero.

(ii) Any finite Morse index solution u is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n_+$. Indeed, there exist $K \geq 1$ and $X_K := \operatorname{span}\{\psi_1, \dots, \psi_K\} \subset C_c^2(\overline{\mathbb{R}^n_+})$ such that $Q_u(\psi) < 0$ for any $\psi \in X_K \backslash \{0\}$. Then, $Q_u(\psi) \geq 0$ for every $\psi \in C_c^2(\overline{\mathbb{R}^n_+} \backslash \mathcal{K})$, where

$$\mathcal{K} := \bigcup_{j=1}^{K} \operatorname{supp}(\psi_j).$$

Now, we can state our main results. For any fixed $a \geq 0$ and $n \geq 5$, we have the following theorem.

Theorem 1.1. *Let $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n_+ \backslash \{0\})$ be a homogeneous, stable solution of (1.1) in $\mathbb{R}^n_+ \backslash \{0\}$, $p \in (\frac{n+4+2a}{n-4}, p_{JL2}(n,a))$ and assume that $|x|^a |u|^{p+1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+ \backslash \{0\})$. Then, $u \equiv 0$.*

Theorem 1.2. *Let $u \in C^4(\overline{\mathbb{R}^n_+})$ be a stable solution of (1.1). If $1 < p < p_{JL2}(n,a)$, then $u \equiv 0$*

Theorem 1.3. *Let $u \in C^4(\overline{\mathbb{R}^n_+})$ be a solution of (1.1) that is stable outside a compact set.*

- *If $1 < p < p_{JL2}(n,0)$, $p \neq \frac{n+4+2a}{n-4}$, then $u \equiv 0$.*
- *If $p = \frac{n+4+2a}{n-4}$, then u has finite energy, i.e.,*

$$\int_{\mathbb{R}^n_+} (\Delta u)^2 = \int_{\mathbb{R}^n_+} |x|^a |u|^{p+1} < +\infty.$$

Here, the representation of $p_{JL2}(n,a)$ in Theorem 1.1 is the fourth-order Joseph–Lundgren exponent given by (2.1) below.

Remark 1.2. From Theorem 1.3 and the fact that any finite Morse index solution is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n_+$, we directly obtain that under the same assumption of Theorem 1.3, there is no finite Morse index solution to (1.1).

The proof of Theorem 1.2 or 1.3 is rather long and contains several technical aspects. The idea of the proof relies on some integral estimates together with blowing down sequence combined with a version of monotonicity formula of equation (1.1). We mention that the monotonicity formula is a powerful tool to understand supercritical elliptic equations or systems. This approach has been used successfully for Lane–Emden equation in [23].

This paper is organized as follows. In Section 1, we establish some finer integral estimates for the solutions of (1.1) which will be the key that we will use in the proofs of Theorems 1.2 and 1.3, and we construct a monotonicity formula which is a crucial tool to handle the supercritical case. Section 2 is devoted to the proof of 1.1. In Section 3, we prove Liouville-type theorem for stable solutions of (1.1), that is, Theorem 1.1. While in Section 4, we prove Theorem 1.3.

In the following, we use $B_r(x)$ to denote the open ball in $\mathbb{R}^n$ centered at x with radius r, we also write $B_r = B_r(0)$. C denotes a generic positive constant, which could be changed from one line to another.

2. Monotonicity formula and integral estimates

In this section, we construct a monotonicity formula and we establish various integral estimates of stable solutions which play an important role in dealing with Theorems 1.2 and 1.3.

To explore the main results in this paper, we need to provide an implicit existence of the fourth-order Joseph–Lundgren exponent for equation (1.1). For any fixed $a > -4$ and $n \geq 5$, we define

$$J_2 = \alpha(\alpha + 2)(n - 2 - \alpha)(n - 4 - \alpha)$$

and

$$\begin{aligned} F_a(\alpha) &= pJ_2 - \frac{n^2(n-4)^2}{16} \\ &= (\alpha + 4 + a)(\alpha + 2)(n - 2 - \alpha)(n - 4 - \alpha) - \frac{n^2(n-4)^2}{16}, \end{aligned}$$

where $\alpha = \frac{4+a}{p-1}$. Note that

$$\Big(p > \frac{n + 4 + 2a}{n - 4}\Big) \Longleftrightarrow \Big(0 < \alpha < \frac{n-4}{2}\Big),$$

F_a is increasing on $(0, \frac{n-4}{2})$. A direct computation finds

$$F_a\Big(\frac{n-4}{2}\Big) = \frac{n + 4 + 2a}{n - 4}\frac{n^2(n-4)^2}{16} - \frac{n^2(n-4)^2}{16} = \frac{2(4 + a)}{n - 4}\frac{n^2(n-4)^2}{16} > 0.$$

We also have

$$F_a(0) = \frac{(n-4)}{16}(-n^3 + 4n^2 + 32(a + 4)n - 64a - 256) = \frac{(n-4)}{16}E_a(n),$$

where

$$E_a(x) = -x^3 + 4x^2 + 32(a + 4)x - 64a - 256.$$

The function E_a satisfies the following properties:

(1) $E_a(5) > 0$, for all $a > -4$,

(2) $E_a''(x) = -6x + 8 < 0$ on $[5, +\infty)$,

(3) $\lim_{x\to+\infty} E_a(x) = -\infty$.

It follows that there exists a unique $x_a \in (5, +\infty)$ such that $E_a(x_a) = 0$ and $E_a(x) > 0$ on $[5, x_a)$. If we denote by $n(a)$ the integer part of x_a, then we have the following.

(i) $\forall n \leq n(a), E_a(n) > 0$. This implies that $F_a(0) > 0$. As a consequence $F_a(\alpha) > 0$ on $(0, \frac{n-4}{2})$.

(ii) $\forall n \geq n(a) + 1$, $E_a(n) < 0$. This yields $F_a(0) < 0$. Then, there exists a unique $\alpha_a \in (0, \frac{n-4}{2})$ such that $F_a(\alpha_a) = 0$.

For any fixed $a > -4$ and $n \geq 5$, we define

$$p_{JL2}(n, a) = \begin{cases} +\infty & \text{if } n \leq n(a), \\ p(n, a) & \text{if } n \geq n(a) + 1, \end{cases} \tag{2.1}$$

where $p(n,a) = \frac{4+a}{\alpha_a} + 1$. Therefore, we find that

$$pJ_2 > \frac{n^2(n-4)^2}{16}$$

for any $\frac{n+4+2a}{n-4} < p < p_{JL2}(n,a)$. In particular, if $a = 0$, then $F_0(\alpha_0 := \frac{4}{p_{JL2}(n,0)-1}) = 0$, where $p_{JL2}(n,0)$ in (2.1) is the fourth order Joseph–Lundgren exponent which is computed by Gazzola and Grunau [12]. See also Harrabi and Zaidi [21] in the study of the sixth-order for $a > 0$. Furthermore, $F_a(\alpha_0) > F_0(\alpha_0) = 0$, then $\alpha_0 > \alpha_a$ for all $n > n(a)$, this implies that $p_{JL2}(n,0) < p_{JL2}(n,a)$ for $a > -4$.

Next, we will establish a monotonicity formula. Equation (1.1) has two important features. It is variational, with the energy functional given by

$$\int \Big(\frac{1}{2}|\Delta u|^2 - \frac{1}{p+1}|x|^a|u|^{p+1}\Big).$$

For $\lambda > 0$, set $B_\lambda^+ = B_\lambda \cap \mathbb{R}^n_+$. Under the scaling transformation

$$u^\lambda(x) = \lambda^{\frac{4+a}{p-1}} u(\lambda x),$$

this suggests that the variation of the rescaled energy

$$\int_{B_1^+} \Big(\frac{1}{2}|\Delta u^\lambda|^2 - \frac{1}{p+1}|x|^a|u^\lambda|^{p+1}\Big).$$

For any given $x \in \mathbb{R}^n_+$, we choose $u \in W^{4,2}_{\mathrm{loc}}(\mathbb{R}^n_+) \cap L^{p+1}_{\mathrm{loc}}(\mathbb{R}^n_+)$ and define

$$\begin{aligned}
E(u,\lambda) = {} & \lambda^{\frac{4(p+1)+2a}{p-1}-n}\Big(\int_{B_\lambda^+} \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|x|^a|u|^{p+1}\Big) \\
& + \frac{4+a}{2(p-1)}\Big(n-2-\frac{4+a}{p-1}\Big)\lambda^{\frac{8+2a}{p-1}+1-n}\int_{\partial B_\lambda^+} u^2 \\
& + \frac{4+a}{2(p-1)}\Big(n-2-\frac{4+a}{p-1}\Big)\frac{d}{d\lambda}\Big(\lambda^{\frac{8+2a}{p-1}+2-n}\int_{\partial B_\lambda^+} u^2\Big) \\
& + \frac{\lambda^3}{2}\frac{d}{d\lambda}\Big[\lambda^{\frac{8+2a}{p-1}+1-n}\int_{\partial B_\lambda^+}\Big(\frac{4}{p-1}\lambda^{-1}u + \frac{\partial u}{\partial r}\Big)^2\Big] \\
& + \frac{1}{2}\frac{d}{d\lambda}\Big[\lambda^{\frac{8+2a}{p-1}+4-n}\int_{\partial B_\lambda^+}\Big(|\nabla u|^2 - |\frac{\partial u}{\partial r}|^2\Big)\Big] \\
& + \frac{1}{2}\lambda^{\frac{8+2a}{p-1}+3-n}\int_{\partial B_\lambda^+}\Big(|\nabla u|^2 - |\frac{\partial u}{\partial r}|^2\Big),
\end{aligned}$$

where derivatives are taken in the sense of distributions. Then, we have the following monotonicity formula.

Proposition 2.1. *Let $n \geq 5, a \geq 0$ and $p > \frac{n+4+2a}{n-4}$, $u \in W^{4,2}_{\mathrm{loc}}(\mathbb{R}^n_+)$ and $|x|^a|u|^{p+1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+)$ be a weak solution of (1.1). Then, $E(u,\lambda)$ is non-decreasing in $\lambda > 0$. Furthermore, there is a constant $C(n,p,a) > 0$ depending only on n, p and a such that*

$$\frac{d}{dr}E(u,\lambda) \geq C(n,p,a)\lambda^{-n+2+\frac{8+2a}{p-1}} \int_{\partial B^+_\lambda} \Big(\frac{4+a}{p-1}\lambda^{-1}u + \frac{\partial u}{\partial r}\Big)^2 dS.$$

Proof. The proof follows the main lines of the demonstration of [22, Theorem 2.1], with small modifications. Since the boundary integrals in $E(u,\lambda)$ only involve second order derivatives of u, the boundary integrals in $\frac{dE}{d\lambda}(u,\lambda)$ only involve third order derivatives of u. Thus, the following calculations can be rigorously verified. Assume that $x = 0$ and that the balls B_λ are all centered at 0. Take

$$\widetilde{E}(\lambda) = \lambda^{\frac{4(p+1)+2a}{p-1}-n} \int_{B^+_\lambda} \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|x|^a|u|^{p+1}.$$

Define

$$v = \Delta u, u^\lambda(x) = \lambda^{\frac{4+a}{p-1}}u(\lambda x), \quad v^\lambda(x) = \lambda^{\frac{4+a}{p-1}+2}v(\lambda x).$$

We still have $v^\lambda = \Delta u^\lambda$, $\Delta v^\lambda = |x|^a|u^\lambda|^{p-1}u^\lambda$, and by differentiating in λ,

$$\Delta\frac{du^\lambda}{d\lambda} = \frac{dv^\lambda}{d\lambda}.$$

Note that differentiation in λ commutes with differentiation and integration in x. A rescaling shows that

$$\widetilde{E}(\lambda) = \int_{B^+_1} \frac{1}{2}(v^\lambda)^2 - \frac{1}{p+1}|x|^a|u^\lambda|^{p+1}.$$

Then,

$$\begin{aligned}\frac{d}{d\lambda}\widetilde{E}(\lambda) &= \int_{B^+_1} v^\lambda\frac{dv^\lambda}{d\lambda} - |x|^a|u^\lambda|^{p-1}u^\lambda\frac{du^\lambda}{d\lambda}\\ &= \int_{B^+_1} v^\lambda\Delta\frac{du^\lambda}{d\lambda} - \Delta v^\lambda\frac{du^\lambda}{d\lambda} = \int_{\partial B^+_1} v^\lambda\frac{\partial}{\partial r}\frac{du^\lambda}{d\lambda} - \frac{\partial v^\lambda}{\partial r}\frac{du^\lambda}{d\lambda}.\end{aligned}$$

Since $u^\lambda = 0$ in $\partial\mathbb{R}^n_+$ for any $\lambda > 0$, then $\frac{du^\lambda}{d\lambda} = 0$ in $\partial\mathbb{R}^n_+$. Therefore, all boundary terms appearing in the integrations by parts vanish under the Dirichlet boundary conditions. So, we get

$$\frac{d}{d\lambda}\widetilde{E}(\lambda) = \int_{\partial B^+_1} \Big(v^\lambda\frac{\partial}{\partial r}\frac{du^\lambda}{d\lambda} - \frac{\partial v^\lambda}{\partial r}\frac{du^\lambda}{d\lambda}\Big). \tag{2.2}$$

In what follows, we express all derivatives of u^λ in the $r = |x|$ variable in terms of derivatives in the λ variable. In the definition of u^λ and v^λ, directly differentiating in λ gives

$$\frac{du^\lambda}{d\lambda}(x) = \frac{1}{\lambda}\Big(\frac{4+a}{p-1}u^\lambda(x) + r\frac{\partial u^\lambda}{\partial r}(x)\Big), \tag{2.3}$$

and

$$\frac{dv^\lambda}{d\lambda}(x) = \frac{1}{\lambda}\Big(\frac{2(p+1)+a}{p-1}v^\lambda(x) + r\frac{\partial v^\lambda}{\partial r}(x)\Big). \tag{2.4}$$

In (2.3), taking derivatives in λ once again, we get

$$\lambda\frac{d^2u^\lambda}{d\lambda^2}(x) + \frac{du^\lambda}{d\lambda}(x) = \frac{4+a}{p-1}\frac{du^\lambda}{d\lambda}(x) + r\frac{\partial}{\partial r}\frac{du^\lambda}{d\lambda}(x). \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.2), we obtain

$$\begin{aligned}\frac{d\widetilde{E}}{d\lambda} &= \int_{\partial B_1^+} v^\lambda\Big(\lambda\frac{d^2u^\lambda}{d\lambda^2} + \frac{p-5-a}{p-1}\frac{du^\lambda}{d\lambda}\Big) - \frac{du^\lambda}{d\lambda}\Big(\lambda\frac{dv^\lambda}{d\lambda} - \frac{2(p+1)+a}{p-1}v^\lambda\Big)\\ &= \int_{\partial B_1^+} \lambda v^\lambda\frac{d^2u^\lambda}{d\lambda^2} + 3v^\lambda\frac{du^\lambda}{d\lambda} - \lambda\frac{du^\lambda}{d\lambda}\frac{dv^\lambda}{d\lambda}. \end{aligned}\tag{2.6}$$

Observe that v^λ is expressed as a combination of x derivatives of u^λ. So, we also transform v^λ into λ derivatives of u^λ. By taking derivatives in r in (2.3) and noting (2.5), we get on ∂B_1^+, that

$$\begin{aligned}\frac{\partial^2u^\lambda}{\partial r^2} &= \lambda\frac{\partial}{\partial r}\frac{\partial u^\lambda}{\partial\lambda} - \frac{p+3+a}{p-1}\frac{\partial u^\lambda}{\partial r}\\ &= \lambda^2\frac{\partial^2u^\lambda}{\partial\lambda^2} + \frac{p-5-a}{p-1}\lambda\frac{du^\lambda}{d\lambda} - \frac{p+3+a}{p-1}\Big(\lambda\frac{du^\lambda}{d\lambda} - \frac{4+a}{p-1}u^\lambda\Big)\\ &= \lambda^2\frac{\partial^2u^\lambda}{\partial\lambda^2} - \frac{8+2a}{p-1}\lambda\frac{du^\lambda}{d\lambda} + \frac{(4+a)(p+3+a)}{(p-1)^2}u^\lambda.\end{aligned}$$

Then, on ∂B_1^+,

$$\begin{aligned}v^\lambda &= \frac{\partial^2u^\lambda}{\partial r^2} + \frac{n-1}{r}\frac{\partial u^\lambda}{\partial r} + \frac{1}{r^2}\Delta_\theta u^\lambda\\ &= \lambda^2\frac{d^2u^\lambda}{d\lambda^2} - \frac{8+2a}{p-1}\lambda\frac{du^\lambda}{d\lambda} + \frac{(4+a)(p+3+a)}{(p-1)^2}u^\lambda\\ &\quad + (n-1)\Big(\lambda\frac{du^\lambda}{d\lambda} - \frac{4+a}{p-1}u^\lambda\Big) + \Delta_\theta u^\lambda\\ &= \lambda^2\frac{d^2u^\lambda}{d\lambda^2} + \Big(n-1-\frac{8+2a}{p-1}\Big)\lambda\frac{du^\lambda}{d\lambda} + \frac{4+a}{p-1}\Big(\frac{4+a}{p-1} - n + 2\Big)u^\lambda + \Delta_\theta u^\lambda.\end{aligned}$$

Here, Δ_θ is the Laplace–Beltrami operator on ∂B_1 and below ∇_θ represents the tangential derivative on ∂B_1. For notational convenience, we also define the constants

$$\alpha = n-1-\frac{8+2a}{p-1}, \quad \beta = \frac{4+a}{p-1}\Big(\frac{4+a}{p-1} - n + 2\Big).$$

Now, (2.6) reads

$$\frac{d}{d\lambda}\widetilde{E}(\lambda) := I_1 + I_2,$$

where

$$I_1 := \int_{\partial B_1^+} \lambda(\lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + \alpha\lambda \frac{du^\lambda}{d\lambda} + \beta u^\lambda) \frac{d^2u^\lambda}{d\lambda^2}$$
$$+ 3\Big(\lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + \alpha\lambda \frac{du^\lambda}{d\lambda} + \beta u^\lambda\Big) \frac{du^\lambda}{d\lambda} - \lambda \frac{du^\lambda}{d\lambda} \frac{d}{d\lambda}\Big(\lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + \alpha\lambda \frac{du^\lambda}{d\lambda} + \beta u^\lambda\Big)$$

and

$$I_2 := \int_{\partial B_1^+} \lambda \Delta_\theta u^\lambda \frac{d^2u^\lambda}{d\lambda^2} + 3\Delta_\theta u^\lambda \frac{du^\lambda}{d\lambda} - \lambda \frac{du^\lambda}{d\lambda} \Delta_\theta \frac{du^\lambda}{d\lambda}.$$

Let $\lambda > 0$. Since $\frac{du^\lambda}{d\lambda} = 0$ in $\partial\mathbb{R}^n_+$, then all boundary terms appearing in the integrations by parts vanish under the Dirichlet boundary conditions; hence, the calculations are even easier. The integral I_2 can be estimated as

$$\begin{aligned}
I_2 &= \int_{\partial B_1^+} -\lambda \nabla_\theta u^\lambda \nabla_\theta \frac{d^2u^\lambda}{d\lambda^2} - 3\nabla_\theta u^\lambda \nabla_\theta \frac{du^\lambda}{d\lambda} + \lambda \Big|\nabla_\theta \frac{du^\lambda}{d\lambda}\Big|^2 \\
&= -\frac{\lambda}{2} \frac{d^2}{d\lambda^2}\Big(\int_{\partial B_1^+} |\nabla_\theta u^\lambda|^2\Big) - \frac{3}{2}\frac{d}{d\lambda}\Big(\int_{\partial B_1^+} |\nabla_\theta u^\lambda|^2\Big) + 2\lambda \int_{\partial B_1^+} \Big|\nabla_\theta \frac{du^\lambda}{d\lambda}\Big|^2 \\
&= -\frac{1}{2} \frac{d^2}{d\lambda^2}\Big(\lambda \int_{\partial B_1^+} |\nabla_\theta u^\lambda|^2\Big) - \frac{1}{2}\frac{d}{d\lambda}\Big(\int_{\partial B_1^+} |\nabla_\theta u^\lambda|^2\Big) + 2\lambda \int_{\partial B_1^+} \Big|\nabla_\theta \frac{du^\lambda}{d\lambda}\Big|^2 \\
&\geq -\frac{1}{2} \frac{d^2}{d\lambda^2}\Big(\lambda \int_{\partial B_1^+} |\nabla_\theta u^\lambda|^2\Big) - \frac{1}{2}\frac{d}{d\lambda}\Big(\int_{\partial B_1^+} |\nabla_\theta u^\lambda|^2\Big).
\end{aligned}$$

Furthermore, a direct calculation implies that

$$\begin{aligned}
I_1 &= \int_{\partial B_1^+} \lambda^3 \Big(\frac{d^2u^\lambda}{d\lambda^2}\Big)^2 + \lambda^2 \frac{d^2u^\lambda}{d\lambda^2}\frac{du^\lambda}{d\lambda} + \beta\lambda u^\lambda \frac{d^2u^\lambda}{d\lambda^2} + 3\beta u^\lambda \frac{du^\lambda}{d\lambda} \\
&\quad + (2\alpha - \beta)\lambda \Big(\frac{du^\lambda}{d\lambda}\Big)^2 - \lambda^3 \frac{du^\lambda}{d\lambda}\frac{d^3u^\lambda}{d\lambda^3} \\
&= \int_{\partial B_1^+} 2\lambda^3 \Big(\frac{d^2u^\lambda}{d\lambda^2}\Big)^2 + 4\lambda^2 \frac{d^2u^\lambda}{d\lambda^2}\frac{du^\lambda}{d\lambda} + (2\alpha - 2\beta)\lambda\Big(\frac{du^\lambda}{d\lambda}\Big)^2 + \frac{\beta}{2}\frac{d^2}{d\lambda^2}[\lambda(u^\lambda)^2] \\
&\quad + \frac{\beta}{2}\frac{d}{d\lambda}(u^\lambda)^2 - \frac{1}{2}\frac{d}{d\lambda}\Big[\lambda^3 \frac{d}{d\lambda}\Big(\frac{du^\lambda}{d\lambda}\Big)^2\Big].
\end{aligned}$$

Here, we have used the relations (writing $f' = \frac{d}{d\lambda} f$, etc.)

$$\lambda f f'' = \Big(\frac{\lambda}{2} f^2\Big)'' - 2ff' - \lambda(f')^2$$

and

$$-\lambda^3 f' f''' = -\Big[\frac{\lambda^3}{2}((f')^2)'\Big]' + 3\lambda^2 f' f'' + \lambda^3 (f'')^2.$$

Since $p > \frac{n+4+2a}{n-4}$, direct calculations show that

$$\alpha - \beta = \Big(n - 1 - \frac{8+2a}{p-1}\Big) - \frac{4+a}{p-1}\Big(\frac{4+a}{p-1} - n + 2\Big) > 1.$$

Consequently,

$$\begin{aligned} 2\lambda^3\Big(\frac{d^2u^\lambda}{d\lambda^2}\Big)^2 &+ 4\lambda^2\frac{d^2u^\lambda}{d\lambda^2}\frac{du^\lambda}{d\lambda} + (2\alpha - 2\beta)\lambda\Big(\frac{du^\lambda}{d\lambda}\Big)^2 \\ &= 2\lambda\Big(\lambda\frac{d^2u^\lambda}{d\lambda^2} + \frac{du^\lambda}{d\lambda}\Big)^2 + (2\alpha - 2\beta - 2)\lambda\Big(\frac{du^\lambda}{d\lambda}\Big)^2 \geq 0. \end{aligned}$$

Then, we conclude that

$$I_1 \geq \int_{\partial B_1^+} \frac{\beta}{2}\frac{d^2}{d\lambda^2}[\lambda(u^\lambda)^2] - \frac{1}{2}\frac{d}{d\lambda}\Big[\lambda^3\frac{d}{d\lambda}\Big(\frac{du^\lambda}{d\lambda}\Big)^2\Big] + \frac{\beta}{2}\frac{d}{d\lambda}(u^\lambda)^2.$$

Now, rescaling back, we can write those λ derivatives in I_1 and I_2 as follows:

$$\int_{\partial B_1^+} \frac{d}{d\lambda}(u^\lambda)^2 = \frac{d}{d\lambda}\Big(\lambda^{\frac{8+2a}{p-1}+1-n}\int_{\partial B_\lambda^+} u^2\Big),$$

$$\int_{\partial B_1^+} \frac{d^2}{d\lambda^2}[\lambda(u^\lambda)^2] = \frac{d^2}{d\lambda^2}\Big(\lambda^{\frac{8+2a}{p-1}+2-n}\int_{\partial B_\lambda^+} u^2\Big),$$

$$\int_{\partial B_1^+} \frac{d}{d\lambda}\Big[\lambda^3\frac{d}{d\lambda}\Big(\frac{du^\lambda}{d\lambda}\Big)^2\Big] = \frac{d}{d\lambda}\Big[\lambda^3\frac{d}{d\lambda}\Big(\lambda^{\frac{8+2a}{p-1}+1-n}\int_{\partial B_\lambda^+}\Big(\frac{4+a}{p-1}\lambda^{-1}u + \frac{\partial u}{\partial r}\Big)^2\Big)\Big],$$

$$\frac{d^2}{d\lambda^2}\Big(\lambda\int_{\partial B_1^+} |\nabla_\theta u^\lambda|^2\Big) = \frac{d^2}{d\lambda^2}\Big[\lambda^{1+\frac{8+2a}{p-1}+2+1-n}\int_{\partial B_\lambda^+}\Big(|\nabla u|^2 - |\frac{\partial u}{\partial r}|^2\Big)\Big],$$

and

$$\frac{d}{d\lambda}\Big(\int_{\partial B_1^+} |\nabla_\theta u^\lambda|^2\Big) = \frac{d}{d\lambda}\Big[\lambda^{\frac{8+2a}{p-1}+2+1-n}\int_{\partial B_\lambda^+}\Big(|\nabla u|^2 - |\frac{\partial u}{\partial r}|^2\Big)\Big].$$

Substituting these into $\frac{d}{d\lambda}E(u,\lambda)$, we finish the proof. ∎

For $\beta > 0$, set $B_\beta^+ = B_\beta \cap \mathbb{R}^n_+$ and $A_\beta^+ = \{x \in \mathbb{R}^n_+, a_1\beta < |x| < a_2\beta\}$ for some $0 < a_1 < a_2$. Let u be a solution of (1.1), which is stable outside a compact set $\mathcal{K} \subset B_{R_0}^+$. For all $R > 4R_0$, we define a family of test functions $\psi = \psi_{(R,R_0)} \in C_c^2(\mathbb{R}^N)$ satisfying

$$\begin{cases} 0 \leq \psi \leq 1 \text{ and } \psi \equiv 0 & \text{if } |x| < R_0 \text{ or } |x| > 2R, \\ \psi \equiv 1 & \text{if } 2R_0 < |x| < R, \\ |\nabla^q\psi| \leq CR_0^{-q} & \text{if } R_0 < |x| < 2R_0, \\ |\nabla^q\psi| \leq CR^{-q} & \text{if } R < |x| < 2R \text{ and } 1 \leq q \leq 4. \end{cases} \tag{2.7}$$

Similarly, if u is a stable solution of (1.1), then $\psi = \psi_{(R)}$, with $R > 0$ verifying (2.7) with $R_0 = 0$; that is, $\psi = 1$ if $|x| < R$. Then, we have the following integral estimates.

Lemma 2.1. *Let $u \in C^4(\overline{\mathbb{R}^n_+})$ be a solution of (1.1), which is stable outside a compact set $\mathcal{K}$. Let $R_0 > 0$ such that $\mathcal{K} \subset B^+_{R_0}$ and set $v = \Delta u$, there hold the following:*

$$\int_{B_R^+} v^2 + \int_{B_R^+} |x|^a |u|^{p+1} \le C_0 + CR^{n - \frac{4(p+1)+2a}{p-1}} \quad \forall R > 4R_0 \tag{2.8}$$

and

$$\int_{B_R^+} v^2 + \int_{B_R^+} |x|^a |u|^{p+1} \le C_0 + CR^{-4} \int_{A_R^+} u^2 + CR^{-2} \int_{A_R^+} |uv| \quad \forall R > 4R_0, \tag{2.9}$$

where C_0 and C are positive constants independent of R.

Proof of (2.8). First, for $\varepsilon \in (0,1)$ and $\eta \in C^2(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n_+} [\Delta(u\eta)]^2 &= \int_{\mathbb{R}^n_+} (u\Delta\eta + 2\nabla u \nabla\eta + \eta\Delta u)^2 \\ &\le (1 + C\varepsilon) \int_{\mathbb{R}^n_+} v^2\eta^2 + \frac{C}{\varepsilon} \int_{\mathbb{R}^n_+} u^2(\Delta\eta)^2 + \frac{C}{\varepsilon} \int_{\mathbb{R}^n_+} |\nabla u|^2 |\nabla\eta|^2.\end{aligned}$$

Using

$$\Delta(u^2) = 2|\nabla u|^2 + 2u\Delta u$$

yields

$$2\int_{\mathbb{R}^n_+} |\nabla u|^2 |\nabla\eta|^2 = \int_{\mathbb{R}^n_+} u^2 \Delta(|\nabla\eta|^2) - 2\int_{\mathbb{R}^n_+} uv|\nabla\eta|^2. \tag{2.10}$$

So, we get

$$\int_{\mathbb{R}^n_+} [\Delta(u\eta)]^2 \le (1 + C\varepsilon) \int_{\mathbb{R}^n_+} v^2\eta^2 + \frac{C}{\varepsilon} \int_{\mathbb{R}^n_+} u^2\big[(\Delta\eta)^2 + |\Delta(|\nabla\eta|^2)|\big] + \frac{C}{\varepsilon} \int_{\mathbb{R}^n_+} |uv||\nabla\eta|^2. \tag{2.11}$$

Take $\eta = \eta^m$ with $m \ge 2$. Apply Cauchy–Schwarz's inequality, we get

$$\int_{\mathbb{R}^n_+} |uv||\nabla\eta^m|^2 \le C\varepsilon^2 \int_{\mathbb{R}^n_+} v^2\eta^{2m} + C_{\varepsilon,m} \int_{\mathbb{R}^n_+} u^2 |\nabla\eta|^4 \eta^{2m-4}. \tag{2.12}$$

Substitute η by ψ^m in (2.11), then from (2.12) and (2.7), we obtain

$$\int_{B_{2R}^+} [\Delta(u\psi^m)]^2 \le C_0 + (1 + C\varepsilon) \int_{B_{2R}^+} v^2\psi^{2m} + C_\varepsilon R^{-4} \int_{A_R^+} u^2,$$

where

$$C_0 = CR^{-4} \int_{A_0^+} u^2,$$

$$A_0^+ = \{x \in \mathbb{R}^n_+, R_0 < |x| < 2R_0\}.$$

Let u be a solution of (1.1), which is stable outside a compact set $\mathcal{K} \subset B^+_{R_0}$. Clearly, $u\psi^m \in H^2_0(B^+_{2R} \setminus B^+_{R_0})$, so after a standard approximation argument the main inequality of stability (1.6) implies that

$$p \int_{B^+_{2R}} |x|^a |u|^{p+1} \psi^{2m} - \int_{B^+_{2R}} (\Delta(u\psi^m))^2 \leq 0 \quad \forall R > 4R_0.$$

Therefore, we conclude that

$$p \int_{B^+_{2R}} |x|^a |u|^{p+1} \psi^{2m} - (1 + C\varepsilon) \int_{B^+_{2R}} v^2 \psi^{2m} \leq C_0 + C_\varepsilon R^{-4} \int_{A^+_R} u^2. \tag{2.13}$$

On the other hand, recall that $u = \frac{\partial u}{\partial x_n} = 0$ in $\partial \mathbb{R}^n_+$, then multiply equation (1.1) by $u\eta^2$, $\eta \in C^2(\mathbb{R}^N)$ and integrate by parts, using again (2.10), we derive

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} \left[v^2\eta^2 - |x|^a |u|^{p+1} \eta^2\right] \\
&= -4 \int_{\mathbb{R}^n_+} \eta v \nabla u \cdot \nabla \eta - 2 \int_{\mathbb{R}^n_+} \eta u v \Delta \eta - 2 \int_{\mathbb{R}^n_+} uv |\nabla \eta|^2 \\
&\leq C\varepsilon \int_{\mathbb{R}^n_+} v^2\eta^2 + C_\varepsilon \int_{\mathbb{R}^n_+} u^2 (\Delta\eta)^2 + C_\varepsilon \int_{\mathbb{R}^n_+} |\nabla u|^2 |\nabla \eta|^2 - 2 \int_{\mathbb{R}^n_+} uv |\nabla \eta|^2 \\
&\leq C\varepsilon \int_{\mathbb{R}^n_+} v^2\eta^2 + C_\varepsilon \int_{\mathbb{R}^n_+} u^2 \left[(\Delta\eta)^2 + |\Delta(|\nabla\eta|^2)\right] + C_\varepsilon \int_{\mathbb{R}^n_+} |uv| |\nabla\eta|^2.
\end{aligned} \tag{2.14}$$

Using the above inequality (where one substitutes η by ψ^m), it follows from (2.12) and (2.7) that

$$(1 - C\varepsilon) \int_{B^+_{2R}} v^2 \psi^{2m} - \int_{B^+_{2R}} |x|^a |u|^{p+1} \psi^{2m} \leq C_0 + C_\varepsilon R^{-4} \int_{A^+_R} u^2. \tag{2.15}$$

Taking $\varepsilon > 0$ small enough, multiplying (2.15) by $\frac{1+2C\varepsilon}{1-C\varepsilon}$, and adding it with (2.13) we then get

$$C\varepsilon \int_{B^+_{2R}} v^2 \psi^{2m} + (p - \frac{1 + 2C\varepsilon}{1 - C\varepsilon}) \int_{B^+_{2R}} |x|^a |u|^{p+1} \psi^{2m} \leq C_0 + C_\varepsilon R^{-4} \int_{A^+_R} u^2.$$

As $p > 1$ and $A^+_R \subset B^+_{2R}$, using $\varepsilon > 0$ small enough, there holds that

$$\int_{B^+_R} v^2 + \int_{B^+_R} |x|^a |u|^{p+1} \leq C_0 + CR^{-4} \int_{A^+_R} u^2.$$

Applying Young's inequality, we deduce then for any $\varepsilon' > 0$ that

$$\int_{B^+_R} v^2 + (1 - \varepsilon') \int_{B^+_R} |x|^a |u|^{p+1} \leq C_0 + CR^{n - \frac{4(p+1)+2a}{p-1}} \quad \forall R > 4R_0.$$

Taking $\varepsilon' > 0$ small enough, the estimate (2.8) is proved. ∎

Proof of (2.9). Invoking now (2.11) where we substitute η by ψ^m, we obtain

$$\int_{B_{2R}^+}[\Delta(u\psi^m)]^2 \le C_0 + (1+C\varepsilon)\int_{B_{2R}^+} v^2\psi^{2m} + C_\varepsilon R^{-4}\int_{A_R^+} u^2 + C_\varepsilon R^{-2}\int_{A_R^+}|uv|.$$

Adopting a similar argument as above where we uses the equality (2.14) and inequality of stability (1.6), we obtain readily the estimate equality (2.9). Thus, Lemma 2.1 is well proved. ∎

3. Proof of Theorem 1.1

In this section, we obtain a nonexistence result for a homogeneous stable solution of (1.1). We have the following lemma.

Lemma 3.1. *Let $n \ge 5, a > 0$, we define*

$$J_1 = (\alpha+2)(n-4-\alpha) + \alpha(n-2-\alpha).$$

If $p \in (\frac{n+4+2a}{n-4}, p_{JL2}(n,a))$, then we have

$$J_1 > 0,\, J_2 > 0,\, pJ_1 > \frac{n(n-4)}{2}$$

and

$$pJ_2 > \frac{n^2(n-4)^2}{16}.$$

Proof. Since

$$p > \frac{n+4+2a}{n-4} > \frac{n+4}{n-4},$$

then

$$J_1 > 0 \quad \text{and} \quad J_2 > 0.$$

For $\frac{n+4+2a}{n-4} < p < p_{JL2}(n,a)$, we get from the definition of $p_{JL2}(n,a)$ that

$$pJ_2 > \frac{n^2(n-4)^2}{16}. \tag{3.1}$$

From (3.1), we obtain

$$2^2p^2J_2 > \left(\frac{1}{2}\right)^2 n^2(n-4)^2. \tag{3.2}$$

Using inequality $\sqrt{xy} \le \frac{1}{2}(x+y)$ for all $x \ge 0, y \ge 0$ with $x = (\alpha+2)(n-4-\alpha)$ and $y = \alpha(n-2-\alpha)$, we derive

$$2^2J_2 < (J_1)^2. \tag{3.3}$$

The last inequality combined with (3.2), yields

$$pJ_1 > \frac{1}{2}n(n-4). \tag{3.4}$$

This finishes the proof of Lemma 3.1. ∎

Let u be a homogeneous solution of (1.1); that is, there exists a $w \in W^{2,2}(\mathbb{S}^{n-1}_+)$ such that in polar coordinates

$$u(r,\theta) = r^{-\frac{4+a}{p-1}} w(\theta).$$

Denote $A^+_R = B^+_{2R} \backslash B^+_R$. Since $u \in W^{2,2}(A^+_1)$ and $|x|^a |u|^{p+1} \in L^1(A^+_1)$, it implies that $w \in W^{2,2}(\mathbb{S}^{n-1}_+) \cap L^{p+1}(\mathbb{S}^{n-1}_+)$. A direct calculation gives

$$\Delta^2_\theta w(\theta) - J_1 \Delta_\theta w(\theta) + J_2 w(\theta) = |w|^{p-1} w \quad \text{in } \mathbb{S}^{n-1}_+, \quad w = \frac{\partial w}{\partial \theta_n} = 0 \quad \text{on } \partial \mathbb{S}^{n-1}_+, \tag{3.5}$$

where

$$J_1 = \Big(\frac{4+a}{p-1} + 2\Big)\Big(n - 4 - \frac{4+a}{p-1}\Big) + \frac{4+a}{p-1}\Big(n - 2 - \frac{4+a}{p-1}\Big)$$

and

$$J_2 = \frac{4+a}{p-1}\Big(\frac{4+a}{p-1} + 2\Big)\Big(n - 4 - \frac{4+a}{p-1}\Big)\Big(n - 2 - \frac{4+a}{p-1}\Big).$$

Because $w \in W^{2,2}(\mathbb{S}^{n-1}_+)$, we can test (3.5) with w to obtain

$$\int_{\mathbb{S}^{n-1}_+} (\Delta_\theta w)^2 + J_1 |\nabla_\theta w|^2 + J_2 w^2 d\theta = \int_{\mathbb{S}^{n-1}_+} |w|^{p+1} d\theta. \tag{3.6}$$

As in [7], for any $\varepsilon > 0$, choose $\eta_\varepsilon \in C^\infty_0((\frac{\varepsilon}{2}, \frac{2}{\varepsilon}))$ such that $\eta_\varepsilon \equiv 1$ in $(\varepsilon, \frac{1}{\varepsilon})$, and

$$r|\eta'_\varepsilon(r)| + r^2 |\eta''_\varepsilon(r)| \le 64 \quad \forall r > 0.$$

Let $\Omega_k = B_{2k/\varepsilon} \backslash B_{\varepsilon/2k}$, since $w \in W^{2,2}(\mathbb{S}^{n-1}_+) \cap L^{p+1}(\mathbb{S}^{n-1}_+)$, $r^{-\frac{n-4}{2}} w(\theta)\eta_\varepsilon(r)$ can be approximated by $C^\infty_0(\Omega_2 \cap \mathbb{R}^n_+)$ functions in $W^{2,2}(\Omega_1 \cap \mathbb{R}^n_+) \cap L^{p+1}(\Omega_1 \cap \mathbb{R}^n_+)$. Hence, in the stability condition for u, we are allowed to choose a test function of the form

$$r^{-\frac{n-4}{2}} w(\theta)\eta_\varepsilon(r).$$

Direct calculations show that

$$\begin{aligned}\Delta(r^{-\frac{n-4}{2}} w(\theta)\eta_\varepsilon(r)) = &-\frac{n(n-4)}{4} r^{-\frac{n}{2}} w(\theta)\eta_\varepsilon(r) + 3r^{-\frac{n}{2}+1} w(\theta)\eta'_\varepsilon(r) \\ &+ r^{-\frac{n}{2}+2} w(\theta)\eta''_\varepsilon(r) + r^{-\frac{n}{2}} \Delta_\theta w(\theta)\eta_\varepsilon(r).\end{aligned} \tag{3.7}$$

Substituting (3.7) into the stability condition for u, we deduce that

$$\begin{aligned} &p\Big(\int_{\mathbb{S}^{n-1}_+} |w|^{p+1} d\theta\Big)\Big(\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2 dr\Big) \\ &\le \Big(\int_{\mathbb{S}^{n-1}_+} \Big((\Delta_\theta w)^2 + \frac{n(n-4)}{2}|\nabla_\theta w|^2 + \frac{n^2(n-4)^2}{16} w^2\Big) d\theta\Big)\Big(\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2 dr\Big) \\ &\quad + O\Big[\int_0^{+\infty} \big(r\eta'_\varepsilon(r)^2 + r^3\eta''_\varepsilon(r)^2 + \eta_\varepsilon(r)|\eta'_\varepsilon(r)| + r\eta_\varepsilon(r)|\eta''_\varepsilon(r)|\big) dr \\ &\qquad \times \int_{\mathbb{S}^{n-1}_+} (|\nabla_\theta w(\theta)|^2 + w(\theta)^2) d\theta\Big]. \end{aligned}$$

Note that

$$\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2 dr \geq |\log\varepsilon|,$$
$$\int_0^{+\infty} (r\eta_\varepsilon'(r)^2 + r^3\eta_\varepsilon''(r)^2 + \eta_\varepsilon(r)|\eta_\varepsilon'(r)| + r\eta_\varepsilon(r)|\eta_\varepsilon''(r)|)dr \leq C$$

for some constant C independent of ε. By letting $\varepsilon \to 0$, we obtain

$$p\int_{\mathbb{S}_+^{n-1}} |w|^{p+1}d\theta \leq \int_{\mathbb{S}_+^{n-1}} (\Delta_\theta w)^2 + \frac{n(n-4)}{2}|\nabla_\theta w|^2 + \frac{n^2(n-4)^2}{16}w^2 d\theta. \tag{3.8}$$

Substituting (3.6) into (3.8), we derive

$$\int_{\mathbb{S}_+^{n-1}} (p-1)(\Delta_\theta w)^2 + \Big(pJ_1 - \frac{n(n-4)}{2}\Big)|\nabla_\theta w|^2 + \Big(pJ_2 - \frac{n^2(n-4)^2}{16}\Big)w^2 d\theta \leq 0.$$

Finally, by Lemma 3.1, we observe that $w \equiv 0$. Then, it follows that $u \equiv 0$.

4. Proof of Theorem 1.2

For the case $1 < p \leq \frac{n+4+2a}{n-4}$, we apply the integral estimates. For the case $\frac{n+4+2a}{n-4} < p < p_{JL2}(n,a)$ with the energy estimates and the desired monotonicity formula we can show that the stable solutions must be homogeneous solutions; hence, by applying the classification of the homogeneous solutions (see Theorem1.1), the solutions must be zero.

Since we assume that u is a stable solution, then the integral estimate (2.8) holds with $C_0 = 0$. We divide the proof into three cases.

Case 1. The subcritical $1 < p < \frac{n+4+2a}{n-4}$.

Applying (2.8), we deduce that

$$\int_{B_R^+} v^2 + \int_{B_R^+} |x|^a|u|^{p+1} \leq CR^{n-\frac{4(p+1)+2a}{p-1}} \to 0 \quad \text{as } R \to +\infty.$$

Consequently, we obtain $u \equiv 0$.

Case 2. The critical $p = \frac{n+4+2a}{n-4}$.

Applying again (2.8), we have

$$\int_{\mathbb{R}_+^n} v^2 + |x|^a|u|^{p+1} < +\infty.$$

So, we get

$$\lim_{R\to+\infty}\int_{A_R^+} v^2 + |x|^a|u|^{p+1} \equiv 0.$$

Now, using Hölder's inequality, we derive that

$$R^{-4}\int_{A_R^+} u^2 \leq CR^{-4}\Big(\int_{A_R^+}|x|^a|u|^{p+1}\Big)^{\frac{2}{p+1}}\Big(\int_{A_R^+}|x|^{\frac{-2a}{p-1}}\Big)^{\frac{p-1}{p+1}}.$$

Therefore, from (2.9), we conclude that

$$\int_{B_R^+} v^2+|x|^a|u|^{p+1} \leq CR^{(n-\frac{2a}{p-1})\frac{p-1}{p+1}-4}\Big(\int_{A_R^+}|x|^a|u|^{p+1}\Big)^{\frac{2}{p+1}}+C\int_{A_R^+} v^2.$$

Under the assumptions $p=\frac{n+4+2a}{n-4}$, tending $R\to+\infty$, we obtain $u\equiv 0$.

Case 3. The supercritical $\frac{n+4+2a}{n-4}<p<p_{JL2}(n,a)$.

We define blowing down sequences

$$u^\lambda(x)=\lambda^{\frac{4+a}{p-1}}u(\lambda x),\quad v^\lambda(x)=\lambda^{\frac{4+a}{p-1}+2}v(\lambda x)\quad \forall\lambda>0.$$

u^λ is also a smooth stable solution of (1.1) on $\mathbb{R}^n_+$. By rescaling (2.8), for all $\lambda>0$ and balls $B_r\subset\mathbb{R}^n$,

$$\int_{B_r^+}(v^\lambda)^2+|x|^a|u^\lambda|^{p+1}\leq Cr^{n-\frac{4(p+1)+2a}{p-1}}.$$

In particular, u^λ are uniformly bounded in $L^{p+1}_{\text{loc}}(\mathbb{R}^n_+)$. By elliptic estimates, u^λ are also uniformly bounded in $W^{2,2}_{\text{loc}}(\mathbb{R}^n_+)$. Hence, up to a subsequence of $\lambda\to+\infty$, we can assume that $u^\lambda\to u^\infty$ weakly in $W^{2,2}_{\text{loc}}(\mathbb{R}^n_+)\cap L^{p+1}_{\text{loc}}(\mathbb{R}^n_+)$. By compactness embedding, one has $u^\lambda\to u^\infty$ strongly in $W^{2,2}_{\text{loc}}(\mathbb{R}^n_+)$. Then, for any ball $B_R^+(0)$, by interpolation between L^q spaces and noting (2.8), for any $q\in[1,p+1)$, as $\lambda\to+\infty$, we have

$$\|u^\lambda-u^\infty\|_{L^q(B_R^+(0))}\leq\|u^\lambda-u^\infty\|^{\mu}_{L^1(B_R^+(0))}\|u^\lambda-u^\infty\|^{1-\mu}_{L^{p+1}(B_R^+(0))}\to 0, \tag{4.1}$$

where

$$\frac{1}{q}=\mu+\frac{1-\mu}{p+1}.$$

That is, $u^\lambda\to u^\infty$ in $L^q_{\text{loc}}(\mathbb{R}^n_+)$ for any $q\in(1,p+1)$.

For any function $\zeta\in C_0^\infty(\mathbb{R}^n_+)$, we have

$$\int_{\mathbb{R}^n_+}\Delta u^\infty\Delta\zeta-|x|^a|u^\infty|^{p-1}u^\infty\zeta=\lim_{\lambda\to\infty}\int_{\mathbb{R}^n_+}\Delta u^\lambda\Delta\zeta-|x|^a|u^\lambda|^{p-1}u^\lambda\zeta,$$

$$\int_{\mathbb{R}^n_+}(\Delta\zeta)^2-p|x|^a|u^\infty|^{p-1}(\zeta)^2=\lim_{\lambda\to\infty}\int_{\mathbb{R}^n_+}(\Delta\zeta)^2-p|x|^a|u^\lambda|^{p-1}(\zeta)^2\geq 0.$$

Thus,

$$u^\infty\in W^{2,2}_{\text{loc}}(\mathbb{R}^n_+)\cap L^{p+1}_{\text{loc}}(\mathbb{R}^n_+)$$

is a stable solution of (1.1).

Now, we can follow exactly the proof of Lemmas 3.1–3.3 in Hu [22], (see also Lemmas 4.4–4.6 in Dávila et al. [7]), to obtain the following lemma.

Lemma 4.1. *We have the following:*

(1) $\lim_{\lambda\to+\infty} E(u,\lambda) < +\infty$,

(2) u^∞ *is homogeneous,*

(3) $\lim_{r\to+\infty} E(u,r) = 0$.

Therefore, by the monotonicity formula, we know that u is homogeneous, then by Proposition 2.1 and using the classification of the homogeneous solutions given by Theorem 1.1, we get $u \equiv 0$. This finishes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

We proceed based on a Pohozaev-type identity, the decay estimates from the doubling lemma [24], the monotonicity formula, and the classification of the homogeneous solutions and stable solutions. The proof is divided into three cases.

Case 1. The subcritical $1 < p < \frac{n+4+2a}{n-4}$.

The proof is based on the following Pohozaev-type identity. More precisely, we start by testing the equation (1.1) against by $\nabla u \cdot x\psi$, where $\psi \in C_c^2(\mathbb{R}^N)$, $0 \le \psi \le 1$, are cut-off functions satisfying

$$\begin{cases} \psi \equiv 1 & \text{if } |x| < R, \quad \psi \equiv 1 \quad \text{if } |x| > 2R, \\ |\nabla^q \psi| \le CR^{-q} & \text{if } x \in A_R = \{R < |x| < 2R\},\ q \le 2. \end{cases} \tag{5.1}$$

Then, in view of the cut-off functions ψ, we can avoid the spherical integrals raised in [3, 25], which are very difficult to control and we have the following lemma.

Lemma 5.1. *Let u be a solution of* (1.1) *and set $v = \Delta u$. Then, for any $\psi \in C_c^2(B_{2R}^+)$,*

$$\begin{aligned} &\frac{n+a}{p+1}\int_{B_{2R}^+} |x|^a|u|^{p+1}\psi - \frac{n-4}{2}\int_{B_{2R}^+} v^2\psi \\ &\quad = -\frac{1}{p+1}\int_{B_{2R}^+} |x|^a|u|^{p+1}(\nabla\psi\cdot x) + \frac{1}{2}\int_{B_{2R}^+}(\nabla\psi\cdot x)v^2 \\ &\qquad - \int_{B_{2R}^+}\big[2v(\nabla u\cdot\nabla\psi) + 2v\nabla^2 u(x,\nabla\psi) + v(\nabla u\cdot x)\Delta\psi\big]. \end{aligned} \tag{5.2}$$

Proof. Let $\psi \in C_c^2(B_{2R}^+)$, multiplying equation (1.1) by $\nabla u \cdot x\psi$ and integrating by parts, we get

$$\begin{aligned} &\int_{B_{2R}^+} |x|^a|u|^{p-1}u(\nabla u\cdot x)\psi \\ &= \int_{B_{2R}^+} \Delta u\Delta(\nabla u\cdot x\psi) = \int_{B_{2R}^+} v\big[(\nabla(v)\cdot x)\psi + 2v\psi + 2\nabla(\nabla u\cdot x)\cdot\nabla\psi + (\nabla u\cdot x)\Delta\psi\big]. \end{aligned}$$

Direct calculation yields

$$\nabla(\nabla u \cdot x) \cdot \nabla\psi = \nabla^2 u(x, \nabla\psi) + (\nabla u \cdot \nabla\psi)$$

and

$$\begin{aligned}\int_{B_{2R}^+} v\big[(\nabla(v) \cdot x)\psi + 2v\psi\big] &= \int_{B_{2R}^+} \frac{\nabla(v^2)}{2} \cdot x\psi + 2\int_{B_{2R}^+} v^2\psi \\ &= \frac{4-n}{2}\int_{B_{2R}^+} v^2\psi - \frac{1}{2}\int_{B_{2R}^+} v^2(\nabla\psi \cdot x).\end{aligned}$$

Moreover,

$$\int_{B_{2R}^+} |x|^a |u|^{p-1} u(\nabla u \cdot x)\psi = -\frac{n+a}{p+1}\int_{B_{2R}^+} |u|^{p+1}\psi - \frac{1}{p+1}\int_{B_{2R}^+} |x|^a |u|^{p+1} x \cdot \nabla\psi.$$

Therefore, (5.2) follows by regrouping the above equalities. ■

We claim then the following lemma.

Lemma 5.2. *Let $u \in C^4(\overline{\mathbb{R}^n_+})$ be a solution of (1.1) which is stable outside a compact set of $\mathbb{R}^n_+$. If $p \in (1, \frac{n+4+2a}{n-4})$, then $|x|^{\frac{a}{p+1}} u \in L^{p+1}(\mathbb{R}^n_+)$, $v \in L^2(\mathbb{R}^n_+)$, we have*

$$\frac{n-4}{2}\int_{\mathbb{R}^n_+} v^2 = \frac{n+a}{p+1}\int_{\mathbb{R}^n_+} |x|^a |u|^{p+1} \tag{5.3}$$

and

$$\int_{\mathbb{R}^n_+} v^2 = \int_{\mathbb{R}^n_+} |x|^a |u|^{p+1}. \tag{5.4}$$

Proof. Using (2.8) and tending $R \to \infty$, we obtain

$$|x|^{\frac{a}{p+1}} u \in L^{p+1}(\mathbb{R}^n_+) \quad \text{and} \quad v \in L^2(\mathbb{R}^n_+). \tag{5.5}$$

By Hölder's inequality, there holds that

$$R^{-4}\int_{A_R^+} |u|^2 \le CR^{(n-\frac{4(p+1)+2a}{p-1})\frac{p-1}{p+1}}\left(\int_{A_R^+} |x|^a |u|^{p+1}\right)^{\frac{2}{p+1}}.$$

On the other hand, by standard scaling argument, there exists $C > 0$ such that for any $R > 0$, any $u \in C^4(A_R^+)$ with $A_R^+ = B_{2R}^+ \backslash B_R^+$,

$$R^{-2}\int_{A_R^+} |\nabla u|^2 \le C\int_{A_R^+} v^2 + CR^{-4}\int_{A_R^+} u^2.$$

Therefore, as p is subcritical, we deduce that

$$CR^{-4}\int_{A_R^+} u^2 + R^{-2}\int_{A_R^+} |\nabla u|^2 \to 0 \quad \text{as } R \to \infty. \tag{5.6}$$

Now, we will estimate the integral

$$\int_{A_R^+} |\nabla^2 u|^2.$$

Since $u\zeta = 0$ on $\partial\mathbb{R}^n_+$, by standard elliptic theory, there exists $C > 0$ such that

$$\int_{A_R^+} |\nabla^2(u\zeta)|^2 \le C\int_{A_R^+} |\Delta(u\zeta)|^2 \le C\int_{A_R^+} \left[u^2|\Delta\zeta|^2 + |\nabla u|^2|\nabla\zeta|^2 + v^2\right]. \tag{5.7}$$

So, we get

$$\begin{aligned}\int_{A_R^+} |\nabla^2 u|^2\zeta^2 &\le C\int_{A_R^+} |\nabla^2(u\zeta)|^2 + C\int_{A_R^+} |\nabla u|^2|\nabla\zeta|^2 + C\int_{A_R^+} u^2(|\nabla\zeta|^4 + |\nabla^2\zeta|^2)\\ &\le C\int_{A_R^+} v^2 + CR^{-4}\int_{A_R^+} u^2 + R^{-2}\int_{A_R^+} |\nabla u|^2. \end{aligned}\tag{5.8}$$

Using (5.5) and (5.6), there holds that

$$\int_{\mathbb{R}^n_+} |\nabla^2 u|^2 < \infty. \tag{5.9}$$

Now, to prove (5.3), we will show that any terms on the right-hand side of (5.2) (denoted by I_R) tends to 0 as $R \to +\infty$. Remark that $\nabla\psi \ne 0$ only in $A_R^+ = B_{2R}^+ \backslash B_R^+$ and $\|\nabla^k\psi\|_\infty \le C_k R^{-k}$, there holds that

$$|I_R| \le C\int_{A_R^+} \left(|x|^a|u|^{p+1} + v^2\right) + \frac{C}{R}\int_{A_R^+} |v||\nabla u| + C\int_{A_R^+} |v||\nabla^2 u|.$$

Thanks to the estimates (5.5)-(5.9) and Hölder's inequality, clearly, $\lim_{R\to\infty} I_R = 0$; hence, we get (5.3).

On the other hand, using $u\psi$ as test function in (1.1), we have

$$\begin{aligned}\int_{B_{2R}^+} v^2\psi - \int_{B_{2R}^+} |x|^a|u|^{p+1}\psi &\le C\int_{B_{2R}^+} |uv||\Delta\psi| + C\int_{B_{2R}^+} |v||\nabla u||\nabla\psi|dx\\ &\le \frac{C}{R^2}\int_{A_R^+} |uv| + \frac{C}{R}\int_{A_R^+} |v||\nabla u|.\end{aligned}$$

Applying Hölder's inequality, (5.5), (5.6) and tending R to infinity, so we obtain (5.4). The proof is completed. ∎

Combining (5.3) and (5.4), there holds that

$$\left(\frac{n-4}{2} - \frac{n+a}{p+1}\right)\int_{A_R^+} |u|^{p+1} = 0.$$

We are done since $n < \frac{4(p+1)+2a}{p-1}$ implies that

$$\frac{n-4}{2} - \frac{n+a}{p+1} < 0.$$

Case 2. The critical $p = \frac{n+4+2a}{n-4}$.

We can proceed as in the proof of equality (5.4) to derive that

$$\int_{\mathbb{R}^n_+} v^2 = \int_{\mathbb{R}^n_+} |x|^a |u|^{p+1} < +\infty.$$

Case 3. The supercritical $\frac{n+4+2a}{n-4} < p < p_{JL2}(n,0)$.

To classify finite Morse index solutions in the supercritical case, applying the doubling lemma in [24], we get the following crucial lemma.

Lemma 5.3. *Let $n \geq 1$, $1 < p < p_{JL2}(n,0)$ and $\tau \in (0,1]$. Let $c \in C^\tau(\overline{B_1^+})$ satisfies*

$$\|c\|_{C^\tau(\overline{B_1^+})} \leq C_1 and c(x) \geq C_2, x \in \overline{B_1^+} \tag{5.10}$$

for some constants $C_1, C_2 > 0$. There exists a constant C, depending on α, C_1, C_2, p, n such that for any stable solution u of

$$\Delta^2 u = c(x)|u|^{p-1}u \quad \text{in } B_1^+ \quad \text{and} \quad u = \frac{\partial u}{\partial x_n} = 0 \quad \text{on } \partial B_1^+, \tag{5.11}$$

u satisfies

$$|u(x)|^{\frac{p-1}{4}} \leq C(1 + \operatorname{dist}^{-1}(x, \partial B_1^+)).$$

Proof. Arguing by contradiction, we suppose that there exist sequences c_k, u_k verifying (5.10)–(5.11) and points y_k such that the functions

$$M_k = |u_k|^{\frac{p-1}{4}}$$

satisfy

$$M_k(y_k) > 2k(1 + \operatorname{dist}^{-1}(y_k, \partial B_1^+)) \geq 2k(\operatorname{dist}^{-1}(y_k, \partial B_1^+)).$$

By the doubling lemma in [24], there exists x_k such that

$$M_k(x_k) \geq M_k(y_k), M_k(x_k) \geq 2k(\operatorname{dist}^{-1}(x_k, \partial B_1^+))$$

and

$$M_k(z) \leq 2M_k(x_k) \quad \forall z \in B_1^+ \text{ such that } |z - x_k| \leq k M_k^{-1}(x_k). \tag{5.12}$$

We have

$$\lambda_k = M_k^{-1}(x_k) \to 0 \quad \text{as } k \to \infty \tag{5.13}$$

due to $M_k(x_k) \geq M_k(y_k) > 2k$.

Next, we let

$$v_k(y) = \lambda_k^{\frac{4}{p-1}} u_k(x_k + \lambda_k y) \quad \text{and} \quad \tilde{c}_k(y) = c_k(x_k + \lambda_k y) \quad \text{for } y \in B_k,\ y_n > -\frac{y_{k,n}}{\lambda_k},$$

where $y_k = (y_{k,1}, \ldots, y_{k,n})$. Then, $v_k(y)$ is the solution of

$$\begin{cases} \Delta^2 v_k(y) = \tilde{c}_k(y)|v_k(y)|^{p-1} v_k(y), & |y| < k, y_n > -\frac{y_{k,n}}{\lambda_k}, \\ v_k(y) = \frac{\partial v_k(y)}{\partial y_n} = 0, & |y| < k, y_n = -\frac{y_{k,n}}{\lambda_k}, \end{cases}$$

with

$$|v_k(0)| = 1 \quad \text{and} \quad |v_k(y)| \leq 2^{\frac{4}{p-1}}, \quad |y| < k, \; y_n > -\frac{y_{k,n}}{\lambda_k}.$$

Two cases may occur as $k \to \infty$, either case (1)

$$\frac{y_{k,n}}{\lambda_k} \to +\infty$$

for a subsequence still denoted as before, or case (2)

$$\frac{y_{k,n}}{\lambda_k} \to c \geq 0.$$

In case (1), after extracting a subsequence, $\tilde{c}_k \to C$ in $C_{\text{loc}}(\mathbb{R}^n)$ with $C > 0$ a constant and we may also assume that $v_k \to v$ in $C^4_{\text{loc}}(\mathbb{R}^n)$, and v is a stable solution of

$$\Delta^2 v = C|v|^{p-1} v \quad \text{in } \mathbb{R}^n \quad \text{and} \quad |v(0)| = 1.$$

By the Liouville-type theorems in [7] for stable solutions, we derive that $v \equiv 0$. This is a contradiction.

In case (2) we can prove that $c > 0$, thus we get a stable solution of (1.1) in $\overline{\mathbb{R}^n_+}$ and $|v(c)| = 1$, which contradict Theorem 1.2 for $1 < p < p_0(n, 4)$. ∎

Proposition 5.1. *Let u be a (positive or sign changing) solution to* (1.1) *which is stable outside a compact set of $\mathbb{R}^n_+$. There exist constants C and R_0 such that*

$$|u(x)| \leq C|x|^{-\frac{4+a}{p-1}} \quad \forall x \in B^+_{R_0}(0)^c, \tag{5.14}$$

$$\sum_{k \leq 3} |x|^{\frac{4+a}{p-1}+k} |\nabla^k u(x)| \leq C \quad \forall x \in B^+_{3R_0}(0)^c. \tag{5.15}$$

Proof. Assume that u is stable outside $B^+_{R_0}$ and $|x_0| > 2R_0$. We denote

$$R = \frac{1}{2}|x_0|$$

and observe that, for all $y \in B^+_1$, $\frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2}$, so that $x_0 + Ry \in B^+_{R_0}(0)^c$. Let us thus define

$$U(y) = R^{\frac{4+a}{p-1}} u(x_0 + Ry).$$

Then, U is a solution of

$$\Delta^2 U = c(y)|U|^{p-1} U \quad \text{in } B^+_1 \quad \text{and} \quad U = \frac{\partial U}{\partial y_n} = 0 \quad \text{on } \partial B^+_1 \text{ with } c(y) = \left| y + \frac{x_0}{R} \right|^a.$$

Notice that $|y + \frac{x_0}{R}| \in [1, 3]$ for all $y \in \overline{B_1^+}$. Moreover,

$$\|c\|_{C^1(\overline{B_1^+})} \leq C(a).$$

Then, applying Lemma 5.3, we have $|U(0)| \leq C$; hence,

$$|u(x_0)| \leq CR^{-\frac{4+a}{p-1}},$$

which yields the inequality (5.14).

Next, we prove the inequality (5.15). For any x_0 with $|x_0| > 3R_0$, take $\lambda = \frac{|x_0|}{2}$ and define

$$\bar{u}(x) = \lambda^{\frac{4+a}{p-1}} u(x_0 + \lambda x).$$

From (5.14), $|\bar{u}| \leq C_0$ in $B_1^+(0)$. Then, standard elliptic estimates give

$$\sum_{k \leq 5} |\nabla^k \bar{u}(0)| \leq C. \qquad \blacksquare$$

Lemma 5.4. *There exists a constant C_2, such that forall $r > 3R_0$, $E(u, r) \leq C_2$.*

Proof. From the monotonicity formula, combining the derivative estimates (5.15), we have

$$\begin{aligned}
E(u, r) \leq\, & Cr^{\frac{4(p+1)+2a}{p-1} - n} \left(\int_{B_r^+} v^2 + |x|^a |u|^{p+1} \right) \\
& + Cr^{\frac{8+2a}{p-1} + 1 - n} \int_{\partial B_r^+} u^2 + Cr^{\frac{8+2a}{p-1} + 2 - n} \int_{\partial B_r^+} |u||\nabla u| \\
& + Cr^{\frac{8+2a}{p-1} + 3 - n} \int_{\partial B_r^+} |\nabla u|^2 \\
& + Cr^{\frac{8+2a}{p-1} + 3 - n} \int_{\partial B_r^+} |u||\nabla^2 u| \\
& + Cr^{\frac{8+2a}{p-1} + 4 - n} \int_{\partial B_r^+} |u||\nabla^2 u| \leq C,
\end{aligned}$$

where C depends on the constant that appeared in (5.15). $\blacksquare$

We claim then the following corollary.

Corollary 5.1. *We have*

$$\int_{(B_{3R_0}^+(0))^c} \frac{(\frac{4+a}{p-1}|x|^{-1} u(x) + \frac{\partial u}{\partial r}(x))^2}{|x|^{n-2-\frac{8+2a}{p-1}}} < +\infty.$$

As before, we define a blowing down sequence

$$u^\lambda(x) = \lambda^{\frac{4+a}{p-1}} u(\lambda x).$$

By Proposition 5.1, u^λ are uniformly bounded in $C^5(B_r^+(0)\backslash B_{1/r}^+(0))$ for any fixed $r>1$. u^λ is stable outside $B_{R_0/\lambda}^+(0)$. There exists a function $u^\infty\in C^6(\mathbb{R}^n\backslash\{0\})$, such that up to a subsequence of $\lambda\to+\infty$, u^λ converges to $u^\infty\in C^4_{\text{loc}}(\mathbb{R}^n_+\backslash\{0\})$. u^∞ is a stable solution of (1.1) in $\mathbb{R}^n_+\backslash\{0\}$.

Using Corollary 5.1, we obtain, for any $r>1$,

$$\begin{aligned}
&\int_{B_r^+\backslash B_{1/r}^+}\frac{(\frac{4+a}{p-1}|x|^{-1}u^\infty(x)+\frac{\partial u^\infty}{\partial r}(x))^2}{|x|^{n-2-\frac{8+2a}{p-1}}}\\
&\quad=\lim_{\lambda\to+\infty}\int_{B_r^+\backslash B_{1/r}^+}\frac{(\frac{4+a}{p-1}|x|^{-1}u^\lambda(x)+\frac{\partial u^\lambda}{\partial r}(x))^2}{|x|^{n-2-\frac{8+2a}{p-1}}}\\
&\quad=\lim_{\lambda\to+\infty}\int_{B_r^+\backslash B_{1/r}^+}\frac{(\frac{4+a}{p-1}|x|^{-1}u(x)+\frac{\partial u}{\partial r}(x))^2}{|x|^{n-2-\frac{8+2a}{p-1}}}=0.
\end{aligned}$$

Hence, u^∞ is homogeneous, and from Theorem 1.1, $u^\infty\equiv 0$. This holds for every limit of u^λ as $\lambda\to+\infty$; thus, we get

$$\lim_{|x|\to+\infty}|x|^{\frac{4+a}{p-1}}|u(x)|=0.$$

From (5.15), we derive

$$\lim_{|x|\to+\infty}\sum_{k\le 4}|x|^{\frac{4+a}{p-1}+k}|\nabla^k u(x)|=0.$$

For $\varepsilon>0$, take an R such that for $|x|>R$,

$$\sum_{k\le 4}|x|^{\frac{4+a}{p-1}+k}|\nabla^k u(x)|\le\varepsilon.$$

Then, for $r\gg R$,

$$\begin{aligned}
E(u,r)&\le Cr^{\frac{4(p+1)+2a}{p-1}-n}\left(\int_{B_R^+(0)}v^2+|x|^a|u|^{p+1}\right)\\
&\quad+C\varepsilon r^{\frac{8+2a}{p-1}+4-n}\int_{B_r^+(0)\backslash B_R^+(0)}|x|^{-\frac{8+2a}{p-1}-4}\\
&\quad+C\varepsilon r^{\frac{8+2a}{p-1}+5-n}\int_{\partial B_r^+(0)}|x|^{-\frac{8+2a}{p-1}-4}\le C(R)\big(r^{\frac{4(p+1)+2a}{p-1}-n}+\varepsilon\big).
\end{aligned}$$

Since $\frac{4(p+1)+2a}{p-1}-n<0$ and ε can be arbitrarily small, we derive $\lim_{r\to+\infty}E(u,r)=0$. Because $\lim_{r\to 0}E(r,u)=0$ (by the smoothness of u), the same argument for stable solutions implies that $u\equiv 0$.

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Abdelbaki Selmi
Department of Mathematics, Northern Border University, Arar, KSA; Département de Mathématiques, Faculté des Sciences de Bizerte, Université de Tunis, Zarzouna, 7021 Bizerte, Tunisia; abdelbaki.selmi@fsb.rnu.tn

Cherif Zaidi
Institut Supérieur des Sciences Appliquées et de Technologie de Kairouan, Avenue Beit El Hikma, 3100 Kairouan, Tunisia; zaidicherif10@gmail.com