Rend. Lincei Mat. Appl. 35 (2024), 17–47
DOI 10.4171/RLM/1032

Analysis. – *Characterizations of Sobolev multiplier spaces and their preduals in terms of Littlewood–Paley and Lusin area functions*, by Keng Hao Ooi, communicated on 8 March 2024.

Abstract. – We characterize Sobolev multiplier spaces and their preduals in terms of Littlewood–Paley and Lusin area functions. The vector-valued and weighted norm estimates of Hardy–Littlewood maximal operator will be used as the main tools in such characterizations.

Keywords. – Sobolev multiplier spaces, Littlewood–Paley functions, Lusin area functions, capacities.

Mathematics Subject Classification 2020. – 42B25 (primary); 31C15 (secondary).

1. Introduction

The Sobolev multiplier spaces and their preduals have been addressed in [13]. Let us recall the definitions of such function spaces. Let $\alpha > 0$, $s > 1$, $n \in \mathbb{N}$, and $\alpha s \le n$. We define the Riesz and Bessel capacities respectively by

$$\operatorname{cap}_{\alpha,s}(E) = \inf\left\{\|f\|_{L^s(\mathbb{R}^n)}^s : f \ge 0,\ I_\alpha * f \ge 1 \text{ on } E\right\}, \quad \alpha s < n,$$
$$\operatorname{Cap}_{\alpha,s}(E) = \inf\left\{\|f\|_{L^s(\mathbb{R}^n)}^s : f \ge 0,\ G_\alpha * f \ge 1 \text{ on } E\right\}, \quad \alpha s \le n,$$

where $I_\alpha(x) = \mathcal{F}^{-1}(|\cdot|^{-\alpha})(x)$, $G_\alpha(x) = \mathcal{F}^{-1}[(1+|\cdot|^2)^{-\alpha/2}](x)$, $x \in \mathbb{R}^n$, are the Riesz and Bessel kernels respectively, and $\mathcal{F}^{-1}$ is the inverse distributional Fourier transform on $\mathbb{R}^n$. By Sobolev multiplier spaces $\dot{M}_p^{\alpha,s}$, and $M_p^{\alpha,s}$, $1 < p < \infty$, we mean the class of locally p-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\dot{M}_p^{\alpha,s}} = \sup_K \left(\frac{\int_K |f(x)|^p\,dx}{\operatorname{cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}} < \infty,$$
$$\|f\|_{M_p^{\alpha,s}} = \sup_K \left(\frac{\int_K |f(x)|^p\,dx}{\operatorname{Cap}_{\alpha,s}(K)}\right)^{\frac{1}{p}} < \infty,$$

where the suprema are taken with respect to all compact subsets $K \subseteq \mathbb{R}^n$ with non-zero capacities. We note that the Sobolev multiplier spaces arise naturally in many super-critical nonlinear PDEs including the Navier–Stokes system (see [6, 14, 15] for examples). It is also shown in [12, Theorem 3.1.4] that the norms $\|f\|_{\dot{M}_p^{\alpha,s}}$ and $\|f\|_{M_p^{\alpha,s}}$ correspond respectively to the best constants $C>0$ of the following trace inequalities:

$$\left(\int_{\mathbb{R}^n}|I_\alpha*\varphi(x)|^s\left|f(x)\right|^p dx\right)^{\frac{1}{p}}\leq C\|\varphi\|_{L^s(\mathbb{R}^n)}^{\frac{s}{p}},$$
$$\left(\int_{\mathbb{R}^n}|G_\alpha*\varphi(x)|^s\left|f(x)\right|^p dx\right)^{\frac{1}{p}}\leq C\|\varphi\|_{L^s(\mathbb{R}^n)}^{\frac{s}{p}},$$

where $\varphi\in L^s(\mathbb{R}^n)$.

Several preduals of the Sobolev multiplier spaces have been characterized in [13]. For the readers' convenience, let us recall the definition of preduals in our context. Given two function spaces X and Y, X is said to be a predual of Y, if for any continuous linear functional $\mathcal{L}$ on X, there is a corresponding function $g\in Y$ such that

$$\mathcal{L}(f)=\int_{\mathbb{R}^n}f(x)g(x)dx,$$

where $f\in X$. We write $X^*\approx Y$ if both X and Y are equipped with quasi-norms and $c^{-1}\|g\|_Y\leq\|\mathcal{L}\|_{X^*}\leq c\|g\|_Y$ for some constant $c>0$. Here we repeat some of the typical preduals of $\dot{M}_p^{\alpha,s}$, while similar assertions regarding the preduals of $M_p^{\alpha,s}$ will be clear in the following context. We have $X^*\approx\dot{M}_p^{\alpha,s}$ for $X=(\dot{M}_p^{\alpha,s})'$, $\dot{N}_q^{\alpha,s}$, and $\dot{B}_q^{\alpha,s}$ (see [13, Theorem 9.3]). They are the spaces with the following finite quantities $\|f\|_X$:

$$\|f\|_{(\dot{M}_p^{\alpha,s})'}=\sup\left\{\left|\int_{\mathbb{R}^n}f(x)g(x)dx\right|:g\in\dot{M}_p^{\alpha,s},\ \|g\|_{\dot{M}_p^{\alpha,s}}\leq 1\right\},$$

$$\|f\|_{\dot{N}_q^{\alpha,s}}=\inf_{w}\left(\int_{\mathbb{R}^n}\left|f(x)\right|^q\omega(x)^{1-q}dx\right)^{\frac{1}{q}},\quad 1/p+1/q=1, \tag{1.1}$$

$$\|f\|_{\dot{B}_q^{\alpha,s}}=\inf\left\{\sum_j|c_j|:f(x)=\sum_j c_ja_j(x)\text{ a.e.}\right\}, \tag{1.2}$$

where the infimum in (1.1) is taken over quasi-everywhere (i.e., except for only a set of zero capacity) defined weights $\omega\geq 0$ with $\|\omega\|_{L^1(\mathrm{cap}_{\alpha,s})}\leq 1$ and

$$\|\omega\|_{L^1(\mathrm{cap}_{\alpha,s})}=\int_0^\infty\mathrm{cap}_{\alpha,s}\left(\{x\in\mathbb{R}^n:\omega(x)>t\}\right)dt,$$

where each a_j in (1.2) satisfies that $\|a_j\|_{L^q(\mathbb{R}^n)}\leq\mathrm{cap}_{\alpha,s}(A_j)^{-1/p}$ and $\{a_j\neq 0\}\subseteq A_j$ for some bounded set A_j. In this case, each a_j is called a block in $\dot{B}_q^{\alpha,s}$. Note that the weights $\omega\geq 0$ in (1.1) can be taken as $\omega\in A_1$ with $[\omega]_{A_1}\leq c(n,\alpha,s)$, where $c(n,\alpha,s)>0$ is a constant depending only on n, α, and s (see [13, Lemma 9.1 and Theorem 9.3]).

In classical analysis, Littlewood–Paley theory serves as an important tool in characterizing certain function spaces. For instance, it is well known that

$$\|f\|_{L^p(\mathbb{R})} \approx \left\| \left(\sum_{j \in \mathbb{Z}} \left| \left(\chi_{\{2^j \le |\xi| < 2^{j+1}\}} \hat{f}(\xi) \right)^{\vee} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})},$$

which can be interpreted as characterizing an $L^p(\mathbb{R})$ function by means of frequency localization on each block $\{2^j \le |\xi| < 2^{j+1}\}$; here $A \approx B$ designates the two-sided estimates $C^{-1}A \le B \le CA$ and the constant $C > 0$ does not depend upon the main parameters in A and B. For higher dimension $n > 1$, one has

$$\|f\|_{L^p(\mathbb{R}^n)} \approx \left\| \left(\sum_{j \in \mathbb{Z}} \left| \left(\hat{\Psi}(2^{-j}\xi) \hat{f}(\xi) \right)^{\vee} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}, \tag{1.3}$$

where Ψ is a sufficiently regular function. Besides that, there are similar Littlewood–Paley characterizations of the real Hardy spaces $H^p(\mathbb{R}^n)$ and the Morrey spaces $\mathcal{M}_p^q(\mathbb{R}^n)$; we refer the readers to [7, Section 2.2.4] and [8] for further details.

In the present paper, we wish to obtain the Littlewood–Paley characterizations of the Sobolev multiplier spaces and their preduals. Our approach will be the “continuous” version rather than (1.3). To begin with, denote by $\dot{\mathcal{Z}}$ the Sobolev multiplier spaces $\dot{M}_p^{\alpha,s}$ and $\dot{\mathcal{Z}}'$ their preduals. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radial Schwartz function with compact supported away from the origin and

$$\int_0^\infty \widehat{\varphi}(t\xi) \frac{dt}{t} = 1, \quad \xi \neq 0.$$

Denote by $\varphi_t(\cdot)$ the L^1-dilation of φ that $\varphi_t(\cdot) = t^{-n}\varphi(\cdot/t)$ for $t > 0$. We will show that

$$\|f\|_X \approx \left\| \left(\int_0^\infty \left| \varphi_t * f(\cdot) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_X \tag{1.4}$$

for $X = \dot{\mathcal{Z}}$ and $\dot{\mathcal{Z}}'$ with $0 \notin \operatorname{supp}(\hat{f})$, where $f \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution.

As before, denote by $\mathcal{Z}$ the Sobolev multiplier spaces $M_p^{\alpha,s}$ and $\mathcal{Z}'$ their preduals. For the case where $X = \mathcal{Z}$ and $\mathcal{Z}'$ in (1.4), we need a different approach. Let $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$ be a compactly supported infinitely differentiable function on $\mathbb{R}^n$ with non-vanishing integral. As usual, denote by $(\varphi_0)_t$ the L^1-dilation of φ_0 for $t > 0$. Let

$$\varphi(\cdot) = \frac{d}{dt}\{(\varphi_0)_t(\cdot)\}_{t=1}.$$

Then there are $\psi_0, \psi \in \mathcal{D}(\mathbb{R}^n)$ such that

$$f(\cdot) = \psi_0 * \varphi_0 * f(\cdot) + \lim_{\varepsilon \to 0} \int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot) \frac{dt}{t}, \quad f \in \mathcal{D}(\mathbb{R}^n), \tag{1.5}$$

where the limit is taken in $\mathcal{D}(\mathbb{R}^n)$ (see [16, Remark 1.7]). We will show that

$$\|f\|_X \approx \|\varphi_0 * f\|_X + \left\| \left(\int_0^1 |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_X \tag{1.6}$$

for $X = \mathcal{Z}$ and $\mathcal{Z}'$.

Note that (1.4) and (1.6) are about the norm estimates of Littlewood–Paley functions. We could also address the topologies counterpart. In classical theory, if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a radial Schwartz function with $\widehat{\varphi}(0) = 0$, then given any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\partial^\gamma(\hat{f})(0) = 0$ for all multi-indices $\gamma \in (\mathbb{N} \cup \{0\})^n$, it holds that

$$\int_\varepsilon^\delta \varphi_t * f(\cdot) \frac{dt}{t} \to f(\cdot), \quad \varepsilon \to 0,\ \delta \to \infty, \tag{1.7}$$

in the topology of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We also have the Calderón reproducing formula that

$$\int_\varepsilon^\delta \psi_t * \psi_t * f(\cdot) \frac{dt}{t} \to f(\cdot), \quad \varepsilon \to 0,\ \delta \to \infty, \tag{1.8}$$

in $L^2(\mathbb{R}^n)$ as long as $\psi \in \mathcal{S}(\mathbb{R}^n)$ is a radial Schwartz function with $\widehat{\psi}(0) = 0$ and

$$\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} = 1, \quad \xi \neq 0. \tag{1.9}$$

We will show that (1.7) and (1.8) hold for the weak* topology of $\dot{\mathcal{Z}}$ and the norm topology of $\dot{\mathcal{Z}}'$ respectively. On the other hand, assuming that $\psi_0, \varphi_0, \psi_t, \varphi_t$ are as in (1.5), then the convergence

$$\psi_0 * \varphi_0 * f(\cdot) + \int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot) \frac{dt}{t} \to f(\cdot), \quad \varepsilon \to 0,$$

holds for the weak*-topology of $\mathcal{Z}$ and the norm topology of $\mathcal{Z}'$.

Readers may notice that there are different types of Littlewood–Paley functions in characterizing $\dot{\mathcal{Z}}$ and $\mathcal{Z}$ (also their associated preduals). A way to view this is due to the homogeneities of the Riesz kernels $I_\alpha(\cdot)$ and the Bessel kernels $G_\alpha(\cdot)$ respectively. Note that

$$I_\alpha(x) = \frac{\gamma_{n,\alpha}}{|x|^{n-\alpha}}, \quad x \neq 0,$$

where $\gamma_{n,\alpha}$ is a constant depending only on n and α, whereas

$$G_\alpha(x) \approx |x|^{\alpha-n}, \quad |x| \to 0,$$

and

$$G_\alpha(x) = O\big(e^{-\frac{|x|}{2}}\big), \quad |x| \to \infty.$$

Note that the standard Hardy–Littlewood maximal function and Calderón–Zygmund operator are bounded on the spaces $\dot{\mathcal{Z}}$ and $\dot{\mathcal{Z}}'$, which are associated with Riesz capacities (see [13, Theorem 9.5]). Meanwhile, only the local Hardy–Littlewood maximal function and truncated Calderón–Zygmund operators, but not the standard one, are bounded on the spaces $\mathcal{Z}$ and $\mathcal{Z}'$, which are associated with Bessel capacities (see [13, Theorem 1.11]). In view of (1.4), $\dot{\mathcal{Z}}$ and $\dot{\mathcal{Z}}'$ are similar to the homogeneous Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}$ defined by

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \Big\| \Big(\sum_{j\in\mathbb{Z}} \big(2^{j\alpha}\big|\Delta_j^{\Psi}(f)\big|\big)^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^n)},$$

where $\Delta_j^{\Psi}(f) = \Psi_{2^{-j}} * f$, and Ψ is a radial Schwartz function on $\mathbb{R}^n$ whose Fourier transform is nonnegative, is supported on the annulus $1 - 1/7 \le |\xi| \le 2$, is equal to one on the smaller annulus $1 \le |\xi| \le 2 - 2/7$, and satisfies

$$\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \quad \xi \neq 0$$

(see [7, Definition 2.2.1]). However, in view of (1.6), $\mathcal{Z}$ and $\mathcal{Z}'$ are similar to the inhomogeneous Triebel–Lizorkin spaces $F_p^{\alpha,q}$ defined by

$$\|f\|_{F_p^{\alpha,q}} = \|S_0(f)\|_{L^p(\mathbb{R}^n)} + \Big\| \Big(\sum_{j=1}^{\infty} \big(2^{j\alpha}\big|\Delta_j^{\Psi}(f)\big|\big)^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^n)},$$

where S_0 is an operator defined by

$$S_0 + \sum_{j=1}^{\infty} \Delta_j^{\Psi} = I,$$

and I is the identity operator, whereas the convergence of the above series is taken in $\mathcal{S}'(\mathbb{R}^n)$ (see [7, Definition 2.2.1]). Roughly speaking, the Sobolev multiplier spaces and their preduals (together with their homogeneous counterpart), through the characterizations in terms of Littlewood–Paley functions, can be viewed as a variant of Triebel–Lizorkin spaces $F_p^{\alpha,q}$ (or $\dot{F}_p^{\alpha,q}$) with the underlying $L^p(\mathbb{R}^n)$ norm taken in the definition of $F_p^{\alpha,q}$ replaced by $\mathcal{Z}$ and $\mathcal{Z}'$ (or $\dot{\mathcal{Z}}$ and $\dot{\mathcal{Z}}'$ for the homogeneous counterpart).

Lusin area and Littlewood–Paley functions always come as a pair in certain function spaces characterizations. Let ψ be as in (1.9). Recall that the Lusin area function $s(\cdot)$ is defined to be

$$s(f)(x) = \Big(\int_0^\infty \int_{|x-y|<t} \big|\psi_t * f(y)\big|^2 dy \frac{dt}{t^{n+1}} \Big)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

It holds that

$$\|f\|_{L^p(\mathbb{R}^n)} \approx \left\|s(f)\right\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty.$$

A similar norm estimate for real Hardy spaces $H^p(\mathbb{R}^n)$ also holds:

$$\|f\|_{H^p(\mathbb{R}^n)} \approx \left\|s(f)\right\|_{L^p(\mathbb{R}^n)}, \quad 0 < p \leq 1$$

(see [18, Chapter XII, Section 4]). We will then give the norm estimates in terms of Lusin area functions for the Sobolev multiplier spaces and their preduals.

In what follows, we denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$ the spaces of Schwartz and compactly supported infinitely differentiable functions respectively, while $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ refer to their canonical dual spaces. For any scalar-valued function f, denote by $f_t(\cdot)$ the L^1-dilation of f that $f_t(\cdot) = t^{-n} f(\cdot/t)$. Furthermore, for any quasi-normed spaces X and Y, we write $X \hookrightarrow Y$ if $\|\cdot\|_Y \leq C\|\cdot\|_X$ for some constant $C > 0$. The notation $A \lesssim B$ will abbreviate the inequality $A \leq CB$ for some constant $C > 0$, $A \gtrsim B$ refers to $B \lesssim A$, and $A \approx B$ simply means that both $A \lesssim B$ and $A \gtrsim B$.

2. Statements of main results: Riesz capacities

In this section, we focus on the Sobolev multiplier spaces $\dot{\mathcal{Z}}$ and their preduals $\dot{\mathcal{Z}}'$ which correspond to the Riesz capacities. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\widehat{\varphi}(0) = 0$. The Littlewood–Paley g-function is defined to be

$$\left(\int_0^\infty \left|\varphi_t * f(\cdot)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

where $f \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 2.1. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be radial. Suppose that $\widehat{\varphi}$ is compactly supported away from the origin and*

$$\int_0^\infty \widehat{\varphi}(t\xi)\frac{dt}{t} = 1, \quad \xi \neq 0.$$

Then

$$f(\cdot) = \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_\varepsilon^\delta \varphi_t * f(\cdot)\frac{dt}{t}, \quad f \in \dot{\mathcal{Z}}',$$

where the limit is taken in $\dot{\mathcal{Z}}'$.

With the aid of the above topological properties, we obtain the two-sided norm estimates of the Littlewood–Paley g-function.

THEOREM 2.2. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be as in Theorem 2.1. Let X be either $\dot{\mathcal{Z}}$ or $\dot{\mathcal{Z}}'$. Suppose that $f \in X$ and $0 \notin \operatorname{supp}(\hat{f})$. Then the following norm estimate holds:*

$$\|f\|_X \approx \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right) \right\|_X.$$

Now we address the issue of Calderón reproducing formula. To this end, let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be radial which satisfies that $\widehat{\psi}(0) = 0$ and

$$\int_0^\infty \left|\widehat{\psi}(t\xi)\right|^2 \frac{dt}{t} = 1, \quad \xi \neq 0. \tag{2.1}$$

Denote

$$\mathcal{S}_0(\mathbb{R}^n) = \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \hat{f} = 0 \text{ on a neighborhood of } 0 \right\},$$

and

$$\mathcal{S}_\infty(\mathbb{R}^n) = \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \partial^\gamma(\hat{f})(0) = 0 \text{ for any multi-index } \gamma \right\}.$$

Clearly, $\mathcal{S}_0(\mathbb{R}^n)$ is a subclass of $\mathcal{S}_\infty(\mathbb{R}^n)$. We obtain a type of Calderón reproducing formula.

THEOREM 2.3. *Let ψ be as in (2.1). For any $f \in \dot{\mathcal{Z}}'$, it holds that*

$$f(\cdot) = \lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \int_\varepsilon^\delta \psi_t * \psi_t * f(\cdot) \frac{dt}{t},$$

where the limit is taken in $\dot{\mathcal{Z}}'$.

Subsequently, with such a ψ in (2.1), we define the Lusin area function by

$$s_\alpha(f)(x) = \left(\int_0^\infty \int_{|x-y|<\alpha t} \left|\psi_t * f(y)\right|^2 (\alpha t)^{-n} dy \frac{dt}{t} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n,\ 1 \le \alpha < \infty,$$

where $f \in \mathcal{S}'(\mathbb{R}^n)$. The sublinear operator $s_1(\cdot)$ is denoted by $s(\cdot)$ for the sake of convenience. The role of $s_\alpha(\cdot)$ in norm estimates can be replaced by $s(\cdot)$ (see Proposition 5.5). The next result gives the norm estimates of $s(\cdot)$.

THEOREM 2.4. *Let X be either $\dot{\mathcal{Z}}$ or $\dot{\mathcal{Z}}'$. Then the estimate*

$$\left\|s(f)\right\|_X \lesssim \|f\|_X$$

holds. For any $f \in L^2(\mathbb{R}^n) \cap X$, it holds that

$$\|f\|_X \lesssim \left\|s(f)\right\|_X.$$

Consequently, the following two-sided estimates

$$\|f\|_X \approx \|s(f)\|_X$$

hold for any $f \in L^2(\mathbb{R}^n) \cap X$.

3. Statements of main results: Bessel capacities

We now turn to address on the Sobolev multiplier spaces $\mathcal{Z}$ and their preduals $\mathcal{Z}'$ which correspond to the Bessel capacities. To begin with, we need a local type Littlewood–Paley function which is constructed in [16]. By [16, Remark 1.7], for any $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$ with non-vanishing integral and

$$\varphi(\cdot) = \frac{d}{dt}\{(\varphi_0)_t(\cdot)\}_{t=1},$$

there are $\psi_0, \psi \in \mathcal{D}(\mathbb{R}^n)$ such that

$$f(\cdot) = \psi_0 * \varphi_0 * f(\cdot) + \lim_{\varepsilon\to 0}\int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot)\frac{dt}{t}, \quad f \in \mathcal{D}(\mathbb{R}^n), \tag{3.1}$$

where the limit is taken in $\mathcal{D}(\mathbb{R}^n)$. Furthermore, the convergence in (3.1) also holds in $\mathcal{D}'(\mathbb{R}^n)$ for any $f \in \mathcal{D}'(\mathbb{R}^n)$. We obtain the two-sided norm estimates in terms of local Littlewood–Paley functions.

Theorem 3.1. *Let* φ_0 *and* φ_t *be as in* (3.1). *Then*

$$\|f\|_X \approx \|\varphi_0 * f\|_X + \left\|\left(\int_0^1 |\varphi_t * f(\cdot)|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_X \tag{3.2}$$

for $X = \mathcal{Z}$ *and* $\mathcal{Z}'$.

The following theorem addresses the topological property.

Theorem 3.2. *Let* φ_0 *and* φ_t *be as in* (3.1). *Assume that* $f \in \mathcal{D}'(\mathbb{R}^n)$ *satisfies*

$$\|\varphi_0 * f\|_{\mathcal{Z}} + \left\|\left(\int_0^1 |\varphi_t * f(\cdot)|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{\mathcal{Z}} < \infty.$$

Then $f \in \mathcal{Z}$ *and*

$$f(\cdot) = \psi_0 * \varphi_0 * f(\cdot) + \lim_{\varepsilon\to 0}\int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot)\frac{dt}{t},$$

where the limit is taken in the weak-topology of* $\mathcal{Z}$.

Now we define a type of Lusin area function which fits into the structures of $\mathcal{Z}$ and $\mathcal{Z}'$. To this end, let φ_0 and φ_t be as in (3.1). For any $f \in \mathcal{D}'(\mathbb{R}^n)$, define

$$S(f)(x) = \left(\int_0^1 \int_{|x-y|<t} \left| \varphi_t * f(y) \right|^2 dy \frac{dt}{t^{n+1}} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n,$$

and

$$S^d(f)(x) = \left(\int_0^1 \sum_{\mathcal{Q} \in \mathcal{Q}_t} \sup_{z \in \mathcal{Q}} \left| \varphi_t * f(z) \right|^2 \chi_{\mathcal{Q}}(x) \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $\mathcal{Q}_t$ is the set of all dyadic cubes $\mathcal{Q}$ with side length $\ell(\mathcal{Q}) = 2^{[\log_2 t]}$, $t > 0$. Note that

$$\begin{aligned} S^d(f)(x) &= \left(\sum_{k=0}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \sum_{\mathcal{Q} \in \mathcal{Q}_t} \sup_{z \in \mathcal{Q}} \left| \varphi_t * f(z) \right|^2 \chi_{\mathcal{Q}}(x) \frac{dt}{t} \right)^{\frac{1}{2}} \\ &= \left\{ \sum_{k=0}^{\infty} \sum_{\mathcal{Q} \in \mathcal{Q}_{2^{-k}}} \left(\int_{2^{-(k+1)}}^{2^{-k}} \sup_{z \in \mathcal{Q}} \left| \varphi_t * f(z) \right|^2 \frac{dt}{t} \right) \chi_{\mathcal{Q}}(x) \right\}^{\frac{1}{2}}, \end{aligned}$$

which yields $S(f)(x) \lesssim S^d(f)(x)$.

THEOREM 3.3. *The estimates*

$$\left\| S(f) \right\|_X \lesssim \| f \|_X \tag{3.3}$$

and

$$\| f \|_X \approx \| \varphi_0 * f \|_X + \left\| S^d(f) \right\|_X \tag{3.4}$$

hold for $X = \mathcal{Z}$ *and* $\mathcal{Z}'$.

4. PRELIMINARIES

We recall the standard Sobolev embedding theorems that

$$\begin{aligned} |E|^{1-\frac{\alpha s}{n}} &\lesssim \mathrm{cap}_{\alpha,s}(E), \quad \alpha s < n, \\ |E| &\lesssim \mathrm{Cap}_{\alpha,s}(E), \quad \alpha s \le n, \end{aligned}$$

which immediately give

$$\| f \|_{\dot{\mathcal{Z}}} \lesssim \| f \|_{L^{\frac{np}{\alpha s},\infty}(\mathbb{R}^n)}, \tag{4.1}$$

$$\| f \|_{\mathcal{Z}} \lesssim \| f \|_{L^\infty(\mathbb{R}^n)}. \tag{4.2}$$

It is also shown in [13, Remark 3.3] that the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in both $\dot{\mathcal{Z}}'$ and $\mathcal{Z}'$. In fact, the dense subset can be taken into a smaller class.

Lemma 4.1. *The class $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot{\mathcal{Z}}'$. The density is also valid for $\mathcal{Z}'$.*

Proof. Given that $f \in \dot{\mathcal{Z}}' \approx \dot{B}_q^{\alpha,s}$, then there exist a sequence $\{c_j\} \in \ell^1$ and a sequence $\{a_j\}$ of blocks in $B_q^{\alpha,s}$ such that

$$f(x) = \sum_j c_j a_j(x) \quad \text{a.e.,} \quad \|f\|_{\mathcal{Z}'} \approx \sum_j |c_j|.$$

It is clear that

$$\left\| f - \sum_{j=1}^{N} c_j a_j \right\|_{\dot{B}_q^{\alpha,s}} \to 0, \quad N \to \infty.$$

To establish the density in question, it suffices to approximate each block a_j by an element of $\mathcal{S}_0(\mathbb{R}^n)$. Since $a_j \in L^q(\mathbb{R}^n)$, we can now appeal to the density of $\mathcal{S}_0(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$ (see [17, Theorem 1, p. 103]). The density for $\mathcal{Z}'$ follows by exactly the same reasoning. ■

Proposition 4.2. *The embedding holds*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

for $X = \dot{\mathcal{Z}}, \mathcal{Z}, \dot{\mathcal{Z}}'$, and $\mathcal{Z}'$.

Proof. The embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow X$ for $X = \dot{\mathcal{Z}}$ and $\mathcal{Z}$ follows by (4.1) and (4.2). On the other hand, for any $f \in \dot{\mathcal{Z}}'$ and $g \in \mathcal{S}(\mathbb{R}^n)$,

$$|\langle f, g\rangle| = \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \lesssim \|f\|_{\dot{\mathcal{Z}}'} \|g\|_{\dot{\mathcal{Z}}}.$$

As we have just proved that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{Z}}$, this implies $\dot{\mathcal{Z}}' \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. A similar argument will prove the embedding for $\mathcal{Z}' \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. ■

Remark 4.3. Note that

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^n)} &= \sup_{\|g\|_{L^\infty(\mathbb{R}^n)} \le 1} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \lesssim \sup_{\|g\|_{\mathcal{Z}} \le 1} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \\ &\approx \|f\|_{\mathcal{Z}'}.\end{aligned}$$

We actually have $\mathcal{Z}' \hookrightarrow L^1(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

Let us recall a basic fact in non-linear potential theory that

$$\begin{aligned}\operatorname{cap}_{\alpha,s}(B_R) &\approx R^{n-\alpha s}, \quad R > 0,\ \alpha s < n, \\ \operatorname{Cap}_{\alpha,s}(B_R) &\approx R^n, \qquad R > 1,\ \alpha s \le n,\end{aligned}$$

where B_R is any ball with radius R.

The next lemma shows that the elements in the Sobolev multiplier spaces cannot have polynomial growths.

LEMMA 4.4. *$\mathcal{P} \notin \mathcal{Z}$ and $\mathcal{P} \notin \dot{\mathcal{Z}}$ for any nonzero polynomial $\mathcal{P}$.*

PROOF. It suffices to show that $\mathcal{P}(x) = x^N \notin \mathcal{Z}$, $N \in \mathbb{N}$. For any $R > 1$, we see that

$$\|\mathcal{P}\|_{\mathcal{Z}}^p \geq \frac{\int_{B_R(0)} |x|^{Np} dx}{\mathrm{Cap}_{\alpha,s}\big(B_R(0)\big)} \gtrsim \frac{R^{Np+n}}{R^n} = R^{Np}.$$

We conclude that $\|\mathcal{P}\|_{\mathcal{Z}} = \infty$ by taking $R \to \infty$. ∎

On the other hand, elements with a certain rate of polynomial decays belong to the preduals of the Sobolev multiplier spaces.

EXAMPLE 4.5. The following rational functions belong to the preduals of the Sobolev multiplier spaces:

$$\frac{1}{\big(1+|\cdot|\big)^N} \in \dot{\mathcal{Z}}', \quad N > n - \alpha s,\ \alpha s < n, \tag{4.3}$$

$$\frac{1}{\big(1+|\cdot|\big)^N} \in \mathcal{Z}', \quad N > n,\ \alpha s \leq n. \tag{4.4}$$

PROOF. We first show the following for (4.4). Express

$$\frac{1}{\big(1+|x|\big)^N} = \frac{1}{\big(1+|x|\big)^N}\chi_{\{|x|<1\}} + \sum_{j=0}^{\infty} \frac{1}{\big(1+|x|\big)^N}\chi_{\{2^j \leq |x| < 2^{j+1}\}}.$$

The term $\frac{1}{(1+|\cdot|)^N}\chi_{\{|x|<1\}}$ belongs to $\mathcal{Z}'$. Let us estimate the second term. Indeed,

$$\begin{aligned}
&\sum_{j=0}^{\infty} \frac{1}{\big(1+|x|\big)^N}\chi_{\{2^j \leq |x| < 2^{j+1}\}} \\
&\quad \leq \sum_{j=0}^{\infty} \frac{1}{2^{jN}}\chi_{\{2^j \leq |x| < 2^{j+1}\}} \\
&\quad = \sum_{j=0}^{\infty} \frac{1}{2^{jN}} \cdot \mathrm{Cap}_{\alpha,s}\big(\{2^j \leq |x| < 2^{j+1}\}\big) \cdot \frac{\chi_{\{2^j \leq |x| < 2^{j+1}\}}}{\mathrm{Cap}_{\alpha,s}\big(\{2^j \leq |x| < 2^{j+1}\}\big)} \\
&\quad \lesssim \sum_{j=0}^{\infty} \frac{1}{2^{j(N-n)}} \cdot \frac{\chi_{\{2^j \leq |x| < 2^{j+1}\}}}{\mathrm{Cap}_{\alpha,s}\big(\{2^j \leq |x| < 2^{j+1}\}\big)}.
\end{aligned}$$

The term $\frac{\chi_{\{2^j \leq |x| < 2^{j+1}\}}}{\mathrm{Cap}_{\alpha,s}(\{2^j \leq |x| < 2^{j+1}\})}$ is a block in $B_q^{\alpha,s}$, while $\sum_{j=0}^{\infty} 2^{-j(N-n)} < \infty$ for $N > n$. We conclude that $1/(1+|\cdot|)^N \in B_q^{\alpha,s} \approx \mathcal{Z}'$ and (4.3) follows similarly. ∎

In the sequel, we will use several types of weighted norm estimates. To this end, let us recall the definitions of Muckenhoupt $\mathcal{A}_p$ weights. Let ω be a locally integrable function on $\mathbb{R}^n$ such that $\omega(x) > 0$ almost everywhere, $x \in \mathbb{R}^n$. We say that $\omega \in \mathcal{A}_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq A < \infty,$$

for every ball $B \subseteq \mathbb{R}^n$. However, $\omega \in \mathcal{A}_1$ if

$$\mathbf{M}\omega(x) \leq A \cdot \omega(x) \text{ almost everywhere, } x \in \mathbb{R}^n,$$

where $\mathbf{M}$ is the Hardy–Littlewood maximal operator defined by

$$\mathbf{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|dy, \quad x \in \mathbb{R}^n.$$

The infimum over all such constants A is denoted by $[\omega]_{\mathcal{A}_p}$. A basic fact in weighted norm theory says that $\mathcal{A}_p \subseteq \mathcal{A}_q$ for $1 \leq p \leq q < \infty$. Moreover, $\omega \in \mathcal{A}_p$ if and only if $\omega^{-\frac{1}{p-1}} \in \mathcal{A}_{p'}$, where $p' = p/(p-1)$, $1 < p < \infty$. The local weights are also defined in [16]. We say that $\omega \in \mathcal{A}_1^{\mathrm{loc}}$ if

$$\mathbf{M}^{\mathrm{loc}}\omega(x) \leq A \cdot \omega(x) \text{ almost everywhere, } x \in \mathbb{R}^n,$$

where $\mathbf{M}^{\mathrm{loc}}$ is the local Hardy–Littlewood maximal operator defined by

$$\mathbf{M}^{\mathrm{loc}}f(x) = \sup_{0<r\leq 1} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|dy, \quad x \in \mathbb{R}^n.$$

As usual, the infimum over all such constants A is denoted by $[\omega]_{\mathcal{A}_1^{\mathrm{loc}}}$. The class $\mathcal{A}_p^{\mathrm{loc}}$, $1 < p < \infty$, is defined analogous to $\mathcal{A}_p$. It is shown in [13, Lemma 8.1] that for each $\omega \in \mathcal{A}_1^{\mathrm{loc}}$ with $[\omega]_{\mathcal{A}_1^{\mathrm{loc}}} \leq \bar{\mathbf{c}}$ and $B = B_{R_0}(x_0)$, there is an $\overline{\omega} \in \mathcal{A}_1$ such that

$$\begin{aligned} \overline{\omega} &= \omega \quad \text{on } B, \\ [\overline{\omega}]_{\mathcal{A}_1} &\leq c(n, R_0, \bar{\mathbf{c}}). \end{aligned}$$

In the sequel, we will assume the strong measurability of the vector-valued functions $x \to \{f_s(x)\}_{s>0}$ which will be satisfied in the later applications.

Lemma 4.6. *For any $\omega \in \mathcal{A}_1^{\mathrm{loc}}$ and $1 < r, q < \infty$, it holds that*

$$\left\|\left(\int_0^\infty \left|\mathbf{M}^{\mathrm{loc}} f_s(\cdot)\right|^r \frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{L^q(\omega)} \lesssim [\omega]_{\mathcal{A}_1^{\mathrm{loc}}}^{c_q} \cdot \left\|\left(\int_0^\infty \left|f_s(\cdot)\right|^r \frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{L^q(\omega)}. \tag{4.5}$$

PROOF. Note that the estimate

$$\left\|\left(\int_0^\infty \left|\mathbf{M} f_s(\cdot)\right|^r \frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{L^q(\overline{\omega})} \lesssim [\overline{\omega}]_{\mathcal{A}_1}^{c_q} \cdot \left\|\left(\int_0^\infty \left|f_s(\cdot)\right|^r \frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{L^q(\overline{\omega})} \tag{4.6}$$

holds, which is known as the continuous version of vector-valued Fefferman–Stein maximal theorem. For a proof of (4.6), we use [5, Theorem 3.16, p. 496]

$$\int \|T(F)(x)\|_B^q \cdot \overline{\omega}(x)dx \lesssim [\overline{\omega}]_{\mathcal{A}_1}^{c_q} \cdot \int \|F(x)\|_A^q \cdot \overline{\omega}(x)dx,$$

where $A = L^r((0,\infty), \frac{ds}{s})$, $B = L_X^r((0,\infty), \frac{ds}{s})$, $X = L^\infty(0,\infty)$, $F(x) = \{f_s(x)\}_{s>0}$,

$$T(F)(x) = \left\|\left\{\left\|\{f_s * \phi_\delta(x)\}_{\delta>0}\right\|_{L^\infty(0,\infty)}\right\}_{s>0}\right\|_{(L^r(0,\infty), \frac{ds}{s})},$$

and $\phi \in \mathcal{S}(\mathbb{R}^n)$ is nonnegative, $\phi \geq 1$ on the unit ball $B_1(0)$. Observe that T can be realized as a Calderón–Zygmund singular integral operator associated with a kernel K satisfying Hörmander's condition that

$$\sup_{y\neq 0} \int_{|x|>2|y|} \|K(x-y) - K(x)\|_{\mathcal{L}(A,B)} dx < \infty,$$

where $\mathcal{L}(A, B)$ is the space of all bounded linear operators from A into B. Having established (4.6), let us show the following for (4.5). Let $\mathcal{Q}$ be a cube with $|\mathcal{Q}| = 1$. There is an $\overline{\omega} \in \mathcal{A}_1$ such that $\overline{\omega} = \omega$ on $3\mathcal{Q}$. We have

$$\begin{aligned}
&\int_{\mathcal{Q}} \left(\int_0^\infty \left|\mathbf{M}^{\mathrm{loc}} f_s(x)\right|^r \frac{ds}{s}\right)^{\frac{q}{r}} \omega(x)dx \\
&\quad\leq \int_{\mathcal{Q}} \left(\int_0^\infty \left|\mathbf{M}^{\mathrm{loc}} (f_s \chi_{3\mathcal{Q}})(x)\right|^r \frac{ds}{s}\right)^{\frac{q}{r}} \omega(x)dx \\
&\quad\leq \int_{\mathbb{R}^n} \left(\int_0^\infty \left|\mathbf{M}^{\mathrm{loc}} (f_s \chi_{3\mathcal{Q}})(x)\right|^r \frac{ds}{s}\right)^{\frac{q}{r}} \overline{\omega}(x)dx \\
&\quad\lesssim [\overline{\omega}]_{\mathcal{A}_1}^{c_q} \cdot \int_{\mathbb{R}^n} \left(\int_0^\infty \left|f_s \chi_{3\mathcal{Q}}(x)\right|^r \frac{ds}{s}\right)^{\frac{q}{r}} \overline{\omega}(x)dx \\
&\quad\lesssim [\omega]_{\mathcal{A}_1^{\mathrm{loc}}}^{c_q} \cdot \int_{3\mathcal{Q}} \left(\int_0^\infty \left|f_s(x)\right|^r \frac{ds}{s}\right)^{\frac{q}{r}} \omega(x)dx.
\end{aligned}$$

Summing over all dyadic cubes $\mathcal{Q}$ with $|\mathcal{Q}| = 1$ yields (4.5). ■

Suppose that $\alpha s \leq n$ and $E \subseteq \mathbb{R}^n$ with $0 < \mathrm{Cap}_{\alpha,s}(E) < \infty$. Then there exists a $V^E \geq \chi_E$ quasi-everywhere and

$$(V^E)^\delta \in \mathcal{A}_1^{\mathrm{loc}}, \tag{4.7}$$

$$[(V^E)^\delta]_{\mathcal{A}_1^{\mathrm{loc}}} \leq c(n,\alpha,s,\delta), \tag{4.8}$$

$$\|(V^E)^\delta\|_{L^1(\mathrm{Cap}_{\alpha,s})} \approx \mathrm{Cap}_{\alpha,s}(E), \tag{4.9}$$

where $\delta \in (1, n/(n-\alpha))$ for $s < 2$ and $\delta \in (s-1, n(s-1)/(n-\alpha s))$ for $s \geq 2$ (see [13, Theorem 3.1 and Lemma 3.2]). Fix a $\delta = \delta(n,\alpha,s)$ in (4.8). Then the constant $c(n,\alpha,s,\delta) = c(n,\alpha,s)$ in (4.8) depends only on n, α, and s, which we will simply denote by $\mathbf{c}$ in the sequel.

By [13, Theorem 9.5], it holds that

$$\|\mathbf{M}f\|_X \lesssim \|f\|_X$$

for $X = \dot{\mathcal{Z}}$, and $\dot{\mathcal{Z}}'$. Moreover, the above boundedness also holds for $X = \mathcal{Z}$ (see [11, Theorem in Section 2.6.3, p. 98]). On the other hand, it holds that

$$\|\mathbf{M}^{\mathrm{loc}}f\|_{\mathcal{Z}'} \lesssim \|f\|_{\mathcal{Z}'}$$

(see [13, Theorem 1.10]). In general, if T is an operator (not necessarily linear or sublinear) such that

$$\int_{\mathbb{R}^n} |T(f)(x)|^q \omega(x)dx \leq C\left(n,q,[\omega]_{\mathcal{A}_1}\right) \int_{\mathbb{R}^n} |f(x)|^q dx, \quad 1 < q < \infty,$$

where $C(n,q,[\omega]_{\mathcal{A}_1}) > 0$ is a constant depending only on n, q, $[\omega]_{\mathcal{A}_1}$, and $C(n,q,\cdot)$ is an increasing function, then

$$\|T(f)\|_X \lesssim \|f\|_X \tag{4.10}$$

for $X = \dot{\mathcal{Z}}$ and $\dot{\mathcal{Z}}'$ (see [13, Theorem 9.5]). This is due to the characterizations that

$$\|f\|_{\dot{\mathcal{Z}}} \approx \sup\left\{\left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} : \omega \in \mathcal{A}_1, [\omega]_{\mathcal{A}_1} \leq \mathbf{c}, \|\omega\|_{L^1(\mathrm{cap}_{\alpha,s})} \leq 1\right\}, \tag{4.11}$$

$$\|f\|_{\dot{\mathcal{Z}}'} \approx \inf\left\{\left(\int_{\mathbb{R}^n} |f(x)|^q \omega(x)^{1-q}dx\right)^{\frac{1}{q}} : \omega \in \mathcal{A}_1, [\omega]_{\mathcal{A}_1} \leq \mathbf{c}, \|\omega\|_{L^1(\mathrm{cap}_{\alpha,s})} \leq 1\right\} \tag{4.12}$$

(see [13, Lemma 9.1 and Theorem 9.3]). We further note that

$$\|f\|_{\mathcal{Z}} \approx \sup\left\{\left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} : \omega \in \mathcal{A}_1^{\mathrm{loc}}, [\omega]_{\mathcal{A}_1^{\mathrm{loc}}} \leq \mathbf{c}, \|\omega\|_{L^1(\mathrm{Cap}_{\alpha,s})} \leq 1\right\}, \tag{4.13}$$

(4.14)

$$\|f\|_{\mathcal{Z}'}\approx\inf\left\{\left(\int_{\mathbb{R}^n}|f(x)|^q\omega(x)^{1-q}dx\right)^{\frac{1}{q}}:\omega\in\mathcal{A}_1^{\mathrm{loc}},\,[\omega]_{\mathcal{A}_1^{\mathrm{loc}}}\leq\mathbf{c},\,\|\omega\|_{L^1(\mathrm{Cap}_{\alpha,s})}\leq 1\right\}$$

(see [13, Theorems 1.2, 1.4, and 1.8]).

The next two propositions address the vector-valued counterpart of (4.10) in terms of maximal function.

Proposition 4.7. *For any* $1<r<\infty$, *the estimate*

$$\left\|\left(\int_0^\infty|\mathbf{M}f_s(\cdot)|^r\frac{ds}{s}\right)^{\frac{1}{r}}\right\|_X\lesssim\left\|\left(\int_0^\infty|f_s(\cdot)|^r\frac{ds}{s}\right)^{\frac{1}{r}}\right\|_X$$

holds for $X=\dot{\mathcal{Z}}$ *and* $\dot{\mathcal{Z}}'$.

Proof. The proof follows by (4.6), (4.11), and (4.12). ■

Proposition 4.8. *For any* $1<r<\infty$, *it holds that*

$$\left\|\left(\int_0^\infty|\mathbf{M}^{\mathrm{loc}}f_s(\cdot)|^r\frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{\mathcal{Z}}\lesssim\left\|\left(\int_0^\infty|f_s(\cdot)|^r\frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{\mathcal{Z}}.$$

Proof. Let $K\subseteq\mathbb{R}^n$ be a compact set with $\mathrm{Cap}_{\alpha,s}(K)>0$. Choose a $(V^K)^\delta\in\mathcal{A}_1^{\mathrm{loc}}$ as in (4.7). By letting $E_t=\{x\in\mathbb{R}^n:|(V^K)^\delta(x)|>t\}$ and using Lemma 4.6, we have

$$\begin{aligned}
&\int_K\left(\int_0^\infty|\mathbf{M}^{\mathrm{loc}}f_s(x)|^r\frac{ds}{s}\right)^{\frac{p}{r}}dx\\
&\leq\int_{\mathbb{R}^n}\left(\int_0^\infty|\mathbf{M}^{\mathrm{loc}}f_s(x)|^r\frac{ds}{s}\right)^{\frac{p}{r}}(V^K)^\delta(x)dx\\
&\lesssim\int_{\mathbb{R}^n}\left(\int_0^\infty|f_s(x)|^r\frac{ds}{s}\right)^{\frac{p}{r}}(V^K)^\delta(x)dx\\
&=\int_0^\infty\left(\int_{E_t}\left(\int_0^\infty|f_s(x)|^r\frac{ds}{s}\right)^{\frac{p}{r}}dx\right)dt\\
&\leq\left\|\left(\int_0^\infty|f_s(\cdot)|^r\frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{\mathcal{Z}}^p\cdot\int_0^\infty\mathrm{Cap}_{\alpha,s}(E_t)dt\\
&=\left\|\left(\int_0^\infty|f_s(\cdot)|^r\frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{\mathcal{Z}}^p\cdot\|(V^K)^\delta\|_{L^1(\mathrm{Cap}_{\alpha,s})}.
\end{aligned}$$

We obtain by (4.9) that

$$\left(\mathrm{Cap}_{\alpha,s}(K)^{-1}\int_K\left(\int_0^\infty\left|\mathbf{M}^{\mathrm{loc}}f_s(x)\right|^r\frac{ds}{s}\right)^{\frac{p}{r}}dx\right)^{\frac{1}{p}}\lesssim\left\|\left(\int_0^\infty|f_s(\cdot)|^r\frac{ds}{s}\right)^{\frac{1}{r}}\right\|_{\mathcal{Z}}.$$

Taking supremum over all compact set $K\subseteq\mathbb{R}^n$ with $\mathrm{Cap}_{\alpha,s}(K)>0$, the result follows. ■

Lemma 4.9. *For any $\varphi:\mathbb{R}^n\to\mathbb{C}$ with*

$$\left|\varphi(x)\right|\lesssim\frac{1}{\left(1+|x|\right)^N},\quad x\in\mathbb{R}^n,\ N>n,$$

it holds that $\|\varphi*f\|_X\lesssim\|f\|_X$ for $X=\dot{\mathcal{Z}},\mathcal{Z},\dot{\mathcal{Z}}'$, and $\mathcal{Z}'$.

Proof. Note that

$$\|\varphi*f\|_X\lesssim\|\mathbf{M}f\|_X\leq\|f\|_X$$

for $X=\dot{\mathcal{Z}}$, $\mathcal{Z}$, and $\dot{\mathcal{Z}}'$. Now we deal with the case that $X=\mathcal{Z}'$. To this end, let $g\in\mathcal{Z}$ with $\|g\|_{\mathcal{Z}}\leq 1$. We have

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\varphi*f(x)g(x)dx\right|&\leq\int_{\mathbb{R}^n}\left|f(x)\right|\int_{\mathbb{R}^n}\left|\varphi(-y)\right|\cdot\left|g(x-y)\right|dy\,dx\\&\lesssim\int_{\mathbb{R}^n}\left|f(x)\right|\int_{\mathbb{R}^n}\frac{1}{\left(1+|y|\right)^N}\cdot\left|g(x-y)\right|dy\,dx\\&\lesssim\int_{\mathbb{R}^n}\left|f(x)\right|\cdot\left\|\frac{1}{\left(1+|\cdot|\right)^N}\right\|_{\mathcal{Z}'}\cdot\left\|g(x-\cdot)\right\|_{\mathcal{Z}}dx\\&\lesssim\|f\|_{L^1(\mathbb{R}^n)}\|g\|_{\mathcal{Z}}\lesssim\|f\|_{\mathcal{Z}'},\end{aligned}$$

where we have used Example 4.5, Remark 4.3, and $\|g(x-\cdot)\|_{\mathcal{Z}}=\|g\|_{\mathcal{Z}}$, which is due to the translation invariant property of the capacity that $\mathrm{Cap}_{\alpha,s}(z+E)=\mathrm{Cap}_{\alpha,s}(E)$ with $z\in\mathbb{R}^n$ and $E\subseteq\mathbb{R}^n$. The result then follows by the $(\dot{\mathcal{Z}},\dot{\mathcal{Z}}')$ duality. ■

The following lemma should be familiar to specialists. We include its proof for the readers' convenience.

Lemma 4.10. *Let $f\in\mathcal{S}'(\mathbb{R}^n)$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$ be as in Theorem 2.1. Assuming that* $\mathrm{supp}(f)$ *is compact and* $0\notin\mathrm{supp}(f)$*, then*

$$f=\lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}}\int_\varepsilon^\delta\varphi_t*f(\cdot)\frac{dt}{t},$$

where the limit is taken in $\mathcal{S}'(\mathbb{R}^n)$.

Proof. Since $\operatorname{supp}(f)$ is compact and $0 \notin \operatorname{supp}(f)$, there is a function $\eta \in \mathcal{D}(\mathbb{R}^n)$ such that $\eta(0) = 0$ and $\eta = 1$ on a neighborhood W of $\operatorname{supp}(f)$. Let $g \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary. It suffices to show that

$$\langle f, \hat{g} \rangle = \lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \int_\varepsilon^\delta \langle \varphi_t * f, \hat{g} \rangle \frac{dt}{t},$$

or, equivalently,

$$\langle \hat{f}, g \rangle = \lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \int_\varepsilon^\delta \langle \widehat{\varphi_t * f}, g \rangle \frac{dt}{t}.$$

Since $\operatorname{supp}(g(1-\eta)) \cap \operatorname{supp}(\hat{f}) = \emptyset$, we have

$$\langle \widehat{\varphi_t * f}, g \rangle = \langle \hat{f}, \widehat{\varphi_t} g \rangle = \langle \hat{f}, \widehat{\varphi_t} g \eta \rangle + \big\langle \hat{f}, \widehat{\varphi_t} g(1-\eta) \big\rangle = \langle \hat{f}, \widehat{\varphi_t} g \eta \rangle.$$

It is routine to check that

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \int_\varepsilon^\delta \widehat{\varphi}(t\cdot) h(\cdot) \frac{dt}{t} = h(\cdot), \quad h \in \mathcal{S}(\mathbb{R}^n),$$

where the limit is taken in $\mathcal{S}(\mathbb{R}^n)$. The result follows by letting $h = g\eta$. ■

In the sequel, we denote $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$ for $a, b \in \mathbb{R}$.

Lemma 4.11. *Assume that $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\widehat{\psi}(0) = \widehat{\varphi}(0) = 0$. Then*

$$\big|\psi_t * \varphi_s * f(x)\big| \lesssim \left(\frac{s}{t} \wedge \frac{t}{s}\right) \mathbf{M} f(x), \quad 0 < t, s < \infty,\ x \in \mathbb{R}^n.$$

Furthermore,

$$\big|\psi_t * \varphi_s * f(x)\big| \lesssim \left(\frac{s}{t} \wedge \frac{t}{s}\right) \mathbf{M}^{\mathrm{loc}} f(x), \quad 0 < t, s \le 1,\ x \in \mathbb{R}^n.$$

Proof. By [7, Appendix B.5, p. 599], one has

$$\big|\psi_t * \varphi_s(x)\big| \lesssim \left(\frac{s}{t} \wedge \frac{t}{s}\right) \cdot \frac{(t \vee s)^{-n}}{\big(1 + (t \vee s)^{-1}|x|\big)^{n+1}}, \quad x \in \mathbb{R}^n.$$

Let $l = t \vee s$ and $P(x) = (1 + |x|)^{-(n+1)}$, $x \in \mathbb{R}^n$. Then

$$\begin{aligned} \big|\psi_t * \varphi_s * f(x)\big| &\le \int_{\mathbb{R}^n} \big|\psi_t * \varphi_s(y)\big| \cdot \big|f(x-y)\big| dy \\ &\lesssim \left(\frac{s}{t} \wedge \frac{t}{s}\right) \int_{\mathbb{R}^n} P_l(y) \cdot \big|f(x-y)\big| dy \\ &= \left(\frac{s}{t} \wedge \frac{t}{s}\right) \cdot \big(P_l * |f|\big)(x) \lesssim \left(\frac{s}{t} \wedge \frac{t}{s}\right) \mathbf{M} f(x). \end{aligned}$$

If $0 < t, s \leq 1$, then $0 < l \leq 1$ and we have

$$\left|\psi_t * \varphi_s * f(x)\right| \lesssim \left(\frac{s}{t} \wedge \frac{t}{s}\right)\mathbf{M}^{\mathrm{loc}} f(x), \quad 0 < t, s \leq 1,\ x \in \mathbb{R}^n. \qquad \blacksquare$$

5. Proofs of main results: Riesz capacities

Proof of Theorem 2.1. The estimate

$$\left\| \int_\varepsilon^\delta \varphi_t * f(\cdot)\frac{dt}{t} \right\|_{L^q(\omega)} \lesssim [\omega]_{\mathcal{A}_1}^{c_q} \cdot \|f\|_{L^q(\omega)} \tag{5.1}$$

holds with implicit constants independent of $0 < \varepsilon < \delta < \infty$. For a proof, let

$$T_{\varepsilon,\delta}(f)(\cdot) = \int_\varepsilon^\delta \varphi_t * f(\cdot)\frac{dt}{t}.$$

We claim that $T_{\varepsilon,\delta}$ is a Calderón–Zygmund singular integral operator associated with a kernel $K_{\varepsilon,\delta}$ that

$$K_{\varepsilon,\delta}(x) = \int_\varepsilon^\delta \varphi_t(x)\frac{dt}{t}.$$

The claim follows by checking that

$$\left|\widehat{K_{\varepsilon,\delta}}(\xi)\right| \lesssim 1, \quad \xi \in \mathbb{R}^n, \tag{5.2}$$

$$\left|K_{\varepsilon,\delta}(x)\right| \lesssim \frac{1}{|x|^n}, \quad x \neq 0, \tag{5.3}$$

$$\left|\nabla K_{\varepsilon,\delta}(x)\right| \lesssim \frac{1}{|x|^{n+1}}, \quad x \neq 0, \tag{5.4}$$

with the implicit constants independent of $0 < \varepsilon < \delta < \infty$. Indeed, assuming that $\operatorname{supp}(\widehat{\varphi}) \subseteq \{a \leq |\xi| \leq b\}$ with $0 < a < b < \infty$, then

$$\left|\widehat{K_{\varepsilon,\delta}}(\xi)\right| = \left|\int_\varepsilon^\delta \widehat{\varphi}(t\xi)\frac{dt}{t}\right| \leq \int_{a|\xi|^{-1}}^{b|\xi|^{-1}} \left|\widehat{\varphi}(t\xi)\right|\frac{dt}{t} \lesssim \|\widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)},$$

so (5.2) follows. Subsequently,

$$\left|K_{\varepsilon,\delta}(x)\right| \leq \left(\sup_{x\in\mathbb{R}^n} \left(1 + |x|\right)^{n+1}\left|\varphi(x)\right|\right) \cdot \int_0^\infty \frac{1}{\left(t + |x|\right)^{n+1}} dt \lesssim \frac{1}{|x|^n},$$

which yields (5.3) and (5.4) holds in a similar fashion. Consequently, (5.1) holds by a standard weighted norm inequality (see [3, Theorem 7.1.1]), and it follows by (4.12) that

$$\left\| \int_\varepsilon^\delta \varphi_t * f(\cdot)\frac{dt}{t} \right\|_{\dot{\mathcal{Z}}'} \lesssim \|f\|_{\dot{\mathcal{Z}}'}, \quad f \in \dot{\mathcal{Z}}'.$$

Now, given any $f \in \dot{\mathcal{Z}}'$ and $\eta > 0$, choose by Lemma 4.1 some $g \in \mathcal{S}_0(\mathbb{R}^n)$ such that $f = g + b$ with $\|b\|_{\dot{\mathcal{Z}}'} < \eta$. Then

$$\begin{aligned}
&\left\| f(\cdot) - \int_\varepsilon^\delta \varphi_t * f(\cdot) \frac{dt}{t} \right\|_{\dot{\mathcal{Z}}'} \\
&\quad \le \left\| g(\cdot) - \int_\varepsilon^\delta \varphi_t * g(\cdot) \frac{dt}{t} \right\|_{\dot{\mathcal{Z}}'} + \|b\|_{\dot{\mathcal{Z}}'} + \left\| \int_\varepsilon^\delta \varphi_t * b(\cdot) \frac{dt}{t} \right\|_{\dot{\mathcal{Z}}'} \\
&\quad \lesssim \left\| g(\cdot) - \int_\varepsilon^\delta \varphi_t * g(\cdot) \frac{dt}{t} \right\|_{\dot{\mathcal{Z}}'} + 2\eta.
\end{aligned}$$

Finally, note that

$$\int_\varepsilon^\delta \varphi_t * g(\cdot) \frac{dt}{t} \to g(\cdot), \quad \varepsilon \to 0,\ \delta \to \infty,$$

in $\mathcal{S}(\mathbb{R}^n)$. The result follows by Proposition 4.2 since

$$\limsup_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \left\| f(\cdot) - \int_\varepsilon^\delta \varphi_t * f(\cdot) \frac{dt}{t} \right\|_{\dot{\mathcal{Z}}'} \le 2\eta,$$

and $\eta > 0$ is arbitrarily given. ■

The next two propositions provide the one-sided estimates of the Littlewood–Paley functions on the Sobolev multiplier spaces and their preduals.

Proposition 5.1. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\widehat{\varphi}(0) = 0$. *Then*

$$\left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_X \lesssim \|f\|_X$$

for $X = \dot{\mathcal{Z}}$ *and* $\dot{\mathcal{Z}}'$.

Proof. A standard fact regarding the weighted Littlewood–Paley theory says that

$$\left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^q(\omega)} \lesssim [\omega]_{\mathcal{A}_1}^{c_q} \cdot \|f\|_{L^q(\omega)}$$

(see [10, Theorem 5.2.2]). Now we recourse to (4.11) and (4.12) to finish the proof. ■

Proposition 5.2. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *be as in Theorem* 2.1. *Then*

$$\|f\|_X \lesssim \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_X \tag{5.5}$$

for $f \in X$ *with* $0 \notin \operatorname{supp}(\hat{f})$, $X = \dot{\mathcal{Z}}$ *and* $\dot{\mathcal{Z}}'$.

Proof. Choose a radial $\tau \in \mathcal{S}_0(\mathbb{R}^n)$ such that $\widehat{\tau} = 1$ on the support of $\widehat{\varphi}$. As a consequence, $\tau_t * \varphi_t = \varphi_t$. We first assume that $f \in \dot{\mathcal{Z}}$ and $g \in \dot{\mathcal{Z}}'$ with $\|g\|_{\dot{\mathcal{Z}}'} \leq 1$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)g(x)dx &= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_\varepsilon^\delta \langle f, \varphi_t * g\rangle \frac{dt}{t} \\
&= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_\varepsilon^\delta \langle \varphi_t * f, \tau_t * g\rangle \frac{dt}{t} \\
&= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_{\mathbb{R}^n} \int_\varepsilon^\delta \varphi_t * f(x)\tau_t * g(x) \frac{dt}{t} dx.
\end{aligned} \tag{5.6}$$

We obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| &\leq \int_{\mathbb{R}^n} \left(\int_0^\infty |\varphi_t * f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_0^\infty |\tau_t * g(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx \\
&\leq \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}} \left\| \left(\int_0^\infty |\tau_t * g(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}'} \\
&\lesssim \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}} \cdot \|g\|_{\dot{\mathcal{Z}}'} \\
&\leq \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}}.
\end{aligned}$$

Hence, (5.5) follows by the $(\dot{\mathcal{Z}}, \dot{\mathcal{Z}}')$ duality for $X = \dot{\mathcal{Z}}$.

Now we let $f \in \dot{\mathcal{Z}}'$ and $g \in \mathcal{S}_0(\mathbb{R}^n)$ and $\|g\|_{\dot{\mathcal{Z}}} \leq 1$. By Theorem 2.1, (5.6) is valid. Hence, we obtain by Proposition 5.1 that

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| &\leq \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}'} \left\| \left(\int_0^\infty |\tau_t * g(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}} \\
&\lesssim \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}'} \cdot \|g\|_{\dot{\mathcal{Z}}} \\
&\leq \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}'}.
\end{aligned}$$

Denote by $C_0(\mathbb{R}^n)$ the space of all compactly supported continuous functions on $\mathbb{R}^n$. Recall the result in [13, Theorem 9.4] that $(\overline{C_0(\mathbb{R}^n)}^{\dot{\mathcal{Z}}})^* \approx \dot{\mathcal{Z}}'$, from which one may easily modify the proof to deduce that

$$\left(\overline{\mathcal{S}_0(\mathbb{R}^n)}^{\dot{\mathcal{Z}}}\right)^* \approx \dot{\mathcal{Z}}'. \tag{5.7}$$

Consequently, the estimate (5.5) follows for $X = \dot{\mathcal{Z}}'$ by the duality (5.7). ∎

PROOF OF THEOREM 2.2. One obtains the result by combining Propositions 5.1 and 5.2. ∎

If we remove the assumption that $0 \notin \operatorname{supp}(\hat{f})$ in Theorem 2.2, then we have only the one-sided estimate.

PROPOSITION 5.3. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be as in Theorem 2.1. Assuming that $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies*

$$\left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}} < \infty,$$

then the limit

$$F(\cdot) = \lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \int_\varepsilon^\delta \varphi_t * f(\cdot) \frac{dt}{t}$$

exists in the weak-topology of $\dot{\mathcal{Z}}$, and it satisfies*

$$\|F\|_{\dot{\mathcal{Z}}} \lesssim \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}}.$$

If, in addition, $f \in \dot{\mathcal{Z}}$, then

$$f(\cdot) = \lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \int_\varepsilon^\delta \varphi_t * f(\cdot) \frac{dt}{t},$$

where the limit is taken in the weak-topology of $\dot{\mathcal{Z}}$, and it holds that*

$$\|f\|_{\dot{\mathcal{Z}}} \lesssim \left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}}.$$

PROOF. Let

$$f_N(\cdot) = \int_{2^{-N}}^{2^N} \varphi_t * f(\cdot) \frac{dt}{t}, \quad N \in \mathbb{N}.$$

Proposition 5.2 yields

$$\|f_N\|_{\dot{\mathcal{Z}}} \lesssim \left\| \left(\int_0^\infty |\varphi_t * f_N(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}}.$$

Assuming that $\operatorname{supp}(\widehat{\varphi}) \subseteq \{a \le |\xi| \le b\}$ with $0 < a < b < \infty$, then $\widehat{\varphi}(s\xi)\widehat{\varphi}(t\xi) = 0$ for $\frac{s}{t} < \frac{a}{b}$ and $\frac{s}{t} > \frac{b}{a}$, $\xi \in \mathbb{R}^n$. Hence,

$$\begin{aligned}
|\varphi_t * f_N(\cdot)| &= \left| \int_{2^{-N}}^{2^N} \varphi_s * \varphi_t * f(\cdot) \frac{ds}{s} \right| \\
&= \left| \int_{2^{-N}}^{2^N} \chi_{\frac{a}{b} \le \frac{s}{t} \le \frac{b}{a}} \cdot \varphi_s * \varphi_t * f(\cdot) \frac{ds}{s} \right|
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{2^{-N}}^{2^N} \chi_{\frac{a}{b} \le \frac{s}{t} \le \frac{b}{a}} \cdot \mathbf{M}(\varphi_t * f)(\cdot) \frac{ds}{s} \\
&\le \mathbf{M}(\varphi_t * f)(\cdot) \int_{\frac{a}{b}\cdot t}^{\frac{b}{a}\cdot t} \frac{ds}{s} \\
&\lesssim \mathbf{M}(\varphi_t * f)(\cdot).
\end{aligned}$$

We have

$$\|f_N\|_{\dot{\mathcal{Z}}} \lesssim \left\| \left(\int_0^\infty \left|\mathbf{M}(\varphi_t * f)(\cdot)\right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}} \lesssim \left\| \left(\int_0^\infty \left|\varphi_t * f(\cdot)\right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}},$$

where the implicit constants are independent of $N \in \mathbb{N}$. Since $(\dot{\mathcal{Z}}')^* \approx \dot{\mathcal{Z}}$ and $\dot{\mathcal{Z}}'$ is separable, by the sequential Banach–Alaoglu theorem, there is a subsequence $\{f_{N_k}\}$ such that $f_{N_k} \to F$ in the weak*-topology of $\dot{\mathcal{Z}}$; that is,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_{N_k}(x) g(x) dx = \int_{\mathbb{R}^n} F(x) g(x) dx, \quad g \in \dot{\mathcal{Z}}'.$$

Using the above, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} F(x) g(x) dx &= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_\varepsilon^\delta \int_{\mathbb{R}^n} F(x) \varphi_t * g(x) dx \frac{dt}{t} \\
&= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_\varepsilon^\delta \int_{\mathbb{R}^n} \varphi_t * F(x) g(x) dx \frac{dt}{t} \\
&= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_{\mathbb{R}^n} \int_\varepsilon^\delta \langle F, \varphi_t(x - \cdot)\rangle \frac{dt}{t} g(x) dx \\
&= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_{\mathbb{R}^n} \lim_{k\to\infty} \int_\varepsilon^\delta \langle f_{N_k}, \varphi_t(x - \cdot)\rangle \frac{dt}{t} g(x) dx \\
&= \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_{\mathbb{R}^n} \int_\varepsilon^\delta \varphi_t * f(x) \frac{dt}{t} g(x) dx.
\end{aligned}$$

We conclude that the limit

$$F(\cdot) = \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_\varepsilon^\delta \varphi_t * f(\cdot) \frac{dt}{t} \tag{5.8}$$

exists in the weak*-topology of $\dot{\mathcal{Z}}$.

As $\{f_{N_k}\}$ converges to F in the weak*-topology of $\dot{\mathcal{Z}}$, we have

$$\|F\|_{\dot{\mathcal{Z}}} \le \liminf_{k\to\infty} \|f_{N_k}\|_{\dot{\mathcal{Z}}} \lesssim \left\| \left(\int_0^\infty \left|\varphi_t * f(\cdot)\right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{\dot{\mathcal{Z}}}.$$

If, in addition, $f \in \dot{\mathcal{Z}}$, then we have by Proposition 5.1 that

$$\left\| \left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t} \right) \right\|_{\dot{\mathcal{Z}}} \lesssim \|f\|_{\dot{\mathcal{Z}}} < \infty.$$

By (5.8), $\widehat{f - F}$ is supported in the origin, and it follows that $f - F \in \dot{\mathcal{Z}}$ is a polynomial. We conclude by Lemma 4.4 that $f = F$. ■

The following proposition gives the inversion formula for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ under the topologies of the preduals $\dot{\mathcal{Z}}'$ and $\mathcal{Z}'$.

Proposition 5.4. *Let ψ be as in (2.1). For any $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, it holds that*

$$f(\cdot) = \lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \int_\varepsilon^\delta \psi_t * \psi_t * f(\cdot) \frac{dt}{t},$$

where the limit is taken in both $\dot{\mathcal{Z}}'$ and $\mathcal{Z}'$.

Proof. Let $\varepsilon, \delta > 0$ be given. Suppose that $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, and then we have

$$\left| f(x) - \int_\varepsilon^\delta \psi_t * \psi_t * f(x) \frac{dt}{t} \right| \leq c(\psi, f, N, n) \left(\varepsilon + \frac{1}{\delta} \right) \frac{1}{(1 + |x|)^N}$$

for each $N \in \mathbb{N}$, where $c(\psi, f, N, n) > 0$ is a constant depending on ψ, f, N, and n (see [4, p. 124]). By choosing an $N > n$, it holds that $1/(1 + |\cdot|)^N \in \dot{\mathcal{Z}}'$ and $\mathcal{Z}'$ by Example 4.5. As a result,

$$\left\| f(\cdot) - \int_\varepsilon^\delta \psi_t * \psi_t * f(\cdot) \frac{dt}{t} \right\|_X \leq c'(\psi, f, N, n) \left(\varepsilon + \frac{1}{\delta} \right) \to 0$$

as $\varepsilon \to 0$, $\delta \to \infty$ for $X = \dot{\mathcal{Z}}'$ and $\mathcal{Z}'$, and the lemma follows. ■

Alternative proof of Proposition 5.4. We know that for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$f(\cdot) = \lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} \int_\varepsilon^\delta \psi_t * \psi_t * f(\cdot) \frac{dt}{t},$$

where the limit is taken in $\mathcal{S}(\mathbb{R}^n)$. Now we recourse to the embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow X$ for $X = \dot{\mathcal{Z}}'$ and $\mathcal{Z}'$ as in Proposition 4.2. ■

Proof of Theorem 2.3. Fix a $1 < q < \infty$ and let $p = \frac{q}{q-1}$ be its Hölder's conjugate. If $\omega \in \mathcal{A}_1 \subseteq \mathcal{A}_q$, then $\omega^{1-p} \in \mathcal{A}_p$. By an argument of $(L^q(\omega^{1-p}), L^p(\omega^{1-p}))$ duality, twice the applications of Hölder inequalities with appropriate indices, and the standard weighted Littlewood–Paley theory, for any $f \in \dot{\mathcal{Z}}'$, we have

$$\begin{aligned}
&\left\| \int_\varepsilon^\delta \psi_t * \psi_t * f(\cdot) \frac{dt}{t} \right\|_{L^q(\omega)} \\
&= \left\| \int_\varepsilon^\delta \psi_t * \psi_t * f(\cdot) \frac{dt}{t} \omega^{p-1}(\cdot) \right\|_{L^q(\omega^{1-p})} \\
&= \sup_{\|h\|_{L^p(\omega^{1-p})} \le 1} \left| \int_{\mathbb{R}^n} \int_\varepsilon^\delta \psi_t * \psi_t * f(x) \frac{dt}{t} h(x) dx \right| \\
&= \sup_{\|h\|_{L^p(\omega^{1-p})} \le 1} \left| \int_{\mathbb{R}^n} \int_\varepsilon^\delta \psi_t * f(x) \psi_t * h(x) \frac{dt}{t} dx \right| \\
&\le \sup_{\|h\|_{L^p(\omega^{1-p})} \le 1} \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \omega(x)^{\frac{1}{q}} \left(\int_0^\infty |\psi_t * h(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \omega(x)^{-\frac{1}{q}} dx \\
&\le \sup_{\|h\|_{L^p(\omega^{1-p})} \le 1} \left\| \left(\int_0^\infty |\psi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^q(\omega)} \left\| \left(\int_0^\infty |\psi_t * h(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p(\omega^{1-p})} \\
&\lesssim [\omega]_{\mathcal{A}_1}^{c_q} \cdot \sup_{\|h\|_{L^p(\omega^{1-p})} \le 1} \left\| \left(\int_0^\infty |\psi_t * f(\cdot)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^q(\omega)} \|h\|_{L^p(\omega^{1-p})} \\
&\lesssim [\omega]_{\mathcal{A}_1}^{c_q} \cdot \|f\|_{L^q(\omega)}.
\end{aligned}$$

We deduce by (4.12) that

$$\left\| \int_\varepsilon^\delta \psi_t * \psi_t * f(\cdot) \frac{dt}{t} \right\|_{\dot{\mathcal{Z}}'} \lesssim \|f\|_{\dot{\mathcal{Z}}'}, \quad f \in \dot{\mathcal{Z}}'.$$

We combine the above with Proposition 5.4 to obtain the result by repeating the argument given in the last part of the proof of Theorem 2.1. ■

Proposition 5.5. *For any $1 < \alpha < \infty$, it holds that*

$$\|s_\alpha(f)\|_X \lesssim \|s(f)\|_X$$

for $X = \dot{\mathcal{Z}}$ and $\dot{\mathcal{Z}}'$.

Proof. If $\omega \in \mathcal{A}_1$, then

$$\begin{aligned}
\|s_\alpha(f)\|_{L^q(\omega)} &\lesssim \alpha^{n(\frac{1}{q}-\frac{1}{2})} [\omega]_{\mathcal{A}_1}^{c_q} \cdot \|s(f)\|_{L^q(\omega)}, \quad 1 < q \le 2, \\
\|s_\alpha(f)\|_{L^q(\omega)} &\lesssim [\omega]_{\mathcal{A}_1}^{c'_q} \cdot \|s(f)\|_{L^q(\omega)}, \quad 2 < q < \infty,
\end{aligned}$$

where $c_q, c'_q > 0$ are constants depending only on $1 < q < \infty$ (see [17, Theorems 1 and 2, Chapter IV]). The result follows by (4.11) and (4.12). ■

PROOF OF THEOREM 2.4. We quote from [9, Theorem 1.2] that

$$\left\|s(f)\right\|_{L^q(\omega)} \lesssim [\omega]_{\mathcal{A}_1}^{c_q} \cdot \|f\|_{L^q(\omega)}, \quad 1 < q < \infty,$$

which yields $\|s(f)\|_X \lesssim \|f\|_X$ for $X = \dot{\mathcal{Z}}$ and $\dot{\mathcal{Z}}'$ by (4.11) and (4.12) respectively. On the other hand, it is computed in [18, (4.3), p. 314] that

$$\|f\|_{L^2(\mathbb{R}^n)} = c_n \left\|s(f)\right\|_{L^2(\mathbb{R}^n)}.$$

By polarization technique, for any $f \in L^2(\mathbb{R}^n) \cap \dot{\mathcal{Z}}'$ and $g \in \mathcal{S}_0(\mathbb{R}^n)$, one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \\
&= c_n\left|\int_{\mathbb{R}^n}\int_0^\infty\int_{\mathbb{R}^n} \psi_t * f(y)\psi_t * g(y)\chi_{|x-y|<t}dy\frac{dt}{t}dx\right| \\
&\lesssim \int_{\mathbb{R}^n}\left(\int_0^\infty\int_{|x-y|<t}\left|\psi_t * f(x)\right|^2 dy\frac{dt}{t}\right)^{\frac{1}{2}}\left(\int_0^\infty\int_{|x-y|<t}\left|\psi_t * g(x)\right|^2 dy\frac{dt}{t}\right)^{\frac{1}{2}}dx \\
&\leq \left\|s(f)\right\|_{\dot{\mathcal{Z}}'}\left\|s(g)\right\|_{\dot{\mathcal{Z}}} \\
&\lesssim \left\|s(f)\right\|_{\dot{\mathcal{Z}}'}\|g\|_{\dot{\mathcal{Z}}}.
\end{aligned}$$

We conclude by the duality (5.7) that $\|f\|_{\dot{\mathcal{Z}}'} \lesssim \|s(f)\|_{\dot{\mathcal{Z}}'}$, whereas the estimate $\|f\|_{\dot{\mathcal{Z}}} \lesssim \|s(f)\|_{\dot{\mathcal{Z}}}$ for $f \in L^2(\mathbb{R}^n) \cap \dot{\mathcal{Z}}$ also holds similarly. ■

6. PROOFS OF MAIN RESULTS: BESSEL CAPACITIES

PROOF OF THEOREM 3.1. We first note that $\|\varphi_0 * f\|_{\mathcal{Z}} \lesssim \|\mathbf{M}f\|_{\mathcal{Z}} \lesssim \|f\|_{\mathcal{Z}}$ and that

$$\begin{aligned}
\|\varphi_0 * f\|_{\mathcal{Z}'} &\approx \sup_{\|g\|_{\mathcal{Z}}\leq 1}\left|\int_{\mathbb{R}^n}\varphi_0 * f(x)g(x)dx\right| \\
&= \sup_{\|g\|_{\mathcal{Z}}\leq 1}\left|\int_{\mathbb{R}^n} f(x)\widetilde{\varphi_0} * g(x)dx\right| \\
&\lesssim \|f\|_{\mathcal{Z}'}\cdot\sup_{\|g\|_{\mathcal{Z}}\leq 1}\|\widetilde{\varphi_0} * g\|_{\mathcal{Z}} \\
&\lesssim \|f\|_{\mathcal{Z}'},
\end{aligned}$$

where $\widetilde{\varphi_0}(x) = \varphi_0(-x)$, $x \in \mathbb{R}^n$. For simplicity, denote

$$\mathcal{G}(f)(\cdot) = \left(\int_0^1\left|\varphi_t * f(\cdot)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}, \quad f \in \mathcal{D}'(\mathbb{R}^n). \tag{6.1}$$

Fix a cube $\mathcal{Q}$ with $|\mathcal{Q}| = 1$. Let $a > 0$ be $\operatorname{supp}(\varphi_0) \cup \operatorname{supp}(\varphi) \subseteq B_a(0)$ and $\tilde{f} = f\chi_{(1+a)\mathcal{Q}}$. Note that $\mathcal{G}(f)$ can be realized as a Calderón–Zygmund singular integral operator so that $\|\mathcal{G}(f)\|_{L^q(\overline{\omega})} \lesssim [\overline{\omega}]_{\mathcal{A}_1}^{c_q} \cdot \|f\|_{L^q(\overline{\omega})}$. Given any $\omega \in \mathcal{A}_1^{\mathrm{loc}}$, we extend ω to an $\overline{\omega} \in \mathcal{A}_1$ such that $\overline{\omega} = \omega$ on $(1+2a)\mathcal{Q}$. Observe that $\mathcal{G}(f) = \mathcal{G}(\tilde{f})$ on $\mathcal{Q}$. We obtain

$$\|\mathcal{G}(f)\|_{L^q_\omega(\mathcal{Q})} \leq \|\mathcal{G}(\tilde{f})\|_{L^q_{\overline{\omega}}(\mathbb{R}^n)} \lesssim [\overline{\omega}]_{\mathcal{A}_1}^{c_q} \cdot \|\tilde{f}\|_{L^q_{\overline{\omega}}(\mathbb{R}^n)} = [\overline{\omega}]_{\mathcal{A}_1}^{c_q} \cdot \|f\|_{L^q_\omega((1+a)\mathcal{Q})}.$$

Summing over all dyadic cubes $\mathcal{Q}$ with $|\mathcal{Q}| = 1$ yields the $\lesssim$ direction of (3.2) by means of (4.13) and (4.14). To prove the $\gtrsim$ directions of (3.2), express $f \in \mathcal{D}'(\mathbb{R}^n)$ as

$$f(\cdot) = \psi_0 * \varphi_0 * f(\cdot) + \lim_{\varepsilon \to 0} \int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot) \frac{dt}{t},$$

where the limit converges in $\mathcal{D}'(\mathbb{R}^n)$. Fix a $g \in \mathcal{D}(\mathbb{R}^n)$ with $\|g\|_{\mathcal{Z}'} \leq 1$. We have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x) g(-x) dx\right| \\
&\leq |\psi_0 * \varphi_0 * f * g(0)| + \lim_{\varepsilon \to 0} \int_\varepsilon^1 |\psi_t * \varphi_t * f * g(0)| \frac{dt}{t} \\
&\leq \int_{\mathbb{R}^n} |\varphi_0 * f(x)| \cdot |\psi_0 * g(-x)| dx + \int_0^1 \int_{\mathbb{R}^n} |\varphi_t * f(x)| \cdot |\psi_t * g(-x)| dx \frac{dt}{t} \\
&\leq \|\varphi_0 * f\|_{\mathcal{Z}} \cdot \|\psi_0 * g\|_{\mathcal{Z}'} + \int_{\mathbb{R}^n} \mathcal{G}(f)(x) \cdot \mathcal{G}(g)(x) dx \\
&\lesssim \|\varphi_0 * f\|_{\mathcal{Z}} + \|\mathcal{G}(f)\|_{\mathcal{Z}},
\end{aligned}$$

which yields the $\gtrsim$ direction of (3.2) for $X = \mathcal{Z}$. For the case where $X = \mathcal{Z}'$, we use the duality similar to (5.7) that $(\overline{\mathcal{D}(\mathbb{R}^n)}^{\mathcal{Z}})^* \approx \mathcal{Z}'$. ∎

Proposition 6.1. *Let ψ_t and φ_t be as in (3.1). Then*

$$\left\|\int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot) \frac{dt}{t}\right\|_X \lesssim \|f\|_X \tag{6.2}$$

for $X = \mathcal{Z}$ and $\mathcal{Z}'$, where the implicit constants are independent of $0 < \varepsilon < 1$.

Proof. In view of (4.13) and (4.14), it suffices to prove that

$$\left\|\int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot) \frac{dt}{t}\right\|_{L^q(\omega)} \lesssim [\omega]_{\mathcal{A}_1^{\mathrm{loc}}}^{c_q} \cdot \|f\|_{L^q(\omega)}$$

with implicit constants independent of $0 < \varepsilon < 1$. Fix a cube $\mathcal{Q}$ with $|\mathcal{Q}| = 1$. Take an $a > 0$ so that $B_a(0)$ contains both $\mathrm{supp}(\psi)$ and $\mathrm{supp}(\varphi)$. Let $\tilde{f} = f\chi_{(1+2a)\mathcal{Q}}$ and

$$T_\varepsilon(f)(\cdot) = \int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot) \frac{dt}{t}.$$

Note that the estimates in the beginning of the proof of Proposition 2.3 already show that $\|T_\varepsilon f\|_{L^q(\varpi)} \lesssim [\varpi]_{\mathcal{A}_1}^{c_q} \cdot \|f\|_{L^q(\varpi)}$ with implicit constants independent of $0<\varepsilon<1$. Now we extend $\omega \in \mathcal{A}_1^{\mathrm{loc}}$ to an $\varpi \in \mathcal{A}_1$ such that $\varpi = \omega$ on $(1+2a)\mathcal{Q}$. Observe that $T_\varepsilon(f) = T_\varepsilon(\tilde{f})$ on $\mathcal{Q}$. We obtain

$$\|T_\varepsilon f\|_{L^q_\omega(\mathcal{Q})} \le \left\|T_\varepsilon(\tilde{f})\right\|_{L^q_{\varpi}(\mathbb{R}^n)} \lesssim [\varpi]_{\mathcal{A}_1}^{c_q} \cdot \|\tilde{f}\|_{L^q_{\varpi}(\mathbb{R}^n)} = [\varpi]_{\mathcal{A}_1}^{c_q} \cdot \|f\|_{L^q_\omega((1+2a)\mathcal{Q})}.$$

Summing over all dyadic cubes $\mathcal{Q}$ with $|\mathcal{Q}|=1$ yields (6.2). ■

Proof of Theorem 3.2. Let

$$f_N(\cdot) = \psi_0 * \varphi_0 * f(\cdot) + \int_{2^{-N}}^1 \psi_t * \varphi_t * f(\cdot)\frac{dt}{t}, \quad N \in \mathbb{N}.$$

Theorem 3.1 yields

$$\|f_N\|_{\mathcal{Z}} \lesssim \|\varphi_0 * f_N\|_{\mathcal{Z}} + \left\|\left(\int_0^1 |\varphi_t * f_N(\cdot)|^2\frac{dt}{t}\right)^{\frac12}\right\|_{\mathcal{Z}}.$$

Furthermore, using Lemma 4.11, we have

$$\begin{aligned}
(6.3)\quad |\varphi_t * f_N(\cdot)| &= \left|\int_{2^{-N}}^1 \varphi_t * \psi_s * \varphi_s * f(\cdot)\frac{ds}{s}\right|, \quad 0<t<1\\
&\lesssim \int_{2^{-N}}^1 \left(\frac{s}{t}\wedge\frac{t}{s}\right)\cdot \mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(\cdot)\frac{ds}{s}\\
&\le \left(\int_{2^{-N}}^1 \left(\frac{s}{t}\wedge\frac{t}{s}\right)\frac{ds}{s}\right)^{\frac12}\left(\int_{2^{-N}}^1 \left(\frac{s}{t}\wedge\frac{t}{s}\right)\left|\mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(\cdot)\right|^2\frac{ds}{s}\right)^{\frac12}\\
&\lesssim \left(\int_0^1 \left(\frac{s}{t}\wedge\frac{t}{s}\right)\left|\mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(\cdot)\right|^2\frac{ds}{s}\right)^{\frac12}.
\end{aligned}$$

Besides that,

$$|\varphi_0 * f_N(\cdot)| \lesssim |\mathbf{M}(\varphi_0 * f)(\cdot)| + \left|\int_{2^{-N}}^1 \psi_t * \varphi_t * \varphi_0 * f(\cdot)\frac{dt}{t}\right|.$$

Using Propositions 6.1 and 4.8, one has

$$\begin{aligned}
\|f_N\|_{\mathcal{Z}} &\lesssim \left\|\mathbf{M}(\varphi_0 * f)\right\|_{\mathcal{Z}} + \left\|\int_{2^{-N}}^1 \psi_t * \varphi_t * \varphi_0 * f(\cdot)\frac{dt}{t}\right\|_{\mathcal{Z}}\\
&\quad + \left\|\left(\int_0^1\int_0^1 \left(\frac{s}{t}\wedge\frac{t}{s}\right)\left|\mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(\cdot)\right|^2\frac{ds}{s}\frac{dt}{t}\right)^{\frac12}\right\|_{\mathcal{Z}}\\
&\lesssim \|\varphi_0 * f\|_{\mathcal{Z}} + \left\|\left(\int_0^1 \left|\mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(\cdot)\right|^2\frac{ds}{s}\right)^{\frac12}\right\|_{\mathcal{Z}}\\
&\lesssim \|\varphi_0 * f\|_{\mathcal{Z}} + \left\|\left(\int_0^1 |\varphi_s * f(\cdot)|^2\frac{ds}{s}\right)^{\frac12}\right\|_{\mathcal{Z}},
\end{aligned}$$

where the implicit constants are independent of $N \in \mathbb{N}$. The rest of the argument follows similarly to the proof of Proposition 5.3. ∎

Proof of Theorem 3.3. Since $S(f)(\cdot) \lesssim S^d(f)(\cdot)$, (3.3) follows by the $\gtrsim$ direction of (3.4). Now we show the following for (3.4). Recall the notation $\mathcal{G}(f)(\cdot)$ in (6.1). Then Theorem 3.1 and $\mathcal{G}(f)(\cdot) \lesssim S^d(f)(\cdot)$ imply that

$$\|f\|_X \lesssim \|\varphi_0 * f\|_X + \left\|S^d(f)\right\|_X.$$

On the other hand, let $f \in \mathcal{D}(\mathbb{R}^n)$ be given. We have

$$f(\cdot) = \psi_0 * \varphi_0 * f(\cdot) + \lim_{\varepsilon\to 0}\int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot)\frac{dt}{t}$$

where the limit is taken in $\mathcal{D}(\mathbb{R}^n)$. For any $x, z \in \mathcal{Q}$, $\mathcal{Q} \in \mathfrak{Q}_t$, and $0 < t < 1$, we obtain

$$\left|\varphi_t * \psi_0 * \varphi_0 * f(z)\right| \lesssim \mathbf{M}^{\mathrm{loc}}(\psi_0 * \varphi_0 * f)(x),$$

and the estimates in (6.3) show that

$$\left|\int_\varepsilon^1 \varphi_t * \psi_s * \varphi_s * f(z)\frac{ds}{s}\right| \lesssim \left(\int_0^1 \left(\frac{s}{t}\wedge\frac{t}{s}\right)\left|\mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(x)\right|^2\frac{ds}{s}\right)^{\frac{1}{2}}, \quad \varepsilon > 0.$$

Since $\mathfrak{Q}_t$ is a disjoint class, it follows that

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_t}\sup_{z\in\mathcal{Q}}\left|\varphi_t * f(z)\right|^2\chi_{\mathcal{Q}}(x)$$
$$\lesssim \left|\mathbf{M}^{\mathrm{loc}}(\psi_0 * \varphi_0 * f)(x)\right|^2 + \int_0^1\left(\frac{s}{t}\wedge\frac{t}{s}\right)\left|\mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(x)\right|^2\frac{ds}{s}.$$

Subsequently, for $X = L^q_\omega(\mathbb{R}^n)$,

$$\begin{aligned}
&\left\|S^d(f)\right\|_X\\
&\le \left\|\mathbf{M}^{\mathrm{loc}}(\psi_0 * \varphi_0 * f)\right\|_X + \left\|\left(\int_0^1\int_0^1\left(\frac{s}{t}\wedge\frac{t}{s}\right)\left|\mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(\cdot)\right|^2\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_X\\
&\lesssim \left\|\mathbf{M}^{\mathrm{loc}}(\psi_0 * \varphi_0 * f)\right\|_X + \left\|\left(\int_0^1\left|\mathbf{M}^{\mathrm{loc}}(\varphi_s * f)(\cdot)\right|^2\frac{ds}{s}\right)^{\frac{1}{2}}\right\|_X\\
&\lesssim \|\psi_0 * \varphi_0 * f\|_X + \left\|\left(\int_0^1\left|\varphi_t * f(\cdot)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_X.
\end{aligned}$$

The above estimates still hold for general $f \in L^q_\omega(\mathbb{R}^n)$ by approximating f with the functions in $\mathcal{D}(\mathbb{R}^n)$ and using Fatou's lemma. Therefore, one can conclude (3.4) by (4.13), (4.14), and Lemma 4.9. ∎

For the sake of completeness, we include the following result which corresponds to the topological property given in Proposition 5.4 with $\dot{\mathcal{Z}}'$ in place of $\mathcal{Z}'$ for $f \in \mathcal{Z}'$.

PROPOSITION 6.2. *Let φ_0 and φ_t be as in (3.1). For any $f \in \mathcal{Z}'$, it holds that*

$$f(\cdot) = \psi_0 * \varphi_0 * f(\cdot) + \lim_{\varepsilon \to 0} \int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot) \frac{dt}{t}, \tag{6.4}$$

where the limit is taken in $\mathcal{Z}'$.

PROOF. Since

$$\psi_0 * \varphi_0 * f(\cdot) + \int_\varepsilon^1 \psi_t * \varphi_t * f(\cdot) \frac{dt}{t} \to f(\cdot)$$

in $\mathcal{D}(\mathbb{R}^n)$, we may use Proposition 4.2 to deduce that (6.4) holds for $f \in \mathcal{D}'(\mathbb{R}^n)$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{Z}'$, (6.4) holds for $f \in \mathcal{Z}'$ by Proposition 6.1. ∎

7. FINAL REMARKS

The Morrey space $L^{p,\lambda}$ for $1 \le p < \infty$, $0 < \lambda \le n$ is defined to be the set of all locally p-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\lambda}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{\lambda-n}\left(\int_{B_r(x)} |f(y)|^p dy\right)^{\frac{1}{p}} < \infty.$$

A Littlewood–Paley characterization for $L^{p,\lambda}$ and its predual $H^{p',\lambda}$ can be obtained through the means in this paper. First of all, part of the results in [1,2] say that

$$\|f\|_{L^{p,\lambda}} \approx \sup\left\{\left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}} : \omega \in \mathcal{A}_1,\ \|\omega\|_{L^1(\Lambda^{(\infty)}_{N-\lambda})} \le 1\right\},$$

$$\|f\|_{H^{p',\lambda}} \approx \inf\left\{\left(\int_{\mathbb{R}^n} |f(x)|^{p'} \omega(x)^{1-p'} dx\right)^{\frac{1}{p'}} : \omega \in \mathcal{A}_1,\ \|\omega\|_{L^1(\Lambda^{(\infty)}_{N-\lambda})} \le 1\right\},$$

where $\Lambda^{(\infty)}_{N-\lambda}$ is the Hausdorff capacity. Therefore, it holds that

$$\|f\|_X \approx \left\|\left(\int_0^\infty |\varphi_t * f(\cdot)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_X, \quad 0 \notin \operatorname{supp}(\hat{f}),$$

for $X = L^{p,\lambda}, H^{p',\lambda}$, and φ is a sufficiently regular function. This is a different approach than the norm estimates of the Littlewood–Paley theory given in [8].

Acknowledgments. – The author would like to thank the anonymous referee(s) for a number of very helpful suggestions.

References

[1] D. R. Adams – J. Xiao, Nonlinear potential analysis on Morrey spaces and their capacities. *Indiana Univ. Math. J.* **53** (2004), no. 6, 1629–1663. Zbl 1100.31009 MR 2106339

[2] D. R. Adams – J. Xiao, Morrey spaces in harmonic analysis. *Ark. Mat.* **50** (2012), no. 2, 201–230. Zbl 1254.31009 MR 2961318

[3] J. Duoandikoetxea, *Fourier analysis*. Grad. Stud. Math. 29, American Mathematical Society, Providence, RI, 2001. Zbl 0969.42001 MR 1800316

[4] M. Frazier – B. Jawerth – G. Weiss, *Littlewood-Paley theory and the study of function spaces*. CBMS Reg. Conf. Ser. Math. 79, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1991. Zbl 0757.42006 MR 1107300

[5] J. García-Cuerva – J. L. Rubio de Francia, *Weighted norm inequalities and related topics*. North-Holland Math. Stud. 116, North-Holland Publishing, Amsterdam, 1985. Zbl 0578.46046 MR 0807149

[6] P. Germain, Multipliers, paramultipliers, and weak-strong uniqueness for the Navier-Stokes equations. *J. Differential Equations* **226** (2006), no. 2, 373–428. Zbl 1104.35033 MR 2237686

[7] L. Grafakos, *Modern Fourier analysis*. 3rd edn., Grad. Texts in Math. 250, Springer, New York, 2014. Zbl 1304.42002 MR 3243741

[8] T. Izumi – Y. Sawano – H. Tanaka, Littlewood-Paley theory for Morrey spaces and their preduals. *Rev. Mat. Complut.* **28** (2015), no. 2, 411–447. Zbl 1316.42028 MR 3344085

[9] A. K. Lerner, On some sharp weighted norm inequalities. *J. Funct. Anal.* **232** (2006), no. 2, 477–494. Zbl 1140.42005 MR 2200743

[10] S. Lu – Y. Ding – D. Yan, *Singular integrals and related topics*. World Scientific Publishing, Hackensack, NJ, 2007. Zbl 1124.42011 MR 2354214

[11] V. G. Maz'ya – T. O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*. Monogr. Stud. Math. 23, Pitman, Boston, MA, 1985. Zbl 0645.46031 MR 0785568

[12] V. G. Maz'ya – T. O. Shaposhnikova, *Theory of Sobolev multipliers*. Grundlehren Math. Wiss. 337, Springer, Berlin, 2009. Zbl 1157.46001 MR 2457601

[13] K. H. Ooi – N. C. Phuc, Characterizations of predual spaces to a class of Sobolev multiplier type spaces. *J. Funct. Anal.* **282** (2022), no. 6, article no. 109348. Zbl 1492.46037 MR 4360359

[14] T. V. Phan – N. C. Phuc, Stationary Navier-Stokes equations with critically singular external forces: existence and stability results. *Adv. Math.* **241** (2013), 137–161. Zbl 1286.35196 MR 3053708

[15] N. C. Phuc, Quasilinear Riccati type equations with super-critical exponents. *Comm. Partial Differential Equations* **35** (2010), no. 11, 1958–1981. Zbl 1248.35084 MR 2754075

[16] V. S. Rychkov, Littlewood-Paley theory and function spaces with A_p^{loc} weights. *Math. Nachr.* **224** (2001), 145–180. Zbl 0984.42011 MR 1821243

[17] J.-O. Strömberg – A. Torchinsky, *Weighted Hardy spaces*. Lecture Notes in Math. 1381, Springer, Berlin, 1989. Zbl 0676.42021 MR 1011673

[18] A. Torchinsky, *Real-variable methods in harmonic analysis*. Pure Appl. Math. 123, Academic Press, Orlando, FL, 1986. Zbl 0621.42001 MR 0869816

Received 22 May 2023,
and in revised form 7 February 2024

Keng Hao Ooi
Department of Mathematics, National Central University
No. 300, Jhongda Rd, 32001 Jhongli City, Taoyuan County, Taiwan (R.O.C.)
kooi1@math.ncu.edu.tw