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Functional Analysis. – *A local surjection theorem with continuous inverse in Banach spaces*, by Ivar Ekeland and Éric Séré, communicated on 8 March 2024.

Dedicated to the memory of our friend Antonio Ambrosetti.

Abstract. – In this paper, we prove a local surjection theorem with continuous right-inverse for maps between Banach spaces, and we apply it to a class of inversion problems with loss of derivatives.

Keywords. – Inverse function theorem, Nash–Moser, loss of derivatives, Ekeland's variational principle.

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1. Introduction

In the recent work [4], we introduced a new algorithm for solving nonlinear functional equations admitting a right-invertible linearization, but with an inverse losing derivatives. These equations are of the form $F(u) = v$ with $F(0) = 0$, v small and given, u small and unknown. The main difference with the classical Nash–Moser algorithm [7, 12] was that, instead of using a regularized Newton scheme, we constructed a sequence $(u_n)_n$ of solutions to Galerkin approximations of the "hard" problem and proved the convergence of $(u_n)_n$ to a solution u of the exact equation. Each u_n was obtained thanks to a topological theorem on the surjectivity of maps between Banach spaces, due to one of us in [3]. However, this topological theorem does not provide the continuous dependence of u_n as a function of v. As a consequence, nothing was said in [4] on the existence of a continuous selection of solutions $u(v)$. Theorem 8 of the present work overcomes this limitation thanks to a variant of the topological argument, stated in Theorem 2.

In the sequel, $\mathcal{L}(X, Y)$ is the space of bounded linear operators between Banach spaces X and Y; the operator norm on this space is denoted by $\|\cdot\|_{X,Y}$. We first restate the result of [3] below for the reader's convenience.

Theorem 1 ([3]). *Let X and Y be Banach spaces. Denote by B the open ball of radius $R > 0$ around the origin in X. Let $f : B \to Y$ be continuous and Gâteaux-differentiable,* *with $f(0) = 0$. Assume that the derivative $Df(x)$ has a right-inverse $L(x)$, uniformly bounded on the ball B_R:*

$$\forall (x,k) \in B \times Y, \quad Df(x)L(x)k = k,$$
$$\sup \left\{ \left\| L(x) \right\|_{Y,X} : \|x\|_X < R \right\} < m.$$

Then, for every $y \in Y$ with $\|y\|_Y < Rm^{-1}$, there is some $x \in B$ satisfying

$$f(x) = y \quad \text{and} \quad \|x\|_X \leq m\|y\|_Y.$$

We recall that in the standard local inversion theorem, one assumes that f is of class C^1, with $Df(0)$ invertible and y small. An explicit bound on $\|y\|_Y$ is provided by the classical Newton–Kantorovich invertibility condition (see [1]) when f is of class C^2. The bound $\|y\|_Y < Rm^{-1}$ of Theorem 1 is much less restrictive than the Newton–Kantorovich condition, at the price of losing uniqueness, even in the case when $L(x)$ is also a left inverse of $Df(x)$. To illustrate this, we consider a finite-dimensional example.

Example A. We take $X = Y = \mathbb{C}$ viewed as a 2-dimensional real vector space and $f(z) = (2+z)^n - 2^n$, for any complex number z in the open disc of center 0 and radius $R = 1$ (here n is a positive integer). In that case, $Df(z)$ is the multiplication by $n(2+z)^{n-1}$ and $L(z)$ is the multiplication by $n^{-1}(2+z)^{1-n}$, so f satisfies the assumptions of Theorem 1 for $R = 1$ and any real number $m > n^{-1}$. Thus, Theorem 1 tells us that the equation $f(z) = Z$ has a solution of modulus less than or equal to $m|Z|$, provided Z has modulus less than m^{-1}. However, uniqueness does not hold. The solutions of the algebraic equation $(2+z)^n - 2^n = Z$ are of the form $z_k = 4ie^{i\frac{k\pi}{n}} \sin \frac{k\pi}{n} + O(2^{-n})$, and for $|Z| > 4\pi$ the three solutions z_0, z_1, z_{-1} lie in the closed disc $\{|z| \leq n^{-1}|Z|\}$ when n is large enough. Yet, there is a unique continuous function g such that $g(0) = 0$ and $f \circ g(Z) = Z$ for all complex numbers Z of modulus less than $1/m$. This continuous selection is $g(Z) = z_0 = 2((1 + 2^{-n}Z)^{\frac{1}{n}} - 1)$ with $(\rho e^{it})^{\frac{1}{n}} = \rho^{1/n}e^{it/n}$, $\forall (\rho, t) \in (0, \infty) \times (-\pi, \pi)$.

This example raises the following question: in the general case, can we select a solution x depending continuously on y, even in infinite dimension and when $Df(x)$ does not have a left inverse? The following theorem gives a positive response, under mild additional assumptions.

Theorem 2. *Let X, Y be two Banach spaces. Denote by B the open ball of radius $R > 0$ around the origin in X. Consider a map $f : B \to Y$ with $f(0) = 0$. We assume the following:*

(i) *f is Lipschitz-continuous and Gâteaux-differentiable on B.*

(ii) *There are a function* $L : B \to \mathcal{L}(Y, X)$*, a constant* $a < 1$ *and, for any* $(x, w) \in B \times Y$*, a positive radius* $\alpha(x, w)$ *such that if* $\|x' - x\|_X < \alpha(x, w)$*, then* $x' \in B$ *and*

$$\left\| \big(Df(x') \circ L(x) - I_Y\big) w \right\|_Y \le a \|w\|_Y .$$

(iii) *There is some* $m < \infty$ *such that*

$$\sup \left\{ \left\| L(x) \right\|_{Y,X} : x \in B \right\} < m.$$

Denote by $B' \subset Y$ *the open ball of radius* $R' := (1 - a)Rm^{-1}$ *and center* 0*. Then, there is a continuous map* $g : B' \to B$ *such that*

$$\forall y \in B', \quad \left\| g(y) \right\|_X \le \frac{m}{1-a} \|y\|_Y \quad \text{and} \quad f \circ g(y) = y.$$

If, in addition, one has the following:

(iv) f *is Fréchet differentiable on* B*,* $Df(x)$ *has a left-inverse for all* $x \in B$ *and there is a non-decreasing function* $\varepsilon : (0, \infty) \to (0, \infty)$ *with* $\lim_{t \to 0} \varepsilon(t) = 0$*, such that for all* x_1*,* x_2 *in* B*,*

$$\left\| f(x_2) - f(x_1) - Df(x_1)(x_2 - x_1) \right\|_Y \le \varepsilon\big(\|x_2 - x_1\|_X\big) \|x_2 - x_1\|_X ;$$

then g *is the unique continuous right-inverse of* f *defined on* B' *and mapping* 0_Y *to* 0_X*.*

REMARK 3. If a function f satisfies the assumptions (i), (ii) and (iii), then, for every $x_0 \in B$, taking the radius $R_{x_0} = R - \|x_0\|_X$, one can apply Theorem 1 to the function

$$z \in B_X(0, R_{x_0}) \mapsto f(x_0 + z) - f(x_0),$$

and one concludes that the restriction of f to $B_X(x_0, R_{x_0})$ has a continuous right-inverse g_{x_0} defined on $B_Y(f(x_0), (1 - a)R_{x_0} m^{-1})$ such that

$$\left\| g_{x_0}(y) - x_0 \right\|_X \le \frac{m}{1-a} \left\| y - f(x_0) \right\|_Y \quad \text{for all } y \in B_Y\big(f(x_0), (1 - a)R_{x_0} m^{-1}\big).$$

If, in addition, f satisfies (iv), then g_{x_0} is the unique continuous right-inverse of f defined on the ball $B_Y(f(x_0), (1 - a)R_{x_0} m^{-1})$ and mapping $f(x_0)$ to x_0.

REMARK 4. Assumption (ii) implies that $Df(x)$ has a right-inverse $\widehat{L}$ such that

$$\|\widehat{L}\|_{Y,X} \le (1 - a)^{-1} \|L\|_{Y,X} .$$

Indeed, taking $P = I_Y - Df(x) \circ L(x)$, one can choose $\widehat{L} := L \circ (\sum_{k=0}^{\infty} P^k)$.

Conversely, Assumption (ii) is satisfied, for instance, if (i) and (ii′) hold true, with the following:

(ii′) For each $x \in B$, $Df(x)$ has a right-inverse $L(x) \in \mathcal{L}(Y, X)$. Moreover, the map $x \to Df(x)$ is continuous for the strong topology of B and the strong operator topology of $\mathcal{L}(X, Y)$: in other words, if $\|x_n - x\|_X \to 0$, then, for any $v \in X$, $\|(Df(x_n) - Df(x))v\|_Y \to 0$.

The function f of Example A satisfies the assumptions (i), (ii′) and (iii). In that finite-dimensional case, f is of course differentiable in the classical sense of Fréchet. Let us give an example for which Fréchet differentiability does not hold.

EXAMPLE B. Let $\phi \in C^1(\mathbf{R}, \mathbf{R})$ with ϕ' bounded on $\mathbf{R}$ and $\inf_{\mathbf{R}} \phi' > 0$. The Nemitskii operator

$$\Phi : u \in L^p(\mathbf{R}) \to \phi \circ u \in L^p(\mathbf{R}), \quad 1 \leq p < \infty,$$

is not Fréchet differentiable when ϕ' is not constant [9, 10]. However, Φ satisfies conditions (i), (ii′) and (iii) for any $r > 0$ and $m > (\inf_{\mathbf{R}} \varphi')^{-1}$. Therefore, Theorem 2 applies to Φ, but the inverse Ψ is easily found without the help of this theorem, as a Nemitskii operator: $\Psi(u) = \psi \circ u$ with $\psi = \phi^{-1}$.

It turns out that any function f satisfying (i) has the *Hadamard differentiability property* which is stronger than the Gâteaux differentiability and which we recall below.

DEFINITION 5. Let X and Y be normed spaces. A map $f : X \to Y$ is called Hadamard differentiable at x, with derivative $Df(x) \in \mathcal{L}(X, Y)$, if, for every sequence $v_n \to v$ in V and every sequence $h_n \to 0$ in $\mathbb{R}$, we have

$$\lim_n \frac{1}{h_n}\big(f(x + h_n v_n) - f(x)\big) = Df(x)v.$$

This notion is weaker than Fréchet differentiability, but in finite dimension, Hadamard and Fréchet differentiability are equivalent. On the other hand, Hadamard differentiability is stronger than Gâteaux differentiability, but if a map f is Gâteaux-differentiable and Lipschitz, then it is Hadamard differentiable (see, e.g., [6]). In particular, the functions f of Theorems 1, 2 are Hadamard differentiable.

Note that the chain rule holds true for Hadamard differentiable functions, while this is not the case with Gâteaux differentiability (see [6]). Hadamard differentiable functions are encountered, for instance, in statistics [6, 13, 14] and in the bifurcation theory of nonlinear elliptic partial differential equations [5].

The paper is organized as follows. In Section 2, we prove Theorem 2. In Section 3, we state the hard surjection theorem with continuous right-inverse (Theorem 8) that can be proved using Theorem 2 and proceeding as in [4]. Finally, under additional assumptions, we state and prove the uniqueness of the continuous right-inverse (Theorem 9).

2. Proof of Theorem 2

In [3], Theorem 1 was proved by applying Ekeland's variational principle in the Banach space X, to the map $x \mapsto \|f(x) - y\|_Y$. This principle provided the existence of an approximate minimizer $\underline{x}$. Assuming that $\|f(\underline{x}) - y\|_Y > 0$ and considering the direction of descent $L(\underline{x})(y - f(\underline{x}))$, a contradiction was found. Thus, $f(\underline{x}) - y$ was necessarily equal to zero and $\underline{x}$ was the desired solution of the equation $f(x) = y$. However, there was no continuous dependence of $\underline{x}$ as a function of y. In order to obtain such a continuous dependence, it is more convenient to solve *all* the equations $f(x) = y$ for all possible values of $y \in B'$ simultaneously, by applying the variational principle in a functional space of continuous maps from B' to X. The drawback is that it is more difficult to construct a direction of descent, as this direction should be a continuous function of y. In order to do so, we use an argument inspired of the classical pseudo-gradient construction for C^1 functionals in Banach spaces [8], which makes use of the paracompactness property of metric spaces.

Consider the space $\mathcal{C}$ of continuous maps $g : B' \to X$ such that $\|y\|^{-1} g(y)$ is bounded on $\dot{B}'$, with the notation $\dot{B}' := B' \setminus \{0\}$. Endowed with the norm

$$\|g\|_{\mathcal{C}} = \sup_{\dot{B}'} \|y\|^{-1} \big\|g(y)\big\|,$$

$\mathcal{C}$ is a Banach space. Consider the function

$$\varphi(g) := \sup_{y \in \dot{B}'} \|y\|^{-1} \big\|f \circ g(y) - y\big\| \quad \text{if } \|g\|_{\mathcal{C}} \leq \frac{m}{1-a},$$
$$\varphi(g) := +\infty \quad \text{otherwise.}$$

The function φ is lower semi-continuous on $\mathcal{C}$ and its restriction to the closed ball $\{g \in \mathcal{C} : \|g\|_{\mathcal{C}} \leq \frac{m}{1-a}\}$ is finite-valued. In addition, we have

$$\varphi(0) = \sup_{\dot{B}'} \|y\|^{-1} \big\|f(0) - y\big\| = 1,$$
$$\varphi(g) \geq 0, \quad \forall g \in \mathcal{C}.$$

Choose some m_0 with

$$\sup\left\{\big\|L(x)\big\|_{Y,X} : x \in B\right\} < m_0 < m.$$

By Ekeland's variational principle [2], there exists some $g_0 \in \mathcal{C}$ such that

$$\varphi(g_0) \leq 1, \tag{2.1}$$

$$\|g_0 - 0\|_{\mathcal{C}} \leq \frac{m_0}{1-a}, \tag{2.2}$$

$$\forall g \in \mathcal{C}, \quad \varphi(g) \geq \varphi(g_0) - \frac{(1-a)\varphi(0)}{m_0} \|g - g_0\|_{\mathcal{C}}. \tag{2.3}$$

Equation (2.2) implies that g_0 maps B' into the open ball of center 0_X and radius $m_0(1-a)^{-1}R' = Rm_0m^{-1} < R$, and the last equation can be rewritten:

$$\forall g \in \mathcal{C}, \quad \varphi(g) \geq \varphi(g_0) - \frac{1-a}{m_0}\|g-g_0\|_{\mathcal{C}}. \tag{2.4}$$

If $\varphi(g_0) = 0$, then $f(g_0(y)) - y = 0$ for all $y \in B'$ and the existence proof is over. If not, then $\varphi(g_0) > 0$ and we shall derive a contradiction. In order to do so, we are going to build a deformation g_t of g_0 which contradicts the optimality property (2.3) of g_0.

Let $a < a' < 1$ be such that

$$\sup\left\{\left\|L(x)\right\|_{Y,X} : x \in B\right\} < \frac{1-a'}{1-a}m_0.$$

We define a continuous map $w : B' \to Y$ by the formula

$$w(y) := y - f \circ g_0(y) \in Y.$$

By the continuity of w, the set

$$\mathcal{V} := \left\{y \in \dot{B}' : \left\|w(y)\right\|_Y < \frac{1}{2}\varphi(g_0)\|y\|_Y\right\}$$

is open in $\dot{B}'$.

Now, Df is bounded since f is Lipschitz-continuous, and L is bounded on B by Assumption (iii). Therefore, combining these bounds with the continuity of w, we see that for each $(x,y) \in B \times (\dot{B}' \setminus \mathcal{V})$, there exists a positive radius $\beta(x,y)$ such that if $(x',y') \in B_X(x,\beta(x,y)) \times B_Y(y,\beta(x,y))$, then $(x',y') \in B \times \dot{B}'$ and

$$\left(\left\|Df(x')\right\|_{X,Y}\left\|L(x)\right\|_{Y,X} + 1 + a'\right)\left\|w(y') - w(y)\right\|_X \leq (a'-a)\left\|w(y)\right\|_X,$$

which implies the inequality

$$a\left\|w(y)\right\|_Y + \left\|\left(Df(x') \circ L(x) - I_Y\right)\left(w(y') - w(y)\right)\right\|_Y \leq a'\left\|w(y')\right\|_Y. \tag{2.5}$$

Let $\gamma(x,y) := \min(\alpha(x,w(y)); \beta(x,y))$ where $\alpha(x,w)$ is the radius introduced in Assumption (ii). Then, this assumption combined with (2.5) implies that

$$\left\|\left(Df(x') \circ L(x) - I_Y\right)w(y')\right\|_Y \leq a'\left\|w(y')\right\|_Y \tag{2.6}$$

for each $(x,y) \in B \times (\dot{B}' \setminus \mathcal{V})$ and all $(x',y') \in B_X(x,\gamma(x,y)) \times B_Y(y,\beta(x,y))$.

Since the set

$$\Omega := \bigcup_{(x,y)\in B\times(\dot{B}'\setminus\mathcal{V})} B_X\big(x,\gamma(x,y)\big) \times B_Y\big(y,\beta(x,y)\big)$$

is a metric space, it is paracompact [11]. Thus, Ω has a locally finite open covering $(\omega_i)_{i\in I}$ where for each $i \in I$,

$$\omega_i \subset B_X\big(x_i, \gamma(x_i, y_i)\big) \times B_Y\big(y_i, \beta(x_i, y_i)\big)$$

for some $(x_i, y_i) \in B \times (\dot{B}' \setminus \mathcal{V})$. In the sequel, we take the norm $\max(\|x\|_X; \|y\|_Y)$ on $X \times Y$. For $(x, y) \in B \times \dot{B}'$, we define

$$\begin{aligned}
\sigma_i(x, y) &:= \operatorname{dist}\big((x, y), (B \times \dot{B}') \setminus \omega_i\big),\\
\theta(x, y) &:= \frac{\sum_{i\in I} \sigma_i(x, y)}{\operatorname{dist}(y, \dot{B}' \setminus \mathcal{V}) + \sum_{i\in I} \sigma_i(x, y)} \in [0, 1].
\end{aligned}$$

Note that $\theta(x, y) = 1$ when $\|w(y)\|_Y \geq \frac{\varphi(g_0)}{2}\|y\|_Y$, and $\theta(x, y) = 0$ when $(x, y) \notin \Omega$.

We are now ready to define

$$\widetilde{L}(x, y) = \Big(\operatorname{dist}(y, \dot{B}' \setminus \mathcal{V}) + \sum_{i\in I} \sigma_i(x, y)\Big)^{-1} \sum_{i\in I} \sigma_i(x, y) L(x_i).$$

One easily checks that $\widetilde{L}$ is locally Lipschitz on $B \times \dot{B}'$. Moreover, it satisfies the same uniform estimate as L:

$$\sup\left\{\big\|\widetilde{L}(x, y)\big\|_{Y,X} : x \in B,\ y \in \dot{B}'\right\} < \frac{1-a'}{1-a} m_0, \tag{2.7}$$

and due to (2.6), it is an approximate inverse of Df "in the direction $w(y)$":

$$\begin{aligned}
&\big\|\big(Df(x) \circ \widetilde{L}(x, y) - \theta(x, y) I_Y\big) w(y)\big\|_Y \\
&\qquad \leq a' \theta(x, y) \big\|w(y)\big\|_Y, \quad \forall (x, y) \in B \times \dot{B}'.
\end{aligned}\tag{2.8}$$

Now, to each $y \in \dot{B}'$, we associate the vector field on B:

$$X_y(x) := \widetilde{L}(x, y) w(y),$$

and we consider the Cauchy problem

$$\begin{cases} \dfrac{dx}{dt} = X_y(x), \\ x(0) = g_0(y). \end{cases}$$

The vector field X_y is locally Lipschitz in the variable $x \in B$, and from (2.7) we have the uniform estimate

$$\sup\left\{\|y\|_Y^{-1} \big\|X_y(x)\big\|_X : x \in B,\ y \in \dot{B}'\right\} < \frac{1-a'}{1-a} m_0 \varphi(g_0) \leq \frac{1-a'}{1-a} m_0.$$

Thus, recalling that $\|g_0(y)\|_X < Rm_0 m^{-1}$, we see that our Cauchy problem has a unique solution $x(t) = g_t(y) \in B$ on the time interval $[0,\tau]$ with $\tau = \frac{m-m_0}{(1-a')m_0}$. In addition, we take $g_t(0) = 0$. This gives us a one-parameter family of functions $g_t : B' \to B$. For $0 < t \le \tau$, g_t satisfies the estimate

$$\sup\left\{\|y\|_Y^{-1}\left\|g_t(y) - g_0(y)\right\|_X : y \in \dot{B}'\right\} < \frac{1-a'}{1-a} m_0\, \varphi(g_0) t. \tag{2.9}$$

Since $\|g_0\|_{\mathcal{C}} \le \frac{m_0}{1-a}$ and $\varphi(g_0) \le 1$, inequality (2.9) implies that

$$\sup_{\dot{B}'} \|y\|_Y^{-1}\left\|g_t(y)\right\|_X < \frac{m}{1-a}, \qquad \forall t \in [0,\tau]. \tag{2.10}$$

We recall that $\widetilde{L}(\cdot,\cdot)$ is locally Lipschitz on $B \times \dot{B}'$ and g_0, w are continuous. So, by Gronwall's inequality, for each $t \in [0,\tau]$ the function g_t is continuous on $\dot{B}'$. We can conclude that $g_t \in \mathcal{C}$, and (2.10) implies that $\varphi(g_t) < \infty$.

Now, to each $(t,y) \in [0,\tau] \times \dot{B}'$, we associate $s_t(y) := \int_0^t \theta(g_u(y), y)\,du$ and we consider the function

$$(t,y) \in [0,\tau] \times \dot{B}' \mapsto h(t,y) := f \circ g_t(y) - y + \big(1 - s_t(y)\big) w(y) \in Y.$$

Since f is Lipschitzian, its Gâteaux-differential $Df(x)$ at any $x \in B$ is also a Hadamard differential, as mentioned in the introduction. This implies that for any function $\gamma : (-1,1) \to B$ differentiable at 0 such that $\gamma(0) = x$, the function $f \circ \gamma$ is differentiable at 0 and the chain rule holds true: $(f \circ \gamma)'(0) = Df(\gamma(0))\gamma'(0)$.

Therefore, using (2.8), we get

$$\begin{aligned}\left\|\frac{\partial}{\partial t} h(t,y)\right\|_Y &= \left\|\big(Df\big(g_t(y)\big) \circ \widetilde{L}\big(g_t(y), y\big) - \theta\big(g_t(y), y\big) I_Y\big) w(y)\right\|_Y \\ &\le a'\theta\big(g_t(y), y\big)\left\|w(y)\right\|_Y.\end{aligned}$$

In addition, $h(0,y) = 0$, so by the mean value theorem,

$$\left\|h(t,y)\right\|_Y \le a' s_t(y)\left\|w(y)\right\|_Y.$$

By the triangle inequality, this implies that

$$\left\|f\big(g_t(y)\big) - y\right\|_Y \le \big(1 - (1-a')s_t(y)\big)\left\|w(y)\right\|_Y. \tag{2.11}$$

We are now ready to get a contradiction. The estimate (2.9) may be written as follows:

$$\frac{1-a}{m_0}\,\|g_t - g_0\|_{\mathcal{C}} < (1-a')t\,\varphi(g_0), \quad \forall t \in (0,\tau]. \tag{2.12}$$

As a consequence of (2.11), if $\|y\|_Y^{-1}\|f(g_t(y)) - y\|_Y \geq \frac{\varphi(g_0)}{2}$ then

$$\|w(y)\|_Y \geq \frac{\varphi(g_0)}{2}\|y\|_Y;$$

hence, $s_t(y) = t$. Thus, the estimate (2.11) implies that for all $(t, y) \in [0, \tau] \times \dot{B}'$,

$$\|y\|_Y^{-1}\left\|f\big(g_t(y)\big) - y\right\|_Y \leq \max\left\{\frac{\varphi(g_0)}{2}; \big(1-(1-a')t\big)\|y\|_Y^{-1}\|w(y)\|_Y\right\}.$$

However, we always have $\|y\|_Y^{-1}\|w(y)\|_Y \leq \varphi(g_0)$, so, with $\tau' := \min\{\tau, \frac{1}{2}\}$, we get

$$\|y\|_Y^{-1}\left\|f\big(g_t(y)\big) - y\right\|_Y \leq \big(1-(1-a')t\big)\varphi(g_0)$$

for all $0 \leq t \leq \tau'$ and $y \in \dot{B}'$. This means that

$$\varphi(g_t) \leq \big(1-(1-a')t\big)\varphi(g_0), \quad \forall t \in [0, \tau']. \tag{2.13}$$

Combining (2.12) with (2.13), we find the following, for $0 < t \leq \tau'$:

$$\varphi(g_t) \leq \varphi(g_0) - (1-a')t\,\varphi(g_0) < \varphi(g_0) - \frac{1-a}{m_0}\|g_t - g_0\|_{\mathscr{C}},$$

which contradicts (2.4). This ends the proof of the existence statement in Theorem 2.

The uniqueness statement is proved by more standard arguments: if g_1 and g_2 are two continuous right-inverses of f such that $g_1(0) = g_2(0) = 0$, then the set

$$Z := \big\{y \in B' \mid g_1(y) = g_2(y)\big\}$$

is nonempty and closed. On the other hand, if $Df(x)$ is left and right invertible, it is an isomorphism. By Remark 4, its inverse $\widehat{L}(x)$ is bounded independently of x. We fix an arbitrary y_0 in Z and we consider a small radius $\rho > 0$ (to be chosen later) such that $B_Y(y_0, \rho) \subset B'$. By continuity of $g_1 - g_2$ at y_0, there is $\eta(\rho) > 0$ such that $\lim_{\rho\to 0}\eta(\rho) = 0$, and, for each y in the ball $B_Y(y_0, \rho)$,

$$\left\|g_2(y) - g_1(y)\right\|_X \leq \eta(\rho).$$

However, we also have $f(g_2(y)) - f(g_1(y)) = y - y = 0$. Thus, using (iv), we find that

$$\left\|Df\big(g_1(y)\big)\big(g_2(y) - g_1(y)\big)\right\|_Y \leq (\varepsilon\circ\eta)(\rho)\left\|g_2(y) - g_1(y)\right\|_Y.$$

Then, multiplying $Df(g_1(y))(g_2(y) - g_1(y))$ on the left by $\widehat{L}(g_1(y))$ and using the uniform bound on $\widehat{L}$, we get a bound of the form

$$\left\|g_2(y) - g_1(y)\right\|_X \leq \xi(\rho)\left\|g_2(y) - g_1(y)\right\|_X$$

with $\lim_{\rho\to 0}\xi(\rho) = 0$. As a consequence, for ρ small enough, one has $g_2(y) - g_1(y) = 0$, so $y \in Z$. This proves that Z is open. By connectedness of B', we conclude that $Z = B'$, so g_1 and g_2 are equal. This ends the proof of Theorem 2.

3. A hard surjection theorem with continuous right-inverse

In this section, we state our hard surjection theorem with continuous right-inverse, and we shortly explain its proof which is a variant of the arguments of [4] in which Theorem 1 is replaced by Theorem 2.

Let $(V_s, \|\cdot\|_s)_{0\le s\le S}$ be a scale of Banach spaces; namely,

$$0 \le s_1 \le s_2 \le S \implies \big[V_{s_2} \subset V_{s_1} \text{ and } \|\cdot\|_{s_1} \le \|\cdot\|_{s_2}\big].$$

We shall assume that to each $\Lambda \in [1,\infty)$ is associated a continuous linear projection $\Pi(\Lambda)$ on V_0, with a range $E(\Lambda) \subset V_S$. We shall also assume that the spaces $E(\Lambda)$ form a nondecreasing family of sets indexed by $[1,\infty)$, while the spaces $\operatorname{Ker}\Pi(\Lambda)$ form a nonincreasing family. In other words,

$$1 \le \Lambda \le \Lambda' \implies \Pi(\Lambda)\Pi(\Lambda') = \Pi(\Lambda')\Pi(\Lambda) = \Pi(\Lambda).$$

Finally, we assume that the projections $\Pi(\Lambda)$ are "smoothing operators" satisfying the following estimates.

Polynomial growth and approximation. *There are constants $A_1, A_2 \ge 1$ such that, for all numbers $0 \le s \le S$, all $\Lambda \in [1,\infty)$ and all $u \in V_s$, we have*

$$\forall t \in [0,S], \quad \big\|\Pi(\Lambda)u\big\|_t \le A_1 \Lambda^{(t-s)^+}\|u\|_s, \tag{3.1}$$

$$\forall t \in [0,s], \quad \big\|\big(1-\Pi(\Lambda)\big)u\big\|_t \le A_2 \Lambda^{-(s-t)}\|u\|_s. \tag{3.2}$$

When the above properties are met, we shall say that $(V_s, \|\cdot\|_s)_{0\le s\le S}$, endowed with the family of projectors $\{\Pi(\Lambda), \Lambda\in[1,\infty)\}$, is a *tame* Banach scale.

Let $(W_s, \|\cdot\|'_s)_{0\le s\le S}$ be another tame scale of Banach spaces. We shall denote by $\Pi'(\Lambda)$ the corresponding projections defined on W_0 with ranges $E'(\Lambda) \subset W_S$, and by A'_i $(i = 1,2,3)$ the corresponding constants in (3.1), (3.2).

We also denote by B_s the unit ball in V_s and by $B'_s(0,r)$ the ball of center 0 and positive radius r in W_s:

$$B_s = \big\{u \in V_s \mid \|u\|_s < 1\big\} \quad \text{and} \quad B'_s(0,r) = \big\{v \in W_s \mid \|v\|'_s < r\big\}.$$

In the sequel, we fix nonnegative constants s_0, m, ℓ and ℓ'. We will assume that S is large enough.

We first recall the definition of Gâteaux differentiability, in a form adapted to our framework.

Definition 6. We shall say that a function $F : B_{s_0+m} \to W_{s_0}$ is *Gâteaux-differentiable* (henceforth G-differentiable) if for every $u \in B_{s_0+m}$, there exists a linear map

$$DF(u) : V_{s_0+m} \to W_{s_0}$$

such that for every $s \in [s_0, S-m]$, if $u \in B_{s_0+m} \cap V_{s+m}$, then $DF(u)$ maps continuously V_{s+m} into W_s, and

$$\forall h \in V_{s+m}, \quad \lim_{t\to 0} \left\| \frac{1}{t}\big[F(u+th) - F(u)\big] - DF(u)h \right\|'_s = 0.$$

Note that, even in finite dimension, a G-differentiable map need not be C^1, or even continuous. However, if $DF : B_{s_0+m} \cap V_{s+m} \to \mathcal{L}(V_{s+m}, W_s)$ is locally bounded, then $F : B_{s_0+m} \cap V_{s+m} \to W_s$ is locally Lipschitz, hence continuous. In the present paper, we are in such a situation.

We now define the notion of S-tame differentiability.

DEFINITION 7.

- We shall say that the map $F : B_{s_0+m} \to W_{s_0}$ is S-tame differentiable if it is G-differentiable in the sense of Definition 6, and, for some positive constant a and all $s \in [s_0, S-m]$, if $u \in B_{s_0+m} \cap V_{s+m}$ and $h \in V_{s+m}$, then $DF(u)h \in W_s$ with the tame direct estimate

$$\big\| DF(u)h \big\|'_s \le a\big(\|h\|_{s+m} + \|u\|_{s+m}\|h\|_{s_0+m}\big).$$

- Then, we shall say that DF is tame right-invertible if there are $b > 0$ and $\ell, \ell' \ge 0$ such that for all $u \in B_{s_0+\max\{m,\ell\}}$, there is a linear map $L(u) : W_{s_0+\ell'} \to V_{s_0}$ satisfying

$$\forall k \in W_{s_0+\ell'}, \quad DF(u)L(u)k = k,$$

and for all $s_0 \le s \le S - \max\{\ell, \ell'\}$, if $u \in B_{s_0+\max\{m,\ell\}} \cap V_{s+\ell}$ and $k \in W_{s+\ell'}$, then $L(u)k \in V_s$, with the tame inverse estimate

$$\big\| L(u)k \big\|_s \le b\big(\|k\|'_{s+\ell'} + \|k\|'_{s_0+\ell'}\|u\|_{s+\ell}\big). \tag{3.3}$$

In the above definition, the numbers m, ℓ, ℓ' represent the loss of derivatives for DF and its right-inverse.

The main result of this section is the following theorem.

THEOREM 8. *Assume that the map $F : B_{s_0+m} \to W_{s_0}$ is S-tame differentiable between the tame scales $(V_s)_{0\le s\le S}$ and $(W_s)_{0\le s\le S}$ with $F(0) = 0$ and that DF is tame right-invertible. Let s_0, m, ℓ, ℓ' be the associated parameters.*

Assume in addition that for each $\Lambda, \Lambda' \in [1, S]$, the map

$$u \in B_{s_0+\max\{m,\ell\}} \cap E(\Lambda) \mapsto \Pi'_{\Lambda'} DF(u) \restriction_{E_\Lambda} \in \mathcal{L}\big(E(\Lambda), E'(\Lambda')\big)$$

is continuous for the norms $\|\cdot\|_{s_0}$ and $\|\cdot\|'_{s_0}$.

Let $s_1 \geq s_0 + \max\{m, \ell\}$ and $\delta > s_1 + \ell'$. Then, for S large enough, there exist a radius $r > 0$ and a continuous map $G : B'_\delta(0, r) \to B_{s_1}$ such that

$$G(0) = 0 \quad \text{and} \quad F \circ G = I_{B'_\delta(0,r)},$$
$$\left\|G(v)\right\|_{s_1} \leq r^{-1}\|v\|'_\delta, \quad \forall v \in B'_\delta(0, r).$$

As mentioned in the introduction, compared with the results of [4], the novelty in Theorem 8 is the continuity of G. To prove this theorem, one repeats with some modifications the arguments of [4] in the case $\varepsilon = 1$ (in that paper, a singularly perturbed problem depending on a parameter ε was dealt with, but for simplicity, we do not consider such a dependence here). With the notation of that paper, let us explain briefly the necessary changes.

We recall that in [4] a vector v was given in $B'_\delta(0, r)$ and the goal was to solve the equation $F(u) = v$. The solution u was the limit of a sequence u_n of approximate solutions constructed inductively. Each u_n was a solution of the projected equation

$$\Pi'_n F(u_n) = \Pi'_{n-1} v, \quad u_n \in E_n.$$

It was found as $u_n = u_{n-1} + z_n$, z_n being a small solution in E_n of an equation of the form $f_n(z) = \Delta_n v + e_n$, with $f_n(z) := \Pi'_n(F(u_{n-1} + z) - F(u_{n-1}))$, $\Delta_n v := \Pi'_{n-1}(1 - \Pi'_{n-2})v$ and $e_n := -\Pi'_n(1 - \Pi'_{n-1})F(u_{n-1})$. The existence of z_n was proved by applying Theorem 1 to the function f_n in a ball $B_{\mathcal{N}_n}(0, R_n)$ (see [4, Section 3.3.2] for precise definitions of E_n, Π'_n, f_n and $\mathcal{N}_n$).

Instead, we construct inductively a sequence of continuous functions

$$G_n : B'_\delta(0, r) \to B_{s_1} \cap E_n$$

such that

$$\Pi'_n F \circ G_n(v) = \Pi'_{n-1} v$$

for all v in $B'_\delta(0, r)$. Each G_n is of the form $G_{n-1} + H_n$ with

$$H_n(v) = g_n\big(\Delta_n v - \Pi'_n(1 - \Pi'_{n-1})F \circ G_{n-1}(v)\big),$$

where g_n is a continuous right-inverse of f_n such that $g_n(0) = 0$, obtained thanks to Theorem 2.

Moreover, under the same conditions on the parameters as in [4], we find that the sequence of continuous functions $(G_n)_n$ converges uniformly on $B'_\delta(0, r)$ for the norm $\|\cdot\|_{s_1}$, and this implies the continuity of their limit $G : B'_\delta(0, r) \to B_{s_1}$. This limit is the desired continuous right-inverse of F. We insist on the fact that the conditions on r are exactly the same as in [4]. Indeed, in order to apply Theorem 2 to f_n, we just have to check assumptions (i), (ii′) and (iii). This is done with exactly the same constraints on the parameters as in [4]. ∎

We end the paper with a uniqueness result, which requires additional conditions.

THEOREM 9. *Suppose that we are under the assumptions of Theorem 8 and that the following two additional conditions hold true:*

- *For each* $u \in B_{s_0+\max(m,\ell)}$,

$$\forall h \in V_{s_0+m+\ell'}, \quad L(u)DF(u)h = h. \tag{3.4}$$

- *For each* $s \in [s_0, S-m]$ *and* $c > 0$*, there is a non-decreasing function*

$$\varepsilon_{s,c} : (0,\infty) \to (0,\infty)$$

such that $\lim_{t\to 0} \varepsilon_{s,c}(t) = 0$ *and, for all* u_1, u_2 *in* $B_{s_0+m} \cap E_{s+m}$ *with* $\|u_1\|_{s+m} \le c$,

$$\begin{aligned} &\big\| F(u_2) - F(u_1) - DF(u_1)(u_2-u_1)\big\|_s \\ &\qquad \le \varepsilon_{s,c}\big(\|u_2-u_1\|_{s+m}\big)\|u_2-u_1\|_{s_0+m}. \end{aligned} \tag{3.5}$$

Let $s_1 \ge s_0 + \max\{2m+\ell', m+\ell\}$*. Then, for any* $S \ge s_1$, $\delta \in [s_0, S]$ *and* $r > 0$*, there is at most one map* $G : B'_\delta(0,r) \to B_{s_0+\max(m,\ell)} \cap W_{s_1}$ *continuous for the norms* $\|\cdot\|'_\delta$ *and* $\|\cdot\|_{s_1}$*, such that*

$$G(0) = 0 \quad \textit{and} \quad F \circ G = I_{B'_\delta(0,r)}. \tag{3.6}$$

REMARK 10. The tame estimate (3.5) is satisfied, in particular, when F is of class C^2 with a classical tame estimate on its second derivative as in [12, (2.11)]. In that special case, for s and c fixed, one has the bound $\varepsilon_{s,c}(t) = O(t)_{t\to 0}$.

In order to prove Theorem 9, we assume that G_1, G_2 both satisfy (3.6), and we introduce the set

$$Z := \{v \in B'_\delta(0,r) \mid G_1(v) = G_2(v)\}.$$

This set is nonempty since it contains 0, and it is closed in $B'_\delta(0,r)$ for the norm $\|\cdot\|'_\delta$ by continuity of $G_1 - G_2$. It remains to prove that it is open.

For that purpose, we fix an arbitrary v_0 in Z and we consider a small radius $\rho > 0$ (to be chosen later) such that $B'_\delta(v_0,\rho) \subset B'_\delta(0,r)$. By continuity of G_1, G_2 at v_0, there is $\eta(\rho) > 0$ such that $\lim_{\rho\to 0}\eta(\rho) = 0$, and, for each v in the ball $B'_\delta(v_0,\rho)$,

$$\big\|G_1(v)\big\|_{s_1} \le \big\|G_1(v_0)\big\|_{s_1} + \eta(\rho) \quad \text{and} \quad \big\|G_2(v) - G_1(v)\big\|_{s_1} \le \eta(\rho).$$

However, we also have $F(G_2(v)) - F(G_1(v)) = v - v = 0$. Thus, imposing $\eta(\rho) \le 1$ and applying (3.5) with $s = s_1 - m$, $c = \|G_1(v_0)\|_{s_1} + 1$ and $u_i = G_i(v)$, $i = 1, 2$, we find that

$$\big\|DF\big(G_1(v)\big)\big(G_2(v) - G_1(v)\big)\|'_{s_1-m} \le (\varepsilon_{s_1-m,c}\circ\eta)(\rho)\big\|G_2(v) - G_1(v)\big\|_{s_0+m}.$$

Then, multiplying $DF(G_1(v))(G_2(v) - G_1(v))$ on the left by $L(G_1(v))$ and using (3.4) and the tame estimate (3.3), we get a bound of the form

$$\left\| G_2(v) - G_1(v) \right\|_{s_1 - \max(m+\ell',\ell)} \leq \xi(\rho) \left\| G_2(v) - G_1(v) \right\|_{s_0+m}$$

with $\lim_{\rho\to 0} \xi(\rho) = 0$. Since $s_1 - \max(m+\ell',\ell) \geq s_0 + m$, we conclude that for ρ small enough, one has $G_2(v) - G_1(v) = 0$, so $v \in Z$. The set Z is thus nonempty, closed and open in $B'_8(0,r)$, so we conclude that $Z = B'_8(0,r)$ and Theorem 9 is proved. ∎

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