Rend. Lincei Mat. Appl. 35 (2024), 77–103
DOI 10.4171/RLM/1034

Calculus of Variations. – *Periodic bounce trajectories in moving domains*, by Jun Hirata, Yoshitaka Matsugi and Kazunaga Tanaka, communicated on 19 April 2024.

Dedicated to the memory of Professor Antonio Ambrosetti.

Abstract. – We study the existence of periodic trajectories in periodically moving domains. Our trajectories bounce at the boundary of moving domains in an elastic way. We prove the existence of infinitely many periodic bounce trajectories.

Keywords. – Periodic solutions, bounce orbits.

Mathematics Subject Classification 2020. – 37J46 (primary); 58E05, 70H12 (secondary).

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open bounded set with boundary $\partial\Omega$ of class C^2 and we consider the situation where Ω moves in $\mathbb{R}^N$ periodically. Precisely, we denote $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and we consider moving domain $\Omega(t)$ defined by

$$\Omega(t) = \big\{R(t)x + c(t);\ x \in \Omega\big\} \quad \text{for } t \in S^1, \tag{1.1}$$

where $R(t)\colon S^1 \to \mathrm{SO}(N)$, $c(t)\colon S^1 \to \mathbb{R}^N$ are given 2π-periodic functions of class C^2. We study the existence of 2π-periodic bounce trajectories of the following Hamiltonian system in the moving domain $\Omega(t)$:

$$\ddot{q} + \nabla\widetilde{V}\big(t, q(t)\big) = 0. \tag{1.2}$$

Here $\widetilde{V}(t,x)\colon \bar{D} \to \mathbb{R}$, $D = \bigcup_{t\in S^1}(\{t\} \times \Omega(t))$ is a given 2π-periodic function of class C^2 and $\cdot = \frac{d}{dt}$, $\nabla = (\partial_{x_1}, \dots, \partial_{x_N})$.

Here we say $q(t)\colon S^1 \to \mathbb{R}^N$ is a 2π-periodic bounce trajectory for (1.2) if

(1) $q(t) \in \bar{\Omega}(t)$ for all $t \in S^1$.

(2) $q(t)$ is of class C^2 except for at most a finitely number of instants $t_1, \dots, t_k \in S^1$ for which $q(t_j) \in \partial\Omega(t_j)$.

(3) $q(t)$ satisfies for $t \in S^1 \setminus \{t_1, \dots, t_k\}$

$$\ddot{q} + \nabla\widetilde{V}\big(t, q(t)\big) = 0. \tag{1.3}$$

(4) For $t \in \{t_1, \dots, t_k\}$, $q(t)$ bounces in an elastic way and the usual law of reflection is satisfied in the dynamic coordinate system. That is, denote $q(t)$ as

$$q(t) = R(t)u(t) + c(t)$$

with $u(t)\colon S^1 \to \bar{\Omega}$. We have for $\bar{t} \in \{t_1, \dots, t_k\}$

$$\begin{aligned}
&u(\bar{t}) \in \partial\Omega, \\
&\dot{u}_\pm(\bar{t}) = \lim_{s\to\pm 0} \dot{u}(\bar{t}+s) \text{ exists}, \\
&\left|\dot{u}_+(\bar{t})\right| = \left|\dot{u}_-(\bar{t})\right|, \\
&\dot{u}_+(\bar{t}) = \dot{u}_-(\bar{t}) - c\nu\big(u(\bar{t})\big),
\end{aligned}$$

where $c \geq 0$ and $\nu(x)$ is the unit outward vector at $x \in \partial\Omega$.

We call $\{t_j\}$ that appeared in (2)–(4) as bounce instants for (1.2).

Our main result is the following theorem.

THEOREM 1.1. *Assume $\Omega \subset \mathbb{R}^N$ is an open bounded set which is symmetric with respect to 0 and $0 \in \Omega$. Let $\Omega(t)$ be a moving domain given in (1.1), which moves 2π-periodically in $\mathbb{R}^N$ and $\widetilde{V}(t,x) \in C^2(\bar{D}, \mathbb{R})$, where $D = \bigcup_{t\in S^1}(\{t\} \times \Omega(t))$. Then (1.2) has infinitely many 2π-periodic bounce trajectories $(q_n)_{n=1}^\infty \in H^1(S^1, \mathbb{R}^N)$ such that*

(1) *each $q_n(t)$ has at most finitely many bounce instants;*

(2) *$\int_0^{2\pi} |\dot{q}_n|^2\, dt \to \infty$ as $n \to \infty$.*

The existence of such bounce trajectories is studied in Benci–Giannoni [5] and Liu–Jiang [11]. In these papers authors consider the existence of periodic bounce trajectories in a fixed domain, that is, $R(t) \equiv I$ and $c(t) \equiv 0$, but without the assumption of symmetry of the domain. In [5], Benci and Giannoni consider the case $\widetilde{V}(t,x)$ is independent of t and find the existence of at least one bounce trajectory with at most $N+1$ bounce instants.

In [11], Liu and Jiang consider the case $\bar{\Omega}$ is C^2 diffeomorphic to the unit ball $B_1 = \{x \in \mathbb{R}^N; |x| \leq 1\}$ and for 2π-periodic function $\widetilde{V}(t,x) \in C^2(S^1 \times \bar{\Omega}, \mathbb{R})$ they showed the existence of infinitely many bounce trajectories using perturbation from symmetry (cf. [1–4, 7, 8, 12, 14–17]). We also refer to [9, 10] for billiard problems.

In both of [5, 11], they study (1.2) through approximation schemes. They introduce penalty functions to (1.2) and first they find periodic solutions of penalized problems. In [5], they introduce a penalty function of form $\varepsilon U(x)$, where $U(x) \in C^2(\Omega, \mathbb{R})$ satisfies for $\delta > 0$ small

$$U(x) = \begin{cases} \frac{1}{\operatorname{dist}(x,\partial\Omega)^2} & \text{for } x \in \Omega \text{ with } \operatorname{dist}(x,\partial\Omega) < \delta, \\ 0 & \text{for } x \in \Omega \text{ far from } \partial\Omega. \end{cases}$$

That is, they introduce a penalty U inside Ω such that $U(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$. First they find a periodic solution q_ε of

$$\ddot{q} + \nabla\widetilde{V}(t,q) + \varepsilon\nabla U(q) = 0$$

and second they take a limit as $\varepsilon \to 0$.

In [11], Liu and Jiang introduce another penalty. After extending $\widetilde{V}(t,x)$ on $S^1 \times \mathbb{R}^N$, they consider a penalty of form $\lambda(x)^n$, where $\lambda(x) \in C^\infty(\mathbb{R}^N, \mathbb{R})$ satisfies $\lambda(x) \in (0,1)$ in Ω, $\lambda(x) > 1$ in $\mathbb{R}^N \setminus \overline{\Omega}$, and $\lambda(x) = |x|^2$ for large $|x|$. First they find a periodic solution of

$$\ddot{q} + \nabla\widetilde{V}(t,q) + \nabla\lambda^n(q) = 0$$

and second they take a limit as $n \to \infty$.

In this paper we take an approach different from those in [5, 11]. For simplicity we explain our idea in the case $R(t) \equiv I$, $c(t) \equiv 0$. For $\delta > 0$ small we set

$$\Omega_\delta = \left\{x \in \mathbb{R}^N;\ \operatorname{dist}(x,\Omega) < \delta\right\}$$

and we consider a function $U_\delta(x) \in C^2(\Omega_\delta, \mathbb{R})$ such that

$$\begin{aligned} &U_\delta(x) = 0 \text{ in } \overline{\Omega}, \quad U_\delta(x) > 0 \text{ in } \Omega_\delta \setminus \overline{\Omega}, \\ &U_\delta(x) \to \infty \quad \text{as } \operatorname{dist}(x, \partial\Omega_\delta) \to 0. \end{aligned}$$

First for $\ell \geq 1$ we find a critical point u_ℓ of

$$I_\ell(u) = \frac{1}{2}\int_0^{2\pi} |\dot{u}|^2\,dt - \int_0^{2\pi} \widetilde{V}(t,u)\,dt - \ell\int_0^{2\pi} U_\delta(u)\,dt \in C^2(\Lambda_\delta, \mathbb{R}),$$

via suitable minimax methods. Here

$$\Lambda_\delta = \left\{u \in H^1(S^1, \mathbb{R}^N);\ u(t) \in \Omega_\delta \text{ for all } t \in S^1\right\}.$$

Second we take a limit as $\ell \to \infty$. As a virtue of this approach, the limit of minimax values can be characterized as minimax values of the limit functional $J_\infty(u)$. Here $J_\infty\colon H^1(S^1, \mathbb{R}^N) \to [-\infty, \infty)$ is defined by

$$J_\infty(u) = \begin{cases} \frac{1}{2}\int_0^{2\pi} |\dot{u}|^2\,dt - \int_0^{2\pi} \widetilde{V}(t,u)\,dt & \text{for } u \in \overline{\Lambda}_0, \\ -\infty & \text{for } u \notin \overline{\Lambda}_0, \end{cases}$$

where $\overline{\Lambda}_0 = \{u \in H^1(S^1, \mathbb{R}^N);\ u(t) \in \overline{\Omega} \text{ for all } t \in S^1\}$. As a special case, this property gives a minimax characterization of a bounce trajectory using $J_\infty(u)$. It is also convenient to apply the ideas in a perturbation from symmetry. A similar idea was used in [15] in a different situation. Approximation schemes in [5, 11] do not have such a property.

REMARK 1.2. An orbit $u_\infty(t)$ with $\#\{t \in S^1;\ u_\infty(t) \in \partial\Omega(t)\} = \infty$ may appear as a limit of critical points u_ℓ of $I_\ell(u)$. We give 2 examples here. Let $N = 2$ and $\Omega = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1\}$, $V(t, x) \equiv 0$.

(1) Assume $R(t) = \left[\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right]$, $c(t) = \rho\left[\begin{smallmatrix} \cos t \\ \sin t \end{smallmatrix}\right]$ with $\rho > 1$. Then $J_\ell(u)$ has a critical point

$$u_\ell(t) = \begin{bmatrix} a_\ell \\ 0 \end{bmatrix},$$

where $a_\ell \in (1, 1+\delta)$ is suitably chosen. We have $a_\ell \to 1$ as $\ell \to \infty$ and $u_\infty(t) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$.

(2) Assume $R(t) \equiv I$ and $c(t) \equiv 0$. Then $J_\ell(t)$ has a critical point

$$u_\ell(t) = a_\ell \begin{bmatrix} \cos t \\ \sin t \end{bmatrix},$$

where $a_\ell \in (1, 1+\delta)$ with $a_\ell \to 1$ as $\ell \to \infty$ and $u_\infty(t) = \left[\begin{smallmatrix} \cos t \\ \sin t \end{smallmatrix}\right]$.

In both examples, we have $u_\infty(t) \in \partial\Omega$ for all t and $u_\infty(t)$ does not satisfy (1.3) at all $t \in S^1$. We also note that $J_\ell(u_\ell)$ stays finite as $\ell \to \infty$ and we will see in Remark 4.8 that the Morse index diverges as $\ell \to \infty$. It is an interesting question to ask in which situation such solutions appear.

2. PRELIMINARIES

Let Ω be an open bounded set with boundary $\partial\Omega$ of class C^2. We assume

$$[-2, 2]^N \subset \Omega. \tag{2.1}$$

Let $h(x)$ be the signed distance function from $\partial\Omega$; that is,

$$h(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{for } x \in \mathbb{R}^N \setminus \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{for } x \in \Omega. \end{cases}$$

We can see that for $\delta_0 > 0$ small, $h(x)$ is of class C^2 and $|\nabla h(x)| = 1$ in $\{x \in \mathbb{R}^N;\ \operatorname{dist}(x, \partial\Omega) \le \delta_0\}$. We also observe that $\nu(x) = \nabla h(x)$ is the unit outward normal vector at $x \in \partial\Omega$.

For $\delta \in [0, \delta_0]$, we set

$$\Omega_\delta = \{x \in \mathbb{R}^N;\ h(x) < \delta\}.$$

We note that $\Omega = \Omega_0 \subset \overline{\Omega}_0 \subset \Omega_\delta$. We extend $\nu(x)$ onto $\overline{\Omega}_{\delta_0}$ by

$$\nu(x) = \psi\big(h(x)\big)\nabla h(x),$$

where $\psi(t) \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\psi(t) = 1$ in $[-\frac{\delta_0}{2}, \delta_0]$, $\psi(t) = 0$ in $(-\infty, -\delta_0]$ and $\psi(t) \in [0,1]$ for all t. We also define $\zeta(t)\colon (-\infty,1) \to \mathbb{R}$ and $\zeta_\delta(t)\colon (-\infty,\delta) \to \mathbb{R}$ by

$$\zeta(t) = \begin{cases} \left(\frac{t}{1-t}\right)^4 & \text{for } t \in [0,1), \\ 0 & \text{for } t \in (-\infty, 0), \end{cases}$$

$$\zeta_\delta(t) = \zeta(t/\delta).$$

And we define $U_\delta(x)\colon \Omega_\delta \to \mathbb{R}$ by

$$U_\delta(x) = \zeta_\delta\big(h(x)\big).$$

Lemma 2.1. *$\zeta_\delta(t)$ and $U_\delta(x)$ have the following properties:*

(i) *For $t \in (0,\delta)$,*

$$\zeta_\delta'(t) = \frac{4}{t(1-t/\delta)}\zeta_\delta(t) \geq \frac{16}{\delta}\zeta_\delta(t), \tag{2.2}$$

$$\zeta_\delta''(t) \geq \frac{3}{t}\zeta_\delta'(t). \tag{2.3}$$

(ii) *For $x \in \Omega_\delta$,*

$$\big(\nabla U_\delta(x), \nu(x)\big) = \big|\nabla U_\delta(x)\big|, \tag{2.4}$$

$$\nabla U_\delta(x) = \big|\nabla U_\delta(x)\big|\nu(x), \tag{2.5}$$

$$U_\delta(x) \leq \frac{1}{4}h(x)\big(1 - h(x)/\delta\big)\big|\nabla U_\delta(x)\big| \leq \frac{\delta}{16}\big|\nabla U_\delta(x)\big|. \tag{2.6}$$

For $x \in \Omega_\delta \setminus \overline{\Omega}$,

$$\nabla^2 U_\delta(x)\big(\nu(x), \nu(x)\big) \geq \left(\frac{3}{h(x)} - C\right)\big|\nabla U_\delta(x)\big|, \tag{2.7}$$

where $C > 0$ is independent of x.

Proof. (i) We compute

$$\begin{aligned} \zeta_\delta'(t) &= \frac{4}{\delta}\frac{(t/\delta)^3}{(1-t/\delta)^5} \\ &= \frac{4}{\delta}\frac{1}{t/\delta(1-t/\delta)}\left(\frac{t/\delta}{1-t/\delta}\right)^4 = \frac{4}{t(1-t/\delta)}\left(\frac{t/\delta}{1-t/\delta}\right)^4 \\ &= \frac{4}{t(1-t/\delta)}\zeta_\delta(t) \leq \frac{16}{\delta}\zeta_\delta(t), \end{aligned}$$

$$\begin{aligned}\zeta_\delta''(t) &= \frac{12}{\delta^2}\frac{(t/\delta)^2}{(1-t/\delta)^5} + \frac{20}{\delta^2}\frac{(t/\delta)^3}{(1-t/\delta)^6} = \frac{4}{\delta^2}\left(\frac{3}{t/\delta} + \frac{5}{1-t/\delta}\right)\frac{(t/\delta)^3}{(1-t/\delta)^5}\\ &= \frac{1}{\delta}\left(\frac{3}{t/\delta} + \frac{5}{1-t/\delta}\right)\zeta_\delta'(t) \geq \frac{3}{t}\zeta_\delta'(t).\end{aligned}$$

Thus (i) is proved.

(ii) First we note $U_\delta(x) = 0$ on $\overline{\Omega}$. We need to show the desired inequalities for $x \in \Omega_\delta \setminus \overline{\Omega}$. Since $\nu(x) = \nabla h(x)$ and $|\nu(x)| = 1$ on $\Omega_\delta \setminus \overline{\Omega}$, we have

$$\nabla U_\delta(x) = \zeta_\delta'(h(x))\nabla h(x) = \zeta_\delta'(h(x))\nu(x).$$

Thus we have $|\nabla U_\delta(x)| = \zeta_\delta'(h(x))$ and $|\nabla U_\delta(x)| = (\nabla U_\delta(x), \nu(x))$. Then (2.6) follows from (2.2). For (2.7), we have for $v \in \mathbb{R}^N$

$$\nabla^2 U_\delta(x)(v,v) = \zeta_\delta''\big(h(x)\big)\big(\nabla h(x), v\big)^2 - \zeta_\delta'\big(h(x)\big)\nabla^2 h(x)(v,v).$$

Setting $v = \nu(x)$, we have from the boundedness of $\nabla^2 h(x)$ in $\overline{\Omega_\delta}$ and (2.3)

$$\begin{aligned}\nabla^2 U_\delta(x)\big(\nu(x),\nu(x)\big) &\geq \zeta_\delta''\big(h(x)\big) - C\zeta_\delta'\big(h(x)\big) \geq \left(\frac{3}{h(x)} - C\right)\zeta_\delta'\big(h(x)\big)\\ &\geq \left(\frac{3}{h(x)} - C\right)\big|\nabla U_\delta(x)\big|.\end{aligned}$$

■

We set $E = H^1(S^1, \mathbb{R}^N)$ and for $u \in E$

$$\begin{aligned}\|u\|_p &= \left(\int_0^{2\pi} |u|^p\,dt\right)^{1/p} \quad \text{for } p \in [1,\infty),\\ \|u\|_\infty &= \operatorname{ess\,sup}\big|u(t)\big|,\\ \|u\|_E &= \big(\|\dot u\|_2^2 + \|u\|_2^2\big)^{1/2}\end{aligned}$$

and for $u, v \in L^2(S^1, \mathbb{R}^N)$, we denote

$$(u,v)_2 = \int_0^{2\pi} uv\,dt.$$

For $\delta \in (0, \delta_0/2]$ we set

$$\begin{aligned}\Lambda_\delta &= \big\{u \in E;\ u(t) \in \Omega_\delta \text{ for all } t \in S^1\big\},\\ \partial\Lambda_\delta &= \big\{u \in E;\ u(t) \in \overline{\Omega_\delta} \text{ for all } t \in S^1 \text{ and } u(t) \in \partial\Omega_\delta \text{ for some } t \in S^1\big\}.\end{aligned}$$

We also set

$$\begin{aligned}\Lambda_0 &= \big\{u \in E;\ u(t) \in \Omega \text{ for all } t \in S^1\big\},\\ \overline{\Lambda_0} &= \big\{u \in E;\ u(t) \in \overline{\Omega} \text{ for all } t \in S^1\big\}.\end{aligned}$$

Since $U_\delta(x)$ satisfies the strong force condition at $\partial\Omega_\delta$, we have the following lemma.

LEMMA 2.2. *Let* $\delta \in (0, \delta_0/2]$. *If* $(u_j)_{j=1}^\infty \subset \Lambda_\delta$ *satisfies* $u_j \rightharpoonup u_0 \in \partial\Lambda_\delta$ *weakly in* E, *then*

$$\int_0^{2\pi} U_\delta(u_j)\,dt \to \infty \quad \text{as } j \to \infty.$$

3. Functional setting and Palais–Smale condition

We extend $\widetilde{V}(t,x)$ on $S^1 \times \mathbb{R}^N$ and set

$$V(t,x) = \widetilde{V}\big(t, R(t)x + c(t)\big) \in C^2(S^1 \times \mathbb{R}^N, \mathbb{R}).$$

Clearly we may assume $V(t,x)$, $V_t(t,x)$, $\nabla V(t,x)$ are bounded on $S^1 \times \mathbb{R}^N$.

Writing $q(t) = R(t)u(t) + c(t)$, where $u(t)\colon S^1 \to \overline{\Omega}$, solutions $q(t)$ of (1.2) can be characterized as critical points of

$$I(u) = \frac{1}{2}\left\|\frac{d}{dt}\big(R(t)u + c(t)\big)\right\|_2^2 - \int_0^{2\pi} V\big(t, u(t)\big)\,dt.$$

To find critical points of $I(u)$, we introduce a penalized functional $I_\ell(u)$ for $\ell \in [1,\infty)$ by

$$I_\ell(u) = \frac{1}{2}\left\|\frac{d}{dt}\big(R(t)u + c(t)\big)\right\|_2^2 - \int_0^{2\pi} V\big(t, u(t)\big)\,dt - \ell\int_0^{2\pi} U_\delta(u)\,dt.$$

We observe $I_\delta(u) \in C^2(\Lambda_\delta, \mathbb{R})$.

To show the Palais–Smale condition for $I_\ell(u)$, we start with fundamental properties of Λ_δ, $\nu(u)$, etc.

LEMMA 3.1. *Let* $\delta \in (0, \delta_0/2]$.

(i) *There exists* $C > 0$ *independent of* $u \in \Lambda_\delta$ *such that*

$$\|u\|_\infty \le C, \quad \big\|\nu(u)\big\|_2 \le C, \quad \left\|\frac{d}{dt}\big(\nu(u)\big)\right\|_2 \le C\big(\|\dot u\|_2 + 1\big) \quad \textit{for all } u \in \Lambda_\delta.$$

(ii) *For* $u \in \Lambda_\delta$, *set* $q(t) = R(t)u + c(t)$. *Then*

$$\|q\|_\infty \le C, \quad \|\dot u\|_2 - C \le \|\dot q\|_2 \le \|\dot u\|_2 + C \quad \textit{for all } u \in \Lambda_\delta.$$

PROOF. (i) We note that $\frac{d}{dt}(\nu(u)) = \nabla\nu(u)\dot u$ and $\nu(x)$ is bounded in $\overline{\Omega_\delta}$. Since $u \in \Lambda_\delta$ implies $u(t) \in \Omega_\delta$ for all t, (i) follows.

(ii) follows from the fact that $\dot q = R(t)\dot u + \dot R(t)u + \dot c(t)$ and $\|R(t)\dot u\|_2 = \|\dot u\|_2$. ∎

Next we show that the Palais–Smale condition holds for $I_\ell(u)$ for small δ.

PROPOSITION 3.2. *There exists $\delta \in (0, \delta_0/2]$ small with the following properties:*

(i) *For any $\ell \geq 1$, $I_\ell(u)\colon \Lambda_\delta \to \mathbb{R}$ satisfies the Palais–Smale condition.*

(ii) *For any $b \in \mathbb{R}$ there exists a constant $C_b > 0$ independent of $\ell \in [1,\infty)$ such that if $u \in \Lambda_\delta$ satisfies*

$$I'_\ell(u) = 0, \quad I_\ell(u) \leq b,$$

then

$$\|\dot{u}\|_2 \leq C_b, \quad \ell\int_0^{2\pi} \big(\nabla U_\delta(u), \nu(u)\big)\,dt \leq C_b, \quad \ell\int_0^{2\pi} U_\delta(u)\,dt \leq C_b.$$

PROOF. We divide the proof into two steps.

Step 1. First we prove (ii) in a slightly general situation. Assume that $\ell \in [1,\infty)$ and $u \in \Lambda_\delta$ satisfies

$$\big\|I'_\ell(u)\big\|_{E^*} \leq 1, \tag{3.1}$$

$$I_\ell(u) \leq b + 1. \tag{3.2}$$

We show the conclusion of (ii) holds for a constant C_b independent of $\ell \in [1,\infty)$.

By Lemma 3.1 (i), we have from (3.1) that

$$I'_\ell(u)\nu(u) \leq \big\|\nu(u)\big\|_E \leq C\big(\|\dot{u}\|_2 + 1\big). \tag{3.3}$$

On the other hand, we have for $q = R(t)u + c(t)$

$$I'_\ell(u)\nu(u) = \Big(\dot{q}, \frac{d}{dt}\big(\nu(u)\big)\Big)_2 - \int_0^{2\pi} \nabla V(t,u)\nu(u)\,dt - \ell\int_0^{2\pi} \nabla U_\delta(u)\nu(u)\,dt. \tag{3.4}$$

By Lemma 3.1 (i), we have

$$\Big|\Big(\dot{q}, \frac{d}{dt}\big(\nu(u)\big)\Big)_2\Big| \leq C\big(\|\dot{u}\|_2^2 + 1\big), \tag{3.5}$$

$$\Big|\int_0^{2\pi} \nabla V(t,u)\nu(u)\,dt\Big| \leq C. \tag{3.6}$$

Combining (3.3)–(3.6), we have

$$\ell\int_0^{2\pi} \big(\nabla U_\delta(u), \nu(u)\big)\,dt \leq C'\big(\|\dot{u}\|_2^2 + 1\big). \tag{3.7}$$

By (2.6) and (2.4), we have

$$\ell \int_0^{2\pi} U_\delta(u)\,dt \le \frac{\delta}{16}\ell \int_0^{2\pi} \big(\nabla U_\delta(u), \nu(u)\big)\,dt \le \delta C''\big(\|\dot u\|_2^2+1\big). \tag{3.8}$$

Thus by (3.2) and Lemma 3.1 (ii),

$$\begin{aligned} b+1 &\ge \frac{1}{2}\|\dot u\|_2^2 - \int_0^{2\pi} V(t,u)\,dt - \ell \int_0^{2\pi} U_\delta(u)\,dt \\ &\ge \left(\frac{1}{2}-\delta C''\right)\|\dot u\|_2^2 - C - \delta C''. \end{aligned}$$

Thus for $\delta > 0$ small, $\|\dot u\|_2$ is uniformly bounded with respect to $\ell \in [1,\infty)$. The uniform boundedness of $\ell \int_0^{2\pi} (\nabla U_\delta(u), \nu(u))\,dt$ and $\ell \int_0^{2\pi} U_\delta(u)\,dt$ follows from (3.8). Thus the results of Step 1 and (ii) of Proposition 3.2 hold.

Step 2: Proof of (i). Next we show (i). We assume that (u_j) satisfies

$$I_\ell(u_j) \to b, \quad \big\|I_\ell'(u_j)\big\|_{E^*} \to 0 \quad \text{as } j \to \infty.$$

We may assume that (3.1)–(3.2) hold for $u = u_j$ for all $j \in \mathbb{N}$. By Step 1,

$$\|u_j\|_E, \quad \ell \int_0^{2\pi} U_\delta(u_j)\,dt$$

is bounded as $j \to \infty$. Extracting a subsequence if necessary, we may assume $u_j \rightharpoonup u_\infty \in \overline{\Lambda_\delta}$ weakly in E. Since $\int_0^{2\pi} U_\delta(u_j)\,dt$ stays bounded, we have $u_\infty \in \Lambda_\delta$ by Lemma 2.2. Thus

$$\inf_{j\in\mathbb{N}} \operatorname{dist}\big(u_j([0,2\pi]), \partial\Omega_\delta\big) > 0$$

and we can show the strong convergence of $(u_j)_{j=1}^\infty$ in the standard way. Thus (ii) is proved. ■

In what follows, we fix small $\delta > 0$, for which Proposition 3.2 holds. Proposition 3.2 enables us to apply minimax methods to find the critical points of $I_\ell(u)$ for $\ell \in [1,\infty)$. Later we apply symmetric mountain pass theorem and take a limit as $\ell \to \infty$.

4. Limit process

In this section we study the behavior of critical points (u_ℓ) of $I_\ell(u)$ as $\ell \to \infty$. We assume that for some sequence $\ell_j \to \infty$ as $j \to \infty$

$$I_{\ell_j}(u_{\ell_j}) \to b, \tag{4.1}$$

$$I_{\ell_j}'(u_{\ell_j}) = 0 \tag{4.2}$$

for some $b \in \mathbb{R}$. For sake of simplicity of notation, we write ℓ instead of ℓ_j.

By Proposition 3.2, there exists $C_b > 0$ such that

$$\|\dot{u}_\ell\|_2 \le C_b, \tag{4.3}$$

$$\ell \int_0^{2\pi} \big(\nabla U_\delta(u_\ell), \nu(u_\ell)\big)\, dt \le C_b, \tag{4.4}$$

$$\ell \int_0^{2\pi} U_\delta(u_\ell)\, dt \le C_b. \tag{4.5}$$

Thus (u_ℓ) is bounded in E and we may assume that $u_\ell \rightharpoonup u_\infty \in E$ weakly in E. By (4.5), we have $u_\infty \in \overline{\Lambda}_0$; i.e., $u_\infty(t) \in \overline{\Omega}$ for all t.

Since $I_\ell'(u_\ell) = 0$, we have $I_\ell'(u_\ell)(R(t)^{-1}\varphi) = 0$ for all $\varphi \in E$; that is, writing $q_\ell = R(t)u_\ell + c(t)$,

$$(\dot{q}_\ell, \dot{\varphi})_2 - \big(\nabla V(t, u_\ell), R(t)^{-1}\varphi\big)_2 - \big(\ell \nabla U_\delta(t, u_\ell), R(t)^{-1}\varphi\big)_2 = 0 \quad \text{for all } \varphi \in E.$$

Thus the following Euler–Lagrange equation holds:

$$\ddot{q}_\ell + R(t)\big(\nabla V(t, u_\ell) + \ell \nabla U_\delta(u_\ell)\big) = 0. \tag{4.6}$$

We have the following lemma.

Lemma 4.1. *(u_ℓ) has the following property:*

$$\ell \int_0^{2\pi} U_\delta(u_\ell)\, dt \to 0 \quad \textit{as } \ell \to \infty.$$

Proof. Since $u_\infty(t) \in \overline{\Omega}$ for all t,

$$\max_{t \in S^1} h\big(u_\ell(t)\big) \to 0.$$

By (2.6) and (2.4),

$$\begin{aligned} \ell \int_0^{2\pi} U_\delta(u_\ell)\, dt &\le \frac{1}{4} \ell \int_0^{2\pi} h(u_\ell)\big(1 - h(u_\ell)/\delta\big)\big(\nabla U_\delta(u_\ell), \nu(u_\ell)\big)\, dt \\ &\le \frac{1}{4} \max_{t \in S^1} h\big(u_\ell(t)\big) \cdot \ell \int_0^{2\pi} \big(\nabla U_\delta(u_\ell), \nu(u_\ell)\big)\, dt \to 0. \end{aligned}$$ ∎

By (2.4) and (4.4), we observe that $\ell \nabla U_\delta(u_\ell)$ is bounded in $L^1(0, 2\pi)$. Thus we may assume

$$\ell \big|\nabla U_\delta(u_\ell)\big| \rightharpoonup \mu \tag{4.7}$$

for a positive finite measure μ on S^1. Since $\nabla U_\delta(x) = 0$ on Ω, we also observe that

$$\operatorname{supp} \mu \subset C(u_\infty) \equiv \big\{t \in S^1;\ u_\infty(t) \in \partial\Omega\big\}. \tag{4.8}$$

Since $\nabla U_\delta(u_\ell) = |\nabla U_\delta(u_\ell)|\nu(u_\ell)$ by (2.5), $\ell\nabla U(u_\ell) \rightharpoonup \mu\nu(u_\infty)$ and thus

$$\ddot{q}_\infty + R(t)\nabla V(t, u_\infty) + \mu R(t)\nu(u_\infty) = 0. \tag{4.9}$$

We set

$$F_\ell(t) = \frac{1}{2}\left|\dot{q}_\ell(t)\right|^2 + V\big(t, u_\ell(t)\big) + \ell U_\delta\big(u_\ell(t)\big) - \big(\dot{q}_\ell(t), \dot{R}(t)u_\ell(t) + \dot{c}(t)\big).$$

Then we have the following lemma.

Lemma 4.2. *(u_ℓ) and (q_ℓ) have the following properties:*

(i) *$(\ddot{u}_\ell)$ is bounded in $L^1(S^1)$ and $(\dot{q}_\ell(t))$ has a strongly convergent subsequence in $L^2(S^1)$ as $\ell \to \infty$.*

(ii) *$F_\ell(t)$ has a strongly convergent subsequence in $W^{1,1}(S^1)$ as $\ell \to \infty$.*

Proof. (i) We recall that $(\ell\nabla U_\delta(u_\ell))$ is bounded in $L^1(S^1)$. Thus by the Euler–Lagrange equation (4.6) we observe that $(\ddot{q}_\ell)$ is bounded in $L^1(S^1)$. Thus $(\dot{q}_\ell)$ is bounded in $W^{1,1}(S^1)$. By the compactness of embedding $W^{1,1}(S^1) \subset L^2(S^1)$, $(\dot{q}_\ell)$ has a strongly convergent subsequence in $L^2(S^1)$.

(ii) Clearly $F_\ell(t)$ has a strongly convergent subsequence in $L^1(S^1)$ by (i) and Lemma 4.1. We compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left[|\dot{q}_\ell|^2\right] &= (\ddot{q}_\ell, \dot{q}_\ell)_2 = (\ddot{q}_\ell, R(t)\dot{u}_\ell + \dot{R}(t)u_\ell + \dot{c}) \\
&= -\big(\nabla V(t, u_\ell) + \ell\nabla U_\delta(u_\ell), \dot{u}_\ell\big) + (\ddot{q}_\ell, \dot{R}u_\ell + \dot{c}), \\
\frac{d}{dt}(\dot{q}_\ell, \dot{R}u_\ell + \dot{c}) &= (\ddot{q}_\ell, \dot{R}u_\ell + \dot{c}) + (\dot{q}_\ell, \dot{R}\dot{u}_\ell + \ddot{R}u_\ell + \ddot{c}).
\end{aligned}$$

Thus

$$\frac{d}{dt}F_\ell(t) = V_t(t, u_\ell) - (\dot{q}_\ell, \dot{R}\dot{u}_\ell + \ddot{R}u_\ell + \ddot{c}).$$

By (i), $(\dot{q}_\ell)$ and $(\dot{u}_\ell)$ have a strongly convergent subsequence in $L^2(S^1)$ and thus $\frac{d}{dt}F_\ell(t)$ also has a strongly convergent subsequence in $L^1(S^1)$. Thus (ii) holds. ∎

The following corollary follows from Lemma 4.1 and Lemma 4.2 (i).

Corollary 4.3. *$I_\ell(u_\ell) \to \frac{1}{2}\|\frac{d}{dt}(Ru_\infty + c)\|_2^2 - \int_0^{2\pi} V(t, u_\infty)\, dt$ as $\ell \to \infty$.*

We also have the following corollary.

Corollary 4.4. *$t \mapsto |\dot{u}_\infty(t)|^2$ is continuous.*

PROOF. We note that

$$\begin{aligned} F_\ell(t) &= \frac{1}{2}|R\dot{u}_\ell + \dot{R}u_\ell + \dot{c}|^2 + V(t,u_\ell) + \ell U_\delta(u_\ell) - (R\dot{u}_\ell + \dot{R}u_\ell + \dot{c}, \dot{R}u_\ell + \dot{c}) \\ &= \frac{1}{2}|\dot{u}_\ell(t)|^2 - \frac{1}{2}|\dot{R}u_\ell + \dot{c}|^2 + V(t,u_\ell) + \ell U_\delta(u_\ell). \end{aligned}$$

Thus by Lemma 4.1

$$F_\ell(t) \to F_\infty(t) \equiv \frac{1}{2}\left|\dot{u}_\infty(t)\right|^2 - \frac{1}{2}|\dot{R}u_\infty + \dot{c}|^2 + V(t,u_\infty)$$

strongly in $W^{1,1}(S^1)$ and $C(S^1)$. Thus $|u_\infty(t)|^2$ is continuous. ■

Next we study non-regular instants of $u_\infty(t)$. Following [5], we say $\bar{t} \in S^1$ is a *non-regular instant* of $u_\infty(t)$ if and only if for any $\rho > 0$ $u_\infty(t)$ does not satisfy

$$\ddot{q}_\infty + R(t)\nabla V(t,u_\infty) = 0 \quad \text{in } (\bar{t}-\rho, \bar{t}+\rho).$$

We have the following proposition.

PROPOSITION 4.5. *Assume that $\bar{t} \in S^1$ is a non-regular instant of $u_\infty(t)$. Then*

(i) $u_\infty(\bar{t}) \in \partial\Omega$.

(ii) *For any $\rho > 0$*

$$\liminf_{\ell\to\infty} \ell \int_{\bar{t}-\rho}^{\bar{t}+\rho} \left|\nabla U_\delta(u_\ell)\right| dt > 0.$$

PROOF. (i) Since $u_\infty(t)$ satisfies (4.9), if $\bar{t} \in S^1$ is a non-regular instant, we have $\bar{t} \in \operatorname{supp}\mu \subset C(u_\infty)$. Thus by (4.8) we have $u_\infty(\bar{t}) \in \partial\Omega$.

(ii) We argue indirectly. If $\bar{t} \in S^1$ satisfies

$$\lim_{\ell\to\infty} \ell \int_{\bar{t}-\rho}^{\bar{t}+\rho} \left|\nabla U_\delta(u_\ell)\right| dt = 0,$$

then by (4.7), $\mu = 0$ on $(\bar{t}-\rho, \bar{t}+\rho)$. Thus $\bar{t}$ is not a non-regular instant of $u_\infty(t)$. ■

The next proposition gives an estimate of Morse index from below by the number of non-regular instants of $u_\infty(t)$. For a critical point u of $I_\ell(u)$, we define the Morse index $i(I_\ell''(u))$ and augmented Morse index $i_0(I_\ell''(u))$ by

$$\begin{aligned} i\left(I_\ell''(u)\right) = \max\big\{ &\dim H;\ H \subset E \text{ is a subspace of } E \text{ such that} \\ &I_\ell''(u)(v,v) < 0 \text{ for all } v \in H \setminus \{0\}\big\}, \\ i_0(I_\ell''(u)) = \max\big\{ &\dim H;\ H \subset E \text{ is a subspace of } E \text{ such that} \\ &I_\ell''(u)(v,v) \le 0 \text{ for all } v \in H\big\}. \end{aligned}$$

PROPOSITION 4.6. *We have the following Morse index estimate for* (u_ℓ)*:*

$$\liminf_{\ell\to\infty} i\big(I''_\ell(u_\ell)\big) \ge \#\{\bar t \in S^1;\ \bar t \text{ is a non-regular instant of } u_\infty\}.$$

PROOF. Let $\bar t$ be a non-regular instant of u_∞. For a given $\rho > 0$ we choose a function

$$\varphi(t) \in C^\infty(S^1, \mathbb{R})$$

such that

$$\begin{aligned} &\varphi(t) = 1 \quad \text{for } |t-\bar t| \le \rho,\\ &\varphi(t) = 0 \quad \text{for } |t-\bar t| \ge 2\rho,\\ &\varphi(t) \in [0,1] \quad \text{for all } t \in S^1. \end{aligned}$$

To show Proposition 4.6, it suffices to show

$$I''_\ell(u_\ell)\big(\varphi\nu(u_\ell), \varphi\nu(u_\ell)\big) \to -\infty \quad \text{as } \ell \to \infty. \tag{4.10}$$

Note that $\nabla^2 U_\delta(u_\ell(t)) = 0$, $\nabla U_\delta(u_\ell(t)) = 0$ if $h(u_\ell(t)) \le 0$ and recall (ii) of Proposition 4.5. By (2.7),

$$\begin{aligned} (4.11)\quad &\ell \int_0^{2\pi} \nabla^2 U_\delta(u_\ell)\big(\varphi\nu(u_\ell), \varphi\nu(u_\ell)\big)\, dt\\ &\ge \ell \int_{\{t;\, h(u_\ell(t))>0\}} \left(\frac{3}{h(u_\ell)} - C\right) \big|\nabla U_\delta(u_\ell)\big| \varphi(t)^2\, dt\\ &\ge \inf_{\{t:\, h(u_\ell(t))>0\}} \left(\frac{3}{h(u_\ell)} - C\right) \cdot \ell \int_{\bar t-\rho}^{\bar t+\rho} \big|\nabla U_\delta(u_\ell)\big|\, dt \to \infty \quad \text{as } \ell \to \infty. \end{aligned}$$

Here we use the fact that $\max_{t\in[0,2\pi]} h(u_\ell(t)) \to 0$ as $\ell \to \infty$, which follows from $u_\ell \to u_\infty \in \overline{\Lambda}_0$.

On the other hand, it is easy to see that

$$\left\|\frac{d}{dt}\big(R(t)u_\ell + c(t)\big)\right\|_2^2, \qquad \int_0^{2\pi} \nabla^2 V(t, u_\ell)\big(\varphi\nu(u_\ell), \varphi\nu(u_\ell)\big)\, dt$$

stay bounded as $\ell \to \infty$. Thus (4.10) follows. ■

We summarize the results obtained in this section.

PROPOSITION 4.7. *Suppose* $(u_\ell) \subset \Lambda_\delta$ *satisfies* (4.1)–(4.2). *After extracting a subsequence, we assume* $u_\ell \rightharpoonup u_\infty$ *weakly in* E. *Then we have the following:*

(i) *Except for non-regular instants for* $u_\infty(t)$,

$$\ddot q_\infty + R(t)\nabla V(t, u_\infty) = 0. \tag{4.12}$$

(ii) $\#\{\bar t \in S^1;\ \bar t \text{ is a non-regular instant of } u_\infty\} \le \liminf_{\ell\to\infty} i\big(I''_\ell(u_\ell)\big)$.

(iii) *Assume that $\bar{t}$ is an isolated non-regular instant for $u_\infty(t)$. Then*

$$\dot{u}_{\infty,\pm}(\bar{t}) = \lim_{s\to\pm 0} \dot{u}_\infty(\bar{t}+s)$$

exists. Moreover,

$$\left|\dot{u}_{\infty,+}(\bar{t})\right| = \left|\dot{u}_{\infty,-}(\bar{t})\right|, \tag{4.13}$$

$$\dot{u}_{\infty,+}(\bar{t}) = \dot{u}_{\infty,-}(\bar{t}) - c\nu\big(u_\infty(\bar{t})\big), \tag{4.14}$$

where $c \geq 0$.

Proof. We need to show (iii). Since $\bar{t}$ is an isolated non-regular instant, for some $\rho > 0$, $u_\infty(t)$ is of class C^2 in $[\bar{t}-\rho, \bar{t}]$ and $[\bar{t}, \bar{t}+\rho]$. Thus $\dot{u}_{\infty,\pm}(\bar{t})$ exists. Then (4.13) follows from Corollary 4.4. Remark that

$$\dot{q}_{\infty,+}(\bar{t}) - \dot{q}_{\infty,-}(\bar{t}) = R(\bar{t})\big(\dot{u}_{\infty,+}(\bar{t}) - \dot{u}_{\infty,-}(\bar{t})\big).$$

By (4.9), we obtain (4.14) with $c \geq 0$. ∎

Remark 4.8. As we stated in Remark 1.2, there are examples in which the limit $u_\infty(t)$ has infinitely many non-regular instants. Let $N = 2$, $\Omega = \{x \in \mathbb{R}^2;\ |x| < 1\}$, $V(t,x) \equiv 0$.

(1) When $R(t) = \left[\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right]$, $c(t) = \rho\left[\begin{smallmatrix} \cos t \\ \sin t \end{smallmatrix}\right]$ with $\rho > 1$. Then $J_\ell'(u_\ell) = 0$ holds if and only if

$$\ddot{q}_\ell + \ell R(t)\nabla U_\delta(u_\ell) = 0,$$

where $q_\ell(t) = R(t)u_\ell(t) + c(t)$. $u_\ell(t) = \left[\begin{smallmatrix} a_\ell \\ 0 \end{smallmatrix}\right]$ is a solution if and only if $a_\ell \in (1, 1+\delta)$ satisfies

$$-(a_\ell + \rho) + \ell\zeta_\delta'(a_\ell - 1) = 0.$$

We note that $\zeta_\delta'(0) = 0$ and $\zeta_\delta'(t) \to \infty$ as $t \to \delta - 0$. Thus for any $\ell \in [1,\infty)$ there exists a solution $a_\ell \in (1, 1+\delta)$. We can easily see that $a_\ell \to 1$ as $\ell \to \infty$ and $u_\infty(t) = \lim_{\ell\to\infty} u_\ell(t) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Clearly $q_\infty(t)$ does not satisfy $\ddot{q}_\infty = 0$ and all $t \in S^1$ are non-regular instants.

(2) Assume $R(t) \equiv I$ and $c(t) \equiv 0$. Then

$$u_\ell(t) = a_\ell \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

is a critical point of $I_\ell(u)$ if and only if $-a_\ell + \ell\zeta_\delta'(a_\ell - 1) = 0$. Such a $a_\ell \in (1, 1+\delta)$ exists and it satisfies $a_\ell \to 1$ as $\ell \to \infty$. Thus

$$u_\ell(t) \to u_\infty(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \quad \text{as } \ell \to \infty.$$

Clearly $q_\infty(t) = u_\infty(t)$ does not satisfy $\ddot{q}_\infty = 0$ and all $t \in S^1$ are non-regular instants.

5. Minimax methods

To define minimax values, first we define an extension of $I_\ell(u)$ to E. We define $J_\ell(u)\colon E \to [-\infty, \infty)$ by

$$J_\ell(u) = \begin{cases} I_\ell(u) & \text{for } u \in \Lambda_\delta, \\ -\infty & \text{for } u \in E \setminus \Lambda_\delta. \end{cases}$$

We note that $J_\ell(u)$ is of class C^2 in its effective domain $\{u \in E;\ J_\ell(u) > -\infty\} = \Lambda_\delta$. We also define $J_\infty(u)\colon E \to [-\infty, \infty)$ by

$$J_\infty(u) = \begin{cases} \frac{1}{2}\left\| \frac{d}{dt}(Ru + c)\right\|_2^2 - \int_0^{2\pi} V(t,u)\,dt & \text{for } u \in \overline{\Lambda}_0, \\ -\infty & \text{for } u \in E \setminus \overline{\Lambda}_0. \end{cases}$$

We can easily see the following lemma.

LEMMA 5.1. *$J_\ell(u)$ has the following limit behavior:*

(i) *For $\ell \in [1, \infty]$, $J_\ell(u)$ is upper semi-continuous; that is, $u_j \to u_\infty$ strongly in E implies*

$$J_\ell(u_\infty) \geq \limsup_{j\to\infty} J_\ell(u_j).$$

(ii) *For each $u \in E$,*

$$\ell \mapsto J_\ell(u);\ [1,\infty] \to [-\infty, \infty)$$

is non-increasing on $[1, \infty]$.

(iii) *$\ell \mapsto J_\ell(u)$ is continuous on $[1, \infty)$ and $J_\ell(u) \to J_\infty(u)$ as $\ell \to \infty$ in the following sense:*

$$\begin{aligned} &J_\ell(u) = J_\infty(u) > -\infty \quad \textit{if } u \in \overline{\Lambda}_0, \\ &J_\ell(u) \to J_\infty(u) = -\infty \quad \textit{if } u \in \Lambda_\delta \setminus \overline{\Lambda}_0, \\ &J_\ell(u) = -\infty \textit{ for all } \ell \in [1,\infty] \quad \textit{if } u \in E \setminus \Lambda_\delta. \end{aligned}$$

(iv) *For any non-empty compact set $K \subset E$ and $\ell \in [1, \infty]$*

$$\max_{u \in K} J_\ell(u)$$

is attained. Here we regard

$$\begin{aligned} &\max_{u\in K} J_\ell(u) = -\infty \quad \textit{if } \ell \in [1,\infty) \textit{ and } K \cap \Lambda_\delta = \emptyset, \\ &\max_{u\in K} J_\infty(u) = -\infty \quad \textit{if } K \cap \overline{\Lambda}_0 = \emptyset. \end{aligned}$$

In what follows, we consider minimax values for $J_\ell(u)$. For $\ell \in [1,\infty)$, $I_\ell(u)$ satisfies the Palais–Smale condition and we can develop the deformation theory to show finite minimax values are critical values. For $J_\ell(u)$ ($\ell \in [1,\infty)$), if minimax values are finite, we can also show that the minimax values are critical values since we may assume under the deformation the level set $\{u \in E;\ J_\ell(u) \in [-\infty, c]\}$ is invariant for $c \in (-\infty,\infty)$ less than minimax values. To find critical points, we apply ideas in the perturbation from symmetry in [1–4, 12, 14, 17]. Here we follow mainly Rabinowitz [12, 13] and Tanaka [17].

Now we assume that $0 \in \Omega$ and Ω is symmetric with respect to 0. We also assume

$$\Omega \subset \left\{x \in \mathbb{R}^N;\ |x| \le L\right\}. \tag{5.1}$$

We choose an orthonormal basis $e_1, e_2, \ldots, e_N, e_{N+1}, \ldots$ in E. We may assume $e_1 = (1, 0, \ldots, 0)$, …, $e_N = (0, \ldots, 0, 1)$ are constant functions.

For $n \in \mathbb{N}$ we set

$$E_n = \operatorname{span}\{e_1, \ldots, e_n\}$$

and choose $0 < R_1 < \cdots < R_n < \cdots$ such that

$$\|u\|_\infty \ge L \text{ for } u \in E_n \text{ with } \|u\|_E \ge R_n. \tag{5.2}$$

We also set

$$D_n = \left\{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n;\ \xi_1^2 + \cdots + \xi_n^2 \le R_n^2\right\},$$
$$\partial D_n = \left\{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n;\ \xi_1^2 + \cdots + \xi_n^2 = R_n^2\right\}.$$

We regard $D_1 \subset D_2 \subset \cdots$ and for $\xi = (\xi_1, \ldots, \xi_n) \in D_n$ we set

$$\gamma_0(\xi) = \sum_{i=1}^n \xi_i e_i.$$

By (5.1)–(5.2), we have $J_\ell(\gamma_0(\xi)) = -\infty$ for all $\xi \in \partial D_n$ and $\ell \in [1,\infty]$.

Following [12], we introduce minimax values b_n^ℓ for $n \in \mathbb{N}$ and $\ell \in [1,\infty]$ as follows:

$$b_n^\ell = \inf_{\gamma \in \Gamma_n} \max_{\xi \in D_n} J_\ell\big(\gamma(\xi)\big),$$

where

$$\Gamma_n = \left\{\gamma \in C(D_n, E);\ \gamma(-\xi) = -\gamma(\xi) \text{ for all } \xi \in D_n,\ \gamma(\xi) = \gamma_0(\xi) \text{ for all } \xi \in \partial D_n\right\}. \tag{5.3}$$

We also define

$$c_n^\ell = \inf_{\gamma \in \Gamma_{n+1}^+} \max_{\xi \in D_{n+1}^+} J_\ell\big(\gamma(\xi)\big),$$

where

$$\begin{aligned} D_{n+1}^+ &= \big\{(\xi_1, \dots, \xi_n, \xi_{n+1}) \in D_{n+1};\ \xi_{n+1} \ge 0\big\}, \\ \Gamma_{n+1}^+ &= \big\{\gamma \in C(D_{n+1}^+, E); \\ &\qquad \gamma|_{D_n} \in \Gamma_n,\ \gamma(\xi) = \gamma_0(\xi) \text{ for } |\xi| = R_{n+1} \text{ or } \xi \in E_n \setminus D_n\big\}. \end{aligned}$$

We note that for any $\ell \in [1, \infty]$, $n \in \mathbb{N}$,

$$\begin{aligned} J_\ell\big(\gamma(\xi)\big) &= -\infty \quad \text{for all } \gamma \in \Gamma_n \text{ and } \xi \in \partial D_n, \\ J_\ell\big(\gamma(\xi)\big) &= -\infty \quad \text{for all } \gamma \in \Gamma_{n+1}^+ \text{ and } \xi \in \partial D_{n+1}^+ \setminus D_n. \end{aligned}$$

We have the following proposition.

PROPOSITION 5.2. *(b_n^ℓ) and (c_ℓ^n) have the following properties:*

(i) *For any $n \in \mathbb{N}$ and $1 \le \ell \le \ell' \le \infty$,*

$$\begin{aligned} -\infty < b_n^\infty \le b_n^{\ell'} \le b_n^\ell \le b_n^1, \\ -\infty < c_n^\infty \le c_n^{\ell'} \le c_n^\ell \le c_n^1, \\ b_n^\ell \le c_{n+1}^\ell. \end{aligned}$$

(ii) *For any $n \in \mathbb{N}$,*

$$b_n^\ell \to b_n^\infty, \quad c_n^\ell \to c_n^\infty \quad \text{as } \ell \to \infty.$$

PROOF. (i) Since $0 = \gamma(0) \in \gamma(D_n)$ for $\gamma \in \Gamma_n$,

$$\max_{\xi \in D_n} J_\infty\big(\gamma(\xi)\big) \ge J_\infty(0) = \frac{1}{2}\|\dot{c}\|_2^2 - \int_0^{2\pi} V(t, 0)\, dt > -\infty.$$

Thus we have $b_n^\infty > -\infty$. Monotonicity on ℓ and $b_n^\ell \le c_n^\ell$ are clear from the definition.

(ii) For any $\gamma \in \Gamma_n$, $K = \gamma(D_n)$ is a compact subset of E and we observe that

$$\ell \mapsto \max_{u \in K} J_\ell(u);\ [1, \infty] \to \mathbb{R}$$

is continuous and monotone non-increasing. Thus $\ell \mapsto b_n^\ell = \inf_{\gamma \in \Gamma_n} \max_{\xi \in D_n} J_\ell(\gamma(\xi))$ is left-continuous on $[1, \infty]$. In particular, $b_n^\ell \to b_n^\infty$ as $\ell \to \infty$. In a similar way, we can show $c_n^\ell \to c_n^\infty$ as $\ell \to \infty$. ■

By the argument in [12], if $b_n^\ell < c_n^\ell$, we can show $J_\ell(u)$ has a critical value greater than b_n^ℓ. Moreover, if $b_n^\infty < c_n^\infty$, we can show $b_n^\ell < c_n^\ell$ for large ℓ. We will show the existence of a sequence $(n_k) \subset \mathbb{N}$ with $b_{n_k}^\infty < c_{n_k}^\infty$.

Next we study the behavior of (b_n^∞) as $n \to \infty$. We will show the following result in Section 6.

LEMMA 5.3. $\liminf_{n\to\infty} \frac{b_n^\infty}{n^2} > 0$.

The following proposition follows from the above lemma.

PROPOSITION 5.4. *There exists a sequence* $(n_k) \subset \mathbb{N}$ *such that*

$$0 < b_{n_k}^\infty + 1 \le c_{n_k}^\infty \quad \textit{for } k = 1, 2, \dots .$$

PROOF. Arguing indirectly, we assume that for some $n_0 \in \mathbb{N}$

$$c_n^\infty \le b_n^\infty + 1 \quad \text{for } n \ge n_0. \tag{5.4}$$

We note that $-u \in \overline{\Lambda}_0$ for $u \in \overline{\Lambda}_0$ since Ω is symmetric with respect to 0. We have for $u \in \overline{\Lambda}_0$

$$\begin{aligned} J_\infty(-u) - J_\infty(u) &= \frac{1}{2}\left(\left\| \frac{d}{dt}(-Ru + c) \right\|_2^2 - \left\| \frac{d}{dt}(Ru + c) \right\|_2^2 \right) \\ &\quad - \int_0^{2\pi} V(t, -u) - V(t, u)\, dt \\ &= -2\left(\frac{d}{dt}(Ru), \dot{c} \right)_2 - \int_0^{2\pi} V(t, -u) - V(t, u)\, dt \\ &= 2(Ru, \ddot{c})_2 - \int_0^{2\pi} V(t, -u) - V(t, u)\, dt. \end{aligned}$$

Thus there exists $C > 0$ such that

$$J_\infty(-u) \le J_\infty(u) + C \quad \text{for all } u \in \overline{\Lambda}_0. \tag{5.5}$$

For $\gamma(\xi) \in \Gamma_{n+1}^+$, we set $\widetilde{\gamma}(\xi) \in \Gamma_{n+1}$ by

$$\widetilde{\gamma}(\xi) = \begin{cases} \gamma(\xi) & \text{for } \xi \in D_{n+1}^+, \\ -\gamma(-\xi) & \text{for } \xi \in -D_{n+1}^+. \end{cases}$$

By (5.5), for any $\gamma \in \Gamma_{n+1}^+$,

$$\max_{\xi \in -D_{n+1}^+} J_\infty\big(\widetilde{\gamma}(\xi)\big) = \max_{\xi \in D_{n+1}^+} J_\infty\big(-\gamma(\xi)\big) \le \max_{\xi \in D_{n+1}^+} J_\infty\big(\gamma(\xi)\big) + C.$$

Thus

$$\begin{aligned} \max_{\xi \in D_{n+1}} J_\infty\big(\widetilde{\gamma}(\xi)\big) &\le \max\Big\{ \max_{\xi \in D_{n+1}^+} J_\infty\big(\widetilde{\gamma}(\xi)\big), \max_{\xi \in -D_{n+1}^+} J_\infty\big(\widetilde{\gamma}(\xi)\big) \Big\} \\ &\le \max_{\xi \in D_{n+1}^+} J_\infty\big(\gamma(\xi)\big) + C. \end{aligned}$$

Since $\gamma \in \Gamma_{n+1}^{+}$ is arbitrary, we have

$$b_{n+1}^{\infty} \leq c_n^{\infty} + C \quad \text{for all } n.$$

Thus under (5.4) we have

$$b_{n+1}^{\infty} \leq b_n^{\infty} + C + 1 \quad \text{for } n \geq n_0,$$

which implies

$$b_n^{\infty} \leq b_{n_0}^{\infty} + (C+1)(n-n_0) \quad \text{for } n \geq n_0.$$

This is a contradiction to Lemma 5.3. ∎

Now we can complete the proof of Theorem 1.1.

END OF THE PROOF OF THEOREM 1.1. By Proposition 5.4, there exists a sequence $(n_k) \subset \mathbb{N}$ such that $b_{n_k}^{\infty} + 1 \leq c_{n_k}^{\infty}$ for all $k \in \mathbb{N}$. We choose $\gamma_{0,n_k} \in \Gamma_{n_k}$ such that

$$\max_{\xi \in D_{n_k}} J_{\infty}\big(\gamma_{0,n_k}(\xi)\big) \leq b_{n_k}^{\infty} + \frac{1}{3}$$

and we set

$$\tilde{\Gamma}_{n_k+1}^{+} = \{\gamma \in C(D_{n_k+1}^{+}, E);\ \gamma|_{D_{n_k}} = \gamma_{0,n_k}\}$$

and for $\ell \in [1,\infty]$

$$\tilde{c}_{n_k}^{\ell} = \inf_{\gamma \in \tilde{\Gamma}_{n_k+1}^{+}} \max_{\xi \in D_{n_k}} J_{\ell}\big(\gamma(\xi)\big).$$

Then we have

$$b_{n_k}^{\infty} + 1 \leq c_{n_k}^{\infty} \leq \tilde{c}_{n_k}^{\infty},$$

$$b_{n_k}^{\ell} + \frac{2}{3} \leq \tilde{c}_{n_k}^{\ell} \quad \text{for sufficiently large } \ell \geq 1.$$

Since the Palais–Smale condition holds for $J_\ell(u)$ with $\ell \in [1,\infty)$ and for all $\gamma \in \tilde{\Gamma}_{n_k+1}^{+}$

$$\max_{\xi \in \partial D_{n_k}^{+}} J_{\ell}\big(\gamma(\xi)\big) = \max_{\xi \in \partial D_{n_k}^{+}} J_{\ell}\big(\gamma_{0,n_k}(\xi)\big) \to \max_{\xi \in \partial D_{n_k}^{+}} J_{\infty}\big(\gamma_{0,n_k}(\xi)\big) < b_{n_k}^{\infty} + \frac{1}{3},$$

there exists a critical point $u_{n_k}^{\ell} \in \Lambda_\delta$ such that

$$J_\ell(u_{n_k}^{\ell}) = \tilde{c}_{n_k}^{\ell}, \quad J_\ell'(u_{n_k}^{\ell}) = 0, \quad i\big(J_\ell''(u_{n_k}^{\ell})\big) \leq n_k \quad \text{for large } \ell \geq 1.$$

For the estimate of the Morse index, see [17].

Noting $\tilde{c}^{\ell}_{n_k} \to \tilde{c}^{\infty}_{n_k} \in [b^{\infty}_{n_k}, \infty)$, after extracting a subsequence if necessary, we may assume

$$u^{\ell}_{n_k} \rightharpoonup u^{\infty}_{n_k} \quad \text{weakly in } E.$$

We can deduce that $u^{\infty}_{n_k}$ is a bounce trajectory and

$$J_\infty(u^{\infty}_{n_k}) = \tilde{c}^{\infty}_{n_k} \geq b^{\infty}_{n_k} \to \infty,$$
$$\#\{\bar{t};\ \bar{t} \text{ is a non-regular instant of } u^{\infty}_{n_k}\} \leq n_k.$$

Thus there exist infinitely many bounce trajectories and the proof of Theorem 1.1 is completed. ∎

6. Proof of Lemma 5.3

6.1. *One-dimensional problem*

To prove Lemma 5.3, first we consider the following 1-dimensional problem:

$$\ddot{u} + W'_\delta(u) = 0 \quad \text{in } \mathbb{R}, \tag{6.1}$$

$$u(t + 2\pi) = u(t) \quad \text{in } \mathbb{R}. \tag{6.2}$$

Here $u(t)\colon S^1 \to \mathbb{R}$ is unknown and $W_\delta(t)\colon (-1-\delta, 1+\delta) \to \mathbb{R}$ is given by

$$W_\delta(t) = \begin{cases} \zeta_\delta(t-1) & \text{for } t \in [1, 1+\delta), \\ 0 & \text{for } t \in [0, 1), \\ W_\delta(-t) & \text{for } t \in (-1-\delta, 0). \end{cases}$$

This is a special case studied in previous sections and a related problem is studied well in Berestycki [6]. See also [17] for related results on Morse indices. Here we give related results to [6].

We set

$$F = H^1(S^1, \mathbb{R}^1),$$
$$\Theta_\delta = \{u \in F;\ u(t) \in (-1-\delta, 1+\delta) \text{ for all } t\},$$
$$H(u) = \frac{1}{2}\|\dot{u}\|_2^2 - \int_0^{2\pi} W_\delta(u)\,dt \in C^2(\Theta_\delta, \mathbb{R}).$$

Solutions of (6.1)–(6.2) are characterized as critical points of $H(u)$.

We have the following lemma.

Lemma 6.1. *Choosing $\delta \in (0, \delta_0/2]$ smaller if necessary, we have the following:*

(i) *For any critical point u of H, $H(u) \geq \frac{1}{4}\|\dot{u}\|_2^2$. In particular, all critical values of $H(u)$ are non-negative.*

(ii) *$H(u)$ satisfies the Palais–Smale condition.*

PROOF. (i) For $t \in (0,\delta)$, we have from (2.2) that

$$W_\delta(1+t) = \zeta_\delta(t) \le \frac{\delta}{16}\zeta'_\delta(t) \le \frac{\delta}{16}\zeta'_\delta(t)(1+t) \le \frac{\delta}{16}W'_\delta(1+t)(1+t).$$

Noting $W_\delta(t) = 0$ for $t \in [-1,1]$, we have

$$W_\delta(t) \le \frac{\delta}{16}W'_\delta(t)t \quad \text{for all } t \in (-1-\delta, 1+\delta).$$

Thus

$$\int_0^{2\pi} W_\delta(u)\,dt \le \frac{\delta}{16}\int_0^{2\pi} W'_\delta(u)u\,dt \quad \text{for all } u \in \Theta_\delta. \tag{6.3}$$

It follows from $H'(u)u = 0$ that

$$\|\dot{u}\|_2^2 = \int_0^{2\pi} W'_\delta(u)u\,dt. \tag{6.4}$$

Thus, by (6.3) and (6.4) we have for a critical point u

$$\begin{aligned} H(u) &= \frac{1}{2}\|\dot{u}\|_2^2 - \int_0^{2\pi} W_\delta(u)\,dt \\ &\ge \frac{1}{2}\|\dot{u}\|_2^2 - \frac{\delta}{16}\int_0^{2\pi} W'_\delta(u)u\,dt \\ &= \left(\frac{1}{2} - \frac{\delta}{16}\right)\|\dot{u}\|_2^2. \end{aligned}$$

Thus for $\delta \in (0,1)$ we have (i). (ii) is a special case of Proposition 3.2. ∎

It is easy to see that solutions of (6.1)–(6.2) are constant or $\frac{2\pi}{k}$-periodic with $k \in \mathbb{N}$. We have the following proposition.

PROPOSITION 6.2. *Constant and periodic solutions have the following properties:*

(i) *For constant solution $u_a(t) \equiv a \in [-1,1]$, we have*

$$H(u_a) = 0, \quad i_0\big(H''(u_a)\big) = 1.$$

(ii) *For $\frac{2\pi}{k}$-periodic solution $u_k(t)$ with minimal period $\frac{2\pi}{k}$ ($k \in \mathbb{N}$), we have*

$$H(u_k) \ge \frac{2k^2}{\pi}, \quad i_0\big(H''(u_k)\big) \le 2k+1.$$

PROOF. (i) Since $H''(0)(h,h) = \|\dot{h}\|_2^2$ for $h \in F$, we have $i_0(H''(u_a)) = 1$.

(ii) Let $u_k(t)$ be a $\frac{2\pi}{k}$-periodic solution of (6.1)–(6.2). We may assume $u_k(0) = 0$ and $u'_k(0) > 0$. We observe that $u_k(\frac{\pi}{k}i) = 0$ for $i = 1, 2, \ldots, 2k$ and $u_k(t)$ takes maximum at $t = \frac{\pi}{2k}$ and $u_k(\frac{\pi}{2k}) > 1$. Thus

$$1 < \int_0^{\frac{\pi}{2k}} \dot{u}_k \, dt \le \left(\frac{\pi}{2k}\right)^{1/2} \|\dot{u}_k\|_{L^2(0,\frac{\pi}{2k})},$$

which implies

$$\|\dot{u}_k\|_2^2 = \|\dot{u}_k\|_{L^2(0,2\pi)}^2 = 4k\|\dot{u}_k\|_{L^2(0,\frac{\pi}{2k})}^2 > \frac{8k^2}{\pi}.$$

By (i) of Lemma 6.1, we have $H(u_k) \ge \frac{2k^2}{\pi}$.

We denote i-th eigenvalue of $-\frac{d^2}{dt^2} - W_\delta'(u_k)$ in $(0, 2\pi)$ under Neumann (periodic respectively) boundary conditions by λ_i^N (λ_i^P respectively).

Differentiating (6.1), we observe that $v(t) = u_k'(t)$ satisfies

$$\begin{aligned} &\ddot{v} + W_\delta'(u_k)v = 0 \quad \text{in } \mathbb{R}, \\ &\dot{v}(0) = \dot{v}(2\pi) = 0 \quad \text{in } \mathbb{R}. \end{aligned} \tag{6.5}$$

Thus $v(t)$ is an eigenfunction of $-\frac{d^2}{dt^2} - W_\delta'(u_k)$ under Neumann boundary condition. Since $v(t)$ has $2k$ zeros in $(0, 2\pi)$, we have $\lambda_{2k+1}^N = 0$. Thus

$$0 = \lambda_{2k+1}^N < \lambda_{2k+2}^N \le \lambda_{2k+2}^P.$$

Thus we have $i_0(H''(u_k)) \le 2k + 1$. ■

6.2. *Estimate for b_n^∞*

Since $[-1-\delta, 1+\delta]^N \subset \Omega$ by (2.1), there exists $C > 0$ such that

$$V(t, x) \le \sum_{i=1}^N W_\delta(x_i) + C \quad \text{for all } x = (x_1, \ldots, x_N). \tag{6.6}$$

Here we regard $W_\delta(t) = \infty$ for $|t| \ge 1 + \delta$.

We define $\widehat{H}(u) \colon E \to [-\infty, \infty)$ by

$$\widehat{H}(u) = \begin{cases} \sum_{i=1}^N H(u_i) & \text{for } u = (u_1, \ldots, u_N) \text{ with } u_i \in \Theta_\delta \text{ for all } i, \\ -\infty & \text{otherwise.} \end{cases}$$

By (6.6), we have for a constant $C > 0$ independent of u

$$\begin{aligned} J_\infty(u) &= \frac{1}{2}\left\|\frac{d}{dt}(Ru + c)\right\|_2^2 - \int_0^{2\pi} V(t, u)\, dt \\ &= \frac{1}{2}\|(R\dot{u} + \dot{R}u + \dot{c})\|_2^2 - \int_0^{2\pi} V(t, u)\, dt \\ &\ge \frac{1}{4}\|\dot{u}\|_2^2 - C \ge \frac{1}{2}\widehat{H}(u) - C \quad \text{for all } u \in \overline{\Lambda}_0. \end{aligned} \tag{6.7}$$

We set for $n \in \mathbb{N}$

$$\beta_n = \inf_{\gamma \in \Gamma_n} \max_{\xi \in D_n} \widehat{H}\big(\gamma(\xi)\big),$$

where Γ_n is defined in (5.3).

Since $\widehat{H}(u)$ is an even functional, symmetric mountain pass values β_n are critical values of $\widehat{H}(u)$. More precisely, we have the following lemma.

LEMMA 6.3. *(β_n) has the following properties:*

(i) $b_n^\infty \geq \frac{1}{2}\beta_n - C$ *for all* n.

(ii) *For* $n \geq N+1$, *we have*

(1) $\beta_n > 0$;

(2) β_n *is a critical value of* $\widehat{H}(u)$;

(3) *there exists a critical point* $w_n \in E$ *such that*

$$0 < \widehat{H}(w_n) \leq \beta_n, \quad i_0\big(\widehat{H}''(w_n)\big) \geq n.$$

PROOF. (i) follows from (6.7).

(ii) (1) In a small neighborhood of 0 in E, we have

$$\widehat{H}(u) = \frac{1}{2}\|\dot{u}\|_2^2.$$

Since $u \mapsto \|\dot{u}\|_2^2$ is positive definite on $E_N^\perp$, we can deduce $\beta_n > 0$ for $n \geq N+1$.

(2) Since $\widehat{H}(u)$ satisfies the Palais–Smale condition and $\widehat{H}(u)$ is even, we can see that $\beta_n > 0$ is a critical value. (3) follows from [17]. ∎

Finally we give an estimate for $\widehat{H}(w_n)$, where w_n appeared in Lemma 6.3 (ii) (3).

LEMMA 6.4. *There exists $c > 0$ independent of n such that*

$$\beta_n \geq cn^2 \quad \text{for } n \geq N+1.$$

PROOF. Let $w_n = (\omega_1, \ldots, \omega_N)$ be a critical point of $\widehat{H}(u)$ obtained in Lemma 6.3 (ii) (3). Then ω_i's are the critical point of $H(u)$ and

$$\sum_{i=1}^{N} H(\omega_i) = \widehat{H}(w_n) \leq \beta_n, \tag{6.8}$$

$$\sum_{i=1}^{N} i_0\big(H''(\omega_i)\big) = i_0\big(\widehat{H}''(w_n)\big) \geq n. \tag{6.9}$$

By (6.9), there exists ω_i such that $i_0(H''(\omega_i)) \geq [\frac{n}{N}]$. Suppose that ω_i is $\frac{2\pi}{k}$-periodic. By Proposition 6.2, we have

$$2k+1 \geq i_0\big(H''(\omega_i)\big) \geq \left[\frac{n}{N}\right],$$

$$H(\omega_i) \geq \frac{2k^2}{\pi}.$$

Thus we have for some constant $c > 0$ independent of n,

$$\widehat{H}(w_n) \geq H(\omega_i) \geq cn^2 \quad \text{for } n \geq N+1.$$

Thus Lemma 6.4 is proved. ■

End of the proof of Lemma 5.3. Combining Lemma 6.3 (i) and Lemma 6.4, we have

$$b_n^\infty \geq \frac{1}{2}cn^2 - C' \quad \text{for } n \geq N+1.$$

Thus Lemma 5.3 is proved. ■

7. Benci–Giannoni's bounce orbit

Finally in this section, we give another approach to show the existence of a bounce trajectory in the setting of the paper [5].

Here we assume $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$ is an open-bounded domain with boundary $\partial\Omega$ of class C^2 (we do not assume symmetry of Ω). For simplicity we assume $V(t,x) \equiv 0$ and we consider the situation $R(t) \equiv I$, $c(t) \equiv 0$.

For small $\delta > 0$ we define Ω_δ, $U_\delta(x)$ as in previous sections and

$$J_\ell(u) = \begin{cases} \frac{1}{2}\|\dot{u}\|_2^2 - \ell \int_0^{2\pi} U_\delta(u)\,dt & \text{for } u \in \Lambda_\delta, \\ -\infty & \text{for } u \in E \setminus \Lambda_\delta, \end{cases}$$

$$J_\infty(u) = \begin{cases} \frac{1}{2}\|\dot{u}\|_2^2 & \text{for } u \in \overline{\Lambda}_0, \\ -\infty & \text{for } u \in E \setminus \overline{\Lambda}_0. \end{cases}$$

We set for $R > 0$

$$Q = \big\{y + re_{N+1};\ y \in E_N,\ \|y\|_E \leq R,\ 0 \leq r \leq R\big\}.$$

We observe that $\{y \in E_N;\ \|y\|_E \geq R\} \cap \Omega_\delta = \emptyset$ for large $R \geq 1$ and $J_1(u) \leq 0$ for all $u \in \partial Q$.

We introduce

$$
\begin{aligned}
&\widetilde{\Gamma} = \{\gamma \in C(Q, E);\ \gamma(u) = u \text{ for all } u \in \partial Q\}, \\
&b^\ell = \inf_{\gamma\in\widetilde{\Gamma}} \max_{u\in Q} J_\ell(\gamma(u)) \quad \text{for } \ell \in [1, \infty].
\end{aligned}
\tag{7.1}
$$

Since $J_\infty(u) = \frac{1}{2}\|\dot{u}\|_2^2$ in a neighborhood of 0, we observe that for small $\rho > 0$

$$
\inf_{u\in S_\rho} J_\infty(u) > 0,
$$

where

$$
S_\rho = \{u \in E_N^\perp;\ \|u\|_E = \rho\}.
$$

Since $\gamma(Q) \cap S_\rho \neq \emptyset$ for all $\gamma \in \widetilde{\Gamma}$, we have $b^\infty > 0$ and thus

$$
\begin{aligned}
&0 < b^\infty \le b^\ell \le b^1 \quad \text{for all } \ell \in [1, \infty), \\
&b^\ell \to b^\infty \quad \text{as } \ell \to \infty.
\end{aligned}
$$

Since $J_\ell(u)$ satisfies the Palais–Smale condition for $\ell \in [1, \infty)$, there exists a sequence (u_ℓ) of critical points of $J_\ell(u)$ such that

$$
J_\ell(u_\ell) = b^\ell \to b^\infty, \quad J_\ell'(u_\ell) = 0, \quad i\big(J_\ell''(u_\ell)\big) \le N + 1.
$$

After extracting a subsequence, we assume $u_\ell \to u_\infty$ weakly in E. From the argument in previous sections, we have the following theorem, which gives a minimax characterization of a bounce trajectory.

THEOREM 7.1. *For b^∞ defined in (7.1), there exists a bounce trajectory $u_\infty(t)$ such that*

$$
\begin{aligned}
&J_\infty(u_\infty) = b^\infty, \\
&u_\infty(t) \textit{ has at most } N + 1 \textit{ non-regular instants}.
\end{aligned}
$$

ACKNOWLEDGMENTS. – The initial research for this paper was started as part of the second author's master thesis, while he was a master course student of Department of Mathematics, Waseda University. The second author thanks Department of Mathematics, Waseda University, for their guidance and hospitality.

FUNDING. – The third author is supported in part by Grant-in-Aid for Scientific Research (18KK0073, 17H02855, 22K03380) of Japan Society for the Promotion of Science.

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Received 7 February 2023

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