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Differential Equations. – *Shape optimization for a nonlinear elliptic problem related to thermal insulation*, by Rosa Barbato, communicated on 19 April 2024.

Abstract. – In this paper, we consider a minimization problem of a nonlinear functional $I_{\beta,p}(D,\Omega)$ related to a thermal insulation problem with a convection term, where Ω is a bounded connected open set in $\mathbb{R}^n$ and $D \subset \bar{\Omega}$ is a compact set. The Euler–Lagrange equation relative to $I_{\beta,p}$ is a p-Laplace equation, $1 < p < \infty$, with a Robin boundary condition with parameter $\beta > 0$. The main aim is to study extremum problems for $I_{\beta,p}(D,\Omega)$, among domains D with given geometrical constraints and $\Omega \setminus D$ of fixed thickness. In the planar case, we show that under perimeter constraint the disk maximizes $I_{\beta,p}$. In the n-dimensional case we restrict our analysis to convex sets showing that the same is true for the ball but under different geometrical constraints.

Keywords. – Shape optimization, optimal insulation, mixed boundary conditions.

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1. Introduction

Insulation problems have interested many researchers and it is a very active field of research as it is related to environmental improvement. In fact, thermal insulation may be applied to save energy and even if it may seem counterintuitive, having too much insulation can have a negative impact on the energy efficiency; furthermore, adding too much insulation is expensive and unsustainable. On this topic, there are many papers, as for instance [1–6, 9–11].

In this paper we consider a problem of this type: let Ω be a bounded connected open set of $\mathbb{R}^n$, $D \subset \bar{\Omega}$ a compact set, $\beta > 0$ a fixed constant. Let

$$(1.1)\quad \begin{aligned} &I_{\beta,p}(D;\Omega) \\ &\quad = \inf\Big\{\int_\Omega |D\phi|^p\,dx + \beta\int_{\partial\Omega}|\phi|^p\,d\mathcal{H}^{n-1},\ \phi \in W^{1,p}(\Omega),\ \phi \geq 1 \text{ in } D\Big\}. \end{aligned}$$

Our aim is to study maximization and minimization problems of $I_{\beta,p}(D,\Omega)$, among domains with given geometrical constraints. In Proposition 3.1 we prove that if Ω has

Lipschitz boundary, then there exists a minimizer u of (1.1) that satisfies

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega \setminus \bar{D} \\ u = 1 & \text{in } D \\ |Du|^{p-2}\frac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right)$$

is the p-Laplace operator, and the functional $I_{\beta,p}(D,\Omega)$ assumes the following form:

$$I_{\beta,p}(D,\Omega) = \beta \int_{\partial\Omega} |u|^{p-2}u \, d\mathcal{H}^{n-1}.$$

In this order of ideas, the case $p = 2$ has been treated in [8]. In this case, the minimization problem arises from a thermal insulation problem; in fact, they consider a domain of given temperature, thermally insulated by a thermal insulator of constant thickness, and they ask for the best or worst choice in terms of heat dispersion.

In the present paper we consider the general case $1 < p < +\infty$.

The plan of the paper is the following: after recalling some well-known facts on convex domains in Section 2, we prove some basic properties of $I_{\beta,p}(D,\Omega)$ in Section 3.

In Sections 4–6 we discuss our main results. First, we consider the domains of the type $\Omega = D + \delta B$, with $\delta > 0$ and B the unit ball in $\mathbb{R}^n$. In particular, in the planar case studied in Section 4, in Theorem 4.1 we prove that if D is an open, bounded, connected set of $\mathbb{R}^n$ with piecewise C^1 boundary, the maximum of $I_{\beta,p}(D, D+\delta B)$ is achieved at the disk having the same perimeter of D. Moreover, in Section 5 we consider the n-dimensional case for convex sets and we obtain in Theorem 5.1 that among the convex sets $\Omega = D + \delta B$, with fixed δ and of given W_{n-1} quermassintegral of D (see Section 2 for the precise definition), the maximum is attained when D is a ball.

Finally in Section 6, we show a counterintuitive behavior of the functional $I_{\beta,p}(D,\Omega)$; indeed we prove in Proposition 6.3 that for suitable values of β, there exists a positive constant δ_0 such that for any bounded domain Ω, with $D \subset \Omega$ and $|\Omega| - |D| < \delta_0$, then

$$I_{\beta,p}(B_R,\Omega) > I_{\beta,p}(B_R,B_R).$$

2. Preliminaries

In this section, we list some basic facts on convex sets (see, for example, [7, 12]). Let K be a nonempty, bounded, convex set in $\mathbb{R}^n$ and let $\delta > 0$. Then the Steiner formula for the volume and the perimeter states that

$$
\begin{aligned}
|K+\delta B| &= \sum_{j=0}^{n}\binom{n}{j} W_j(K)\delta^j \\
&= |K| + nW_1(K)\delta + \frac{n(n-1)}{2}W_2(K)\delta^2 + \cdots + \omega_n\delta^n,
\end{aligned}
$$

$$
\begin{aligned}
P(K+\delta B) &= n\sum_{j=0}^{n-1}\binom{n-1}{j} W_{j+1}(K)\delta^j \\
&= P(K) + n(n-1)W_2(K)\delta + \cdots + n\omega_n\delta^{n-1},
\end{aligned}
\tag{2.1}
$$

where B is the unit ball in $\mathbb{R}^n$ centered at the origin, ω_n is its measure, and $K+\delta B$ stands for the Minkowski sum.

The coefficients $W_j(K)$ are the so-called quermassintegrals of K. In particular W_0 is the volume of K, $W_1 = \frac{P}{n}$ and $W_n = \omega_n$.

It follows that

$$
\lim_{\delta\to 0+} \frac{P(K+\delta B) - P(K)}{\delta} = n(n-1)W_2(K). \tag{2.2}
$$

If K has C^2 boundary, with nonzero Gaussian curvature, the quermassintegrals can be rewritten in terms of the principal curvatures of ∂K. Indeed, in such a case

$$
W_i(K) = \frac{1}{n}\int_{\partial K} H_{i-1}(x)\,d\mathcal{H}^{n-1}, \quad i = 1,\ldots,n. \tag{2.3}
$$

Meanwhile, H_j denotes the j-th normalized elementary symmetric function of the principal curvatures of ∂K; that is, $H_0 = 1$, and

$$
H_j(x) = \binom{n-1}{j}^{-1} \sum_{1\le i_1\le\cdots\le i_j\le n-1} k_{i_1}(x)\cdots k_{i_j}(x), \quad j = 1,\ldots,n-1,
$$

where $k_1(x)\cdots k_{n-1}(x)$ are the principal curvatures at a point $x\in\partial K$. In particular, by (2.2) and (2.3) follows also that

$$
\lim_{\delta\to 0+} \frac{P(K+\delta B) - P(K)}{\delta} = (n-1)\int_{\partial K} H_1(x)\,d\mathcal{H}^{n-1},
$$

where $H_1(x)$ is the mean curvature of ∂K at a point x.

The Alexandrov–Fenchel inequalities state that

$$
\left(\frac{W_j(K)}{\omega_n}\right)^{\frac{1}{n-j}} \ge \left(\frac{W_i(K)}{\omega_n}\right)^{\frac{1}{n-i}}, \quad 0\le i<j\le n-1, \tag{2.4}
$$

where the inequality becomes an equality if and only if K is a ball.

In what follows, we will use the Alexandrov–Fenchel inequalities for particular values of i and j. When $i = 0$ and $j = 1$, we have the classical isoperimetric inequality:

$$P(K) \geq n\omega_n^{\frac{1}{n}} |K|^{1-\frac{1}{n}}.$$

Moreover, if $i = k - 1$, and $j = k$, we have

$$W_k(K) \geq \omega_n^{\frac{1}{n-k+1}} W_{k-1}(K)^{\frac{n-k}{n-k+1}}.$$

Now we denote by K^* a ball such that $W_{n-1}(K) = W_{n-1}(K^*)$ and then by the Alexandrov–Fenchel inequalities (2.4), for $0 \leq i < n-1$ it holds that

$$\left(\frac{W_i(K^*)}{\omega_n}\right)^{\frac{1}{n-i}} = \frac{W_{n-1}(K^*)}{\omega_n} = \frac{W_{n-1}(K)}{\omega_n} \geq \left(\frac{W_i(K)}{\omega_n}\right)^{\frac{1}{n-i}};$$

hence,

$$W_i(K) \leq W_i(K^*), \quad 0 \leq i \leq n-1. \tag{2.5}$$

Now consider the two-dimensional case. Let D be an open, bounded, connected set in $\mathbb{R}^2$; we denote by D^* the disk having the same perimeter of D. If $\Omega = D + \delta B$ and $\Omega_* = D^* + \delta_* B$, where B is the disk centered at the origin, the Steiner formulae become

$$|\Omega| = |D| + P(D)\delta + \pi\delta^2, \qquad P(\Omega) = P(D) + 2\pi\delta,$$
$$|\Omega_*| = |D^*| + P(D^*)\delta_* + \pi\delta_*^2, \quad P(\Omega_*) = P(D^*) + 2\pi\delta_*.$$

Let us observe that, in our context, if we ask that the area of the insulating material $\Omega \setminus \overline{D}$ remains constant, then

$$|\Omega| - |D| = P(D)\delta + \pi\delta^2 = |\Omega_*| - |D^*| = P(D^*)\delta_* + \pi\delta_*^2.$$

Since $P(D) = P(D^*)$, then $\delta = \delta_*$ and, as byproduct, $P(\Omega) = P(\Omega_*)$. On the contrary, if $\delta = \delta_*$, then $|\Omega| - |D| = |\Omega_*| - |D^*|$.

If D is a general bounded domain, with piecewise C^1 boundary, it holds that

$$|\Omega| \leq |D| + P(D)\delta + \pi\delta^2, \quad P(\Omega) \leq P(D) + 2\pi\delta; \tag{2.6}$$

hence $\delta = \delta_*$ implies

$$|\Omega| - |D| \leq |\Omega_*| - |D^*|,$$

which means that if we fix the perimeter $P(D)$ and the thickness δ, the area of the insulating material increases.

3. The variational problem

Given a compact set D in $\mathbb{R}^n$ and a bounded connected open set Ω of $\mathbb{R}^n$ with $D \subset \bar{\Omega}$, we are interested in studying

$$\begin{aligned}(3.1)\quad & I_{\beta,p}(D;\Omega)\\ &= \inf\left\{\int_\Omega |D\phi|^p\,dx + \beta\int_{\partial^*\Omega}|\phi|^p\,d\mathcal{H}^{n-1},\ \phi\in W^{1,p}(\Omega),\ \phi\geq 1 \text{ in } \bar{D}\right\}\end{aligned}$$

with $\beta > 0$. The following result holds.

Proposition 3.1. *If Ω is a bounded, connected, open set of $\mathbb{R}^n$ with Lipschitz boundary, and D is a compact set with $D \subset \Omega$, then there exists a unique positive minimizer $u \in W^{1,p}(\Omega)$ of (3.1) which satisfies*

$$(3.2)\qquad \begin{cases}\Delta_p u = 0 & \text{in } \Omega\setminus\bar{D},\\ u = 1 & \text{in } D,\\ |Du|^{p-2}\frac{\partial u}{\partial \nu} + \beta|u|^{p-2}u = 0 & \text{on } \partial\Omega.\end{cases}$$

Moreover the functional can be written as

$$(3.3)\qquad I_{\beta,p}(D,\Omega) = \beta\int_{\partial\Omega} u^{p-1}\,d\mathcal{H}^{n-1},$$

where u is the solution to (3.2).

Proof. The proof follows by a standard application of a calculus of variation's argument. We sketch the proof for completeness.

Let u_n be a minimizing sequence for $I_{\beta,p}(\Omega,D)$. Then u_n is bounded in $W^{1,p}$, $\exists u_{n_k}\to u$ weakly in $W^{1,p}(\Omega)$, strongly in L^p and a.e in Ω. Then by semicontinuity, u is a minimizer of $I_{\beta,p}(\Omega,D)$. Moreover, $|u|$ still being a minimizer, we can assume $u\geq 0$. By Harnack inequality, $u>0$ in Ω. Furthermore, by convexity, u is the unique minimizer and it satisfies

$$\begin{cases}\Delta_p u = 0 & \text{in } \Omega,\\ u = 1 & \text{in } \bar{D},\\ |Du|^{p-2}\frac{\partial u}{\partial \nu} + \beta u^{p-1} = 0 & \text{on } \partial\Omega.\end{cases}$$

Let $W^{1,p}_{\bar{D}}(\Omega)$ be the closure in $W^{1,p}(\Omega)$ of $\{\phi|_\Omega : \phi\in C^\infty(\mathbb{R}^n) \text{ with } \bar{D}\cap\operatorname{supp}\phi=\emptyset\}$. Hence, the minimizer u is such that $u-1\in W^{1,p}_{\bar{D}}(\Omega)$ and it is a solution to (3.2). In particular,

$$\int_{\Omega\setminus\bar{D}}|Du|^{p-2}DuD\phi\,dx+\beta\int_{\partial\Omega}u^{p-1}\phi\,d\mathcal{H}^{n-1}=0$$

for any $\phi\in W^{1,p}_{\bar{D}}(\Omega)$.

So, if we take $\phi = u - 1 \in W^{1,p}_{D}(\Omega)$,

$$\int_\Omega |Du|^{p-2} Du(Du)\,dx + \beta \int_{\partial\Omega} u^{p-1} u(u-1)\,d\mathcal{H}^{n-1} = 0;$$

that is,

$$\int_\Omega |Du|^p\,dx + \beta \int_{\partial\Omega} (u^p - u^{p-1})\,d\mathcal{H}^{n-1} = 0,$$

and this implies (3.3). ∎

4. The planar case

In this section we consider the case where Ω is the Minkowski sum

$$\Omega = D + \delta B,$$

B is the unit ball centered at the origin, and δ is a positive constant, representing the thickness of $\Omega \setminus D$. Then we set

$$I_{\beta,p,\delta}(D) = I_{\beta,p}(D, D + \delta B).$$

Theorem 4.1. *Let D be an open, bounded, connected set of $\mathbb{R}^2$ with piecewise C^1 boundary. Then*

$$I_{\beta,p,\delta}(D) \le I_{\beta,p,\delta}(D^*),$$

where D^* is the disk having the same perimeter of D.

Proof. Let v be the radial minimizer of $I_{\beta,p,\delta}(D^*)$ and $\Omega_* = D^* + \delta B$. Given $R > 0$ the radius of D^*, we denote

$$v_m = v(R + \delta) = \min_{\Omega_*} v$$

and

$$\max_{\Omega_*} v = v(R) = 1.$$

As v is radial, the modulus of the gradient of v is constant on the level lines of v.

So we can consider the function

$$g(t) = |Dv|_{v=t}, \quad v_m < t \le 1$$

and

$$w(x) = G\big(R + d(x)\big), \quad x \in \Omega, \text{ where } G^{-1}(t) = R + \int_t^1 \frac{1}{g(s)}\,ds$$

and $d(x)$ is the distance of a point x from D. In particular, v is decreasing and the function g can be zero only for $v = 1$. This means that G^{-1} is decreasing; therefore so is G and then $w \in W^{1,p}(\Omega)$. So we get

$$\max_{\Omega} w = w|_{\partial D} = 1 = G(R),$$
$$w_m = \min_{\Omega} w = w|_{\partial\Omega} = G(R+\delta) = v_m,$$
$$|Dw|_{w=t} = |Dv|_{v=t} = g(t), \quad w_m \le t \le 1.$$

Hence, w is a test function. Then

$$I_{\beta,p,\delta}(D) \le \int_{\Omega\setminus D} |Dw|^p \, dx + \beta \int_{\partial\Omega} |w|^p \, d\mathcal{H}^1.$$

Let

$$E_t = \{x \in \Omega : w(x) > t\} = \{x \in \Omega : d(x) < G^{-1}(t)\} = D + G^{-1}(t)B$$

and let

$$B_t = \{x \in \Omega_* : v(x) > t\}.$$

By Steiner formula (2.6) we get

$$P(\Omega) \le P(D) + 2\pi\delta,$$

so

$$P(E_t) \le P(D) + 2\pi G^{-1}(t) = P(D^*) + 2\pi G^{-1}(t) = 2\pi\big(R + G^{-1}(t)\big) = P(B_t)$$

for every $t \in]w_m, 1]$. Hence,

$$\int_{w=t} |Dw| d\mathcal{H}^1 = \int_{w=t} g(t) d\mathcal{H}^1 = g(t)P(E_t) \le g(t)P(B_t)$$
$$= \int_{v=t} |Dv| d\mathcal{H}^1, \quad w_m < t \le 1.$$

Then, using the co-area formula,

$$\int_{\Omega\setminus\bar{D}} |Dw|^p dx = \int_{w_m}^{1} dt \int_{w=t} |Dw|^{p-1} d\mathcal{H}^1 \tag{4.1}$$
$$= \int_{w_m}^{1} \big[g(t)\big]^{p-1} P(E_t) dt \le \int_{w_m}^{1} \big[g(t)\big]^{p-1} P(B_t) dt$$
$$= \int_{\Omega\setminus\bar{D}_*} |Dv|^p dx.$$

Since by construction $w = w_m = v_m$ on $\partial\Omega$ and $P(\Omega) = P(\Omega_*)$, we have

$$\int_{\partial\Omega} |w|^p d\mathcal{H}^1 = |w_m|^p P(\Omega) = |v_m|^p P(\Omega_*) = \int_{\partial\Omega_*} |v|^p d\mathcal{H}^1. \tag{4.2}$$

Hence, by (4.1) and (4.2) it holds that

$$\begin{aligned} I_{\beta,p,\delta}(D) &\le \int_{\Omega\setminus D} |Dw|^p dx + \beta \int_{\partial\Omega} |w|^p d\mathcal{H}^1 \\ &\le \int_{\Omega_*\setminus D^*} |Dv|^p dx + \beta \int_{\partial\Omega_*} |v|^p d\mathcal{H}^1 = I_{\beta,p,\delta}(D^*). \end{aligned}$$ ■

5. The n-dimensional case

Now we prove that in higher dimension ($n \ge 3$) balls still maximize $I_{\beta,p,\delta}$, but our result finds its natural generalization in the class of convex domains D.

Theorem 5.1. *Let D be an open, bounded, convex set of $\mathbb{R}^n$. Then*

$$I_{\beta,p,\delta}(D) \le I_{\beta,p,\delta}(D^*),$$

where D^* is the ball having the same W_{n-1} quermassintegral of D; that is, $W_{n-1}(D) = W_{n-1}(D^*)$.

Proof. Let v be the radial minimizer of $I_{\beta,p,\delta}(D^*)$ and let $\Omega_* = D^* + \delta B$. Since $\Omega = D + \delta B$, the Steiner formula for the perimeter (2.1) and Alexandrov–Fenchel inequalities (2.5) imply $P(\Omega) \le P(\Omega^*)$.

Let us denote $v_m = v(R+\delta) = \min_{\Omega^*} v$ and $\max_{\Omega_*} v = v(R) = 1$. As v is radial, the modulus of the gradient of v is constant on the level lines of v.

Now we consider the function

$$g(t) = |Dv|_{v=t}, \quad v_m < t \le 1$$

and

$$w(x) = G\big(R + d(x)\big), \quad x \in \Omega, \text{ where } G^{-1}(t) = R + \int_t^1 \frac{1}{g(s)} ds,$$

and $d(x)$ is the distance of a point x from D.

By construction $w \in W^{1,p}(\Omega)$ and with G decreasing one has

$$\begin{aligned} \max_{\Omega} w &= w|_{\partial D} = 1 = G(R), \\ w_m &= \min_{\Omega} w = w|_{\partial\Omega} = G(R+\delta) = v_m, \\ |Dw|_{w=t} &= |Dv|_{v=t} = g(t), \quad w_m \le t \le 1. \end{aligned}$$

Then

$$I_{\beta,p,\delta}(D) \leq \int_{\Omega\setminus D} |Dw|^p \, dx + \beta \int_{\partial\Omega} |w|^p \, d\mathcal{H}^{n-1}.$$

Let

$$E_t = \{x \in \Omega : w(x) > t\} = \{x \in \Omega : d(x) < G^{-1}(t)\} = D + G^{-1}(t)B$$

and

$$B_t = \{x \in \Omega_* : v(x) > t\}.$$

With $W_{n-1}(D) = W_{n-1}(D^*)$, using the Steiner formula and (2.5), we get for $t \in]w_m, 1]$ and $\rho = G^{-1}(t)$ that

$$\begin{aligned} P(E_t) = P(D + \rho B) &= n \sum_{n=0}^{n-1} \binom{n-1}{k} W_{k+1}(D)\rho^k \\ &\leq \sum_{n=0}^{n-1} \binom{n-1}{k} W_{k+1}(D^*)\rho^k = P(D^* + \rho B) = P(B_t). \end{aligned}$$

Hence,

$$\begin{aligned} \int_{w=t} |Dw| d\mathcal{H}^{n-1} &= g(t)P(E_t) \leq g(t)P(B_t) \\ &= \int_{v=t} |Dv| d\mathcal{H}^{n-1}, \quad w_m < t \leq 1; \end{aligned}$$

then, using co-area formula,

$$\begin{aligned} \int_{\Omega\setminus D} |Dw|^p dx &= \int_{w_m}^1 dt \int_{w=t} |Dw|^{p-1} d\mathcal{H}^{n-1} \\ &= \int_{w_m}^1 [g(t)]^{p-1} P(E_t) dt \leq \int_{w_m}^1 [g(t)]^{p-1} P(B_t) dt \\ &= \int_{\Omega\setminus D_*} |Dv|^p dx. \end{aligned}$$

Since by construction $w = w_m = v_m$ on $\partial\Omega$ and $P(\Omega) \leq P(\Omega_*)$, we have

$$\int_{\partial\Omega} |w|^p d\mathcal{H}^{n-1} = |w_m|^p P(\Omega) = |v_m|^p P(\Omega_*) = \int_{\partial\Omega_*} |v|^p d\mathcal{H}^{n-1}.$$

So,

$$\begin{aligned} I_{\beta,p,\delta}(D) &\leq \int_{\Omega\setminus D} |Dw|^p dx + \beta \int_{\partial\Omega} |w|^p d\mathcal{H}^{n-1} \\ &\leq \int_{\Omega_*\setminus D^*} |Dv|^p dx + \beta \int_{\partial\Omega_*} |v|^p d\mathcal{H}^{n-1} = I_{\beta,p,\delta}(D^*). \end{aligned}$$

■

6. Remarks

There is a counterintuitive behavior of the functional $I_{\beta,p,\delta}(D,\Omega)$ when Ω and D are concentric balls, which can be seen in the next two propositions.

Proposition 6.1. *Let B_R be the ball of radius $R>0$ centered at the origin. If $\beta \geq [\frac{n-1}{R(p-1)}]^{p-1}$, then $I_{\beta,p,\delta}(B_R)$ is decreasing in δ. When $\beta < [\frac{n-1}{R(p-1)}]^{p-1}$, then $I_{\beta,p,\delta}(B_R)$ is increasing for $\delta < (\frac{n-1}{p-1})\frac{1}{\beta^{\frac{1}{p-1}}} - R$ and decreasing for $\delta > (\frac{n-1}{p-1})\frac{1}{\beta^{\frac{1}{p-1}}} - R$.*

Proof. If $D = B_R$ is the ball of radius $R>0$ centered at the origin, then obviously $\Omega = B_R + \delta B = B_{R+\delta}$ and the minimum of (3.1) is the radial function

$$u(r) = \begin{cases} 1 - \gamma_1^{\frac{1}{p-1}}(\frac{p-1}{p-n})R^{\frac{p-n}{p-1}}\left[(\frac{r}{R})^{\frac{p-n}{p-1}} - 1\right] & p \neq n \\ 1 - \gamma_1^{\frac{1}{n-1}}\log\frac{r}{R} & p = n \end{cases} \tag{6.1}$$

for a suitable constant γ_1. This follows from the fact that

$$r^{n-1}\Delta_p u = \frac{d}{dr}\big(r^{n-1}|u'(r)|^{p-2}u'(r)\big) = 0,$$

but u is decreasing and positive, so

$$u'(r) = -\frac{\gamma_1^{\frac{1}{p-1}}}{r^{\frac{n-1}{p-1}}}, \qquad r \in [R, R+\delta].$$

If $p \neq n$, then integrating by parts and keeping in mind that $u(R) = 1$ it holds that

$$u(r) = -\gamma_1^{\frac{1}{p-1}}\left(\frac{p-1}{p-n}\right)\left[r^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}\right] + 1.$$

If $p = n$,

$$u(r) = -\gamma_1^{\frac{1}{n-1}}\log\frac{r}{R} + 1.$$

Now, we can find γ_1 using the boundary conditions.

If $p \neq n$, then using Robin condition on $\partial\Omega$ it holds that

$$-\big(-u'(R+\delta)\big)^{p-1} + \beta\big(u(R+\delta)\big)^{p-1} = 0.$$

By using the explicit expression of u,

$$\gamma_1^{\frac{1}{p-1}}(R+\delta)^{-\frac{n-1}{p-1}} = \beta^{\frac{1}{p-1}}\left(-\gamma_1^{\frac{1}{p-1}}\left(\frac{p-1}{p-n}\right)R^{\frac{p-n}{p-1}}\left(1 - \left(\frac{R+\delta}{R}\right)^{\frac{p-n}{p-1}}\right) + 1\right)$$

and we have that

$$\gamma_1 = \frac{\beta}{\left[(R+\delta)^{-\frac{n-1}{p-1}} + \left(\frac{p-1}{p-n}\right)\beta^{\frac{1}{p-1}}R^{\frac{p-n}{p-1}}\left(\left(\frac{R+\delta}{R} - 1\right)^{\frac{p-n}{p-1}}\right)\right]^{p-1}}.$$

In particular,

$$\int_{\partial(\Omega\setminus D)} \big(|Du|^{p-2}Du\cdot\nu\big)d\mathcal{H}^{n-1} = 0,$$

and using the divergence theorem,

$$-\int_{\Omega\setminus D}|Du|^p + \int_{\partial(\Omega\setminus D)} u\big(|Du|^{p-2}Du\cdot\nu\big) = 0,$$

$$-\int_{\Omega\setminus D}|Du|^p - \int_{\partial\Omega}\beta u^p + \int_{\partial D} u\big(|Du|^{p-2}Du\cdot\nu\big) = 0.$$

This implies that

$$I_{\beta,p,\delta}(B_R) = \int_{\partial B_R}|Du|^{p-2}\frac{\partial u}{\partial\nu}\,d\mathcal{H}^{n-1}.$$

Let us observe that

$$I_{\beta,p,\delta}(B_R) = n\omega_n\gamma_1$$

and we have

$$\partial_\delta\big[I_{\beta,p,\delta}(B_R)\big] < 0$$

if

$$\partial_\delta\left[(R+\delta)^{\frac{1-n}{p-1}} + \beta^{\frac{1}{p-1}}\left(\frac{p-1}{p-n}\right)R^{\frac{p-n}{p-1}}\left(\left(\frac{R+\delta}{R}\right)^{\frac{p-n}{p-1}}-1\right)\right] > 0$$

and this is true if

$$\delta > \left(\frac{n-1}{p-1}\right)\frac{1}{\beta^{\frac{1}{p-1}}} - R.$$

On the other hand, when $p = n$ we have

$$\gamma_1 = \frac{\beta}{\left[\frac{1}{(R+\delta)} + \beta^{\frac{1}{n-1}}\log\frac{R+\delta}{R}\right]^{n-1}}$$

and

$$I_{\beta,n,\delta}(B_R) = \frac{n\omega_n\beta}{\left[\frac{1}{(R+\delta)} + \beta^{\frac{1}{n-1}}\log\frac{R+\delta}{R}\right]^{n-1}}.$$

So

$$\partial_\delta\big[I_{\beta,n,\delta}(B_R)\big] < 0$$

if

$$\partial_\delta\left[\frac{1}{(R+\delta)} + \beta^{\frac{1}{n-1}}\log\left(1+\frac{\delta}{R}\right)\right] > 0$$

and this is true if and only if

$$\delta > \frac{1}{\beta^{\frac{1}{n-1}}} - R,$$

and the proposition is proved. ■

To show the next result, we first need the following lemma, using the following notation.

Let $D = B_R \subset \Omega$, and denote

$$\begin{aligned} P &= P(B_R) = n\omega_n R^{n-1}, \qquad V = |B_R| = \omega_n R^n, \\ \Delta P &= P(\Omega) - P(B_R), \qquad \Delta V = |\Omega| - |B_R|. \end{aligned}$$

LEMMA 6.2. *Let $D = B_R$ be the ball of radius $R > 0$ centered at the origin such that $D \subset \Omega$. For any $\delta_0 > 0$, there exists a constant*

$$C = \frac{n\omega_n R^{n-1}}{\delta_0}\left[\left(1 + \frac{\delta_0}{\omega_n R^n}\right)^{1-\frac{1}{n}} - 1\right]$$

such that if $\Delta V \le \delta_0$, it holds that

$$\Delta P \ge C \Delta V.$$

We refer to [8] for the proof.

Now, we want to prove that, in the regime β "small", if the thickness δ is below a certain threshold value, $I_{\beta,p}(B_R, \Omega)$ is greater than $I_{\beta,p}(B_R, B_R)$.

PROPOSITION 6.3. *Let $D = B_R$ be the ball of radius $R > 0$ centered at the origin and $\beta < [\frac{n-1}{R(p-1)}]^{p-1}$. Then there exists a positive constant δ_0 such that for any bounded domain Ω, with $D \subset \Omega$ and $|\Omega| - |D| < \delta_0$, then*

$$I_{\beta,p}(B_R, \Omega) > I_{\beta,p}(B_R, B_R).$$

PROOF. Let u be the minimizer of $I_{\beta,p}(B_R, \Omega)$. Consider

$$\Sigma = \Omega \setminus B_R, \quad \Gamma_m = \partial\Omega \setminus \partial B_R, \quad \Gamma_t = \partial\{u > t\} \setminus \partial B_R, \quad \Gamma_1 = \partial B_R \cap \Omega$$

and

$$p(t) = P\big(\{u > t\} \cap \Sigma\big), \quad \text{for a.e. } t > 0.$$

Our aim is to prove that

$$I_{\beta,p}(B_R; B_R) = \beta P(B_R) < I_{\beta,p}(B_R; \Omega) = \int_\Omega |Du|^p dx + \beta \int_{\partial\Omega} |u|^p d\mathcal{H}^{n-1}$$

or equivalently

$$\mathcal{H}^{n-1}(\Gamma_1) < \frac{1}{\beta}\int_\Omega |Du|^p dx + \int_{\Gamma_0} |u|^p d\mathcal{H}^{n-1}.$$

Indeed by the co-area formula and Fubini theorem, we have

$$\begin{aligned}(6.2)\qquad & \int_0^1 t^{p-1}p(t)dt \\ &= \int_0^1 t^{p-1}P\big(\{u>t\}\cap\Sigma\big)dt = \int_0^1\Big(\int_{\Gamma_1} t^{p-1}d\mathcal{H}^{n-1}\Big)dt \\ &\quad + \int_0^1\Big(\int_{\Gamma_t\cap\Omega} t^{p-1}d\mathcal{H}^{n-1}\Big)dt + \int_0^1\Big(\int_{\Gamma_t\cap\partial\Omega} t^{p-1}d\mathcal{H}^{n-1}\Big)dt \\ &= \frac{\mathcal{H}^{n-1}(\Gamma_1)}{p} + \frac{1}{p}\int_{\Gamma_0} u^p d\mathcal{H}^{n-1} + \int_\Omega u^{p-1}|Du|dx.\end{aligned}$$

From Lemma 6.2 we know that for $|\Sigma| < \delta_0$, with δ_0 fixed,

$$p(t) - 2\mathcal{H}^{n-1}(\Gamma_1) \geq C\mu(t)$$

and then

$$\begin{aligned}\int_0^1 t^{p-1}p(t)dt &\geq 2\int_0^1 t^{p-1}\mathcal{H}^{n-1}(\Gamma_1)dt + C\int_0^1 t^{p-1}\mu(t)dt \\ &= \frac{2}{p}\mathcal{H}^{n-1}(\Gamma_1) + \frac{C}{p}\int_\Omega u^p dx,\end{aligned}$$

where C is the constant of Lemma 6.2. Combining this with (6.2), we have

$$\frac{2}{p}\mathcal{H}^{n-1}(\Gamma_1) + \frac{C}{p}\int_\Omega u^p dx \leq \frac{\mathcal{H}^{n-1}(\Gamma_1)}{p} + \frac{1}{p}\int_{\Gamma_0} u^p d\mathcal{H}^{n-1} + \int_\Omega u^{p-1}|Du|dx.$$

On the other hand, by the Young inequality,

$$\int_\Omega u^{p-1}|Du|dx \leq \frac{p-1}{p\varepsilon^{\frac{1}{p-1}}}\int_\Omega u^p dx + \frac{\varepsilon}{p}\int_\Omega |Du|^p dx,$$

we get the following estimate:

$$\begin{aligned}\mathcal{H}^{n-1}(\Gamma_1) + C\int_\Omega u^p dx &\leq \int_{\Gamma_0} u^p d\mathcal{H}^{n-1} + \frac{p-1}{\varepsilon^{\frac{1}{p-1}}}\int_\Omega u^p dx + \varepsilon\int_\Omega |Du|^p dx \\ &= \int_{\Gamma_0} u^p d\mathcal{H}^{n-1} + \frac{p}{\varepsilon^{\frac{1}{p-1}}}\int_\Omega u^p dx - \frac{1}{\varepsilon^{\frac{1}{p-1}}}\int_\Omega u^p dx \\ &\quad + \varepsilon\int_\Omega |Du|^p dx,\end{aligned}$$

and choosing $\varepsilon = \frac{1}{\beta}$ it holds that

$$\mathcal{H}^{n-1}(\Gamma_1) + \big[C - \beta^{\frac{1}{p-1}}(p-1)\big] \int_\Omega u^p \, dx \le \int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1} + \frac{1}{\beta} \int_\Omega |Du|^p \, dx.$$

Therefore, as $R < \frac{n-1}{(p-1)\beta^{\frac{1}{p-1}}}$, for δ_0 sufficiently small, the constant C is larger than $\beta^{\frac{1}{p-1}}(p-1)$ and the thesis follows. ■

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