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Galois lines for a canonical curve of genus 4, I: Non-skew cyclic lines

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Abstract – Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4 over an algebraically closed field k of characteristic 0. For a line $l \subset \mathbb{P}^3$, we consider the projection $\pi_l: C \to \mathbb{P}^1$ with center l and the extension of the function fields $\pi_l^*: k(\mathbb{P}^1) \hookrightarrow k(C)$. A line l is assumed to be *cyclic* for C, if the extension $k(C)/\pi_l^*(k(\mathbb{P}^1))$ is cyclic. A line l is assumed to be *non-skew*, if $C \cap l \neq \emptyset$, i.e., $\deg \pi_l < \deg C = 6$. We investigate the number of non-skew cyclic lines for C. As main results, we explicitly give the equation of C in the particular case in which C has two cyclic trigonal morphisms; we prove that the number of cyclic lines with $\deg \pi_l = 4$ is at most 1, and the number of cyclic lines with $\deg \pi_l = 5$ is at most 1.

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1. Introduction

Let $C \subset \mathbb{P}^3$ be a nonsingular nondegenerate projective curve over an algebraically closed field k. For a line $l \subset \mathbb{P}^3$, we consider the projection $\pi_l: C \to \mathbb{P}^1$ with center l, and the extension of the function fields $\pi_l^*: k(\mathbb{P}^1) \hookrightarrow k(C)$. We denote $k(C)$ by K and $\pi_l^*(k(\mathbb{P}^1))$ by K_l.

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Definition 1.1 ([11]). We refer to l as a *Galois line* if the extension K/K_l is Galois. We denote $\{\sigma \in \mathrm{Aut}(C) \mid \pi_l \circ \sigma = \pi_l\}$ by G_l, which is isomorphic to $\mathrm{Gal}(K/K_l)$, and name it the *Galois group for a Galois line* l. If G_l is isomorphic to a cyclic group C_m of order m (resp. a dihedral group D_m of order $2m$, a symmetric group S_m on m letters,...), then the Galois line l is referred to as a C_m*-line* or a *cyclic line* (resp. D_m*-line*, S_m*-line*,...).

When l is not Galois, let L_l be the Galois closure of K/K_l. Yoshihara [11] proved that if l is general in the Grassmannian $\mathbb{G}(1,\mathbb{P}^3)$, then the Galois group $\mathrm{Gal}(L_l/K_l)$ is the full symmetric group. More generally, in the case that a curve $C \subset \mathbb{P}^r$ is irreducible and nondegenerate, Pirola and Schlesinger [10] proved the following: the locus of projective linear subspaces M of dimension $r-2$ such that the Galois groups given by the projections $\pi_M : C \dashrightarrow \mathbb{P}^1$ with center M are not isomorphic to the full symmetric group, has codimension at least 2 in $\mathbb{G}(r-2,\mathbb{P}^r)$.

A line l is said to be *skew* if $C \cap l = \emptyset$, i.e., $\deg \pi_l = \deg C$. In [11], Yoshihara investigated various properties of skew Galois lines. In particular, he proved that the number of skew Galois lines for an irrational C is finite, and that the number of skew Galois lines for C is at most one if $\deg C$ is a prime greater than or equal to 5. He also studied the skew Galois lines for curves of low degree. In [2,6,12], Yoshihara, Duyaguit, and Kanazawa studied the number and arrangement of skew Galois lines for an elliptic space curve, which is a $(2,2)$-complete intersection in $\mathbb{P}^3$. In addition, Fukasawa and Higashine [4], and Fukasawa [3] determined the arrangement of all Galois lines for the Giulietti–Korchmáros curve and the Artin–Schreier–Mumford curve.

Our main theorem is as follows.

Theorem 1.2 (Corollary 3.3, Theorems 3.6, 3.9). *Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4 over an algebraically closed field k of characteristic zero.*

(I) *If C has two trigonal morphisms and both of them are cyclic, then C is projectively equivalent to the curve defined by*

$$\begin{cases} XW = YZ, \\ Z(W-Z)(W+Z) = Y^3 + cXY^2 - 9X^2Y - cX^3 \end{cases} \qquad (c \in k),$$

where $(X : Y : Z : W)$ are homogeneous coordinates on $\mathbb{P}^3$.

(II) *The number of C_4-lines equals 0 or 1.*

(III) *The number of C_5-lines equals 0 or 1.*

We note that in (I) of Theorem 1.2, there are only one or two trigonal morphisms of C, and there are infinitely many lines such that the projections with these center lines are same as each trigonal morphism (Propositions 2.3, 2.4 and 2.5).

In this paper, we assume that k is algebraically closed and $\mathrm{char}(k) = 0$, and we study the number of non-skew cyclic lines for a canonical curve $C \subset \mathbb{P}^3$ of genus 4. We note that C is a $(2,3)$-complete intersection, and that a cyclic line l for C is non-skew if and only if l is a C_3-, C_4- or C_5-line, in this setting. We provide a similar study on skew cyclic lines in a subsequent work [9]. We give preliminary results in Section 2 and the detailed results of our main theorem in Section 3. We prove these results in Section 4. We present concrete examples of non-skew cyclic lines in Section 5.

2. Preliminaries

For a point $P \in C$, we denote by $H(P)$ the Weierstrass semigroup of P, that is,

$$H(P) = \{n \in \mathbb{Z}_{>0} \mid \text{ there exists } f \in k(C) \text{ such that } (f)_\infty = nP\}, \tag{1}$$

where $(f)_\infty$ is the divisor of the poles of f. We denote by $\langle n_1, n_2, n_3, \ldots \rangle$ the numerical semigroup generated by non-negative integers $n_1, n_2, n_3, \ldots$

Definition 2.1 ([7]). A Weierstrass point $P \in C$ is referred to as a *weak Galois–Weierstrass point*, if there exists a Galois covering $f: C \to \mathbb{P}^1$ such that P is a total ramification point of f. We denote by $\mathrm{degGW}(P)$ the set

$$\{n \in \mathbb{Z}_{>0} \mid \text{there exists a Galois covering } f: C \to \mathbb{P}^1 \text{ such that } P \text{ is a total ramification point of } f \text{ and } n = \deg f\}.$$

We note that a Weierstrass point P is a weak Galois–Weierstrass point if and only if $\mathrm{degGW}(P) \neq \emptyset$. Theorem 2.2 is required in the proof of Theorem 3.9.

Theorem 2.2 ([8]). *Let C be a nonsingular projective curve. Let a and b be coprime integers such that $3 < a + 1 < b$. Then, the number of weak Galois–Weierstrass points $P \in C$ with $H(P) = \langle a, b \rangle$ and $b \in \mathrm{degGW}(P)$ is equal to 0 or 1.*

Hereafter, we assume that $C \subset \mathbb{P}^3$ is a canonical curve of genus 4. The following are well-known facts, which play fundamental roles in this paper.

Proposition 2.3 ([1, page 118], [5, page 298]). *The curve C is a $(2,3)$-complete intersection, that is, the homogeneous ideal $I(C)$ of C is generated by a quadratic form Q and a cubic form F. The degree of C equals 6. The surface zero locus of Q is a unique quadric surface that contains C. The gonality $\mathrm{gon}(C)$ of C equals 3. If $\mathrm{rank}\, Q = 3$, then C has a unique trigonal morphism $C \to \mathbb{P}^1$, which is given by the projection from the vertex of the surface $Q = 0$. If $\mathrm{rank}\, Q = 4$, then C has exactly two trigonal morphisms $C \to \mathbb{P}^1$.*

Because $C \subset \mathbb{P}^3$ is a canonical curve, for every $\sigma \in \mathrm{Aut}(C)$, there exists a unique projective transformation $T \in \mathrm{Aut}(\mathbb{P}^3)$ such that $T(C) = C$ and $T|_C = \sigma$. Namely, we have $\mathrm{Aut}(C) \hookrightarrow \mathrm{Aut}(\mathbb{P}^3) \cong \mathrm{PGL}(4, k)$. Therefore, we represent automorphisms of C by square matrices of 4 rows and columns.

Let $l \subset \mathbb{P}^3$ be a line. Considering that $\deg C = 6$ and C is not hyperelliptic, we have $3 \leq \deg \pi_l \leq 6$, where $\pi_l : C \to \mathbb{P}^1$ is the projection with center l.

Proposition 2.4. *If $g^1_3 : C \to \mathbb{P}^1$ is a trigonal morphism, then there exists a line l such that $g^1_3 = \pi_l$.*

On trigonal morphisms $g^1_3 : C \to \mathbb{P}^1$, we note that $\pi_{l_1} = \pi_{l_2}$ does not imply that $l_1 = l_2$, where $\pi_{l_1} = \pi_{l_2}$ means an equal up to an isomorphism of codomains, that is, there exists an isomorphism $T : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\pi_{l_2} = T \circ \pi_{l_1}$. We note that by taking a suitable projective transformation, we may assume that the quadric that contains C is expressed as $XW - YZ = 0$ or $Y^2 - XW = 0$. If the quadric is given by $XW - YZ = 0$, then the projections with centers $X = Y = 0$ and $X = Z = 0$ give two different trigonal morphisms. If the quadric is given by $Y^2 - XW = 0$, then the projection with center $X = Y = 0$ gives a trigonal morphism. Hence, in both cases, without loss of generality, we may assume that a trigonal morphism C is given by the projection with center $X = Y = 0$. Proposition 2.5 makes clear which line gives the trigonal morphism that coincides with π_l, where l is expressed as $X = Y = 0$. Proposition 2.5 is only used in Section 5 and not in the proofs of the results in Section 3.

Proposition 2.5. *Let $l' \subset \mathbb{P}^3$ be a line.*

(I) *Assume that C is defined by the quadric $XW - YZ = 0$ and a cubic $F = 0$. Let the line l be given by $X = Y = 0$, which gives a trigonal morphism π_l. Then, $\pi_l = \pi_{l'}$ if and only if l' is given by $\alpha X + \beta Z = \alpha Y + \beta W = 0$ for some $(\alpha : \beta) \in \mathbb{P}^1$.*

(II) *Assume that C is defined by the quadric $Y^2 - XW = 0$ and a cubic $F = 0$. Let the line l be given by $X = Y = 0$, which gives a trigonal morphism π_l. Then, $\pi_l = \pi_{l'}$ if and only if l' is given by $\alpha X + \beta Y = \alpha Y + \beta W = 0$ for some $(\alpha : \beta) \in \mathbb{P}^1$.*

However, if $\deg \pi_l \geq 4$, then π_l uniquely determines the center l.

Proposition 2.6. *Assume $\deg \pi_l \geq 4$. Then, $\pi_l = \pi_{l'}$ (up to an isomorphism of the codomains $\mathbb{P}^1$ of π_l and $\pi_{l'}$) if and only if $l = l'$.*

3. Results

All throughout this section, we will assume $C \subset \mathbb{P}^3$ to be a canonical curve of genus 4 over an algebraically closed field k of characteristic 0. We present some results for C_3-lines, C_4-lines and C_5-lines in each subsection.

3.1 – C_3-*lines*

The function field of C that has a cyclic trigonal morphism is expressed as follows. (Proposition 3.1 is required in the proof of Theorem 3.2.)

PROPOSITION 3.1. *Let* $\pi_l \colon C \to \mathbb{P}^1$ *be a cyclic trigonal morphism. Then, one of the following holds.*

(I) *There exist* $x, y \in k(C)$ *such that* $k(C) = k(x, y)$ *and*

$$y^3 = \prod_{i=1}^{5}(x - c_i), \tag{2}$$

where $c_1, \ldots, c_5 \in k$ *are mutually distinct. In this case,* C *has a unique trigonal morphism* π_l*, which is given by the function* x.

(II) *There exist* $x, y \in k(C)$ *such that* $k(C) = k(x, y)$ *and*

$$y^3 = \prod_{i=1}^{3}(x - c_i) \cdot \prod_{i=4}^{5}(x - c_i)^2, \tag{3}$$

where $c_1, \ldots, c_5 \in k$ *are mutually distinct. In this case,* C *has exactly two trigonal morphisms. One trigonal morphism is* π_l*, which is given by the function* x*, and the other one is the morphism given by the function* $y/((x - c_4)(x - c_5))$.

Note that the number of trigonal morphisms of C equals 1 or 2. (See Propositions 2.3, 2.4 and 2.5.) The function field of C that has two cyclic trigonal morphisms is expressed as follows. (Theorem 3.2 is proved using Proposition 3.1, and is required in the proof of Corollary 3.3.)

THEOREM 3.2. *Let* $C \subset \mathbb{P}^3$ *be a canonical curve of genus* 4*. Assume that* C *has two trigonal morphisms. Then, both of the trigonal morphisms are cyclic if and only if there exist* $x, y \in k(C)$ *such that* $k(C) = k(x, y)$ *and*

$$y^3 = (x^3 + cx^2 - 9x - c)(x - 1)^2(x + 1)^2, \tag{4}$$

where $c \in k$*,* $x^3 + cx^2 - 9x - c$ *does not have multiple factor and does not have common factor with* $(x - 1)(x + 1)$.

When $C \subset \mathbb{P}^3$ has two cyclic trigonal morphisms, the curve C and the C_3-lines are expressed as follows. (Corollary 3.3 is proved using Proposition 3.1 and Theorem 3.2.)

Corollary 3.3. *Assume that C has two trigonal morphisms. Both of the trigonal morphisms are cyclic if and only if by taking a suitable projective transformation of $\mathbb{P}^3$, C can be expressed as*

$$\begin{cases} XW = YZ, \\ Z(W-Z)(W+Z) = Y^3 + cXY^2 - 9X^2Y - cX^3, \end{cases} \tag{5}$$

where $c \in k$ validate $Y^3 + cXY^2 - 9X^2Y - cX^3$ not having multiple factors and common factor with $(Y-X)(Y+X)$. Furthermore, if C is defined by equations (5), then $l_{1,(\alpha:\beta)} : \alpha X + \beta Z = \alpha Y + \beta W = 0$, where $(\alpha : \beta) \in \mathbb{P}^1$, and $l_{2,(\gamma:\delta)} : \gamma X + \delta Y = \gamma Z + \delta W = 0$, where $(\gamma : \delta) \in \mathbb{P}^1$, are all the C_3-lines. The Galois groups of lines $l_{1,(\alpha:\beta)}$ and $l_{2,(\gamma:\delta)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix},$$

respectively.

3.2 – C_4-*lines*

The function field of C that has a C_4-line is expressed as follows. (Proposition 3.4 is required in the proof of Corollary 3.5.)

Proposition 3.4. *There exists a C_4-line l if and only if there exist $x, y \in k(C)$ such that $k(C) = k(x, y)$ and*

$$y^4 = (x-c_1)(x-c_2)(x-c_3)(x-c_4)^2, \tag{6}$$

where $c_1, \ldots, c_4 \in k$ are mutually distinct. The projection $\pi_l \colon C \to \mathbb{P}^1$ is given by the function x.

When $C \subset \mathbb{P}^3$ has a C_4-line, the curve C and the C_4-line are expressed as follows. (Corollary 3.5 is proved using Propositions 2.3 and 3.4, and is required in the proof of Theorem 3.6.)

COROLLARY 3.5. *There exists a C_4-line l if and only if by taking a suitable projective transformation of $\mathbb{P}^3$, C can be expressed as*

$$\begin{cases} Z^2 = YW, \\ XW^2 = (Y - b_1X)(Y - b_2X)(Y - b_3X), \end{cases} \tag{7}$$

where $b_1, b_2, b_3 \in k$ are mutually distinct and not equal to 0, and l is given by $X = Y = 0$. Moreover, the Galois group G_l is generated by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where i is a primitive fourth root of unity.

From Corollary 3.5, we see that a C_4-line meets C in one point. We conclude this part by giving a bound on the number of C_4-lines. In the proof of Theorem 3.6, we observe the shifts of the ramification points of C_4-coverings by the actions of the Galois groups. By using Weierstrass semigroups, we distinguish two types of the ramification points. The Weierstrass semigroup is calculated from the defining equation of C in Proposition 3.4 and Corollary 3.5. (Theorem 3.6 is proved using Propositions 2.3, 2.6, 3.4 and Corollary 3.5.)

THEOREM 3.6. *Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4. Then, the number of C_4-lines for C equals 0 or 1.*

3.3 – C_5-*lines*

The function field of C that has a C_5-line is expressed as follows. (Proposition 3.7 is required in the proof of Corollary 3.8.)

PROPOSITION 3.7. *There exists a C_5-line l if and only if there exist $x, y \in k(C)$ such that $k(C) = k(x, y)$ and*

$$y^5 = (x - c_1)(x - c_2)(x - c_3), \tag{8}$$

where $c_1, c_2, c_3 \in k$ are mutually distinct. The projection $\pi_l : C \to \mathbb{P}^1$ is given by the function x.

When C has a C_5-line, the curve C and the C_5-line are expressed as follows. (Corollary 3.8 is proved using Propositions 2.3 and 3.7.)

COROLLARY 3.8. *There exists a C_5-line l if and only if by taking a suitable projective transformation of $\mathbb{P}^3$, C can be expressed as*

$$\begin{cases} Y^2 = XW, \\ YW^2 = (Z - c_1X)(Z - c_2X)(Z - c_3X), \end{cases} \tag{9}$$

where $c_1, c_2, c_3 \in k$ are mutually distinct, and l is given by $X = Z = 0$. Moreover, the Galois group G_l is generated by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix},$$

where ζ is a primitive fifth root of unity.

Regarding the number of C_5-lines, we have the following theorem. (Theorem 3.9 is proved using Theorem 2.2 and Proposition 3.7.)

THEOREM 3.9. *Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4. Then, the number of C_5-lines for C equals 0 or 1.*

4. Proofs

We prove the theorems, propositions and corollaries that are stated in the above sections.

4.1 – *Proofs of propositions in Section 2*

In this subsection, we prove propositions in Section 2.

PROOF OF PROPOSITION 2.4. Let $g_3^1 : C \to \mathbb{P}^1$ be a trigonal morphism. Then, there exists an effective divisor D such that $g_3^1 = \Phi_{|D|}$. According to the Riemann–Roch theorem, we have $\dim_k H^0(C, \mathcal{O}_C(K_C - D)) > 0$, and thus $|D| \subset |K_C|$. Because $C \subset \mathbb{P}^3$ is a canonical curve, g_3^1 is given by a projection from some line l. ■

PROOF OF PROPOSITION 2.5. We prove the "if" part of (I). From $XW = YZ$ on C, we observe that $(\alpha Y + \beta W)/(\alpha X + \beta Z) = Y/X$ on C. Thus, $\pi_l = \pi_{l'}$ holds.

We prove the "only if" part of (I). Let l' be a line such that $l' \neq l$ and $\pi_{l'} = \pi_l$ up to an isomorphism of $\mathbb{P}^1$. Take a hyperplane H_s of the form $Y = sX$, where $s \in k$ is general. Then $l \subset H_s$ and $C \cap H_s = (l \cap C) \cup \{R_1, R_2, R_3\}$, where R_1, R_2, R_3 are three mutually distinct points in $C \setminus (l \cap C)$. We have $(XW - YZ = 0) \cap H_s = l \cup l_s$, where l_s is a line defined by $W - sZ = Y - sX = 0$. Because $C \subset (XW - YZ = 0)$, we have that $\{R_1, R_2, R_3\} \subset l_s$. Because $\pi_l = \pi_{l'}$, there exists a hyperplane H'_s such that $l_s \cup l' \subset H'_s$. Note that $H_s \neq H'_s$, because $l \neq l'$ and $s \in k$ is general. Note that $C \cap l \cap l' = \emptyset$. Indeed, if $C \cap l \cap l' \neq \emptyset$, then $C \cap l \cap l' \subset l_s$, so $C \cap l_s$ consists of four points, which contradicts the fact that C is not hyperelliptic. Because $l_s \subset H'_s$, H'_s is defined by $W - sZ = u(Y - sX)$ for some $u \in k$. For $t \in k \setminus \{s\}$, we consider similar hyperplanes, that is, $H_t = \{Y = tX\}$ and $H'_t = \{W - tZ = v(Y - tX)\}$ for some $v \in k$. Then, $l' = H'_s \cap H'_t$. Note that l' is also non-skew. Let P' be a point in $l' \cap C$. Because $l : X = Y = 0$ and $C \cap l \cap l' \neq \emptyset$, the point P' is expressed as $(1 : y : z : yz)$ or $(x : 1 : xw : w)$. Because $P' \in H'_s \cap H'_t$, if $P' = (1 : y : z : yz)$, then $yz - sz = u(y - s)$ and $yz - tz = v(y - t)$. Because s, t are general, we may assume that $y - s \neq 0$ and $y - t \neq 0$. Then, $u = (yz - sz)/(y - s) = z$ and $v = (yz - tz)/(y - t) = z$; thus, $u = v = z$. If $P' = (x : 1 : xw : w)$, then we also have $u = v = w$, by the same argument. Therefore, l' is defined as $W - sZ - u(Y - sX) = W - tZ - u(Y - tX) = 0$ ($u = z$ or w) for general s, t, that is, $Z - uX = W - uY = 0$. Including the case that $l = l'$, if $\pi_l = \pi_{l'}$, then $l' : \alpha X + \beta Z = \alpha Y + \beta W = 0$.

The proof of (II) follows similarly. ∎

Proof of Proposition 2.6. The "if" part is trivial. We prove the "only if" part. Assume that $\pi_l = \pi_{l'}$, $n := \deg \pi_l \geq 4$ and $l \neq l'$. Let H be a general hyperplane containing l. Then, we may assume that $l' \not\subset H$ and $C \cap H = (C \cap l) \cup \{R_1, \ldots, R_n\}$, where $R_1, \ldots, R_n \notin l \cup l'$, which are mutually distinct n points. Because $\pi_l = \pi_{l'}$ up to an isomorphism of $\mathbb{P}^1$, there exists a hyperplane H' such that $C \cap H' = (C \cap l') \cup \{R_1, \ldots, R_n\}$. Hence, $R_1, \ldots, R_n \in H \cap H'$. Note that $H \neq H'$, because $l \neq l'$. Thus, $\tilde{l} := H \cap H'$ is a line, and $R_1, \ldots, R_n \in \tilde{l}$. Consider the projection $\pi_{\tilde{l}} \colon C \to \mathbb{P}^1$. The degree of $\pi_{\tilde{l}}$ equals $6 - n \leq 2$, which contradicts the fact that C is not hyperelliptic. ∎

4.2 – *Proofs of the results on C_3-lines in Section 3*

We prove the theorem, proposition and corollary stated in Subsection 3.1.

Proof of Proposition 3.1. Because π_l is a triple cyclic covering, there exist $x, y \in k(C)$ such that $k(C) = k(x, y)$ and

$$y^3 = \prod_{i=1}^{t} (x - c_i)^{d_i}, \tag{10}$$

where $c_1, \ldots, c_t \in k$ are mutually distinct. The trigonal morphism π_l is given by the function x. If $d_i \geq 3$, then we replace $y/(x - c_i)^j$ with a new y, where j is the maximum integer not greater than $d_i/3$. We may assume in this way that $d_i = 1$ or 2 ($i = 1, 2, \ldots, t$). As a consequence, we split the factors with exponent 1 and the factors with exponent 2. Namely,

$$y^3 = \prod_{i=1}^{s}(x - c_i) \cdot \prod_{i=s+1}^{t}(x - c_i)^2. \tag{11}$$

We may assume that the point $x = \infty \in \mathbb{P}^1$ is a branch point of π_l. Then, the degree of the right-hand side of equation (11) is not a multiple of 3. Because π_l has 6 branch points, we have $t = 5$. If $s = 0, 1$ or 2, then we replace $\prod_{i=1}^{5}(x - c_i)/y$ with a new y. We may assume that $s = 3, 4$ or 5. However, if $s = 4$, then the degree of the right-hand side of equation (11) is equal to 6. Therefore, we have $s = 3$ or 5; that is, we have equations (2) and (3). Let $P_\infty \in C$ be the ramification point of π_l that satisfies $x(P_\infty) = \infty$. If equation (2) holds, then $H(P_\infty) = \langle 3, 5\rangle$, because $(x)_\infty = 3P_\infty$ and $(y)_\infty = 5P_\infty$ (for the definition of $H(P_\infty)$, see equation (1)). Thus, C has only one trigonal pencil, which is $|3P_\infty|$. Indeed, if C has two distinct trigonal pencils $|D| := |3P_\infty|$ and $|E|$, where $D \not\sim E$, then $D + E \sim K_C$ by the Riemann–Roch theorem. Because $K_C \sim 2D$, we have $E \sim D$, which is a contradiction. If equation (3) holds, then C has two trigonal morphisms, which are given by the functions x and $y/((x - c_4)(x - c_5))$. ∎

PROOF OF THEOREM 3.2. We prove the "only if" part. Based on Proposition 3.1, there exist $x, y \in k(C)$ such that $k(C) = k(x, y)$ and equation (3) holds. C has two trigonal morphisms π_l, one is given by x and is cyclic, and the other is given by $z := y/((x - c_4)(x - c_5))$. We denote the second trigonal morphism as h_3^1. By considering a suitable projective transformation of $\mathbb{P}^1$, we may assume that $c_4 = 1$ and $c_5 = -1$. Then, we have that $(x - 1)(x + 1)z^3 = (x - c_1)(x - c_2)(x - c_3)$, that is,

$$x^3 - (c_1 + c_2 + c_3 + z^3)x^2 + (c_1c_2 + c_2c_3 + c_3c_1)x + (-c_1c_2c_3 + z^3) = 0. \tag{12}$$

Because every ramification point of the triple cyclic covering h_3^1 is a total ramification point, every multiple root of equation (12) is a triple root. Note that $z = \infty \in \mathbb{P}^1$ is not a branch point of h_3^1. Indeed, we can find the three points on C with $z = \infty$ from equation (3). Considering that h_3^1 has 6 total ramification points, equation (12) has triple roots for 6 values of z. Let $\lambda \in k$ be the triple root for a value $z = z_0 \in k$. Then, $(x - \lambda)^3 = x^3 - (c_1 + c_2 + c_3 + z_0^3)x^2 + (c_1c_2 + c_2c_3 + c_3c_1)x + (-c_1c_2c_3 + z_0^3)$. Hence,

$$\begin{cases} 3\lambda = c_1 + c_2 + c_3 + z_0^3, \\ 3\lambda^2 = c_1c_2 + c_2c_3 + c_3c_1, \\ \lambda^3 = c_1c_2c_3 - z_0^3. \end{cases} \tag{13}$$

From equations (13), we have:

$$(c_1c_2 + c_2c_3 + c_3c_1 + 9)z_0^3 = 9c_1c_2c_3 - (c_1c_2 + c_2c_3 + c_3c_1)(c_1 + c_2 + c_3).$$

This implies that if $c_1c_2 + c_2c_3 + c_3c_1 + 9 \neq 0$, then equation (12) has triple roots for only three values of z, which is a contradiction. Hence, we have $c_1c_2 + c_2c_3 + c_3c_1 = -9$. Using equations (13), we have $\lambda = \pm\sqrt{3}\,i$, where i is a primitive fourth root of unity, and $c_1c_2c_3 = -(c_1 + c_2 + c_3)$. Let $c := c_1c_2c_3$. Subsequently, we obtain equation (4).

For the "if" part, assume that x and y satisfy equation (4). We have two trigonal morphisms, which are given by the functions x and $z := y/((x-1)(x+1))$. Evidently, the triple extension $k(x, y)/k(x)$ is cyclic. On $k(x, z)/k(z)$, we have the minimal equation

$$x^3 + (c - z^3)x^2 - 9x + (-c + z^3) = 0. \tag{14}$$

Let D be the discriminant of equation (14). Using computer calculations, we have

$$D = (2i(z^6 - 2cz^3 + c^2 + 27))^2.$$

In particular, $\sqrt{D} \in k(z)$. We observe that the triple extension $k(x, z)/k(z)$ is cyclic. ∎

In the proof of Theorem 3.2, we have that if x and y satisfy equation (4), then two triple extensions $k(x)(y)/k(x)$ and $k(z)(x)/k(z)$ are cyclic, where $z := y/((x-1)(x+1))$. We note that the Galois group of $k(x)(y)/k(x)$ (resp. $k(z)(x)/k(z)$) is generated by the automorphism given by $y \mapsto \omega y$ (resp. $x \mapsto (x-3)/(x+1)$), where ω is a primitive third root of unity. Corollary 3.3 follows immediately from Theorem 3.2 and the lemma below. (Although (I) of Lemma 4.1 below is not necessary for the proof of Corollary 3.3, we write it here because it is a similar claim to (II) and used in some examples in Section 5.)

LEMMA 4.1. *We have the following:*

(I) *If C satisfies* (I) *in Proposition* 3.1, *then by considering a suitable projective transformation of $\mathbb{P}^3$, we may assume that C is defined by*

$$\begin{cases} Y^2 = XW, \\ Z^3 = YW^2 + a_4XW^2 + a_3XYW + a_2X^2W + a_1X^2Y + a_0X^3, \end{cases} \tag{15}$$

where $a_0, \ldots, a_4 \in k$.

(II) *If C satisfies* (II) *in Proposition* 3.1*, then by considering a suitable projective transformation of* $\mathbb{P}^3$*, we may assume that C is defined by*

$$\begin{cases} XW = YZ, \\ Z(W - c_4Z)(W - c_5Z) = (Y - c_1X)(Y - c_2X)(Y - c_3X). \end{cases} \tag{16}$$

Proof. We prove (I). Assume that C satisfies (I) in Proposition 3.1. Let $P_\infty \in C$ be the total ramification point of the morphism given by x such that $x(P_\infty) = \infty$. Then, we have $(x)_\infty = 3P_\infty$ and $(y)_\infty = 5P_\infty$. According to the Riemann–Roch theorem, $K_C \sim 6P_\infty$. The morphism $C \ni P \mapsto (1 : x(P) : y(P) : x^2(P)) \in \mathbb{P}^3$ is a canonical embedding. The image of this canonical embedding is expressed as equations (15) for some $a_0, \ldots, a_4 \in k$.

We prove (II). Assume that C satisfies (II) in Proposition 3.1. Let P_∞ and $P_i \in C$ $(i = 1, \ldots, 5)$ be the total ramification points of the morphism given by x such that $x(P_\infty) = \infty$ and $x(P_i) = c_i$. Let $z := y/((x - c_4)(x - c_5))$. Then, we have $(x)_\infty = 3P_\infty$ and $(z)_\infty = P_\infty + P_4 + P_5$. According to the Riemann–Roch theorem, $K_C \sim 4P_\infty + P_4 + P_5$. The morphism $C \ni P \mapsto (1 : x(P) : z(P) : x(P)z(P)) \in \mathbb{P}^3$ is a canonical embedding. The image of this canonical embedding is expressed as equations (16). ■

4.3 – *Proofs of the results on C_4-lines in Section 3*

We prove the theorem, proposition and corollary stated in Subsection 3.2.

Proposition 3.4 follows from an argument similar to that of Proposition 3.1. Corollary 3.5 follows immediately from Proposition 3.4 and the following lemma.

Lemma 4.2. *Assume that $k(C) = k(x, y)$ and x and y satisfy equation* (6)*. Then, by considering a suitable projective transformation of* $\mathbb{P}^3$*, we may assume that $C \subset \mathbb{P}^3$ is defined by equations* (7)*. In particular, C has only one trigonal morphism.*

Proof. Let P_∞ be the ramification point of the morphism $\pi_l : C \to \mathbb{P}^1$ given by x such that $x(P_\infty) = \infty$. Let $z := y^2/(x - c_4)$. Then, $(x - c_4)_\infty = 4P_\infty$, $(y)_\infty = 5P_\infty$, and $(z)_\infty = 6P_\infty$. According to the Riemann–Roch theorem, $K_C \sim 6P_\infty$. The morphism $C \ni P \mapsto (1 : x(P) - c_4 : y(P) : z(P)) \in \mathbb{P}^3$ provides a canonical embedding. The image of the canonical embedding is given by equations (7), where $b_i = c_i - c_4$ $(i = 1, 2, 3)$. Based on Proposition 2.3, C has only one trigonal system. ■

In the proof of Theorem 3.6, we use the data on the Weierstrass semigroups.

Lemma 4.3. *Assume that $k(C) = k(x, y)$ and x and y satisfy equation* (6)*. Let P_i $(i = \infty, 1, 2, 3, 4', 4'')$ be all the ramification points of the morphism $\pi_l : C \to \mathbb{P}^1$, which* *is given by x, such that $x(P_\infty) = \infty$, $x(P_i) = c_i$ $(i = 1, 2, 3)$, and $x(P_{4'}) = x(P_{4''}) = c_4$. Then, $H(P_\infty) = \langle 4, 5, 6\rangle$, $H(P_i) = \langle 4, 6, 7, 9\rangle$ $(i = 1, 2, 3)$, and $H(P_{4'}) = H(P_{4''}) = \langle 4, 5, 6\rangle$ or $\langle 4, 6, 7, 9\rangle$.*

Proof. We showed that $H(P_\infty) = \langle 4, 5, 6\rangle$ in the proof of Lemma 4.2. Let σ be the automorphism of C that is induced by $\sigma^*\colon x \mapsto x$, $y \mapsto iy$, where i is a primitive fourth root of unity. We have $G_l = \langle\sigma\rangle$. The nonsingular projective curve $C/\langle\sigma^2\rangle$ is an elliptic curve, and P_i $(i = \infty, 1, 2, 3, 4', 4'')$ are the ramification points of the double covering $C \to C/\langle\sigma^2\rangle$. Hence, we have $4, 6 \in H(P_i)$; thus, $H(P_i) = \langle 4, 5, 6\rangle$ or $\langle 4, 6, 7, 9\rangle$ $(i = \infty, 1, 2, 3, 4', 4'')$. Considering that $(y/(x - c_i)^2)_\infty = 7P_i$, $H(P_i) = \langle 4, 6, 7, 9\rangle$ $(i = 1, 2, 3)$. ∎

Proof of Theorem 3.6. We assume by contradiction that two C_4-lines l_1 and l_2 $(l_1 \neq l_2)$ exist for C. We will show that there exists a third C_4-line, and that this gives a contradiction.

Based on Corollary 3.5, we may assume that C is given by equation (7) and $l_1 : X = Y = 0$. Let $P_i \in C$ $(i = \infty, 1, 2, 3, 4', 4'')$ be the ramification points of π_{l_1}, such as those in Lemma 4.3. Note that $l_1 \cap C = \{P_\infty\}$ and $H(P_\infty) = \langle 4, 5, 6\rangle$. Moreover, by Corollary 3.5 again, $l_2 \cap C$ also consists of one point; we denote it as $\bar{P}$. We have $H(\bar{P}) = \langle 4, 5, 6\rangle$. Because $\pi_{l_1} = \Phi_{|4P_\infty|}$ and Proposition 2.6, we have $P_\infty \neq \bar{P}$.

Claim 4.4. The number of C_4-lines for C equals 3. Let l_3 be the third C_4-line. We have $l_1 \cap C = \{P_\infty\}$, $l_2 \cap C = \{P_{4'}\}$ and $l_3 \cap C = \{P_{4''}\}$ (or $l_2 \cap C = \{P_{4''}\}$, $l_3 \cap C = \{P_{4'}\}$). In particular, $H(P_\infty) = H(P_{4'}) = H(P_{4''}) = \langle 4, 5, 6\rangle$.

Proof of Claim 4.4. Let $G := \langle \sigma \in \mathrm{Aut}(C) \mid \mathrm{ord}\,\sigma = 4\rangle$. Let $g^1_3\colon C \to \mathbb{P}^1$ be the trigonal morphism. Because C has only one trigonal system, we have a group homomorphism $\varphi\colon \mathrm{Aut}(C) \to \mathrm{Aut}(\mathbb{P}^1)$ such that $g^1_3 \circ \sigma = \varphi(\sigma) \circ g^1_3$ for every $\sigma \in \mathrm{Aut}(C)$. Let Ker and Im be the kernel and the image of $\varphi|_G$, respectively. We have the exact sequence $1 \to \mathrm{Ker} \to G \to \mathrm{Im} \to 1$. Because for $\sigma \in G$, $\sigma \in \mathrm{Ker}$ if and only if $g^1_3 \circ \sigma = g^1_3$, we observe that $\mathrm{Ker} \cong 1$ or C_3. Because $G_{l_i} \cong C_4$, $\varphi|_{G_{l_i}}\colon G_{l_i} \to \mathrm{Im}$ $(i = 1, 2)$ is injective. Moreover, $\varphi(G_{l_1}) \neq \varphi(G_{l_2})$. Indeed, if $\varphi(G_{l_1}) = \varphi(G_{l_2})$, then we have the exact sequence $1 \to C_3 \to \langle G_{l_1}, G_{l_2}\rangle \xrightarrow{\varphi} C_4 \to 1$. The order of $\langle G_{l_1}, G_{l_2}\rangle$ equals 12. Hence, $\langle G_{l_1}, G_{l_2}\rangle \cong C_{12}$, $C_2 \times C_6$, D_6, $C_3 \times C_4$ or A_4, where D_6 and A_4 are the dihedral group of order 12 and the alternating group on 4 letters, respectively. This contradicts the fact that $\langle G_{l_1}, G_{l_2}\rangle$ contains two C_4 subgroups, namely, G_{l_1} and G_{l_2}. Because $\mathrm{Im} \subset \mathrm{Aut}(\mathbb{P}^1)$ is a finite group and Im contains two C_4 subgroups, we have $\mathrm{Im} \cong S_4$, where S_4 is the symmetric group on 4 letters. Considering that S_4 contains three C_4 subgroups, the number of C_4-lines is at most 3. Note that $l_2 \cap C = \langle\bar{P}\rangle$ and $H(\bar{P}) = \langle 4, 5, 6\rangle$. Let σ_1 be a generator of G_{l_1}. If $\bar{P}$ is not a ramification point of π_{l_1}, then the points $\bar{P}, \sigma_1(\bar{P}), \sigma_1^2(\bar{P}), \sigma_1^3(\bar{P})$ are mutually distinct, and lines $l_2, \sigma_1(l_2), \sigma_1^2(l_2), \sigma_1^3(l_2)$ are also mutually distinct. These four lines are C_4-lines, which is a contradiction. Hence, $\bar{P}$ is a ramification point of π_{l_1}. Based on $H(\bar{P}) = \langle 4, 5, 6 \rangle$ and Lemma 4.3, we may assume that $\bar{P} = P_{4'}$. Let $l_3 := \sigma_1(l_2)$. Based on the intersection with C, it is clear that l_3 is different from l_1 and l_2. We have that l_3 is a C_4-line, $l_3 \cap C = \{P_{4''}\}$ and $H(P_{4''}) = \langle 4, 5, 6 \rangle$. This concludes the proof of Claim 4.4. □

Let S_i $(i = 1, 2, 3)$ be the set

$$\{P \in C \mid P \text{ is a total ramification point of } \pi_{l_i} \text{ and } H(P) = \langle 4, 6, 7, 9 \rangle\}.$$

Let P be a point in S_2. Then, because $\pi_{l_1} = \Phi_{|4P_i|}$ $(i = 1, 2, 3)$, $\pi_{l_1} \neq \pi_{l_2}$ and $\pi_{l_2} = \Phi_{|4P|}$, P is not a ramification point of π_{l_1}. Hence, $S_1 \cap S_2 = \emptyset$. We also have that $S_2 \cap S_3 = \emptyset$ and $S_3 \cap S_1 = \emptyset$. The number of elements in $S_1 \cup S_2 \cup S_3$ is nine. However, because P is not a ramification point of π_{l_1}, we observe that $P, \sigma_1(P), \sigma_1^2(P), \sigma_1^3(P)$ are mutually distinct. For $\sigma \in \mathrm{Aut}(C)$, $\sigma(P)$ is a total ramification point of $\pi_{\sigma(l_2)}$, $\sigma(l_2)$ is a C_4-line, and $H(\sigma(P)) = \langle 4, 6, 7, 9 \rangle$. Based on Claim 4.4, $\sigma(P) \in S_1 \cup S_2 \cup S_3$. Hence, by considering the placement in the cyclic covering π_{l_1}, we see that the number of elements in $S_1 \cup S_2 \cup S_3$ must be expressed as $4r + 3$ for some r. This is a contradiction. This concludes Theorem 3.6. ■

4.4 – *Proofs of the results on C_5-lines in Section 3*

We prove the theorem, proposition and corollary stated in Subsection 3.3.

Proposition 3.7 follows from an argument similar to that of Proposition 3.1. Corollary 3.8 follows immediately from Proposition 3.7 and the following lemma.

Lemma 4.5. *Assume that $k(C) = k(x, y)$, and x and y satisfy equation* (8). *Then, according to a suitable projective transformation of $\mathbb{P}^3$, we may assume that $C \subset \mathbb{P}^3$ is defined by equations* (9). *In particular, C has only one trigonal morphism.*

Proof. Let P_∞ be the ramification point of the morphism $\pi_l\colon C \to \mathbb{P}^1$, which is given by x, such that $x(P_\infty) = \infty$. Then, $(x)_\infty = 5P_\infty$, $(y)_\infty = 3P_\infty$. Based on the Riemann–Roch theorem, $K_C \sim 6P_\infty$. The morphism $C \ni P \mapsto (1 : y(P) : x(P) : y^2(P)) \in \mathbb{P}^3$ provides a canonical embedding. The image of the canonical embedding is given by equations (9). The rank of $Y^2 - XW$ is equal to 3. Based on Proposition 2.3, C has only one trigonal system. ■

It should be noted that if there exists a C_5-line l, then $C \cap l = \{P_\infty\}$ and $H(P_\infty) = \langle 3, 5 \rangle$, where P_∞ is the point stated in the proof of Lemma 4.5.

Proof of Theorem 3.9. We assume by contradiction that there exist two C_5-lines for C. Let l_1 and l_2 be the two C_5-lines, and P_1 and P_2 be the points such that $C \cap l_i = \{P_i\}$ $(i = 1, 2)$. Then, P_1 and P_2 are weak Galois–Weierstrass points with $H(P_i) = \langle 3, 5\rangle$ and $5 \in \mathrm{degGW}(P_i)$ $(i = 1, 2)$. Based on Theorem 2.2, we have $P_1 = P_2$ $(=: P)$. Let $G := \langle G_{l_1}, G_{l_2}\rangle \subset \mathrm{Aut}(C)$. Then, for every $\sigma \in G$, we have that $\sigma(P) = P$. Hence, G is a cyclic group. Considering that $l_1 \neq l_2$, we have $G_{l_1} \neq G_{l_2}$. Thus, G contains two C_5 subgroups, which contradicts the fact that G is cyclic. ■

5. Examples

We present some examples of the case in which $\deg \pi_l = 3$. We note that C has only one or two trigonal morphisms. The following is an example in which both of the trigonal morphisms are cyclic.

Example 5.1. Let C be a nonsingular projective curve such that $k(C) = k(x, y)$ and

$$y^3 - x(x-3)(x+3)(x-1)^2(x+1)^2 = 0. \tag{17}$$

The polynomial stated on the left-hand side of equation (17) is irreducible. Let $z := y/((x-1)(x+1))$. Two trigonal morphisms are given by functions x and z. We denote these as g_3^1 and h_3^1, respectively. Using the Riemann–Hurwitz formula, the genus of C equals 4. Because we have an automorphism of order 3,

$$\mathrm{Aut}\, k(C) \ni \sigma_1^*: x \longmapsto x,\ y \longmapsto \omega y,\ z \longmapsto \omega z,$$

where ω is a primitive cubic root of unity, the trigonal morphism g_3^1 is cyclic. Considering that we have an automorphism of order 3,

$$\mathrm{Aut}\, k(C) \ni \sigma_2^*: x \longmapsto \frac{x-3}{x+1},\ y \longmapsto \frac{-8y}{(x+1)^3},\ z \longmapsto z,$$

the trigonal morphism h_3^1 is cyclic. Based on the same canonical embedding in the proof of (II) of Lemma 4.1, $C \subset \mathbb{P}^3$ is defined as follows:

$$\begin{cases} XW = YZ, \\ Z(W-Z)(W+Z) = Y(Y-3X)(Y+3X). \end{cases} \tag{18}$$

(I) The trigonal morphism g_3^1 is given by the projection π_{l_1}, where $l_1: X = Y = 0$. Hence, l_1 is a C_3-line. Based on Proposition 2.5, lines $l_{1,(\alpha:\beta)} : \alpha X + \beta Z =$

$\alpha Y + \beta W = 0$, where $(\alpha : \beta) \in \mathbb{P}^1$, are all the lines that hold $\pi_{l_1} = \pi_{l_{1,(\alpha:\beta)}}$. Thus, these are also C_3-lines. Let

$$\sigma_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}.$$

We have $\sigma_1 \in \mathrm{Aut}(C)$, $\mathrm{ord}(\sigma_1) = 3$, and $\pi_{l_1} \circ \sigma_1 = \pi_{l_1}$.

(II) The trigonal morphism h_3^1 is given by the projection π_{l_2}, where $l_2 : X = Z = 0$. Hence, l_2 is a C_3-lines. Based on Proposition 2.5, lines $l_{2,(\alpha:\beta)} : \alpha X + \beta Y = \alpha Z + \beta W = 0$, where $(\alpha : \beta) \in \mathbb{P}^1$, are all the lines that hold $\pi_{l_2} = \pi_{l_{2,(\alpha:\beta)}}$. Thus, these are also C_3-lines. Let

$$\sigma_2 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix}.$$

We have $\sigma_2 \in \mathrm{Aut}(C)$, $\mathrm{ord}(\sigma_2) = 3$, and $\pi_{l_2} \circ \sigma_2 = \pi_{l_2}$.

The following is an example that shows that one of two trigonal morphisms is cyclic and the other is not Galois.

EXAMPLE 5.2. Let C be a nonsingular projective curve such that $k(C) = k(x, y)$ and

$$y^3 - (x^3 - 1)x^2(x + 1)^2 = 0. \tag{19}$$

The polynomial stated on the left-hand side of equation (19) is irreducible. Let $z := y/(x(x + 1))$. Two trigonal morphisms are given by functions x and z. We denote these as g_3^1 and h_3^1, respectively. Using the Riemann–Hurwitz formula, the genus of C equals 4. Using the same canonical embedding in the proof of (II) of Lemma 4.1, $C \subset \mathbb{P}^3$ is defined as follows:

$$\begin{cases} XW = YZ, \\ ZW(W + Z) = Y^3 - X^3. \end{cases} \tag{20}$$

(I) Based on the same argument in Example 5.1 (I), we have $l_1 : X = Y = 0$ as C_3-line and lines $l_{1,(\alpha:\beta)} : \alpha X + \beta Z = \alpha Y + \beta W = 0$, where $(\alpha : \beta) \in \mathbb{P}^1$ are also C_3-lines.

(II) The trigonal morphism h_3^1 is given by the projection π_{l_2}, where $l_2 : X = Z = 0$. The line l_2 is not a Galois line. Indeed, the minimal equation of $k(x,z)/k(z)$ is

$$x^3 - z^3x^2 - z^3x - 1 = 0. \tag{21}$$

For $z = z_0 \in k$ such that $z_0^3 = \sqrt{9 + 6\sqrt{3}}$, from computer calculations, we can observe that equation (21) on x has a double root and a simple root. Because every ramification point of C_3 covering is totally ramified, h_3^1 is not Galois.

The following is an example in which C has a unique trigonal morphism that is cyclic.

Example 5.3. Let C be a nonsingular projective curve such that $k(C) = k(x, y)$ and

$$y^3 - x^5 + 1 = 0. \tag{22}$$

The polynomial stated on the left-hand side of equation (22) is irreducible. Using the Riemann–Hurwitz formula, the genus of C equals 4. We have a trigonal morphism g_3^1 given by the function x. Because we have an automorphism of order 3

$$\operatorname{Aut} k(C) \ni \sigma^*: x \longmapsto x,\ y \longmapsto \omega y,$$

where ω is a primitive cubic root of unity, the trigonal morphism g_3^1 is cyclic. Based on the same canonical embedding in the proof of (I) of Lemma 4.1, $C \subset \mathbb{P}^3$ is defined as follows:

$$\begin{cases} Y^2 = XW, \\ Z^3 = YW^2 - X^3. \end{cases}$$

Because $\operatorname{rank}(Y^2 - XW) = 3$, C has only one trigonal morphism. The trigonal morphism g_3^1 is given by the projection π_l, where $l : X = Y = 0$. Thus, l is a C_3-line. Based on Proposition 2.5, $l_{(\alpha:\beta)} : \alpha X + \beta Y = \alpha Y + \beta W = 0$, where $(\alpha : \beta) \in \mathbb{P}^1$, are all the lines that hold $\pi_l = \pi_{l_{(\alpha:\beta)}}$. Thus, these are also C_3-lines. Let

$$\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $\sigma \in \operatorname{Aut}(C)$, $\operatorname{ord}(\sigma) = 3$, and $\pi_l \circ \sigma = \pi_l$.

The following is an example of a curve C which has a C_4-line.

Example 5.4. Let C be a nonsingular projective curve such that $k(C) = k(x, y)$ and

$$y^4 - (x^3 - 1)x^2 = 0. \tag{23}$$

The polynomial stated on the left-hand side of equation (23) is irreducible. Let $g_4^1 : C \to \mathbb{P}^1$ be the morphism that is given by x. Using the Riemann–Hurwitz formula, the genus of C equals 4. Considering that we have an automorphism of order 4,

$$\operatorname{Aut} k(C) \ni \sigma^* : x \longmapsto x,\ y \longmapsto iy,$$

where i is a primitive fourth root of unity, the morphism g_4^1 is cyclic. Based on the same canonical embedding as that in the proof of Lemma 4.2, $C \subset \mathbb{P}^3$ is defined as follows:

$$\begin{cases} Z^2 = YW, \\ XW^2 = Y^3 - X^3. \end{cases} \tag{24}$$

The morphism g_4^1 is given by the projection π_l, where $l : X = Y = 0$. The line l is a C_4-line. Let

$$\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We have $\sigma \in \operatorname{Aut}(C)$, $\operatorname{ord}(\sigma) = 4$, and $\pi_l \circ \sigma = \pi_l$.

The following is an example that C has a C_5-line.

Example 5.5. Let C be a nonsingular projective curve such that $k(C) = k(x, y)$ and

$$y^5 - x^3 + 1 = 0. \tag{25}$$

The polynomial stated on the left-hand side of equation (25) is irreducible. Let $g_5^1 : C \to \mathbb{P}^1$ be the morphism that is given by x. Using the Riemann–Hurwitz formula, the genus of C equals 4. Considering that we have the automorphism of order 5,

$$\operatorname{Aut} k(C) \ni \sigma^* : x \longmapsto x,\ y \longmapsto \zeta y,$$

where ζ is a primitive fifth root of unity, the morphism g_5^1 is cyclic. Based on the same canonical embedding in the proof of Lemma 4.5, $C \subset \mathbb{P}^3$ is defined as follows:

$$\begin{cases} Y^2 = XW, \\ YW^2 = Z^3 - X^3. \end{cases} \tag{26}$$

The morphism g^1_5 is given by the projection π_l, where $l : X = Z = 0$. The line l is a C_5-line. Let

$$\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix}.$$

We have $\sigma \in \mathrm{Aut}(C)$, $\mathrm{ord}(\sigma) = 5$, and $\pi_l \circ \sigma = \pi_l$.

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