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Galois lines for a canonical curve of genus 4, II: Skew cyclic lines

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Abstract – Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4 over an algebraically closed field k of characteristic zero. For a line l, we consider the projection $\pi_l: C \to \mathbb{P}^1$ with center l and the extension of the function fields $\pi_l^*: k(\mathbb{P}^1) \hookrightarrow k(C)$. A line l is referred to as a *cyclic line* if the extension $k(C)/\pi_l^*(k(\mathbb{P}^1))$ is cyclic. A line $l \subset \mathbb{P}^3$ is said to be *skew* if $C \cap l = \emptyset$. We prove that the number of skew cyclic lines is equal to 0, 1, 3 or 9. We determine curves that have nine skew cyclic lines.

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1. Introduction and the main theorem

Yoshihara [10] investigated various properties of skew Galois lines (for the definition, see below) for nondegenerate nonsingular curves C in $\mathbb{P}^3$. He proved that the number of skew Galois lines for an irrational C is finite, and that the number of skew Galois lines for C is at most one if $\deg C$ is a prime and $\deg C \geq 5$. He also studied the defining equations of curves C of low degrees that have skew Galois lines. In addition, Yoshihara et al. [2, 7, 11], studied the number and arrangement of skew Galois lines for elliptic space curves. Fukasawa and Higashine [4] and subsequent work by Fukasawa [3] determined the arrangement of all the Galois lines for the Giulietti–Korchmáros

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curve and for the Artin–Schreier–Mumford curve, respectively. More recently, in [8], we studied the number of non-skew cyclic lines for canonical curves of genus 4. As a continuation of this work [8], in this study, we investigate the number of skew cyclic lines for canonical curves of genus 4. We would like to note Kuribayashi et al. [9], however we will not use it in the present paper. By giving generators with respect to linear representations in the vector space of holomorphic differentials, they presented a complete classification of automorphism groups for compact Riemann surfaces of genera 3 and 4.

Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4 over an algebraically closed field k of characteristic 0, which is a $(2, 3)$-complete intersection in $\mathbb{P}^3$. A line $l \subset \mathbb{P}^3$ is said to be *skew* if $C \cap l = \emptyset$. For a line l, we consider the projection $\pi_l\colon C \to \mathbb{P}^1$ with center l and the extension of the function fields $\pi_l^*\colon k(\mathbb{P}^1) \hookrightarrow k(C)$. Because $\deg C = 6$, we have $\deg \pi_l \leq 6$, and if l is skew, then we have $\deg \pi_l = 6$. We refer to a line l as a *Galois line* if the extension is Galois. We refer to the Galois line l as a C_6*-line* (resp. S_3*-line*) if the Galois group is isomorphic to the cyclic group C_6 of order 6 (resp. the symmetric group S_3 on 3 letters). We note that l is a skew cyclic line if and only if l is a C_6-line, in the setting of this paper. In [8], we explicitly gave the equations of C in the particular case in which C has two cyclic trigonal morphisms; we prove that the number of cyclic lines with $\deg \pi_l = 4$ is at most 1; and the number of cyclic lines with $\deg \pi_l = 5$ is at most 1. Our main theorem of the present paper is as follows.

THEOREM. *Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4 over an algebraically closed field of characteristic 0. Then, the number of C_6-lines equals 0, 1, 3 or 9. Moreover, if there exist nine C_6-lines for C, then C is projectively equivalent to the curve defined by one of the following:*

$$\begin{cases} XY - Z^2 = 0, \\ X^3 + Y^3 + W^3 = 0, \end{cases} \tag{1}$$

or

$$\begin{cases} X^2 + Y^2 + Z^2 = 0, \\ XYZ + W^3 = 0, \end{cases} \tag{2}$$

where $(X : Y : Z : W)$ are homogeneous coordinates on $\mathbb{P}^3$.

In Section 2, we present selected preliminary results. The proof of the theorem is provided in Section 3. In Sections 4 and 5, we determine all the C_6-lines for curves defined by equations (1) and (2). Section 6 presents examples of curves that have only one or three C_6-lines.

In the present paper, we assume that the base field k is algebraically closed and $\mathrm{char}(k) = 0$. For a line l, "skew" means "skew with respect to C", and also C_6-line means "with respect to C", and the reference to C will always be tacitly assumed. For the Galois line l, we denote $\{\sigma \in \mathrm{Aut}(C) \mid \pi_l \circ \sigma = \pi_l\}$ by G_l, which is isomorphic to the Galois group. We denote by C_m the cyclic group of order m; by D_m the dihedral group of order $2m$; by A_m the alternating group on m letters; by S_m and the symmetric group on m letters.

2. Preliminaries

Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4. Let $(X : Y : Z : W)$ be homogeneous coordinates on $\mathbb{P}^3$. The following are well-known facts.

Proposition 2.1 ([1, page 118], [6, page 298]). *The curve C is a $(2,3)$-complete intersection; that is, the homogeneous ideal $I(C) \subset k[X,Y,Z,W]$ of C is generated by a quadratic form Q and cubic form F. The degree of C is 6. The surface $Q = 0$ is a unique quadric surface that contains C. The gonality $\mathrm{gon}(C)$ of C is equal to 3. If $\mathrm{rank}\, Q = 3$, then C has a unique trigonal morphism $C \to \mathbb{P}^1$, which is given by the projection from the vertex of the surface $Q = 0$. If $\mathrm{rank}\, Q = 4$, then C has exactly two trigonal morphisms $C \to \mathbb{P}^1$.*

Let $l \subset \mathbb{P}^3$ be a line and $\pi_l : C \to \mathbb{P}^1$ the projection with center l. Because $\deg C = 6$ and C is not hyperelliptic, we have $3 \le \deg \pi_l \le 6$. A line l is skew if and only if $\deg \pi_l = 6$. If $\deg \pi_l \ge 4$, then π_l uniquely determines the center l.

Proposition 2.2 ([8]). *Assume $\deg \pi_l \ge 4$. Then, $\pi_l = \pi_{l'}$ (up to an isomorphism of the codomains $\mathbb{P}^1$ of π_l and $\pi_{l'}$), if and only if $l = l'$.*

We have a canonical representation $\mathrm{Aut}(C) \hookrightarrow \mathrm{GL}(\Gamma(C, \Omega^1)) \cong \mathrm{GL}(4,k)$, where Ω^1 is the sheaf of regular 1-forms on C. As $C \subset \mathbb{P}^3$ is a canonical curve, we also have $\mathrm{Aut}(C) \hookrightarrow \mathrm{Aut}(\mathbb{P}^3) \cong \mathrm{PGL}(4,k)$. That is, for every $\sigma \in \mathrm{Aut}(C)$, there exists a unique projective transformation $T : \mathbb{P}^3 \to \mathbb{P}^3$ such that $T(C) = C$ and $T|_C = \sigma$. We express the elements in $\mathrm{Aut}(C)$ as the projective transformations of $\mathbb{P}^3$.

Proposition 2.3. *There exist a quadratic form $Q \in k[X,Y,Z,W]$ and a cubic form $F \in k[X,Y,Z,W]$ with $I(C) = (Q,F)$ such that $\sigma(Q = 0) = (Q = 0)$ and $\sigma(F = 0) = (F = 0)$ for any $\sigma \in \mathrm{Aut}(C) \subset \mathrm{Aut}(\mathbb{P}^3)$.*

Proof. There exists a unique quadric $Q = 0$ that contains C. Clearly, $\sigma(Q = 0) = (Q = 0)$. Let

$$I_3 := \{F \in k[X, Y, Z, W] \mid F \text{ is a cubic form}, C \subset (F = 0)\} \cup \{0\},$$
$$J := \{(aX + bY + cZ + dW)Q \mid a, b, c, d \in k\},$$

and let $G \subset \mathrm{GL}(4, k)$ be a finite group isomorphic to $\mathrm{Aut}(C)$ via the natural quotient map $\mathrm{GL}(4, k) \twoheadrightarrow \mathrm{PGL}(4, k)$. Then, $\dim_k I_3 = 5$, $\dim_k J = 4$, $J \subsetneq I_3$, and G acts linearly on I_3 and J. Because $\mathrm{char}(k) = 0$, according to Maschke's theorem, the representation $G \to \mathrm{GL}(I_3)$ is completely reducible. Thus, there exists $F \in I_3 \setminus J$ such that $(A^* F)/F \in k \setminus \{0\}$ for any $A \in G$. ∎

Proposition 2.4 ([10]). *Assume that there exists a C_6-line l. Then, by taking a suitable projective transformation of $\mathbb{P}^3$, we may assume that l is defined by $X = Y = 0$, and a generator σ of $G_l \subset \mathrm{Aut}(\mathbb{P}^3)$ is expressed by a diagonal matrix with diagonal components $1, 1, \alpha, \beta$ ($\alpha, \beta \in k \setminus \{0\}$), and $(\mathrm{ord}(\alpha), \mathrm{ord}(\beta)) = (3, 6), (2, 3)$, or $(2, 6)$. That is, we may assume*

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix},$$

where ζ is a primitive sixth root of the unity.

Proof. Most of the claims are proved in the proof of [10, Theorem 4.5] (see Claim 7 on pages 466-467 of [10]). We only have to verify the following: the diagonal matrix with diagonal components $1, 1, \zeta^4, \zeta$ is unsuitable for a generator σ of G_l. Indeed, if σ is such an automorphism, then using Proposition 2.3, Q will be reducible. ∎

In Proposition 2.4, we note that the position of the line l and the form of the generator σ are specified simultaneously. From the following argument we see that this is possible: first, we fix the position of the line l to be $X = Y = 0$; next, from $\pi_l = \pi_l \circ \sigma$, we find the conditions that the representation matrix of σ must satisfy; finally by using a projective transformation that does not change the position of l, we diagonalize the representation matrix of σ.

Definition 2.5. We say that a C_6-line l is *of type* $(3, 6)$ (resp. *of type* $(2, 3)$, *of type* $(2, 6)$) if a generator of $G_l \subset \mathrm{Aut}(\mathbb{P}^3)$ can be represented as a matrix with eigenvalues $1, 1, \alpha, \beta$ with $(\mathrm{ord}(\alpha), \mathrm{ord}(\beta)) = (3, 6)$ (resp. $(2, 3)$, $(2, 6)$).

Corollary 2.6. *We assume that there exists a C_6-line l. Let $Q \in k[X, Y, Z, W]$ be a quadratic form such that the quadric surface $Q = 0$ contains C. Then,* rank $Q = 3$. *Hence, there exists only one trigonal morphism $g^1_3 : C \to \mathbb{P}^1$, which is given by the projection from the vertex of $Q = 0$.*

Proof. The quadric $Q = 0$ containing C satisfies $\sigma(Q = 0) = (Q = 0)$ for any $\sigma \in G_l$. From Proposition 2.4, we see that rank $Q = 3$. ■

For σ as stated in Proposition 2.4, we note that $\mathrm{Fix}(\sigma) := \{P \in \mathbb{P}^3 \mid \sigma(P) = P\}$ consists of a line $Z = W = 0$ and two points $(0 : 0 : 1 : 0), (0 : 0 : 0 : 1)$, and $l : X = Y = 0$ passes through these two points. Hence, we can immediately see the following.

Proposition 2.7. *Let l_1 and l_2 be distinct C_6-lines for C. Then, $G_{l_1} \neq G_{l_2}$ as subgroups of* $\mathrm{Aut}(C)$.

On S_3-lines, we have the following proposition. Proposition 2.8 is not used in the proof of our main theorem, but is required for the calculations in Sections 4 and 5. In Sections 4 and 5, we will determine not only C_6-lines but also S_3-lines for curves concretely defined by Equations (1) and (2).

Proposition 2.8 (Proof of [10, Theorem, 4.5]). *Let l be an S_3-line for C. Then, by taking a suitable projective transformation, we may assume that $l : X = Y = 0$, and G_l is generated by the following two elements:*

$$\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{pmatrix} \quad \textit{and} \quad \tau := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where ω is a primitive cubic root of the unity.

Since the proof of Proposition 2.8 is not stated in [10] as it is obvious, we present it here.

Proof. Let σ and τ be automorphisms of C such that $G_l = \langle \sigma, \tau \rangle$, where $\sigma^3 = \tau^2 = \mathrm{id}_C$ and $\tau\sigma\tau = \sigma^2$. By taking a suitable projective transformation, we may assume that l is defined by $X = Y = 0$. Because $\pi_l \circ \sigma = \pi_l$ and $\pi_l \circ \tau = \pi_l$, we have that σ and τ are represented as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

Because $\sigma^3 = \mathrm{id}_C$, σ is diagonalizable. We may assume that

$$\sigma = \begin{pmatrix} I & O \\ O & A \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} I & O \\ L & M \end{pmatrix},$$

where L and M are some 2×2 matrices,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \text{ or } \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

By using $\tau^2 = \mathrm{id}_C$ and $\tau\sigma\tau = \sigma^2$, we infer that $L + ML = O$, $M^2 = I$, $L + MAL = O$ and $MAM = A^2$. We have

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad L = O, \quad \text{and} \quad M = \begin{pmatrix} 0 & c \\ 1/c & 0 \end{pmatrix}$$

for some $c \in k \setminus \{0\}$. By taking the projective transformation that is represented by the diagonal matrix with diagonal elements 1, 1, c and 1, we have the representations of σ and τ as stated in the proposition. ■

For σ and τ as stated as in Proposition 2.8, we note that $\mathrm{Fix}(\sigma) := \{P \in \mathbb{P}^3 \mid \sigma(P) = P\}$ consists of a line $Z = W = 0$ and two points $(0:0:1:0)$, $(0:0:0:1)$, and l passes through these two points. The set $\mathrm{Fix}(\tau) := \{P \in \mathbb{P}^3 \mid \tau(P) = P\}$ consists of a hyperplane $Z - W = 0$ and a point $(0:0:-1:1)$, and l passes through the point.

Assume that $\operatorname{rank} Q = 3$, where the quadric $Q = 0$ contains C. Because the trigonal morphism $g_3^1\colon C \to \mathbb{P}^1$ is unique, for any $\sigma \in \mathrm{Aut}(C)$, there exists $A_\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ such that $g_3^1 \circ \sigma = A_\sigma \circ g_3^1$. Let G be a subgroup of $\mathrm{Aut}(C)$. Let $\varphi\colon G \to \mathrm{Aut}(\mathbb{P}^1)$ be the map $\sigma \mapsto A_\sigma$, which is a homomorphism between the groups. Let $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ be the kernel and image of φ, respectively. We denote the inclusion $\operatorname{Ker}\varphi \hookrightarrow G$ as ψ. We have a short exact sequence

$$1 \longrightarrow \operatorname{Ker}\varphi \xrightarrow{\psi} G \xrightarrow{\varphi} \operatorname{Im}\varphi \longrightarrow 1. \tag{3}$$

The short exact sequence (3) and Proposition 2.9 play central roles in the proof of our main theorem.

PROPOSITION 2.9. *We have the following:*

(I) *The group* $\operatorname{Im}\varphi$ *is isomorphic to one of the following groups:* C_m $(m \in \mathbb{Z}_{>0})$, D_m $(m \in \mathbb{Z}_{>0})$, A_4, S_4 *or* A_5.

(II) *The three conditions "*$\operatorname{Ker}\varphi \neq 1$*", "*$\operatorname{Ker}\varphi \cong C_3$*", and "*$g_3^1$ *is cyclic" are equivalent.*

PROOF. Because $\operatorname{Im}\varphi \subset \operatorname{Aut}(\mathbb{P}^1)$ is finite, (I) is well known. As $\operatorname{Ker}\varphi = \{\sigma \in G \mid g_3^1 \circ \sigma = g_3^1\}$, we see that (II) holds. ∎

On automorphism groups of a plane quadric curve, we have the following proposition. Proposition 2.10 is required in the proof of our main theorem.

PROPOSITION 2.10. *Let* $V \subset \mathbb{P}^2$ *be the curve defined by* $XY = Z^2$, *which is isomorphic to* $\mathbb{P}^1$.

(I) *Let* $S_4 \subset \operatorname{Aut}(V) \subset \operatorname{Aut}(\mathbb{P}^2)$ *be the symmetric group on four letters. Then, by taking a suitable projective transformation, we can assume that* $S_4 = \langle \rho, \tau \rangle$,

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix},$$

where i *is a primitive fourth root of the unity.*

(II) *Let* $D_m \subset \operatorname{Aut}(V) \subset \operatorname{Aut}(\mathbb{P}^2)$ $(m \geq 2)$ *be the dihedral group of order* $2m$. *Then, by taking a suitable projective transformation, we can assume that* $D_m = \langle \rho, \tau \rangle$,

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_m^2 & 0 \\ 0 & 0 & \zeta_m \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ζ_m *is a primitive* m*th root of the unity.*

PROOF. We may assume that the group $S_4 \subset \operatorname{Aut}(\mathbb{P}^1)$ (resp. $D_m \subset \operatorname{Aut}(\mathbb{P}^1)$) is generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} 1 & 0 \\ 0 & \zeta_m \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

The form of the matrices in the proposition comes from the images of these generators via the embedding $\mathbb{P}^1 \ni (x_0 : x_1) \mapsto (x_0^2 : x_1^2 : x_0 x_1) \in \mathbb{P}^2$. ∎

3. Proof of the main theorem

In this section, we prove the main theorem. Note that, if there exists a C_6-line, then C has a unique trigonal morphism $g^1_3\colon C \to \mathbb{P}^1$ by Corollary 2.6. Let us consider the short exact sequence (3) for

$$G := \langle \sigma \in \mathrm{Aut}(C) \mid \sigma \in G_l \text{ for some } C_6\text{-line } l \rangle.$$

The map φ defined just before the sequence (3) will be used many times with the group G defined here.

We give an overview of the proof. We will assume that there exist at least two C_6-lines, and discuss the proof in the following two cases: there exists at least one C_6-line of type $(3,6)$; there does not exist a C_6-line of type $(3,6)$. It will be important that g^1_3 is cyclic in both cases (Propositions 3.1 and 3.2). In the case that there exists a C_6-line of type $(3,6)$, we can determine the defining equations of the curve C concretely (Lemma 3.4). Once the curve C is given by the concrete equations, it is possible to find all the Galois lines completely (Section 4). In the case that there does not exist a C_6-line of type $(3,6)$, we will consider the short exact sequence (3). The group $\mathrm{Ker}\,\varphi$ and homomorphisms φ and ψ are easy to understand, and it is known what groups can be isomorphic to the group $\mathrm{Im}\,\varphi$ (Proposition 2.9). We will discuss the proof for each group that may be $\mathrm{Im}\,\varphi$, and we will find $\mathrm{Im}\,\varphi \cong D_2, D_3$ or S_4 (Lemmas 3.6–3.10). In the case that $\mathrm{Im}\,\varphi \cong S_4$, we can determine the defining equations of the curve C concretely (Lemma 3.12), and find all the Galois lines completely (Section 5). In the cases that $\mathrm{Im}\,\varphi \cong D_2, D_3$, we can determine the defining equations of C roughly, and we will see that the number of C_6-lines is equal to 3 (Lemma 3.13).

The two propositions below provide sufficient conditions for g^1_3 to be cyclic.

Proposition 3.1. *Assume that there exists a C_6-line l of type $(2,3)$ or $(2,6)$. Let σ_l be a generator of G_l. Then,* $\mathrm{Ker}\,\varphi = \langle \sigma_l^2 \rangle$*, and* $\mathrm{ord}(\varphi(\sigma_l)) = 2$*. In particular, the trigonal morphism g^1_3 is cyclic.*

Proof. By Proposition 2.4, using a suitable projective transformation, we may assume that σ_l is expressed as the diagonal matrix with diagonal components $1, 1, -1, \zeta^2$ or $1, 1, -1, \zeta$, where ζ is a primitive sixth root of the unity. The quadric $Q = 0$ that contains C has the vertex $R := (0:0:0:1)$. The trigonal morphism g^1_3 is given by the projection π_R with center R. Because $\pi_R \circ \sigma_l^2 = \pi_R$, we have $\sigma_l^2 \in \mathrm{Ker}\,\varphi$. Use Proposition 2.9. ∎

Proposition 3.2. *Assume that there exist two C_6-lines. Then, the trigonal morphism g^1_3 is cyclic.*

PROOF. Let l_1 and l_2 be two C_6-lines for C. We assume that $\operatorname{Ker}\varphi = 1$. Then, $G \cong \operatorname{Im}\varphi \cong C_m, D_m, A_4, S_4$, or A_5. This contradicts the fact that G includes two cyclic groups, G_{l_1} and G_{l_2}, of order 6. Therefore, $\operatorname{Ker}\varphi \neq 1$. Use Proposition 2.9. ■

We assume that there exist two C_6-lines for C. Let $P_1, \dots, P_6$ be all the ramification points of the cyclic trigonal morphism g^1_3.

LEMMA 3.3. *There exists a hyperplane $H \subset \mathbb{P}^3$ such that $\{P_1, \dots, P_6\} \subset H$.*

PROOF. By [8, Proposition 3.1], there exist $x, y \in k(C)$ such that $k(C) = k(x, y)$ and $y^3 = \prod_{j=1}^5 (x - c_j)$. We can assume that $x(P_j) = c_j$ $(j = 1, \dots, 5)$ and $x(P_6) = \infty$. Then, $(x - c_j) = 3P_j - 3P_6$ $(j = 1, \dots, 5)$ and $(y) = P_1 + \cdots + P_5 - 5P_6$. By using the Riemann–Roch theorem, it is clear that $K_C \sim 6P_6$. Thus, $K_C \sim P_1 + \cdots + P_6$. Because $C \subset \mathbb{P}^3$ is a canonical curve, this concludes the lemma. ■

LEMMA 3.4. *Assume that there exists a C_6-line of type* $(3, 6)$ *and that the trigonal morphism g^1_3 is cyclic. Then, C is projectively equivalent to the curve defined by equations* (1).

PROOF. Let l be a C_6-line of type $(3, 6)$. We assume that $G_l = \langle \sigma_l \rangle$ and

$$\sigma_l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix},$$

where ζ denotes a primitive sixth root of the unity. By using Proposition 2.3 and considering a suitable projective transformation, we can determine the defining equation of C as follows:

$$\begin{cases} Q = b(X, Y)Z + W^2 = 0, \\ F = X^3 + Y^3 + Z^3 = 0, \end{cases} \tag{4}$$

where $b(X, Y) = X - aY$ $(a \in k)$ or Y. If $b(X, Y) = Y$, then C is projectively equivalent to the curve defined by equations (1). Assume that $b(X, Y) = X - aY$. Let us show $a = 0$. The vertex of quadric $Q = 0$ is $R := (a : 1 : 0 : 0)$. The trigonal morphism $g^1_3 : C \to \mathbb{P}^1$ is given by the projection $\pi_R : (X : Y : Z : W) \mapsto (X - aY : Z : W)$. Let $P \in C$ be a ramification point of g^1_3. Then, $Z(P) \neq 0$. Indeed, if $Z(P) = 0$, then $P = (\zeta^{2j+1} : 1 : 0 : 0)$, where $j = 0, 1$ or 2. However, $(\zeta^{2j+1} : 1 : 0 : 0)$ is not a ramification point of g^1_3. Let $\pi_R(P) = (c : 1 : \sqrt{-c})$, where $c \in k$. A point in $C \cap \pi_R^{-1}(\pi_R(P))$ is $(ay + c : y : 1 : \sqrt{-c})$, where $y \in k$ satisfies

$$(ay + c)^3 + y^3 + 1 = 0. \tag{5}$$

Note that $a^3 + 1 \neq 0$, because C is nonsingular. As P is a total ramification point of g^1_3, equation (5) has a triple root. In other words, there exists $\beta \in k$ such that

$$(a^3 + 1)(y - \beta)^3 = (a^3 + 1)y^3 + 3a^2cy^2 + 3ac^2y + c^3 + 1.$$

Then, we have

$$\begin{cases} -3\beta(1 + a^3) = 3a^2c, \\ 3\beta^2(1 + a^3) = 3ac^2, \\ -\beta^3(1 + a^3) = c^3 + 1. \end{cases} \tag{6}$$

If $a \neq 0$, then equations (6) do not have a root β. Hence, $a = 0$ and C is projectively equivalent to the curve defined by equations (1). ■

We note that as in the proof of Lemma 3.4, for the curve defined by equations (1), there exists a C_6-line of type $(3, 6)$ and g^1_3 is cyclic. The number of C_6-lines of the curve defined by equations (1) will be calculated later in Section 4. In the discussion of Section 4 we do not use the results in Section 3. From Proposition 3.2, Lemma 3.4, and Section 4, we have the following result.

Proposition 3.5. *Assume that there exist two C_6-lines and one of them is of type* $(3, 6)$. *Then, C is projectively equivalent to the curve defined by equations* (1). *There are exactly nine C_6-lines and exactly one S_3-line for C. We have that* $\mathrm{Aut}(C) \cong C_3 \times D_6$.

Proof. From the assumption that there exist two C_6-lines, by using Proposition 3.2, the trigonal morphism g^1_3 is cyclic. Combining this with the assumption that there exists a C_6-line of type $(3, 6)$, by using Lemma 3.4, we have that C is projective equivalent to the curve defined by equations (1). By the results in Section 4, we have $\mathrm{Aut}(C)$ and the number of skew Galois lines. ■

Hereafter, in this section, we continue to prove our main theorem, except in the case that C is projectively equivalent to the curve defined by equations (1). That is, we assume that there exist at least two C_6-lines for C, and every C_6-line is not of type $(3, 6)$.

Lemma 3.6. *We have that* $\mathrm{Im}\,\varphi \ncong A_5$.

Proof. Assume that $\mathrm{Im}\,\varphi \cong A_5$. Then, $|G| = 180$. However, the Hurwitz theorem states $|G| = 84(g - 1), 48(g - 1), 40(g - 1), \ldots = 252, 144, 120, \ldots$; thus, this is a contradiction. ■

Lemma 3.7. *We have that* $\mathrm{Im}\,\varphi \ncong A_4$ *or* C_m.

Proof. From Proposition 3.1, $\operatorname{Im}\varphi$ is generated by some elements of order 2. However, A_4 and C_m $(m \geq 3)$ are not generated by elements of order 2. If $\operatorname{Im}\varphi \cong C_2$, then, G does not include two C_6 subgroups, because the order of G equals 6. ■

Lemma 3.8. *If* $\operatorname{Im}\varphi \cong D_m$, *then* $m \leq 6$.

Proof. Let $Q = 0$ be the quadric that contains C, where the rank of the quadratic Q equals 3, and R be the vertex of the quadric $Q = 0$. Then, the cyclic trigonal morphism g_3^1 is given by the projection π_R with center R. All the ramification points $P_1, \ldots, P_6$ of g_3^1 are on a hyperplane $H = 0$. Because $g_3^1 = \Phi_{|3P_j|}$ $(j = 1, \ldots, 6)$, for any $\sigma \in \operatorname{Aut}(C)$, $\sigma(\{P_1, \ldots, P_6\}) = \{P_1, \ldots, P_6\}$. Thus, $\sigma((Q = H = 0)) = (Q = H = 0)$, where $Q = H = 0$ is a plane quadric curve. We can regard that $g_3^1 = \pi_R|_C : C \to (Q = H = 0) \cong \mathbb{P}^1$ and $\varphi : G \ni \sigma \mapsto \sigma|_{Q=H=0} \in \operatorname{Im}\varphi \subset \operatorname{Aut}(Q = H = 0)$. Because $\operatorname{Im}\varphi$ acts on the set $\{P_1, \ldots, P_6\} \subset (Q = H = 0)$ faithfully, we determine that the order of each element in $\operatorname{Im}\varphi$ is at most 6. This concludes that $m \leq 6$. ■

By Lemmas 3.6, 3.7, and 3.8, we have $\operatorname{Im}\varphi \cong D_m$ $(2 \leq m \leq 6)$ or S_4.

Lemma 3.9. *The maximum number of* C_6*-lines is nine. If there exist nine* C_6*-lines, then* $\operatorname{Im}\varphi \cong S_4$.

Proof. Let $l_1, l_2, \ldots$ be all the C_6-lines for C, which are of type $(2, 3)$ or $(2, 6)$. Let σ_j $(j = 1, 2, \ldots)$ be a generator of G_{l_j}. By Propositions 2.7 and 3.1, $\varphi(\sigma_1), \varphi(\sigma_2), \ldots$ are mutually distinct elements in $\operatorname{Im}\varphi$ and are of order 2. The number of elements of order 2 in S_4 (resp. D_6, D_5, D_4, D_3, D_2) equals 9 (resp. $7, 5, 5, 3, 3$). This now concludes the lemma. ■

Lemma 3.10. *We have that* $\operatorname{Im}\varphi \not\cong D_4, D_5, D_6$.

Proof. Assume that $\operatorname{Im}\varphi \cong D_4$. Because the rank of the quadric $Q = 0$ that contains C equals 3, by taking a suitable projective transformation, we may assume that $Q = XY - Z^2$. From Lemma 3.3, all the ramification points $P_1, \ldots, P_6$ of the cyclic trigonal morphism g_3^1 are on some hyperplane $H = 0$. By taking a suitable projective transformation that does not change Q, we may assume $H = W$. Note that we can take such a projective transformation because $(0 : 0 : 0 : 1) \notin H$. By using Proposition 2.10 and the same argument as in the proof of Lemma 3.8, we may assume that:

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \right\rangle,$$

where ω (resp. i) is a primitive cubic (resp. fourth) root of the unity and $\lambda_1, \lambda_2 \in k \setminus \{0\}$. By using Proposition 2.3, we find a cubic form $F \in k[X, Y, Z, W] \setminus \{0\}$, such that the cubic surface $F = 0$ contains C. By the condition $\sigma(F = 0) = (F = 0)$ for any $\sigma \in G$, we have $F = a(X^2 + Y^2)Z + W^3$, $F = a(X^2 - Y^2)Z + W^3$, or $F = aXYZ + bZ^3 + W^3$, where $a, b \in k$. The curves defined by $Q = XY - Z^2 = 0$ and $F = a(X^2 + Y^2)Z + W^3 = 0$ are projectively equivalent to the curve defined by equations (2), and thus, $\operatorname{Im}\varphi \cong S_4$. The curves defined by $Q = XY - Z^2 = 0$ and $F = a(X^2 - Y^2)Z + W^3 = 0$ are also projectively equivalent to the curve defined by equations (2). The curves defined by $Q = XY - Z^2 = 0$ and $F = aXYZ + bZ^3 + W^3 = 0$ have singular points $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$. Hence, we see that $\operatorname{Im}\varphi \not\cong D_4$.

By using the same argument as above, we also see that $\operatorname{Im}\varphi \not\cong D_5$.

Assume that $\operatorname{Im}\varphi \cong D_6$. From the same argument as above, we see that C must be projectively equivalent to the curve defined by equations (1). Then, there exists a C_6-line for C of type $(3, 6)$. However, this is a contradiction. This concludes $\operatorname{Im}\varphi \not\cong D_6$. ■

Remark 3.11. To prove our main theorem, we have discussed the proof above with the assumption that there is no C_6-line of type $(3, 6)$, which is stated just after Proposition 3.5. If we allow the existence of C_6-lines of type $(3, 6)$, then by the same argument as in the proof of Lemma 3.10, we see the following: if a canonical curve $C \subset \mathbb{P}^3$ of genus 4 satisfies the conditions "there exists a unique trigonal morphism g^1_3", "g^1_3 is cyclic", and "$\operatorname{Im}\varphi \cong D_6$", then C is projectively equivalent to the curve defined by equations (1).

Hence, $\operatorname{Im}\varphi \cong D_2$, D_3, or S_4.

Lemma 3.12. *Assume that* $\operatorname{Im}\varphi \cong S_4$. *Then, C is projectively equivalent to the curve defined by equations* (2). *Hence, there exist nine C_6-lines (see Section 5).*

Proof. We may assume that the ramification points $P_1, \ldots, P_6$ of the trigonal morphism g^1_3 are on the hyperplane $W = 0$ and the quadric $Q = 0$ that contains C is $XY - Z^2 = 0$. By using Proposition 2.10, we can assume that

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \right\rangle,$$

where ω (resp. i) is a primitive cubic (resp. fourth) root of the unity and $\lambda_1, \lambda_2 \in k \setminus \{0\}$. By the same argument as in the proof of Lemma 3.10, C must be defined by

$$\begin{cases} XY - Z^2 = 0, \\ c(X^2 - Y^2)Z + W^3 = 0, \end{cases}$$

where $c \in k$. Then, C is projectively equivalent to the curve defined by equations (2). ∎

Note that we do not use the results in Section 3 in the discussion of Section 5.

Lemma 3.13. *If* $\operatorname{Im}\varphi \cong D_2$ *or* D_3*, then the number of* C_6*-lines equals* 3.

Proof. If $\operatorname{Im}\varphi \cong D_2$ or D_3, then the number of C_6-lines is at most three because the group $\operatorname{Im}\varphi$ contains only three elements of order 2.

Assume that $\operatorname{Im}\varphi \cong D_2$. We may assume that all the ramification points $P_1, \dots, P_6$ of the trigonal morphism g^1_3 are on the hyperplane $W = 0$ and the quadric $Q = 0$ that contains C is $XY - Z^2 = 0$. By using Proposition 2.10, we can assume that

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \right\rangle,$$

where ω is a primitive cubic root of the unity and $\lambda_1, \lambda_2 \in k \setminus \{0\}$. By the same argument as in the proof of Lemma 3.10, C must be projectively equivalent to the curve defined by

$$\begin{cases} XY - Z^2 = 0, \\ (X^3 + Y^3) + c(X + Y)Z^2 + W^3 = 0, \end{cases} \tag{7}$$

or

$$\begin{cases} XY - Z^2 = 0, \\ (X^2 + Y^2)Z + cZ^3 + W^3 = 0, \end{cases} \tag{8}$$

where $c \in k$. Then, the three lines $X = Y = 0$, $X + Y = Z = 0$, $X - Y = Z = 0$ are C_6-lines. Indeed, if C is defined by equations (7) (resp. equations (8)), then we have automorphisms of order 6 as follows:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&-1&0\\0&0&0&\omega\end{pmatrix}$$

$$\left(\text{resp. }\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix}\right).$$

Thus, the number of C_6-lines is at least three.

Assume that $\mathrm{Im}\,\varphi \cong D_3$. According to the above argument, C must be projectively equivalent to the curve defined by

$$\begin{cases} XY - Z^2 = 0, \\ (X^3 + Y^3) + cZ^3 + W^3 = 0, \end{cases} \tag{9}$$

where $c \in k$. Then, the three lines $X + Y = Z = 0$, $X + \omega Y = Z = 0$, $X + \omega^2 Y = Z = 0$ are C_6-lines. Indeed, we have automorphisms of order 6 as follows:

$$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&\omega&0&0\\\omega^2&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&\omega^2&0&0\\\omega&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix}.$$

Thus, the number of C_6-lines is at least three. ∎

The proof of our main theorem is now complete.

4. Example: Galois lines for the curve defined by equations (1)

In this section, let C be the nonsingular projective curve such that $k(C) = k(x, y)$, and

$$x^6 + y^3 + 1 = 0. \tag{10}$$

The polynomial on the left-hand side of equation (10) is irreducible. Let $g^1_3 \colon C \to \mathbb{P}^1$ be the trigonal morphism given by the function x. Then, g^1_3 is a cyclic triple covering, and there exist 6 branch points. By using the Riemann–Hurwitz formula, we have that the genus of C is equal to 4. Let $(x)_\infty = D$ be the divisor of poles of x. Then, $(x^2)_\infty = (y)_\infty = 2D$. Therefore, $\dim_k H^0(C, \mathcal{O}_C(2D)) \geq 4$. By using the Riemann–Roch theorem, we have that $K_C \sim 2D$. The morphism $C \ni P \mapsto (1 : x^2(P) : x(P) : y(P)) \in \mathbb{P}^3$ is a canonical embedding. The image of this canonical embedding is expressed as equations (1). We regard C as the canonical curve defined by equations (1).

We can identify nine C_6-lines and one S_3-line, as indicated in Tables 1 and 2. Because $\deg \pi_{l_j} = 6$, $\sigma_j \in \mathrm{Aut}(C)$, $\mathrm{ord}(\sigma_j) = 6$, and $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ $(j = 1, \dots, 9)$, it is clear that the lines $l_1, \dots, l_9$ are C_6-lines. As $\deg \pi_{l_{10}} = 6$, $\sigma_{10}, \tau_{10} \in \mathrm{Aut}(C)$, $\langle \sigma_{10}, \tau_{10} \rangle \cong S_3$, $\pi_{l_{10}} \circ \sigma_{10} = \pi_{l_{10}}$, and $\pi_{l_{10}} \circ \tau_{10} = \pi_{l_{10}}$, the line l_{10} is clearly an S_3-line.

Let $R := (0 : 0 : 0 : 1)$, which is the vertex of the quadric $XY - Z^2 = 0$. The projection $\pi_R \colon C \to (XY - Z^2 = W = 0) \cong \mathbb{P}^1 \subset (W = 0) \cong \mathbb{P}^2$ yields the unique trigonal morphism g_3^1. We have that g_3^1 is cyclic,

$$\rho := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \in \mathrm{Aut}(C),\ \mathrm{ord}(\rho) = 3 = \deg g_3^1, \text{ and } \pi_R \circ \rho = \pi_R.$$

The ramification points of g_3^1 are

$$\begin{aligned} P_1 &:= (1 : -1 : i : 0), & P_2 &:= (1 : -1 : -i : 0), \\ P_3 &:= (1 : -\omega : i\omega^2 : 0), & P_4 &:= (1 : -\omega : -i\omega^2 : 0), \\ P_5 &:= (1 : -\omega^2 : i\omega : 0), & P_6 &:= (1 : -\omega^2 : -i\omega : 0), \end{aligned}$$

where ω (resp. i) is a primitive cubic (resp. fourth) root of the unity. Because $g_3^1 = \Phi_{|3P_j|}$ $(j = 1, \dots, 6)$, we have $\mathrm{Aut}(C)$ acts on $\{P_1, \dots, P_6\}$. Thus, $\sigma(W = 0) = (W = 0)$ for any $\sigma \in \mathrm{Aut}(C) \subset \mathrm{Aut}(\mathbb{P}^3)$.

Because g_3^1 is a unique trigonal morphism, a unique $A_\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ exists for any $\sigma \in \mathrm{Aut}(C)$ such that $g_3^1 \circ \sigma = A_\sigma \circ g_3^1$. We denote the map $\sigma \mapsto A_\sigma$ as $\varphi \colon \mathrm{Aut}(C) \to \mathrm{Aut}(\mathbb{P}^1)$, which is a homomorphism between the groups. Note that $\sigma(W = 0) = (W = 0)$, and g_3^1 is obtained by using the projection $\pi_R \colon (X : Y : Z : W) \mapsto (X : Y : Z)$. By considering $\varphi(\sigma) = A_\sigma$ as an automorphism of the quadric plane curve $(XY - Z^2 = W = 0) \subset (W = 0) \cong \mathbb{P}^2$, we see that φ is expressed as follows:

$$\sigma = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \longmapsto \sigma' = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

Line l	Defining equation of l	G_l	Generators of G_l
l_1	$X = Y = 0$	C_6	$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_2	$X + Y = Z = 0$	C_6	$\sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_3	$X + \omega Y = Z = 0$	C_6	$\sigma_3 = \begin{pmatrix} 0 & \omega & 0 & 0 \\ \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_4	$X + \omega^2 Y = Z = 0$	C_6	$\sigma_4 = \begin{pmatrix} 0 & \omega^2 & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_5	$X - Y = Z = 0$	C_6	$\sigma_5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_6	$X - \omega Y = Z = 0$	C_6	$\sigma_6 = \begin{pmatrix} 0 & -\omega & 0 & 0 \\ -\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_7	$X - \omega^2 Y = Z = 0$	C_6	$\sigma_7 = \begin{pmatrix} 0 & -\omega^2 & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_8	$X = W = 0$	C_6	$\sigma_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
l_9	$Y = W = 0$	C_6	$\sigma_9 = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
ω is a primitive cubic root of the unity.			

Table 1. C_6-lines for the curve defined by Equations (1)

Line l	Defining equation of l	G_l	Generators of G_l
l_{10}	$Z = W = 0$	S_3	$\sigma_{10} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\tau_{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
ω is a primitive cubic root of the unity.			

TABLE 2. S_3-lines for the curve defined by equations (1)

where σ' is regarded as an element of $\mathrm{Aut}(XY - Z^2 = W = 0) \subset \mathrm{Aut}(\mathbb{P}^2)$. Let $\mathrm{Ker}\,\varphi$ and $\mathrm{Im}\,\varphi$ be the kernel and image of φ, respectively. We have the short exact sequence (3) for $\mathrm{Aut}(C)$, and $\mathrm{Ker}\,\varphi = \langle \rho \rangle$.

CLAIM 4.1. We have that $\mathrm{Im}\,\varphi \cong D_6$, which is the dihedral group of order 12.

PROOF. From Proposition 2.9, $\mathrm{Im}\,\varphi$ is isomorphic to C_m, D_m, A_4, S_4, or A_5. Let σ_j $(j = 1, \dots, 10)$ be the automorphism provided in Tables 1 and 2. Because the order of $\varphi(\sigma_8)$ is equal to 6, we see that $\mathrm{Im}\,\varphi \cong C_m$ or D_m, where m is a multiple of 6. Because $\varphi(\sigma_1) \neq \varphi(\sigma_2)$, and the orders of both $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ are equal to 2, we have $\mathrm{Im}\,\varphi \cong D_m$. Note that $\mathrm{Aut}(C)$ acts on the set $\{P_1, \dots, P_6\}$. Let $\sigma \in \mathrm{Aut}(C)$. If $\sigma(P_j) = P_j$ for every P_j $(j = 1, \dots, 6)$, then $\varphi(\sigma)$ is the identity. Thus, the order of $\varphi(\sigma)$ is at most 6. This concludes that $\mathrm{Im}\,\varphi \cong D_6$. ∎

We have an exact sequence $1 \to C_3 \xrightarrow{\psi} \mathrm{Aut}(C) \xrightarrow{\varphi} D_6 \to 1$. The order of $\mathrm{Aut}(C)$ is 36. Let $G := \langle \rho, \sigma_2, \sigma_8 \rangle$.

CLAIM 4.2. We have that $\mathrm{Aut}(C) = G \cong C_3 \times D_6$.

PROOF. Because of the exact sequence $1 \to C_3 \xrightarrow{\psi} G \xrightarrow{\varphi} D_6 \to 1$, $G = \mathrm{Aut}(C)$. We show that there is a left-inverse of ψ. For $\sigma \in G$, we have a unique matrix representation M_σ such that $M_\sigma^*(XY - Z^2) = XY - Z^2$ and the $(4, 4)$-component of M_σ is $1, \omega$, or ω^2. We denote the $(4, 4)$-component of M_σ as λ_σ. Let $\psi' \colon G \to \mathrm{Ker}\,\varphi \cong C_3$ be as follows:

$$\sigma = M_\sigma \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_\sigma \end{pmatrix}.$$

Because ψ' is a homomorphism between groups, and $\psi' \circ \psi = \mathrm{id}$, this concludes that $G \cong C_3 \times D_6$. ∎

The group $\mathrm{Aut}(C) \cong C_3 \times D_6$ has only ten C_6 subgroups:

$$\langle\sigma_1\rangle, \ldots, \langle\sigma_9\rangle, \left\langle \overline{\sigma} := \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

As $\overline{\sigma}$ has no multiple eigenvalues, $\langle\overline{\sigma}\rangle$ is not a Galois group associated with a Galois line. Therefore, the number of C_6-lines is equal to 9. The group $\mathrm{Aut}(C) \cong C_3 \times D_6$ has only six S_3 subgroups: $\langle\sigma_m, \tau_n\rangle$ for $(m, n) = (0, 0), (0, 1), (1, 0), (1, 1), (2, 0)$ and $(2, 1)$, where

$$\sigma_m = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^m \end{pmatrix} \quad \text{and} \quad \tau_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & (-1)^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Proposition 2.8, the lines that might be S_3-lines are $l_8 : X = W = 0$, $l_9 : Y = W = 0$, and $l_{10} : Z = W = 0$. However, l_8 and l_9 are C_6-lines. The line l_{10} is the only one S_3-line.

REMARK 4.3. Let $P' := (1 : 0 : 0 : 0)$, which is the point at which lines l_9 and l_{10} intersect. By the projection $\pi_{P'}: (X : Y : Z : W) \mapsto (Y : Z : W)$ with center P', we have a singular plane curve $T_6 : Y^6 + Z^6 + Y^3W^3 = 0$ as the image $\pi_{P'}(C)$. The points $(0 : 1 : 0) = \pi_{P'}(l_9)$ and $(1 : 0 : 0) = \pi_{P'}(l_{10})$ are outer Galois points for T_6 with Galois groups C_6 and S_3, respectively. The plane curves $T_{2m} : Y^{2m} + Z^{2m} + Y^mW^m = 0$ are examples of curves that are known to have two outer Galois points with Galois groups C_{2m} and D_m (See [5]).

5. Example: Galois lines for the curve defined by equations (2)

In this section, let C be the nonsingular projective curve such that $k(C) = k(x, y)$, and

$$y^6 + x^2(x^2 + 1) = 0. \tag{11}$$

The polynomial on the left-hand side of equation (11) is irreducible. Let $g_6^1: C \to \mathbb{P}^1$ be the cyclic morphism of degree 6 given by the function x. By using the Riemann–Hurwitz formula, we have that the genus of C is equal to 4. Let P_∞, $P_{\infty'}$, P_0, $P_{0'}$,

P_i, P_{-i} be six points on C such that $x(P_\infty) = x(P_{\infty'}) = \infty$, $x(P_0) = x(P_{0'}) = 0$, $x(P_i) = i$, and $x(P_{-i}) = -i$, where i is a primitive fourth root of the unity. Because $(x) = 3P_0 + 3P_{0'} - 3P_\infty - 3P_{\infty'}$, $(y) = P_0 + P_{0'} + P_i + P_{-i} - 2P_\infty - 2P_{\infty'}$, and $(x-i) = 6P_i - 3P_\infty - 3P_{\infty'}$, we have

$$\left(\frac{y^3}{x(x-i)}\right)_\infty = 3P_i \quad \text{and} \quad \left(\frac{y}{x-i}\right)_\infty = 5P_i.$$

Hence, the Weierstrass semigroup of P_i is $H(P_i) = \langle 3, 5\rangle$ (for the definition of Weierstrass semigroup, see [8, Equation (1)]). Thus, C is not hyperelliptic and $K_C \sim 6P_i \sim 3P_\infty + 3P_{\infty'}$. Because $1, y^3/(x(x-i)), y/(x-i), 1/(x-i)$ are linearly independent over k, the morphism $C \ni P \mapsto (x^2(P) : y^3(P) : x(P) : -x(P)y(P)) \in \mathbb{P}^3$ is a canonical embedding. The image of this embedding is expressed as equations (2). We regard C as the canonical curve defined by equations (2).

We can find nine C_6-lines and four S_3-lines, as in Tables 3 and 4. Because $\deg \pi_{l_j} = 6$, $\sigma_j \in \mathrm{Aut}(C)$, $\mathrm{ord}(\sigma_j) = 6$ and $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ $(j = 1, \ldots, 9)$, we see that the lines $l_1, \ldots, l_9$ are C_6-lines. As $\deg \pi_{l_j} = 6$, $\sigma_j, \tau_j \in \mathrm{Aut}(C)$, $\langle \sigma_j, \tau_j\rangle \cong S_3$, $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ and $\pi_{l_j} \circ \tau_j = \pi_{l_j}$ $(j = 10, \ldots, 13)$, we see that the lines $l_{10}, \ldots, l_{13}$ are S_3-lines.

Claim 5.1. We have that $\mathrm{Aut}(C) \cong C_3 \times S_4$.

Proof. We have the following automorphisms of C:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where ω is a primitive cubic root of the unity. The group generated by these four elements, which is a subgroup of $\mathrm{Aut}(C)$, is isomorphic to $C_3 \times S_4$. By considering the short exact sequence (3) for $G = \mathrm{Aut}(C)$, we have that $1 \to C_3 \to \mathrm{Aut}(C) \xrightarrow{\varphi} \mathrm{Im}\,\varphi \to 1$ and $\mathrm{Im}\,\varphi \cong C_m, D_m, A_4, S_4$, or A_5. By using the same argument as in the proof of Lemma 3.8 or Claim 4.1, if $\mathrm{Im}\,\varphi \cong C_m$ or D_m, then $m \le 6$. Because $C_3 \times S_4 \subset \mathrm{Aut}(C)$, we see that $\mathrm{Im}\,\varphi \cong S_4$ and $\mathrm{Aut}(C) \cong C_3 \times S_4$. ■

Because the group $C_3 \times S_4$ contains exactly nine C_6 subgroups and exactly four S_3 subgroups, this concludes that the lines in Tables 3 and 4 are all the C_6-lines and all the S_3-lines, respectively.

Line l	Defining equation of l	G_l	Generators of G_l
l_1	$X = Y = 0$	C_6	$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_2	$Y = Z = 0$	C_6	$\sigma_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_3	$X = Z = 0$	C_6	$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_4	$X + Y = Z = 0$	C_6	$\sigma_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_5	$X - Y = Z = 0$	C_6	$\sigma_5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_6	$X + Z = Y = 0$	C_6	$\sigma_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_7	$X - Z = Y = 0$	C_6	$\sigma_7 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_8	$X = Y + Z = 0$	C_6	$\sigma_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_9	$X = Y - Z = 0$	C_6	$\sigma_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$

ω is a primitive cubic root of the unity.

TABLE 3. C_6-lines for the curve defined by equations (2)

Line l	Defining equation of l	G_l	Generators of G_l
l_{10}	$X+Y+Z=W=0$	S_3	$\sigma_{10}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$, $\tau_{10}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
l_{11}	$X-Y+Z=W=0$	S_3	$\sigma_{11}=\begin{pmatrix}0&-1&0&0\\0&0&-1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$, $\tau_{11}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
l_{12}	$-X+Y+Z=W=0$	S_3	$\sigma_{12}=\begin{pmatrix}0&-1&0&0\\0&0&1&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}$, $\tau_{12}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
l_{13}	$X+Y-Z=W=0$	S_3	$\sigma_{13}=\begin{pmatrix}0&1&0&0\\0&0&-1&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}$, $\tau_{13}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

TABLE 4. S_3-lines for the curve defined by equations (2)

6. Other examples

In this section, we present two examples of canonical curves of genus 4, which have exactly one C_6-line and exactly three C_6-lines, respectively.

Example 6.1. Let $C \subset \mathbb{P}^3$ be the curve defined by

$$\begin{cases} Q := YZ - W^2 = 0, \\ F := X^3 - X^2Y - XY^2 + Z^3 = 0. \end{cases}$$

Then, C is a canonical curve of genus 4. The line $l : X = Y = 0$ is a C_6-line. Indeed,

$$\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & -\omega^2 \end{pmatrix}$$

(where ω is a primitive cubic root of the unity) satisfies $\sigma \in \mathrm{Aut}(C)$, $\pi_l \circ \sigma = \pi_l$, and $\mathrm{ord}(\sigma) = 6 = \deg \pi_l$. Because rank $Q = 3$, the trigonal morphism g_3^1 is unique, and g_3^1 is obtained by the projection π_R with center $R := (1 : 0 : 0 : 0)$, which is the vertex of $Q = 0$. Because $\pi_R^{-1}((1 : 1 : 1))$ consists of only two points $(1 : 1 : 1 : 1)$ and $(-1 : 1 : 1 : 1)$, we see that g_3^1 is not Galois. From Proposition 3.2, the number of C_6-lines equals one.

Example 6.2. Let $C \subset \mathbb{P}^3$ be the curve defined by

$$\begin{cases} Q := XY - Z^2 = 0, \\ F := X^3 + Y^3 + Z^3 + W^3 = 0. \end{cases}$$

Then, C is a canonical curve of genus 4. The lines $l_1 : X + Y = Z = 0$, $l_2 : X + \omega Y = Z = 0$, and $l_3 : X + \omega^2 Y = Z = 0$ are C_6-lines of type $(2, 3)$. Indeed,

$$\sigma_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & \omega & 0 & 0 \\ \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \text{ and } \sigma_3 := \begin{pmatrix} 0 & \omega^2 & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

(where ω is a primitive cubic root of the unity) satisfy $\sigma_j \in \mathrm{Aut}(C)$, $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ and $\mathrm{ord}(\sigma_j) = 6 = \deg \pi_{l_j}$ $(j = 1, 2, 3)$.

We show that all the C_6-lines for C of types $(2,3)$ or $(2,6)$ are the three lines l_1, l_2 and l_3. Here, we explain how to find the C_6-lines of types $(2,3)$ or $(2,6)$. Let

$$P_1 := (1 : \zeta^2 : \zeta : 0), \qquad P_2 := (1 : \zeta^4 : \zeta^2 : 0),$$
$$P_3 := (1 : \zeta^8 : \zeta^4 : 0), \qquad P_4 := (1 : \zeta : \zeta^5 : 0),$$
$$P_5 := (1 : \zeta^5 : \zeta^7 : 0), \qquad P_6 := (1 : \zeta^7 : \zeta^8 : 0),$$

where ζ is a primitive ninth root of the unity. Because rank $Q = 3$, from Proposition 2.1, there exists a unique trigonal morphism $g^1_3 : C \to \mathbb{P}^1$. From Proposition 3.1 (or 3.2), g^1_3 is cyclic. Points $P_1, \dots, P_6$ are all the ramification points of g^1_3. Let $H(3P_m + 3P_n) \subset \mathbb{P}^3$ (resp. $H(6P_m) \subset \mathbb{P}^3$) ($P_m, P_n \in \{P_1, \dots, P_6\}$) be the hyperplane that defines the divisor $3P_m + 3P_n$ (resp. $6P_m$) on C. Let l be a C_6-line of type $(2,3)$ or $(2,6)$. From Proposition 3.1, the projection $\pi_l : C \to \mathbb{P}^1$ is the composition of $g^1_3 : C \to \mathbb{P}^1$ and some morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2. Thus, $P_1, \dots, P_6$ are ramification points of π_l. At least two fibers of π_l are formed as $3P_m + 3P_n$, where $P_m \neq P_n$ and $P_m, P_n \in \{P_1, \dots, P_6\}$. In other words, there exist four mutually distinct points $P_{m_1}, P_{m_2}, P_{m_3}, P_{m_4} \in \{P_1, \dots, P_6\}$ such that $H(3P_{m_1} + 3P_{m_2}) \cap H(3P_{m_3} + 3P_{m_4}) = l$. Moreover, we have $l \subset H(3P_{m_5} + 3P_{m_6})$ or $l \subset H(6P_{m_5}) \cap H(6P_{m_6})$, where $\{P_{m_1}, \dots, P_{m_6}\} = \{P_1, \dots, P_6\}$. By using this fact, we search for lines that might be C_6-lines of types $(2,3)$ or $(2,6)$.

For example, let $l_{1234} \subset \mathbb{P}^3$ be the line $H(3P_1 + 3P_2) \cap H(3P_3 + 3P_4)$. Because $H(3P_1 + 3P_2)$ and $H(3P_3 + 3P_4)$ are defined by $\zeta^3 X + Y - (\zeta + \zeta^2)Z = 0$ and $X + Y - (\zeta^4 + \zeta^5)Z = 0$, respectively, we have

$$R_{1234} := (-\zeta(1+\zeta) : \zeta(1+\zeta)(1+\zeta^3) : 1 : 0) \in l_{1234}.$$

The hyperplanes $H(3P_5 + 3P_6)$, $H(6P_5)$, and $H(6P_6)$ are defined by $\zeta^6 X + Y - (\zeta^7 + \zeta^8)Z = 0$, $\zeta^5 X + Y - 2\zeta^7 Z = 0$, and $\zeta^7 X + Y - 2\zeta^8 Z = 0$, respectively. We see that $R_{1234} \notin H(3P_5 + 3P_6)$, $R_{1234} \notin H(6P_5)$, and $R_{1234} \notin H(6P_6)$. Thus, $l_{1234} \not\subset H(3P_5 + 3P_6)$, $l_{1234} \not\subset H(6P_5)$, and $l_{1234} \not\subset H(6P_6)$. This concludes that l_{1234} is not a C_6-line of type $(2,3)$ or $(2,6)$. By using the same argument as above and computer calculations, we check whether $l_{m_1 m_2 m_3 m_4}$ can be a C_6-line of type $(2,3)$ or $(2,6)$ for every line $l_{m_1 m_2 m_3 m_4} := H(3P_{m_1} + 3P_{m_2}) \cap H(3P_{m_3} + 3P_{m_4})$. Then, we see that only three lines l_{1236}, l_{1423}, l_{1625} might be C_6-lines of types $(2,3)$ or $(2,6)$, which are C_6-lines l_3, l_2, l_1, respectively.

According to Sections 4 and 5, seven C_6-lines of types $(2,3)$ or $(2,6)$ exist for the curve defined by equations (1), and nine C_6-lines of types $(2,3)$ or $(2,6)$ exist for the curve defined by equations (2). Thus, C is not projectively equivalent to the curves defined by equations (1) or (2). From our main theorem, we see that all the C_6-lines for C are the three lines l_1, l_2, and l_3.

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