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Fibrantly generated weak factorization systems

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Abstract – We prove that, assuming Vopěnka's principle, every small projectivity class in an accessible category is accessible. This conclusion is not provable in ZFC alone, and in fact carries large cardinal strength.

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1. Introduction

Injectivity in locally presentable categories $\mathcal{K}$ is well understood (see [3]). For instance, every small injectivity class is accessible and accessibly embedded to $\mathcal{K}$. Closely related to small injectivity classes are cofibrantly generated weak factorization systems which permeate abstract homotopy theory (see, e.g., [14]). Much less is known about the dual concept of projectivity. Similarly for the related concept of a fibrantly generated weak factorization system. Our main result is that, assuming Vopěnka's principle, every small projectivity class in a locally presentable category is accessible and accessibly embedded. On the other hand, assuming $\mathrm{V} = \mathrm{L}$, free abelian groups form a small projectivity class which is not accessible.

We provide two proofs of our main result – while one uses the concept of purity in accessible categories and is valid in every accessible category $\mathcal{K}$ with pushouts, the

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other is more set-theoretical and applies to every accessible category. We also show that our main result is equivalent to the fact that every object of a locally presentable category (in fact of every accessible category with pushouts) is μ-pure injective for some regular cardinal μ. Finally, we explain the relation of fibrantly generated weak factorization systems to cotorsion theories that are generated by sets.

All needed facts about locally presentable and accessible categories can be found in [3].

2. Fibrant generation

Let $\mathcal{K}$ be a category and $f: A \to B$, $g: C \to D$ morphisms such that in each commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

there is a diagonal $d: B \to C$ with $df = u$ and $gd = v$. One says that g has the *right lifting property* w.r.t. f and that f has the *left lifting property* w.r.t. g. We write $f \,\square\, g$. For a class $\mathcal{X}$ of morphisms of $\mathcal{K}$ we put

$$\mathcal{X}^{\square} = \{g \mid g \text{ has the right lifting property w.r.t. each } f \in \mathcal{X}\}$$

and

$$^{\square}\mathcal{X} = \{f \mid f \text{ has the left lifting property w.r.t. each } g \in \mathcal{X}\}.$$

We will consider pairs $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in a locally presentable category $\mathcal{K}$ such that $\mathcal{R} = \mathcal{L}^{\square}$ and $\mathcal{L} = {}^{\square}\mathcal{R}$. We will call these pairs *saturated*. A saturated pair is called *cofibrantly generated* if $\mathcal{R} = \mathcal{X}^{\square}$ for a set $\mathcal{X}$ and *fibrantly generated* if $\mathcal{L} = {}^{\square}\mathcal{Y}$ for a set $\mathcal{Y}$.

A saturated pair $(\mathcal{L}, \mathcal{R})$ is called a *weak factorization system* if every morphism of $\mathcal{K}$ factorizes as an $\mathcal{L}$-morphism followed by an $\mathcal{R}$-morphism. Every cofibrantly generated saturated pair in a locally presentable category is a weak factorization system [6].

Problem 2.1. Is every fibrantly generated pair $(\mathcal{L}, \mathcal{R})$ a weak factorization system?

In what follows, 0 will denote an initial object of $\mathcal{K}$ and 1 a terminal one. An object K is *$\mathcal{L}$-injective* if $K \to 1$ is in $\mathcal{L}^{\square}$ and *$\mathcal{R}$-projective* if $0 \to K$ is in ${}^{\square}\mathcal{R}$. $\mathcal{L}$-Inj or $\mathcal{R}$-Proj denote the full subcategories of $\mathcal{K}$ consisting of $\mathcal{L}$-injectives or $\mathcal{R}$-projectives, respectively. A *small projectivity class* is a class of the form $\mathcal{R}$-Proj, where $\mathcal{R}$ is a set. A class of morphisms $\mathcal{L}$ is *left-cancellable* if $gf \in \mathcal{L}$ implies that $f \in \mathcal{L}$. In a category with products, an *$\mathcal{L}$-cogenerator* is a set $\mathcal{G}$ of objects such that, for every object K, the canonical morphism

$$\gamma_K \colon K \longrightarrow K^* = \prod_{C \in \mathcal{G}} C^{\mathcal{K}(K,C)}$$

is in $\mathcal{L}$ (see [1]).

Proposition 2.2. *Let $\mathcal{L}$ be left-cancellable and assume that $\mathcal{K}$ has products and an $\mathcal{L}$-injective $\mathcal{L}$-cogenerator. Then $(\mathcal{L}, \mathcal{L}^{\square})$ is a fibrantly generated weak factorization system.*

Proof. Following [2, Proposition 1.6], $(\mathcal{L}, \mathcal{L}^{\square})$ is a weak factorization system. Let $\mathcal{Y} = \{C \to 1 \mid C \in \mathcal{G}\}$ where $\mathcal{G}$ is an $\mathcal{L}$-injective $\mathcal{L}$-cogenerator. Clearly, $\mathcal{Y} \subseteq \mathcal{L}^{\square}$, hence $\mathcal{L} \subseteq {}^{\square}\mathcal{Y}$. On the other hand, let $f \colon K \to L$ be in ${}^{\square}\mathcal{Y}$. Since K^* is ${}^{\square}\mathcal{Y}$-injective, there is $g \colon L \to K^*$ such that $gf = \gamma_K$. Since $\mathcal{L}$ is left-cancellable, $f \in \mathcal{L}$. ■

Example 2.3. (1) Let $\mathcal{L}$ consist of embeddings in the category **Pos** of posets. Then $\mathcal{L}$-injectives are complete lattices and a two-element chain 2 is an $\mathcal{L}$-injective $\mathcal{L}$-cogenerator. In fact, isotone maps $K \to 2$ correspond to down-sets in K. If a, b are incomparable elements of K then there exists a down-set $Z \subseteq K$ such that $a \in Z$ and $a \notin Z$. Hence γ_K is an embedding. Since $\mathcal{L}$ is left-cancellable, following Proposition 2.2, $(\mathcal{L}, \mathcal{L}^{\square})$ is a weak factorization system fibrantly generated by $2 \to 1$.

(2) In every category with an injective cogenerator the class Mono of monomorphisms forms a fibrantly generated weak factorization system $(\mathrm{Mono}, \mathrm{Mono}^{\square})$. This includes every Grothendieck topos and every Grothendieck abelian category. In particular, the category R-**Mod** of R-modules.

(3) Let **Ban** be the category of Banach spaces and linear maps of norm ≤ 1. Let $p \colon \mathbb{C} \to 1$. Following the Hahn–Banach theorem, $\mathcal{L} = {}^{\square}p$ is the class of linear isometries. $\mathcal{L}$-injective Banach spaces are precisely Banach spaces $C(X)$ where X is an extremally disconnected compact Hausdorff space and **Ban** has enough $\mathcal{L}$-injectives (see [9]). In particular, $\mathbb{C}$ is $\mathcal{L}$-injective. On the other hand, if $f \colon A \to B$ is not an isometry, witnessed by a vector x of norm 1 (i.e. $\|fx\| < 1$), take $g \colon A \to \mathbb{C}$ of norm 1, such that $|gx| = 1$ (by the Hahn–Banach theorem). Supposing $g = hf$, we get

$$1 = |gx| = |hfx| \leq \|h\| \cdot \|fx\| < \|h\|,$$

which is a contradiction. Hence $\mathbb{C}$ is an $\mathcal{L}$-injective $\mathcal{L}$-cogenerator. Since $\mathcal{L}$ is left cancellable, $(\mathcal{L}, \mathcal{L}^{\square})$ is a fibrantly generated weak factorization system.

Remark 2.4. Whenever $\mathcal{L} = {}^{\square}\mathcal{Y}$ for a set $\mathcal{Y}$ consisting of morphisms of the form $C \to 1$, then $\mathcal{L}$ is left-cancellable.

Remark 2.5. Let $(\mathcal{L}, \mathcal{R})$ be a saturated pair in a locally presentable category $\mathcal{K}$ cofibrantly generated by $\mathcal{X} \subseteq \mathcal{L}$. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorization system and $\mathcal{L}$ consists of retracts of *cellular* morphisms, i.e., transfinite compositions of pushouts of morphisms from $\mathcal{X}$. This is a consequence of a *small object argument* (see [6]).

One cannot expect this for fibrantly generated saturated pairs $(\mathcal{L}, \mathcal{R})$ in a locally presentable category. But, if $(\mathcal{L}, \mathcal{R})$ is a weak factorization system fibrantly generated by $\mathcal{Y}$, then $\mathcal{R}$ consists of retracts of transfinite cocompositions of pullbacks of elements of $\mathcal{Y}$; the authors of [5] call the latter *$\mathcal{Y}$-Postnikov towers*. They show in [5, Subsection 3.1] that the model category of non-negatively graded chain complexes of vector spaces is fibrantly generated.

A *cotorsion theory* is a pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules such that

$$\mathcal{C} = \mathcal{F}^{\perp} = \{C \mid \mathrm{Ext}^1(F, C) = 0 \text{ for all } F \in \mathcal{F}\}$$

and

$$\mathcal{F} = {}^{\perp}\mathcal{C} = \{F \mid \mathrm{Ext}^1(F, C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is *generated by a set* if there is a set $\mathcal{X}$ such that $\mathcal{F} = {}^{\perp}\mathcal{X}$, and if this holds, we say $\mathcal{X}$ *generates the cotorsion theory* $(\mathcal{F}, \mathcal{C})$. It is called *cogenerated by a set* if there is a set $\mathcal{Y}$ such that $\mathcal{C} = \mathcal{Y}^{\perp}$, and if this holds, we say that $\mathcal{Y}$ *cogenerates the cotorsion theory* $(\mathcal{F}, \mathcal{C})$.[1]

$\mathcal{F}$-monomorphisms are monomorphisms whose cokernel is in $\mathcal{F}$ and *$\mathcal{C}$-epimorphisms* are epimorphisms whose kernel is in $\mathcal{C}$.

Remark 2.6. For a cotorsion theory $(\mathcal{F}, \mathcal{C})$, ($\mathcal{F}$-Mono, $\mathcal{C}$-Epi) is a saturated pair ([16, Proposition 3.1]). Moreover, $\mathcal{C} =$ ($\mathcal{F}$-Mono)-Inj and $\mathcal{F} =$ ($\mathcal{C}$-Epi)-Proj. If a cotorsion theory $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set, then ($\mathcal{F}$-Mono, $\mathcal{C}$-Epi) is a cofibrantly generated weak factorization system ([16, Remark 3.2]).

Proposition 2.7. *A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is generated by a set if and only if the saturated pair* ($\mathcal{F}$-Mono, $\mathcal{C}$-Epi) *is fibrantly generated.*

Proof. Let $(\mathcal{F}, \mathcal{C})$ be generated by a set $\mathcal{Y}$. Following [19, Lemma 2.1], there is a set $\mathcal{Z}_0$ of $\mathcal{Y}$-epimorphisms such that $\mathcal{F} = \mathcal{Z}_0$-Proj. Let $\mathcal{Z} = \mathcal{Z}_0 \cup \{p\}$, where

(1) The definitions of "generated by a set" and "cogenerated by a set" are sometimes reversed in the literature.

$p\colon C \to 1$ for an injective cogenerator C. Following Example 2.3(2), ${}^{\square}\mathcal{Z} \subseteq$ Mono. Following the dual of part I of the proof of [15, Lemma 4.4], ${}^{\square}\mathcal{Z} = \mathcal{F}$-Mono.

Conversely, let ($\mathcal{F}$-Mono, $\mathcal{C}$-Epi) be fibrantly generated by $\mathcal{Y}$. Let $\mathcal{Z}$ consist of kernels C of epimorphisms $f\colon A \to B$ from $\mathcal{Y}$. Assume that $\mathrm{Ext}^1(X, C) = 0$ for every $C \in \mathcal{Z}$. Consider the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(X, C) \longrightarrow \mathrm{Hom}(X, A) \longrightarrow \mathrm{Hom}(X, B) \longrightarrow \mathrm{Ext}^1(X, C) \longrightarrow \cdots$$

induced by f. Since $\mathrm{Ext}^1(X, C) = 0$, $\mathrm{Hom}(X, f)\colon \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$ is surjective. Hence $X \in \mathcal{F}$. We have proved that $\mathcal{Z}$ generates $(\mathcal{F}, \mathcal{C})$. ■

Remark 2.8. Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion theory generated by a set such that $\mathcal{F}$ is either closed under pure submodules or $\mathcal{C}$ consists of modules of finite injective dimension and the pair is hereditary. Then, assuming V = L, $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set (see [18, Theorems 1.3 and 1.7]). Following Proposition 2.7 and Remark 2.6, the fibrantly generated saturated pair ($\mathcal{F}$-Mono, $\mathcal{C}$-Epi) is also cofibrantly generated. Hence, it is a weak factorization system.

3. Accessibility

Theorem 3.1. *Assuming Vopěnka's principle, every object in a locally presentable category is μ-pure injective for some regular cardinal μ.*

Proof. Let $\mathcal{K}$ be locally λ-presentable. Suppose K is not μ-pure injective for any regular cardinal μ. Hence, for every regular cardinal $\mu \geq \lambda$, there is a μ-pure monomorphism $f_\mu\colon K \to L_\mu$ which does not split. Indeed, if every μ-pure monomorphism $f\colon K \to L$ splits, then K is μ-pure injective because, given a μ-pure monomorphism $h\colon A \to B$ and $g\colon A \to K$ then a pushout

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\ K & \xrightarrow[\ \overline{h}\ ]{} & P \end{array}$$

yields a μ-pure monomorphism $\overline{h}$ (see [4, Proposition 15]). Since $\overline{h}$ splits, there is $s\colon P \to K$ such that $s\overline{h} = \mathrm{id}_K$. Then $s\overline{g}h = s\overline{h}g = g$.

If a morphism $f\colon K \to L$ is a μ-pure monomorphism for all μ then f is a split monomorphism. Thus there is a sequence of regular cardinals

$$\mu_0 < \mu_1 < \cdots \mu_i < \cdots$$

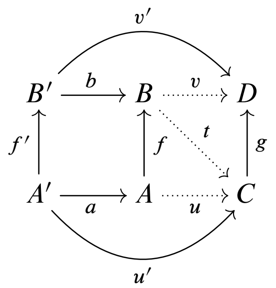
indexed by ordinals and μ_i-pure monomorphisms $f_i\colon K \to L_i$ such that f_i is not μ_j-pure for all $j > i$. Thus, there is no morphism $h\colon f_i \to f_j$, $i < j$ in $K \downarrow \mathcal{K}$. Indeed, having $h\colon L_i \to L_j$ with $hf_i = f_j$, then f_i is μ_i-pure (see [3, Remarks 2.28]). It contradicts Vopěnka's principle (see [3, Lemma 6.3]). ∎

Example 3.2. (1) Assume that there is a regular cardinal μ such that every abelian group is μ-pure injective. Consider a μ-pure epimorphism $f\colon A \to B$ in **Ab** and its kernel $g\colon C \to A$. Then g is a μ-pure monomorphism and, since C is μ-pure injective, g splits. Thus, f splits, which contradicts [8, Lemma 5.10].

(2) Assume $0^{\#}$ does not exist and let R be a ring which is not right perfect. Then the R-module $R^{(\omega)}$ is not μ-pure injective for any regular cardinal μ (see the proof of [19, Proposition 1.5]).

Lemma 3.3. *Let $\mathcal{K}$ be a locally presentable category, μ be a regular cardinal, and $\mathcal{Y}$ be a class of morphisms which are μ-pure injective in $\mathcal{K}^{\to}$. Then ${}^{\square}\mathcal{Y}$ is closed under μ-pure subobjects in $\mathcal{K}^{\to}$.*

Proof. Let $f\colon A \to B$ be in ${}^{\square}\mathcal{Y}$ and $(a, b)\colon f' \to f$ be a μ-pure monomorphism where $f'\colon A' \to B'$. Consider $g\colon C \to D$ in $\mathcal{Y}$ and $(u', v')\colon f' \to g$. Since g is μ-pure injective in $\mathcal{K}^{\to}$, there is $(u, v)\colon f \to g$ in $\mathcal{K}^{\to}$ such that $(u, v)(a, b) = (u', v')$. Thus there is $t\colon B \to C$ such that $tf = u$ and $gt = v$.



Hence $tbf' = tfa = ua = u'$ and $gtb = vb = v'$. Thus, $f' \in {}^{\square}\mathcal{Y}$. ∎

Theorem 3.4. *Let $(\mathcal{L}, \mathcal{R})$ be a fibrantly generated saturated pair in a locally presentable category $\mathcal{K}$. Then, assuming Vopěnka's principle, $\mathcal{L}$ is an accessible and accessibly embedded subcategory of $\mathcal{K}^{\to}$.*

Proof. Let $(\mathcal{L}, \mathcal{R})$ be a saturated pair fibrantly generated by a set $\mathcal{Y}$ and assume Vopěnka's principle. Since $\mathcal{Y}$ is a set, Theorem 3.1 implies there is a regular cardinal μ such that all members of $\mathcal{Y}$ are μ-pure injective in $\mathcal{K}^{\to}$. Then Lemma 3.3 implies that $\mathcal{L}$ is closed under μ-pure subobjects. Hence $\mathcal{L}$ is accessible and accessibly embedded to $\mathcal{K}^{\to}$ (see [3, Theorem 6.17]). ∎

Corollary 3.5. *Assuming Vopěnka's principle, every small projectivity class $\mathcal{P}$ in a locally presentable category $\mathcal{K}$ is accessible and accessibly embedded to $\mathcal{K}$.*

Proof. Let $\mathcal{P} = \mathcal{S}$-Proj for a set $\mathcal{S}$. Following Theorem 3.4, ${}^{\square}\mathcal{S}$ is accessible and accessibly embedded. Hence the same holds for $\mathcal{P}$. ■

Remark 3.6. (1) On the other hand, Theorem 3.4 follows from Corollary 3.5. Indeed, $f \,\square\, g$ where $g\colon C \to D$ iff f is projective to $(\mathrm{id}_C, g)\colon \mathrm{id}_C \to g$.

(2) Moreover, Theorem 3.1 follows from Theorem 3.4. Indeed, let $\mathcal{K}$ be locally presentable and K be in $\mathcal{K}$. Consider $\mathcal{L} = {}^{\square}t$ where $t\colon K \to 1$. Following Theorem 3.4, there is a regular cardinal μ such that $\mathcal{L}$ is μ-accessible and closed under μ-directed colimits in $\mathcal{K}^{\to}$. Since $\mathcal{L}$ contains split monomorphisms, it contains μ-pure monomorphisms (see [3, Proposition 2.30]). Thus, K is μ-pure injective.

Example 3.7. (1) Let $(\mathcal{F}, \mathcal{I})$ be the cotorsion theory in **Ab** generated by $\mathbb{Z}$, i.e., $\mathcal{F}$ is the class of Whitehead groups. Following [19, Lemma 2.1] it is a small projectivity class. Assuming V = L, $\mathcal{F}$ is the class of free groups and it is not accessible (see [12, Chapter VII]).

(2) It follows from Remark 3.6 (2) and Example 3.2 (2) that Theorem 3.4 needs large cardinals. Note that V = L implies that $0^{\#}$ does not exist.

Theorem 3.8. *Let $(\mathcal{L}, \mathcal{R})$ be a cofibrantly generated weak factorization system in a locally presentable category $\mathcal{K}$. Suppose $(\mathcal{L}, \mathcal{R})$ is also fibrantly generated by a set $\mathcal{Y}$ of morphisms, such that each member of $\mathcal{Y}$ is μ-pure injective in $\mathcal{K}^{\to}$ for some regular cardinal μ. Then, assuming the existence of a proper class of almost strongly compact cardinals, $\mathcal{L}$ is accessible and accessibly embedded to $\mathcal{K}^{\to}$.*

Proof. Following [17, Corollary 3.3], $\mathcal{L}$ is a full image of an accessible functor. Moreover, it is closed under μ-pure subobjects in $\mathcal{K}^{\to}$ for some regular cardinal μ (see Lemma 3.3). The result follows from [7, Theorem 3.2]. ■

Remark 3.9. A weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ is *accessible* if the factorization functor $\mathcal{K}^{\to} \to \mathcal{K}^{\to\to}$ is accessible. Every cofibrantly generated weak factoriztion system in a locally presentable category is accessible. Since [17, Corollary 3.3] is valid even for accessible weak factorization systems, in Theorem 3.8, the weak factorization system $(\mathcal{L}, \mathcal{R})$ could only be accessible instead of cofibrantly generated.

Remark 3.10. All results of this section are valid in every accessible category $\mathcal{K}$ with pushouts. It suffices to do this in Theorem 3.1, which follows from [3, Remark 2.30].

4. Another proof of Theorem 3.4

We sketch our original proof of Theorem 3.4. While this version is less succinct than the proof given above, it may be more accessible to logicians. Moreover, in contrast to Remark 3.10, it does not need pushouts. All set-theoretic terminology used here agrees with [13].

Assume Vopěnka's principle and that $\mathcal{K}$ is an accessible category. By Remark 3.6 (1) above and [3, Corollary 6.10], it suffices to show that every small projectivity class is accessible.

By [3, Theorem 5.35], $\mathcal{K}$ is equivalent to the category of models of some basic L_μ-theory T for some regular cardinal μ. From now on we will identify $\mathcal{K}$ with this category of models. By routine induction on formula complexity, it can be seen that *if* $\mathfrak{N} = (N, \in)$ is a Σ_1-elementary substructure of the universe of sets, T and its signature are both elements and subsets of $\mathfrak{N}$, and $\mathfrak{N}$ happens to be closed under sequences of length μ, *then* whenever K is a model of T and $K \in \mathfrak{N}$, it makes sense to form the restriction $K \restriction \mathfrak{N}$ (with underlying set $K \cap N$), and this restriction is also model of T. Similarly, if $f : L \to K$ is a morphism in $\mathfrak{N}$ (i.e., $f : L \to K$ is a morphism in $\mathcal{K}$ and f, L, and K are all elements of $\mathfrak{N}$), then its restriction $f \restriction \mathfrak{N} : L \restriction \mathfrak{N} \to K \restriction \mathfrak{N}$ is a morphism in the category.

Suppose $\mathcal{S}$ is a set of morphisms in $\mathcal{K}$ and $\mathcal{P} = \mathcal{S}$-Proj. By [10, Corollary A.2], there is an inaccessible cardinal $\kappa > \mu$ with the following property. For every set b there is an $\mathfrak{N} = (N, \in)$ such that:

(1) $\mathfrak{N}$ is a Σ_1-elementary substructure of the universe of sets, T and its signature are both elements and subsets of $\mathfrak{N}$, and $\mathfrak{N}$ is closed under μ-sequences (hence the comments above regarding restrictions to $\mathfrak{N}$ are applicable). [2]

(2) $b \in \mathfrak{N}$ and $|\mathfrak{N}| < \kappa$.

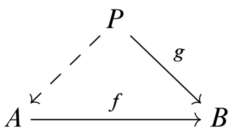
(3) $|\bigcup \mathcal{S}| < \kappa$ and $\mathfrak{N} \cap \kappa$ is transitive. This implies that $s \restriction \mathfrak{N} = s$ for every $s \in \mathfrak{N} \cap \mathcal{S}$.

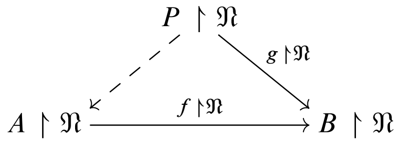
(4) ($\mathfrak{N}$ reflects membership in $\mathcal{P}$) For every $K \in \mathfrak{N}$, $K \in \mathcal{P}$ if and only if $K \restriction \mathfrak{N} \in \mathcal{P}$.

(5) ($\mathfrak{N}$ reflects existence of fill-ins) Whenever $g : P \to B$ and $f : A \to B$ are morphisms in $\mathfrak{N}$, there is a fill-in for one of the following diagrams if and only if there is a

([2]) The statement of [10, Corollary A.2] does not include closure of $\mathfrak{N}$ under μ sequences, but the proof there easily arranges such closure. Namely, in the proof of Corollary A.1, if κ is chosen larger than μ, and the λ_0 is then chosen to be of cofinality $> \mu$, then H_{λ_0} is closed under μ sequences (and hence so is the $j[H_{\lambda_0}]$ from that proof, since the critical point of j is larger than μ).

fill-in for the other:





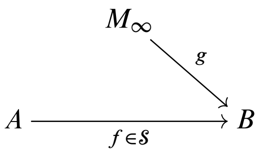
We first claim that every element of $\mathcal{P}$ is a κ-directed colimit of $< \kappa$-sized members of $\mathcal{P}$. This has nothing to do with projectivity classes, but simply uses that $\mathfrak{N}$ reflects membership in the class $\mathcal{P}$. Suppose $P \in \mathcal{P}$, and let θ be a regular cardinal such that $P \in H_\theta$, where H_θ denotes the collection of sets of hereditary cardinality less than θ. Let U denote the set of $\mathfrak{N} \in P_\kappa(H_\theta)$ such that $P \in \mathfrak{N}$, and $\mathfrak{N}$ has the properties listed above (here P is playing the role of the b in the list of properties). In particular, each $\mathfrak{N} \in U$ reflects membership in $\mathcal{P}$, so $P \restriction \mathfrak{N} \in \mathcal{P}$ for every $\mathfrak{N} \in U$. By the assumptions on κ, U is a stationary subset of $P_\kappa(H_\theta)$. It follows that $\{P \restriction \mathfrak{N} \mid \mathfrak{N} \in U\}$ is a κ-directed collection (under inclusion) of members of $\mathcal{P}$, each of size $< \kappa$, with union P.

Finally, we show that $\mathcal{P}$ is closed under κ-directed colimits. Suppose

$$\mathcal{D} = \langle \pi_{i,j} \colon P_i \longrightarrow P_j \mid i \leq j \in I \rangle$$

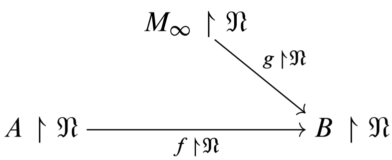
is a κ-directed system of members of $\mathcal{P}$. Since κ is inaccessible, it is sharply stronger than μ, and hence $\mathcal{K}$ is closed under κ-directed colimits. So, $\mathcal{D}$ has a colimit in $\mathcal{K}$, which will be denoted M_∞.

Suppose, toward a contradiction, that $M_\infty \notin \mathcal{P} = \mathcal{S}$-Proj, as witnessed by some diagram



for which there is no completion from M_∞ into A. Then there is a Σ_1-elementary substructure $\mathfrak{N}$ of the universe with the properties listed above such that $\mathcal{D}$, f, and g are elements of $\mathfrak{N}$ (here the ordered tuple $(\mathcal{D}, f, g)$ is playing the role of the b). By property (5), the diagram

(4.1)



has no completion from $M_\infty \restriction \mathfrak{N}$ into $A \restriction \mathfrak{N}$. By property (3), the bottom row of diagram (4.1) is simply the map $f\colon A \to B$. In summary, the diagram

(4.2)
$$\begin{array}{ccc} & M_\infty \restriction \mathfrak{N} & \\ & & \searrow^{g \restriction \mathfrak{N}} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

has no completion from $M_\infty \restriction \mathfrak{N}$ into A.

On the other hand, since $|\mathfrak{N}| < \kappa$ and I is κ-directed, there is an $i^* \in I$ above all members of $\mathfrak{N} \cap I$. Consider the colimit map

$$\pi_{i^*,\infty}\colon P_{i^*} \longrightarrow M_\infty.$$

Next we show there is a morphism

$$e\colon M_\infty \restriction \mathfrak{N} \longrightarrow P_{i^*}$$

such that

(4.3) $\pi_{i^*,\infty} e$ is the inclusion map from $M_\infty \restriction \mathfrak{N}$ into M_∞.

The map e is defined as follows: by elementarity of $\mathfrak{N}$, any member of $M_\infty \restriction \mathfrak{N}$ is of the form $\pi_{k,\infty}(x_k)$ for some $k \in I$ and some $x_k \in P_k$, where both k and x_k are *elements* of $\mathfrak{N}$. Then define $e(\pi_{k,\infty}(x_k)) := \pi_{k,i^*}(x_k)$. It is routine to verify this does not depend on the choice of $k \in I \cap \mathfrak{N}$ or of $x_k \in P_k \cap \mathfrak{N}$; see [11] for a similar argument (using directedness of I and elementarity of $\mathfrak{N}$). And clearly (4.3) holds, since

$$\pi_{i^*,\infty}(e(\pi_{k,\infty}(x_k))) = \pi_{i^*,\infty}(\pi_{k,i^*}(x_k)) = \pi_{k,\infty}(x_k).$$

To see that e is a morphism, suppose $\rho < \mu$, $\boldsymbol{z} = \langle z_\xi \mid \xi < \rho \rangle$ is a sequence of members of $M_\infty \restriction \mathfrak{N}$, and $\dot{h}$ is a ρ-ary function symbol in the signature. Note that $\dot{h} \in \mathfrak{N}$ because the signature is contained (as a subset) in $\mathfrak{N}$, and the sequence $\boldsymbol{z}$ is an element of $\mathfrak{N}$ by the μ-closure of $\mathfrak{N}$. It follows by Σ_1-elementarity of $\mathfrak{N}$ in the universe, and κ-directedness of I, that there is some $k^* \in I \cap \mathfrak{N}$ and some sequence $\boldsymbol{z}^* = \langle z^*_\xi \mid \xi < \rho \rangle$ of elements of P_{k^*} such that $\pi_{k^*,\infty}(z^*_\xi) = z_\xi$ for each $\xi < \rho$, the entire sequence $\boldsymbol{z}^*$ is an element of $\mathfrak{N}$, and each z^*_ξ is an element of $\mathfrak{N}$. It follows that $e(z_\xi) = \pi_{k^*,i^*}(z^*_\xi)$ for each ξ, and that $h^{P_{k^*}}(\boldsymbol{z}^*)$ is an element of $\mathfrak{N}$. Then

$$\begin{aligned} e(h^{M_\infty}(\boldsymbol{z})) &= e(\pi_{k^*,\infty}(\overbrace{h^{P_{k^*}}(\boldsymbol{z}^*)}^{\in \mathfrak{N} \cap P_{k^*}})) = \pi_{k^*,i^*}(h^{P_{k^*}}(\boldsymbol{z}^*)) \\ &= h^{P_{i^*}}(\langle \underbrace{\pi_{k^*,i^*}(z^*_\xi)}_{e(z_\xi)} \mid \xi < \rho \rangle), \end{aligned}$$

where the second equality is by the definition of e.

Consider the diagram:

$$\begin{array}{ccccc} M_\infty \upharpoonright \mathfrak{N} & \xrightarrow{e} & P_{i^*} & \xrightarrow{\pi_{i^*,\infty}} & M_\infty \\ & & & & \downarrow{\scriptstyle g} \\ & A & \xrightarrow[f]{\qquad\qquad} & & B \end{array}$$

Since $f \in \mathcal{S}$ and $P_{i^*} \in \mathcal{S}$-Proj, there exists a morphism $\tau\colon P_{i^*} \to A$ such that

$$f\tau = g\pi_{i^*,\infty}.$$

Set $\tau' := \tau e$, which yields the following commutative diagram:

$$\begin{array}{ccccc} M_\infty \upharpoonright \mathfrak{N} & \xrightarrow{e} & P_{i^*} & \xrightarrow{\pi_{i^*,\infty}} & M_\infty \\ {\scriptstyle \tau'}\downarrow & & & & \downarrow{\scriptstyle g} \\ A & & \xrightarrow[f]{\qquad\qquad} & & B \end{array} \tag{4.4}$$

By (4.3), the top row of diagram (4.4) is just the inclusion map from $M_\infty \upharpoonright \mathfrak{N}$ into M_∞. Hence, τ' is a completion of the diagram (4.2), yielding a contradiction.

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