Rend. Sem. Mat. Univ. Padova 152 (2024), 167–170
DOI 10.4171/RSMUP/144

On a complex collar neighbourhood theorem

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Abstract – In this note we review and add an extra precision to the proof of our collar neighbourhood theorem for strictly pseudoconvex complex manifolds with boundary.

Mathematics Subject Classification (2020) – Primary 57T15; Secondary 14M15.

Keywords – CR-manifold, Levi form.

The purpose of this note is to review the complex collar neighbourhood theorem of our paper [6] and add an extra precision to its proof. This is motivated by the importance of the subject, which has attracted the attention of several authors, (see e.g. [2–4]). The two main ingredients are a local *extension* (or *realizability*) result (see [5]) and an ingenious use of the Zorn lemma which, in the non-compact case, is a substitute for the *bumping* technique of [1].

Our result is the following.

Theorem 1. *Let Ω be a paracompact smooth manifold of real dimension $2n$, D an open domain in Ω with a smooth boundary $M = bD$ and $J_0\colon TM \to TM$ a smooth almost complex structure on Ω, with the following properties:*

- *J_0 is formally integrable on $\bar{D} = D \cup M$;*
- *$M = bD$ is strictly pseudoconvex for J_0.*

Then we can find an open neighbourhood ω of $\bar{D}$ in Ω, and an integrable almost complex structure $J\colon T\omega \to T\omega$ such that $J|_D = J_0|_D$.

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The proof consists of several steps.

(1) We consider the family $\mathfrak{X}$ consisting of pairs (X, J), where X is an open set with $D \subseteq X \subseteq \Omega$ and $J : TX \to TX$ is an integrable almost-complex structure on X with $J|_D = J_0|_D$.

(2) We take the quotient $\widetilde{\mathfrak{X}} = \mathfrak{X}/_\sim$ by setting

$$(X_1, J_1) \sim (X_2, J_2) \Leftrightarrow \begin{cases} X_1 \cap M = X_2 \cap M, \\ \exists\, G^{\text{open}} \text{ with } X_1 \cap M \subset G \subseteq X_1 \cap X_2 \text{ and } J_1|_G = J_2|_G. \end{cases}$$

Denote by $[X, J]$ the equivalence class of $(X, J) \in \mathfrak{X}$.

(3) On $\widetilde{\mathfrak{X}}$ we introduce the order relation "$\prec$" by setting

$$[X_1, J_1] \prec [X_2, J_2] \Leftrightarrow \begin{cases} X_1 \cap M \subsetneqq X_2 \cap M, \\ \exists\, G^{\text{open}} \text{ with } X_1 \cap M \subset G \subseteq X_1 \cap X_2 \text{ and } J_1|_G = J_2|_G. \end{cases}$$

(4) We prove that every chain in $(\widetilde{\mathfrak{X}}, \prec)$ has an upper bound.

(5) We prove that a maximal element of $(\widetilde{\mathfrak{X}}, \prec)$ is of the form $[X, J]$ for an $(X, J) \in \mathfrak{X}$ with $M \subset X$.

While the other points are explained in full detail in [6], the proof of the fifth was somehow sketchy. For this reason we provide in the lemma below a more detailed proof of this item. Lemma 2 replaces and gives more precision to the construction of $\widetilde{D}$ on [6, p. 26].

Lemma 2. *Let $(\widetilde{D}, J)$ be the pair consisting of an open subset $\widetilde{D}$ of Ω and an integrable complex structure J on $\widetilde{D}$. Then we can find an open subset B of Ω with the following properties:*

- *$D \subseteq B \subseteq \widetilde{D}$;*
- *$B \cap M = \widetilde{D} \cap M$;*
- *the closure of B in Ω, with the restriction of J, is a complex manifold with a strictly pseudoconvex boundary.*

Proof. Fix a smooth retraction $\pi : \Omega \setminus D \to M$ and take a locally finite open covering by relatively compact coordinate patches $\mathcal{U} = \{U_i\}_{i \in I}$ of M in Ω such that $\pi(U_i \setminus D) \subset U_i$ for all $i \in I$. On each U_i we have smooth real coordinates $x_i^1, \dots, x_i^{2n}$, in which the complex structure J is represented by a matrix J_i having real-valued smooth coefficients. For a multiindex α we denote by D_i^α the higher-order derivative $(\partial/\partial x_i^1)^{\alpha_1} \cdots (\partial/\partial x_i^{2n})^{\alpha_{2n}}$. Fix a Riemannian distance "dist" on Ω and let

$$\delta(q) = \operatorname{dist}(q, M \setminus \widetilde{D}).$$

Since $M \setminus \widetilde{D}$ is closed, this is a continuous function on Ω. Next we define an open subset X of Ω by fixing a smooth partition of unity $\{\phi_i\}_{i \in I}$ subordinated to $\mathcal{U}$, a sequence $\{\chi_\nu\}$ of smooth real-valued functions with

$$\begin{cases} \chi_\nu(q) = 1 & \text{if } \delta(q) < e^{-\nu}, \\ \chi_\nu(q) = 0 & \text{if } \delta(q) > 2e^{-\nu}, \end{cases}$$

and setting, for $U = \bigcup_i U_i$ [the expression below makes sense as the matrix J_i is defined on a neighbourhood of the support of ϕ_i and the sum is locally finite in $(\widetilde{D} \cap U) \setminus D$],

$$X = D \cup \Big\{q \in (\widetilde{D} \cap U) \setminus D \;\Big|\; \textstyle\sum_{i,\alpha} \sup|\chi_{|\alpha|}(q)\phi_i(q)(D_i^\alpha J_i(q) - D_i^\alpha J_i(\pi(q)))| < \delta(q)\Big\}.$$

Clearly, X satisfies the first two conditions on B set in the statement of the lemma. To complete the proof, we fix a locally finite open covering $\mathcal{V} = \{V_k\}_{k \in K}$ of $M \setminus \widetilde{D}$ by relatively compact open subsets and a positive partition of unity $\{\psi_k\}_{k \in K}$ subordinated to $\mathcal{V}$. If $D = \{\rho < 0\}$ for a real-valued smooth function on Ω with $d\rho(q) \neq 0$ on M, then we can choose a sequence of positive $\{\varepsilon_k\}$ such that, for

$$\rho_B(q) = \rho(q) - \sum_k \varepsilon_k \psi_k(q) \quad \text{and} \quad B = \{\rho_B < 0\},$$

we have that B is an open subset of Ω with a smooth boundary in Ω,

$$D \subseteq B \subseteq X,$$

and the pair (B, J) defines a complex manifold with strictly pseudoconvex boundary ∂B in Ω. ■

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Manoscritto pervenuto in redazione il 21 marzo 2022.