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Abelian surfaces and the non-Archimedean Hodge-$\mathcal{D}$-conjecture – The semi-stable case

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Abstract – If X is a smooth projective variety over $\mathbb{R}$, the Hodge $\mathcal{D}$-conjecture of Beilinson asserts the surjectivity of the regulator map to Deligne cohomology with real coefficients. It is known to be false in general, but is true in some special cases like Abelian surfaces and $K3$-surfaces – and still expected to be true when the variety is defined over a number field. We prove an analogue of this for Abelian surfaces at a non-Archimedean place where the surface has bad reduction. Here, the Deligne cohomology is replaced by a certain Chow group of the special fibre. The case of good reduction is harder and was first studied by Spiess (1999) in the case of products of elliptic curves and by the author in general (Sreekantan, 2014). The case of bad reduction was also studied by the author in Sreekantan (2008).

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1. Introduction

1.1 – *The Hodge $\mathcal{D}$-conjecture for Abelian surfaces*

Let A be an Abelian surface over a p-adic local field K and $\mathcal{A}$ be a semi-stable model over the ring of integers $\mathcal{O} = \mathcal{O}_K$. Let $\mathcal{A}_v$ be the special fibre over the closed point v – which we assume is semi-stable, namely a union of divisors with normal crossings whose components are smooth. The aim of this paper is to prove a non-Archimedean analogue of the Hodge-$\mathcal{D}$-conjecture for such Abelian surfaces.

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This conjecture states that the map

$$\mathrm{CH}^2(A,1)\otimes\mathbb{Q}\xrightarrow{\partial}\mathrm{PCH}^1(\mathcal{A}_v)\otimes\mathbb{Q}$$

is surjective. Here $\mathrm{PCH}^1(\mathcal{A}_v)$ is a certain sub-quotient of the Chow group of the special fibre. This group has the property that

$$\dim_{\mathbb{Q}}\mathrm{PCH}^1(\mathcal{A}_v)\otimes\mathbb{Q}=-\operatorname{ord}_{s=1}L_p(H^2(A),s),$$

where $L_p(H^2(A),s)$ is the local L-factor at p. This group can hence be viewed as a p-adic version of the real Deligne cohomology – which has that property with respect to the Archimedean factor – and hence the map ∂ can be viewed as a p-adic version of the regulator map.

When $\mathcal{A}_v$ is smooth, that is, p is a prime of good reduction, the group $\mathrm{PCH}^1(\mathcal{A}_v)$ is simply $\mathrm{CH}^1(\mathcal{A}_v)$. This case was studied in [8, 10]. When A is a product of (non-isogenous) elliptic curves and p is a prime of semi-stable reduction for both, this was studied in [9]. This paper essentially closes the chapter – proving it in the remaining case of semi-stable reduction of simple Abelian surfaces.

Note that the group $\mathrm{CH}^2(A,1)\otimes\mathbb{Q}$ has many different avatars – it is the same as the $\mathcal{K}$-cohomology group $H^1_{\mathrm{Zar}}(A,\mathcal{K}_2)\otimes\mathbb{Q}$ and the motivic cohomology group $H^3_{\mathcal{M}}(A,\mathbb{Q}(2))$.

2. The target space of the boundary map

Let X be a smooth proper variety over a local field K and $\mathcal{O}$ the ring of integers of K with closed point v and generic point η.

By a model $\mathcal{X}$ of X we mean a flat proper scheme $\mathcal{X}\to\mathrm{Spec}(\mathcal{O})$, together with an isomorphism of the generic fibre X_η with X. Let Y be the special fibre $\mathcal{X}_v=\mathcal{X}\times\mathrm{Spec}(k(v))$ and let $i\colon Y\hookrightarrow\mathcal{X}$ denote the inclusion map. We will always also make the assumption that the model is strictly semi-stable, which means that it is a regular model and the fibre Y is a divisor with normal crossings whose irreducible components are smooth, have multiplicity one and intersect transversally.

2.1 – *Consani's double complex*

Consani [4] defined a double complex of Chow groups of the components of the special fibre with a monodromy operator N, following the work of Steenbrink [11] and Bloch–Gillet–Soulé [2]. Using this complex she was able to relate the higher Chow group of the special fibre at a semi-stable prime to the regular Chow groups of the components. This relation is what is used in defining the group PCH.

Let $Y = \bigcup_{i=1}^{t} Y_i$ be the special fibre of dim n with Y_i its irreducible components. For $I \subset \{1, \dots, t\}$, define

$$Y_I = \bigcap_{i \in I} Y_i.$$

Let $r = |I|$ denote the cardinality of I. Define

$$Y^{(r)} := \begin{cases} \mathcal{X} & \text{if } r = 0, \\ \coprod_{|I|=r} Y_I & \text{if } 1 \le r \le n, \\ \emptyset & \text{if } r > n. \end{cases}$$

For u and t with $1 \le u \le t < r$, define the map

$$\delta(u) \colon Y^{(t+1)} \to Y^{(t)}$$

as follows. Let $I = (i_1, \dots, i_{t+1})$ with $i_1 < i_2 < \cdots < i_{t+1}$. Let $J = I - \{i_u\}$. This gives an embedding $Y_I \to Y_J$. Putting these together induces the map $\delta(u)$. Let $\delta(u)_*$ and $\delta(u)^*$ denote the corresponding maps on Chow homology and cohomology respectively. They further induce the Gysin and restriction maps on the Chow groups.

Define

$$\gamma := \sum_{u=1}^{r+1} (-1)^{u-1} \delta(u)_*$$

and

$$\rho := \sum_{u=1}^{r+1} (-1)^{u-1} \delta(u)^*.$$

These maps have the properties that

- $\gamma^2 = 0$,
- $\rho^2 = 0$,
- $\gamma \cdot \rho + \rho \cdot \gamma = 0$.

2.2 – *The group* PCH

Let a, q be two integers with $q - 2a > 0$. We have

$$\mathrm{PCH}^{q-a-1}(Y, q-2a-1)$$
$$= \begin{cases} \dfrac{\mathrm{Ker}(i^* i_* \colon \mathrm{CH}_{n\text{-}a}(Y^{(1)}) \to \mathrm{CH}^{a+1}(Y^{(1)}))}{\mathrm{Im}(\gamma \colon \mathrm{CH}_{n\text{-}a}(Y^{(2)}) \to \mathrm{CH}_{n\text{-}a}(Y^{(1)}))} \otimes \mathbb{Q} & \text{if } q-2a = 1, \\ \dfrac{\mathrm{Ker}(\gamma \colon \mathrm{CH}_{n\text{-}(q\text{-}a\text{-}1)}(Y^{(q\text{-}2a)}) \to \mathrm{CH}_{n\text{-}(q\text{-}a\text{-}1)}(Y^{(q\text{-}2a\text{-}1)}))}{\mathrm{Im}(\gamma \colon \mathrm{CH}_{n\text{-}(q\text{-}a\text{-}1)}(Y^{(q\text{-}2a+1)}) \to \mathrm{CH}_{n\text{-}(q\text{-}a\text{-}1)}(Y^{(q\text{-}2a)}))} \otimes \mathbb{Q} & \text{if } q-2a > 1. \end{cases}$$

Here n is the dimension of Y. Note that if $q-2a>1$ and Y is non-singular, this group is 0, while if Y is singular and semi-stable, the Parshin–Soulé conjecture implies that this group is $\mathrm{CH}^{q-a-1}(Y,q-2a-1)\otimes\mathbb{Q}$. If $q-2a=1$ and Y is non-singular, the group is $\mathrm{CH}^a(Y)\otimes\mathbb{Q}$. Our interest is in the remaining case, namely when $q-2a=1$ and Y is singular.

The "real" Deligne cohomology has the property that its dimension is the order of the pole of the Archimedean factor of the L-function at a certain point on the left of the critical point. The group $\mathrm{PCH}^1(Y)$ is expected to have a similar property. Let F^* be the geometric Frobenius and $N(v)$ the number of elements of $k(v)$. The local L-factor of the $(q-1)$st cohomology group is then

$$L_v(H^{q-1}(X),s)=(\det(I-F^*N(v)^{-s}\mid H^{q-1}(\bar{X},\mathbb{Q}_\ell)^I))^{-1}.$$

Theorem 2.1 (Consani). *Let v be a place of semi-stable reduction. Assuming the weight-monodromy conjecture, the Tate conjecture for the components and the injectivity of the cycle class map on the components Y_I, the Parshin–Soulé conjecture and that F^* acts semi-simply on $H^*(\bar{X},\mathbb{Q}_\ell)^I$, we have*

$$\dim_{\mathbb{Q}}\mathrm{PCH}^{q-a-1}(Y,q-2a-1)=-\operatorname{ord}_{s=a}L_v(H^{q-1}(X),s):=d_v.$$

Proof. See [4, Cor. 3.6]. ∎

From this point of view the group $\mathrm{PCH}^{q-a-1}(Y,q-2a-1)$ can be viewed as a non-Archimedean analogue of the "real" Deligne cohomology. Since the L-factor at a prime of good reduction does not have a pole at $s=a$ when $q-2a>1$, the Parshin–Soulé conjecture can be interpreted as the statement that this non-Archimedean Deligne cohomology has the correct dimension, namely 0, even at a prime of good reduction.

As is clear from the definition, the group PCH depends on the choice of the semi-stable model of X. However, Consani's theorem says that the dimension does not. So, to a large extent, one can work with any semi-stable model. Perhaps the correct definition is one obtained by taking a limit of semi-stable models as in the work of Bloch–Gillet–Soulé [2] on non-Archimedean Arakelov theory.

2.3 – *Elements of the higher Chow group*

From this point on we specialize to the case when X is a surface and, further, $n=2$, $q=3$ and $a=1$. We will be interested in the group $\mathrm{CH}^2(X,1)$ and the map to $\mathrm{PCH}^1(Y):=\mathrm{PCH}^1(Y,0)$. This is related to the order of the pole of the L-function of $H^2(X)$ at $s=1$. Soon we will further specialize to the case when X is an Abelian surface.

Let X be a surface over a field K. The group $\mathrm{CH}^2(X,1)$ has the following presentation [7]. It is generated by formal sums of the type

$$\sum_i (C_i, f_i),$$

where C_i are curves on X and f_i are $\bar{K}$-valued functions on the C_i satisfying the cocycle condition

$$\sum_i \operatorname{div} f_i = 0.$$

Relations in this group are give by the tame symbol of pairs of functions on X.

There are some elements of this group coming from the product structure

$$\mathrm{CH}^1(X_L) \otimes \mathrm{CH}^1(X_L,1) \longrightarrow \mathrm{CH}^2(X_L,1) \xrightarrow{\mathrm{Nm}_K^L} \mathrm{CH}^2(X,1).$$

Here Nm_K^L is the norm map from a finite extension L of K. The image of this group as L runs through all finite extensions of K is called the subgroup of *decomposable elements*, $\mathrm{CH}^2_{\mathrm{dec}}(X,1)$

A theorem of Bloch [1] says that $\mathrm{CH}^1(X_L,1)$ is simply L^*, where L is the field of definition of X_L, so such an element looks like a sum of elements of the type (C,a), where C is a curve on X_L and a is in L^*. The group of *indecomposable elements* is the quotient group

$$\mathrm{CH}^2(X,1)_{\mathrm{ind}} = \mathrm{CH}^2(X,1)/\mathrm{CH}^2_{\mathrm{dec}}(X,1).$$

In general, it is hard to find elements in this group.

The group $\mathrm{CH}^2(X,1)\otimes\mathbb{Q}$ has several avatars – it is the same as the $\mathcal{K}$-cohomology group $H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)\otimes\mathbb{Q}$ and the motivic cohomology group $H^3_{\mathcal{M}}(X,\mathbb{Q}(2))$.

2.4 – *The boundary map*

The usual Beilinson regulator maps the higher Chow group to the real Deligne cohomology. In the non-Archimedean context, it appears that the boundary map

$$\partial\colon \mathrm{CH}^2(X,1) \longrightarrow \mathrm{PCH}^1(Y)$$

plays a similar role. It is defined as

$$\partial\Big(\sum_i (C_i, f_i)\Big) = \sum_i \operatorname{div}_{\bar{C}_i}(\bar{f}_i)$$

where $\overline{f_i}$ is the function f_i on the closure $\overline{C_i}$ of C_i in the semi-stable model $\mathcal{X}$ of X. By the cocycle condition, the "horizontal divisor", namely, the closure $\sum_i \overline{\mathrm{div}_{C_i}(f_i)}$ of $\sum_i \mathrm{div}_{C_i}(f_i)$, is 0, and so the boundary is supported on the special fibre. Further, since the boundary ∂ of an element is the sum of divisors of functions, it lies in $\mathrm{Ker}(i^* i_*)$.

For a decomposable element of the form (C, a), the boundary map is particularly simple to compute:

$$\partial((C, a)) = \mathrm{ord}_v(a)\mathcal{C}_v,$$

where $\mathcal{C}_v$ is the special fibre of a model $\mathcal{C}$. In particular, a cycle in the special fibre which is not the restriction of the closure of a cycle in the generic fibre cannot appear in the boundary of the subgroup of decomposable elements.

3. Semi-stable reduction of Abelian surfaces

3.1 – *Types of semi-stable reductions*

If A is an Abelian surface over a local field, Kulikov and Persson–Pinkham classified the possible semi-stable degenerations. For a surface X the dual graph of its special fibre is defined as follows. It is the simplicial complex with one vertex v_i for every component Y_i. The simplex $[v_{i_1}, \dots, v_{i_k}]$ lies in the simplicial complex if and only if $Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_k} \neq \emptyset$.

THEOREM 3.1 (Kulikov–Persson–Pinkham [5]). *If $\mathcal{A}$ is the Néron model of an Abelian surface – so that $K_{\mathcal{A}} = 0$ – the possible semi-stable special fibres are the following:*

Type 1 – *$\mathcal{A}_p$ is smooth and the dual graph is a point.*

Type 2 – *$\mathcal{A}_p = \bigcup_i Y_i$, where $\bigcup_i Y_i$ is a cycle of elliptic ruled surfaces such that adjacent surfaces $Y_i \cap Y_j$ intersect at elliptic curves E_{ij}. All the elliptic surfaces are isomorphic. The dual graph is S^1.*

Type 3 – *$\mathcal{A}_p$ is a cycle of rational surfaces, each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ such that the dual graph is topologically $S^1 \times S^1$. The double curves are "-1 hexagons", there are six components in every double curve and each component is a -1 curve.*

3.2 – *The group* $\mathrm{PCH}^1(\mathcal{A}_p)$

We want to study the boundary of the map

$$\mathrm{CH}^2(A, 1) \otimes \mathbb{Q} \xrightarrow{\partial} \mathrm{PCH}^1(\mathcal{A}_p) \otimes \mathbb{Q}.$$

In this case the target space is

$$\mathrm{PCH}^1(\mathcal{A}_p) = \frac{\mathrm{Ker}(i^*i_*\colon \mathrm{CH}_1(Y^{(1)}) \to \mathrm{CH}^2(Y^{(1)}))}{\mathrm{Im}(\gamma\colon \mathrm{CH}_1(Y^{(2)}) \to \mathrm{CH}_1(Y^{(1)}))} \otimes \mathbb{Q}$$

with dimension

$$\dim_{\mathbb{Q}} \mathrm{PCH}^1(\mathcal{A}_p) = -\operatorname{ord}_{s=1} L_v(H^2(A), s) := d_v.$$

We have to study each case separately.

3.2.1. Type 1 degenerations of Abelian surfaces. This is the case when the special fibre is a smooth Abelian surface and it was studied in [10]. Here, the target space of the boundary map is simply $\mathrm{CH}^1(\mathcal{A}_p) \otimes \mathbb{Q}$ and the rank of this space is at least 2. We showed that there exists a new element of the higher Chow group of A for each new element of $\mathrm{CH}^1(\mathcal{A}_p)$. The argument here is quite subtle and uses deformation theory. This includes the case when the special fibre is a product of two elliptic curves.

3.2.2. Type 2 degenerations of Abelian surfaces. In this case the special fibre is a cycle of elliptic ruled surfaces Y_i. These surfaces intersect at elliptic curves

$$Y_i \cap Y_j = \begin{cases} E_{ij} \text{ an elliptic curve}, & j = i \pm 1, \\ \emptyset, & |j-i| > 1, \end{cases}$$

which are the bases of the elliptic ruled surfaces. All the elliptic curves are isomorphic. Here we have

$$Y^{(j)} = \begin{cases} \mathcal{A}_p, & j = 0, \\ \bigsqcup_i Y_i, & j = 1, \\ \bigsqcup_i Y_i \cap Y_{(i+1)} = \bigsqcup_i E_{i(i+1)}, & j = 2, \\ \emptyset, & j > 2. \end{cases}$$

As the conjectures upon which it is conditional hold in this case, we can apply Consani's theorem. Hence the dimension of the group $\mathrm{PCH}^1(\mathcal{A}_p) \otimes \mathbb{Q}$ is the order of vanishing of the local L-factor at v of the L-function of $H^2(A)$. This dimension can be computed using the analogue of the Clemens–Schmid exact sequence due to [2] and turns out to be 2. Hence we need to construct 2 higher Chow cycles.

One of them can be constructed as follows. If $\mathcal{D}$ is a cycle in $\mathrm{CH}_1(\mathcal{A})$ then the restriction of $\mathcal{D}$ to $\mathcal{A}_p$, which is $\mathcal{D}_p$, lies in $\mathrm{PCH}^1(\mathcal{A}_p)$. This is because

$$i^*i_*(\mathcal{D}_p) = i^*(\mathcal{D})$$

and

$$i^*(\mathcal{D}) = \bigoplus_j (\mathcal{D} \cap \operatorname{div}(\pi))|_{Y_j},$$

so $\mathcal{D} \cap \operatorname{div}(\pi)$ is the divisor of the function π restricted to $\mathcal{D}$, hence is 0 in $\mathrm{CH}^2(\mathcal{A})$, and so maps to 0 in $\mathrm{CH}^2(Y^{(1)})$. Hence the restriction of a generic cycle always lies in the group $\mathrm{PCH}^1(\mathcal{A}_p)$. This bounds one of the generators of the group $\mathrm{PCH}^1(\mathcal{A}_p)$. The conjecture predicts that there is a second element of the higher Chow group which bounds the other generator of this group. We will construct this cycle in the next section.

3.2.3. Type 3 degenerations of Abelian surfaces. In this case the individual components are $\mathbb{P}^2$ blown up at the vertices of a triangle – which we will denote by $\widehat{\mathbb{P}}^2$. This results in a "-1 hexagon", where three of the six sides are the strict transforms of the edges of the triangle and the other three are the exceptional fibres. They are all (-1)-curves on $\widehat{\mathbb{P}}^2$ and if two components intersect, they intersect along one of these curves:

$$Y^{(r)} = \begin{cases} \mathcal{A}_p, & r = 0, \\ \bigsqcup_i Y_i = \bigsqcup \widehat{\mathbb{P}}^2_i, & r = 1, \\ \bigsqcup_i Y_i \cap Y_j = \bigsqcup_{i,j} \mathbb{P}^1_{i,j}, & r = 2, \\ \bigsqcup_{i,j,k} Y_{i,j,k} = \bigsqcup_{i,j,k} \mathbb{P}^0_{i,j,k}, & r = 3, \\ \emptyset, & r > 3. \end{cases}$$

Here one knows from the analogue of the Clemens–Schmid exact sequence that the dimension is 3. Hence the conjecture predicts that there are three elements of the higher Chow group. One of them is the boundary of a decomposable element coming from the genus 2 curve on the generic fibre. We will show that there are at least two other elements which are linearly independent.

4. Higher Chow cycles

4.1 – *Collino's construction*

Collino [3] constructed a higher Chow cycle on a principally polarized Abelian surface A as follows. Since A is principally polarized, $A = \mathrm{Jac}(C)$ where C is a genus 2 curve. Let P and Q be two ramification points on C. There is a function f on C with divisor

$$\operatorname{div}(f) = 2P - 2Q.$$

Let C_P and C_Q be the images of the curve C under the maps ι_P and ι_Q, where

$$\iota_x(y) = y - x,$$

and let f_P and f_Q be the function f being thought of as a function on C_P and C_Q respectively. Then

$$\mathrm{div}(f_P) = 2(0) - 2(Q - P) \quad \text{and} \quad \mathrm{div}(f_Q) = 2(P - Q) - 2(0).$$

Since $P - Q$ is a two torsion point on the Abelian surface, $P - Q = Q - P$. Hence the element

$$\Xi_{P,Q} = (C_P, f_P) + (C_Q, f_Q)$$

satisfies the co-cycle condition

$$\mathrm{div}(f_P) + \mathrm{div}(f_Q) = 0,$$

hence is an element of the higher Chow group $\mathrm{CH}^2(A, 1)$.

4.2 – *Surjectivity*

Our conjecture states that the boundary map in the localization sequence is surjective. We show that the element of Collino's described above, with suitable choices of points P and Q, suffices to show surjectivity in the cases when the special fibre of the Abelian surface is singular, as well as in the case when the special fibre is a product of elliptic curves.

We do this by computing the boundary of the element in terms of the components of the regular minimal model of the curve C. For that we need the theorems of Parshin [6] on minimal models of genus 2 curves. In all the cases, the computation of the boundary is done as follows. Suppose the special fibre

$$\mathcal{C}_p = \bigcup_i X_i.$$

Then

$$\mathrm{div}(\bar{f}) = \mathcal{H} + \sum_i a_i X_i,$$

where $\bar{f}$ is the function f on the closure $\mathcal{C}$ and $\mathcal{H}$ is the horizontal divisor $\overline{\mathrm{div}(f)}$. To compute the a_j we do the following. We know that the decomposable element (C, p^k) has boundary $\mathrm{div}(p^k) = k\mathcal{C}_p = k(\sum_i X_i)$. Hence

$$\mathrm{div}(\bar{f}) - \mathrm{div}(p^{a_j}) = \mathcal{H} + \sum_i (a_i - a_j) X_i$$

and, in particular, X_j is not in the support. The degree of a divisor of a function on a curve on an algebraic surface which is not contained in the support is 0. Hence restricting this to X_j gives us an equation

$$(\mathcal{H} \cdot X_j) + \sum_i (a_i - a_j)(X_i \cdot X_j) = 0.$$

Using that and what we know about the intersection numbers $(X_i \cdot X_j)$ gives us a linear equation among the a_i not including a_j

However, we can simplify our calculations using the following observation. If X is a component of the special fibre $\mathcal{C}_p$ then $(X \cdot \mathcal{C}_p) = 0$. So equivalently we have the equation

$$(\mathcal{H} \cdot X_j) + \sum_i a_i (X_i \cdot X_j) = 0,$$

though here we have to use what we know about the self-intersection $(X_j \cdot X_j)$.

Repeating this with the different components gives us a system of simultaneous equations in the a_i which we can solve quite easily. We get as many equations as components this way and so the space of solutions is one-dimensional. Sometimes it is convenient to make a choice of the coefficient of one of the components in order to get a “nice” description of the boundary.

From the Néron mapping property, the map $\iota_x : C \to A$ extends to a map, which we will also denote by ι_x,

$$\iota_x : \mathcal{C}^{\mathrm{ns}} \to \mathcal{A}^0,$$

where $\mathcal{C}^{\mathrm{ns}}$ is the curve $\mathcal{C}$ with the singular points removed and $\mathcal{A}^0$ is the Néron model of the Jacobian. The special fibre of the Néron model of the Jacobian is the group $\mathcal{A}_p \setminus \bigcup_{i,j,i \neq j} (Y_i \cap Y_j)$, where $\mathcal{A}_p = \bigcup Y_i$ is the special fibre of a minimal regular model with components Y_i. Each component of the special fibre is an extension of an Abelian variety by a power of $\mathbb{G}_m$ – so in our case it is either an Abelian surface, an extension of an elliptic curve E by $\mathbb{G}_m$ or $\mathbb{G}_m \times \mathbb{G}_m$. We choose a particular component where the closure of the zero section lies and define that to be the identity component. The set of components has the structure of a finite Abelian group.

The element in the higher Chow group that we consider is $\Xi_{P,Q}$. The boundary of the element $\Xi_{P,Q}$ is the closure of

$$\iota_P(\mathrm{div}(\bar{f})) + \iota_Q(\mathrm{div}(\bar{f}))$$

in the special fibre $\mathcal{A}_p$. Since the horizontal cycles cancel, one has

$$\partial(\Xi_{P,Q}) = \sum_i a_i (\iota_P(X_i) + \iota_Q(X_i)).$$

The curves $\iota_P(X_i)$ and $\iota_Q(X_i)$ are linearly equivalent in $\mathrm{PCH}^1(Y)$, hence the boundary is

$$\partial(\Xi_{P,Q}) = \sum_i 2a_i(\iota_P(X_i))$$

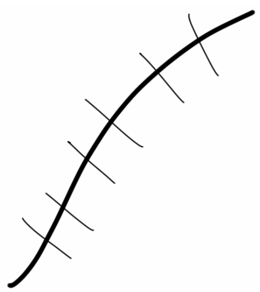
in $\mathrm{PCH}^1(Y) \otimes \mathbb{Q}$.

Hence, what we have to show is that for a suitable choice of P and Q one obtains a new cycle in $\mathrm{PCH}^1(Y)$ and in the case of type 3, we show that for different choices of P and Q we can get two new cycles.

We now do a case-by-case analysis. There are seven cases of minimal regular models of genus 2 curves. In all that follows, let $\mathcal{C}$ be in the minimal regular model of a genus 2 curve C and $\mathcal{C}_p$ the special fibre. We use the notation of [6]. In the pictures, the bold lines correspond to the curves and the thin lines indicate where the Weierstrass points lie.

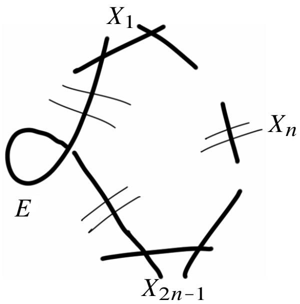
Case	Type of Jacobian	Rank of PCH^1
I	1	≥ 2
II	2	2
III	3	3
IV	1	≥ 3
V	2	2
VI	3	3
VII	3	3

4.2.1. Case I.



In this case the curve $\mathcal{C}$ reduces to a smooth genus 2 curve and the Jacobian is of type 1. Since the special fibre has only one component, one has $\mathrm{div}(\bar{f}_P) = \mathcal{H} + a\mathcal{C}_p$. Computing the intersection with $\mathcal{C}_p$ shows that $a = 0$. Hence Collino's cycle has no boundary here – but the decomposable element can be used to bound $\mathcal{C}_p$. However, the dimension of $\mathrm{PCH}^1(Y) \otimes \mathbb{Q} = \mathrm{CH}^1(Y) \otimes \mathbb{Q}$ is at least 2, owing to the existence of the Frobenius endomorphism. The conjecture predicts there are at least *two* higher Chow cycles. Further, it is usually the case that the Picard number of the special fibre is strictly larger than that of the generic fibre. In those cases, decomposable cycles will not suffice to prove surjectivity. Collino's cycle has boundary 0 so does not work. Hence we have to find new indecomposable cycles. This is the content of [10]. The idea there was to "deform" a rational curve corresponding to the extra cycle to construct a new element. Curiously, this is the hardest case.

4.2.2. Case II.



In this case the stable model of the curve is a genus 1 curve with a node and the Jacobian is of type 2. The special fibre $\mathcal{C}_p$ of the regular minimal model consists of a genus 1 curve E and a chain of $(2n-1)$ rational curves X_i meeting E at two points α and β, where $n \geq 2$ is an integer. One has $X_i^2 = -2 = E^2$. Finally, the closure of four of the Weierstrass points meets E and the remaining two meet the middle component X_n.

Choose P and Q such that P meets E and Q meets X_n. One has

$$\operatorname{div}(\bar{f}_P) = \mathcal{H} + aE + \sum_{i=1}^{2n-1} b_i X_i.$$

To compute a and b_i we modify by a decomposable element and intersect with E and X_i. As we remarked, we can simply consider the restriction to E or X_i. For reasons of symmetry one has

$$\begin{aligned} b_i &= b_{2n-i}, \\ 0 &= (\operatorname{div}(\bar{f}_P)\cdot E) - = 2-2a+b_1+b_{2n-1} = 2-2a+2b_1 \Rightarrow b_1 = a-1, \\ 0 &= (\operatorname{div}(\bar{f}_P)\cdot X_1) = a-2b_1+b_2 \Rightarrow b_2 = b_1-1 = a-1-1 = a-2. \end{aligned}$$

Continuing in this manner one can see that

$$b_k = a-k, \quad 1 \leq k \leq (n-1)$$

and, by symmetry,

$$b_{2n-k} = b_k, \quad 1 \le k \le (n-1).$$

Finally,

$$0 = -2 + 2b_{n-1} - 2b_n = -2 + 2(a - (n-1)) - 2b_n \Rightarrow b_n = a - n.$$

So one has

$$\operatorname{div}(\bar{f}_P) = \mathcal{H} + aE - \left(\sum_{i=1}^{n-1}(i-a)(X_i + X_{2n-i}) + (n-a)X_n\right).$$

A similar calculation shows that

$$\operatorname{div}(\bar{f}_Q) = -\mathcal{H} + aE - \left(\sum_{i=1}^{n-1}(i-a)(X_i + X_{2n-i}) + (n-a)X_n\right),$$

so combining these two, the boundary of the element $\Xi_{P,Q}$ in the Néron special fibre is

$$\begin{aligned}\partial(\Xi_{P,Q}) &= \iota_P(\operatorname{div}(\bar{f}_P)) + \iota_Q(\operatorname{div}(\bar{f}_Q))\\ &= 2a\iota_P(E) - 2\left(\sum_{i=1}^{n-1}(i-a)(\iota_P(X_i) + \iota_P(X_{2n-i})) + (n-a)\iota_P(X_n)\right).\end{aligned}$$

A different choice of Weierstrass points will either change sign, if the roles of P and Q are reversed, or have boundary 0, if P and Q lie on the same component.

Each component of the Néron special fibre is a non-split extension of E by $\mathbb{G}_m$ and the group of components is isomorphic to $\mathbb{Z}/(2n-1)\mathbb{Z}$. Each X_i is isomorphic to $\mathbb{G}_m$ and its closure in the special fibre of the degenerate Abelian surface is a $\mathbb{P}^1$. The special fibre of the closure of the curve C_P is a copy of E in one component with a chain of $\mathbb{P}^1$'s meeting E at two different points which are translates of each other. Hence the boundary of the decomposable element (C, p^a) is

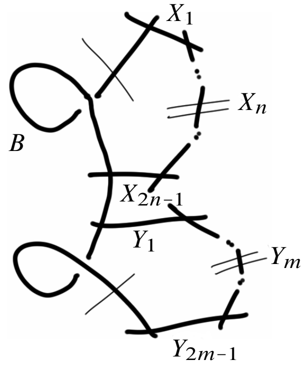
$$\partial((C, p^a)) = a\mathcal{C}_p = a\left(\iota_P(E) + \sum_{i=1}^{2n-1}\iota_P(X_i)\right).$$

Adding twice this to our computation of the boundary of $\Xi_{P,Q}$ gives that, up to a decomposable element, the boundary of $\Xi_{P,Q}$ in $\mathrm{PCH}^1(\mathcal{A}_p) \otimes \mathbb{Q}$ is

$$\partial(\Xi_{P,Q}) = -2\left(\sum_{i=1}^{n-1} i(\iota_P(X_i) + \iota_P(X_{2n-i})) + n(\iota_P(X_n))\right).$$

This can be seen to be non-zero by intersecting with E, for instance. In particular, it is not a multiple of $\mathcal{C}_p$. Hence the two elements C_p and $\partial(\Xi_{P,Q})$ are linearly independent and therefore generate $\mathrm{PCH}^1(Y) \otimes \mathbb{Q}$.

4.2.3. Case III.



In this case the stable model is a genus 0 curve with two nodes and the Jacobian is of type 3. This can be viewed as the case when the elliptic curve in Case II degenerates to a nodal curve. Here, the special fibre $\mathcal{C}_p$ of the regular minimal model consists of a genus 0 curve B and two chains of rational curves X_i, $1 \leq i \leq 2n-1$ and Y_j, $1 \leq j \leq 2m-1$. The closures of two of the Weierstrass points meet B as well as X_n and Y_m. One has $X_i^2 = Y_j^2 = -2$ and $B^2 = -4$.

In this case there are essentially three different elements we can construct. Suppose P, Q and R are three Weierstrass points whose closures lie on B, X_n and Y_m respectively. Then one has the elements $\Xi_{P,Q}$, $\Xi_{P,R}$ and $\Xi_{Q,R}$. However, it is easy to see that $\Xi_{P,Q} - \Xi_{P,R} = \Xi_{Q,R}$ in $\mathrm{CH}^2(X, 1)$ as they differ by the tame symbol of a pair of functions.

Suppose P lies on B and Q lies on X_n. Then an analysis similar to what is done above shows, up to the boundary of a decomposable element,

$$\partial(\Xi_{P,Q}) = 2a\left(B + \sum_{j=1}^{2m-1} Y_j\right) - 2\left(\sum_{i=1}^{n-1}(i-a)(X_i + X_{2n-i}) + (n-a)X_n\right).$$

Similarly, if P lies on B and R lies on Y_m one has

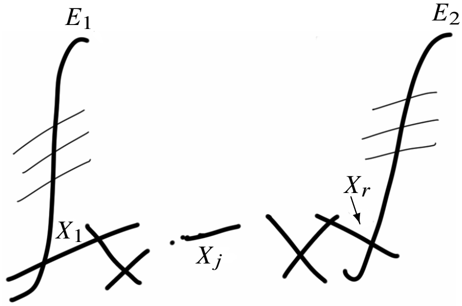
$$\partial(\Xi_{P,R}) = 2a\left(B + \sum_{i=1}^{2n-1} X_i\right) - 2\left(\sum_{j=1}^{m-1}(j-a)(Y_j + Y_{2m-j}) + (m-a)Y_m\right).$$

The boundary of a decomposable element (C, p^a) is, like before,

$$\partial((C, p^a)) = a\left(B + \sum_{i=1}^{2n-1} X_i + \sum_{j=1}^{2m-1} Y_i\right).$$

Intersecting with X_n and Y_m, for instance, shows that the boundaries are linearly independent and hence they generate the group $\mathrm{PCH}^1(Y) \otimes \mathbb{Q}$. So once again, the boundary map is surjective.

4.2.4. Case IV.



In this case the stable model is a union of two elliptic curves meeting at a point and the closures of three of the Weierstrass points lie on each elliptic curve. Here the Jacobian is smooth, hence is of type 1. The regular minimal model consists of the two elliptic curves along with a chain of rational curves joining them. The elliptic curves E_i satisfy $E_i^2 = -1$ while the rational curves X_j, $1 \leq j \leq r$ satisfy $X_j^2 = -2$.

The case of the generic fibre being the product of elliptic curves was studied by Spiess [8]. While he constructed a particular element using an irreducible genus 2 curve and two elliptic curves in the generic fibre, in fact one can use the element $\Xi_{P,Q}$ constructed above, with P and Q being chosen such that their closures lie on different components. Then if the divisor of $\bar{f}_P$ is

$$\operatorname{div}(\bar{f}_P) = b_0 E_1 + \sum_{j=1}^{r} b_j X_j + b_{r+1} E_2,$$

using a calculation similar to that above shows

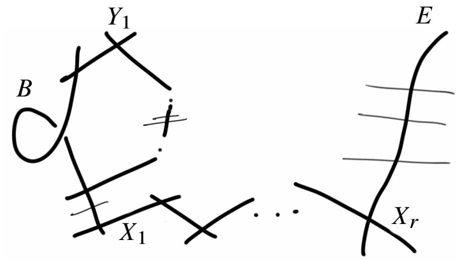
$$b_j = b_0 - 2j.$$

A convenient choice of b_0 is $r+1$ as in that case we have $b_j = (r+1-2j) = -b_{r+1-j}$ and the boundary is

$$\partial(\Xi_{P,Q}) = 2(r+1)(E_1 - E_2) + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} 2(r+1-2j)(X_j - X_{r+1-j}),$$

where $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a. Under the map to the Jacobian the X_j's map to a point and so the cycle maps to $2(r+1)(E_1 - E_2)$, which is clearly not a multiple of $E_1 + E_2$. Hence the boundary of the decomposable cycle, along with this cycle, generates the group $\mathrm{PCH}^1(C_p) \otimes \mathbb{Q}$.

4.2.5. Case V.



In this case the stable model is a union on an elliptic curve with a nodal rational curve and the Jacobian is of type 2. The minimal regular model consists of the elliptic curve E along with a chain of rational curves X_i satisfying $X_i^2 = -2$, linking E with a rational curve B. One has $B^2 = -3$ and $E^2 = -1$. Further, there is a chain of rational curves Y_j, $1 \leq j \leq 2m-1$ linking two points of B, with $Y_j^2 = -2$. This is essentially the case when one of the elliptic curves in Case IV degenerates to a nodal rational curve and corresponds to the case when the extension class of the elliptic surface is trivial – the surface is a product $E \times \mathbb{P}^1$.

Here, three of the closures of the Weierstrass points lie on E, one of the points lies on B and finally two lie on Y_m. We choose P and Q such that the closure of P lies on E and the closure of Q lies on B. Calculating as before, suppose

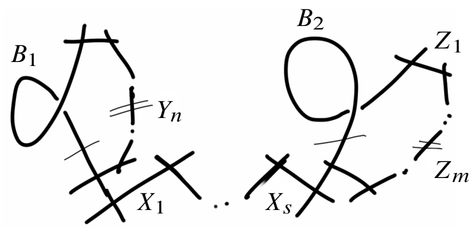
$$\mathrm{div}(\bar{f_P}) = \mathcal{H} + aE + \sum_{i=1}^{r} b_i X_i + cB + \sum_{j=1}^{2m-1} d_j Y_j .$$

One has $b_i = a - 2i$, $c = a - 2(r+1)$ and $d_j = a - 2(r+1)$ for all j, where we use the fact that by symmetry $d_j = d_{2m-j}$. So the boundary is

$$\partial(\Xi_{P,Q}) = 4aE - 2\Bigg(\sum_{i=1}^{r}(2i-a)X_i + (2(r+1)-a)\Bigg(B + \sum_{j=1}^{2m-1} Y_j\Bigg)\Bigg).$$

There is another element we can consider – when Q meets the component Y_m instead of B – but a similar calculation shows that the boundary is the same. If P lies on B and Q lies on Y_m or any such combination, the boundary can be seen to be 0.

4.2.6. Case VI.



In this case the stable model is the union of two nodal rational curves meeting at a point. The minimal regular model consists of two rational curves B_1 and B_2 each with a chain of rational curves Y_j, $1 \leq j \leq 2n-1$ and Z_k, $1 \leq k \leq 2m-1$. The curves B_i are also linked by a chain of rational curves X_i, $1 \leq i \leq s$. Finally, $X_i^2 = Y_j^2 = Z_k^2 = -2$ and $B_1^2 = B_2^2 = -3$. A pair of the closures of the Weierstrass points meet the curves Y_n and Z_m each and the remaining two meet B_1 and B_2.

This is the case when both elliptic curves in Case IV degenerate to nodal rational curves. We studied this in [9] – when the generic fibre was assumed to be a product of two non-isogenous elliptic curves. However, one can use the element above to prove surjectivity in more generality. Here, the Jacobian is of type 3 so one expects two new elements.

To get the first we choose P and Q such that their closures lie on B_1 and B_2 respectively. As before, one has

$$\operatorname{div}(\bar{f}_P) = b_1 B_1 + \sum_{i=1}^{s} c_i X_i + \sum_{j=1}^{2n-1} d_j Y_j + \sum_{k=1}^{2m-1} e_k Z_k + b_2 B_2.$$

Calculating as before, we have

$$-3b_1 + c_1 + d_1 + d_{2n-1} + 2 = 0,$$
$$b_1 - 2d_1 + d_2 = 0.$$

In general we have

$$d_{j+1} = 2d_j - d_{j-1}, \quad 2 \leq j \leq 2n-2.$$

Hence, adding a decomposable element so that $b_1 = 0$, we see that $d_i = 0$ for all i and $c_1 = -2$. Further calculation shows that

$$c_{i+1} = 2c_i - c_{i-1}, \quad 2 \leq i \leq s-1.$$

Using this we have $c_i = -2i$. We also have

$$b_2 - 2c_s + c_{s-1} = 0,$$

which shows that $b_2 = -2(s+1)$. Finally, using symmetry to say that $e_i = e_{2m-i}$, we have

$$\begin{aligned} -3b_2 + c_s + 2e_1 - 2 &= 0, \\ e_{k-1} - 2e_k + e_{k+1} &= 0, \quad 1 < k < 2m-1, \\ e_{2m-2} - 2e_{2m-1} + b_2 &= 0, \end{aligned}$$

which shows that $e_k = -2(s+1)$ for all k. Hence the divisor is

$$\mathrm{div}(\bar{f}_P) = \sum_{i=1}^{s} (-2i)X_i - 2(s+1)\left(B_2 + \sum_{k=1}^{2m-1} Z_k\right).$$

Finally, adding a decomposable element we see that $\partial(\Xi_{P,Q})$ can be written more symmetrically as

$$2\left((s+1)\left(B_1 + \sum_{j=1}^{2n-1} Y_j\right)\right) + 2\left(\sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (s+1-2i)(X_i - X_{s+1-i})\right)$$
$$- 2\left((s+1)\left(B_2 + \sum_{k=1}^{2m-1} Z_k\right)\right).$$

To get the second new element we choose P and Q such that the closure of P lies on B_1 as before, but the closure of Q lies on Z_m. As before we can assume $b_1 = 0$ and the same calculation as above holds to show $d_j = 0$ and $c_i = -2i$ and $b_2 = -2(s+1)$.

The first difference is that we have

$$c_s - 3b_2 + 2e_1 = 0,$$

hence

$$e_1 = -2(s+1) - 1.$$

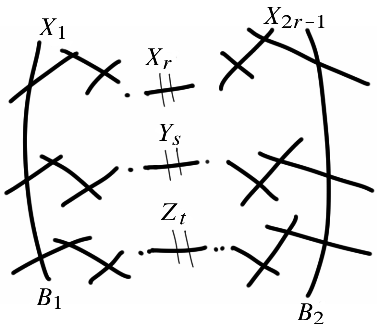
Using

$$\begin{aligned} &b_2 - 2e_1 + e_2 = 0 \\ &e_{k+1} = 2e_k - e_{k-1}, \quad 1 \le k \le m-1, \\ &e_{m-k} = e_{m+k}, \qquad 1 \le k \le m-1, \end{aligned}$$

shows $e_k = -2(s+1) - k$ for $1 \le k \le m$ and so one has

$$\operatorname{div}(\bar{f_P}) = \sum_{i=1}^{s} -2iX_i - 2(s+1)B_2 - \Big(\sum_{k=1}^{m-1}(2(s+1)+k)(Z_k + Z_{m+k}) + (2(s+1)+m)Z_m\Big).$$

4.2.7. Case VII.



In this case the stable model of the special fibre consists of two smooth rational curves B_1 and B_2 meeting at three points. The minimal regular model consists of the two smooth curves along with three chains of rational curves X_i, $1 \le i \le 2r-1$, Y_j, $1 \le j \le 2s-1$ and Z_k, $1 \le k \le 2t-1$ lying over the three points of intersection. A pair each of the closures of the Weierstrass points meet the curves X_r, Y_s and Z_t. One has $B_i^2 = -3$ and $X_i^2 = Y_j^2 = Z_k^2 = -2$.

Once again the Jacobian is of type 3 and so one expects two new elements. To get the first we choose Weierstrass points P, Q and R such that the closure of P lies on X_r, the closure of Q lies on Y_s and the closure of R lies on Z_t. Then, if f_P is as before, we have

$$\operatorname{div}(\bar{f_P}) = b_1B_1 + b_2B_2 + \sum a_iX_i + \sum c_jY_j + \sum d_kZ_k.$$

Assume $b_1 = 0$. Then one has $a_1 + c_1 + d_1 = 0$. Intersecting the divisor with X_1 shows that $b_1 - 2a_1 + a_2 = 0$. Continuing in this manner, intersecting with the X_i's one gets

$$a_i = \begin{cases} ia_1, & 1 \le i \le r, \\ ia_1 - 2(i-r), & r \le i \le 2r-1, \end{cases}$$
$$b_2 = 2ra_1 - 2r.$$

Similarly, using Y_j's one gets

$$c_j = \begin{cases} jc_1, & 1 \leq j \leq s, \\ jc_1 + 2(j-s), & s \leq j \leq 2s-1, \end{cases}$$
$$b_2 = 2sc_1 + 2s,$$

and using Z_k's one has

$$d_k = kd_1, \quad 1 \leq i \leq 2k-1,$$
$$b_2 = 2kd_1.$$

Solving this system of simultaneous equations shows $a_1 = 1, c_1 = -1$ and $b_1 = b_2 = d_1 = 0$. Hence one has

$$\partial(\Xi_{P,Q}) = 2\Big(rX_r + \sum_{i=1}^{r-1} i(X_i + X_{2r-i})\Big) - 2\Big(sY_s + \sum_{j=1}^{s-1} j(Y_j + Y_{2s-j})\Big).$$

A similar calculation works for the elements $\Xi_{Q,R}$ and $\Xi_{P,R}$. Clearly one has

$$\Xi_{P,R} = \Xi_{P,Q} + \Xi_{Q,R},$$

hence there are essentially two elements one can construct in this manner.

To check that these elements are linearly independent we intersect with X_r, Y_s and Z_t. Then $(\partial(\Xi_{P,Q}) \cdot X_r) = -2$ while $(X_r \cdot \partial\Xi_{Q,R}) = 0$, hence they are not linearly equivalent. Further, $\partial(\Xi_{P,Q}) \neq 0$. Similarly, intersecting with Y_s shows that $\partial(\Xi_{Q,R}) \neq 0$.

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References

[1] S. Bloch, Algebraic cycles and higher K-theory. *Adv. in Math.* **61** (1986), no. 3, 267–304. Zbl 0608.14004 MR 852815

[2] S. Bloch – H. Gillet – C. Soulé, Non-Archimedean Arakelov theory. *J. Algebraic Geom.* **4** (1995), no. 3, 427–485. Zbl 0866.14011 MR 1325788

[3] A. Collino, Griffiths' infinitesimal invariant and higher K-theory on hyperelliptic Jacobians. *J. Algebraic Geom.* **6** (1997), no. 3, 393–415. Zbl 0895.19001 MR 1487220

[4] C. Consani, Double complexes and Euler L-factors. *Compos. Math.* **111** (1998), no. 3, 323–358. Zbl 0932.14011 MR 1617133

[5] R. Friedman – D. R. Morrison, The birational geometry of degenerations: An overview. In *The birational geometry of degenerations (Cambridge, Mass., 1981)*, pp. 1–32, Progr. Math. 29, Birkhäuser, Boston, MA, 1983. Zbl 0493.00005 MR 690262

[6] A. N. Parşin, Minimal models of curves of genus 2, and homomorphisms of abelian varieties defined over a field of finite characteristic. *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 67–109. Zbl 0246.14007 MR 316456

[7] D. Ramakrishnan, Regulators, algebraic cycles, and values of L-functions. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, pp. 183–310, Contemp. Math. 83, American Mathematical Society, Providence, RI, 1989. Zbl 0694.14002 MR 991982

[8] M. Spiess, On indecomposable elements of K_1 of a product of elliptic curves. *K-Theory* **17** (1999), no. 4, 363–383. Zbl 0952.14008 MR 1706121

[9] R. Sreekantan, A non-Archimedean analogue of the Hodge-$\mathscr{D}$-conjecture for products of elliptic curves. *J. Algebraic Geom.* **17** (2008), no. 4, 781–798. Zbl 1149.14007 MR 2424927

[10] R. Sreekantan, Higher Chow cycles on Abelian surfaces and a non-Archimedean analogue of the Hodge-$\mathscr{D}$-conjecture. *Compos. Math.* **150** (2014), no. 4, 691–711. Zbl 1309.19010 MR 3200674

[11] J. Steenbrink, Limits of Hodge structures. *Invent. Math.* **31** (1975/76), no. 3, 229–257. Zbl 0303.14002 MR 429885

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