Rend. Sem. Mat. Univ. Padova 152 (2024), 171–228
DOI 10.4171/RSMUP/134

Algebraic theory of formal regular-singular connections with parameters

Phùng Hô Hai (*) – João Pedro dos Santos (**) –
Phạm Thanh Tâm (***)

Abstract – This paper is divided into two parts. The first is a review, through categorical lenses, of the classical theory of regular-singular differential systems over $C(\!(x)\!)$ and $\mathbb{P}^1_C \setminus \{0, \infty\}$, where C is algebraically closed and of characteristic zero. It aims to read the existing classification results as an equivalence between regular-singular systems and representations of the group $\mathbb{Z}$. In the second part, we deal with regular-singular connections over $R(\!(x)\!)$ and $\mathbb{P}^1_R \setminus \{0, \infty\}$, where $R = C[\![t_1, \ldots, t_r]\!]/I$. The picture we offer shows that regular-singular connections are equivalent to representations of $\mathbb{Z}$, now over R.

Mathematics Subject Classification (2020) – Primary 12H05; Secondary 13N15, 18M05.

Keywords – Regular-singular connections, Tannakian categories, group schemes.

1. Introduction

This paper is an outgrowth of our study of regular-singular connections through the past years. It is divided into two parts which although thematically close, are distinct in originality. Indeed, Part I is a patient revision of classical theory ([10, Chapter 4], [34, 43], [28, Section 16]) of regular-singular connections (or differential systems) in a more categorical setting, *plus* an exposition of a more recent original contribution

(*) *Indirizzo dell'A.*: Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam; phung@math.ac.vn

(**) *Indirizzo dell'A.*: Institut de Mathématiques de Jussieu – Paris Rive Gauche, 4 place Jussieu, Case 247, 75252 Paris Cedex 5, France; joao_pedro.dos_santos@yahoo.com

(***) *Indirizzo dell'A.*: Department of Mathematics, Hanoi Pedagogical University 2, Vinh Phuc, Vietnam; phamthanhtam@hpu2.edu.vn

of Deligne [12, §15]. Part II is a study of the theory of regular-singular connections on $R((x))$ and on $\mathbb{P}^1_R \setminus \{0, \infty\}$, where R is a certain complete local ring. The method behind Part II comes in great part from [24] and it is hoped that it will be a means to grasp [24] in a less complex-analytic setting.

The classical theory of formal regular-singular connections presents roughly two classifications of these objects: one by reducing each system to one with constant coefficients ([10, 28, 43]), and one by means of tensor products of unipotent and rank-one connections [34]. As beautiful as they are, these classifications tend to give an incomplete picture due to the lack of categorical structures and equivalences. For example, although systems of differential equations with constant coefficients play a fundamental role, their natural properties are seldom addressed. Our take on the matter, accomplished in Part I, is to use [11] *as a guiding principle* and obtain an equivalence between formal regular-singular systems and representations of the "fundamental group," which is $\mathbb{Z}$. As far as we know, this point of view is adopted, over $\mathbb{C}$, only in [42]. In addition, under this mindset, we are able to comment on the important theory of Deligne's tensor product of categories. Our approach to the theory of connections on $\mathbb{P}^1 \setminus \{0, \infty\}$ follows the same path, but its structuring is facilitated by the formal case.

Part II contains new material on formal differential modules whose ring of constants is a complete local ring. Our original motivation for writing down this piece of work was to give a less technical and algebraic version of our paper [24] which, nevertheless, would allow us to see the main ideas in it. To wit, an abstract picture stemming from [24] is the following: Let C be an algebraically closed field of characteristic zero, R a noetherian, local and complete C-algebra with maximal ideal $\mathfrak{r}$ and residue field C. We now give ourselves two R-linear categories $\mathcal{C}$ and $\mathcal{C}'$; denote by $\mathcal{C}_n$ and $\mathcal{C}'_n$ the full subcategories of objects "annihilated by $\mathfrak{r}^{n+1}$." Now, suppose that $\mathcal{C}_0 \simeq \mathcal{C}'_0$. We wish to conclude that $\mathcal{C} \simeq \mathcal{C}'$. The strategy is to promote $\mathcal{C}_0 \simeq \mathcal{C}'_0$ into an equivalence $\mathcal{C}_n \simeq \mathcal{C}'_n$ for all n and then to "pass to the limit." (Needless to say, this is only reasonable in certain cases.) Part II of the present work goes through this idea in the special case where $\mathcal{C}$ is the category of regular-singular formal connections and $\mathcal{C}'$ is the category of representations of the abstract group $\mathbb{Z}$. The equivalence between $\mathcal{C}_0$ and $\mathcal{C}'_0$ is derived here from the results of Part I, while in [24] we relied on [11].

Let us now review the remaining sections separately. In what follows, C is an algebraically closed field of characteristic zero and for any C-algebra R, we let ϑ stand for the derivation of $R((x)) = R[[x]][x^{-1}]$ defined by $\vartheta \sum a_k x^k = \sum k a_k x^k$.

Section 2 serves to introduce basic notation and definitions: especially important are the *logarithmic connections* and the *regular-singular* ones over $C((x))$; see Definitions 2.1 and 2.2. Section 3 covers basic facts on *Euler connections*, which correspond to differential systems of the form $\vartheta y = Ay$ in which A is a matrix with entries on C

(Definition 3.1). The approach is *categorical* and we study the *Euler functor* from the category of "endomorphisms" to the category of logarithmic connections (Definition 3.2). Most findings contain little more than simple remarks on spectral analysis of linear operators in finite dimension.

Section 4 brings to light one of the main actors in the whole theory: the *residue endomorphism* of a logarithmic connection. Most results of this part are well known, although not phrased in our language (see Theorems 4.1 and 4.2). But not all is referencing, and in Proposition 4.4 we show, motivated by our categorical take, how to limit "the size of poles" between an arrow of logarithmic models in terms of the difference of the exponents. Later, this plays an important role when dealing with regular-singular connections "depending on parameters" (e.g. the proof of Theorem 9.1). The section ends with the construction of preferred logarithmic models of regular-singular connections (Theorem 4.5); we name these Deligne–Manin models, but many other names are in the literature (canonical extensions, τ-extensions, etc.).

Section 5 revisits Manin's elegant paper [34] with the intention of presenting its gist as an equivalence between the categories of representations of $\mathbb{Z}$ and regular-singular connections. It begins by using classical results to prove a fundamental structural theorem of [34] and then goes on to study *unipotent* (Section 5.2) and *diagonalizable* (Section 5.3) regular-singular connections. The former category is then proved to be equivalent to the category of unipotent endomorphism (see Theorem 5.4); this allows us to observe that unipotent regular-singular connections amount to representations of the additive group (Corollary 5.5). We go on to exhibit an equivalence between the category of diagonalizable connections and representations of the diagonal group scheme whose group of characters is $C/\mathbb{Z}$. Calling on set theory, we note that $C/\mathbb{Z} \simeq C^{\times}$, which puts us in an ideal position to establish an equivalence between regular-singular connections and representations of $\mathbb{Z}$. This final goal is obtained by means of the Deligne tensor product of abelian categories. This construction is a delicate piece of category theory so that some of the necessary results are to be written down in a separate work [15]. Here we content ourselves with a brief presentation of the definitions and fundamental results (Section 5.4). In Section 5.5, all is put together to arrive at the conclusion motivating the section, which is Corollary 5.14.

With Section 6 we end Part I with a review of an equivalence between regular-singular connections on $C((x))$ and on $\mathbb{P}^1 \setminus \{0, \infty\}$ (Theorem 6.4). Mostly we follow the ideas in [12, Sections 15.28–36] in proving the key non-trivial point: all regular-singular connections on $\mathbb{P}^1 \setminus \{0, \infty\}$ are "Euler connections"; see Proposition 6.5. From that and the knowledge obtained in the previous sections, the desired equivalence follows without much effort.

We now begin to review the sections pertaining to Part II. In Section 7 we fix a certain finite-dimensional C-algebra Λ and start exploring the notion of objects in

C-linear categories carrying an action of Λ (Definition 7.1). This is to be applied to categories of regular-singular connections and we show that most results from Part I carry over to this context. See for example the existence of Deligne–Manin models stated in Theorems 7.8 and 7.12. Let us draw the reader's attention to the notion of freeness in relation to[1] Λ (see Definitions 7.2 and 7.10), which plays a key role in the rest of the paper.

In Section 8, after fixing a *complete local noetherian C-algebra R* having residue field C, we begin the study of *regular-singular connections over $R((x))$*. One of the most relevant concepts in this case is our definition of *residues and exponents* (Definition 8.6) stating that "exponents should be indifferent to reduction modulo the maximal ideal of R." In particular, exponents are elements of C. This definition allows us to prove Theorem 8.10, the analogue of Theorem 4.1, which shows that Euler connections still play a central role in this theory. Then, applying ideas around the theme of Hensel's lemma, we explain how to lift the Jordan decomposition of an endomorphism between R-modules (Corollary 8.12), which in turn allows us to deduce Theorem 8.16, paralleling Theorem 4.2 in the present context. At this point, our assumptions on the $R[[x]]$-modules are in many places strong – they are to be free – and improvements appear in Section 9. We also draw attention to Theorem 8.18 and Remark 8.20. In the former result, we present a criterion for a connection over $R((x))$ to underlie a flat $R((x))$-module. Since the fibres of Spec $R((x)) \to$ Spec R are not generally of finite type over the residue field, the proof of Theorem 8.18 relies on a beautiful result of Y. André, which we re-prove swiftly in Remark 8.20.

Section 9 contains the first main result, Corollary 9.7. It shows the equivalence

$$(*)\qquad \begin{matrix}\text{regular-singular connections}\\ \text{over } R((x))\end{matrix} \xrightarrow{\sim} R\text{-representations of } \mathbb{Z},$$

thus obtaining the exact analogue of Deligne–Manin's theory from Section 5. (No assumption is made on the nature of the $R((x))$ or R-modules underlying connections or representations.) The heart of the matter is the existence of certain preferred logarithmic models (Deligne–Manin) for regular-singular connections over $R((x))$ and these are obtained in Theorem 9.1. The proof of this result relies on the fact that we are able to "pass to the limit" of the models obtained previously – since $R[[x]]$ is a complete local ring – to construct a suitable logarithmic model. Such a limit process is only possible since exponents do not change from "truncation to truncation" and since the "size of

(1) In [24], we used the expression "relatively to", but after more careful study, we prefer to write "in relation to", as will be done in this paper.

the pole" of a given arrow is controlled by the differences of exponents (Proposition 4.4). To see what can easily go wrong, the reader should read Counterexample 9.3. Once the logarithmic models of Theorem 9.1 have been shown to exist, we are then able to apply a limit process to arrive at the equivalence ($*$).

The paper then ends with Section 10, which shows a second main result: the restriction functor

$$\begin{matrix}\text{regular-singular connections} \\ \text{over } \mathbb{P}_R \setminus \{0, \infty\}\end{matrix} \xrightarrow{\text{restriction}} \begin{matrix}\text{category of regular-singular} \\ \text{connections over } R((x))\end{matrix}$$

is an equivalence. (See Theorem 10.1.) The proof is based on the previous techniques, with one important modification: the fact that modules over $R[\![x]\!]$ are constructed from limits leaves room for Grothendieck's GFGA, stating that coherent modules over $\mathbb{P}_R$ are constructed by limits of coherent modules over the truncations of $\mathbb{P}_R$ modulo the maximal ideal.

Finally, let us call the reader's attention to some important works on "differential structures depending on parameters" which have appeared in recent times: these are [20, 21, 37–39]. At the end of the introduction in [24], the reader will find a brief summary of some of the ideas behind these works.

Notation and conventions

(1) In this text, C stands for an algebraically closed field of characteristic zero.

(2) Given a (commutative and unital) ring R, we let $R((x))$ stand for $R[\![x]\!][x^{-1}]$ and $\vartheta\colon R((x)) \to R((x))$ the derivation defined by

$$\vartheta \sum a_n x^n = x\frac{d}{dx}\sum a_n x^n = \sum n a_n x^n.$$

(3) We let $\mathrm{M}_{m\times n}(R)$, respectively $\mathrm{M}_n(R)$, stand for the associative ring of $m \times n$ matrices, respectively $n \times n$ matrices, with entries in a ring R.

(4) For a prime ideal $\mathfrak{p}$ in a ring R, we let $\boldsymbol{k}(\mathfrak{p})$ stand for the residue field of the local ring $R_{\mathfrak{p}}$.

(5) If $A\colon V \to V$ is an endomorphism of vector space over C, we let Sp_A stand for the set of its eigenvalues. Given ϱ an eigenvalue, $\mathbf{G}(A, \varrho)$ denotes the generalized eigenspace of A associated to ϱ.

(6) For an abstract group or group scheme G, we let $\mathrm{Rep}_C(G)$ stand for the category of finite-dimensional C-linear representations of G.

(7) Throughout the text, τ stands for a subset of C such that the natural map $\tau \to C/\mathbb{Z}$ is bijective.

(8) If A and B are subsets of C, we denote by $A \ominus B$ the set $\{a - b\colon a \in A,\ b \in B\}$.

Part I

2. Definitions, terminology and basic results

For the convenience of the reader and to ease referencing, we recall some standard definitions.

Definition 2.1. The *category of connections*, $\mathbf{MC}(C((x))/C)$, has for

objects those couples (M, ∇) consisting of a finite-dimensional $C((x))$-space and a C-linear endomorphism $\nabla\colon M \to M$, called *the derivation*, satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$, and the

arrows from (M, ∇) to (M', ∇') are $C((x))$-linear morphisms $\varphi\colon M \to M'$ such that $\nabla'\varphi = \varphi\nabla$.

The category of *logarithmic connections*, $\mathbf{MC}_{\log}(C[[x]]/C)$, has for

objects those couples $(\mathcal{M}, \nabla)$ consisting of a finite $C[[x]]$-module and a C-linear endomorphism, called the *derivation*, $\nabla\colon \mathcal{M} \to \mathcal{M}$ satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$, and

arrows from $(\mathcal{M}, \nabla)$ to $(\mathcal{M}', \nabla')$ are $C[[x]]$-linear morphisms $\varphi\colon \mathcal{M} \to \mathcal{M}'$ such that $\nabla'\varphi = \varphi\nabla$.

As is well known, $\mathbf{MC}(C((x))/C)$ is an abelian category: subobjects, respectively quotients, will be called subconnections, respectively quotient connections. Also, when speaking of the rank of a connection, we shall mean the dimension of the underlying $C((x))$-vector space. The category $\mathbf{MC}_{\log}(C[[x]]/C)$ is also abelian.

We possess an evident C-linear functor

$$\gamma\colon \mathbf{MC}_{\log}(C[[x]]/C) \longrightarrow \mathbf{MC}(C((x))/C).$$

Definition 2.2. An object $M \in \mathbf{MC}(C((x))/C)$ is said to be *regular-singular* if it is isomorphic to a certain $\gamma(\mathcal{M})$. The full category of $\mathbf{MC}(C((x))/C)$ whose objects are regular-singular will be denoted by $\mathbf{MC}_{\mathrm{rs}}(C((x))/C)$.

Given $M \in \mathbf{MC}_{\mathrm{rs}}(C((x))/C)$, any object $\mathcal{M} \in \mathbf{MC}_{\log}(C[[x]]/C)$ such that $\gamma(\mathcal{M}) \simeq M$ is called a *logarithmic model of* M. In the case that the model $\mathcal{M}$ is, in addition, a *free* $C[[x]]$-module, we shall speak of a logarithmic *lattice*.

It is not hard to see that any object in $\mathbf{MC}_{\mathrm{rs}}(C((x))/C)$ admits a logarithmic lattice; indeed, if $\mathcal{M}$ is a logarithmic model, then $\mathcal{M}_{\mathrm{tors}} = \{m \in \mathcal{M} : xm = 0\}$ is stable under ϑ and $\mathcal{M}/\mathcal{M}_{\mathrm{tors}}$ is the desired logarithmic lattice.

Given $(\mathcal{M}, \nabla)$ and $(\mathcal{M}', \nabla')$ in $\mathbf{MC}_{\log}(C[\![x]\!]/R)$, the $C[\![x]\!]$-module $\mathcal{M} \otimes_{C[\![x]\!]} \mathcal{M}'$ becomes a logarithmic connection by means of

$$\begin{aligned} \nabla \otimes \nabla' \colon \mathcal{M} \otimes \mathcal{M}' &\longrightarrow \mathcal{M} \otimes \mathcal{M}', \\ \sum m_i \otimes m_i' &\longmapsto \sum_i \nabla(m_i) \otimes m_i' + m_i \otimes \nabla'(m_i'). \end{aligned}$$

We then obtain in $\mathbf{MC}_{\log}(C[\![x]\!]/C)$ the structure of a C-linear tensor category which gives $\mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)$ the structure of a C-linear tensor category. (Note that in $\mathbf{MC}_{\log}$ we do not always have "duals.") Similar constructions then allow us to obtain the next proposition, which is explicitly written down in [42, Lemma 3.10]. See also the proof of Proposition 8.3 further ahead.

Proposition 2.3. *The category* $\mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)$ *is an abelian subcategory of* $\mathbf{MC}(C(\!(x)\!)/C)$ *which is stable under direct sums, duals and tensor products. Furthermore, given* $(M, \nabla) \in \mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)$ *and a subobject* $(M', \nabla') \subset (M, \nabla)$*, respectively a quotient* $(M, \nabla) \to (M'', \nabla'')$*, then both* (M', ∇') *and* (M'', ∇'') *are regular-singular.* ■

Of course, not all objects of $\mathbf{MC}_{\log}(C[\![x]\!]/C)$ have "duals."

Example 2.4 (Twisted models). Let $\delta \in \mathbb{Z}$. Write $\mathbb{1}(\delta)$ for the $C[\![x]\!]$-submodule of $C(\!(x)\!)$ generated by $x^{-\delta}$. Clearly $\vartheta(\mathbb{1}(\delta)) \subset \mathbb{1}(\delta)$ and in this way, whenever $\delta \leq 0$, we obtain a subobject of $(C[\![x]\!], \vartheta)$. More generally, for any $\mathcal{M} \in \mathbf{MC}_{\log}(C[\![x]\!]/C)$, we obtain a new logarithmic connection $\mathcal{M}(\delta)$ by defining $\mathcal{M}(\delta) = \mathbb{1}(\delta) \otimes \mathcal{M}$.

Example 2.5. Let $(\mathcal{M}, \nabla)$ and $(\mathcal{M}', \nabla')$ be objects from $\mathbf{MC}_{\log}(C[\![x]\!]/C)$ and on the $C[\![x]\!]$-module $\mathcal{H} := \mathrm{Hom}_{C[\![x]\!]}(\mathcal{M}, \mathcal{M}')$ let us define

$$D \colon \mathcal{H} \longrightarrow \mathcal{H}, \quad h \longmapsto \nabla' \circ h - h \circ \nabla.$$

This defines a logarithmic connection called the *internal "Hom."* In analogous fashion, we can defined the internal "Hom" for two connections.

By means of the canonical isomorphism

$$\mathrm{Hom}_{C[\![x]\!]}(\mathcal{M}, \mathcal{M}') \underset{C[\![x]\!]}{\otimes} C(\!(x)\!) \simeq \mathrm{Hom}_{C(\!(x)\!)}(\gamma\mathcal{M}, \gamma\mathcal{M}'),$$

we see that the internal "Hom" constructed from two regular-singular connections is also regular-singular.

3. Euler connections

The simplest class of examples of logarithmic connections is given by "Euler" connections (the name is inspired by [10, Chapter 4, Section 5]; it is also adopted by [28, Example 15.9]). In this section we shall write **MC** and $\mathbf{MC}_{\log}$ in place of $\mathbf{MC}(C((x))/C)$ and $\mathbf{MC}_{\log}(C[\![x]\!]/C)$.

Definition 3.1 (Euler connections). Let V be a finite-dimensional vector space over C and $A \in \mathrm{End}_C(V)$. The Euler logarithmic connection associated to the couple (V, A) is defined by the couple $(C[\![x]\!] \otimes_C V, D_A)$, where $D_A(f \otimes v) = \vartheta(f) \otimes v + f \otimes Av$. Notation: $\mathrm{eul}(V, A)$.

Since Euler connections play a prominent role in the theory, let us spend some more time studying them.

Definition 3.2. Let **End** be the category whose

objects are couples (V, A) consisting of a finite-dimensional C-space V and a C-linear endomorphism $A\colon V \to V$, and whose

arrows from (V, A) and (V', A') are C-linear morphisms $\varphi\colon V \to V'$ such that $A'\varphi = \varphi A$.

Needless to say, letting $\mathfrak{e} = C$ be the one-dimensional Lie algebra, **End** is none other than $\mathrm{Rep}_C(\mathfrak{e})$. In particular, **End** comes with the canonical structure of an abelian, C-linear *tensor* category [7, §3, Nos. 1–2]. (Its unit object is $(C, 0)$.) Moreover, for any couple (V, A) and (V', A') in **End**, we can produce an "internal Hom" $\mathrm{Hom}_C((V, A), (V', A'))$ [7, §3, No. 3, Proposition 3] by endowing $\mathrm{Hom}_C(V, V')$ with the endomorphism

$$H_{A,A'}\colon \mathrm{Hom}_C(V, V') \longrightarrow \mathrm{Hom}_C(V, V'), \quad \varphi \longmapsto A'\varphi - \varphi A.$$

With these properties in sight, we now have a functor

$$\mathrm{eul}\colon \mathbf{End} \longrightarrow \mathbf{MC}_{\log};$$

it is obviously C-linear, exact and faithful. In addition, eul is a tensor functor (the tensor structure on $\mathbf{MC}_{\log}$ is explained in Section 2).

As it should, the obvious morphism of $C[\![x]\!]$-module

$$\mathrm{eul}(\mathrm{Hom}_C((V, A), (V', A'))) \longrightarrow \mathrm{Hom}_{C[\![x]\!]}(C[\![x]\!] \otimes V, C[\![x]\!] \otimes V')$$

defines an isomorphism in $\mathbf{MC}_{\log}$, where the right-hand-side has the "internal Hom" logarithmic connection (cf. Example 2.5).

We end this section by studying the influence of eul on Hom sets.

LEMMA 3.3. *The following claims are true:*

(1) *Suppose that* Sp_A *contains no negative integer. Then any horizontal section of* $\mathrm{eul}(V, A)$ *has the form* $1 \otimes v$ *with* $v \in \mathrm{Ker}(A)$.

(2) *Let* (V, A) *and* (V', A') *have the following property: the difference* $\mathrm{Sp}_{A'} \ominus \mathrm{Sp}_A$ *contains no negative integer. Then each arrow* $\Phi\colon \mathrm{eul}(V, A) \to \mathrm{eul}(V', A')$ *is of the form* $\mathrm{id} \otimes \varphi\colon C[\![x]\!] \otimes_C V \to C[\![x]\!] \otimes_C V'$ *for a certain* $\varphi\colon V \to V'$ *such that* $A'\varphi = \varphi A$. *In addition, if* $\mathrm{id} \otimes \psi = \Phi$, *then* $\varphi = \psi$. *Said otherwise, the natural arrow*

$$\mathrm{Hom}_{\mathbf{E}nd}((V, A), (V', A')) \longrightarrow \mathrm{Hom}_{\mathbf{MC}_{\log}}(\mathrm{eul}(V, A), \mathrm{eul}(V', A'))$$

is bijective.

PROOF. (1) For each $v \in \mathrm{Ker}(A)$, the element $1 \otimes v \in \mathrm{eul}(V, A)$ is clearly horizontal. Conversely, let $\sum_n x^n \otimes v_n$ be horizontal. Then

$$0 = \sum_n x^n \otimes (Av_n + nv_n).$$

This shows that $v_0 \in \mathrm{Ker}(A)$. In addition, if $n > 0$, the equation $Ac = -nc$ cannot have a non-zero solution in V, and hence $v_n = 0$.

(2) Let $\Phi\colon \mathrm{eul}(V, A) \to \mathrm{eul}(V', A')$ be a non-zero arrow in $\mathbf{MC}_{\log}$ and regard it as a non-zero horizontal element of

$$\mathrm{Hom}(\mathrm{eul}(V, A), \mathrm{eul}(V;, A')) \simeq \mathrm{eul}(\mathrm{Hom}_C(V, V'), H_{A,A'}).$$

The assumption on the spectra together with a classical result from linear algebra shows that $H_{A,A'}$ cannot have a negative integer as eigenvalue: indeed, if $T \neq 0$ is such that $A'T - TA = -kT$, then $\mathrm{Sp}_{A'+k\mathrm{I}} \cap \mathrm{Sp}_A \neq \varnothing$ [43, Theorem 4.1, p. 19], which forces $-k \in \mathrm{Sp}_{A'} \ominus \mathrm{Sp}_A$. By part (1), it follows that $\Phi \in C[\![x]\!] \otimes \mathrm{Hom}_C(V, V')$ comes from an element $\varphi \in \mathrm{Hom}_C(V, V')$ such that $0 = H_{A,A'}(\varphi)$. The fact that φ is unique follows from the faithfulness of eul. ∎

4. Basic results in the theory of regular-singular connections

We shall continue to write $\mathbf{MC}_{\log}$ instead of $\mathbf{MC}_{\log}(C[\![x]\!]/C)$ and $\mathbf{MC}_{\mathrm{rs}}$ instead of $\mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)$.

4.1 – *The residue and its applications*

Given $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}$, the very definition of the Leibniz rule ensures $\nabla(x\mathcal{M}) \subset x\mathcal{M}$, so that we obtain, by passage to the quotient, a C-linear endomorphism

$$\mathrm{res}(\nabla)\colon \mathcal{M}/(x) \longrightarrow \mathcal{M}/(x),$$

called the *residue* of ∇. The set of eigenvalues of $\mathrm{res}(\nabla)$ is named the set of *exponents* of ∇ and will be denoted by $\mathrm{Exp}(\nabla)$.

The relevance of the set of exponents is visible through the following central results. Their proofs are to be found in the classics [10] or [43].

Theorem 4.1 ([10, Chapter 4, Theorem 4.1, p. 119] or [43, Theorem 5.1, p. 21]). *Let $\mathcal{M}$ be a free $C[\![x]\!]$-module of finite rank affording a logarithmic connection $\nabla\colon \mathcal{M} \to \mathcal{M}$ such that no two of its exponents differ by a* positive integer *(e.g. they all lie in τ). Then $(\mathcal{M}, \nabla) \simeq \mathrm{eul}(\mathcal{M}/(x); \mathrm{res}(\nabla))$.* ■

Theorem 4.2 ("Shearing"; cf. [10, Chapter 4, Section 4, Lemma, p. 120] or [43, Section 17.1]). *Let (E, ∇_E) be an object of $\mathbf{MC}_{\mathrm{rs}}$. Then it is possible to find a logarithmic lattice $(\mathcal{E}, \nabla_{\mathcal{E}})$ for (E, ∇_E) such that all exponents of $\nabla_{\mathcal{E}}$ lie in τ.* ■

Corollary 4.3. *Let $(M, \nabla) \in \mathbf{MC}_{\mathrm{rs}}$. Then there exists a finite-dimensional vector space V and $A \in \mathrm{End}_C(V)$ such that*

(1) *all eigenvalues of A are in τ and*

(2) *$M \simeq \gamma\mathrm{eul}(V, A)$.* ■

Another relevant feature of regular-singular connections unfolded by the exponents is the following:

Proposition 4.4. *Let $\phi\colon E \to F$ be an arrow of $\mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)$. Let $\mathcal{E}$ and $\mathcal{F}$ be models for E and F and assume that $\mathcal{F}$ is in fact a lattice. We abuse notation and write ϑ for all derivations in sight (viz. $E \to E$, $\mathcal{E} \to \mathcal{E}$, etc):*

(1) *Let $\varrho \in \mathrm{Exp}(\mathcal{E})$ and let $s \in \mathcal{E}$ be such that*

$$(\vartheta - \varrho)^{\mu}(s) \in x\mathcal{E}$$

for a certain $\mu \in \mathbb{N}$. Then, for all $k \in \mathbb{Z}$, we have

$$(\vartheta - (\varrho + k))^{\mu}(x^k\phi(s)) = x^{k+1}\phi(\mathcal{E}).$$

(2) *Let δ be the largest integer in $\mathrm{Exp}(\mathcal{F}) \ominus \mathrm{Exp}(\mathcal{E})$. Then $x^{\delta}\phi(\mathcal{E}) \subset \mathcal{F}$. In particular, adopting the notation of Example 2.4, there exists $\Phi\colon \mathcal{E} \to \mathcal{F}(\delta)$ from $\mathbf{MC}_{\log}$ such that $\gamma\Phi = \phi$.*

(3) *Suppose that* $\mathrm{Exp}(\mathcal{F}) \ominus \mathrm{Exp}(\mathcal{E})$ *contains no positive integer. Then the natural arrow*

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Hom}(E, F)$$

is bijective.

Proof. (1) Using the formula

$$[\vartheta - (\varrho + i)]^{\mu} x^i = x^i (\vartheta - \varrho)^{\mu},$$

it follows that

$$\begin{aligned}[\vartheta - (\varrho + k)]^{\mu}(x^k \phi(s)) &= x^k (\vartheta - \varrho)^{\mu}(\phi(s)) \\ &= x^k \phi[(\vartheta - \varrho)^{\mu}(s)] \\ &\in x^{k+1}\phi(\mathcal{E}).\end{aligned}$$

(2) If $x^\delta \phi(\mathcal{E}) \subset \mathcal{F}$ we have nothing to do. Then let $k > \delta$ be such that $x^k \phi(\mathcal{E}) \subset \mathcal{F}$. We choose $\varrho \in \mathrm{Exp}(\mathcal{E})$ and $s \in \mathcal{E} \setminus x\mathcal{E}$ such that $(\vartheta - \varrho)^{\mu}(s) \in x\mathcal{E}$. By the previous item,

$$(\vartheta - (\varrho + k))^{\mu}(x^k \phi(s)) \in x^{k+1}\phi(\mathcal{E}) \subset x\mathcal{F}.$$

Since $x^k \phi(s) \in \mathcal{F}$ and $\varrho + k$ cannot be an eigenvalue of $\mathrm{res}_{\mathcal{F}}$, it follows that $x^k \phi(s) \in x\mathcal{F}$, which means that $x^{k-1}\phi(s) \in \mathcal{F}$ because $\mathcal{F}$ has no x-torsion.

Now let μ_α be the multiplicity of the exponent α and write

$$\mathcal{E}/x\mathcal{E} \simeq \bigoplus_{\alpha \in \mathrm{Exp}} \mathrm{Ker}(\mathrm{res}_{\mathcal{E}} - \alpha)^{\mu_\alpha}.$$

For any $t \in \mathcal{E}$, we have

$$t = \sum_{\alpha} s_\alpha + x t',$$

where $(\vartheta - \alpha)^{\mu_\alpha}(s_\alpha) \in x\mathcal{E}$ for each α and $t' \in \mathcal{E}$. As a consequence, $x^{k-1}\phi(s_\alpha) \in \mathcal{F}$ and we conclude that

$$x^{k-1}\phi(t) \in \mathcal{F}.$$

Proceeding by induction, we conclude that $x^\delta \phi(\mathcal{E}) \subset \mathcal{F}$.

(3) This follows easily from the previous item and the observation that an arrow $\phi\colon \mathcal{E} \to \mathcal{F}$ which induces $0\colon E \to F$ must be trivial as $\mathcal{F} \to F$ is injective. ■

Putting together Corollary 4.3 and Proposition 4.4 (3) we arrive at the following theorem:

Theorem 4.5 (Deligne–Manin lattices; [11, Proposition II.5.4]). *Let $M \in \mathbf{MC}_{\mathrm{rs}}$ be given. There exists a logarithmic* lattice *$\mathcal{M}$ for M having all its exponents in τ. In addition, if $\mathcal{M}' \in \mathbf{MC}_{\log}$ is another logarithmic lattice for M with all exponents in τ, then there exists a* unique *isomorphism $\varphi\colon \mathcal{M} \to \mathcal{M}'$ rendering the diagram*

$$\begin{array}{ccc} \gamma(\mathcal{M}) & \xrightarrow{\sim} & M \\ & \searrow^{\gamma(\varphi)} & \uparrow \wr \\ & & \gamma(\mathcal{M}') \end{array}$$

commutative. ■

5. Manin's theory revisited

Manin [34] gives a classification of objects in $\mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)$ using certain specific models (M^{ξ} and $M^{(a)}$ in his notation). We wish to rewrite his results in the light of Euler connections (Section 3), categories, functors and group schemes. The strategy of this undertaking is to break up the category of regular-singular connections into those which are unipotent and those which are diagonal.

As before, we write here

$$\mathbf{MC}, \quad \mathbf{MC}_{\mathrm{rs}} \quad \text{and} \quad \mathbf{MC}_{\log}$$

instead of

$$\mathbf{MC}(C(\!(x)\!)/C), \quad \mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C) \quad \text{and} \quad \mathbf{MC}_{\log}(C[\![x]\!]/C).$$

5.1 – *Jordan blocks*

For each $\lambda \in C$ and each positive integer r, let

$$U_{r,\lambda} = \begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots & \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \cdots & 0 & 1 & \lambda \end{pmatrix}$$

be the *Jordan matrix* of size r and eigenvalue λ. Let $J_r(\lambda)$ be the object $(C^r, U_{r,\lambda})$ of **End** and, for a multi-index of positive integers $\mathbf{r} = (r_1, \ldots, r_n)$, let

$$J_{\mathbf{r}}(\lambda) = J_{r_1}(\lambda) \oplus \cdots \oplus J_{r_n}(\lambda).$$

With this notation, Jordan's decomposition theorem and Theorem 4.3 immediately prove the following result:

THEOREM 5.1 (Cf. [34, Theorem 4]). *Let $M \in \mathbf{MC}_{\mathrm{rs}}$ be given and suppose that M is indecomposable and of dimension r. Then $M \simeq \gamma(\mathrm{eul}\, J_r(\lambda))$ for a certain $\lambda \in \tau$.* ∎

5.2 – *Unipotent objects*

In an abelian tensor category (in the sense of [14, Definition 1.15, p. 118]), an object is *unipotent* if it has a filtration whose graded pieces are isomorphic to the unit object (see for example [41, Definition 1.1.9]). Let $\mathbf{MC}_{\mathrm{rs}}^{u}$ and $\mathbf{End}^{u}$ be the categories of unipotent objects in $\mathbf{MC}_{\mathrm{rs}}$ and $\mathbf{End}$. According to [41, Proposition 1.2.1, p. 521], both $\mathbf{MC}_{\mathrm{rs}}^{u}$ and $\mathbf{End}^{u}$ are *abelian*. (This can, of course, be verified directly without much effort.) Another straightforward exercise is to show that $\mathbf{MC}_{\mathrm{rs}}^{u}$ and $\mathbf{End}^{u}$ are tensor subcategories of $\mathbf{MC}_{\mathrm{rs}}$ and $\mathbf{End}$, respectively.

The following simple lemmas will be employed below.

LEMMA 5.2. *Let $(V, A) \in \mathbf{End}$ be given. The following conditions are equivalent:*

(1) *(V, A) is unipotent.*

(2) *A is nilpotent.*

(3) *The spectrum of A is $\{0\}$.* ∎

LEMMA 5.3. *Let E be a unipotent object of $\mathbf{MC}_{\mathrm{rs}}$ and $\psi: E \to Q$ an epimorphism in $\mathbf{MC}_{\mathrm{rs}}$. Then Q is also a unipotent.* ∎

With this vocabulary at hand, we now have the following theorem:

THEOREM 5.4. *The functor*

$$\gamma\mathrm{eul}\colon \mathbf{End}^{u} \longrightarrow \mathbf{MC}_{\mathrm{rs}}^{u}$$

is an equivalence.

PROOF. Let us choose τ such that $\tau \cap \mathbb{Z} = \{0\}$. If (V, A) and (W, B) are such that Sp_A and Sp_B are contained in τ, then Lemma 3.3 (2) and subsequently Proposition 4.4(3) ensure that the natural arrows

$$\begin{aligned}\mathrm{Hom}_{\mathbf{End}}((V, A), (W, B)) &\longrightarrow \mathrm{Hom}_{\mathbf{MC}_{\log}}(\mathrm{eul}(V, A), \mathrm{eul}(W, B)) \\ &\longrightarrow \mathrm{Hom}_{\mathbf{MC}_{\mathrm{rs}}}(\gamma\mathrm{eul}(V, A), \gamma\mathrm{eul}(W, B))\end{aligned}$$

are bijections. Because of Lemma 5.2 and the choice of τ, this fact proves that $\gamma\mathrm{eul}$ is fully faithful when restricted to $\mathbf{End}^{u}$.

Let $M \in \mathbf{MC}_{\mathrm{rs}}$ be non-zero, unipotent and indecomposable. By Theorem 5.1, there exists $\lambda \in \tau$ and $r > 0$ such that $M \simeq \gamma\mathrm{eul}(J_r(\lambda))$. Unipotency ensures the existence of a non-trivial arrow

$$\varphi\colon \gamma\mathrm{eul}(J_1(0)) \longrightarrow \gamma\mathrm{eul}(J_r(\lambda)).$$

From the bijection

$$\mathrm{Hom}(J_1(0), J_r(\lambda)) \longrightarrow \mathrm{Hom}(\gamma\mathrm{eul}(J_1(0)), \gamma\mathrm{eul}(J_r(\lambda)))$$

mentioned before, we conclude that $\mathrm{Hom}(J_1(0), J_r(\lambda)) \neq 0$. This shows that $\lambda = 0$ and consequently $J_r(\lambda)$ is unipotent in **End**. Hence M belongs to the essential image of $\gamma\mathrm{eul}$. In general, we note that any object of $\mathbf{MC}_{\mathrm{rs}}^u$ can be decomposed into a direct sum of indecomposable objects and that these constituents are unipotent because of Lemma 5.3. ∎

The task of describing the category $\mathbf{MC}_{\mathrm{rs}}^u$ now benefits from a well-known fact from the theory of algebraic groups.

If $\mathbb{G}_a = \mathrm{Spec}\, C[t]$ is the additive group scheme, [16, Chapter II, Section 2, no. 2.1 (a), p. 178] explains that there exists an equivalence

$$\mathrm{lev}_1\colon \mathrm{Rep}_C(\mathbb{G}_a) \longrightarrow \mathbf{End}^u$$

defined by associating to any representation $\rho\colon \mathbb{G}_a \to \mathrm{GL}(V)$ the *nilpotent* endomorphism

$$\log(\rho(1))\colon V \longrightarrow V.$$

(The logarithm of a unipotent endomorphism is defined as usual [6, Chapter II, §6, no. 1, p. 51].) We derive the following corollary:

Corollary 5.5. *The composition*

$$\mathrm{Rep}_C(\mathbb{G}_a) \xrightarrow{\mathrm{lev}_1} \mathbf{End}^u \xrightarrow{\gamma\mathrm{eul}} \mathbf{MC}_{\mathrm{rs}}^u$$

is an equivalence. ∎

5.3 – *Diagonalizable regular-singular connections*

Definition 5.6. A connection (E, ∇) in $\mathbf{MC}_{\mathrm{rs}}$ is diagonalizable if it is the direct sum of one-dimensional regular-singular connections. The full subcategory of all diagonalizable regular-singular connections will be denoted by $\mathbf{MC}_{\mathrm{rs}}^{\ell}$.

Obviously $\mathbf{MC}_{\mathrm{rs}}^{\ell}$ is C-linear, stable under tensor products and duals in $\mathbf{MC}_{\mathrm{rs}}$. In addition, it is a standard exercise in the theory of representations of associative rings to prove that $\mathbf{MC}_{\mathrm{rs}}^{\ell}$ is an abelian subcategory of $\mathbf{MC}$ (and hence an abelian subcategory of $\mathbf{MC}_{\mathrm{rs}}$). Indeed, letting $\mathcal{D}$ stand for the ring of differential operators, $\mathbf{MC}$ is the category of left $\mathcal{D}$-modules whose dimension over $C((x))$ is finite, and the fact that $\mathbf{MC}_{\mathrm{rs}}^{\ell}$ is an abelian subcategory is a straightforward consequence of the study of semi-simple modules made in [9, Chapter VIII, §4]; see in particular Corollary 3 of no. 1 on p. 52.

Now let

$$\mathbb{X} = \left\{ \begin{array}{c} \text{isomorphisms classes} \\ \text{of rank one objects in } \mathbf{MC}_{\mathrm{rs}} \end{array} \right\}$$

and endow $\mathbb{X}$ with the group structure induced by the tensor product. It is not hard to see that

$$C \longrightarrow \mathbb{X}, \quad \lambda \longmapsto \text{isomorphism class of } \gamma\mathrm{eul}(C, \lambda)$$

defines an isomorphism

$$C/\mathbb{Z} \xrightarrow{\sim} \mathbb{X}; \tag{5.1}$$

see [34, Theorem 3, p. 120]. Indeed, let $(\mathcal{L}, \nabla) \in \mathbf{MC}_{\log}$ be such that $\mathcal{L} = C[\![x]\!] \cdot \ell$ is free of rank one. Then, if $\nabla(\ell) = a\ell$ and $\ell' = p\ell$ with $p \in C((x))^{\times}$, we see that $\nabla(\ell') = (a + p^{-1}\vartheta p)\ell'$. The desired result is a consequence of the fact that

$$C((x))^{\times} \longrightarrow C[\![x]\!], \quad b \longmapsto \frac{\vartheta b}{b}$$

establishes an isomorphism of groups $C((x))^{\times} \xrightarrow{\sim} \mathbb{Z} + xC[\![x]\!]$.

Write $\mathrm{Diag}(\mathbb{X})$ for the diagonalizable affine group scheme having $\mathbb{X}$ as a group of characters. Said otherwise,

$$\mathrm{Diag}(\mathbb{X}) = \mathrm{Spec}\, C[\mathbb{X}],$$

where $C[\mathbb{X}]$ is the group algebra; cf. [16, Chapter II, §1, no. 2.8, 154ff] or [29, Part I, Chapter 2, Section 5]. As is well known, the tensor category $\mathrm{Rep}_C(\mathrm{Diag}(\mathbb{X}))$ can be identified with the tensor category of $\mathbb{X}$-graded finite-dimensional vector spaces [16, Chapter II, §2, no. 2.5, p. 177]. Hence, from now on, given $V \in \mathrm{Rep}_C(\mathrm{Diag}(\mathbb{X}))$, we shall write V_ξ for the component of degree ξ.

For each $\xi \in \mathbb{X}$, let $\hat{\xi} \in C$ be such that $\hat{\xi} + \mathbb{Z}$ corresponds, under the isomorphism (5.1), to ξ. Then, for each $V \in \mathrm{Rep}_C(\mathrm{Diag}(\mathbb{X}))$, we put

$$\mathbb{L}(V) = C((x)) \otimes_C V$$

and endow it with the derivation D_V obtained from

$$D_V(1 \otimes v_\xi) = \hat{\xi} \cdot (1 \otimes v_\xi), \quad v_\xi \in V_\xi.$$

Obviously, the map $\mathbb{L}$ gives rise to a C-linear additive functor

$$\mathbb{L}\colon \mathrm{Rep}_C(\mathrm{Diag}(\mathbb{X})) \longrightarrow \mathbf{MC}^{\ell}_{\mathrm{rs}}.$$

It is perhaps useful to note that if C_ξ is the $\mathbb{X}$-graded vector space with a copy of C in degree ξ and zero elsewhere, then $\mathbb{L}(C_\xi) = \gamma\mathrm{eul}(C, \hat{\xi})$.

Proposition 5.7. *The functor* $\mathbb{L}$ *is a* C*-linear tensor equivalence.*

Proof. The only point requiring close examination is the tensor nature of $\mathbb{L}$. For that, given $\xi, \eta \in \mathbb{X}$, define $k(\xi, \eta) \in \mathbb{Z}$ by

$$\widehat{\xi + \eta} = \hat{\xi} + \hat{\eta} + k(\xi, \eta). \tag{5.2}$$

Now let V and W be objects of $\mathbf{vect}_{\mathbb{X}}$ and define an arrow of $C(\!(x)\!)$-spaces

$$C(\!(x)\!) \otimes_C (V \otimes_C W) \xrightarrow{\Phi_{VW}} (C(\!(x)\!) \otimes V) \underset{C(\!(x)\!)}{\otimes} (C(\!(x)\!) \otimes W)$$

by imposing that

$$1 \otimes_C (v_\xi \otimes w_\eta) \longmapsto x^{k(\xi,\eta)} \cdot [(1 \otimes v_\xi) \otimes (1 \otimes w_\eta)]$$

whenever $v_\xi \in V_\xi$ and $w_\eta \in W_\eta$. Because of equation (5.2), Φ_{VW} is an isomorphism in $\mathbf{MC}$. Three lengthy but straightforward verifications ensure that the couple $(\mathbb{L}, \Phi)$ is a tensor functor: indeed, the associativity constraint is a consequence of

$$k(\xi, \eta + \zeta) + k(\eta, \zeta) = k(\xi + \eta, \zeta) + k(\xi, \eta),$$

the commutative constraint of

$$k(\xi, \eta) = k(\eta, \xi),$$

and the identity constraint of $\hat{0} \in \mathbb{Z}$. ■

Using basic cardinal arithmetic, we derive another simple description of $\mathbb{X} = C/\mathbb{Z}$ which is well known in the case $C = \mathbb{C}$.

Lemma 5.8. *The abelian groups* $C/\mathbb{Z}$ *and* $C^\times$ *are (non-canonically) isomorphic. In particular,* $\mathbb{X}$ *and* $C^\times$ *are isomorphic.*

Proof. Let $\mu \subset C^\times$ be the subgroup of roots of unity; it is a divisible group and hence there exists an isomorphism $C^\times \simeq \mu \oplus (C^\times/\mu)$. Similarly, $C/\mathbb{Z} \simeq (\mathbb{Q}/\mathbb{Z}) \oplus (C/\mathbb{Q})$. Since $\mu \simeq \mathbb{Q}/\mathbb{Z}$, we only need to show that $C/\mathbb{Q} \simeq C^\times/\mu$.

Now, $C/\mathbb{Q}$ is a $\mathbb{Q}$-vector space as is $C^\times/\mu$ and we prove that any $\mathbb{Q}$-basis of $C/\mathbb{Q}$ has the same cardinal as a $\mathbb{Q}$-basis of $C^\times/\mu$.

Following [8], write $\mathrm{Card}(S)$ to denote the cardinal of a set S. We need a simple result, which we are unfortunately unable to find in the literature.

Claim. *For any infinite-dimensional $\mathbb{Q}$-vector space V with basis B, the equality* $\mathrm{Card}(V) = \mathrm{Card}(\mathcal{B})$ *holds.*

As $\mathrm{Card}(\mathcal{B}) \le \mathrm{Card}(V)$, we only need to show that $\mathrm{Card}(\mathcal{B}) \ge \mathrm{Card}(V)$. Let $\mathfrak{F}$ be the set of finite subsets of $\mathcal{B}$ and for each $F \in \mathfrak{F}$, write V_F for the vector space generated by F. Clearly,

$$\mathrm{Card}(V) \le \mathrm{Card}\Big(\coprod_F V_F\Big).$$

Since $\mathrm{Card}(V_F) = \mathrm{Card}(\mathbb{Q})$ [8, Corollary 2, Chapter III, §6, no. 3] and $\mathrm{Card}(\mathbb{Q}) \le \mathrm{Card}(\mathfrak{F})$, we conclude, with the help of [8, Corollary 3, Chapter III, §6, no. 3], that

$$\mathrm{Card}\Big(\coprod_F V_F\Big) = \mathrm{Card}(\mathfrak{F}).$$

Finally, let $\mathfrak{F}_n$ be the subset of $\mathfrak{F}$ consisting of those subsets with cardinal bounded by n. Clearly, $\mathrm{Card}(\mathcal{B})^n \ge \mathrm{Card}(\mathfrak{F}_n)$, which shows that $\mathrm{Card}(\mathcal{B}) \ge \mathrm{Card}(\mathfrak{F}_n)$ [8, Corollary 4, Chapter III, §6, no. 3] and consequently that $\mathrm{Card}(\mathcal{B}) \ge \mathrm{Card}(\mathfrak{F})$. The claim is settled.

To end the proof, we note that $\mathrm{Card}(C/\mathbb{Q}) \cdot \mathrm{Card}(\mathbb{Q}) = \mathrm{Card}(C)$ [8, Proposition 9, Chapter III, §5, no. 8], and hence $\mathrm{Card}(C/\mathbb{Q}) = \mathrm{Card}(C)$ [8, Corollary 4, Chapter III, §6, no. 3]. Likewise, $\mathrm{Card}(C^\times/\mu) = \mathrm{Card}(C^\times)$ so that $\mathrm{Card}(C^\times/\mu) = \mathrm{Card}(C)$. ∎

5.4 – *The Deligne tensor product*

In order to put the findings of Sections 5.2 and 5.3 together—this is the theme of Section 5.5—we require Deligne's theory of the tensor product of C-linear abelian categories; see [13, Section 5] and [32]. Since the amount of material necessary to explain this theory and the pertinent results is disproportionate to the rest of this text, we shall dedicate [15] to the matter. On the other hand, for the convenience of the reader, we present a summary.

In what follows, $\Bbbk$ is any field. Let $A_1, \ldots, A_n$ and X be $\Bbbk$-linear abelian categories and write

$$\mathbf{Rex}(A_1, \ldots, A_n : X)$$

for the category of functors

$$F : A_1 \times \cdots \times A_n \longrightarrow X,$$

which are $\Bbbk$-multilinear and right-exact in each variable.

Definition 5.9 ([13, Section 5], [32, Definition 1]). Given $\Bbbk$-linear abelian categories A and B, a couple (P, T) consisting of a $\Bbbk$-linear *abelian* category P and a functor $T \in \mathbf{Rex}(A, B : P)$ is called a *Deligne tensor product* of A and B if the following holds. For each $\Bbbk$-linear abelian category X, the functor

$$\mathbf{Rex}(P : X) \longrightarrow \mathbf{Rex}(A, B : X), \quad F \longmapsto F \circ T, \tag{5.3}$$

is an equivalence.

Example 5.10. Let G and H be group schemes over $\Bbbk$. It then follows that the usual tensor product of vector spaces

$$\mathrm{Rep}_{\Bbbk}(G) \times \mathrm{Rep}_{\Bbbk}(H) \longrightarrow \mathrm{Rep}_{\Bbbk}(G \times H)$$

is a Deligne tensor product. See [15].

As argued by [32, p. 208], the drawback of Definition 5.9 is the requirement that P be abelian, while the properties involved speak solely of *right exactness*. For that reason, [32] employs a weaker version of the tensor product (the Kelly tensor product) and then studies the cases where the Kelly tensor product is a Deligne tensor product. This allows [32] to give a complete proof of Deligne's existence theorem [13, Proposition 5.13] (see [32, Proposition 22]), affirming that if A and B are *categories with length* (cf. [32, p. 217] for the definition), then the Deligne tensor product exists.

The question concerning the transport of tensor structures in the theory of the Deligne tensor product is in order. This is dealt with in [13, Sections 5.16–17], but we found that [13] has two omissions: First, the verification of the various functorial commutativity constraints for coherence is left to the reader at the beginning of [13, Proposition 5.17]. Second, nowhere in [13] is a discussion to be found on the monoidal nature of the functors obtained from monoidal functors via the equivalence (5.3). We explain these matters in more detail.

Let A and B be $\Bbbk$-linear abelian categories. Let $(A, \otimes_A, \mathbb{1}_A)$ and $(B, \otimes_B, \mathbb{1}_B)$ define symmetric monoidal structures [33, Chapter VII, Section 1] on each one of them, and assume that, in addition, we have $\otimes_A \in \mathbf{Rex}(A, A : A)$ and $\otimes_B \in \mathbf{Rex}(B, B : B)$. On $A \times B$, let us introduce an evident structure of symmetric monoidal category:

$$\otimes_{AB} \in \mathbf{Rex}(A, B, A, B : A \times B),$$

is given by

$$(a, b) \otimes_{AB} (a', b') = (a \otimes_A a', b \otimes_B b').$$

Suppose that (P, T) is a Deligne tensor product for A and B. As explained in [15], we then have an equivalence

$$(-) \circ T^{\times n} \colon \mathbf{Rex}(\underbrace{P, \dots, P}_{n} : X) \longrightarrow \mathbf{Rex}(\underbrace{A, B, \dots, A, B}_{2n} : X)$$

for each $n \geq 1$. Letting

$$\otimes_P \in \mathbf{Rex}(P, P : P)$$

correspond to $T \circ \otimes_{AB}$ under equation (5.3), we then have a natural isomorphism

$$\mu \colon \otimes_P \circ T^2 \overset{\mu}{\Longrightarrow} T \circ \otimes_{AB}.$$

In addition, letting $\mathbb{1}_P = T(\mathbb{1}_A, \mathbb{1}_B)$, it then follows that $(P, \otimes_P, \mathbb{1}_P)$ is a symmetric monoidal category and T is a monoidal functor. These details are verified in [15]. (Needless to say, the difficulty is ensuring *coherence* of the monoidal structure.)

Finally, let $F \colon A \times B \to X$ be any $\Bbbk$-bilinear functor which is right exact in each variable. Suppose that, giving $A \times B$ the symmetric monoidal structure explained above, F is monoidal. Then a functor $\bar{F} \in \mathbf{Rex}(A, B : P)$ corresponding to F under equation (5.3) is also monoidal.

5.5 – *Conclusions*

Let $(\mathfrak{T}, \boxtimes)$ be the Deligne tensor product of $\mathbf{MC}^{\ell}_{\mathrm{rs}}$ and $\mathbf{MC}^{u}_{\mathrm{rs}}$. If

$$P \colon \mathbf{MC}^{\ell}_{\mathrm{rs}} \times \mathbf{MC}^{u}_{\mathrm{rs}} \longrightarrow \mathbf{MC}_{\mathrm{rs}}$$

is the obvious tensor product, we obtain through the equivalence (5.3) a right-exact C-linear functor

$$\bar{P} \colon \mathfrak{T} \longrightarrow \mathbf{MC}_{\mathrm{rs}}$$

and a natural isomorphism

$$\bar{P} \circ \boxtimes \overset{\sim}{\Longrightarrow} P \tag{5.4}$$

in $\mathbf{Rex}(\mathbf{MC}^{\ell}_{\mathrm{rs}}, \mathbf{MC}^{u}_{\mathrm{rs}} : \mathbf{MC}_{\mathrm{rs}})$. In addition, Section 5.4 ensures that $\bar{P}$ is a tensor functor and [13, Proposition 5.13 (vi), p. 148] that $\bar{P}$ is also left exact.

Theorem 5.11 (The categorical Manin equivalence). *The above-defined functor $\overline{P}$ is an equivalence of C-linear abelian tensor categories.*

Proof. *Essential surjectivity:* It suffices to show that any indecomposable $M \in \mathbf{MC}_{\mathrm{rs}}$ belongs to the essential image. According to Theorem 5.1, $M \simeq \gamma\mathrm{eul}(J_r(\lambda))$. Using that $J_r(\lambda) \simeq J_1(\lambda) \otimes J_r(0)$, from equation (5.4) we obtain

$$\begin{aligned}\gamma\mathrm{eul}(J_r(\lambda)) &\simeq \gamma\mathrm{eul}(J_1(\lambda)) \otimes \gamma\mathrm{eul}(J_r(0))\\ &= \overline{P}[\gamma\mathrm{eul}(J_1(\lambda)) \boxtimes \gamma\mathrm{eul}(J_r(0))].\end{aligned}$$

Full faithfulness: Let $L, L' \in \mathbf{MC}_{\mathrm{rs}}^{\ell}$ and $U, U' \in \mathbf{MC}_{\mathrm{rs}}^{u}$ so that we have an arrow induced by $P_{(L,U),(L',U')}$:

$$\mathrm{Hom}_{\mathbf{MC}_{\mathrm{rs}}^{\ell}}(L, L') \otimes_C \mathrm{Hom}_{\mathbf{MC}_{\mathrm{rs}}^{u}}(U, U') \longrightarrow \mathrm{Hom}_{\mathbf{MC}_{\mathrm{rs}}}(L \otimes U, L' \otimes U'). \tag{5.5}$$

That (5.5) is an isomorphism if $L = \mathbb{1}$ and $U' = \mathbb{1}$ is easily verified. Indeed, in this case, if $L' \not\simeq \mathbb{1}$, then $\mathrm{Hom}_{\mathbf{MC}_{\mathrm{rs}}^{\ell}}(\mathbb{1}, L') = 0$ and $\mathrm{Hom}_{\mathbf{MC}_{\mathrm{rs}}}(U, L') = 0$, which implies that both sides in (5.5) vanish; if $\alpha\colon \mathbb{1} \xrightarrow{\sim} L'$, then $\mathrm{Hom}(\mathbb{1}, L') = C\alpha$, and using the natural isomorphism

$$\mathrm{Hom}(\mathbb{1} \otimes U, L' \otimes \mathbb{1}) \xrightarrow{\sim} \mathrm{Hom}(U, \mathbb{1}),$$

we may identify (5.5) with the arrow which maps $\alpha \otimes \varphi$ to φ. Making use of duals, we conclude that (5.5) is an isomorphism for all L, L', U and U'. Then, employing [13, Proposition 5.13 (v)], we conclude that

$$\overline{P}_{L\boxtimes U, L'\boxtimes U'}\colon \mathrm{Hom}_{\mathfrak{T}}(L \boxtimes U, L' \boxtimes U') \longrightarrow \mathrm{Hom}_{\mathbf{MC}_{\mathrm{rs}}}(\overline{P}(L \boxtimes U), \overline{P}(L' \boxtimes U'))$$

is an isomorphism.

Now let $\mathfrak{T}_0$ be the full subcategory of $\mathfrak{T}$ whose objects are finite direct sums of objects of the form $L \boxtimes U$. The previous argument shows that $\overline{P}$ when restricted to $\mathfrak{T}_0$ is fully faithful.

Now let X and X' be arbitrary objects in $\mathfrak{T}$. We can then find two exact sequences,

$$K \xrightarrow{\iota} Y \xrightarrow{\pi} X \longrightarrow 0$$

and

$$K' \xleftarrow{\pi'} Y' \xleftarrow{\iota'} X' \longleftarrow 0,$$

in which Y, Y', K and K' are in $\mathfrak{T}_0$. This is because each element in $\mathfrak{T}$ is the target of an epimorphism from an object of $\mathfrak{T}_0$; see [32, p. 212]. That of the second exact sequence is a consequence of the fact that $\mathfrak{T}$ is the category of representations of a group scheme (Example 5.10), and hence any object of $\mathfrak{T}$ is the source of a monomorphism to an object of $\mathfrak{T}_0$.

Let $a\colon \overline{P}(X) \to \overline{P}(X')$ be given; as $\overline{P}|_{\mathfrak{T}_0}$ is full, there exist $c_0\colon K \to K'$ and $b_0\colon Y \to Y'$ such that

$$\begin{CD}
\overline{P}(K) @>{\overline{P}(\iota)}>> \overline{P}(Y) @>{\overline{P}(\pi)}>> \overline{P}(X) @>>> 0\\
@V{\overline{P}(c_0)}VV @VV{\overline{P}(b_0)}V @VV{a}V @.\\
\overline{P}(K') @<<{\overline{P}(\pi')}< \overline{P}(Y') @<<{\overline{P}(\iota')}< \overline{P}(X') @<<< 0
\end{CD}$$

commutes. As $\overline{P}|_{\mathfrak{T}_0}$ is faithful, it is the case that

$$\begin{CD}
K @>{\iota}>> Y\\
@V{c_0}VV @VV{b_0}V\\
K' @<<{\pi'}< Y'
\end{CD}$$

commutes. Faithfulness of $\overline{P}|_{\mathfrak{T}_0}$ ensures also that $b_0\iota = 0$, since $\overline{P}(b_0\iota) = 0$. Then there exists $d\colon X \to Y'$ such that $d\pi = b_0$. Since $\overline{P}(\pi' d\pi) = 0$, we can say that $\pi' d\pi = 0$. Hence, there exists $a_0\colon X \to X'$ rendering

$$\begin{CD}
K @>{\iota}>> Y @>{\pi}>> X @>>> 0\\
@V{c_0}VV @VV{b_0}V @VV{a_0}V @.\\
K' @<<{\pi'}< Y' @<<{\iota'}< X' @<<< 0
\end{CD}$$

commutative. As ι' and $\overline{P}(\iota')$ are monomorphisms, and π and $\overline{P}(\pi)$ are epimorphisms, we see that $\overline{P}(a_0) = a$ and that a_0 is unique with such a property. ∎

From now on, the group scheme

$$\boxed{\mathfrak{Z} = \mathrm{Diag}(\mathbb{X}) \times \mathbb{G}_a}$$

will play a relevant role.

Translating the equivalences described in Corollary 5.5, in Proposition 5.7 and in Theorem 5.11, and applying Example 5.10, we arrive at the following corollary:

Corollary 5.12. *There exists an equivalence of C-linear abelian tensor categories*

$$\Phi\colon \mathrm{Rep}_C(\mathfrak{Z}) \longrightarrow \mathbf{MC}_{\mathrm{rs}}$$

having the following properties:

(1) *Let* $\xi \in \mathbb{X}$ *induce* $\chi\colon \mathfrak{Z} \to \mathbb{G}_m$ *via* $\mathrm{pr}_1\colon \mathfrak{Z} \to \mathrm{Diag}(\mathbb{X})$. *Then* $\Phi(\chi)$ *lies in the class* ξ.

(2) *Let* $\rho\colon \mathbb{G}_a \to \mathrm{GL}(V)$ *induce* $\sigma\colon \mathfrak{Z} \to \mathrm{GL}(V)$ *via* $\mathrm{pr}_2\colon \mathfrak{Z} \to \mathbb{G}_a$. *Then*

$$\Phi(\sigma) \simeq \gamma\mathrm{eul}(V, \log\rho(1)).$$ ∎

We now set out to identify $\mathrm{Diag}(\mathbb{X}) \times \mathbb{G}_a$ with the algebraic envelope of the abstract group $(\mathbb{Z}, +)$. Let us recall what this means.

Given an abstract group Γ, there exists an affine group scheme Γ^{aff} (over C) and an arrow

$$u\colon \Gamma \longrightarrow \Gamma^{\mathrm{aff}}(C)$$

such that, for any *algebraic* group scheme G, the natural map

$$\begin{aligned}\mathrm{Hom}(\Gamma^{\mathrm{aff}}, G) &\longrightarrow \mathrm{Hom}(\Gamma, G(C)),\\ \rho &\longmapsto \rho(C)\circ u\end{aligned}$$

is bijective. We know of three ways of constructing Γ^{aff}: by means of the main theorem of Tannakian theory [14, Theorem 2.11], by means of Freyd's adjoint functor theorem [33, Theorem 2, Chapter V, Section 6] or by means of Hochschild–Mostow's method [26, p. 1140], [1, p. 72]. In the case $\Gamma = \mathbb{Z}$, the construction is folkloric, but the only concrete references we were able to find were [42, Section 5.3], which is not really what we want, and [4, Example 1, p. 23], which is imprecise (there is no need for $\widehat{\mathbb{Z}}$ to appear in their conclusion).

Lemma 5.13. *Let* $\alpha\colon \mathbb{X} \xrightarrow{\sim} C^\times$ *be an isomorphism. Define*

$$\begin{aligned} f\colon \mathbb{Z} &\longrightarrow \mathrm{Diag}(\mathbb{X})(C) = \mathrm{Hom}(\mathbb{X}, C^\times),\\ f(k)\colon \xi &\longmapsto \alpha(\xi)^k,\end{aligned}$$

and $\iota\colon \mathbb{Z} \to \mathbb{G}_a(C)$ *as being the evident inclusion. Then*

$$(f,\iota)\colon \mathbb{Z} \longrightarrow \mathfrak{Z}(C)$$

is the affine envelope of $\mathbb{Z}$. *In particular, there exists a tensor equivalence of* C*-linear categories*

$$\Theta\colon \mathrm{Rep}_C(\mathfrak{Z}) \longrightarrow \mathrm{Rep}_C(\mathbb{Z})$$

such that we have the following properties:

(1) *Let* $\xi \in \mathbb{X}$ *induce* $\chi\colon \mathfrak{Z} \to \mathbb{G}_m$ *via* $\mathfrak{Z} \to \mathrm{Diag}(\mathbb{X})$. *Then* $\Theta(\chi)$ *corresponds to the representation defined by* $1 \mapsto \alpha(\xi) \in C^\times$.

(2) *Let* $\rho\colon \mathbb{G}_a \to \mathrm{GL}(V)$ *induce* $\sigma\colon \mathfrak{Z} \to \mathrm{GL}(V)$ *via* $\mathfrak{Z} \to \mathbb{G}_a$. *Then* $\Theta(\sigma)$ *corresponds to the representation defined by* $1 \mapsto \rho(1) \in \mathrm{GL}(V)$.

PROOF. Let Λ be an abelian group, $h\colon \mathbb{Z} \to \mathrm{Diag}(\Lambda)(C)$ a morphism and $h_1\colon C[\Lambda] \to C$ the image of $1 \in \mathbb{Z}$ under h. The morphism of abstract groups

$$\Lambda \xrightarrow{h_1} C^\times \xrightarrow{\alpha^{-1}} \mathbb{X}$$

gives us a morphism of group schemes $h^\natural\colon \mathrm{Diag}(\mathbb{X}) \to \mathrm{Diag}(\Lambda)$. Clearly

$$h^\natural(C) \circ f = h.$$

Now let U be an algebraic unipotent group scheme and $h\colon \mathbb{Z} \to U(C)$ a morphism of abstract groups. Using [44, Theorem 8.3 and Exercise 11 of Chapter 9] plus the fact that $\mathbb{G}_a$ has no non-trivial subgroup schemes, there exists a morphism $g\colon \mathbb{G}_a \to U$ such that $g(1) = h(1)$. This of course just means that $g\iota = h$.

Now let G be any algebraic group scheme and $\rho\colon \mathbb{Z} \to G(C)$ a morphism. The closure of $\mathrm{Im}(\rho)$ is an abelian group scheme [44, Section 4.3, Theorem] and as such can be decomposed into a diagonalizable and a unipotent part [44, Theorem 9.5]. The previous claims then establish what we want. ■

Considering an inverse tensor equivalence to Θ [40, Chapter II, Section 4.4] and employing Corollary 5.12, we arrive at the following corollary:

COROLLARY 5.14. *The C-linear abelian tensor category* $\mathbf{MC}_{\mathrm{rs}}$ *is equivalent to* $\mathrm{Rep}_C(\mathbb{Z})$. *More precisely, following Lemma 5.8, let us fix an isomorphism*

$$\alpha\colon \mathbb{X} \xrightarrow{\sim} C^\times.$$

Then there exists an equivalence of tensor C-linear categories

$$\Psi_\alpha\colon \mathrm{Rep}_C(\mathbb{Z}) \longrightarrow \mathbf{MC}_{\mathrm{rs}}$$

having the following properties:

(1) *If $L \in \mathrm{Rep}_C(\mathbb{Z})$ has dimension one and is defined by letting $1 \in \mathbb{Z}$ act as $\lambda \in C^\times$, then $\Psi_\alpha(L)$ belongs to the class $\alpha^{-1}(\lambda)$.*

(2) *If $V \in \mathrm{Rep}_C(\mathbb{Z})$ is defined by the unipotent automorphism $u\colon V \to V$, then*

$$\Psi_\alpha(V) \simeq \gamma\mathrm{eul}(V, \log(u)).$$

■

For the sake of readability, let us state Corollary 5.14 “in the other direction” and in the case where C is the field of complex numbers, and

$$\alpha(\text{class of } \gamma\mathrm{eul}(\mathbb{C}, \lambda)) = e^{2\pi \mathrm{i}\lambda}.$$

(Consequently, if $\eta_\lambda\colon \mathbb{Z} \to \mathbb{C}^\times$ is defined by $k \mapsto e^{2\pi \mathrm{i} k\lambda}$, then $\Psi_\alpha(\eta_\lambda) \simeq \gamma\mathrm{eul}(\mathbb{C}, \lambda)$.)

Corollary 5.15. *Let α be defined as before. Then there exists an equivalence of $\mathbb{C}$-linear tensor categories*

$$\Omega_\alpha : \mathbf{MC}_{\mathrm{rs}} \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{C}}(\mathbb{Z})$$

such that to each endomorphism $A: V \to V$, we have

$$\Omega_\alpha(\gamma\mathrm{eul}(V, A)) = \begin{array}{l}\textit{the representation of } \mathbb{Z} \textit{ on } V\\ \textit{defined by } k \mapsto e^{2\pi k \mathrm{i} A}.\end{array}$$

In other words, the "exponential" is an inverse to Ψ_α.

Proof. We construct Ω_α as an inverse equivalence to Ψ_α following the proof of [33, Chapter IV, Section 4, Theorem 1]. That this inverse equivalence is *automatically* a tensor functor is verified by the considerations in [40, Chapter II, Section 4.4].

In what follows, for a given $(V, A) \in \mathbf{End}$, we shall write η_A to mean the representation $k \mapsto e^{2\pi k \mathrm{i} A}$ of $\mathbb{Z}$. Let $(V, A) \in \mathbf{End}$ be given. We are required to show that $\Psi_\alpha(\eta_A) \simeq \gamma\mathrm{eul}(V, A)$. Assume first that (V, A) is indecomposable as an object of $\mathbf{End}$; this implies in particular that A has a single eigenvalue λ. This being so, $A = \lambda I + N$, with N nilpotent. Then $e^{2\pi \mathrm{i} A} = e^{2\pi \mathrm{i}\lambda} e^{2\pi \mathrm{i} N}$, which shows that $\eta_A \simeq \eta_\lambda \otimes \eta_N$. Then, by Corollary 5.14,

$$\Psi_\alpha(\eta_A) \simeq \underbrace{\Psi_\alpha(\eta_\lambda)}_{\simeq \gamma\mathrm{eul}(\mathbb{C},\lambda)} \otimes \underbrace{\Psi_\alpha(\eta_N)}_{\simeq \gamma\mathrm{eul}(V,N)}$$

and we conclude that

$$\Psi_\alpha(\eta_A) \simeq \gamma\mathrm{eul}(V, \lambda I + N).$$

The case in which (V, A) is not indecomposable is treated by considering a decomposition into direct sums and we conclude that $\Psi_\alpha(\eta_A) \simeq \gamma\mathrm{eul}(V, A)$, as wanted. ∎

Remark 5.16. It is possible, if the ground field is $\mathbb{C}$, to obtain Corollary 5.14 using the universal Picard–Vessiot extension [42, Chapter 10, Section 2, 262ff].

6. Connections on $\mathbb{P} \setminus \{0, \infty\}$ after [12, Sections 15.28–36]

The theory of regular-singular connections over the ring $C[x^\pm] = C[x, x^{-1}]$ works in close analogy with that of $C((x))$. In this section we review it following Deligne.

Let $\mathbb{P}$ stand for the projective line obtained by gluing

$$\mathbb{A}_0 := \operatorname{Spec} C[x] \quad \text{and} \quad \mathbb{A}_\infty := \operatorname{Spec} C[y]$$

along the open subsets $\operatorname{Spec} C[x^\pm]$ and $\operatorname{Spec} C[y^\pm]$ via the isomorphism $x = y^{-1}$. As suggested by notation, $0 \in \mathbb{P}$ is the point (x) of $\mathbb{A}_0$ and $\infty \in \mathbb{P}$ the point (y) of $\mathbb{A}_\infty$.

Note that $\vartheta\colon C[x] \to C[x]$ can be extended to a global section of the tangent sheaf, call it ϑ also, on $\mathbb{P}$.

Definition 6.1. (1) We let $\mathbf{MC}(C[x^{\pm}]/C)$ be the category whose

objects are couples (M, ∇) consisting of a $C[x^{\pm}]$-module of finite type and a C-linear endomorphism $\nabla\colon M \to M$ satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$ and

arrows between (M, ∇) and (M', ∇') are just $C[x^{\pm}]$-linear maps $\varphi\colon M \to M'$ satisfying $\nabla'\varphi = \varphi\nabla$.

It is called the category of *connections on* $\mathbb{P} \setminus \{0, \infty\}$ *or on* $C[x^{\pm}]$.

(2) We let $\mathbf{MC}_{\log}(\mathbb{P}/C)$ be the category whose

objects are couples $(\mathcal{M}, \nabla)$ consisting of a coherent $\mathcal{O}_{\mathbb{P}}$-module and a C-linear endomorphism $\nabla\colon \mathcal{M} \to \mathcal{M}$ satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$ on all open subsets and

arrows between $(\mathcal{M}, \nabla)$ and $(\mathcal{M}', \nabla')$ are $\mathcal{O}_{\mathbb{P}}$-linear maps $\varphi\colon \mathcal{M} \to \mathcal{M}'$ satisfying $\nabla'\varphi = \varphi\nabla$.

It is called the category of *logarithmic connections on* $\mathbb{P}$.

(3) We let

$$\gamma_{\mathbb{P}}\colon \mathbf{MC}_{\log}(\mathbb{P}/C) \longrightarrow \mathbf{MC}(C[x^{\pm}]/C)$$

be the obvious functor. (If convenient we shall write simply γ.) A connection (M, ∇) in $\mathbf{MC}(C[x^{\pm}]/C)$ is *regular-singular* if $\gamma_{\mathbb{P}}(\mathcal{M}) \simeq M$ for a certain $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}/C)$; in this case, any such $\mathcal{M}$ is a *logarithmic model* of M. In the case that $\mathcal{M}$ is in addition a locally free $\mathcal{O}_{\mathbb{P}}$-module, we call $\mathcal{M}$ a *logarithmic lattice*.

(4) The full subcategory of $\mathbf{MC}(C[x^{\pm}]/C)$ having regular-singular connections as objects is denoted by $\mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$.

Remark 6.2. A fundamental result for an object (M, ∇) from $\mathbf{MC}(C[x^{\pm}]/C)$ is that M is automatically a *free* $C[x^{\pm}]$-module. (That it is a projective module can be found in [30, Proposition 8.9] for instance, but we offer a short proof of a more general fact in Remark 8.20 below.) In addition, proceeding as discussed after Definition 2.2, we can always ensure the existence of *logarithmic lattices* for objects in $\mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$.

Example 6.3. Let $(V, A) \in \mathbf{End}$ (see Definition 3.2). We let $\mathrm{eul}_{\mathbb{P}}(V, A) \in \mathbf{MC}_{\log}(\mathbb{P}/C)$ be the couple $(\mathcal{O}_{\mathbb{P}} \otimes_C V, D_A)$, where $D_A(f \otimes v) = \vartheta f \otimes v + f \otimes Av$ on any open subset of $\mathbb{P}$. This construction gives rise to a functor $\mathrm{eul}_{\mathbb{P}}\colon \mathbf{End} \to \mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$.

The canonical inclusions $C[x^{\pm}] \to C(\!(x)\!)$ and $C[x] \to C[\![x]\!]$ produce C-linear exact tensor functors

$$\mathbf{r}_0\colon \mathbf{MC}_{\log}(\mathbb{P}/C) \longrightarrow \mathbf{MC}_{\log}(C[\![x]\!]/C), \quad \mathcal{M} \longmapsto C[\![x]\!] \underset{C[x]}{\otimes} \mathcal{M}(\mathbb{A}_0)$$

and

$$\mathbf{r}_0\colon \mathbf{MC}(C[x^{\pm}]/C) \longrightarrow \mathbf{MC}(C(\!(x)\!)/C), \qquad M \longmapsto C(\!(x)\!) \underset{C[x^{\pm}]}{\otimes} M.$$

It should be noted that if $\mathrm{eul}_{\mathbb{P}}(V, A)$ is as in Example 6.3, then $\mathbf{r}_0(\mathrm{eul}_{\mathbb{P}}(V, A))$ is simply $\mathrm{eul}(V, A)$, as in Definition 3.1.

In entirely analogous fashion, we have functors "$\mathbf{r}_\infty$" with targets $\mathbf{MC}_{\log}(C[\![y]\!]/C)$ and $\mathbf{MC}(C(\!(y)\!)/C)$. Note, on the other hand, that $\mathbf{r}_\infty(\mathrm{eul}_{\mathbb{P}}(V, A))$ then corresponds to $\mathrm{eul}(V, -A)$ as $\vartheta(y) = -y$.

The relation between $\mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$ and $\mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)$ is given by the following theorem:

THEOREM 6.4. *The functor $\mathbf{r}_0$ induces an equivalence between categories of regular-singular connections.*

In [12, Sections 15.28–36], Deligne offers a proof of this result by constructing an inverse functor and in studying [12], we obtained the following sequence of thoughts (which is possibly not exactly what Deligne had in mind).

PROPOSITION 6.5 (Regular-singular connections are "Euler"). *The functor $\gamma\mathrm{eul}\colon \mathbf{End} \to \mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$ is essentially surjective. More precisely, given $(M, \nabla) \in \mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$, there exists $(\mathfrak{M}, A) \in \mathbf{End}$ and an isomorphism $\gamma\mathrm{eul}_{\mathbb{P}}(\mathfrak{M}, A) \simeq (M, \nabla)$. In addition, A can be chosen to have no two distinct eigenvalues differing by a positive integer.*

PROOF. This is mostly spectral theory of the connection operator. Let $(M, \nabla) \in \mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$ be given. There exists a *finite* $C[x]$-submodule $\mathcal{M}$ of M which is invariant under ∇ and generates M as a $C[x^{\pm}]$-module. Note that $\mathcal{M}$ is necessarily free. For each $k \in \mathbb{Z}$, we define $\mathcal{M}^{(k)} = x^k\mathcal{M}$ to obtain a *decreasing*, *separated* and *exhaustive* filtration of M.

Given $k < \ell$, let us write $\mathcal{M}^{(k,\ell)}$ for the quotient $\mathcal{M}^{(k)}/\mathcal{M}^{(\ell)}$ (this is a finite-dimensional C-space) and $\nabla_{k,\ell}$ for the C-linear map induced by ∇ on it. Since multiplication by x^k induces an isomorphism of C-spaces $\mathcal{M}^{(0,1)} \simeq \mathcal{M}^{(k,k+1)}$, we can show that

$$\mathrm{Sp}(\nabla_{k,k+1}) = \{k\} \oplus \mathrm{Sp}(\nabla_{0,1}).$$

From the exact sequence

$$0 \longrightarrow \mathcal{M}^{(k+1,k+2)} \longrightarrow \mathcal{M}^{(k,k+2)} \longrightarrow \mathcal{M}^{(k,k+1)} \longrightarrow 0$$

we derive

$$\begin{aligned}\mathrm{Sp}(\nabla_{k,k+2}) &= \mathrm{Sp}(\nabla_{k+1,k+2}) \cup \mathrm{Sp}(\nabla_{k,k+1}) \\ &= (\mathrm{Sp}(\nabla_{0,1}) \oplus \{k+1\}) \cup (\mathrm{Sp}(\nabla_{0,1}) \oplus \{k\}),\end{aligned}$$

which in all generality gives

$$\begin{aligned}\mathrm{Sp}(\nabla_{k,\ell}) &= \mathrm{Sp}(\nabla_{k,k+1}) \cup \cdots \cup \mathrm{Sp}(\nabla_{\ell-1,\ell}) \\ &= \bigcup_{j=k}^{\ell-1} \mathrm{Sp}(\nabla_{0,1}) \oplus \{j\}.\end{aligned}$$

We now require the following lemma:

LEMMA 6.6. *The following claims are true:*

(1) *The spectral set* $\mathrm{Sp}(\nabla)$ *is contained in* $\bigcup_{k\in\mathbb{Z}}\{k\} \oplus \mathrm{Sp}(\nabla_{0,1})$ *and is invariant under the action of* $\mathbb{Z}$ *on* $\boldsymbol{C}$.

(2) *Let* $\varrho \in \mathrm{Sp}(\nabla)$. *Then there exists a couple of integers* $k < \ell$ *such that* $\mathbf{G}(\nabla, \varrho) \subset \mathcal{M}^{(k)}$ *and* $\mathbf{G}(\nabla, \varrho) \cap \mathcal{M}^{(\ell)} = (0)$. *In particular,* $\dim \mathbf{G}(\nabla, \varrho) < \infty$.

PROOF. (1) Let $\varrho \in \mathrm{Sp}(\nabla)$ and let $m \in M$ be an eigenvector. Let $k \in \mathbb{Z}$ be such that $m \in \mathcal{M}^{(k)} \setminus \mathcal{M}^{(k+1)}$. Then

$$\varrho \in \mathrm{Sp}(\nabla_{k,k+1}) = \mathrm{Sp}(\nabla_{0,1}) \oplus \{k\}.$$

This shows the inclusion. The final statement is a consequence of the fact that if m is an eigenvector for the eigenvalue ϱ, then $x^k m$ is an eigenvector for $k + \varrho$.

(2) Consider I_ϱ the set of all $k \in \mathbb{Z}$ such that $\varrho \in \mathrm{Sp}(\nabla_{0,1}) \oplus \{k\}$. Clearly I_ϱ is finite; let $\mu = \min I_\varrho$ and $\nu = \max I_\varrho$. For a given $m \in \mathbf{G}(\nabla, \varrho) \setminus \{0\}$, there exists $k \in \mathbb{Z}$ such that $m \subset \mathcal{M}^{(k)} \setminus \mathcal{M}^{(k+1)}$. Hence, $\varrho \in \mathrm{Sp}(\nabla_{k,k+1})$ so that $k \in I_\varrho$. This implies that

$$\mu \le k \le \nu.$$

Hence, $m \in \mathcal{M}^{(\mu)}$ while $m \notin \mathcal{M}^{(\nu+1)}$. ∎

Finding a logarithmic lattice for M and looking at the space of sections with poles on 0 and ∞, we can construct an increasing and exhaustive filtration of M by *finite-dimensional* vector spaces which is, in addition, stable under ∇. It then follows that

$$M = \bigoplus_{\varrho \in \mathrm{Sp}(\nabla)} \mathbf{G}(\nabla, \varrho).$$

Let us now select a finite set $S \subset \mathrm{Sp}(\nabla_{0,1})$ such that

$$\mathrm{Sp}(\nabla) = \bigsqcup_{k \in \mathbb{Z}} S \oplus \{k\}.$$

Write

$$\mathfrak{M} = \bigoplus_{\varrho \in S} \mathbf{G}(\varrho, \nabla);$$

this is a finite-dimensional space because of Lemma 6.6. As multiplication by x^k induces isomorphisms

$$\mathbf{G}(\nabla, \varrho) \xrightarrow{\sim} \mathbf{G}(\nabla, \varrho + k),$$

we have

$$M = \bigoplus_{k \in \mathbb{Z}} x^k \mathfrak{M} = C[x^{\pm}] \otimes_C \mathfrak{M}.$$

Let $A \colon \mathfrak{M} \to \mathfrak{M}$ be the restriction of ∇. We then see that

$$M \simeq \gamma_{\mathbb{P}} \mathrm{eul}(\mathfrak{M}, A).$$

In addition, by construction, no two distinct elements of $\mathrm{Sp}(A) = S$ can differ by a non-zero integer. ∎

Proof of Theorem 6.4. We know that $\mathbf{r}_0$ is faithful since the $C[x^{\pm}]$-module of any object in $\mathbf{MC}(C[x^{\pm}]/C)$ is free (Remark 6.2). Essential surjectivity is an immediate consequence of Corollary 4.3 and Example 6.3. We consider fullness. Let (M, ∇) and (M', ∇') in $\mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$ be given. Because of Proposition 6.5, we may assume that

$$(M, \nabla) = (\mathcal{O}_{\mathbb{P}} \otimes_C V, D_A) \quad \text{and} \quad (M', \nabla') = (\mathcal{O}_{\mathbb{P}} \otimes_C V', D_{A'}),$$

where $A \colon V \to V$ and $A' \colon V' \to V'$ have no two distinct eigenvalues differing by an integer. The result is then a consequence of the explicit determination of $\mathrm{Hom}(\mathrm{eul}(V, A), \mathrm{eul}(V', A'))$ made in Lemma 3.3 and Proposition 4.4. ∎

Let us now express these findings using the notion of exponents.

Definition 6.7. Let $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}/C)$ be given. Define its set of exponents, $\mathrm{Exp}(\mathcal{M})$, as

$$\mathrm{Exp}(\mathcal{M}) = \mathrm{Exp}(\mathbf{r}_0\mathcal{M}) \cup \mathrm{Exp}(\mathbf{r}_\infty\mathcal{M}).$$

With this definition, we can fix certain preferred logarithmic models.

Theorem 6.8 (Deligne–Manin models). *Let $M \in \mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)$. Then there exists a logarithmic* lattice $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}/C)$ *for M whose exponents are all on τ. In addition, if $\mathcal{M}'$ is another logarithmic lattice for M with all exponents on τ, then there exists a unique isomorphism $\varphi\colon \mathcal{M} \to \mathcal{M}'$ rendering the diagram*

$$\begin{array}{ccc} \gamma_{\mathbb{P}}(\mathcal{M}) & \xrightarrow{\ \sim\ } & M \\ & \underset{\gamma_{\mathbb{P}}(\varphi)}{\searrow} & \uparrow \wr \\ & & \gamma_{\mathbb{P}}(\mathcal{M}') \end{array}$$

commutative.

Proof. There exists, by Corollary 4.3, an object $(V, A) \in \mathbf{End}$ with $\mathrm{Sp}_A \subset \tau$ and an isomorphism $u_0\colon \gamma\mathrm{eul}(V, A) \xrightarrow{\sim} \mathbf{r}_0(M)$. Let $\mathbf{M}_0 = \mathrm{eul}_{\mathbb{P}}(V, A)$; this is an object of $\mathbf{MC}_{\log}(\mathbb{P}/C)$. Since $\mathbf{r}_0(\gamma_{\mathbb{P}}(\mathbf{M}_0)) = \gamma\mathrm{eul}(V, A)$, Theorem 6.4 produces an isomorphism $\tilde{u}_0\colon \gamma_{\mathbb{P}}(\mathbf{M}_0) \xrightarrow{\sim} M$ such that $\mathbf{r}_0(\tilde{u}_0) = u_0$. Similarly, we obtain $(W, B) \in \mathbf{End}$, $u_\infty\colon \gamma\mathrm{eul}(W, B) \xrightarrow{\sim} \mathbf{r}_\infty(M)$, $\mathbf{M}_\infty = \mathrm{eul}_{\mathbb{P}}(W, B)$ and $\tilde{u}_\infty\colon \gamma_{\mathbb{P}}(\mathbf{M}_\infty) \xrightarrow{\sim} M$. From this we derive an isomorphism from $\mathbf{MC}(C[x^{\pm}]/C)$:

$$v\colon \gamma_{\mathbb{P}}(\mathbf{M}_0) \xrightarrow{\sim} \gamma_{\mathbb{P}}(\mathbf{M}_\infty)$$

and hence an object $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}/C)$ with the properties required in the statement.

Now let $\mathcal{M}$ and $\mathcal{M}'$ be as in the statement; we possess an isomorphism in $\mathbf{MC}(C[x^{\pm}]/C)$: $f\colon \gamma_{\mathbb{P}}(\mathcal{M}) \xrightarrow{\sim} \gamma_{\mathbb{P}}(\mathcal{M}')$. Write $\mathcal{M}_0 = \mathcal{M}(\mathbb{A}_0)$ and $\mathcal{M}'_0 = \mathcal{M}'(\mathbb{A}_0)$ so that we have an isomorphism of $C[x^{\pm}]$-modules

$$f\colon \mathcal{M}_0 \underset{C[x]}{\otimes} C[x^{\pm}] \xrightarrow{\sim} \mathcal{M}'_0 \underset{C[x]}{\otimes} C[x^{\pm}].$$

Going over to $C(\!(x)\!)$ and using Theorem 4.5, we conclude that

$$f\left(\mathcal{M}_0 \underset{C[x]}{\otimes} C[\![x]\!]\right) \subset \mathcal{M}'_0 \underset{C[x]}{\otimes} C[\![x]\!],$$

and this allows us extend f to a morphism of $\varphi_0\colon \mathcal{M}_0 \to \mathcal{M}'_0$ of $C[x]$-modules. Note that φ_0 is the *unique* such extension and that it is automatically compatible with the derivations; all this is because $\mathcal{M}'_0 \to \mathcal{M}'_0 \otimes C[x^{\pm}]$ is injective. In addition, working with the inverse of f, we conclude that φ_0 is an isomorphism. The same reasoning can be applied to $\mathcal{M}_\infty = \mathcal{M}(\mathbb{A}_\infty)$ and $\mathcal{M}'_\infty = \mathcal{M}'(\mathbb{A}_\infty)$ and the proof is concluded. ∎

A difference between the construction given in Theorem 6.8 and that of Section 4 is that the logarithmic model is not necessarily of the form $\mathrm{eul}_{\mathbb{P}}(V, A)$: indeed, these are *free* $\mathcal{O}_{\mathbb{P}}$-modules. Here is an illustration.

EXAMPLE 6.9. Let $C = \mathbb{C}$, $\tau = \{z \in \mathbb{C}: 0 \le \mathbf{Re}(z) < 1\}$ and $M = \gamma\mathrm{eul}_{\mathbb{P}}(\mathbb{C}, \frac{1}{2})$. In this case, $\mathcal{B} = \mathrm{eul}_{\mathbb{P}}(\mathbb{C}, \frac{1}{2})$ is not what we look for since $\mathrm{Exp}(\mathbf{r}_\infty\mathcal{B}) = \{-\frac{1}{2}\}$. Let us now consider $\mathcal{M} = \mathcal{O}_{\mathbb{P}}(\infty)$, which we understand as being defined by $\mathcal{M}(\mathbb{A}_0) = \mathbb{C}[x] \cdot \mathbf{m}_0$ and $\mathcal{M}(\mathbb{A}_\infty) = \mathbb{C}[y] \cdot \mathbf{m}_\infty$ subjected to the relation $\mathbf{m}_\infty = x^{-1}\mathbf{m}_0$. Now define $\nabla|\mathbb{A}_0$ by $\nabla\mathbf{m}_0 = \frac{1}{2}\mathbf{m}_0$, so that $\nabla(\mathbf{m}_\infty) = -\frac{1}{2}\mathbf{m}_\infty$ and hence $\mathrm{Exp}(\mathcal{M}) = \{\frac{1}{2}\}$.

Part II

We shall now concentrate on the study which gives the title to this paper: regular-singular connections depending on parameters.

7. Connections with an action of a ring

We fix a commutative C-algebra Λ whose dimension as a vector space is finite. The following definition is basic:

DEFINITION 7.1. Let $\mathcal{C}$ be a C-linear category. We define $\mathcal{C}_{(\Lambda)}$ as the category whose

objects are couples (c, α) with $c \in \mathcal{C}$ and $\alpha: \Lambda \to \mathrm{End}(c)$ is a morphism of rings, and an

arrow from (c, α) to (c', α') is a morphism $\varphi: c \to c'$ such that $\alpha'(\lambda) \circ \varphi = \varphi \circ \alpha(\lambda)$ for all $\lambda \in \Lambda$.

To ease terminology, we shall also refer to objects in $\mathcal{C}_{(\Lambda)}$ as objects of $\mathcal{C}$ with an action of Λ and usually abandon the arrow to the ring of endomorphism from notation. In this case, the endomorphism obtained from $\lambda \in \Lambda$ will come with no distinctive graphical symbol.

DEFINITION 7.2. Let $M \in (C[\![x]\!]\text{-}\mathbf{mod})_{(\Lambda)}$. We say that M is *free in relation to* Λ if there exists a Λ-module V, an isomorphism of $C[\![x]\!]$-modules $\psi: C[\![x]\!] \otimes_C V \to M$ such that, for each $\lambda \in \Lambda$, $f \in C[\![x]\!]$ and $v \in V$, we have

$$\psi(f \otimes \lambda v) = \lambda(\psi(f \otimes v)).$$

REMARK 7.3. One easily sees that the canonical arrow $\Lambda \otimes_C C[\![x]\!] \to \Lambda[\![x]\!]$ is an isomorphism and hence we may identify $(C[\![x]\!]\text{-}\mathbf{mod})_{(\Lambda)}$ with $\Lambda[\![x]\!]\text{-}\mathbf{mod}$. Then a

$C[\![x]\!]$-module with the action of Λ is free in relation to Λ if and only if, as a $\Lambda[\![x]\!]$-module, it is of the form $\Lambda[\![x]\!] \otimes_\Lambda V$ for some Λ-module V. The reason for working with $C[\![x]\!]$-modules with an action of Λ instead of with $\Lambda[\![x]\!]$-modules is justified by the fact that we wish to rely on the theories of connections over $C(\!(x)\!)$ and $C[\![x]\!]$.

Here is the first useful property stemming from the definition:

LEMMA 7.4. *Let $M \in (C[\![x]\!]\text{-}\mathbf{mod})_{(\Lambda)}$ be free in relation to Λ. Then M is a free $C[\![x]\!]$-module.* ∎

Another key property is the following:

LEMMA 7.5. *Let $M \in (C[\![x]\!]\text{-}\mathbf{mod})_{(\Lambda)}$ be free in relation to Λ. Then, for each ideal $\mathfrak{I} \subset \Lambda$, the $C[\![x]\!]$-module $M/\mathfrak{I}M$ is also free in relation to Λ. In particular, $M/\mathfrak{I}M$ is a free $C[\![x]\!]$-module.* ∎

We now begin to apply the definition of objects with an action of Λ to categories of connections.

EXAMPLE 7.6. The category $\mathbf{End}_{(\Lambda)}$ consists of couples (V, A) where V is a Λ-module and A is an endomorphism of Λ-modules.

EXAMPLE 7.7. The simplest way of constructing objects in $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(\Lambda)}$ is by means of Euler connections. Let V be a finite Λ-module, $A\colon V \to V$ a C-linear endomorphism and $\mathrm{eul}(V, A)$ the associated Euler connection. Now assume that A is, in addition, Λ-linear (so that $(V, A) \in \mathbf{End}_{(\Lambda)}$). Then, for each $\lambda \in \Lambda$, the endomorphism $[\lambda]\colon C[\![x]\!] \otimes_C V \to C[\![x]\!] \otimes_C V$ defined by $[\lambda](f \otimes v) = f \otimes \lambda v$ is horizontal and gives $\mathrm{eul}(V, A)$ the structure of an object from $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(\Lambda)}$. Clearly, $C[\![x]\!] \otimes_C V \in (C[\![x]\!]\text{-}\mathbf{mod})_{(\Lambda)}$ is free in relation to Λ.

THEOREM 7.8 (Deligne–Manin lattices). *Let $M \in \mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)_{(\Lambda)}$. There exists a logarithmic* lattice *$\mathcal{M} \in \mathbf{MC}_{\log}(C[\![x]\!]/C)$ for M and an action of Λ on it such that*

(1) *all exponents of $\mathcal{M}$ lie on τ;*

(2) *the isomorphism $\gamma(\mathcal{M}) \simeq M$ is compatible with the Λ-actions;*

(3) *$\mathcal{M}$ is free in relation to Λ;*

(4) *in fact, $\mathcal{M}$ and its Λ action can be chosen to be of the form* $\mathrm{eul}(V, A)$*, where $(V, A) \in \mathbf{End}_{(\Lambda)}$ is as in Example 7.7.*

Finally, if

$$\varphi\colon M \longrightarrow N$$

is an arrow of $\mathbf{MC}_{\mathrm{rs}}(C((x))/C)_{(\Lambda)}$ *and* $\mathcal{N} \in \mathbf{MC}_{\log}(C[\![x]\!]/C)$ *is a logarithmic lattice of* N *affording an action of* Λ *and having properties* (1)–(3)*, then there exists a* unique $\tilde{\varphi}\colon \mathcal{M} \to \mathcal{N}$ *in* $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(\Lambda)}$ *rendering*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\tilde{\varphi}} & \mathcal{N} \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

commutative.

Proof. By shearing (Theorem 4.2) there exists a logarithmic lattice $\mathcal{M}$ of M whose exponents are all on τ. By Theorem 4.1 we can say that $\mathcal{M} = \mathrm{eul}(V, A)$, where $A\colon V \to V$ is an endomorphism of the finite-dimensional C-space V. Note that $\mathrm{Sp}_A \subset \tau$.

Using Proposition 4.4, the natural morphism

$$\mathrm{End}_{\mathbf{MC}_{\log}}(\mathrm{eul}(V, A)) \longrightarrow \mathrm{End}_{\mathbf{MC}_{\mathrm{rs}}}(M)$$

is bijective. Hence, we obtain a morphism of rings $\Lambda \to \mathrm{End}_{\mathbf{MC}_{\log}}(\mathrm{eul}(V, A))$; this gives an action of Λ on $\mathrm{eul}(V, A)$ and condition (2) is tautologically fulfilled.

In order to show that $\mathrm{eul}(V, A)$ is free in relation to Λ, we remark that, due to Lemma 3.3 (2), for each $\lambda \in \Lambda$, the arrow

$$\lambda\colon C[\![x]\!] \otimes_C V \longrightarrow C[\![x]\!] \otimes_C V$$

in $\mathbf{MC}_{\log}(C[\![x]\!]/C)$ is of the form $1 \otimes \lambda$ for an arrow $\lambda\colon V \to V$ such that $\lambda \circ A = A \circ \lambda$. We therefore obtain an action of Λ on V. We have therefore shown that properties (1)–(4) hold.

Let N and $\mathcal{N}$ be as in the statement. The existence of an arrow $\tilde{\varphi}\colon \mathcal{M} \to \mathcal{N}$ from $\mathbf{MC}_{\log}(C[\![x]\!]/C)$ fitting into the commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\tilde{\varphi}} & \mathcal{N} \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

is guaranteed by Proposition 4.4 (3). (Recall that as $C[\![x]\!]$-modules, $\mathcal{M}$ and $\mathcal{N}$ are free.) That $\tilde{\varphi}$ is unique and respects the actions of Λ is a simple consequence of the fact that $\mathcal{N} \to N$ is an injection. ∎

We end this section by showing that what was said before about formal connections is, modified accordingly, valid for regular-singular connections on $C[x^{\pm}]$ (considered in Section 6). We start with an immediate consequence of Theorem 6.4.

Corollary 7.9. *The natural functor*

$$\mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)_{(\Lambda)} \longrightarrow \mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)_{(\Lambda)}$$

deduced from $\mathbf{r}_0$ *is an equivalence.* ∎

Before stating the next result, let us put forward the analogue of Definition 7.2.

Definition 7.10. A coherent $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{M}$ with an action of Λ is *locally free in relation to* Λ if there exists a finite Λ-module V and an isomorphism $\mathcal{M}(\mathbb{A}_0) \simeq V \otimes_C \mathcal{O}(\mathbb{A}_0)$, resp. $\mathcal{M}(\mathbb{A}_\infty) \simeq V \otimes_C \mathcal{O}(\mathbb{A}_\infty)$, such that, under these isomorphisms, the action of Λ is given by means of its action on V.

Remark 7.11. Obviously, if $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}/\mathbb{C})_{(\Lambda)}$ is locally free in relation to Λ, then it is a locally free $\mathcal{O}_{\mathbb{P}}$-module.

Theorem 7.12 (Deligne–Manin models). *Let* $M \in \mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)_{(\Lambda)}$. *There exists a logarithmic* lattice $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}/C)$ *endowed with an action of* Λ *such that*

(1) *all exponents of* $\mathcal{M}$ *lie on* τ*;*

(2) *the canonical isomorphism*

$$\gamma_{\mathbb{P}}(\mathcal{M}) \xrightarrow{\sim} M$$

is compatible with Λ*-actions;*

(3) $\mathcal{M}$ *is locally free in relation to* Λ.

Proof. This is much the same as the proof of Theorem 6.8, except that we make the following replacements. The use of Corollary 4.3 is replaced by that of Theorem 7.8. The use of Theorem 6.4 is replaced by that of Corollary 7.9. ∎

Note that the statement of Theorem 7.12 leaves out the uniqueness properties analogous to those in Theorem 7.8. The verification of these occupies the following lines.

Let $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}/C)_{(\Lambda)}$ and $\delta \in \mathbb{Z}$. Let $\mathcal{M}(\delta)$ stand for the logarithmic connection obtained by gluing $x^{-\delta}\mathcal{M}(\mathbb{A}_0)$ and $y^{-\delta}\mathcal{M}(\mathbb{A}_\infty)$ via the isomorphism $x = y^{-1}$. In the possession of this definition, we have an analogue of Proposition 4.4:

Proposition 7.13. *Let* $\varphi\colon E \to F$ *be an arrow of* $\mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)_{(\Lambda)}$. *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be logarithmic models for* E *and* F *and assume that* $\mathcal{F}$ *is in fact a lattice. Let* δ

be the largest integer in $\mathrm{Exp}(\mathcal{F}) \ominus \mathrm{Exp}(\mathcal{E})$. *Then* $x^\delta\varphi(\mathcal{E}) \subset \mathcal{F}$. *In particular, there exists a unique* $\Phi\colon \mathcal{E} \to \mathcal{F}(\delta)$ *from* $\mathbf{MC}_{\log}(\mathbb{P}/C)_{(\Lambda)}$ *such that* $\gamma_{\mathbb{P}}(\Phi) = \varphi$.

Proof. Let us write $(-)_0$ and $(-)_\infty$ for sections over $\mathbb{A}_0$ and $\mathbb{A}_\infty$. Similarly to the proof of Proposition 4.4, we obtain $x^\delta\varphi(\mathcal{E}_0) \subset \mathcal{F}_0$ and $y^\delta\varphi(\mathcal{E}_\infty) \subset \mathcal{F}_\infty$. As $\mathcal{F}$ is locally free, we extend φ to $\Phi\colon \mathcal{E} \to \mathcal{F}(\delta)$, an arrow of $\mathbf{MC}_{\log}(\mathbb{P}/C)$. As the restrictions $\mathcal{F}(\delta)_0 \to F$ and $\mathcal{F}(\delta)_\infty \to F$ are injective, we conclude that Φ is an arrow of $\mathbf{MC}_{\log}(\mathbb{P}/C)_{(\Lambda)}$. Obviously $\gamma_{\mathbb{P}}(\Phi) = \varphi$. The injectivity of $\mathcal{F}(\delta)_0 \to F$ and $\mathcal{F}(\delta)_\infty \to F$ again ensures that Φ is unique. ∎

Theorem 7.14. *Let* $\varphi\colon M \to N$ *be an arrow of* $\mathbf{MC}_{\mathrm{rs}}(C[x^\pm]/C)_{(\Lambda)}$. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be logarithmic models for* M *and* N *affording an action of* Λ, *and having properties* (1)–(3) *of Theorem* 7.12. *Then there exists a* unique $\Phi\colon \mathcal{M} \to \mathcal{N}$ *in* $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(\Lambda)}$ *satisfying*

$$\gamma_{\mathbb{P}}(\Phi) = \varphi.$$

Proof. This is much the same as the last part of the proof of Theorem 7.8, except that we make the following replacement. The use of Proposition 4.4 (3) is replaced by that of Proposition 7.13. ∎

8. Formal connections with parameters in a ring: Basic results

We let R be a complete local noetherian C-algebra with residue field C and maximal ideal $\mathfrak{r}$. The C-algebras $R/\mathfrak{r}^{k+1}$ will be abbreviated to R_k. We let ϑ stand for the R-linear derivation on $R[\![x]\!]$ defined by $\vartheta \sum a_n x^n = \sum a_n n x^n$, as well as its extension to $R(\!(x)\!) = R[\![x]\!][x^{-1}]$. Finally, in developing our arguments, we shall find it convenient to identify $R[\![x]\!]/\mathfrak{r}^{k+1}R[\![x]\!]$ and $R_k[\![x]\!]$ via the canonical morphism [36, Theorem 8.11, p. 61]. (Note also that this identification is possible by replacing $\mathfrak{r}^{k+1}$ with any given ideal of R.)

We begin by recycling the definitions appearing in Section 2.

Definition 8.1. (1) We let $\mathbf{MC}(R(\!(x)\!)/R)$, the category of *$R$-linear connections*, be the category whose objects are couples (M, ∇) consisting of a finite $R(\!(x)\!)$-module and an R-linear endomorphism $\nabla\colon M \to M$ satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$, and whose arrows are defined by imitating Definition 2.1.

(2) We let $\mathbf{MC}_{\log}(R[\![x]\!]/R)$, the category of *$R$-linear logarithmic connections*, be the category whose objects are couples $(\mathcal{M}, \nabla)$ consisting of a finite $R[\![x]\!]$-module

and an R-linear endomorphism $\nabla\colon \mathcal{M} \to \mathcal{M}$ satisfying Leibniz's rule $\nabla(fm) = \vartheta(f)m + f\nabla(m)$, and whose arrows are defined by imitating Definition 2.1. Whenever no confusion is possible, we omit reference to ∇ in the notation.

(3) We denote by

$$\gamma\colon \mathbf{MC}_{\log}(R[\![x]\!]/R) \longrightarrow \mathbf{MC}(R(\!(x)\!)/R)$$

the obvious functor and define $\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$, the category of *regular-singular* connections, as being the full subcategory of $\mathbf{MC}(R(\!(x)\!)/R)$ whose objects are (isomorphic to an object) in the image of γ.

(4) Given $M \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$, any object $\mathcal{M} \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$ for which there is an isomorphism $\gamma(\mathcal{M}) \simeq M$ is said to be a *logarithmic model* of M.

(5) A logarithmic model $\mathcal{M}$ of M is called *x-pure* if multiplication by x is injective on $\mathcal{M}$.

It comes as no surprise that $\mathbf{MC}(R(\!(x)\!)/R)$ is an abelian category such that the forgetful functor to $R(\!(x)\!)$-**mod** is exact.

Given $(\mathcal{M},\nabla) \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$, it is clear that the $R[\![x]\!]$-module $\bigcup_k (0 : x^k)_{\mathcal{M}} = \{m \in \mathcal{M} : x^k m = 0\}$ is stable under ∇, so that, taking the quotient, we have the following lemma:

Lemma 8.2. *Each $M \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ has an x-pure logarithmic model.* ∎

This simple result can be improved; see Theorem 9.1 below. But its utility is promptly manifest.

Proposition 8.3. *The full subcategory $\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ of $\mathbf{MC}(R(\!(x)\!)/R)$ is stable under quotients and subobjects.*

Sketch of proof. Let $N \in \mathbf{MC}(R(\!(x)\!)/R)$ be a subobject of $M \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$. Let $\mathcal{M}$ be an *x-pure* logarithmic model for M (cf. Lemma 8.2). Then $\mathcal{N} := \mathcal{M} \cap N$ is an x-pure logarithmic model of N. Quotients are treated using models for the kernel. ∎

Furthermore, given $(\mathcal{M},\nabla)$ and $(\mathcal{M}',\nabla')$ in $\mathbf{MC}_{\log}(R[\![x]\!]/R)$, their tensor product $\mathcal{M} \otimes_{R[\![x]\!]} \mathcal{M}'$ gives rise to an object of $\mathbf{MC}_{\log}(R[\![x]\!]/R)$ by decreeing that

$$\nabla \otimes \nabla'(m \otimes m') = \nabla(m) \otimes m' + m \otimes \nabla'(m').$$

It is then the case that $\mathbf{MC}_{\log}(R[\![x]\!]/R)$ becomes an R-linear tensor category and $\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ is an R-linear abelian tensor category.

Example 8.4 (Twisted models). For each $\delta \in \mathbb{Z}$, let $\mathbb{1}(\delta)$ denote the *free* $R[\![x]\!]$-submodule of $R(\!(x)\!)$ generated by $x^{-\delta}$. Clearly, $\mathbb{1}(\delta)$ is invariant under ϑ and we obtain in this way an x-pure logarithmic model for the trivial object $(R(\!(x)\!), \vartheta)$. We define analogously, for each $\mathcal{M} \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$, the object $\mathcal{M}(\delta)$ as being $\mathbb{1}(\delta) \otimes \mathcal{M}$.

We now explore further immediate similarities between this theory and the classical one.

Example 8.5. Let $\mathbf{End}_R$ be the category whose objects are couples (V, A) consisting of a finite R-module V and an R-linear endomorphism $A\colon V \to V$, and whose arrows are given as in Definition 3.2. Given $(V, A) \in \mathbf{End}_R$, let $D_A\colon R[\![x]\!] \otimes_R V \to R[\![x]\!] \otimes_R V$ be defined by

$$D_A(f \otimes v) = \vartheta f \otimes v + f \otimes Av.$$

This gives rise to an R-linear functor

$$\mathrm{eul}\colon \mathbf{End}_R \longrightarrow \mathbf{MC}_{\log}(R[\![x]\!]/R)$$

analogous to the one in Definition 3.1.

Let $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$ and note that

$$\mathrm{res}_\nabla\colon \mathcal{M}/(x) \longrightarrow \mathcal{M}/(x), \tag{8.1}$$

given by

$$\mathrm{res}_\nabla(m + (x)) = \nabla(m) + (x),$$

is R-linear.

Definition 8.6 (Residue and exponents). The R-linear map (8.1) is called the *residue* of ∇. If

$$\overline{\mathrm{res}}_\nabla\colon \mathcal{M}/(\mathfrak{r}, x) \longrightarrow \mathcal{M}/(\mathfrak{r}, x)$$

stands for the C-linear morphism obtained from res_∇ by reduction modulo $\mathfrak{r}$, we call the set $\mathrm{Sp}_{\overline{\mathrm{res}}_\nabla}$ the set of *exponents* of ∇; it will be denoted by $\mathrm{Exp}(\mathcal{M}, \nabla)$, $\mathrm{Exp}(\nabla)$ or $\mathrm{Exp}(\mathcal{M})$ if no confusion is likely.

Remark 8.7. It should be highlighted that the *exponents belong to* C. The reason for taking this path is, from a practical viewpoint, justified by the fact that we are able to prove the results we wanted with it. But it is important to throw more light on our choice. While explaining either this work or [24] to others, the question "Why not take, in the case that R is a domain, the exponents in a quotient field of R?"

frequently appeared. This is certainly a possible path and when we started this theory, our exponents (in Definition 8.6) were called *reduced exponents*. Then at some point it became clear that (a) reduced exponents were the ones controlling the theory and leading to Corollary 9.7, our main result; (b) in taking limits, we need non-reduced rings; (c) in taking limits, it is important to have the exponents being constant while "reducing"; see Corollary 8.12. We then decided that the reduced exponents deserved a prominent name. On the other hand, in different situations, our definition may be insufficient; see Remark 8.17 below.

Let us now start by recalling the following lemma:

Lemma 8.8 ([43, Chapter II, Problem 4.1]). *Let m and n be positive integers, A an element of $\mathrm{M}_m(C)$, and B an element of $\mathrm{M}_n(C)$. Let*

$$f\colon \mathrm{M}_{m\times n}(C) \longrightarrow \mathrm{M}_{m\times n}(C)$$

be the linear map defined by $X \mapsto AX - XB$. Then $\mathrm{Sp}_f = \mathrm{Sp}_A \ominus \mathrm{Sp}_B$. In particular, if no two distinct eigenvalues of A differ by an integer, then the linear transformation $\nu\mathrm{id} - \mathrm{ad}_A\colon \mathrm{M}_m(C) \to \mathrm{M}_m(C)$ is invertible for each $\nu \in \mathbb{Z} \setminus \{0\}$. ∎

A direct application of Lemma 8.8 and Nakayama's lemma shows the following:

Corollary 8.9. *Let $A \in \mathrm{M}_n(R)$ be such that its reduction modulo $\mathfrak{r}$, call it $\bar{A} \in \mathrm{M}_n(C)$, has no two distinct eigenvalues differing by an integer. Then, for any $\nu \in \mathbb{Z} \setminus \{0\}$, the R-linear morphism $\nu\mathrm{id} - \mathrm{ad}_A\colon \mathrm{M}_n(R) \to \mathrm{M}_n(R)$ is bijective.* ∎

Theorem 8.10 (Cf. Theorem 4.1). *Let $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$ be such that $\mathcal{M}$ is a free $R[\![x]\!]$-module and no two distinct exponents of ∇ differ by an integer. Then $(\mathcal{M}, \nabla)$ is isomorphic to $\mathrm{eul}(\mathcal{M}/(x), \mathrm{res}_\nabla)$.*

Said otherwise, consider a differential system

$$\vartheta \boldsymbol{y} = A\boldsymbol{y}$$

defined by $A \in \mathrm{M}_r(R[\![x]\!])$ such that $A(0)$ modulo $\mathfrak{r}$ has no two distinct eigenvalues differing by an integer. Then there exists $P \in \mathrm{GL}_r(R[\![x]\!])$ such that, writing $\boldsymbol{y} = P\boldsymbol{z}$, we arrive at the system

$$\vartheta \boldsymbol{z} = B\boldsymbol{z}$$

in which $B \in \mathrm{M}_r(R)$.

Proof. One proceeds as in [43, Sections 4.2, 4.3 and 5.1], but substitute the use of Wasow's Theorem 4.1 by our Corollary 8.9. ∎

We now move to shearing techniques which allow us to eliminate the hypothesis on the exponents in Theorem 8.10. We begin by setting up the necessary linear algebra.

PROPOSITION 8.11. *Let Λ be a commutative C-algebra which is a finite-dimensional C-space. Let $\mathfrak{n} \subset \Lambda$ be a nilpotent ideal, V a finite Λ-module and $A\colon V \to V$ a Λ-linear arrow. Considering A as a C-linear endomorphism, write $\varrho_1, \ldots, \varrho_r$ for its distinct eigenvalues and let*

$$V = \mathbf{G}(A, \varrho_1) \oplus \cdots \oplus \mathbf{G}(A, \varrho_r)$$

be the decomposition into generalized eigenspaces:

(1) *Each $\mathbf{G}(A, \varrho_j)$ is invariant under Λ and $\mathfrak{n} \cdot \mathbf{G}(A, \varrho_j) \neq \mathbf{G}(A, \varrho_j)$.*

(2) *Write $\overline{V} = V/\mathfrak{n}V$ and $\overline{\mathbf{G}(A, \varrho_j)}$ for the image of $\mathbf{G}(A, \varrho_j)$ in $\overline{V}$. Then $\overline{\mathbf{G}(A, \varrho_j)} \neq 0$.*

(3) *Let $\bar{A}$ be the endomorphism of $\overline{V}$ induced by A. Then the space $\overline{\mathbf{G}(A, \varrho_j)}$ is the generalized eigenspace of $\bar{A}$ associated to ϱ_j and $\mathrm{Sp}_{\bar{A}} = \mathrm{Sp}_A$.*

PROOF. By definition,

$$\mathbf{G}(A, \varrho_j) = \bigcup_n \mathrm{Ker}(A - \varrho_j \mathrm{id})^n,$$

so that for every $\lambda \in \Lambda$, we have $\lambda \mathbf{G}(A, \varrho_j) \subset \mathbf{G}(A, \varrho_j)$. Since $\mathbf{G}(A, \varrho_j) \neq 0$, we know that $\mathfrak{n} \cdot \mathbf{G}(A, \varrho_j) \neq \mathbf{G}(A, \varrho_j)$. This establishes (1). To prove (2), we note that $\mathfrak{n}V = \bigoplus_j \mathfrak{n}\mathbf{G}(A, \varrho_j)$ and hence $\mathbf{G}(A, \varrho_j)/\mathfrak{n}\mathbf{G}(A, \varrho_j) \xrightarrow{\sim} \overline{\mathbf{G}(A, \varrho_j)}$. Also, as a consequence, we arrive at the direct sum decomposition

$$\overline{V} = \overline{\mathbf{G}(A, \varrho_1)} \oplus \cdots \oplus \overline{\mathbf{G}(A, \varrho_r)}. \tag{8.2}$$

The nilpotence of $\bar{A} - \varrho_j \mathrm{id}$ when restricted to $\overline{\mathbf{G}(A, \varrho_j)}$ now shows that ϱ_j is the only eigenvalue of $\bar{A}$ on $\overline{\mathbf{G}(A, \varrho_j)}$ and that

$$\overline{\mathbf{G}(A, \varrho_j)} \subset \mathbf{G}(\bar{A}, \varrho_j).$$

Let us fix $j_0 \in \{1, \ldots, r\}$ and show that $\overline{\mathbf{G}(A, \varrho_{j_0})} \supset \mathbf{G}(\bar{A}, \varrho_{j_0})$. Suppose that $\bar{w} \in \overline{V}$ is annihilated by $(\bar{A} - \varrho_{j_0}\mathrm{id})^m$ and write it as $\bar{v}_1 + \cdots + \bar{v}_r$ with $\bar{v}_j \in \overline{\mathbf{G}(A, \varrho_j)}$. Since $\bar{v}_j \in \overline{\mathbf{G}(A, \varrho_j)}$, there exists $n_j \in \mathbb{N}$ such that $(\bar{A} - \varrho_j \mathrm{id})^{n_j}(\bar{v}_j) = 0$. We now choose $\mu = \max\{m, n_1, \ldots, n_r\}$ and then find $P, Q \in C[T]$ such that

$$P(T) \cdot (T - \varrho_{j_0})^{\mu} = 1 + Q(T) \cdot \prod_{j \neq j_0} (T - \varrho_j)^{\mu}.$$

Hence,

$$0 = \bar{w} + Q(\bar{A}) \cdot \prod_{j \neq j_0} (\bar{A} - \varrho_j \mathrm{id})^{\mu}(\bar{w}).$$

Now

$$Q(\bar{A}) \cdot \prod_{j \neq j_0} (\bar{A} - \varrho_j \mathrm{id})^{\mu}(\bar{w}) = \underbrace{Q(\bar{A}) \cdot \prod_{j \neq j_0} (\bar{A} - \varrho_j \mathrm{id})^{\mu}(\bar{v}_{j_0})}_{\in \overline{\mathbf{G}(A, \varrho_{j_0})}},$$

which shows $\bar{w} \in \overline{\mathbf{G}(A, \varrho_{j_0})}$. Finally, (8.2) is the decomposition of $\bar{V}$ into generalized eigenspaces. ∎

The previous result also allows us to grasp the utility of our definition of exponents.

Corollary 8.12. *The following claims are true:*

(1) *Let Λ be a C-algebra which is a finite-dimensional vector space and $\mathfrak{n} \subset \Lambda$ a nilpotent ideal. Let $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(C[\![x]\!]/C)_{(\Lambda)}$ and define $\mathcal{M}|_{\mathfrak{n}} = \mathcal{M}/\mathfrak{n}$. Then ∇ gives rise to $\nabla|_{\mathfrak{n}} \colon \mathcal{M}|_{\mathfrak{n}} \to \mathcal{M}|_{\mathfrak{n}}$ and the couple $(\mathcal{M}|_{\mathfrak{n}}, \nabla|_{\mathfrak{n}})$ is an object of $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(\Lambda/\mathfrak{n})}$ which has the same set of exponents as $(\mathcal{M}, \nabla)$.*

(2) *Let $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$ and $k \in \mathbb{N}$ be given. Define $\mathcal{M}|_k := \mathcal{M}/\mathfrak{r}^{k+1}$. Then this is a $C[\![x]\!]$-module of finite type (since it is a finite $R_k[\![x]\!]$-module). Let $\nabla|_k \colon \mathcal{M}|_k \to \mathcal{M}|_k$ be induced by ∇. Then $(\mathcal{M}|_k, \nabla|_k)$ is an object of $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$ and* $\mathrm{Exp}(\nabla) = \mathrm{Exp}(\nabla|_k)$. ∎

Another useful consequence of Proposition 8.11 is the following corollary:

Corollary 8.13 (Lifting of Jordan decomposition). *Let V be an R-module and $A \colon V \to V$ be an R-endomorphism. Denote by $\bar{A} \colon \bar{V} \to \bar{V}$ the C-linear endomorphism obtained by reducing A modulo* $\mathfrak{r}$.

Then there exist R-submodules $\{V(\varrho) \colon \varrho \in \mathrm{Sp}_{\bar{A}}\}$ of V enjoying the following properties:

(1) *The R-module V is the direct sum of $\{V(\varrho) \colon \varrho \in \mathrm{Sp}_{\bar{A}}\}$.*

(2) *Each $V(\varrho)$ is stable under A.*

(3) *If $\overline{V(\varrho)}$ stands for the image of $V(\varrho)$ in $\bar{V} = V/\mathfrak{r}V$, then $\overline{V(\varrho)} = \mathbf{G}(\bar{A}, \varrho)$.*

In addition, if V is free, then each $V(\varrho)$ is also free.

Proof. Let ϱ be fixed. For a given $k \in \mathbb{N}$, let $A_k \colon V_k \to V_k$ be the R_k-linear endomorphism obtained by reducing A modulo $\mathfrak{r}^{k+1}$. The eigenvalues of A_k will always mean those of the associated C-linear endomorphism of V_k. Applying Proposition 8.11 to the case $\Lambda = R_{k+1}$ and $\mathfrak{n} = \mathfrak{r}^{k+1} \cdot R_{k+1}$, we obtain that $\mathrm{Sp}_{A_{k+1}} = \mathrm{Sp}_{A_k}$.

By induction, $\mathrm{Sp}_{A_k} = \mathrm{Sp}_{A_0} = \mathrm{Sp}_{\bar{A}}$. In addition, we also know that the canonical arrow

$$\mathbf{G}(A_{k+1}, \varrho) \longrightarrow \mathbf{G}(A_k, \varrho)$$

is a surjective morphism of R_{k+1}-modules whose kernel is $\mathrm{r}^{k+1}\mathbf{G}(A_{k+1}, \varrho)$. Now we define

$$V(\varrho) = \varprojlim_k \mathbf{G}(A_k, \varrho),$$

which is considered as an $R = \varprojlim_k R_k$-module. According to [22, 0_{I}, Proposition 7.2.9, p. 65], the natural projection $V(\varrho) \to \mathbf{G}(A_k, \varrho)$ is surjective and has kernel $\mathrm{r}^{k+1}V(\varrho)$.

Using the inclusions $\mathbf{G}(A_k, \varrho) \to V_k$, we obtain an injective arrow of R-modules

$$u\colon \bigoplus_{\varrho} V(\varrho) \longrightarrow \varprojlim_k V_k \quad (\simeq V).$$

In addition, reducing u modulo r and employing the fact that $V(\varrho)/\mathrm{r}V(\varrho) \simeq \mathbf{G}(A_0, \varrho)$, Nakayama's lemma [36, Theorem 2.2, p. 8] tells us that u is surjective.

The verification of the final assertion is clear: because $V(\varrho)$ is a direct summand of V, we can infer that $V(\varrho)$ is projective and of finite type, hence free. ■

In the case that the module V appearing in the statement of Corollary 8.13 is *free*, we have the following (probably well-known) consequence:

Corollary 8.14. *Let $A \in \mathrm{M}_n(R)$ be given and denote by $\{\varrho_1, \dots, \varrho_r\}$ the spectrum of $\bar{A} \in \mathrm{M}_n(C)$. Then there exist*

(1) *$P \in \mathrm{GL}_n(R)$,*

(2) *a partition $n = n_1 + \dots + n_r$ and*

(3) *matrices*

$$U(1) \in \mathrm{M}_{n_1}(R), \dots, U(n_r) \in \mathrm{M}_{n_r}(R)$$

such that

$$P^{-1}AP = \left(\begin{array}{c|c|c} U(1) & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & U(n_r) \end{array}\right),$$

and, for every i, the image of $U(n_i)$ in $\mathrm{M}_{n_i}(C)$ is a generalized Jordan matrix with eigenvalue ϱ_i. ■

Remarks 8.15. (a) Corollary 8.14 should be compared with [43, Theorem 25.1]. In fact, it is not difficult to show that this result holds under the weaker assumption that R is only strictly Henselian. Indeed, the Hensel property allows us to lift the factorization of the characteristic polynomial of A and one proceeds by showing that the kernels of the various factors evaluated at A produce a direct sum decomposition.

(b) There is a substantial literature on the problem of similarity of matrices over rings; see e.g. [23] and references in there.

Once in possession of these properties, we can follow the shearing technique in [43] to prove the following theorem::

THEOREM 8.16. *Let* $(\mathcal{M}, \nabla_{\mathcal{M}}) \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$ *be such that* $\mathcal{M}$ *is a* free $R[\![x]\!]$-*module and let* (M, ∇_M) *be the regular-singular connection associated to* $(\mathcal{M}, \nabla_{\mathcal{M}})$. *Then there exists an object* $(W, B) \in \mathbf{Eul}_R$, *with* W *a free* R-*module, such that* $(M, \nabla_M) \simeq \gamma\mathrm{eul}(W, B)$. *In addition, the eigenvalues of the endomorphism of* $W/(x, \mathfrak{r})$ *defined by* B *all belong to* τ.

Said otherwise, consider a differential system

$$\vartheta \boldsymbol{y} = A\boldsymbol{y}$$

defined by $A \in \mathrm{M}_r(R[\![x]\!])$. *There exists* $P \in \mathrm{GL}_r(R(\!(x)\!))$ *such that, writing* $\boldsymbol{y} = P\boldsymbol{z}$, *we arrive at the system*

$$\vartheta \boldsymbol{z} = B\boldsymbol{z}$$

in which B *belongs to* $\mathrm{M}_r(R)$ *and its image in* $\mathrm{M}_r(C)$ *only has eigenvalues lying in* τ.

PROOF. Because of Nakayama's lemma [36, Theorem 2.2, p. 8] (and the fact that $R[\![x]\!]$ is local), a set of elements of $\mathcal{M}$ which is mapped to a basis of $\mathcal{M}/(x)$ is necessarily a basis of $\mathcal{M}$. According to Corollary 8.14, there exists a basis $\boldsymbol{m} = \{m_i\}_{i=1}^r$ of $\mathcal{M}$ such that the basis

$$\bar{\boldsymbol{m}} = \{m_i + (x)\}_{i=1}^r$$

of $\mathcal{M}/(x)$ has the following properties:

(a) the matrix of $\mathrm{res}_{\mathcal{M}} \colon \mathcal{M}/(x) \to \mathcal{M}/(x)$ with respect to $\boldsymbol{m}$ has the form

$$\left(\begin{array}{c|c} J_{11} & 0 \\ \hline 0 & J_{22} \end{array}\right),$$

where $J_{11} \in \mathrm{M}_q(R)$ and $J_{22} \in \mathrm{M}_{r-q}(R)$. (Here, $q \in \{1, \ldots, r\}$ is a positive integer. In the case that $q = r$, we say only that $\mathrm{res}_{\mathcal{M}} = J_{11}$.)

(b) If $\bar{J}_{11} \in \mathrm{M}_q(C)$ and $\bar{J}_{22} \in \mathrm{M}_{r-q}(C)$ stand for the images of J_{11} and J_{22} respectively, then $\mathrm{Sp}_{\bar{J}_{11}} = \{\varrho\}$ and $\varrho \notin \mathrm{Sp}_{\bar{J}_{22}}$.

Hence, the matrix of $\nabla_{\mathcal{M}}$ with respect to $\boldsymbol{m}$ is

$$\left(\begin{array}{c|c} J_{11} + x\Psi_{11} & x\Psi_{12} \\ \hline x\Psi_{21} & J_{22} + x\Psi_{22} \end{array}\right),$$

where $\Psi_{11} \in \mathrm{M}_q(R[\![x]\!])$ and $\Psi_{22} \in \mathrm{M}_{r-q}(R[\![x]\!])$.

Let us now define $\boldsymbol{m}' = \{m'_1, \dots, m'_r\} \subset M$ by

$$m'_j = \begin{cases} xm_j & \text{if } j \in \{1, \dots, q\}, \\ m_j & \text{if } j \in \{q+1, \dots, r\}, \end{cases}$$

which is to say that the base-change matrix from $\boldsymbol{m}$ to $\boldsymbol{m}'$ is

$$\left(\begin{array}{c|c} x & 0 \\ \hline 0 & I \end{array}\right).$$

Clearly,

$$\mathcal{M}' = \sum_{j=1}^{r} R[\![x]\!] \cdot m'_j$$

is a free $R[\![x]\!]$-module such that $\mathcal{M}'[1/x] = M$. In addition, the matrix of ∇_M with respect to $\boldsymbol{m}'$ is

$$\left(\begin{array}{c|c} 1/x & 0 \\ \hline 0 & I \end{array}\right) \cdot \left(\begin{array}{c|c} x & 0 \\ \hline 0 & 0 \end{array}\right)$$
$$+ \left(\begin{array}{c|c} 1/x & 0 \\ \hline 0 & I \end{array}\right) \cdot \left(\begin{array}{c|c} J_{11} + x\Psi_{11} & x\Psi_{12} \\ \hline x\Psi_{21} & J_{22} + x\Psi_{22} \end{array}\right) \cdot \left(\begin{array}{c|c} x & 0 \\ \hline 0 & I \end{array}\right),$$

which equals

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) + \left(\begin{array}{c|c} J_{11} + x\Psi_{11} & \Psi_{12} \\ \hline x^2\Psi_{21} & J_{22} + x\Psi_{22} \end{array}\right).$$

Hence, with respect to the basis $\{m'_j + (x)\}$ of $\mathcal{M}'/(x)$, we have

$$\mathrm{res}_{\mathcal{M}'} = \left(\begin{array}{c|c} J_{11} + I & \Psi_{12} \\ \hline 0 & J_{22} \end{array}\right),$$

and the exponents of $\mathcal{M}'$ are $\{\varrho + 1\} \cup \mathrm{Sp}_{\bar{J}_{22}}$. Analogously, if we define $\boldsymbol{m}'' = \{m''_1, \dots, m''_r\} \subset M$ by

$$m''_j = \begin{cases} x^{-1}m_j & \text{if } j \in \{1, \dots, q\}, \\ m_j & \text{if } j \in \{q+1, \dots, r\}, \end{cases}$$

and

$$\mathcal{M}'' = \sum_{j=1}^{r} R[\![x]\!] \cdot m_j'',$$

we obtain a logarithmic model $(\mathcal{M}'', \nabla_M)$ such that $\mathrm{Exp}_{\mathcal{M}''} = \{\varrho - 1\} \cup \mathrm{Sp}_{\bar{J}_{22}}$.

By induction, we are able to find a logarithmic model $(\mathcal{M}^+, \nabla^+)$ of M such that $\mathcal{M}^+$ is free and $\mathrm{Exp}_{\mathcal{M}^+} \subset \tau$. Theorem 8.10 now finishes the proof. ■

Remark 8.17. As mentioned in Remark 8.7, our definition of exponents can be inadequate in certain contexts. Suppose that we set out to obtain a "normalization" result like Theorem 8.16 in the following setting. Let $\mathfrak{o}$ be a noetherian C-algebra which is also a domain and define $\mathbf{MC}_{\mathrm{rs}}(\mathfrak{o}(\!(x)\!)/\mathfrak{o})$ along the lines of Definition 8.1 (3). Given $(M, \nabla) \in \mathbf{MC}_{\mathrm{rs}}(\mathfrak{o}(\!(x)\!)/\mathfrak{o})$ such that M is a *free* $\mathfrak{o}$-module, is it possible to find an analogue of the combination of Theorem 8.16 and Corollary 8.14?

Here we recommend [3, Sections 8.3–4]. Their exponents [3, Proposition-Definition 7.6.1] are elements in an extension of $\mathrm{Frac}(\mathfrak{o})$ following the classical construction (cf. Theorems 4.1 and 4.2). From there, André, Baldassarri and Cailotto go on to show that the exponents of (M, ∇) do indeed belong to some integral extension $\mathfrak{o}'$ of $\mathfrak{o}$ and that a "Jordan decomposition" can be achieved over $\mathfrak{o}'(\!(x)\!)$ provided that the differences of exponents are in C [3, Theorem 8.4.2]. This gives another approach to Theorem 8.16.

We end this section with a capital result, Theorem 8.18, concerning the structure of the $R(\!(x)\!)$-module underlying an object of $\mathbf{MC}(R(\!(x)\!)/R)$:

Theorem 8.18. *Let (M, ∇) be an object of $\mathbf{MC}(R(\!(x)\!)/R)$. Then M is flat as an $R(\!(x)\!)$-module if and only if M is R-flat.*

Since the ring $R(\!(x)\!)$ is not $\mathfrak{r}$-adically complete and since the fibres of $\mathrm{Spec}\, R(\!(x)\!) \to \mathrm{Spec}\, R$ may fail to be of finite type over a field, the argument delivering Theorem 8.18 cannot be a direct adaptation of known results, e.g. [31, Lemma 2.4.2, p. 40], [17, p. 82] or [18, Proposition 5.1.1]. (We profit to note at this point that in the proof of [18, Proposition 5.1.1], we need to employ the "fibre-by-fibre flatness criterion" [22, IV_3, 11.3.10, p. 138] and not the "local flatness criterion.") We then need the following theorem, which will also find future applications.

Theorem 8.19. *Let $M \in \mathbf{MC}(R(\!(x)\!)/R)$ be given. Let $\mathfrak{p}$ be a prime ideal of R, S the quotient ring $R/\mathfrak{p}$ and L its field of fractions. Then the $L \otimes_R R(\!(x)\!)$-module*

$$M|_{\mathfrak{p}} := (L \otimes_R R(\!(x)\!)) \underset{R(\!(x)\!)}{\otimes} M$$

is flat.

Proof. Since $R[x]/\mathfrak{p}R[x] \simeq S[x]$, the Artin–Rees lemma ensures that $R[\![x]\!]/\mathfrak{p}R[\![x]\!] \simeq S[\![x]\!]$ [36, Theorem 8.11, p. 61]; inverting x, we conclude that $S \otimes_R R(\!(x)\!) \xrightarrow{\sim} S(\!(x)\!)$. As a consequence, $S \otimes_R M$ is an object of $\mathbf{MC}(S(\!(x)\!)/S)$. Hence, we only need to show that for any $N \in \mathbf{MC}(S(\!(x)\!)/S)$, the $L \otimes_S S(\!(x)\!)$-module $L \otimes_S N$ is flat. Using [2, Theorem 2.5.2.1, p. 713] (see also Remark 8.20), it is enough to show that $L \otimes_S S(\!(x)\!)$ has no ideal invariant under ϑ other than (0) and (1). Then let $J \subset L \otimes_S S(\!(x)\!)$ be a non-zero ideal invariant under ϑ. Since $L \otimes_S S(\!(x)\!)$ is a localization of $S[\![x]\!]$ – note that $S(\!(x)\!)$ is a localization of $S[\![x]\!]$ and $L \otimes_S S(\!(x)\!)$ is a localization of $S(\!(x)\!)$ – we conclude that J is the extension of $I := J \cap S[\![x]\!]$; clearly I is equally stable under ϑ. What we are looking for is a consequence of the following claim:

Claim. *Let $I \subset S[\![x]\!]$ be a ϑ-invariant ideal. Then there exists an ideal $\mathfrak{a} \subset S$ such that*

$$\mathfrak{a} \cdot S(\!(x)\!) = I \cdot S(\!(x)\!).$$

Proof. Let $f_1, \dots, f_n$ be generators of I. We conclude that the vector $\boldsymbol{f} = {}^\top(f_1, \dots, f_n)$ satisfies a differential equation

$$\vartheta \boldsymbol{y} = A\boldsymbol{y},$$

where $A \in \mathrm{M}_n(S[\![x]\!])$. Let us now suppose that $\tau \cap \mathbb{Z} = \{0\}$. There exists

$$P \in \mathrm{GL}_n(S(\!(x)\!))$$

such that, if $\boldsymbol{f} = P\boldsymbol{g}$, then

$$\vartheta \boldsymbol{g} = B\boldsymbol{g} \tag{8.3}$$

with $B \in \mathrm{M}_n(S)$ a matrix whose image in $\mathrm{M}_n(C)$ only has eigenvalues in τ (Theorem 8.16). Since $P \in \mathrm{GL}_n(S(\!(x)\!))$, letting $\boldsymbol{g} = {}^\top(g_1, \dots, g_n)$, we have

$$\sum_{i=1}^{n} S(\!(x)\!) g_i = I \cdot S(\!(x)\!).$$

Let us now write

$$\boldsymbol{g} = \sum_{i \geq i_0} \boldsymbol{g}_i x^i.$$

It then follows from (8.3) that $B\boldsymbol{g}_i = i\boldsymbol{g}_i$ for each $i \geq i_0$. Given $k \in \mathbb{N}$, let

$$B_k \colon S_k^{\oplus n} \longrightarrow S_k^{\oplus n}$$

stand for the C-linear endomorphism defined by B. Since $\mathrm{Sp}_{B_k} = \mathrm{Sp}_{B_0}$ (cf. Proposition 8.11) and $\mathrm{Sp}_{B_0} \cap \mathbb{Z} = \{0\}$, we conclude that, if $i \neq 0$, then the image of $\boldsymbol{g}_i$ in $S_k^{\oplus n}$ vanishes. As k is arbitrary, this implies that $\boldsymbol{g}_i = 0$ for $i \neq 0$ and hence $\boldsymbol{g} \in S^n$. The ideal $\mathfrak{a}$ envisaged in the statement is hence obtained. ■

The proof of the claim, and hence that of the theorem, is finished. ■

PROOF OF THEOREM 8.18. One applies the previous result and the fibre-by-fibre flatness criterion [22, IV_3, 11.3.10, p. 138]. ■

REMARK 8.20. We have employed above a theorem from [2] in order to prove Theorem 8.19. Here is a self-contained result which gives what we want.

Let A be a ring, Ω an A-module and $\mathrm{d}\colon A \to \Omega$ a derivation. Given an A-module M, we define a connection on M as being an additive map $\nabla\colon M \to M \otimes \Omega$ such that $\nabla(am) = a\nabla(m) + m \otimes \mathrm{d}a$. Let $A[\Omega] = A \oplus \Omega$ and give it the structure of a ring by decreeing that $\omega\omega' = 0$ for $\omega, \omega' \in \Omega$. Let $\iota\colon A \to A[\Omega]$ be the obvious inclusion and $t\colon A \to A[\Omega]$ the map defined by $a \mapsto a + \mathrm{d}a$; both are morphisms of rings. Using a connection ∇ on M, we arrive at an isomorphism of $A[\Omega]$-modules

$$A[\Omega] \underset{t,A}{\otimes} M \xrightarrow{\sim} M \underset{A,\iota}{\otimes} A[\Omega] \tag{8.4}$$

which reduces to the identity modulo Ω [5, Proposition 2.9].

Let us suppose that M is of finite type and let Fitt_r be the rth Fitting ideal of M [19, Corollary-Definition 20.4]. By a fundamental property of these ideals [19, Corollary 20.5], the isomorphism in (8.4) says that $t(\mathrm{Fitt}_r)A[\Omega] = \iota(\mathrm{Fitt}_r)A[\Omega]$. This implies the inclusion

$$\mathrm{d}(\mathrm{Fitt}_r) \subset \mathrm{Fitt}_r \cdot \Omega.$$

We say that an ideal $I \subset A$ is d-invariant if $\mathrm{d}(I) \subset I\Omega$. Therefore, imposing that the only d-invariant ideals of A are (0) and (1) and employing [19, Proposition 20.8], we conclude that either $M = 0$, or M is projective of constant rank. (Note that if $\mathrm{Spec}\, A$ is disconnected, then there are immediately d-invariant ideals other than (0) and (1), so constancy of the rank is appropriate.)

9. Logarithmic models for connections from $\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$

Let $k \in \mathbb{N}$. For each $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(R[\![x]\!]/R)$, the arrow

$$\nabla\colon \mathcal{M}/\mathfrak{r}^{k+1} \longrightarrow \mathcal{M}/\mathfrak{r}^{k+1}$$

gives rise to an object of $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$ and this construction produces a functor

$$\bullet|_k : \mathbf{MC}_{\log}(R[\![x]\!]/R) \longrightarrow \mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}.$$

Analogously, we obtain a functor

$$\bullet|_k : \mathbf{MC}(R(\!(x)\!)/R) \longrightarrow \mathbf{MC}(C(\!(x)\!)/C)_{(R_k)}$$

and these two fit into a commutative diagram (up to natural isomorphism)

$$\begin{array}{ccc} \mathbf{MC}_{\log}(R[\![x]\!]/R) & \xrightarrow{\gamma} & \mathbf{MC}(R(\!(x)\!)/R) \\ \bullet|_k \downarrow & & \downarrow \bullet|_k \\ \mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)} & \xrightarrow[\gamma]{} & \mathbf{MC}(C(\!(x)\!)/C)_{(R_k)}. \end{array}$$

In particular, if $M \in \mathbf{MC}(R(\!(x)\!)/R)$ is regular-singular, then $M|_k$ is also regular-singular.

THEOREM 9.1 (Deligne–Manin models). *Any $M \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ possesses a logarithmic model $\mathcal{M}$ such that, for every $k \in \mathbb{N}$, the object*

$$\mathcal{M}|_k \in \mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$$

enjoys the following properties:

(1) *All its exponents lie in τ.*

(2) *It is free in relation to R_k.*

(3) *The isomorphism $\gamma(\mathcal{M}|_k) \simeq M|_k$ is compatible with the action of R_k.*

Put otherwise, $\mathcal{M}|_k$ is a Deligne–Manin model in the sense of Theorem 7.8.

PROOF. Let us begin with a piece of commutative algebra which is fundamental to our argument: the ring $R[\![x]\!]$ is $\mathfrak{r}$-adically complete [36, Exercises 8.6 and 8.2]. This allows us to construct $R[\![x]\!]$-modules by taking limits.

Step 1: Putting Deligne–Manin models of truncations together. For each k, let

$$\begin{array}{c}\mathcal{M}_k \text{ be a Deligne–Manin logarithmic} \\ \text{model of } M|_k \in \mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)_{(R_k)},\end{array}$$

as obtained in Theorem 7.8. By definition, the exponents of $\mathcal{M}_k$ are all on τ. Note that $\mathcal{M}_{k+1}|_k$, regarded as an object of $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$, is a logarithmic lattice for $M|_k$ enjoying all the properties described in Theorem 7.8. (To see that the exponents remain unchanged, see Corollary 8.12.) We can therefore, by Theorem 7.8, find an isomorphism

$$\varphi_k : \mathcal{M}_{k+1}|_k \xrightarrow{\sim} \mathcal{M}_k,$$

in the category $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$, such that

$$\begin{array}{ccc} \mathcal{M}_{k+1}|_k & \xrightarrow{\varphi_k} & \mathcal{M}_k \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ (M|_{k+1})|_k & \xrightarrow[\text{can.}]{} & M|_k \end{array}$$

commutes. Because of [22, 0_{I}, Proposition 7.2.9],

$$\mathcal{M} := \varprojlim_k \mathcal{M}_k$$

is a finite $R[\![x]\!]$-module since, as mentioned before, $R[\![x]\!] \simeq \varprojlim_k R_k[\![x]\!]$. Furthermore, for each k, the natural arrow $\mathcal{M}/\mathfrak{r}^{k+1} \to \mathcal{M}_k$ is an isomorphism by [22, 0_{I}, Proposition 7.2.9]. Using the derivations on the various $\mathcal{M}_k$, we construct a derivation ∇ on $\mathcal{M}$: we have therefore produced an element of $\mathbf{MC}_{\log}(R[\![x]\!]/R)$. Clearly, for any given $k \in \mathbb{N}$, the object $\mathcal{M}|_k \in \mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$ enjoys properties (1), (2) and (3) of the statement.

Step 2: Showing that the previously constructed logarithmic connection is a model. This is not automatic since all we know for the moment is the existence of a compatible family of isomorphisms

$$\mathcal{M}[x^{-1}]/\mathfrak{r}^{k+1} \xrightarrow{\sim} M/\mathfrak{r}^{k+1}.$$

These *do not* necessarily give us an isomorphism of $R(\!(x)\!)$-modules $\mathcal{M}[x^{-1}] \simeq M$.

For that, let $\mathbb{M}$ be an x-pure logarithmic model for M (cf. Lemma 8.2). Then $\mathbb{M}|_k$ is a logarithmic model for $M|_k$ (but we do not have much more to say about it). According to Corollary 8.12 (2) and Proposition 4.4 (2), there exists an integer $\delta \geq 0$ such that the dotted arrow in

$$\begin{array}{ccc} \mathbb{M}|_k & \overset{\psi_k}{\dashrightarrow} & \mathcal{M}(\delta)|_k \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ M|_k & = & M|_k \end{array}$$

can be found for each k. (The definition of $\mathcal{M}(\delta)$ is given in Example 8.4.) Note that ψ_k is automatically an arrow of $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$.

As $R[\![x]\!]$ is $\mathfrak{r}$-adically complete, we then derive an arrow, now in $\mathbf{MC}_{\log}(R[\![x]\!]/R)$,

$$\psi\colon \mathbb{M} \longrightarrow \mathcal{M}(\delta)$$

inducing ψ_k for each k. We contend that $\psi[x^{-1}]\colon \mathbb{M}[x^{-1}] \to \mathcal{M}(\delta)[x^{-1}]$ is an isomorphism. Since $\psi[x^{-1}]/\mathfrak{r}^{k+1}$ is an isomorphism for each k, we conclude that the $\mathfrak{r}R(\!(x)\!)$-adic completion of $\psi[x^{-1}]$ is an isomorphism. Hence, $\psi[x^{-1}]$ is an isomorphism *on a neighbourhood of the closed fibre of* Spec $R(\!(x)\!) \to$ Spec R (apply [22, 0_{I}, Corollary 7.3.3] and [22, 0_{I}, Corollary 7.3.7] to the cokernel and kernel of $\psi[x^{-1}]$). This implies that the kernel and cokernel of $\psi[x^{-1}]$, which are objects of $\mathbf{MC}(R(\!(x)\!)/R)$, vanish on an open neighbourhood of the closed fibre of Spec $R(\!(x)\!) \to$ Spec R. Using Theorem 8.19 and then Lemma 9.2 below, we can infer that the kernel and cokernel of $\psi[x^{-1}]$ are trivial and $\psi[x^{-1}]$ is an isomorphism and $\mathcal{M}(\delta)$ is a model for M. ∎

The following result was employed in verifying Theorem 9.1 and will also be useful in establishing Theorem 9.6 to come.

Lemma 9.2. *Let $R \to \mathcal{O}$ be a faithfully flat morphism of noetherian rings whose fibre rings are domains. Let M be an $\mathcal{O}$-module of finite type such that for each $\mathfrak{p} \in$ Spec R, the fibre $M \otimes_{\mathcal{O}} (\mathcal{O} \otimes_R \boldsymbol{k}(\mathfrak{p}))$ is a flat $\mathcal{O} \otimes_R \boldsymbol{k}(\mathfrak{p})$-module. Assume that $M_{\mathfrak{P}_0} = 0$ for one prime $\mathfrak{P}_0 \in$ Spec $\mathcal{O}$ above $\mathfrak{r}$. Then $M = 0$.*

Proof. Let $U = \{\mathfrak{P} \in \mathrm{Spec}\,\mathcal{O}\colon M_{\mathfrak{P}} = 0\}$ be the complement of the support of M; it is an open and non-empty subset of Spec $\mathcal{O}$. Let $\mathfrak{P} \in U$ and write $\mathfrak{p}$ for its image in Spec R. Now, if $\mathfrak{Q} \in$ Spec $\mathcal{O}$ is also above $\mathfrak{p}$, we can say that $M_{\mathfrak{Q}} = 0$. Indeed, $M \otimes_{\mathcal{O}} \boldsymbol{k}(\mathfrak{P}) = 0$ and hence the projective $\mathcal{O} \otimes_R \boldsymbol{k}(\mathfrak{p})$-module $M \otimes_R \boldsymbol{k}(\mathfrak{p})$ vanishes. Then $M \otimes_R \boldsymbol{k}(\mathfrak{Q})$ vanishes as well and $M_{\mathfrak{Q}} = 0$. Now we note that the image of U in Spec R is open [35, Section 6.H, Theorem 7, pp. 46–47] and contains the closed point $\mathfrak{r}$, which means that the image of U is Spec R. We conclude that $U =$ Spec $\mathcal{O}$. ∎

Let us dig further into the method of proof of Theorem 9.1. In it, we dealt with an object $(M, \nabla) \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ and, for each $k \in \mathbb{N}$, a logarithmic model $\mathcal{M}_k$ of $(M, \nabla)|_k$ to conclude that the $\mathcal{M}_k$ could be used to construct a logarithmic model of M. We now show that the hypothesis that (M, ∇) is regular-singular is necessary.

Counterexample 9.3. Let $R = C[\![t]\!]$, $M = R(\!(x)\!) \cdot \mathbf{m}$ and define $\nabla(\mathbf{m}) = (t/x) \cdot \mathbf{m}$; this gives us an object $(M, \nabla) \in \mathbf{MC}(R(\!(x)\!)/R)$. (It is not difficult to prove that (M, ∇) is not regular-singular.) Let $\mathcal{M}_k = (R_k[\![x]\!], \vartheta) \in \mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$. Let

$$e_k := \sum_{j=0}^{k} \frac{t^j x^{-j}}{j!} \in R(\!(x)\!).$$

Then, in $(M,\nabla)|_k$, the element $e_k\mathbf{m}$ satisfies $\nabla(e_k\mathbf{m}) = 0$. Hence, $\mathcal{M}_k := (R_k[\![x]\!], \vartheta)$ is a logarithmic model for $(M,\nabla)|_k$, but $(R[\![x]\!], \vartheta)$ is not a logarithmic model for (M,∇).

In passing, we observe that the Deligne–Manin models in Theorem 9.1 have a remarkable property if the regular-singular connection underlies a flat R-module.

Corollary 9.4. *Let $(M,\nabla) \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ be given. Then, if M is R-flat, it is the case that the logarithmic model $\mathcal{M}$ from Theorem 9.1 is* free *as an $R[\![x]\!]$-module.*

Proof. Let k be fixed. We shall show that $\mathcal{M}|_k$ is flat over $R_k[\![x]\!]$ and then apply the local flatness criterion [36, Theorem 22.3, p. 174] to ensure flatness of $\mathcal{M} \simeq \varprojlim \mathcal{M}|_k$; this in turn shows that $\mathcal{M}$ is free since $R[\![x]\!]$ is local. Since M is R-flat, we note that it is also $R(\!(x)\!)$-flat (Theorem 8.18) and therefore $M|_k$ is also $R_k(\!(x)\!)$-flat.

By assumption, we can write $\mathcal{M}|_k \simeq R_k[\![x]\!] \otimes_{R_k} V_k$ for a certain R_k-module V_k. Then $R_k(\!(x)\!) \otimes_{R_k} V_k \simeq M|_k$ is $R_k(\!(x)\!)$-flat. Because $R_k \to R_k(\!(x)\!)$ is faithfully flat (flatness follows from the flatness of $R_k \to R_k[\![x]\!]$) we conclude that V_k is R_k-flat [36, p. 46]. Hence, $\mathcal{M}|_k$ is flat. ∎

In possession of Theorem 9.1, we are now able to interpret the category $\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$ as a category of representations echoing Corollary 5.14. We need a definition.

Definition 9.5. We let $\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)^\wedge$ stand for the category whose

objects are families $\{(M_k, \varphi_k)\}_{k\in\mathbb{N}}$, where $M_k \in \mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)_{(R_k)}$ and $\varphi_k\colon M_{k+1}|_k \to M_k$ are isomorphisms in $\mathbf{MC}_{\mathrm{rs}}(C(\!(x)\!)/C)_{(R_k)}$ and

arrows between $\{(M_k, \varphi_k)\}_{k\in\mathbb{N}}$ and $\{(N_k, \psi_k)\}_{k\in\mathbb{N}}$ are compatible sequences

$$\{\alpha_k\colon M_k \to N_k\} \in \prod_k \mathrm{Hom}_{\mathbf{MC}_{(R_k)}}(M_k, N_k).$$

Theorem 9.6. *The natural functor*

$$\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R) \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)^\wedge,$$
$$(M,\nabla) \longmapsto \{(M,\nabla)|_k\}_k$$

is an equivalence.

Proof. We start by showing *essential surjectivity*. To ease notation, we omit reference to the derivations. Let

$$\{M_k, \varphi_k\}_{k\in\mathbb{N}} \in \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)^\wedge.$$

Let $\mathcal{M}_k$ be the logarithmic *lattice* constructed from M_k as in Theorem 7.8. Note that $\mathcal{M}_{k+1}|_k \in \mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$ is a logarithmic lattice for $M_{k+1}|_k$ which satisfies all conditions of Theorem 7.8. Hence,

$$\varphi_k \colon M_{k+1}|_k \xrightarrow{\sim} M_k$$

can be extended to an *isomorphism*

$$\Phi_k \colon \mathcal{M}_{k+1}|_k \xrightarrow{\sim} \mathcal{M}_k$$

in $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$.

Define

$$\mathcal{M} = \varprojlim_k \mathcal{M}_k.$$

As an $R[\![x]\!] = \varprojlim_k R_k[\![x]\!]$-module, it is of finite type. The projection $\mathcal{M} \to \mathcal{M}_k$ has kernel $\mathfrak{r}^{k+1}\mathcal{M}$ [22, $0_{\rm I}$, Proposition 7.2.9]. Therefore, $\mathcal{M}$ gives rise to an object of $\mathbf{MC}_{\log}(R[\![x]\!]/R)$. Let $M = \gamma(\mathcal{M})$. Then M is an object of $\mathbf{MC}_{\rm rs}(R(\!(x)\!)/R)$ whose image in $\mathbf{MC}_{\rm rs}(R(\!(x)\!)/R)^\wedge$ is $\{M_k, \varphi_k\}$.

We now prove *fullness*. Let M and N be objects of $\mathbf{MC}_{\rm rs}(R(\!(x)\!)/R)$ and pick Deligne–Manin models $\mathcal{M}$ and $\mathcal{N}$ of M and N as in Theorem 9.1. For each k, let

$$\varphi_k \colon M|_k \longrightarrow N|_k$$

be an arrow in $\mathbf{MC}_{\rm rs}(C(\!(x)\!)/C)_{(R_k)}$ and suppose that $\varphi_{k+1}|_k = \varphi_k$. Because of Theorem 7.8, there exists an arrow in $\mathbf{MC}_{\log}(C[\![x]\!]/C)_{(R_k)}$, $\tilde{\varphi}_k \colon \mathcal{M}|_k \to \mathcal{N}|_k$, extending φ_k. In addition, uniqueness of the extension forces $\tilde{\varphi}_{k+1}|_k$ to coincide with $\tilde{\varphi}_k$ after all the necessary identifications. Hence, there exists $\tilde{\varphi} \colon \mathcal{M} \to \mathcal{N}$ such that $\tilde{\varphi}|_k = \tilde{\varphi}_k$, which establishes the existence of $\varphi \colon M \to N$ inducing each φ_k.

Finally, we establish *faithfulness*. Let $\varphi \colon M \to N$ be such that $\varphi_k \colon M|_k \to N|_k$ is null; we conclude that $I = \mathrm{Im}(\varphi) \subset \bigcap_k \mathfrak{r}^k\mathcal{N}$. By Nakayama's lemma [36, Theorem 2.2, p. 8], there exists $a \equiv 1 \mod \mathfrak{r}$ such that $aI = 0$. Hence, $I_{\mathfrak{p}} = 0$ if $\mathfrak{p} \in \operatorname{Spec} R(\!(x)\!)$ is above $\mathfrak{r}$. Now $I \in \mathbf{MC}(R(\!(x)\!)/R)$ and hence Theorem 8.19 followed by Lemma 9.2 proves that $I = 0$. ■

Now let

$$\Phi_\alpha \colon \mathrm{Rep}_C(\mathbb{Z}) \longrightarrow \mathbf{MC}_{\rm rs}(C(\!(x)\!)/C)$$

be a tensor equivalence as in Corollary 5.14; it produces obvious equivalences

$$\Phi_\alpha \colon \mathrm{Rep}_C(\mathbb{Z})_{(R_k)} \xrightarrow{\sim} \mathbf{MC}_{\rm rs}(C(\!(x)\!)/C)_{(R_k)}$$

of R_k-linear categories. Following the pattern established in Definition 9.5, we introduce the category $\mathrm{Rep}_R(\mathbb{Z})^\wedge$. With little effort it can be proved that $\mathrm{Rep}_R(\mathbb{Z})^\wedge$ is equivalent to $\mathrm{Rep}_R(\mathbb{Z})$. We hence arrive at the following corollary:

Corollary 9.7. *The composition*

$$\mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R) \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)^{\wedge} \longrightarrow \mathrm{Rep}_R(\mathbb{Z})^{\wedge} \simeq \mathrm{Rep}_R(\mathbb{Z})$$

is an equivalence of R-linear tensor categories. ■

10. Connections on $\mathbb{P}_R \setminus \{0, \infty\}$

In what follows, $\mathbb{P}_R$ stands for the projective line over R; it is covered by the two affine open subsets $\mathbb{A}_0 = \operatorname{Spec} R[x]$ and $\mathbb{A}_\infty = \operatorname{Spec} R[y]$, and $x = y^{-1}$ on $\mathbb{A}_0 \cap \mathbb{A}_\infty = \mathbb{P}_R \setminus \{0, \infty\}$.

Following the pattern of Definition 6.1, we introduce the category of *connections on* $\mathbb{P}_R \setminus \{0, \infty\}$*, or on* $R[x^{\pm}]$, of *logarithmic connections on* $\mathbb{P}_R$ and of *regular-singular connections*; we denote them respectively by

$$\mathbf{MC}(R[x^{\pm}]/R), \quad \mathbf{MC}_{\log}(\mathbb{P}_R/R) \ \text{ and } \ \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R).$$

Letting

$$\mathbf{MC}_{\log}(\mathbb{P}_R/R) \xrightarrow{\mathbf{r}_0} \mathbf{MC}_{\log}(R[\![x]\!]/R) \quad \text{and} \quad \mathbf{MC}_{\log}(\mathbb{P}_R/R) \xrightarrow{\mathbf{r}_\infty} \mathbf{MC}_{\log}(R[\![y]\!]/R)$$

stand for the obvious functors, we define the *exponents* of $(\mathcal{M}, \nabla) \in \mathbf{MC}_{\log}(\mathbb{P}_R/R)$ as the set of exponents of either $\mathbf{r}_0\mathcal{M}$ or $\mathbf{r}_\infty\mathcal{M}$ (cf. Definition 8.6).

Denote by

$$\gamma_{\mathbb{P}} \colon \mathbf{MC}_{\log}(\mathbb{P}_R/R) \longrightarrow \mathbf{MC}(R[x^{\pm}]/R)$$

the functor which associates to $(\mathcal{E}, \nabla)$ its restriction to $\mathbb{P}_R \setminus \{0, \infty\}$. Given $M \in \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$, any $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}_R/R)$ such that $\gamma_{\mathbb{P}}(\mathcal{M}) \simeq M$ is called a *logarithmic model* of M.

Note that if $M \in \mathbf{MC}(R[x^{\pm}]/R)$ is regular-singular, then

$$M|_k = M/\mathfrak{r}^{k+1}M \in \mathbf{MC}(C[x^{\pm}]/C)_{(R_k)}$$

is also regular-singular for any given $k \in \mathbb{N}$.

We now complete the picture drawn in Section 9 by analysing regular-singular connections on $R[x^{\pm}]$. We aim for the following theorem:

Theorem 10.1. *The restriction*

$$\mathbf{r}_0 \colon \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R) \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)$$

is an equivalence.

Its proof will follow with little effort from Theorem 10.2 below. This, in turn, requires the category

$$\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)^{\wedge},$$

whose definition parallels Definition 9.5 (the details are left to the reader). Let

$$\mathbf{r}_0^{\wedge}\colon \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)^{\wedge} \longrightarrow \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)^{\wedge}$$

be the obvious functor. Because of Corollary 7.9, we know that $\mathbf{r}_0^{\wedge}$ is an equivalence.

Theorem 10.2. *The natural functor*

$$\begin{aligned} \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R) &\longrightarrow \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)^{\wedge}, \\ (M,\nabla) &\longmapsto \{(M,\nabla)|_k\}_k \end{aligned}$$

is an equivalence.

Assuming the veracity of this result, we can give the following proof:

Proof of Theorem 10.1. This follows from the commutative diagram of categories

$$\begin{array}{ccc} \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R) & \xrightarrow{\mathbf{r}_0} & \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R) \\ {\scriptstyle \text{Theorem 10.2}}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle \text{Theorem 9.6}} \\ \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)^{\wedge} & \xrightarrow[\mathbf{r}_0^{\wedge}]{} & \mathbf{MC}_{\mathrm{rs}}(R(\!(x)\!)/R)^{\wedge} \end{array}$$

and the fact that $\mathbf{r}_0^{\wedge}$ is an equivalence. ∎

Let us now start the verification of Theorem 10.2. Simple facts come first.

Lemma 10.3. *Any $M \in \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$ allows a logarithmic model $\mathcal{M}$ such that $\mathcal{M}(\mathbb{A}_0)$ has no x-torsion and $\mathcal{M}(\mathbb{A}_\infty)$ no y-torsion.*

Proof. Let $\mathcal{N}$ be any logarithmic model. The submodule of x-torsion in $\mathcal{N}(\mathbb{A}_0)$ is invariant under ϑ. The submodule of y-torsion in $\mathcal{N}(\mathbb{A}_\infty)$ is invariant under ϑ. We can therefore take the quotients to produce the required model. ∎

Lemma 10.4. *Let $E \in \mathbf{MC}(R[x^{\pm}]/R)$ be given. Then, for each $\mathfrak{p} \in \operatorname{Spec} R$, the $\boldsymbol{k}(\mathfrak{p})[x^{\pm}]$-module $\boldsymbol{k}(\mathfrak{p})[x^{\pm}] \underset{R[x^{\pm}]}{\otimes} E$ is locally free.*

Proof. See either [30, Proposition 8.9] or Remark 8.20. ∎

We are unfortunately unable to find a proof of Theorem 10.2 based simply on Corollary 7.9 and the equivalence $\mathbf{r}_0^\wedge$. Hence, we shall need to go through the arguments used to establish Theorem 9.6 (the analogue of Theorem 10.2 in the formal case) and adapt them. Luckily, there are no major modifications, except that the process of taking the limit allowed by $\mathfrak{r}$-adic completeness of $R[\![x]\!]$ needs to be replaced by Grothendieck's GFGA theorem for sheaves on $\mathbb{P}_R$. See [27] for a complete proof of this result and [25, Section 3.2] for a valuable outline. Note that this is also the technique employed in [24], which renders the matter technically more demanding.

When employing GFGA in this context, we are hindered by the following difficulty. Say that $\mathcal{M}$ is a coherent $\mathcal{O}_{\mathbb{P}_R}$-module such that, for every $k \in \mathbb{N}$, the $\mathcal{O}_{\mathbb{P}}$-module (with action of R_k) $\mathcal{M}|_k := \mathcal{M}/\mathfrak{r}^{k+1}$ carries a logarithmic connection $\nabla_k \colon \mathcal{M}|_k \to \mathcal{M}|_k$, and that, in addition, the natural isomorphisms

$$\mathcal{M}|_{k+1} \xrightarrow{\sim} \mathcal{M}|_k$$

are compatible with the logarithmic connections. Since $\mathcal{M}$ *is not* the sheaf $\varprojlim_k \mathcal{M}_k$, we need to ask whether it is possible to endow $\mathcal{M}$ with a logarithmic connection $\nabla \colon \mathcal{M} \to \mathcal{M}$ inducing the various ∇_k. The answer is yes, as we now explain.

Let $\mathcal{E}$ be a coherent $\mathcal{O}_{\mathbb{P}_R}$-module and introduce $J\mathcal{E}$ as being the sheaf of R-modules $\mathcal{E} \oplus \mathcal{E}$. Endow it with the structure of an $\mathcal{O}_{\mathbb{P}_R}$-module by

$$a \cdot (e, e') = (ae, ae' + \vartheta(a)e).$$

Write $p \colon J\mathcal{E} \to \mathcal{E}$ for the projection onto the first factor. It is not hard to see that $J\mathcal{E}$ remains coherent and that a logarithmic connection is none other than an $\mathcal{O}_{\mathbb{P}_R}$-linear arrow

$$\sigma \colon \mathcal{E} \longrightarrow J\mathcal{E}$$

such that $p\sigma = \mathrm{id}$. Indeed, if $p\sigma = \mathrm{id}$, then $\sigma = (\mathrm{id}, \nabla)$, where ∇ is a logarithmic connection. We now return to the question raised above and state it as a lemma for future referencing.

Lemma 10.5. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathbb{P}_R}$-module such that, for every $k \in \mathbb{N}$, the $\mathcal{O}_{\mathbb{P}}$-module (with action of R_k) $\mathcal{M}|_k := \mathcal{M}/\mathfrak{r}^{k+1}$ carries a logarithmic connection $\nabla_k \colon \mathcal{M}|_k \to \mathcal{M}|_k$, and that, in addition, the natural isomorphisms*

$$\mathcal{M}|_{k+1} \xrightarrow{\sim} \mathcal{M}|_k$$

are compatible with these connections. Then $\mathcal{M}$ carries a logarithmic R-linear connection ∇ inducing ∇_k for each k. In addition, if $\mathcal{N}$ is an object of $\mathbf{MC}_{\log}(\mathbb{P}_R/R)$ and $\Phi \colon \mathcal{M} \to \mathcal{N}$ is an arrow of coherent $\mathcal{O}_{\mathbb{P}_R}$-modules such that $\Phi|_k \colon \mathcal{M}|_k \to \mathcal{N}|_k$ lies in $\mathbf{MC}_{\log}(\mathbb{P}/C)_{(R_k)}$ for each k, then Φ is actually an arrow of $\mathbf{MC}_{\log}(\mathbb{P}_R/R)$.

Proof. Let $\sigma_k\colon \mathcal{M}|_k \to J\mathcal{M}|_k$ be defined by $\sigma_k = (\mathrm{id}, \nabla_k)$. We then obtain, by GFGA, an arrow $\sigma\colon \mathcal{M} \to J\mathcal{M}$ such that $p\sigma = \mathrm{id}$, that is, a logarithmic connection. The final claim is also proved with similar techniques. ∎

We can now give the first step towards Theorem 10.2.

Theorem 10.6 (Deligne–Manin models). *Let $M \in \mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$. There exists a unique logarithmic model $\mathcal{M}$ of M such that, for every $k \in \mathbb{N}$, the object*

$$\mathcal{M}|_k \in \mathbf{MC}_{\log}(\mathbb{P}/C)_{(R_k)},$$

enjoys the following properties:

(1) *All its exponents lie in τ.*

(2) *It is free in relation to R_k.*

(3) *The isomorphism $\gamma_{\mathbb{P}}(\mathcal{M}|_k) \simeq M|_k$ is compatible with the action of R_k.*

Put otherwise, $\mathcal{M}|_k$ is a Deligne–Manin model in the sense of Theorem 7.12.

Proof. This is much the same as the proof of Theorem 9.1 and we shall give only some indications of how to replace the arguments in its proof for the present context.

For Step 1. The use of Theorem 7.8 is replaced by that of Theorems 7.12 and 7.14. The use of the $\mathfrak{r}$-adic completeness of $R[\![x]\!]$ is replaced by GFGA *supplemented* by Lemma 10.5. We then arrive at an object $\mathcal{M} \in \mathbf{MC}_{\log}(\mathbb{P}_R/R)$.

For Step 2. We replace Lemma 8.2 by Lemma 10.3 in finding a convenient logarithmic model $\mathbb{M}$ for M. We then replace Proposition 4.4 and Corollary 8.12 by Proposition 7.13. To continue, we employ GFGA and Lemma 10.5 instead of completeness of $R[\![x]\!]$ and Lemma 10.4 instead of Theorem 8.19. ∎

Proof of Theorem 10.2. *Essential surjectivity.* Let $\{M_k, \varphi_k\}_{k\in\mathbb{N}}$ be in $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)^{\wedge}$. For each k, let $\mathcal{M}_k \in \mathbf{MC}_{\log}(\mathbb{P}/C)_{(R_k)}$ be a Deligne–Manin lattice for M_k (cf. Theorem 7.12). Because of Theorem 7.14, the isomorphisms

$$\varphi_k\colon M_{k+1}|_k \xrightarrow{\sim} M_k$$

may be extended to isomorphisms

$$\Phi_k\colon \mathcal{M}_{k+1}|_k \xrightarrow{\sim} \mathcal{M}_k$$

of $\mathbf{MC}_{\log}(\mathbb{P}_C/C)_{(R_k)}$.

By GFGA, there exists a coherent sheaf $\mathcal{M}$ on $\mathbb{P}_R$ and isomorphisms $\mathcal{M}|_k \simeq \mathcal{M}_k$ such that the natural transition isomorphisms correspond to the Φ_k above. Lemma 10.5

now shows that $\mathcal{M}$ comes with a logarithmic connection and we arrive at an object of $\mathbf{MC}_{\log}(\mathbb{P}_R/R)$. Then $M = \gamma_{\mathbb{P}}(\mathcal{M})$ is an object in $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$ satisfying $M|_k \simeq M_k$ for each $k \in \mathbb{N}$.

Fullness. Let M and N be objects of $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$. For each $k \in \mathbb{N}$, let

$$\varphi_k \colon M|_k \longrightarrow N|_k$$

be an arrow in $\mathbf{MC}_{\mathrm{rs}}(C[x^{\pm}]/C)_{(R_k)}$ and suppose that $\varphi_{k+1}|_k = \varphi_k$. Pick Deligne–Manin models $\mathcal{M}$ and $\mathcal{N}$ of M and N as in Theorem 10.6. By Theorem 7.14, there exists, for any given k, an arrow $\Phi_k\colon \mathcal{M}|_k \to \mathcal{N}|_k$ in $\mathbf{MC}_{\log}(\mathbb{P}/C)_{(R_k)}$ such that $\gamma_{\mathbb{P}}(\Phi_k) = \varphi_k$. In addition, uniqueness of the extension forces $\Phi_{k+1}|_k = \Phi_k$ for each k. By GFGA, there exists an arrow $\Phi\colon \mathcal{M} \to \mathcal{N}$ of coherent $\mathcal{O}_{\mathbb{P}_R}$-modules satisfying $\Phi|_k = \Phi_k$ for each $k \in \mathbb{N}$. From Lemma 10.5, we can also affirm that Φ is an arrow of $\mathbf{MC}_{\log}(\mathbb{P}_R/R)$. The arrow $\varphi = \gamma_{\mathbb{P}}(\Phi)$ lies in $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$ and induces φ_k for each $k \in \mathbb{N}$.

Faithfulness. Let $\varphi\colon M \to N$ be an arrow in $\mathbf{MC}_{\mathrm{rs}}(R[x^{\pm}]/R)$ such that $\varphi_k\colon M|_k \to N|_k$ is null for all $k \in \mathbb{N}$. We conclude that $I = \mathrm{Im}(\varphi) \subset \bigcap_k \mathfrak{r}^k N$. By Nakayama's lemma [36, Theorem 2.2, p. 8], there exists $a \equiv 1 \bmod \mathfrak{r}$ such that $aI = 0$. Hence, $I_{\mathfrak{P}} = 0$ if $\mathfrak{P} \in \operatorname{Spec} R[x^{\pm}]$ is above $\mathfrak{r}$. To show that $I = 0$, we only require Lemmas 9.2 and 10.4. ∎

Acknowledgements – The second-named author wishes to thank I. Lopez-Franco for instructing him on the theory of the Deligne tensor product. He also wishes to thank the CNRS for granting him research leave from September to December 2020, when some parts of this text were elaborated. All authors thank the referee for this journal for calling our attention to [3].

Funding – The research of the first-named and the third-named authors is funded by the Vietnam Academy of Science and Technology under grant number NVCC01.01/22-23.

References

[1] E. Abe, *Hopf algebras*. Cambridge Tracts in Math. 74, Cambridge University Press, Cambridge-New York, 1980. Zbl 0476.16008 MR 594432

[2] Y. André, Différentielles non commutatives et théorie de Galois différentielle ou aux différences. *Ann. Sci. École Norm. Sup. (4)* **34** (2001), no. 5, 685–739. Zbl 1010.12004 MR 1862024

[3] Y. André – F. Baldassarri – M. Cailotto, *De Rham cohomology of differential modules on algebraic varieties*. Progr. Math. 189, Birkhäuser/Springer, Cham, 2020. Zbl 1437.14029 MR 4328338

[4] H. Bass – A. Lubotzky – A. R. Magid – S. Mozes, The proalgebraic completion of rigid groups. *Geom. Dedicata* **95** (2002), 19–58. Zbl 1059.20036 MR 1950883

[5] P. Berthelot – A. Ogus, *Notes on crystalline cohomology*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1978. Zbl 0383.14010 MR 0491705

[6] N. Bourbaki, *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitres III: Groupes de Lie*. Actualités Sci. Indust. 1349, Hermann, Paris, 1972. Zbl 0244.22007 MR 0573068

[7] N. Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie. Chapitre 1*. Springer, Berlin, 2007. Zbl 1120.17001

[8] N. Bourbaki, *Theory of sets*. Elements of Mathematics (Berlin), Springer, Berlin, 2004. Zbl 1061.03001 MR 2102219

[9] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples*. Springer, Berlin, 2012. Zbl 1245.16001 MR 3027127

[10] E. A. Coddington – N. Levinson, *Theory of ordinary differential equations*. McGraw-Hill, New York-Toronto-London, 1955. Zbl 0064.33002 MR 0069338

[11] P. Deligne, *Équations différentielles à points singuliers réguliers*. Lecture Notes in Math. 163, Springer, Berlin-New York, 1970. Zbl 0244.14004 MR 0417174

[12] P. Deligne, Le groupe fondamental de la droite projective moins trois points. In *Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987)*, pp. 79–297, Math. Sci. Res. Inst. Publ. 16, Springer, New York, 1989. Zbl 0742.14022 MR 1012168

[13] P. Deligne, *Catégories tannakiennes*. In *The Grothendieck Festschrift, Vol. II*, pp. 111–195, Progr. Math. 87, Birkhäuser Boston, Boston, MA, 1990. Zbl 0727.14010 MR 1106898

[14] P. Deligne – J. Milne, Tannakian categories. In *Hodge cycles, motives, and Shimura varieties*, pp. 101–228, Lecture Notes in Math. 900, Springer, Berlin-New York, 1982. Zbl 0477.14004

[15] J. P. dos Santos, A complement to Deligne's tensor product of categories. To appear.

[16] M. Demazure – P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeurs, Paris; North-Holland, Amsterdam, 1970. Zbl 0203.23401 MR 0302656

[17] J. P. P. dos Santos, The behaviour of the differential Galois group on the generic and special fibres: A Tannakian approach. *J. Reine Angew. Math.* **637** (2009), 63–98. Zbl 1242.12005 MR 2599082

[18] N. D. Duong – P. H. Hai, Tannakian duality over Dedekind rings and applications. *Math. Z.* **288** (2018), no. 3-4, 1103–1142. Zbl 1400.14121 MR 3778991

[19] D. Eisenbud, *Commutative algebra*. Grad. Texts in Math. 150, Springer, New York, 1995. Zbl 0819.13001 MR 1322960

[20] L. Fiorot – T. Monteiro Fernandes – C. Sabbah, Relative regular Riemann-Hilbert correspondence. *Proc. Lond. Math. Soc. (3)* **122** (2021), no. 3, 434–457. Zbl 1473.14037 MR 4230060

[21] L. Fiorot – T. Monteiro Fernandes – C. Sabbah, Relative regular Riemann–Hilbert correspondence II. *Compos. Math.* **159** (2023), no. 7, 1413–1465. Zbl 07700866 MR 4603633

[22] A. Grothendieck, in collaboration with J. Dieudonné, Éléments de géométrie algébrique. *Publ. Math. IHES* **8**, **11** (1961), **17** (1963), **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[23] R. M. Guralnick, Similarity of matrices over local rings. *Linear Algebra Appl.* **41** (1981), 161–174. Zbl 0478.15017 MR 649724

[24] P. H. Hai – J. P. dos Santos, Regular-singular connections on relative complex schemes. To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*.

[25] D. Harbater, Patching and Galois theory. In *Galois groups and fundamental groups*, pp. 313–424, Math. Sci. Res. Inst. Publ. 41, Cambridge University Press, Cambridge, 2003. Zbl 1071.14029 MR 2012220

[26] G. Hochschild – G. D. Mostow, Pro-affine algebraic groups. *Amer. J. Math.* **91** (1969), 1127–1140. Zbl 0213.22701 MR 255690

[27] L. Illusie, Grothendieck's existence theorem in formal geometry. In *Fundamental algebraic geometry*, pp. 179–233, Math. Surveys Monogr. 123, American Mathematical Society, Providence, RI, 2005. Zbl 1085.14001 MR 2223409

[28] Y. Ilyashenko – S. Yakovenko, *Lectures on analytic differential equations*. Grad. Stud. Math. 86, American Mathematical Society, Providence, RI, 2008. Zbl 1186.34001 MR 2363178

[29] J. C. Jantzen, *Representations of algebraic groups*. Pure Appl. Math. 131, Academic Press, Boston, MA, 1987. Zbl 0654.20039 MR 899071

[30] N. M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.* (1970), no. 39, 175–232. Zbl 0221.14007 MR 291177

[31] N. M. Katz, *Exponential sums and differential equations*. Ann. of Math. Stud. 124, Princeton University Press, Princeton, NJ, 1990. Zbl 0731.14008 MR 1081536

[32] I. López Franco, Tensor products of finitely cocomplete and abelian categories. *J. Algebra* **396** (2013), 207–219. Zbl 1305.18042 MR 3108080

[33] S. Mac Lane, *Categories for the working mathematician*. 2nd edn., Grad. Texts in Math. 5, Springer, New York, 1998. Zbl 0906.18001 MR 1712872

[34] Y. Manin, Moduli fuchsiani. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **19** (1965), 113–126. Zbl 0166.04301 MR 180581

[35] H. Matsumura, *Commutative algebra*. 2nd edn., Math. Lect. Not. Ser. 56, Benjamin/Cummings, Reading, Mass., 1980. Zbl 0441.13001 MR 575344

[36] H. Matsumura, *Commutative ring theory*. 2nd edn., Cambridge Stud. Adv. Math. 8, Cambridge University Press, Cambridge, 1989. Zbl 0666.13002 MR 1011461

[37] T. Monteiro Fernandes – C. Sabbah, On the de Rham complex of mixed twistor $\mathcal{D}$-modules. *Int. Math. Res. Not. IMRN* (2013), no. 21, 4961–4984. Zbl 1312.14059 MR 3123671

[38] T. Monteiro Fernandes – C. Sabbah, Riemann-Hilbert correspondence for mixed twistor $\mathcal{D}$-modules. *J. Inst. Math. Jussieu* **18** (2019), no. 3, 629–672. Zbl 1440.14101 MR 3936645

[39] N. Nitsure, Moduli of semistable logarithmic connections. *J. Amer. Math. Soc.* **6** (1993), no. 3, 597–609. Zbl 0807.14007 MR 1182671

[40] N. Saavedra Rivano, *Catégories tannakiennes*. Lecture Notes in Math. 265, Springer, Berlin-New York, 1972. Zbl 0241.14008 MR 0338002

[41] A. Shiho, Crystalline fundamental groups. I. Isocrystals on log crystalline site and log convergent site. *J. Math. Sci. Univ. Tokyo* **7** (2000), no. 4, 509–656. Zbl 0984.14009 MR 1800845

[42] M. van der Put – M. F. Singer, *Galois theory of linear differential equations*. Grundlehren Math. Wiss. 328, Springer, Berlin, 2003. Zbl 1036.12008 MR 1960772

[43] W. Wasow, *Asymptotic expansions for ordinary differential equations*. Robert E. Krieger, Huntington, NY, 1976. MR 0460820

[44] W. C. Waterhouse, *Introduction to affine group schemes*. Grad. Texts in Math. 66, Springer, New York-Berlin, 1979. Zbl 0442.14017 MR 547117

Manoscritto pervenuto in redazione il 29 marzo 2022.