

A self-similar set with non-locally connected components

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Abstract. Luo, Rao and Xiong [Topol. Appl. 322 (2022), 108271] conjectured that if a planar self-similar iterated function system with the open set condition does not involve rotations or reflections, then every connected component of the attractor is locally connected. We create a homogeneous counterexample of Lalley–Gatzouras type, which disproves this conjecture.

1. Introduction

There have been a number of advances in the field of fractal geometry that improve our understanding of the fractal dimensions of fractal sets and measures, but much less is known about the topology. Even some fundamental topological properties such as the connectedness and local connectedness have not been well studied. Hata [4] related the connectedness of a self-similar set to the connectedness of some graph structure of the corresponding iterated function system (or simply IFS). When the self-similar set enjoys nice separation conditions, the graph structure can be thoroughly identified. In fact, this often provides not only an approach for the verification of the connectedness ([1, 2, 12]), but also useful information about the connected components of the original set ([6, 10, 15, 16]).

The local connectedness of fractal sets and their subsets is far more intricate and poorly understood. Recall that a space X is said to be locally connected at $x \in X$ if for every neighborhood U of x , there exists a connected neighborhood V of x contained in U . If X is locally connected at each of its points, it is said to be *locally connected*. A pioneering result was given by Hata [4] stating that if a self-similar set is connected, then it must be locally and path connected. In [9], Luo, Rao and Xiong further demonstrated that a self-similar set is locally connected if and only if it has only finitely many connected components. There also exists some work [7, 8, 11, 13, 14] concerning the local connectedness of self-similar or self-affine tiles and related sets, e.g., their boundaries, the closure of their disjoint interiors or their complements.

Mathematics Subject Classification 2020: 28A80 (primary); 54D05 (secondary).

Keywords: self-similar sets, connected components, locally connected.

In addition to the local connectedness of a given self-similar set, it is also interesting to ask whether its connected components are locally connected. Luo, Rao and Xiong [9] showed that every connected component of any fractal square (or generalized Sierpiński carpet) is locally connected. They also provided an example where this property does not hold when the IFSs involve rotations. The idea can be summarized as follows: consider a carpet-like self-similar set in $[0, 1]^2$ that contains infinitely many vertical line segments, including the left and right sides of $[0, 1]^2$, and require these segments to belong to different components. If there are also two similitudes in the IFS such that the image of $[0, 1]^2$ under these two maps is exactly a pair of “up-down” adjacent small squares, where the top one includes a 90° rotation. Note that the rotation transforms the left and right sides of $[0, 1]^2$ into top and bottom sides of its image. Due to the adjacency, we have a horizontal line segment gluing together infinitely many vertical segments, and the component containing them cannot be locally connected.

Since all the prior examples require the corresponding IFS to either involve rotations or to violate the open set condition (for the definition, please see [3]), Luo, Rao and Xiong proposed the following conjecture.

Conjecture 1 ([9]). *Let Φ be a self-similar IFS on \mathbb{R}^2 and K its attractor. If Φ satisfies the open set condition and if every similitude in Φ involves neither rotation nor reflection, then every component of K is locally connected.*

We are able to construct a homogeneous self-similar IFS on \mathbb{R}^2 satisfying the open set condition (even the convex open set condition, that is, the invariant open set is a convex set) without using any rotations or reflections, such that some particular connected component of the attractor is not locally connected. As a result, only very specific self-similar sets that are generated from a strict grid structure or totally disconnected can have the nice property that every connected component is locally connected.

2. The counterexample

Consider the planar self-similar IFS $\Phi = \{\varphi_i(\mathbf{x}) = \frac{1}{6}\mathbf{x} + a_i\}_{i=1}^{24}$, where

$$\begin{aligned} a_1 &= (0, 0), a_2 = (0, \frac{1}{6}), a_3 = (0, \frac{1}{3}), a_4 = (\frac{1}{24}, \frac{1}{2}), a_5 = (\frac{1}{12}, \frac{2}{3}), \\ a_6 &= (\frac{1}{8}, \frac{5}{6}), a_7 = (\frac{1}{6}, 0), a_8 = (\frac{5}{24}, \frac{1}{6}), a_9 = (\frac{1}{4}, \frac{1}{3}), a_{10} = (\frac{7}{24}, \frac{1}{2}), \\ a_{11} &= (\frac{1}{3}, \frac{2}{3}), a_{12} = (\frac{3}{8}, \frac{5}{6}), a_{13} = (\frac{5}{12}, 0), a_{14} = (\frac{11}{24}, \frac{1}{6}), a_{15} = (\frac{1}{2}, \frac{1}{3}), \\ a_{16} &= (\frac{13}{24}, \frac{1}{2}), a_{17} = (\frac{7}{12}, \frac{2}{3}), a_{18} = (\frac{5}{8}, \frac{5}{6}), a_{19} = (\frac{5}{6}, 0), a_{20} = (\frac{5}{6}, \frac{1}{6}), \\ a_{21} &= (\frac{5}{6}, \frac{1}{3}), a_{22} = (\frac{5}{6}, \frac{1}{2}), a_{23} = (\frac{5}{6}, \frac{2}{3}), a_{24} = (\frac{5}{6}, \frac{5}{6}). \end{aligned}$$

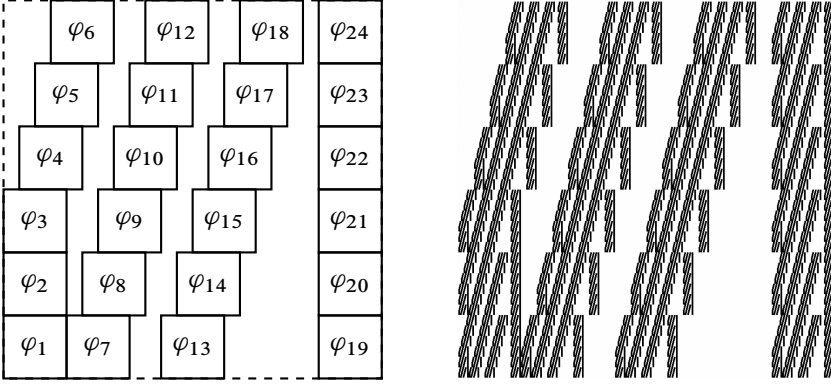


Figure 1. An illustration of the IFS Φ and the attractor K

Please see Figure 1 for an illustration. It is easy to check that $\bigcup_{i=1}^{24} \varphi_i([0, 1]^2) \subset [0, 1]^2$ and $\varphi_1([0, 1]^2), \dots, \varphi_{24}([0, 1]^2)$ have disjoint interiors. In particular, Φ satisfies the (convex) open set condition. Denote by K the attractor associated with Φ . Such a self-similar set is sometimes referred to as a *Lalley–Gatzouras carpet* ([5]).

We first prove that there are an infinite number of long, “well separated” line segments all hitting the left side of the unit square, which is the key ingredient.

Let C_1 be the Cantor set generated by the IFS $\{\frac{1}{6}x + \frac{1}{6}, \frac{1}{6}x + \frac{5}{12}\}$. Recalling the translation vectors a_7, a_{13} , we have $C_1 \times \{0\} \subset K$. For $x \in C_1$, write

$$\ell_x = \{(x + \frac{y}{4}, y) : y \in [0, 1]\}$$

to be the line segment passing through $(x, 0)$ and of slope 4. Note that the other endpoint of ℓ_x is $(x + \frac{1}{4}, 1)$.

Lemma 2. *The union $\bigcup_{x \in C_1} \ell_x$ is contained in K .*

Proof. Fix any $x \in C_1$ and $y \in [0, 1]$. Write $x = \sum_{t=1}^{\infty} x_t 6^{-t}$ where $x_t \in \{1, \frac{5}{2}\}$, and write $y = \sum_{t=1}^{\infty} y_t 6^{-t}$ where $y_t \in \{0, 1, \dots, 5\}$. A direct calculation gives us

$$\varphi_{6\lfloor x_t \rfloor + y_t + 1}(0, 0) = \left(\frac{x_t}{6} + \frac{y_t}{24}, \frac{y_t}{6}\right), \quad t \geq 1.$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi_{6\lfloor x_1 \rfloor + y_1 + 1} \circ \dots \circ \varphi_{6\lfloor x_n \rfloor + y_n + 1}(0, 0) &= \lim_{n \rightarrow \infty} \sum_{t=1}^n 6^{-t+1} \cdot \left(\frac{x_t}{6} + \frac{y_t}{24}, \frac{y_t}{6}\right) \\ &= \left(\sum_{t=1}^{\infty} x_t 6^{-t} + \frac{1}{4} \sum_{t=1}^{\infty} y_t 6^{-t}, \sum_{t=1}^{\infty} y_t 6^{-t}\right) = \left(x + \frac{y}{4}, y\right). \end{aligned}$$

That is to say, $(x + \frac{y}{4}, y) \in K$. Since x, y are arbitrary, $\bigcup_{x \in C_1} \ell_x \subset K$. ■

From the construction and the illustration in Figure 1, it is not hard to see that there is a vacant strip (avoiding K) between each pair of different segments in $\{\ell_x : x \in C_1\}$. We omit the tedious computation here. In particular, each pair of these segments belong to different connected components of K .

Next, let us look at the left side of $[0, 1]^2$. Let C_2 be the Cantor set generated by the IFS $\{\frac{1}{6}x, \frac{1}{6}x + \frac{1}{6}\}$. Recalling the translation vectors a_1, a_2 , we have $\{0\} \times C_2 \subset K$. For $z = \sum_{t=1}^{\infty} z_t 6^{-t} \in C_2$ (where $z_t \in \{0, 1\}$), it is convenient to use the following notations:

- $x(z) = \sum_{t=1}^{\infty} (-\frac{3}{2}z_t + \frac{5}{2})6^{-t}$. Since we have $-\frac{3}{2}z_t + \frac{5}{2} = \frac{5}{2}$ when $z_t = 0$ and $-\frac{3}{2}z_t + \frac{5}{2} = 1$ when $z_t = 1$, the map $z \mapsto x(z)$ is a bijection between C_2 and C_1 . Roughly speaking, $x(z)$ can be thought of as the element in C_1 that is, in a certain sense, the “inverse” of z .
- $E_z = \bigcup_{i=3}^6 \varphi_i(\ell_{x(z)})$.
- $\sigma(z) = \sum_{t=1}^{\infty} z_{t+1}6^{-t}$ (so σ behaves like the usual left shift on symbolic spaces).
- $L_0(z) = \varphi_2(\ell_{\frac{1}{6}x(\sigma(z))+\frac{1}{6}}) \cup \varphi_1(E_{\sigma(z)})$. Recall that $\frac{1}{6}x + \frac{1}{6}$ is a map in the IFS of C_1 and hence $\ell_{\frac{1}{6}x(\sigma(z))+\frac{1}{6}}$ is well defined.
- $L_1(z) = \varphi_2(E_{\sigma(z)})$.

Lemma 3. *For every $z \in C_2$, all of $E_z, L_0(z)$ and $L_1(z)$ are line segments in K .*

Recall that the two endpoints of ℓ_x are $(x, 0)$ and $(x + \frac{1}{4}, 1)$, respectively.

Proof. By the previous lemma, $E_z, L_0(z), L_1(z)$ are of course subsets of K . So it suffices to show that they are all segments. For $3 \leq i \leq 5$, $\varphi_i(\ell_{x(z)})$ has an endpoint

$$\varphi_i\left(x(z) + \frac{1}{4}, 1\right) = \left(\frac{x(z)}{6} + \frac{1}{24}, \frac{1}{6}\right) + \left(\frac{i-3}{24}, \frac{i-1}{6}\right) = \left(\frac{x(z)}{6} + \frac{i}{24} - \frac{1}{12}, \frac{i}{6}\right)$$

while $\varphi_{i+1}(\ell_{x(z)})$ has an endpoint

$$\varphi_{i+1}(x(z), 0) = \left(\frac{x(z)}{6}, 0\right) + \left(\frac{(i+1)-3}{24}, \frac{(i+1)-1}{6}\right) = \left(\frac{x(z)}{6} + \frac{i}{24} - \frac{1}{12}, \frac{i}{6}\right).$$

So they share a common point. Due to the “same slope” fact, $E_z = \bigcup_{i=3}^6 \varphi_i(\ell_{x(z)})$ is a line segment and hence is $L_1(z)$. Similarly, $\varphi_2(\ell_{\frac{1}{6}x(\sigma(z))+\frac{1}{6}})$ has an endpoint

$$\varphi_2\left(\frac{x(\sigma(z))}{6} + \frac{1}{6}, 0\right) = \left(\frac{x(\sigma(z))}{36} + \frac{1}{36}, \frac{1}{6}\right)$$

while $\varphi_1(E_{\sigma(z)})$ has an endpoint

$$\varphi_1\varphi_6\left(x(\sigma(z)) + \frac{1}{4}, 1\right) = \varphi_1\left(\frac{x(\sigma(z))}{6} + \frac{1}{6}, 1\right) = \left(\frac{x(\sigma(z))}{36} + \frac{1}{36}, \frac{1}{6}\right),$$

that is, they share a common point. As a result, $L_0(z)$ is a line segment. ■

Moreover, we can conclude from the above lemma that:

- E_z has endpoints

$$\varphi_6\left(x(z) + \frac{1}{4}, 1\right) = \left(\frac{x(z)}{6} + \frac{1}{6}, 1\right) \text{ and } \varphi_3(x(z), 0) = \left(\frac{x(z)}{6}, \frac{1}{3}\right). \quad (1)$$

- $L_0(z)$ has endpoints $\left(\frac{x(\sigma(z))}{36} + \frac{5}{72}, \frac{1}{3}\right)$ and $\left(\frac{x(\sigma(z))}{36}, \frac{1}{18}\right)$;
- $L_1(z)$ has endpoints $\left(\frac{x(\sigma(z))}{36} + \frac{1}{36}, \frac{1}{3}\right)$ and $\left(\frac{x(\sigma(z))}{36}, \frac{2}{9}\right)$;
- The above two indicate that for $i \in \{0, 1\}$, $L_i(z)$ has endpoints

$$\left(\frac{x(\sigma(z))}{36} - \frac{i}{24} + \frac{5}{72}, \frac{1}{3}\right) \text{ and } \left(\frac{x(\sigma(z))}{36}, \frac{i}{6} + \frac{1}{18}\right). \quad (2)$$

Another useful fact is

$$x(z) - \frac{x(\sigma(z))}{6} = \frac{1}{6} \cdot \left(-\frac{3}{2}z_1 + \frac{5}{2}\right). \quad (3)$$

The lemma below shows that K contains infinitely many line segments hitting the left side of $[0, 1]^2$.

Lemma 4. *If we denote by ℓ^z the line passing through $(0, z)$ and of slope 4, then $\bigcup_{z \in C_2} (\ell^z \cap [0, 1]^2) \subset K$.*

Proof. Fix any $z = \sum_{t=1}^{\infty} z_t 6^{-t} \in C_2$, where $z_t \in \{0, 1\}$. We claim that

$$\ell^z \cap [0, 1]^2 = \{(0, z)\} \cup E_z \cup \bigcup_{n=1}^{\infty} \varphi_{z_1 \dots z_{n-1}}(L_{z_n}(\sigma^{n-1}(z))),$$

where $\varphi_{z_1 \dots z_{n-1}} := \varphi_{z_1} \circ \dots \circ \varphi_{z_{n-1}}$ and φ_{\emptyset} denotes the identity map. Combining this with Lemma 3 finishes the proof. Since all of the above segments have the same slope, it suffices to show the following three facts.

Fact 1. E_z and $L_{z_1}(z)$ share a common endpoint. In fact, just recall (1), (2) and note that by (3),

$$\begin{aligned} \frac{x(z)}{6} - \left(\frac{x(\sigma(z))}{36} - \frac{z_1}{24} + \frac{5}{72}\right) &= \frac{1}{6} \left(x(z) - \frac{x(\sigma(z))}{6}\right) + \frac{z_1}{24} - \frac{5}{72} \\ &= \frac{1}{36} \cdot \left(-\frac{3}{2}z_1 + \frac{5}{2}\right) + \frac{z_1}{24} - \frac{5}{72} = 0. \end{aligned}$$

Fact 2. For all $n \geq 1$, $\varphi_{z_1 \dots z_{n-1}}(L_{z_n}(\sigma^{n-1}(z)))$ and $\varphi_{z_1 \dots z_n}(L_{z_{n+1}}(\sigma^n(z)))$ share a common endpoint. Again recall (1), (2). Then note that the former segment has an endpoint

$$\varphi_{z_1 \dots z_{n-1}}\left(\frac{x(\sigma^n(z))}{36}, \frac{z_n}{6} + \frac{1}{18}\right) = \left(\frac{x(\sigma^n(z))}{6^{n+1}}, \frac{z_n}{6^n} + \frac{1}{3 \cdot 6^n}\right) + \varphi_{z_1 \dots z_{n-1}}(0, 0), \quad (4)$$

while the latter one has an endpoint

$$\begin{aligned} & \varphi_{z_1 \dots z_n} \left(\frac{x(\sigma^{n+1}(z))}{36} - \frac{z_{n+1}}{24} + \frac{5}{72}, \frac{1}{3} \right) \\ &= \left(\frac{x(\sigma^{n+1}(z))}{6^{n+2}} - \frac{z_{n+1}}{4 \cdot 6^{n+1}} + \frac{5}{2 \cdot 6^{n+2}}, \frac{1}{3 \cdot 6^n} \right) + \varphi_{z_1 \dots z_n}(0, 0). \end{aligned}$$

Since

$$\varphi_{z_1 \dots z_n}(0, 0) - \varphi_{z_1 \dots z_{n-1}}(0, 0) = \varphi_{z_1 \dots z_{n-1}} \left(0, \frac{z_n}{6} \right) - \varphi_{z_1 \dots z_{n-1}}(0, 0) = \left(0, \frac{z_n}{6^n} \right),$$

those two endpoints have the same y -coordinate. Moreover, by (3), the difference between their x -coordinates is

$$\begin{aligned} & \frac{x(\sigma^n(z))}{6^{n+1}} - \left(\frac{x(\sigma^{n+1}(z))}{6^{n+2}} - \frac{z_{n+1}}{4 \cdot 6^{n+1}} + \frac{5}{2 \cdot 6^{n+2}} \right) \\ &= \frac{1}{6^{n+1}} \cdot \left(x(\sigma^n(z)) - \frac{x(\sigma^{n+1}(z))}{6} \right) + \frac{z_{n+1}}{4 \cdot 6^{n+1}} - \frac{5}{2 \cdot 6^{n+2}} \\ &= \frac{1}{6^{n+2}} \cdot \left(-\frac{3}{2}z_{n+1} + \frac{5}{2} \right) + \frac{z_{n+1}}{4 \cdot 6^{n+1}} - \frac{5}{2 \cdot 6^{n+2}} = 0, \end{aligned}$$

that is, they also have the same x -coordinate.

Fact 3. One of the endpoints of $\varphi_{z_1 \dots z_{n-1}}(L_{z_n}(\sigma^{n-1}(z)))$ tends to $(0, z)$ as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} \varphi_{z_1 \dots z_n}(0, 0) = (0, z)$, this follows immediately from (4). ■

Lemma 5. *There exists some connected component of K that is not locally connected.*

Proof. Due to the existence of $\varphi_{19}, \dots, \varphi_{24}$, the right side $\ell_* := \{1\} \times [0, 1]$ is contained in K . By Lemma 4, $\varphi_1(\ell_*) \cup \varphi_7(\bigcup_{z \in C_2} (\ell^z \cap [0, 1]^2))$ belong to one connected component \mathcal{C} of K . However, $\varphi_7\varphi_4(\bigcup_{x \in C_1} \ell_x) \subset \varphi_7(\bigcup_{z \in C_2} (\ell^z \cap [0, 1]^2)) \subset \mathcal{C}$ are infinitely many parallel line segments that are mutually “well separated” (recalling this property of $\{\ell_x : x \in C_1\}$). Moreover, for every $x \in C_1$, there is a sequence $\{x_n\} \subset C_1$ such that the distance between ℓ_{x_n} and ℓ_x goes to 0 as $n \rightarrow \infty$. Therefore, \mathcal{C} is not locally connected at any interior point of $\varphi_7\varphi_4(\ell_x)$. ■

Acknowledgements. This research was conducted during a visit to Professor Huo-Jun Ruan at Zhejiang University. I would like to express my gratitude for his warm hospitality and helpful discussions on the topic of local connectedness. I am also grateful to the anonymous referee for his/her valuable comments.

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Received 7 April 2024.

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