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Uniform decay of unimodal states and non-local Minami estimates for a class of Fermionic Anderson models with deterministic potentials

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Abstract. We prove Anderson localization for a class of interactive Fermionic Hamiltonians in a deterministic (including some quasi-periodic) disordered external potential on a one-dimensional lattice. As in our earlier paper on 1-particle Hamiltonians (Chulaevsky, 2014), and in contrast to a recent work (J. Bourgai and I. Kachkovskiy, 2019), the sampling function on the phase space of the dynamical system generating the external potential is not even continuous. As a complement to the parametric analysis of the eigenpairs, we also prove some analogs of the Minami estimate for pairs of eigenvalues in arbitrarily placed intervals, not necessarily nested, or close to/distant from each other.

1. Introduction

Structure of the paper

(1) Description of the model and main results (Sections 1–3).

(2) The linear KAM (Kolmogorov–Arnold–Moser) inductive procedure and the proof of uniform localization of eigenfunctions (Sections 4–5).

(3) Parametric smoothness of the approximate and exact eigenpairs (Section 6).

(4) Parametric exclusion of the "small denominators" (Section 7).

(5) Generalized ("non-local") Minami-type estimates in the parameter space and concluding remarks (Sections 8–9).

We study spectral properties of finite-difference operators arising as Hamiltonians of N-body Fermionic quantum systems on $\mathbb{Z}^d$ with a nontrivial interaction of infinite range, subject to the common external potential. Our goal is two-fold. Firstly, we extend to the interactive quantum systems the Kolmogorov–Arnold–Moser (KAM)

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techniques used in [11] where a uniform exponential localization was established for a class of deterministic (including quasi-periodic) potentials. Secondly, building on a recent work [13], we establish for the first time "non-local" analogs of the Minami-type estimates for interactive particle systems.

The main results are stated in Sections 2 (Theorem 2.1) and 8 (Theorem 8.1). The class of the external potentials, introduced and studied in earlier works [10, 11] in the framework of disordered single-particle quantum systems, provides a rare opportunity to gain an insight into what an "ideal," viz., uniform localization of eigenfunctions (ULE), does look like. Long ago, the authors of [17, 18] made a fairly general constatation: *ULE fails for many models* (cf. [18, Appendix 3]). The nature of this phenomenon can be understood, at least on a heuristic level, with the help of a simple example where a full-fledged Anderson-type random Hamiltonian in an infinite lattice is replaced with its counterpart acting in the two-dimensional space $\ell^2(\llbracket 1,2\rrbracket)$ (here and below, we use the notation $\llbracket a,b\rrbracket := [a,b]\cap\mathbb{Z}$):

$$H_\varepsilon^{(2)}(\omega) = \begin{pmatrix} v_1(\omega) & \varepsilon \\ \varepsilon & v_2(\omega)\end{pmatrix}, \quad \varepsilon > 0,$$

where $v_x(\omega)$, $x\in\llbracket 1,2\rrbracket$, are IID (independent and identically distributed random variables on some probability space $(\Omega,\mathfrak{F},\mathbb{P})$, say, with the standard uniform distribution $\mathrm{Unif}([0,1])$. Within the event $\{\omega : |v_1(\omega)-v_2(\omega)|\geq\delta\}$, with a fixed $\delta>0$, the two eigenvectors of $H_\varepsilon^{(2)}(\omega)$ converge to the basis vectors $(1,0)$ and $(0,1)$ as $\varepsilon\to 0$. In the traditional terminology of Anderson localization, the eigenfunctions of $H_\varepsilon^{(2)}(\omega)$ are localized near their "localization centers," viz. the points 1 and 2. However, once the nonrandom parameter $\varepsilon>0$ is fixed, for any arbitrarily small $c>0$, the event $\{\omega : |v_1(\omega)-v_2(\omega)|\leq c\}$ has a positive probability. With $c\ll\varepsilon$, the two eigenfunctions are close to those for $v_1=v_2$, and the latter are "completely delocalized," as shows an elementary calculation. While in the former case, the eigenfunctions are "unimodal" functions on $\llbracket 1,2\rrbracket$, in the latter one, they are bi-modal, and this phenomenon, having a nonzero probability, can be reproduced on any large scale, although its rigorous parametric analysis, obviously, becomes much more complicated. On the other hand, if the ultimate freedom (independence) of variations of the potential is replaced with a strongly constraint disorder, e.g., generated by a quasi-periodic function on a lattice, one has a chance of constructing an ergodic ensemble of the potential samples in such a way that excessively strong "resonances," harmful to localization, never appear.

Even within the class of random potentials with a strongly constrained, quasi-periodic disorder, an inevitable appearance of multi-modal eigenstates has been observed, for example, in [20, 38], in the one-dimensional models with quasi-periodic potential of the form $V(x,\omega)=\cos(\omega+x\alpha)$, $\omega\in\Omega=\mathbb{T}=\mathbb{R}/\mathbb{Z}$, $\alpha\notin 2\pi\mathbb{Q}$. See also the recent papers [22, 24, 27, 34, 35].

However, a deterministic onset of Anderson localization with unimodal states was observed in a class of quasi-periodic systems [2], extending the well-known Maryland model [19], in the strong disorder regime. While the potentials considered in [2, 19], as well as those from [10, 11], are certainly quite exceptional, they provide some useful "laboratory" where a very detailed parametric analysis of various quantities turns out to be possible. In the present work, we extend this "laboratory" to the class of interactive quantum systems. Note also that ULE was observed in a large class of so-called limit-periodic potentials; cf. [16] and the references therein.

Among more recent papers, we mention in particular [23,25,27–30] where various aspects of the strongest forms of localization have been studied in great detail.

Apart from the localization analysis of eigenfunctions, we also focus on the correlation measures of the eigenvalues studied in the pioneering works [3, 21, 37] and subsequent papers; cf., e.g., [15, 32] and the references therein. Starting from [37], one usually assesses the probability for $n \geq 1$ eigenvalues to fall in the same interval $I \subset \mathbb{R}$, or in nested intervals, or in otherwise close intervals. Following a recent work [13], we consider the case of two eigenvalues $\boldsymbol{\lambda}', \boldsymbol{\lambda}''$, and assess the probability of the events $\{\boldsymbol{\lambda}' \in I', \boldsymbol{\lambda}'' \in I''\}$, with arbitrarily placed intervals $I', I'' \subset \mathbb{R}$. Extensions to any $n \geq 2$ are also possible (see the discussion in [13]).

In several ways, the class of deterministic potentials considered initially in [10, 11], as well as a larger one from [9], is complementary to those studied in the deep works [5, 6, 8], viz. generated by analytic hull functions on a torus.

Recently, an interactive particle system in a potential generated by an analytic hull was considered in [7] (see in this connection the book [4] and the paper [26]). Compared to [7], the hull functions on $\mathbb{T}$ generating the external potential in our model are not even continuous, let alone smooth, and the resulting potential $x \mapsto V(x)$ is not necessarily almost-periodic, for we allow for a richer class of the underlying dynamical systems generating the disorder (see the hypotheses (UPA) and (DIV) below).

1.1. Assumptions

The configuration space of our model, i.e., the space where the interacting Fermi particles evolve, is the lattice $\mathbb{Z}$. An adaptation to lattices of arbitrary dimension $d \geq 2$ is discussed in Section 9 (Paragraph D). We address first the particular case of 2-particle Fermi systems, for the sake of clarity and notational simplicity, but, as is explained in Section 9 (Paragraph B), an extension to any number of particles $\mathfrak{N} \geq 2$ is fairly straightforward, especially in one dimension, where a rigorous definition of the Hamiltonian and formulation of the main results do not require a rather technical construction.

Instead of working with a restriction of the Hamiltonian to the subspace of antisymmetric functions, we use an alternative but equivalent construction. Specifically, the quantum configuration space of the Fermionic 2-particle system is

$$\mathcal{Z}_2 = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 < x_2\}.$$

Ordering the particle positions amounts to declaring the pairs (x, y) and (y, x) equivalent, indistinguishable, and then selecting one representative per equivalence class $\{(x, y), (y, x)\}$. Removing the pairs with identical positions corresponds to the Fermi quantum statistics. In the general case where $d > 1$, one needs to introduce the so-called *symmetric powers of graphs;* cf. Section 9 (Paragraph D).

It is readily seen that an equivalent construction of the configuration space $\mathcal{Z}_2$ is given by the set of functions $\mathbf{n}\colon \mathbb{Z} \mapsto \{0, 1\}$ with $\operatorname{card}\operatorname{supp}\mathbf{n} = 2$, i.e., such that $\sum_{z\in\mathbb{Z}} \mathbf{n}(z) = 2$. Specifically, $\mathbf{x} = (x_1, x_2)$ corresponds to the function

$$z \mapsto \mathbf{n}_{\mathbf{x}}(z) := \mathbf{1}_{x_1}(z) + \mathbf{1}_{x_2}(z)$$

which we call an *occupation numbers* function. To inverse the mapping $\mathbf{x} \mapsto \mathbf{n}_{\mathbf{x}}$, it suffices to take $\operatorname{supp}\mathbf{n}$, which has cardinality 2, and sort it in the increasing order, thus obtaining a point $\mathbf{x} = (x_1, x_2)$ with $x_1 < x_2$. We set $\Pi\mathbf{x} := \{x_1, x_2\}$. In fact, as explained in Section 9, one can slightly reformulate our construction so that the subsets $\{x_1, x_2\} \subset \mathbb{Z}$ of cardinality 2 become themselves the points of the Fermionic 2-particle configuration space, and then the notation $\Pi\mathbf{x}(= \{x_1, x_2\})$ becomes redundant.

The subset $\mathcal{Z}_2 \subset \mathbb{Z}^2$ is endowed with the natural graph structure inherited from $\mathbb{Z}^2$, and this provides the canonical graph Laplacian $\mathbf{\Delta}$ on $\mathcal{Z}_2$.

Remark 1.1. The early papers on multi-particle Anderson localization [1, 14, 31] operated with configurations of *distinguishable particles*. As a result, one had to work with some *pseudo*-metrics in the N-particle configuration space. In particular, the decay estimates with respect to the so-called Hausdorff pseudo-metrics did not allow one to prove localization in any *bounded* spatial domain, no matter how large, thus severely reducing the significance of the first results to the physical models. Moreover, one had to deal formally with "phantom" resonant tunneling processes not corresponding to the physical reality of the modeled quantum systems. Had the same setup been used in the present work, there would (or might) be multi-modal states with multiple "localization centers." Owing to the restriction to a subset of the configuration space corresponding to a specific quantum statistics (Fermi–Dirac's, in the present paper), we can operate with a bona fide, natural distance in the 2-particle quantum space and prove a uniform localization with unimodal eigenstates.

We often make use of balls of some radius $R \geq 0$ in $\mathbb{Z}$ (relative to the distance in $\mathbb{R} \supset \mathbb{Z}$) and in $\mathcal{Z}_2$ (relative to the max-norm distance in $\mathbb{Z}^2 \supset \mathcal{Z}_2$): for $x \in \mathbb{Z}$ and

$\mathbf{x} \in \boldsymbol{\mathcal{Z}}_2$ we set

$$\mathrm{B}_R(x) := \{y \in \mathbb{Z} : |y - x| \le R\}, \quad \mathbf{B}_R(\mathbf{x}) := \{\mathbf{y} \in \boldsymbol{\mathcal{Z}}_2 : |\mathbf{y} - \mathbf{x}| \le R\}.$$

Occasionally, we denote by $\mathcal{B}_R(\cdot)$ the R-neighborhood of a given subset $A \subset \mathbb{Z}$:

$$\mathcal{B}(A) := \bigcup_{a \in A} \mathrm{B}_R(a), \quad R \ge 0. \tag{1.1}$$

We assume that a compactly supported two-body interaction potential $\mathfrak{u}\colon \mathbb{N}^* \to \mathbb{R}$ is given:

$$\exists \mathfrak{r}_0 \ge 1 \ \ \forall x > \mathfrak{r}_0 \quad \mathfrak{u}(x) = 0. \tag{1.2}$$

The interaction in our model is given by the operator of multiplication by the function

$$\boldsymbol{\mathcal{Z}}_2 \ni \mathbf{x} = (x_1, x_2) \mapsto \mathbf{U}(\mathbf{x}) := \mathfrak{u}(x_2 - x_1).$$

Adaptation to the potentials $\mathfrak{u}(\cdot)$ of infinite range is discussed in Section 9 (Paragraph C). The single-particle external potential has the same general form as in [11], viz.

$$\begin{gathered} x \mapsto V(x, \omega, \vartheta) = v(T^x \omega, \vartheta), \\ v(\omega, \vartheta) = \sum_{n \ge 1} a_n \sum_{k=1}^{K_n} \vartheta_{n,k} \chi_{n,k}(\omega), \quad a_n = \mathrm{e}^{-n^2}, \ K_n := 2^n, \\ \chi_{n,k} = \mathbf{1}_{C_{n,k}}, \quad C_{n,k} = \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right), \ k \in [\![1, K_n]\!], \end{gathered} \tag{1.3}$$

except for the functions $\chi_{n,k}$ which were the orthogonal Haar's wavelets in [11] (see Remark 1.2). Here, ω is an element of the phase space Ω, endowed with the structure a probability space $(\Omega, \mathfrak{B}^\Omega, \mathbb{P}^\Omega)$, of a conservative dynamical system $T\colon \mathbb{Z} \times \Omega \to \Omega$. For clarity, we assume as in [11] that $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $\mathbb{P}^\Omega$ is the Haar measure on the Borel σ-algebra $\mathfrak{B}^\Omega$. What is crucial to the proof of localization in the model considered here, is that T satisfies, as in [11], the properties of uniform power-law aperiodicity (UPA),

(UPA) $\exists A, C_A > 0$ such that, for all $\omega \in \Omega$ and $x, y \in \mathbb{Z}$ with $x \ne y$, one has

$$\mathrm{dist}_\Omega(T^x \omega, T^y \omega) \ge C_A^{-1} |x - y|^{-A},$$

and of tempered rate of divergence of trajectories:

(DIV) $\exists A', C_{A'} > 0$ such that, for all $\omega, \omega' \in \Omega$ and $x \in \mathbb{Z} \setminus \{0\}$

$$\mathrm{dist}_\Omega(T^x \omega, T^x \omega') \le C_{A'} |x|^{A'} \, \mathrm{dist}_\Omega(\omega, \omega').$$

For the rotations of the torus, (DIV) holds trivially, since T^x are isometries, and (UPA) becomes a Diophantine condition for the irrational frequency.

We also assume a particular form of the parameters $\{\vartheta_{n,k}, n \in \mathbb{N}^*, k \in [\![1, K_n]\!]\}$:

(UNI) the factors $\vartheta_{n,k}$ in (1.3) form a family of IID (independent and identically distributed) random variables on an auxiliary probability space $(\Theta, \mathfrak{B}^{\Theta}, \mathbb{P}^{\Theta})$ with the common uniform distribution Unif([0, 1]).

Remark 1.2. To avoid any confusion, it is to be stressed that, regardless of the choice of $\vartheta_{\bullet,\bullet}$, the function $\omega \mapsto v_{\vartheta}(\omega) := v(\omega, \vartheta)$ generates a deterministic (e.g., quasi-periodic) spatial external potential, with no "hidden" strong randomness. In fact, the localization that we prove could never be *uniform* in presence of a fully developed, non-deterministic random disorder, no matter how masterly hidden (see the discussion in [18]).

In the paper [11], $\chi_{n,k}$ with a fixed $n \ge 1$ were assumed to be the orthogonal Haar wavelets on $\Omega = \mathbb{T}$, and it was mentioned that a simpler choice with

$$\chi_{n,k} = \mathbf{1}_{C_{n,k}}, \quad C_{n,k} = [(k-1)2^{-n}, k2^{-n}), \; k \in [\![1, K_n]\!], \; K_n := 2^n,$$

requires only a minor technical adaptation. (Here and below, $[\![a, b]\!]$ stands for the integer interval $[a, b] \cap \mathbb{Z}$.) The above form of the functions $\chi_{\bullet,\bullet}$ also provides some notational simplifications, and so we work here with the non-orthogonal functions $\chi_{\bullet,\bullet}$, but our main results remain valid for the Haar's wavelets $\chi_{\bullet,\bullet}$.

1.2. Some measure-theoretic constructions and conventions

It is convenient to fix a particular realization of the space Θ:

$$\Theta = \bigtimes_{n \in \mathbb{N}} \; \bigtimes_{l \in [\![1, K_n]\!]} [0, 1],$$

so that $\vartheta_{n,k}$ are just the coordinates, or projections, of ϑ. Given a partition $\mathbb{N} = \mathcal{I} \sqcup \mathcal{J}$ (or into a greater finite disjoint union of subsets of $\mathbb{N}$), one can introduce a decomposition of the elements $\vartheta = (\vartheta_{\mathcal{I}}, \vartheta_{\mathcal{J}}) \in \Theta_{\mathcal{I}} \times \Theta_{\mathcal{J}}$. We will be mostly concerned with the case where the elements of the partition of $\mathbb{N}$, like $\mathcal{I}$ and $\mathcal{J}$, are intervals of $\mathbb{N}$. Specifically, for every equality/order relation $\# \in \{$"$\neq$", "$=$", "$\le$", "$\ge$", "$<$", "$>$"$\}$, we can consider the factors $\Theta_{\# n}$ of the infinite product Θ, with $n \in \mathbb{N}$; for example, $\Theta_{<n} = \bigtimes_{n'<n} \bigtimes_{l \in [\![1, K_{n'}]\!]} [0, 1]$. The cylindrical σ-algebra on $\Theta_{\# n}$, generated by the Lebesgue σ-algebra on each factor [0, 1], will be denoted $\mathfrak{B}^{\Theta}_{\# n}$. In other words,

$$\mathfrak{B}_{\# n} = \sigma[\vartheta_{n',k} : n' \# n, k \in [\![1, K_{n'}]\!]], \tag{1.4}$$

where the conditions inside the brackets $\sigma[\cdot]$ specify the Θ-random variables $\vartheta_{\bullet,\bullet}$ generating the corresponding σ-algebra . As the reader shall see, on each step $j \in \mathbb{N}$

of the inductive procedure, we actually work with finite-dimensional sections of Θ,

$$\Theta_{\leq n} = \bigtimes_{n' \in [\![1,n]\!]} \bigtimes_{l \in [\![1,K_{n'}]\!]} [0,1], \quad \Theta_{=n} = \bigtimes_{l \in [\![1,K_n]\!]} [0,1],$$

with a suitably chosen $n = n(j)$. By a slight abuse of notations, the σ-algebras defined in (1.4) will occasionally be identified with their embeddings into $\mathfrak{B}^{\Theta}$ generated by the natural projections $\Theta \to \Theta_{\# n}$, e.g., $\mathcal{P}_{\leq n}: \Theta \to \Theta_{\leq n}$ defined by $\mathcal{P}_{\leq n}: (\vartheta_{\leq n}, \vartheta_{>n}) \mapsto \vartheta_{\leq n}$. The most significant measure-theoretic work will be performed, on each induction step, with the components of the form $\vartheta_{<n}$, $\vartheta_{\leq n}$, and ϑ_n. The finite dimensionality of the sections $\Theta_{=n}$ will allow us in Section 6 to treat the "disorder-to-eigenvalues" mappings as smooth, not just measurable functions, with efficiently controllable smooth inverses, and thus establish the crucial measure-theoretic estimates on the "small denominators" inevitably appearing in the KAM (Kolmogorov–Arnold–Moser) procedure, the staple of our approach to uniform localization. In a more general context, the idea of smooth inversion of a suitably restricted "disorder-to-eigenvalues" has been employed in our prior work [12].

Following a recent work [13], we make use of the parametric analysis of the approximate/exact eigenvalues from Section 6 and prove an analog of the well-known Minami estimates for pairs of eigenvalues in arbitrarily located (and not necessarily identical or nested) pairs of intervals of the spectral axis.

1.3. Formal definition of the Hamiltonian

The operator of total external (i.e., particle-media) interaction energy can be defined with the help of the particle positions x_k of a configuration $\mathbf{x} = (x_1, x_2) \in \boldsymbol{\mathcal{Z}_2}$:

$$\mathbf{V}(\mathbf{x}, \omega, \vartheta) = \sum_{1 \leq k \leq 2} v(T^{x_k}\omega, \vartheta).$$

Since we are using the representation of $\boldsymbol{\mathcal{Z}_2}$ by the sites (x', x'') with a specific order of x' and x'', such a definition, with *numbered* particles, is unambiguous, but only in dimension $d = 1$. A better, more invariant representation of the external energy $\mathbf{V}$ makes use of the occupation numbers:

$$\mathbf{V}(\mathbf{x}, \omega, \vartheta) = \sum_{z \in \mathbb{Z}} \mathbf{n}_{\mathbf{x}}(z) v(T^z\omega, \vartheta).$$

Now, we are ready to define the 2-particle Fermionic Hamiltonian:

$$\begin{aligned} \mathbf{H}_\varepsilon(\omega, \vartheta) &= -\varepsilon\boldsymbol{\Delta} + \mathbf{U} + \mathbf{V}(\cdot, \omega, \vartheta) \\ &= H_\varepsilon(\omega, \vartheta) \otimes \mathbf{1} + \mathbf{1} \otimes H_\varepsilon(\omega, \vartheta) + \mathbf{U}, \end{aligned} \tag{1.5a}$$

$$H_\varepsilon(\omega, \vartheta) = -\varepsilon\Delta + V(\cdot, \omega, \vartheta), \tag{1.5b}$$

where Δ is the graph Laplacian on $\mathbb{Z}$, and $V(x, \omega, \vartheta) = v(T^x\omega, \vartheta)$.

Occasionally, we make use of the approximate eigenpairs $(\varphi_x^j(\omega,\vartheta), \lambda_x^j(\omega,\vartheta))$ of the single-particle counterpart $H_\varepsilon(\omega,\vartheta)$ of $\mathbf{H}_\varepsilon(\omega,\vartheta)$, with the same external potential $V(x,\omega,\vartheta)$ as in (1.3), constructed in [11]. Most importantly, $\varphi_x^j(\omega,\vartheta)$ and $\lambda_x^j(\omega,\vartheta)$ appear in the analysis of the two-particle AEF/AEV associated with the configurations $\mathbf{x}$ such that diam $\Pi\mathbf{x} \equiv |x_1 - x_2| > CL_j$ with a suitable $C > 0$; cf. (4.5). We do not repeat their construction, nor do we prove the probabilistic bounds on the subsets of Θ to be excluded in the course of the inductive procedure relative to the 1-particle Hamiltonian $H_\varepsilon(\omega,\vartheta)$. All the required information can be found in [11].

The inner product in the Hilbert spaces $\ell^2(\boldsymbol{\mathcal{Z}}_2)$ and $\ell^2(\mathbb{Z})$ is denoted by $\langle \cdot \mid \cdot \rangle$.

2. Main results

Theorem 2.1. *Consider the Hamiltonian $\mathbf{H}_\varepsilon$ defined by (1.5), and assume (UPA), (DIV), and (UNI). There exists some $\varepsilon_* > 0$ with the following properties.*

For any $\varepsilon \in (0, \varepsilon_*)$, there exists a subset $\Theta^{(\infty)}(\varepsilon) \subset \Theta$ such that

$$\mathbb{P}^\Theta\{\Theta^{(\infty)}(\varepsilon)\} \geq 1 - C\varepsilon^{1/4}$$

and the following holds for all $(\omega,\vartheta) \in \Omega \times \Theta^{(\infty)}(\varepsilon)$.

(A) $\mathbf{H}_\varepsilon(\omega;\vartheta)$ *has a simple pure point spectrum.*

(B) *For any $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_2$, there is exactly one eigenfunction, denoted $\boldsymbol{\varphi}_\mathbf{x}(\cdot\,;\omega;\vartheta)$, with $|\boldsymbol{\varphi}_\mathbf{x}(\mathbf{x};\omega;\vartheta)|^2 > \frac{1}{2}$. We say that $\mathbf{x}$ is the* localization center *of $\boldsymbol{\varphi}_\mathbf{x}$. Moreover, the family $\{\boldsymbol{\varphi}_\mathbf{x}(\mathbf{x};\omega;\vartheta), \mathbf{x} \in \boldsymbol{\mathcal{Z}}_2\}$ is an orthonormal basis in $\ell^2(\boldsymbol{\mathcal{Z}}_2)$.*

(C) *For all $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_2$, the eigenfunctions $\boldsymbol{\varphi}_\mathbf{x}$ decay uniformly exponentially fast away from their respective localization centers:*

$$\forall \mathbf{y} \in \boldsymbol{\mathcal{Z}}_2 \quad |\boldsymbol{\varphi}_\mathbf{x}(\mathbf{y};\omega;\vartheta)| \leq \mathrm{e}^{-m(\varepsilon)|\mathbf{x}-\mathbf{y}|}, \quad m(\varepsilon) = \varepsilon^{1/4} > 0.$$

We often say that functions satisfying (B) are *unimodal.*

Observe that, while the parameter $\vartheta \in \Theta$ is restricted to $\Theta^{(\infty)}(\varepsilon)$, the assertions (A)–(C) hold for *all* $\omega \in \Omega$ and not just for $\mathbb{P}^\Omega$-almost all.

The last, functional-analytic part of the proof of Theorem 2.1, based on a scale induction, occupies Sections 4–5. It relies upon the eigenvalue concentration estimates established in Section 7. As was already mentioned in [10, 11], the deterministic and uniform (with respect to $\omega \in \Omega$) exponential decay of all eigenfunctions implies an exponential decay of the averaged eigenfunction correlators (cf., e.g., [18, 33, 39]):

$$\sup_{\omega\in\Omega}\ \sup_{t\in\mathbb{R}} |\langle \mathbf{1}_\mathbf{x} \,|\mathrm{e}^{-\mathrm{i}t\mathbf{H}(\omega;\vartheta)}|\, \mathbf{1}_\mathbf{y}\rangle| \leq \mathrm{Const}\, \mathrm{e}^{-m(\varepsilon)|\mathbf{x}-\mathbf{y}|}.$$

In Section 8, we prove an analog of the Minami estimate (cf. [37]) extended to a more general case where a pair of eigenvalues $(\boldsymbol{\lambda}', \boldsymbol{\lambda}'')$ is restricted to the product $I' \times I''$ of arbitrarily placed intervals $I', I'' \subset \mathbb{R}$.

3. Phase-space analysis

Partitions of the torus. Recall that we have introduced the partitions $\mathcal{C}_n$, $n \geq 1$, of $\Omega = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ into $K_n = 2^n$ adjacent intervals $C_{n,k}$:

$$C_{n,k} = [l_{n,k}2^{-n}, (l_{n,k}+1)2^{-n}), \quad l_{n,k} \in [\![0, 2^n - 1]\!]. \tag{3.1}$$

$C_{n,k}$ is uniquely identified by an index sequence $(\hat{k}_0, \dots, \hat{k}_{n-1}, \hat{k}_n)$, with $\hat{k}_n = k$, labeling the $(n+1)$ intervals $C_{i,\hat{k}_i} \supset C_{n,k}$, $0 \leq i \leq n$, of the partitions preceding or equal to $\mathcal{C}_n$. Here and below, $\hat{k}_i(\omega)$ stands for the unique index such that

$$\omega \in C_{i,\hat{k}_i(\omega)}.$$

Piecewise-constant approximants of $\boldsymbol{\omega \mapsto v(\omega, \vartheta)}$. For each $N \geq 0$, consider the N-th partial sum of the series $v(\omega; \vartheta)$ defined in (1.3):

$$v_N(\omega; \vartheta) = \sum_{n=0}^{N} a_n \sum_{k=1}^{K_n} \vartheta_{n,k}\chi_{n,k}, \quad a_n = \mathrm{e}^{-n^2}.$$

The random variables $\vartheta \mapsto v_N(\omega; \vartheta)$ on $(\Theta, \mathfrak{B}, \mathbb{P}^\Theta)$ are correlated via the values $(\vartheta_{n',\bullet}, n' < n)$. However, for any fixed n, the family $\{\vartheta_{n,k}, k = 1, \dots, K_n\}$ (the "n-th generation") is independent, by construction, and so are generations with different n. We shall see that the amplitudes $\vartheta_{n,k}$ bring enough "innovation" into the n-th generation of the functions $\varphi_{n,k}$ for the localization to occur.

We shall need a simple estimate of $\|v - v_N\|_\infty := \sup_{\omega\in\Omega} \|v - v_N\|_{L^\infty(\Theta)}$:

$$\|v - v_N\|_\infty \leq a_N \sum_{i \geq 1} \mathrm{e}^{-(N+i)^2 + N^2} \leq C\mathrm{e}^{-N} a_N. \tag{3.2}$$

It is crucial to our proofs to have the right-hand side *much smaller* than the width a_N of the probability distribution of the random coefficients $a_N\vartheta_{N,k} \sim \mathrm{Unif}[0, a_N]$). Let

$$\hat{\mathrm{n}}(L) = \lceil \ln^2(L) \rceil. \tag{3.3}$$

A more optimal choice for $L \mapsto \hat{\mathrm{n}}(L)$ would be, as in [10, 11], $\hat{\mathrm{n}}(L) = C' \ln(L)$ with a suitable $C' > 0$, but the above makes even clearer that, by (UPA), for $L \geq L_0$ with L_0 large enough, for all $u \in \mathbb{Z}^d$ and $\omega \in \Omega$, all the points of a large finite trajectory

$\{T^x\omega, x \in \mathrm{B}_{L^2}(u)\}$ are separated by the partition $\mathcal{C}_{\hat{n}(L)}$. As in [10,11], this proves to be a satisfactory replacement for a "strong randomness" (e.g., independence) of the values of the external potential V. As we shall see, this allows one to eliminate, at the price of suitable parameter exclusions in Θ, the excessively small denominators well before they might appear in the inductive construction of the localized eigenstates.

4. KAM induction

We work with square-summable (most often, compactly supported) functions on a specific graph $\boldsymbol{\mathcal{Z}}_{\mathbf{2}}$, but some important notions can be introduced in a wider framework of a countable graph $\mathcal{G}$ endowed with the canonical graph-distance $\mathrm{d}_{\mathcal{G}}(\cdot,\cdot)$.

Definition 4.1. Let $f \in \ell^2(\mathcal{G}) \setminus \{0\}$. A site $x \in \mathcal{G}$ is called a *localization center* of f if and only if $|f(x)| = \|f\|_\infty$. The set of all localization centers of f is denoted $\widehat{X}(f)$.

Lemma 4.1. *For any* $f \in \ell^2(\mathcal{G}) \setminus \{0\}$*, one has* $0 < \operatorname{card} \widehat{X}(f) < +\infty$.

Definition 4.2. Let be given $m > 0$ and $f \in \ell^2(\mathcal{G})$ with $\|f\|_2 = 1$. f is called *uniformly m-localized* if and only if

(i) f has a localization center $\hat{x}$ such that $|f(\hat{x})|^2 > \frac{1}{2}$;

(ii) for all $y \in \mathcal{G} \setminus \{x\}$, one has $|f(y)| \le \mathrm{e}^{-m\mathrm{d}_{\mathcal{G}}(x,y)}$.

Clearly, if $\|f\|_2 = 1$ and $|f(x)|^2 > \frac{1}{2}$ for some $x \in \mathcal{G}$, then $\widehat{X}(f) = \{x\}$, so we will say sometimes that such a function f is *uniformly m-localized at the point* $x \in \mathcal{G}$.

Before going to the next definition, note that the additive group $\mathbb{Z}$ acts (non-transitively) on $\boldsymbol{\mathcal{Z}}_{\mathbf{2}} \subset \mathbb{Z}^2$ by the shifts $S^a\colon (x_1, x_2) \mapsto (x_1 + a, x_2 + a)$, $a \in \mathbb{Z}$.

Definition 4.3. Let a dynamical system $T\colon \mathbb{Z} \times \Omega \to \Omega$, some set A, and an action $\mathbf{S}$ of the group $\mathbb{Z}$ on A be given. A mapping $F\colon \mathbb{Z}^2 \times \Omega \to A$ is called *T-covariant* if and only if

$$\forall \mathbf{z} \in \mathbb{Z}^2 \;\; F(S^a\mathbf{z}, \omega) = \mathbf{S}^a F(\mathbf{z}, T^a\omega).$$

It is often convenient to write $F_{\mathbf{z}}(\omega)$ instead of $F(\mathbf{z}, \omega)$.

We shall need three kinds of covariant mappings:

(i) scalar mappings $(\mathbf{z}, \omega) \mapsto g_{\mathbf{z}}(\omega) \in \mathbb{R}$ such that $g_{S^a\mathbf{z}}(\omega) = g_{\mathbf{z}}(T^a\omega)$, $\mathbf{z} \in \boldsymbol{\mathcal{Z}}_{\mathbf{2}}$;

(ii) vector-valued mappings $(\mathbf{z}, \omega) \mapsto f_{\mathbf{z}}(\cdot, \omega)$, with the implicit argument $(\cdot)$ in $\boldsymbol{\mathcal{Z}}_{\mathbf{2}}$ and values in $\ell^2(\boldsymbol{\mathcal{Z}}_{\mathbf{2}})$, such that $f_{S^a\mathbf{z}}(\mathbf{x}, \omega) = f_{\mathbf{z}}(S^a\mathbf{x}, T^a\omega)$, $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_{\mathbf{2}}$; here $\mathbf{S}^a$ acts by unitary transformations in $\ell^2(\boldsymbol{\mathcal{Z}}_{\mathbf{2}})$ (shifts S^a of the argument $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_{\mathbf{2}}$);

(iii) matrix-valued mappings with T-covariant column-vectors.

As in [11], apart from the conventional Hilbert norm in $\ell^2(\mathcal{Z}_2)$, we often use the vector norms $\|\cdot\|_{\mathbf{x}}$ and the matrix norms $|||\cdot|||$, both depending upon the parameter $m = m(\varepsilon) > 0$ figuring in (4.17), defined as follows:

$$\|f\|_{\mathbf{x}} = \sum_{\mathbf{y}\in\mathcal{Z}_2} \mathrm{e}^{m|\mathbf{y}-\mathbf{x}|}|f(\mathbf{y})|, \quad \mathbf{x} \in \mathcal{Z}_2,$$
$$|||\mathbf{A}||| = \sup_{\mathbf{x}\in\mathcal{Z}_2} \sum_{\mathbf{y}\in\mathcal{Z}_2} \mathrm{e}^{m|\mathbf{y}-\mathbf{x}|}|\mathbf{A}_{\mathbf{yx}}|.$$

Their efficiency in the context of exponential localization analysis has been demonstrated long ago in [2].

4.1. Induction hypotheses

Introduce an integer sequence $(L_j)_{j\in\mathbb{N}}$ (with $L_0 > 2\mathrm{r}_0$, cf. (1.2)) and decaying positive sequences $(\epsilon_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, $(\beta_j)_{j\in\mathbb{N}}$ of the form

$$L_j = L_0 q^j, \quad \epsilon_j = \epsilon_0^{q^j}, \quad q = 3/2, \quad \epsilon_0 \equiv \epsilon_0(\varepsilon) := \varepsilon^{1/4}, \tag{4.1}$$
$$\delta_j = a_{\hat{\mathrm{n}}_j}\beta_j, \quad \beta_j = \mathrm{e}^{-\hat{\mathrm{n}}_j}, \quad \hat{\mathrm{n}}_j := \hat{\mathrm{n}}(L_j) \ \text{(cf. (3.3))}. \tag{4.2}$$

For notational brevity, we often use notations like $\epsilon_i^{b\pm}$ as shortcuts for $\epsilon_i^{b\pm c}$ with $c > 0$ that can be chosen (before the induction starts) as small as necessary. For future use, we stress that any bounded factors can be absorbed in $\epsilon_i^{b\pm c}$, i.e., $\mathrm{O}[1]\epsilon_i^{b\pm} \le \epsilon_i^{b\pm}$.

Notations like $\mathrm{O}[\cdot]$, $\mathrm{o}[\cdot]$ usually refer to $\varepsilon \downarrow 0$, and $\varepsilon \downarrow 0 \iff \epsilon_0(\varepsilon) \downarrow 0$ (cf. (4.1)). We call *spread* of a matrix (A_{ab}) the quantity $\mathrm{SPR}[A] \in \mathbb{N}$ such that $A_{ab} = 0$ whenever $|a - b| > \mathrm{SPR}[A]$. (This terminology is not traditional but convenient.)

To prove the main results, we have to establish by induction, for every $j \in \mathbb{N}$, the validity of the set $\mathrm{K}(L_j)$ of hypotheses (K1)–(K10) presented below. Admittedly, this presentation is long and quite technical but, on the bright side, it provides one with a wealth of technical features of the approximate and exact spectral data.

Stochastic supports. It is readily seen from the explicit formulae for the approximate eigenvalues $\boldsymbol{\lambda}_{\mathbf{x}}^0$ and approximate eigenfunctions $\boldsymbol{\varphi}_{\mathbf{x}}^0$, specified below in the hypothesis (K4) (cf. (4.7)–(4.6)), that $\boldsymbol{\varphi}_{\mathbf{x}}^0$ do not depend at all upon the random potential $V(\cdot,\omega,\vartheta)$, while $\boldsymbol{\lambda}_{\mathbf{x}}^0$ are measurable functions of exactly two values of the potential, $V(x_1,\omega,\vartheta)$ and $V(x_2,\omega,\vartheta)$. In this simple case, the finite subset $\{x_1,x_2\} \equiv \Pi\mathbf{x} \subset \mathbb{Z}$ is what we call the *stochastic support* of the mapping $f = \boldsymbol{\lambda}_{\mathbf{x}}^0$. To formalize this notion in a more general context where f may be scalar, as $\boldsymbol{\lambda}_{\mathbf{x}}^i$, or vector/matrix-valued, we introduce, as in [11], the following definition.

Definition 4.4. Let $(\mathcal{Y}, \mathfrak{F}_{\mathcal{Y}})$ be a measurable space, and consider the measurable space $(\mathcal{X}, \mathfrak{F}_{\mathcal{X}})$ where $\mathcal{X} = \mathbb{R}^{\mathbb{Z}}$ and $\mathfrak{F}_{\mathcal{X}}$ is the corresponding cylindrical σ-algebra. For any $A \subset \mathbb{Z}$, let $\mathfrak{F}_{[A]}$ be the cylindrical σ-algebra of $\mathbb{R}^A$, canonically identified with the corresponding sub-algebra of $\mathfrak{F}_{\mathcal{X}}$, so that, in particular, $\mathfrak{F}_{[\mathbb{Z}]} = \mathfrak{F}_{\mathcal{X}}$.

The stochastic support of a measurable mapping $f\colon (\mathcal{X}, \mathfrak{F}^{\mathcal{X}}) \to (\mathcal{Y}, \mathfrak{F}^{\mathcal{Y}})$ is the minimal subset $\mathcal{S}(f) \subset \mathbb{Z}$ such that $f\colon (\mathcal{X}, \mathfrak{F}^{\mathcal{X}}_A) \to (\mathcal{Y}, \mathfrak{F}^{\mathcal{Y}})$ with $A = \mathcal{S}(f)$ is measurable.

Finiteness of the stochastic supports of the approximate eigenpairs, while being very convenient if not crucial to our techniques, comes at a price: the approximate eigenbases are approximately but *not exactly* orthogonal. Nevertheless, their precision rapidly improves as $j \uparrow +\infty$, because they rapidly converge to an *exact eigen*base.

K(L_j). For all $i \in [\![0, j]\!]$, there exist some sets $\Theta^i \in \mathfrak{B}_{\le \hat{\mathrm{n}}_i}$, $\hat{\mathrm{n}}_i := \hat{\mathrm{n}}(L_i)$ (cf. (1.4)), with $\mathbb{P}^{\Theta}\big\{\Theta \setminus \Theta^i\big\} \le \epsilon_i^{0+}$ and such that the following holds true.

(K1) For any $0 \le i \le j$ and all $\vartheta \in \widehat{\Theta}^i := \cap_{0 \le l \le i} \Theta^l$, the following T-covariant mappings from $\Omega = \mathbb{T}^\nu$ to $\mathrm{M{\small AT}}(\mathbb{Z}, \mathbb{R})$, parameterized by ϑ, are well defined:

$$
\begin{aligned}
&(\omega, \vartheta) \mapsto \mathbf{\Phi}^i(\omega, \vartheta), \quad \mathbf{\Phi}^i_{\mathbf{y}\mathbf{x}}(\omega, \vartheta) =: \boldsymbol{\varphi}^i_{\mathbf{x}}(\mathbf{y}, \omega, \vartheta), \\
&(\omega, \vartheta) \mapsto \mathbf{\Psi}^i(\omega, \vartheta), \quad \mathbf{\Psi}^i_{\mathbf{y}\mathbf{x}}(\omega, \vartheta) =: \boldsymbol{\psi}^i_{\mathbf{x}}(\mathbf{y}, \omega, \vartheta), \qquad (4.3)\\
&(\omega, \vartheta) \mapsto \mathbf{\Lambda}^i(\omega, \vartheta), \quad \mathbf{\Lambda}^i_{\mathbf{y}\mathbf{x}}(\omega, \vartheta) =: \delta_{\mathbf{y}\mathbf{x}} \boldsymbol{\lambda}^i_{\mathbf{x}}(\omega, \vartheta). \qquad (4.4)
\end{aligned}
$$

Furthermore, for any $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_2$ with $\operatorname{diam} \Pi\mathbf{x} \equiv |x_1 - x_2| > 4L_j$, one has

$$
\begin{aligned}
\boldsymbol{\varphi}^i_{\mathbf{x}}(\omega, \vartheta) &= \varphi^i_{x_1}(\omega, \vartheta) \otimes \varphi^i_{x_2}(\omega, \vartheta), \\
\boldsymbol{\lambda}^i_{\mathbf{x}}(\omega, \vartheta) &= \lambda^i_{x_1}(\omega, \vartheta) + \lambda^i_{x_2}(\omega, \vartheta),
\end{aligned} \qquad (4.5)
$$

where $(\varphi^i_\bullet, \lambda^i_\bullet)$ approximate eigenpairs of the 1-particle Hamiltonian studied in [11], with the external potential defined in (1.3). The discrepancies relative to the approximate eigenpairs $(\varphi^i_\bullet, \lambda^i_\bullet)$ are denoted $\psi^i_\bullet$.

(K2) The matrix $\mathbf{\Phi}^i(\omega, \vartheta)$ has the form

$$
\mathbf{\Phi}^i(\omega, \vartheta) = \mathbf{1} + \widetilde{\mathbf{D}}^i(\omega, \vartheta), \quad \|\!|\widetilde{\mathbf{D}}^i(\omega, \vartheta)|\!\| \le \frac{1}{4} - \frac{1}{4^{i+2}},
$$

so it is invertible by the Neumann series, and its columns form a Riesz basis.

The following relations hold, for all $0 \le i \le j$.

(K3) The matrices $\mathbf{\Phi}^i$, $\mathbf{\Psi}^i$ and Λ^i satisfy the identity $\mathbf{H}\mathbf{\Phi}^i = \mathbf{\Phi}^i \mathbf{\Lambda}^i + \mathbf{\Psi}^i$.

(K4) For $i = 0$ and any $\mathbf{x} = (x_1, x_2) \in \boldsymbol{\mathcal{Z}}_2$, one has

$$\boldsymbol{\varphi}^0_{\mathbf{x}}(\omega, \vartheta) = \varphi^0_{x_1} \otimes \varphi^0_{x_2} = \mathbf{1}_{\mathbf{x}}\,, \tag{4.6}$$

$$\boldsymbol{\lambda}^0_{\mathbf{x}}(\omega, \vartheta) = \mathbf{U}(\mathbf{x}) + \mathbf{V}(\mathbf{x}, \omega, \vartheta), \tag{4.7}$$

$$\boldsymbol{\psi}^0_{\mathbf{x}}(\omega, \vartheta) = \psi^0_{x_1} \otimes \mathbf{1}_{x_2} + \mathbf{1}_{x_1} \otimes \psi^0_{x_2}. \tag{4.8}$$

Once the interaction potential $\mathfrak{u}$ is fixed, for all $i \in [\![1, j]\!]$, the objects $\#^0_\bullet$, with $\# \in \{\lambda, \boldsymbol{\Lambda}, \varphi, \boldsymbol{\Phi}, \psi, \boldsymbol{\Psi}\}$, are determined by the matrix Λ^0 with $\Lambda^0_{yx} = \delta_{yx}\lambda^0_x$.

(K5) The column-vectors of the discrepancy matrices $\boldsymbol{\Psi}^i_\bullet$ obey

$$\sup_{\omega\in\Omega} \|\boldsymbol{\psi}^i_{\mathbf{x}}(\omega, \vartheta)\|_{\mathbf{x}} \le \epsilon_i^{\frac{4}{3}+}. \tag{4.9}$$

(K6) For any $\mathbf{x}$, $\boldsymbol{\psi}^i_{\mathbf{x}}$ is "almost orthogonal" to the AEF $\boldsymbol{\varphi}^i_{\mathbf{x}}$:

$$\sup_{\mathbf{x}\in\boldsymbol{\mathcal{Z}}_2} |\langle \boldsymbol{\psi}^i_{\mathbf{x}} \mid \boldsymbol{\varphi}^i_{\mathbf{x}}\rangle| \le \epsilon_i^{2+}. \tag{4.10}$$

(K7) The AEF $\boldsymbol{\varphi}^i_{\mathbf{x}}$ have compact support, of size uniformly bounded in $\mathbf{x}$,

$$\forall \mathbf{x} \in \boldsymbol{\mathcal{Z}}_2 \quad \operatorname{supp} \boldsymbol{\varphi}^i_{\mathbf{x}} \subset \mathbf{B}_{L_i}(\mathbf{x}), \tag{4.11}$$

and since $\mathbf{H}$ is a second-order finite-difference operator, this implies

$$\forall \mathbf{x} \in \boldsymbol{\mathcal{Z}}_2 \quad \operatorname{supp} \boldsymbol{\psi}_{\mathbf{x}} \subset \mathbf{B}_{L_i+1}(\mathbf{x}). \tag{4.12}$$

(K8) For all $\mathbf{x}, \mathbf{y}$ in $\boldsymbol{\mathcal{Z}}_2$ with $\operatorname{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) \le 8L_i^2$, one has

$$\inf_{\omega\in\Omega} |\boldsymbol{\lambda}^i_{\mathbf{y}}(\omega, \vartheta) - \boldsymbol{\lambda}^i_{\mathbf{x}}(\omega, \vartheta)| \ge 4\delta_i = \epsilon_i^{0+}. \tag{4.13}$$

(K9) The objects $\boldsymbol{\lambda}^i_{\mathbf{x}}, \boldsymbol{\varphi}^i_{\mathbf{x}}, \boldsymbol{\psi}^i_{\mathbf{x}}$ have bounded stochastic supports:

$$\mathcal{S}(\boldsymbol{\lambda}^i_{\mathbf{x}}) \cup \mathcal{S}(\boldsymbol{\varphi}^i_{\mathbf{x}}) \cup \mathcal{S}(\boldsymbol{\psi}^i_{\mathbf{x}}) \subset \mathrm{B}_{L_i}(\mathbf{x}).$$

(K10) For all $0 \le i \le j - 1$, one has

$$\sup_{\mathbf{x}} |\boldsymbol{\lambda}^{i+1}_{\mathbf{x}} - \boldsymbol{\lambda}^i_{\mathbf{x}}| \le \epsilon_i^{2+} \le \epsilon_{i+1}^{\frac{4}{3}+}, \tag{4.14}$$

$$\sup_{\mathbf{x}} \|\boldsymbol{\varphi}^{i+1}_{\mathbf{x}} - \boldsymbol{\varphi}^i_{\mathbf{x}}\|_{\mathbf{x}} \le \epsilon_i^{2+}. \tag{4.15}$$

Apart from the operator family $\mathbf{H}_\varepsilon(\omega, \vartheta)$, the fundamental objects of the inductive procedure are $\boldsymbol{\Lambda}^j$ and $\boldsymbol{\Phi}^j$, while $\boldsymbol{\Psi}^j$ and $\mathbf{F}^j$ are derived from $\boldsymbol{\Lambda}^j$, $\boldsymbol{\Phi}^j$ and $\mathbf{H}_\varepsilon$.

Remark 4.1. We will start the induction step by showing that $\boldsymbol{\Phi}^j$ is "almost orthogonal," so the Gram matrix $\mathbf{C}^i = (\boldsymbol{\Phi}^j)^\top \boldsymbol{\Phi}^j$ of the Riesz basis $\boldsymbol{\varphi}^j_\bullet$ is close to $\mathbf{1}$:

$$|\!|\!|(\boldsymbol{\Phi}^j)^\top \boldsymbol{\Phi}^j - \mathbf{1}\,|\!|\!| \le \epsilon_j^{1+}. \tag{4.16}$$

4.2. The base of induction

We assume that $0 < \varepsilon \le 1/16$, and set

$$m(\varepsilon) := \ln \varepsilon^{-1/4} \xrightarrow[\varepsilon\to 0]{} +\infty. \tag{4.17}$$

Recall that we have introduced in (4.1) the sequence $\epsilon_i = \epsilon_0^{q^i}$, $q = 3/2$, $i \ge 0$.

Relations (4.6)–(4.7) merely define the column-vectors $\boldsymbol{\varphi}^0_{\mathbf{x}}$ of the matrix $\boldsymbol{\Phi}^0$, serving as approximate eigenfunctions (AEF) of $\mathbf{H}$ with approximate eigenvalues (AEV) $\boldsymbol{\lambda}^0_{\mathbf{x}}$, providing the diagonal entries of the diagonal matrix $\boldsymbol{\Lambda}^0$. A simple calculation (cf. [13, Eqn. (3.11)]) shows that the column-vectors $\boldsymbol{\psi}^0_{\mathbf{x}}$ of the matrix $\boldsymbol{\Psi}^0$ given by (4.8) are the correct discrepancies for the approximate eigenpairs $(\boldsymbol{\varphi}^0_{\mathbf{x}}, \boldsymbol{\lambda}^0_{\mathbf{x}})$. Also, it is readily seen from (4.8) that $\operatorname{supp} \boldsymbol{\psi}^0_{\mathbf{x}} \cap \operatorname{supp} \boldsymbol{\varphi}^0_{\mathbf{x}} = \operatorname{supp} \boldsymbol{\psi}^0_{\mathbf{x}} \cap \{\mathbf{x}\} = \varnothing$, whence $(\boldsymbol{\psi}^0_{\mathbf{x}}, \boldsymbol{\varphi}^0_{x}) \equiv 0$, which is stronger than (K6) with $i = 0$.

Among the implicit exponents of the form $b^{\pm}$ introduced in the first paragraph of Section 4.1, the one figuring in (4.13) is quite important, so now we denote it by σ and specify its relations to other key quantities. Specifically, denote for brevity $\hat{\mathrm{n}}_0 = \hat{\mathrm{n}}(L_0) = \ln^2(L_0) > 1$ (cf. (3.3)), and assume that

$$\varepsilon \le \mathrm{e}^{-8\sigma^{-1}\hat{\mathrm{n}}_0^2} = \mathrm{e}^{-8\sigma^{-1}\ln^4(L_0)},$$

then, with $\delta_0 = a_{\hat{\mathrm{n}}_0}\beta_0 = \mathrm{e}^{-\hat{\mathrm{n}}_0^2 - \hat{\mathrm{n}}}$ (cf. (4.2)) and $\epsilon_0(\varepsilon) = \varepsilon^{1/4}$ (cf. (4.1)), we have

$$\delta_0(\varepsilon) > \mathrm{e}^{-2\hat{\mathrm{n}}^2} \ge \varepsilon^{\sigma/4} = \epsilon_0^{\sigma}(\varepsilon). \tag{4.18}$$

As the reader can see, (4.18) can be improved, but it is already sufficient.

Now, assess the norms of the discrepancies $\boldsymbol{\psi}^0_{\bullet}$. The interaction operator $\mathbf{U}$ is diagonal in the basis of vectors $\mathbf{1}_{\mathbf{x}} = \mathbf{1}_{x_1} \otimes \mathbf{1}_{x_2}$, thus $\mathbf{U}(\mathbf{x})\boldsymbol{\varphi}^0_{\mathbf{x}}$ cancels out in the difference $(-\boldsymbol{\Delta} + \mathbf{V} + \mathrm{U})\boldsymbol{\varphi}^0_{\mathbf{x}} - \boldsymbol{\lambda}^0_{\mathbf{x}}\boldsymbol{\varphi}^0_{\mathbf{x}}$, and so for each $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_2$, it suffices to check if (4.8) provides the correct discrepancy of the AEF $\boldsymbol{\varphi}^0_{\mathbf{x}}$ relative to the reduced operator $-\boldsymbol{\Delta} + \mathbf{V} = H^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes H^{(2)}$, $H^{(k)} = -\Delta + V(x_k, \cdot, \cdot)$. A simple calculation shows that (4.8) is correct, thus on account of $\varepsilon^{1/2} \le \frac{1}{4}$,

$$\|\boldsymbol{\psi}^0_{\mathbf{x}}\|_{\mathbf{x}} = \varepsilon \sum_{|\mathbf{y}-\mathbf{x}|=1} \mathrm{e}^{m|\mathbf{y}-\mathbf{x}|} \le 4\mathrm{e}^m \varepsilon = 4\varepsilon^{1/2} \cdot \mathrm{e}^m \varepsilon^{1/2} \le 1 \cdot \epsilon^{-\frac{1}{4}+\frac{1}{2}} = \epsilon^{\frac{1}{4}}.$$

Taking ε sufficiently small, one can have both ϵ_0 arbitrary small and the m-norm estimate (4.9) from (K5) with $i = 0$ holding with $m > 0$ as large as one pleases.

4.3. The inductive step

Below we sometimes use for brevity the notation $a(j) \lesssim b(j)$ for quantities dependent upon the scale L_j, meaning that $a(j) \le Cb(j)$ for some finite constant C and all

$j \geq 0$. The subscript ε in H_ε will be often omitted, firstly, for brevity, and secondly, to avoid using in the same formulae the amplitude ε from (1.5) and the smallness parameters ϵ_j depending upon it. The transpose of a matrix A is denoted $A^\intercal$.

Theorem 4.2. *For any* $j \geq 0$, $\mathrm{K}(L_j)$ *implies* $\mathrm{K}(L_{j+1})$.

Proof. Fix $j \geq 0$ and assume $\mathrm{K}(L_j)$.

Step 1. *The Gram matrix.* Let us show that the Gram matrix $\mathbf{C}^j = (\boldsymbol{\Phi}^j)^\intercal \boldsymbol{\Phi}^j$ of the Riesz basis $\{\boldsymbol{\varphi}^j_\bullet\}$ is close to $\mathbf{1}$, viz. $\mathbf{C}^j = \mathbf{1} + \mathbf{D}^j$, $|||\mathbf{D}^j||| = [\epsilon_j^{1-}]$. It will imply the convergence of Neumann's series for $(\mathbf{1} + \mathbf{D}^j)^{-1}$, so

$$(\mathbf{C}^j)^{-1} = (\mathbf{1} + \mathbf{D}^j)^{-1} = \mathbf{1} - \mathbf{D}^j + [|||\mathbf{D}^j|||^2] = \mathbf{1} + [\epsilon_j^{1-}].$$

Case 1. Assume that $\operatorname{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) \leq 8L_j^2$, so $|\boldsymbol{\lambda}^j_\mathbf{y} - \boldsymbol{\lambda}^j_\mathbf{x}| \geq 4\delta_j$ by (K8). By symmetry of $\mathbf{H}$, we have

$$\begin{aligned}|\mathbf{C}^j_{\mathbf{yx}}| = |\langle \boldsymbol{\varphi}^j_\mathbf{y} \mid \boldsymbol{\varphi}^j_\mathbf{x} \rangle| &\leq \frac{|\langle \boldsymbol{\varphi}^j_\mathbf{y} \mid \boldsymbol{\psi}^j_\mathbf{x} \rangle + \langle \boldsymbol{\psi}^j_\mathbf{y} \mid \boldsymbol{\varphi}^j_\mathbf{x} \rangle|}{|\boldsymbol{\lambda}^j_\mathbf{x} - \boldsymbol{\lambda}^j_\mathbf{y}|} \\ &\leq \frac{|\langle \boldsymbol{\varphi}^j_\mathbf{y} \mid \boldsymbol{\psi}^j_\mathbf{x} \rangle| + |\langle \boldsymbol{\psi}^j_\mathbf{y} \mid \boldsymbol{\varphi}^j_\mathbf{x} \rangle|}{4\delta_j}.\end{aligned} \tag{4.19}$$

The inductive hypotheses (4.15) and (4.6)–(4.8) imply that $\|\boldsymbol{\varphi}^j_\mathbf{x}\|_\mathbf{x} \leq 1 + \sum_i \epsilon_i \leq 2$, while $\|\boldsymbol{\psi}^j_\mathbf{x}\|_\mathbf{x} \leq \epsilon_j^{1+}$ by (4.9). Thus, on account of (K8) (cf. (4.11)–(4.12)), we have,

$$|\langle \boldsymbol{\psi}^j_\mathbf{y} \mid \boldsymbol{\varphi}^j_\mathbf{x} \rangle| + |\langle \boldsymbol{\varphi}^j_\mathbf{x} \mid \boldsymbol{\psi}^j_\mathbf{y} \rangle| \lesssim L_j^{d-1} \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} \delta_j^{-1} \epsilon_j^{1+} \lesssim \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} \epsilon_j^{1+}. \tag{4.20}$$

Recalling that $\mathbf{C}^j_{xx} = \|\boldsymbol{\varphi}^j_\mathbf{x}\|^2 = 1$ by (K1) (here $\|\cdot\|$ is the ℓ^2-norm), we get

$$\mathbf{C}^j_{yx} = \begin{cases} 1, & \text{if } \mathbf{x} = \mathbf{y}; \\ \mathbf{D}^j_{\mathbf{yx}}, \quad |\mathbf{D}^j_{\mathbf{yx}}| \lesssim \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} \epsilon_j^{1+}, & \text{if } 0 < |\mathbf{x} - \mathbf{y}| \leq 2L_j; \\ 0, & \text{if } |\mathbf{x} - \mathbf{y}| > 2L_j. \end{cases} \tag{4.21}$$

To complement the Case 1, it remains to consider the pairs of sites $(\mathbf{x}, \mathbf{y})$ with

$$\operatorname{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) > 8L_j^2. \tag{4.22}$$

Case 2. *Two factorized states.* Assume that (4.22) holds true and, in addition,

$$\min \{\operatorname{diam} \Pi\mathbf{x}, \operatorname{diam} \Pi\mathbf{y}\} > L_j^2. \tag{4.23}$$

Then $\boldsymbol{\varphi}^j_\mathbf{x} = \varphi^j_{x_1} \otimes \varphi^j_{x_2}$, $\boldsymbol{\varphi}^j_\mathbf{y} = \varphi^j_{y_1} \otimes \varphi^j_{y_2}$ by (K1) (cf. (4.5)), thus, by the assumed normalization of the AEF,

$$|\langle \boldsymbol{\varphi}^j_\mathbf{x} \mid \boldsymbol{\varphi}^j_\mathbf{y} \rangle| = |\langle \varphi^j_{x_1} \mid \varphi^j_{y_1} \rangle| |\langle \varphi^j_{x_2} \mid \varphi^j_{y_2} \rangle| \leq \min_{k=1,2} |\langle \varphi^j_{x_k} \mid \varphi^j_{y_k} \rangle|.$$

By definition of the max-norm, $0 \neq |\mathbf{x} - \mathbf{y}| = |x_{\tilde{k}} - y_{\tilde{k}}|$ for some $\tilde{k} \in \{1, 2\}$, thus

$$\mathbf{C}^j_{\mathbf{yx}} = |\langle \boldsymbol{\varphi}^j_{\mathbf{x}} \mid \boldsymbol{\varphi}^j_{\mathbf{y}} \rangle| \le |\langle \varphi^j_{x_{\tilde{k}}} \mid \varphi^j_{y_{\tilde{k}}} \rangle| = |\mathrm{C}^j_{y_{\tilde{k}} x_{\tilde{k}}}|$$

so the required bound follows, for we have by induction (cf. also [11])

$$\mathrm{C}^j_{yx} = \begin{cases} 1, & \text{if } x = y; \\ \mathrm{D}^j_{yx}, \quad |\mathrm{D}^j_{yx}| \lesssim \mathrm{e}^{-m|x-y|} \epsilon_j^{1+}, & \text{if } 0 < |x - y| \le L_j; \\ 0, & \text{if } |x - y| > L_j. \end{cases} \tag{4.24}$$

Case 3. Finally, assume again (4.22) is true, but the opposite to (4.23) holds:

$$\min \{\operatorname{diam} \Pi\mathbf{x}, \operatorname{diam} \Pi\mathbf{y}\} \le L_j^2.$$

Without loss of generality, we can assume that $\operatorname{diam} \Pi\mathbf{x} \le L_j^2$, for the roles of $\mathbf{x}$ and $\mathbf{y}$ in (4.22)–(4.23) are symmetric. Then, for at least one value $k' \in \{1, 2\}$, we have $\mathrm{d}(y_{k'}, \Pi\mathbf{x}) = \mathrm{d}(\Pi\mathbf{y}, \Pi\mathbf{x}) > 2L_j^2$; otherwise, we would have a contradiction:

$$8L_j^2 \le \operatorname{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) \le \mathrm{d}(y_1, \Pi\mathbf{x}) + \operatorname{diam} \Pi\mathbf{x} + \mathrm{d}(\Pi\mathbf{x}, y_2) \le L_j^2 + 4L_j^2.$$

Letting $k'' = 2 - k'$, so that $\{k', k''\} = \{1, 2\}$ and $\Pi\mathbf{y} = \{y_{k'}, y_{k''}\}$, consider the following two alternatives.

Case 3a. $\mathrm{d}(y_{k''}, \Pi\mathbf{x}) > L_j^2$. One has $|\mathbf{y} - \mathbf{x}| > L_j^2$, so $(\operatorname{supp} \boldsymbol{\varphi}_{\mathbf{x}}) \cap (\operatorname{supp} \boldsymbol{\varphi}_{\mathbf{y}}) = \varnothing$ by (K7) (cf. (4.11)), thus $\mathbf{C}^j_{\mathbf{yx}} = \langle \boldsymbol{\varphi}^j_{\mathbf{y}} \mid \boldsymbol{\varphi}^j_{\mathbf{x}} \rangle = 0$.

Case 3b. $\mathrm{d}(y_{k''}, \Pi\mathbf{x}) \le L_j^2$. One has

$$\operatorname{diam} \Pi\mathbf{y} = |y_{k'} - y_{k''}| \ge \mathrm{d}(y_{k'}, \Pi\mathbf{x}) - \mathrm{d}(y_{k''}, \Pi\mathbf{x}) \ge 2L_j^2 - L_j^2 = L_j^2,$$

hence $\boldsymbol{\varphi}^j_{\mathbf{y}} = \varphi^j_{y_1} \otimes \varphi^j_{y_2}$, by (K1). Expand $\boldsymbol{\varphi}^j_{\mathbf{x}}$ in a finite sum,

$$\boldsymbol{\varphi}^j_{\mathbf{x}} = \sum_{\mathbf{z}} \boldsymbol{\varphi}^j_{\mathbf{x}}(\mathbf{z})\, \mathbf{1}_{\mathbf{z}} = \sum_{\mathbf{z} \in \mathcal{S}_{\mathbf{x}}} \boldsymbol{\varphi}^j_{\mathbf{x}}(\mathbf{z})\, \mathbf{1}_{z_1} \otimes \mathbf{1}_{z_2}\,, \ \mathcal{S}_{\mathbf{x}} := \operatorname{supp} \boldsymbol{\varphi}^j_{\mathbf{x}},$$

and observe that

$$\mathrm{d}(y_{k'}, \Pi\mathcal{S}_{\mathbf{x}}) \ge \mathrm{d}(y_{k'}, \Pi\mathbf{x}) - \operatorname{diam} \operatorname{supp} \boldsymbol{\varphi}^j_{\mathbf{x}} \ge L_j^2 - L_j \ge \tfrac{1}{2} L_j^2.$$

Since $\{k', k''\} = \{1, 2\}$, we can write $\Pi\mathbf{z} = \{z_1, z_2\} = \{z_{k'}, z_{k''}\}$. Furthermore, $\mathbf{z} \in \mathcal{S}_{\mathbf{x}}$ implies $\Pi\mathbf{z} = \{z_{k'}, z_{k''}\} \subset \Pi\mathcal{S}_{\mathbf{x}}$, thus

$$\begin{aligned} \mathbf{C}^j_{\mathbf{yx}} &= \langle \boldsymbol{\varphi}^j_{\mathbf{y}} \mid \boldsymbol{\varphi}^j_{\mathbf{x}} \rangle = \sum_{\mathbf{z} \in \mathcal{S}_{\mathbf{x}}} \boldsymbol{\varphi}^j_{\mathbf{x}}(\mathbf{z}) \langle \varphi^j_{y_1} \otimes \varphi^j_{y_2} \mid \mathbf{1}_{z_1} \otimes \mathbf{1}_{z_2} \rangle \\ &= \sum_{\mathbf{z} \in \mathcal{S}_{\mathbf{x}}} \boldsymbol{\varphi}^j_{\mathbf{x}}(\mathbf{z}) \varphi^j_{y_{k'}}(z_{k'}) \varphi^j_{y_{k''}}(z_{k''}) = 0, \end{aligned}$$

for we have $|y_{k'} - z| > \frac{1}{2} L_j^2 > \operatorname{diam} \operatorname{supp} \varphi^j_{y_{k'}}$ for all $z \in \Pi\mathcal{S}_{\mathbf{x}}$.

Conclusion of Step 1. Collecting the estimates of $|\mathbf{C}^j_{\mathbf{yx}}|$ obtained in Cases 1–3, we have $\mathbf{C}^j = \mathbf{1} + \mathbf{D}^j$ where, on account of $\epsilon_j = \epsilon_0^{q^j}$ and $L_j = L_0 q^j$,

$$|||\mathbf{D}^j||| = \sup_{\mathbf{z}\in\mathcal{Z}_2} \sum_{\mathbf{t}\neq\mathbf{z}:\mathbf{C}^j_{\mathbf{tz}}\neq 0} \mathrm{e}^{m|z|}|\mathbf{C}^j_{\mathbf{tz}}| \lesssim L_j^2 \epsilon_j^{1^+} \le \epsilon_j^{1^+}. \tag{4.25}$$

Thus, it follows from $((\boldsymbol{\Phi}^j)^\top - (\boldsymbol{\Phi}^j)^{-1})\boldsymbol{\Phi}^j = \mathbf{D}^j$ and $|||(\boldsymbol{\Phi}^j)^{-1}||| \le 2$ that

$$(\boldsymbol{\Phi}^j)^\top - (\boldsymbol{\Phi}^j)^{-1} = \mathbf{D}^j(\boldsymbol{\Phi}^j)^{-1}, \tag{4.26}$$

$$|||(\boldsymbol{\Phi}^j)^\top - (\boldsymbol{\Phi}^j)^{-1}||| \le 2|||\mathbf{D}^j||| \le \epsilon_j^{1^+}. \tag{4.27}$$

By (K3), the matrix $\boldsymbol{\Phi}^j = \mathbf{1} + \widetilde{\mathbf{D}}^j$ is invertible by Neumann series, and

$$|||(\boldsymbol{\Phi}^j)^{-1} - \mathbf{1}||| \le \sum_{k\ge 1} |||\widetilde{\mathbf{D}}^j|||^k \le \frac{1}{3},$$

whence

$$\max_{0\le i\le j} \max[|||\boldsymbol{\Phi}^i|||, |||(\boldsymbol{\Phi}^i)^\top|||, |||(\boldsymbol{\Phi}^i)^{-1}|||] \le 2. \tag{4.28}$$

Step 2. *Expansion of the discrepancy vectors.* Introduce a matrix

$$\widetilde{\mathbf{Q}}^j_{zx} = (\boldsymbol{\Phi}^j)^\top \boldsymbol{\Psi}^j, \quad \widetilde{\mathbf{Q}}^j_{\mathbf{y},\mathbf{x}} = \langle \varphi^j_{\mathbf{y}} \mid \psi^j_{\mathbf{x}} \rangle, \tag{4.29}$$

serving as a convenient approximant for $\mathbf{F}^j = (\boldsymbol{\Phi}^j)^{-1}\boldsymbol{\Psi}^j$ with $|||\cdot|||$-accuracy of $\epsilon_j^{1^+}$ (cf. (4.26)–(4.27)), and its truncated version $\mathbf{Q}^j$ given by

$$\mathbf{Q}^j_{\mathbf{yx}} := \widetilde{\mathbf{Q}}^j_{\mathbf{yx}} \mathbf{1}_{|\mathbf{y}-\mathbf{x}|\le cL_j} \tag{4.30}$$

with $0 < c < \frac{1}{3}$, e.g., $c = \frac{1}{4}$; the exact value is of little importance. Note that

$$\forall C, c > 0 \;\; \forall m \ge \frac{C \ln \epsilon_0^{-1}}{cL_0} \quad \mathrm{e}^{-mcL_i} \le (\epsilon_0^{q^i})^C = \epsilon_i^C.$$

so for any $c > 0$, the norm $|||\mathbf{Q}^j - \widetilde{\mathbf{Q}}^j|||$ can be made smaller than, say, ϵ_j^4. Now, we can assess the $|||\cdot|||$-norm and the spread of the truncated matrix $\mathbf{Q}^j$:

$$|||\mathbf{Q}^j||| \le |||\widetilde{\mathbf{Q}}^j||| \le |||(\boldsymbol{\Phi}^j)^\top||| \cdot |||\boldsymbol{\Psi}^j||| \le \epsilon_j^{1^+}, \tag{4.31}$$

$$\textsc{Spr}[\mathbf{Q}^j] \le cL_{j+1}.$$

On account of (4.28), we have a similar $|||\cdot|||$-norm bound on $\mathbf{F}^j$:

$$|||\mathbf{F}^j||| \le |||(\boldsymbol{\Phi}^j)^{-1}||| \cdot |||\boldsymbol{\Psi}^j||| \le \epsilon_j^{1^+}. \tag{4.32}$$

Step 3. *new basis.* Due to the length scale growth, $L_j \rightsquigarrow L_{j+1} = qL_j$ with $q > 1$, we must redefine the area $\mathcal{Z}_2^{j,\mathrm{fact}} \subset \mathcal{Z}_2$ where the AEF $\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}$ with $\mathbf{x} \in \mathcal{Z}_2^{j,\mathrm{fact}}$ can be chosen in a factorized form, $\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}(\mathbf{y}) = f(y_1) \otimes g(y_2)$, with the precision required on the step $j+1$. As before, we prefer to be on the safe side and do not define $\mathcal{Z}_2^{j,\mathrm{fact}}$ in an optimal way: for some $\mathbf{x} \in \mathcal{Z}_2 \setminus \mathcal{Z}_2^{j,\mathrm{fact}}$, it may or might be possible, too, to construct $\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}$ in a tensor-product form. However, as we shall see in the Case 2 below, an alternative procedure applies equally well to the actually entangled and factorizable AEF with localization centers $\mathbf{x}$ having a reasonably bounded diam $\Pi\mathbf{x}$.

The principal reason for a special treatment of $\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}$ with diam $\Pi\mathbf{x}$ "too large" is that, for $|x_2 - x_1|$ large enough, the phase points $T^{x_2}\omega$ and $T^{x_1}\omega$ may too close to each other, resulting in abnormally small denominators that one would be unable to avoid by ϑ-parameter exclusion (cf. Step 9 and Section 7). The specifics will become clear when we turn to the analysis of the Case 2.

Case 1. diam $\Pi\mathbf{x} > 4L_{j+1}$ *(tensor-factorized states).* By (K1), the AEF $\boldsymbol{\varphi}_{\mathbf{x}}^{j}$ have the form $\varphi_{x_1}^{j} \otimes \varphi_{x_2}^{j}$. Define the new AEF and AEV,

$$\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} = \varphi_{x_1}^{j+1} \otimes \varphi_{x_2}^{j+1}, \quad \boldsymbol{\lambda}_{\mathbf{x}}^{j+1} = \lambda_{x_1}^{j+1} + \lambda_{x_2}^{j+1}, \tag{4.33}$$

where $(\varphi_z^{j+1}, \lambda_z^{j+1})$ are the approximate eigenpairs of the 1-particle Hamiltonian $H_\varepsilon = -\varepsilon\Delta + V(\cdot,\omega,\vartheta)$ constructed on the step $j+1$ of the KAM induction, carried in essentially the same (but simpler) way as we do for the 2-particle Hamiltonian. Formally, speaking, one should have provided here a parallel scale induction, but this task has already been completed in [11, 13], and it only remains to assess the discrepancy terms $\boldsymbol{\psi}_{\mathbf{x}}^{j+1}$ relative to $(\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}, \boldsymbol{\lambda}_{\mathbf{x}}^{j+1})$. To that end, we use the bounds on the 1-particle discrepancies $\psi_\bullet^{j+1}$ available from [11, 13]. Since $|x_2 - x_1| > 4L_{j+1}$, one has $\mathrm{u}(|z_1 - z_2|) = 0$ for all pairs (z_1, z_2) with $z_1, z_2 \in \operatorname{supp} \varphi_{x_k}^{j+1}$, whence (cf. (1.5))

$$\mathbf{H}\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} = (-\boldsymbol{\Delta} + \mathbf{V})\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} = (H^1 \otimes \mathbf{1} + \mathbf{1} \otimes H^1)\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}.$$

Therefore, on account of (4.33), we have

$$(\mathbf{H} - \boldsymbol{\lambda}_{\mathbf{x}}^{j+1})\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} = \boldsymbol{\psi}_{\mathbf{x}}^{j+1} := \psi_{x_1}^{j+1} \otimes \varphi_{x_2}^{j+1} + \varphi_{x_1}^{j+1} \otimes \psi_{x_2}^{j+1}. \tag{4.34}$$

The norm-bound on ψ_z^{j+1} of the form (K5) (cf. (4.9)) is proved as in [11]. It is worth mentioning that the bound *actually proved* there was (cf. [11, Eqn. (3.41)])

$$|||\psi_z^{j+1}||| \le \epsilon_j^{2-} \ll \epsilon_{j+1}^{1+},$$

although it was *stated* in a weaker form. Since $\|\varphi_\bullet^{j+1}\| = 1$, we infer from (4.34)

$$|||\boldsymbol{\psi}_{\mathbf{x}}^{j+1}||| \le 2\epsilon_{j+1}^{1+} \le \epsilon_{j+1}^{1+},$$

absorbing, once again, the factor 2 in an implicit exponent 1^+.

Case 2. $\operatorname{diam}\Pi\mathbf{x} \le 4L_{j+1}$ *((possibly) entangled states).* Define a matrix $\mathbf{M}^{j+1}$, setting $\mathbf{M}^{j+1}_{\mathbf{yx}} := 0$ if $\mathbf{y} = \mathbf{x}$ or $\mathbf{Q}^{j}_{\mathbf{yx}} = 0$ (which occurs for distant $\mathbf{x}$ and $\mathbf{y}$), otherwise

$$\mathbf{M}^{j+1}_{\mathbf{yx}} := (\boldsymbol{\lambda}^{j}_{\mathbf{x}} - \boldsymbol{\lambda}^{j}_{\mathbf{y}})^{-1}\mathbf{Q}^{j}_{\mathbf{yx}}. \tag{4.35}$$

The entries $\mathbf{M}^{j+1}_{\mathbf{yx}}$ in (4.35) are indeed well defined, owing to the hypothesis (K8) (cf. (4.13)). (Actually, the possibility of having uncontrollable small denominators in (4.35) is always eliminated on Step 8 well before they appear for the first time in the definition of $\mathbf{M}^{j+1}_{\mathbf{yx}}$.) Clearly, the matrix $\mathbf{M}^{j+1}$ defines an operator on the space of compactly supported functions on $\boldsymbol{\mathcal{Z}}_{\mathbf{2}}$, on which one has (cf. (4.13) and (4.31))

$$\begin{aligned} |||\mathbf{M}^{j+1}||| &\le \epsilon_j^{1^+}\delta_j^{-1} \le \epsilon_j^{1^+}, \\ \mathrm{S{\small PR}}[\mathbf{M}^{j+1}] &= \mathrm{S{\small PR}}[\mathbf{Q}^{j}] \le cL_{j+1}, \end{aligned} \tag{4.36}$$

so $\mathbf{M}^{j+1}$ defines also a bounded operator in $\ell^2(\boldsymbol{\mathcal{Z}}_{\mathbf{2}})$. The columns of the matrix

$$\tilde{\boldsymbol{\Phi}}^{j+1} := \boldsymbol{\Phi}^{j}\left(\mathbf{1} + \mathbf{M}^{j+1}\right)$$

form a Riesz basis in $\ell^2(\boldsymbol{\mathcal{Z}}_{\mathbf{2}})$, because both $\boldsymbol{\Phi}^{j}$ and $(\mathbf{1} + \mathbf{M}^{j+1})$ are boundedly invertible. Denoting these column-vectors by $\tilde{\boldsymbol{\varphi}}^{j+1}_{\bullet}$, we have

$$\tilde{\boldsymbol{\varphi}}^{j+1}_{x} = \boldsymbol{\varphi}^{j}_{x} + \sum_{\mathbf{z}\neq\mathbf{x}} \mathbf{M}^{j+1}_{\mathbf{zx}}\boldsymbol{\varphi}^{j}_{\mathbf{z}}, \quad \mathbf{x} \in \boldsymbol{\mathcal{Z}}_{\mathbf{2}}. \tag{4.37}$$

Normalization of $\tilde{\boldsymbol{\varphi}}^{j+1}_{\bullet}$, producing $\boldsymbol{\varphi}^{j+1}_{\bullet}$, is performed on Step 7. By the inequalities $\mathrm{S{\small PR}}[AB] \le \mathrm{S{\small PR}}[A] + \mathrm{S{\small PR}}[B]$ and $\mathrm{S{\small PR}}[A + B] \le \max[\mathrm{S{\small PR}}[A], \mathrm{S{\small PR}}[B]]$, one has

$$\begin{aligned} \mathrm{S{\small PR}}[\tilde{\boldsymbol{\Phi}}^{j+1}] &\le \mathrm{S{\small PR}}[\boldsymbol{\Phi}^{j}] + \mathrm{S{\small PR}}[\mathbf{M}^{j+1}] \le 2cL_j + 1 \le 3cL_j < L_{j+1}, \\ \operatorname{supp}\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}} &\cup \mathcal{S}[\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}] \subset \mathrm{B}_{3cL_j}(\mathbf{x}) < L_{j+1}. \end{aligned} \tag{4.38}$$

with $c < 1/3$. By expansion in the Neumann series, convergent by (4.36), we have

$$(\mathbf{1} + \mathbf{M}^{j+1})^{-1} = \mathbf{1} - \mathbf{M}^{j+1} + (\mathbf{M}^{j+1})^2 - (\mathbf{M}^{j+1})^3(1 + \mathbf{M}^{j+1})^{-1}, \tag{4.39}$$

so the inverse $(\mathbf{1} + \mathbf{M}^{j+1})^{-1}$ can be replaced with $\mathbf{1} - \mathbf{M}^{j+1} + (\mathbf{M}^{j+1})^2$ with accuracy $O[|||\mathbf{M}^{j+1}|||^3]$. The explicit inversion formula (4.39) will be used later.

Step 4. *action of* $\mathbf{H} \equiv \mathbf{H}_\varepsilon$ *on* $\tilde{\boldsymbol{\varphi}}^{j+1}_{\bullet}$. By definition of $\tilde{\mathbf{Q}}^{j}$ and $\mathbf{D}^{j}$, we have

$$\mathrm{Ad}_{\boldsymbol{\Phi}^{j}}[\mathbf{H}] = \boldsymbol{\Lambda}^{j} + \tilde{\mathbf{Q}}^{j} - \mathbf{D}^{j}\mathbf{F}^{j}.$$

Straightforward calculations making use of the identities $[\boldsymbol{\Lambda}^{j}, \mathbf{M}^{j+1}] = -\mathbf{Q}^{j}$ (cf. (4.35)) and (4.39), give rise to the representation

$$\mathrm{Ad}_{\tilde{\boldsymbol{\Phi}}^{j+1}}[\mathbf{H}] = (\mathbf{1} + \mathbf{M}^{j+1})^{-1}(\boldsymbol{\Phi}^{j})^{-1}\mathbf{H}\boldsymbol{\Phi}^{j}(\mathbf{1} + \mathbf{M}^{j+1}) = \boldsymbol{\Lambda}^{j} + \mathbf{W}^{j+1} + \mathbf{Z}^{j+1},$$

where

$$\begin{aligned}\mathbf{W}^{j+1} &= [\mathbf{Q}^j, \mathbf{M}^{j+1}] + \mathbf{D}^j\mathbf{F}^j + (\mathbf{M}^{j+1})^2\mathbf{\Lambda}^j - \mathbf{M}^{j+1}\mathbf{\Lambda}^j\mathbf{M}^{j+1}, \qquad (4.40)\\ \mathbf{Z}^{j+1} &= -\mathbf{M}^{j+1}\mathbf{Q}^j\mathbf{M}^{j+1} + [\mathbf{D}^j\mathbf{F}^j, \mathbf{M}^{j+1}] - \mathbf{M}^{j+1}\mathbf{D}^j\mathbf{F}^j\mathbf{M}^{j+1}\\ &\quad + (\mathbf{M}^{j+1})^2\mathbf{\Lambda}^j\mathbf{M}^{j+1} - (\mathbf{M}^{j+1})^2\mathbf{D}^j\mathbf{F}^j(\mathbf{1}+\mathbf{M}^{j+1})\\ &\quad - (\mathbf{M}^{j+1})^3(\mathbf{1}+\mathbf{M}^{j+1})^{-1}(\mathbf{\Lambda}^j + \widetilde{\mathbf{Q}}^j - \mathbf{D}^j\mathbf{F}^j)(\mathbf{1}+\mathbf{M}^{j+1}). \qquad (4.41)\end{aligned}$$

Equivalently,

$$(\widetilde{\mathbf{\Phi}}^{j+1})^{-1}\mathbf{H}\widetilde{\mathbf{\Phi}}^{j+1} = \mathbf{\Lambda}^{j+1} + \mathbf{F}^{j+1}, \qquad (4.42)$$

with $\mathbf{\Lambda}^{j+1}$ and $\mathbf{F}^{j+1}$ defined by their matrix elements:

$$\begin{aligned}\mathbf{\Lambda}^{j+1}_{\mathbf{yx}} &= \mathbf{\Lambda}^{j}_{\mathbf{yx}} + \delta_{\mathbf{yx}}\mathbf{W}^{j+1}_{\mathbf{xx}}, \qquad (4.43)\\ \mathbf{F}^{j+1}_{\mathbf{yx}} &= (1-\delta_{\mathbf{yx}})\mathbf{W}^{j+1}_{\mathbf{yx}} + \mathbf{Z}^{j+1}_{\mathbf{yx}}. \qquad (4.44)\end{aligned}$$

Now, we are ready to define the new approximate eigenvalues $\boldsymbol{\lambda}^{j+1}_{\bullet}$:

$$\boldsymbol{\lambda}^{j+1}_{\mathbf{x}} := \mathbf{\Lambda}^{j+1}_{\mathbf{xx}} = \boldsymbol{\lambda}^{j}_{\mathbf{x}} + \mathbf{W}^{j+1}_{\mathbf{xx}}. \qquad (4.45)$$

It follows from (4.31), (4.36), (4.21), and (4.32) that

$$|||\mathbf{W}^{j+1}||| \lesssim |||\mathbf{Q}^j|||\,|||\mathbf{M}^{j+1}||| + |||\mathbf{\Lambda}^j|||\,|||\mathbf{M}^{j+1}|||^2 + |||\mathbf{D}^j|||\,|||\mathbf{F}^j||| \le \epsilon_j^{2^+}, \qquad (4.46)$$

thus

$$\sup_{\mathbf{x}} |\boldsymbol{\lambda}^{j+1}_{\mathbf{x}} - \boldsymbol{\lambda}^{j}_{\mathbf{x}}| \le \sup_{\mathbf{x}} |\mathbf{W}^{j+1}_{\mathbf{xx}}| \le \epsilon_j^{2^+} \le \epsilon_{j+1}^{\frac{4}{3}^+}.$$

An equivalent form of (4.42) is

$$\mathbf{H}\widetilde{\mathbf{\Phi}}^{j+1} = \widetilde{\mathbf{\Phi}}^{j+1}\mathbf{\Lambda}^{j+1} + \mathbf{\Psi}^{j+1}, \quad \mathbf{\Psi}^{j+1} := \widetilde{\mathbf{\Phi}}^{j+1}\mathbf{F}^{j+1}, \qquad (4.47)$$

and on account of $|||\widetilde{\mathbf{\Phi}}^{j+1}||| \le 2$, one has

$$|||\widetilde{\mathbf{\Phi}}^{j+1}\mathbf{Z}^{j+1}||| \lesssim |||\mathbf{F}^{j+1}|||\,|||\mathbf{M}^{j+1}|||^2 + |||\mathbf{F}^{j+1}|||\,|||\mathbf{D}^j|||\,|||\mathbf{M}^{j+1}||| + |||\mathbf{M}^{j+1}|||^3.$$

Since $\mathbf{\Psi}^{j+1} = \mathbf{H}\widetilde{\mathbf{\Phi}}^{j+1} - \widetilde{\mathbf{\Phi}}^{j+1}\mathbf{\Lambda}^{j+1}$, where $\mathrm{SPR}[\mathbf{H}] = 1$, $\mathrm{SPR}[\mathbf{\Lambda}^{j+1}] = 0$, we have

$$\mathrm{SPR}[\mathbf{\Psi}^{j+1}] \le \mathrm{SPR}[\widetilde{\mathbf{\Phi}}^{j+1}] + 1.$$

Step 5. *norm of the discrepancy.* Collecting the bounds $|||\widetilde{\mathbf{\Phi}}^{j+1}||| \le 2$, $|||\mathbf{Q}^j||| \le \epsilon_j^{1^+}$ (cf. (4.31)), $|||\mathbf{M}^{j+1}||| \le \epsilon_j^{1^+}$ (cf. (4.36)), $|||\mathbf{D}^j||| \le \epsilon_j^{1^+}$ (cf. (4.25)), $|||\mathbf{F}^j||| \le \epsilon_j^{1^+}$ (cf. (4.32)), and $|||\mathbf{F}^{j+1}||| \le |||\mathbf{W}^{j+1}||| + |||\mathbf{Z}^{j+1}|||$, we get

$$|||\mathbf{Z}^{j+1}||| \lesssim |||\mathbf{D}^j|||\,|||\mathbf{F}^j|||\,|||\mathbf{M}^{j+1}||| + |||\mathbf{M}^{j+1}|||^2|||\mathbf{F}^j||| + |||\mathbf{M}^{j+1}|||^3 \le \epsilon_j^{3^+}. \qquad (4.48)$$

Recalling $\mathbf{F}^{j+1}_{yx} = (1-\delta_{yx})\mathbf{W}^{j+1}_{yx} + \mathbf{Z}^{j+1}_{yx}$ (cf. (4.44)) and (4.46), it follows that

$$|||\mathbf{F}^{j+1}||| \le \epsilon_j^{2^+}, \quad |||\mathbf{\Psi}^{j+1}||| = |||\widetilde{\mathbf{\Phi}}^{j+1}\mathbf{F}^{j+1}||| \le \epsilon_j^{2^+} \le \epsilon_{j+1}^{\frac{4}{3}^+}.$$

Step 6. *perturbations of the AEF.* By $\widetilde{\mathbf{\Phi}}^{j+1} = \mathbf{\Phi}^j(\mathbf{1} + \mathbf{M}^{j+1})$, we have

$$\|\widetilde{\mathbf{\Phi}}^{j+1} - \mathbf{\Phi}^j\| \le \|\mathbf{\Phi}^j\| \|\mathbf{M}^{j+1}\| \le 2\|\mathbf{M}^{j+1}\| \le \epsilon_j^{1+}, \tag{4.49a}$$
$$\|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{x}}^j\|_{\mathbf{x}} \le \epsilon_j^{1+}. \tag{4.49b}$$

Now, we prepare for the proof of the assertion (4.10) with $i = j + 1$ (to be completed on Step 7). By definition of $\boldsymbol{\psi}_{\bullet}^{j+1}$ and $\mathbf{F}^{j+1}$ (cf. (4.47) and (4.43)),

$$\boldsymbol{\psi}_{\mathbf{x}}^{j+1} = \sum_{\mathbf{z}} ((1 - \delta_{\mathbf{zx}} \mathbf{W}_{\mathbf{zx}}^{j+1}) + \mathbf{Z}_{\mathbf{zx}}^{j+1}) \tilde{\boldsymbol{\varphi}}_{\mathbf{z}}^{j+1}.$$

Since $\|\tilde{\boldsymbol{\varphi}}_{\mathbf{y}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{y}}^j\|_{\mathbf{x}} \le \epsilon_j^{1-}$ for all $\mathbf{y} \in \mathcal{Z}_2$, and $|(\boldsymbol{\varphi}_{\mathbf{x}}^j, \boldsymbol{\varphi}_{\mathbf{z}}^j)| \le \epsilon_j^{1+}$ for $\mathbf{z} \neq \mathbf{x}$ by (4.20), we also have $|\langle \tilde{\boldsymbol{\varphi}}_x^{j+1} \mid \tilde{\boldsymbol{\varphi}}_z^{j+1} \rangle| \le \epsilon_j^{1+}$, yielding

$$\left| \sum_{\mathbf{z}} (1 - \delta_{\mathbf{zx}}) \mathbf{W}_{\mathbf{zx}}^{j+1} (\tilde{\boldsymbol{\varphi}}_{\mathbf{z}}^{j+1}, \tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}) \right| \lesssim L_j^d \|\mathbf{W}^{j+1}\| \epsilon_j^{1+} \le \epsilon_j^{3+} \le \epsilon_{j+1}^{2+}. \tag{4.50}$$

By the norm estimate (4.48), we have, with $\epsilon_{j+1} = \epsilon_j^q = \epsilon_j^{3/2}$,

$$\left| \sum_{\mathbf{z}} \mathbf{Z}_{\mathbf{zx}}^{j+1} \langle \tilde{\boldsymbol{\varphi}}_{\mathbf{z}}^{j+1} \mid \tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} \rangle \right| \lesssim L_j^d \|\mathbf{Z}^{j+1}\| \le \epsilon_j^{3+} \le \epsilon_{j+1}^{2+}. \tag{4.51}$$

Collecting (4.50)–(4.51), we get $|\langle \boldsymbol{\psi}_x^{j+1} \mid \tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} \rangle| \le \epsilon_{j+1}^{2+}$. The proof of (4.10) will be completed at the next step.

Step 7. *normalization of the ACE.* Introduce the normalized AEF

$$\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} := \|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\|^{-1} \tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}$$

(as before, $\|\cdot\|$ stands for the ℓ^2-norm). Thus, $\mathcal{S}[\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}] = \mathcal{S}[\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}] \subset \mathrm{B}_{L_{j+1}}(\mathbf{x})$. Since $\|\boldsymbol{\varphi}_{\bullet}^j\| = 1$, we have

$$\|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\|^2 - 1 = 2(\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{x}}^j, \boldsymbol{\varphi}_{\mathbf{x}}^j) + \|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{x}}^j\|^2.$$

Recalling $(1 - \delta_{\mathbf{zx}}) \mathbf{C}_{\mathbf{zx}}^j = O[\epsilon_j^{1+}]$, it follows from (4.37) and (4.24) that

$$|(\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{x}}^j, \boldsymbol{\varphi}_{\mathbf{x}}^j)| \le \sum_{\mathbf{z} \neq \mathbf{x}} |\mathbf{M}_{\mathbf{zx}}^{j+1}| |\mathbf{C}_{\mathbf{zx}}^j| \lesssim L_j^d \epsilon_j^{1+} \cdot \epsilon_j^{1+} \le \epsilon_j^2,$$

and $\|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{x}}^j\| \le \epsilon_j^{1+}$ (cf. (4.49)), thus $|\|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\| - 1| \le \epsilon_j^{2-}$, and (4.10) with $i = j + 1$ follows from the bounds

$$\|\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} - \tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\|_{\mathbf{x}} \le \epsilon_j^{2+}, \tag{4.52a}$$
$$\|\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{x}}^j\|_{\mathbf{x}} \le \|\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} - \tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\|_{\mathbf{x}} + \|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{x}}^j\|_{\mathbf{x}} \le \epsilon_j^{2+}, \tag{4.52b}$$
$$|\langle \boldsymbol{\psi}_{\mathbf{x}}^{j+1} \mid \boldsymbol{\varphi}_{\mathbf{x}}^{j+1} \rangle| \le \epsilon_{j+1}^{2-}. \tag{4.52c}$$

Step 8. The assertion (K2) is proved in the same way as in [11, Section III, Step 8].

Step 9. *The spectral spacings.* Fix a pair of configurations $\mathbf{x}, \mathbf{y} \in \boldsymbol{\mathcal{Z}}_{\mathbf{2}}$ such that one has $\operatorname{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) \le L_{j+1}^2$, and consider the following two alternatives.

Case I. $L_j^2 < \operatorname{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) \le L_{j+1}^2$. In this case, we define the moment $\hat{\mathfrak{l}}(\mathbf{x}, \mathbf{y}) := j + 1$ when a lower bound on $|\boldsymbol{\lambda}_{\mathbf{x}}^{\bullet} - \boldsymbol{\lambda}_{\mathbf{y}}^{\bullet}|$ is established for the first time, at the price of exclusion of some subset of Θ. Before this moment, i.e., for $0 \le i < \hat{\mathfrak{l}}(\mathbf{x}, \mathbf{y})$, we have no effective control of $|\boldsymbol{\lambda}_{\mathbf{x}}^{i} - \boldsymbol{\lambda}_{\mathbf{y}}^{i}|$: the latter may be abnormally small. On the other hand, $|\boldsymbol{\lambda}_{\mathbf{x}}^{i} - \boldsymbol{\lambda}_{\mathbf{y}}^{i}|^{-1}$ never appears in the inductive procedure until the moment $\hat{\mathfrak{l}}(\mathbf{x}, \mathbf{y})$.

As discussed in Case II below, the value of the spacing $|\boldsymbol{\lambda}_{\mathbf{x}}^{\hat{\mathfrak{l}}(\mathbf{x},\mathbf{y})} - \boldsymbol{\lambda}_{\mathbf{y}}^{\hat{\mathfrak{l}}(\mathbf{x},\mathbf{y})}|$ is essentially preserved on all subsequent induction steps $j > \hat{\mathfrak{l}}(\mathbf{x}, \mathbf{y})$. More precisely, denoting temporarily for brevity $\hat{\mathfrak{l}} = \hat{\mathfrak{l}}(\mathbf{x}, \mathbf{y})$, one can guarantee that

$$\forall j > \hat{\mathfrak{l}} \quad |\boldsymbol{\lambda}_{\mathbf{x}}^{j} - \boldsymbol{\lambda}_{\mathbf{y}}^{j}| \ge |\boldsymbol{\lambda}_{\mathbf{x}}^{\hat{\mathfrak{l}}} - \boldsymbol{\lambda}_{\mathbf{y}}^{\hat{\mathfrak{l}}}| - \mathrm{o}[\delta_{\hat{\mathfrak{l}}}], \tag{4.53}$$

provided $\varepsilon > 0$ is small enough. To make the right-hand side of inequality (4.53) compatible with hypothesis (K8) for all $j \ge \hat{\mathfrak{l}}$, we shall exclude from Θ a subset on which a stronger lower bound $|\boldsymbol{\lambda}_{\mathbf{x}}^{\hat{\mathfrak{l}}} - \boldsymbol{\lambda}_{\mathbf{y}}^{\hat{\mathfrak{l}}}| > 5\delta_{\hat{\mathfrak{l}}}$ fails, and assess the $\mathbb{P}^{\Theta}$-probability of this *larger* subset to be excluded.

This task requires some deviation from the functional-analytic flow of arguments presented in this section, and so we postpone it to Section 7. Specifically, it follows from Theorem 7.2 that there exists a subset $\overline{\Theta}^{j+1} = \Theta \setminus \Theta^{j+1} \in \mathfrak{B}_{\le \hat{\mathfrak{n}}_j}$ such that

$$\forall \vartheta \in \widehat{\Theta}^{j} \cap \Theta^{j+1} \quad |\boldsymbol{\lambda}_{\mathbf{x}}^{j+1} - \boldsymbol{\lambda}_{\mathbf{y}}^{j+1}| > 5\delta_{j+1}, \tag{4.54a}$$

$$\mathbb{P}^{\Theta}\{\overline{\Theta}^{j+1}\} \le \epsilon_{j+1}^{0+}. \tag{4.54b}$$

This completes the treatment of Case I, and, as was said, makes a sufficient provision for the bounds $|\boldsymbol{\lambda}_{\mathbf{x}}^{i} - \boldsymbol{\lambda}_{\mathbf{y}}^{i}| \ge 4\delta_i$, for all $i \ge j + 1$, as per (K8).

Remark 4.2. It is worth emphasizing that the proof Theorem 7.2 relies on the parametric analysis carried out in Section 6, and the parametric smoothness properties of the approximate eigenpairs $(\boldsymbol{\varphi}_{\bullet}^{i}, \boldsymbol{\lambda}_{\bullet}^{i})$ are also proved by induction in $i \ge 0$. Further, we shall see that the induction in Section 6 can be carried out in parallel with the one in the present Section 4, and it is driven by latter: once the eigenpairs $(\boldsymbol{\varphi}_{\bullet}^{i}, \boldsymbol{\lambda}_{\bullet}^{i})$ are constructed for $0 \le i \le j$ (viz. Steps 1–8), their smoothness properties can be established as in Sections 6–7, and so the Step 9 can be completed, too. Note also that the recourse to the parametric smoothness analysis is not required in the 1-particle model for the proof of uniform localization per se, as it can be replaced with simpler arguments (cf. [11]). However, already the case of $\mathfrak{N} = 2$ particles is more challenging.

Case II. $\mathrm{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) \le L_j^2$. In this case, the difference between the AEV associated with the configurations $\mathbf{x}$ and $\mathbf{y}$ has already been assessed on some previous induction step $\hat{\mathrm{l}}(\mathbf{x},\mathbf{y}) \le j$, falling in the category treated in Case I. After the exclusion of a suitable subset of Θ, one has $|\boldsymbol{\lambda}_{\mathbf{x}}^{\hat{\mathrm{l}}}(\omega,\vartheta) - \boldsymbol{\lambda}_{\mathbf{y}}^{\hat{\mathrm{l}}}|(\omega,\vartheta) > 5\delta_{\hat{\mathrm{l}}}$ for all remaining $\vartheta \in \Theta$. Therefore, by the bounds on the perturbations $|\boldsymbol{\lambda}_{\mathbf{x}}^{j+1} - \boldsymbol{\lambda}_{\mathbf{x}}^{j}|$ and $|\boldsymbol{\lambda}_{\mathbf{y}}^{j+1} - \boldsymbol{\lambda}_{\mathbf{y}}^{j}|$, we get

$$
\begin{aligned}
|\boldsymbol{\lambda}_{\mathbf{x}}^{j+1} - \boldsymbol{\lambda}_{\mathbf{y}}^{j+1}| &\ge |\boldsymbol{\lambda}_{\mathbf{x}}^{\hat{\mathrm{l}}} - \boldsymbol{\lambda}_{\mathbf{y}}^{\hat{\mathrm{l}}}| - \sum_{\hat{\mathrm{l}}(\mathbf{x},\mathbf{y}) \le i \le j} (|\boldsymbol{\lambda}_{\mathbf{x}}^{i+1} - \boldsymbol{\lambda}_{\mathbf{x}}^{i}| + |\boldsymbol{\lambda}_{\mathbf{y}}^{i+1} - \boldsymbol{\lambda}_{\mathbf{y}}^{i}|) \\
&\ge 5\delta_{\hat{\mathrm{l}}(\mathbf{x},\mathbf{y})} - \sum_{i \ge \hat{\mathrm{l}}(\mathbf{x},\mathbf{y})} \epsilon_i^2 \ge 5\delta_{\hat{\mathrm{l}}(\mathbf{x},\mathbf{y})} - \mathrm{o}[\delta_{\hat{\mathrm{l}}(\mathbf{x},\mathbf{y})}] \ge 4\delta_{\hat{\mathrm{l}}(\mathbf{x},\mathbf{y})}.
\end{aligned}
\tag{4.55}
$$

It is to be stressed that the lower bound in (4.55) is *uniform* in $j+1 > \hat{\mathrm{l}}(\mathbf{x},\mathbf{y})$, derived at any step $j+1$ from the bound on $|\boldsymbol{\lambda}_{\mathbf{x}}^{\hat{\mathrm{l}}} - \boldsymbol{\lambda}_{\mathbf{y}}^{\hat{\mathrm{l}}}|$, with the help of a j-dependent partial sum of a convergent series with general term $\epsilon_i^{2+} < \epsilon_i^2$.

The properties (K7) for $i = 0$ follow directly from the explicit formulae for $\boldsymbol{\varphi}_\bullet^0$ and $\boldsymbol{\psi}_\bullet^0$ (cf. Section 4.2). ■

Summary of the inductive step. For the reader's convenience, we provide the references to the stages in the proof where each of the inductive hypotheses is proved.

(K1) Steps 3 and 4	(K2) Steps 3 and 8	(K3) Step 4
(K4) Section 4.2	(K5) Step 5	(K6) Steps 6 and 7
(K7) Step 3	(K8) Step 9	(K9) Step 3, (4.38)
(K10) Steps 6 and 7		

5. Proof of the main theorem

Theorem 2.1 can be proved now in the same way as [11, Section IV, Theorem 1]. In fact, the concluding arguments used in the aforementioned paper apply to any self-adjoint operator family in the Hilbert space $\ell^2(\mathcal{G})$, where $\mathcal{G}$ is a countable graph, for which the properties (K1)–(K10) are proved for all $j \in \mathbb{N}$.

A. The existence of the norm-limits $\boldsymbol{\varphi}_{\mathbf{x}}(\omega,\vartheta) = \lim_{j\to+\infty} \boldsymbol{\varphi}_{\mathbf{x}}^{j}(\omega,\vartheta)$ follows from the perturbation estimates (4.15).

The completeness of the family $\{\boldsymbol{\varphi}_{\mathbf{x}},\ \mathbf{x} \in \boldsymbol{\mathcal{Z}}_2\}$ follows from the norm-convergence of the transformation operators $\boldsymbol{\Phi}^j$ along with their inverses. Taking the limit $j \to \infty$ in (4.16), we see that this family is orthonormal. Moreover, taking the limit $j \to \infty$

in the equation

$$\mathbf{H}_\varepsilon \boldsymbol{\varphi}_{\mathbf{x}}^j = \boldsymbol{\lambda}_{\mathbf{x}}^j \boldsymbol{\varphi}_{\mathbf{x}}^j + \boldsymbol{\psi}_{\mathbf{x}}^j \quad \text{where } \|\boldsymbol{\psi}_{\mathbf{x}}^j\|_{\mathbf{x}} \xrightarrow[j\to\infty]{} 0,$$

we see that $\{\boldsymbol{\varphi}_{\mathbf{x}}(\omega, \vartheta), \mathbf{x} \in \mathcal{Z}_2\}$ is an eigenbasis of $\mathbf{H}_\varepsilon(\omega, \vartheta)$, so the latter has pure point spectrum. Its simplicity follows from the property (K8) by taking the limit $j \to +\infty$.

B. The unimodality of all eigenfunctions $\boldsymbol{\varphi}_{\mathbf{x}}$ follows easily from the norm-perturbation estimates (4.15) in (K10), since $\boldsymbol{\varphi}_{\mathbf{x}}^0 = \mathbf{1}_{\mathbf{x}}$ and $\boldsymbol{\varphi}_{\mathbf{x}} = \boldsymbol{\varphi}_x^0 + \sum_{j\geq 0}(\boldsymbol{\varphi}_{\mathbf{x}}^{j+1} - \boldsymbol{\varphi}_{\mathbf{x}}^j)$, hence for ε small enough, $\|\boldsymbol{\varphi}_{\mathbf{x}}(\mathbf{x})\| > 1/\sqrt{2}$ for all $\mathbf{x} \in \mathcal{Z}_2$.

C. The exponential decay of $\boldsymbol{\varphi}_{\mathbf{x}}$ follows from the $\|\cdot\|_{\mathbf{x}}$-convergence of $(\boldsymbol{\varphi}_{\mathbf{x}}^j)_{j\geq 0}$.

6. Parametric smoothness of the eigenpairs

The results of this section prepare the ground for an application of Proposition 8.2 to the proof of Theorem 2.1, as well as for the proof of Theorem 8.1. We have to analyze the regularity of Θ-probability distributions of pairs of eigenvalues $(\boldsymbol{\lambda}_{\mathbf{x}_1}, \boldsymbol{\lambda}_{\mathbf{x}_2})$. Such an analysis is required in Section 4 to complete the Step 9 within the inductive step. This is achieved with the help of the approximate eigenvalues $\boldsymbol{\lambda}_{\mathbf{x}_1}^i$ and $\boldsymbol{\lambda}_{\mathbf{x}_2}^i$ by induction in i. The induction requires also a regularity analysis of $\boldsymbol{\varphi}_x^i$ and $\boldsymbol{\psi}_x^i$. Lemma 6.1 sets the base of induction in i, and the induction step is covered by Lemma 6.2.

6.1. General setting and the principal lemmata

The objects $\#_{\mathbf{x}}^i$ (with "#" standing for "$\boldsymbol{\lambda}$", "$\boldsymbol{\varphi}$", or "$\boldsymbol{\psi}$") are considered as functions of the Θ-random variables $\vartheta_{n,k}$. For any $\mathbf{x}$, the objects $\#_{\mathbf{y}}^i$ with $\mathbf{y}$ close to $\mathbf{x}$ are impacted by independent $\vartheta_{n,k}$, with n large enough, so the dependence of $\#_{\mathbf{y}}^i$ upon a single variable $\vartheta_{n,k}$ can be studied with the help of the one-parameter families $V(x; t_z) := V(x) + t_z \mathbf{1}_z(x)$, $z \in \mathbb{Z}$, or two-parameter families $V(x; t_z, t_u) := V(x) + t_z \mathbf{1}_z(x) + t_u \mathbf{1}_u(x)$, $z, u \in \mathbb{Z}$, $z \neq u$.

We focus on the approximate eigenpairs, for these objects are most important to the proofs of the main theorem, but the reader can see that the limiting, exact eigenpairs are just as smooth, owing to the uniform perturbation estimates.

Remark 6.1. The analytic treatment of the smoothness properties of the approximate eigenpairs presented here is an adaptation of the one introduced in our recent work [13] where a 1-particle version of $\mathbf{H}_\varepsilon$ was studied. As in [13], it can be shown that, for any $M \geq 1$ and all $\varepsilon \in (0, \varepsilon_*(M))$ with some $\varepsilon_*(M) > 0$, the approximate (hence exact) eigenvalues and eigenfunctions admit the derivatives of order N.

We use a shortcut ∂_z for $\mathrm{d}/\mathrm{d}t_z$. The explicit formulae for the objects $\#^0_\bullet$ show that

$$\begin{aligned} \forall z \in \mathbb{Z} \ \ \forall \mathbf{x} \in \boldsymbol{\mathcal{Z}}_2 \quad \partial_z \boldsymbol{\lambda}^0_{\mathbf{x}} &= \delta_{z,x_1} + \delta_{z,x_2}, \\ \forall z \in \mathbb{Z} \quad \partial_z \mathbf{\Phi}^0 &= \partial_z \mathbf{F}^0 = \partial_z \mathbf{\Psi}^0 = 0. \end{aligned} \tag{6.1}$$

It is convenient to introduce the matrices $\mathbf{S}^i$, $i \ge 0$, with matrix elements

$$\mathbf{S}^i_{\mathbf{yx}} := \begin{cases} (\boldsymbol{\lambda}^i_{\mathbf{x}} - \boldsymbol{\lambda}^i_{\mathbf{y}})^{-1}, & \text{if } \mathbf{M}^i_{\mathbf{yx}} \neq 0 \text{ and } \boldsymbol{\lambda}^i_{\mathbf{x}} \neq \boldsymbol{\lambda}^i_{\mathbf{y}}; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 6.1. *Consider the functions* $t \mapsto \boldsymbol{\lambda}^0_{\mathbf{z}}[\Lambda^0(\omega,\vartheta) + t\,\mathbf{1}_u]$, $\mathbf{z} \in \boldsymbol{\mathcal{Z}}_2$, $t \in \mathbb{R}$, *and assume that* $|\boldsymbol{\lambda}^0_{\mathbf{x}}(\omega,\vartheta) - \boldsymbol{\lambda}^0_{\mathbf{y}}(\omega,\vartheta)| \ge \delta_0$ *for any pair of non-identical configurations* $\mathbf{x}, \mathbf{y}$ *with* $\operatorname{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) \le 8L_0^2$. *Then*

$$|\partial_u \boldsymbol{\lambda}^1_{\mathbf{x}} - \partial_u \boldsymbol{\lambda}^0_{\mathbf{x}}| \le \epsilon_0^{2-}, \tag{6.2}$$

$$\|\partial_u \tilde{\boldsymbol{\varphi}}^1_{\mathbf{x}} - \partial_u \boldsymbol{\varphi}^0_{\mathbf{x}}\|_{\mathbf{x}} \le \epsilon_0^{1-},$$

$$\|\partial_u \boldsymbol{\psi}^1_{\mathbf{x}}\|_{\mathbf{x}} \le \epsilon_0^{2-}. \tag{6.3}$$

By $\Lambda^0 + t\,\mathbf{1}_u$ we mean the matrix with the entries $\Lambda^0_{yx} + t\delta_{yx}\delta_{yu}$.

Lemma 6.2. *Under the hypotheses of Lemma 6.1, fix some* $j \ge 1$ *and consider the functions* $t \mapsto \boldsymbol{\lambda}^i_{\mathbf{z}}[\Lambda^0(\omega,\vartheta) + t\,\mathbf{1}_u]$, $0 \le i \le j$. *Assume that, for any* $i \in [\![1, j]\!]$ *and any configurations* $\mathbf{x} \neq \mathbf{y}$ *with* $\operatorname{diam}(\Pi\mathbf{x} \cup \Pi\mathbf{y}) \le 8L_i^2$, *one has*

$$|\boldsymbol{\lambda}^i_{\mathbf{x}}(\omega,\vartheta) - \boldsymbol{\lambda}^i_{\mathbf{y}}(\omega,\vartheta)| \ge \delta_i,$$

$$|\partial_u \boldsymbol{\lambda}^i_{\mathbf{x}} - \partial_u \boldsymbol{\lambda}^{i-1}_{\mathbf{x}}| \le \epsilon_{i-1}^{2-}, \tag{6.4}$$

$$\|\partial_u \boldsymbol{\varphi}^i_{x} - \partial_u \boldsymbol{\varphi}^{i-1}_{\mathbf{x}}\|_{\mathbf{x}} \le \epsilon_{i-1}^{1-}, \tag{6.5}$$

$$\|\partial_u \boldsymbol{\psi}^i_{\mathbf{x}}\|_{\mathbf{x}} \le \epsilon_{i-1}^{2-}. \tag{6.6}$$

Assume also (6.2)–(6.3). *Then* (6.4)–(6.6) *hold true for* $i = j + 1$.

Note that by (K9), $\operatorname{diam} \mathcal{S}[\boldsymbol{\lambda}^i_\bullet] \le L_i$, thus, with "#" $\in$ {"$\boldsymbol{\lambda}$", "$\boldsymbol{\varphi}$", "$\boldsymbol{\psi}$"},

$$\mathcal{S}[\#^i_{\mathbf{x}}] \cap \mathcal{S}[\#^i_{\mathbf{y}}] = \varnothing \quad \text{for } |\mathbf{x} - \mathbf{y}| > 2L_i.$$

Let

$$\tilde{\mathrm{l}}(R) := \lceil (\ln(R) - \ln(2L_0))/\ln q \rceil, \qquad R > 0. \tag{6.7}$$

It is readily seen that $\tilde{\mathrm{l}}(R) := \min\{i \ge 0 : 2L_i \ge R\}$.

Lemma 6.3. *For any* $u \in \mathbb{Z}$ *and* $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_2$ *with* $\operatorname{dist}(\Pi\mathbf{x}, u) = R \ge 1$, *on has*

$$\forall j \ge \tilde{\mathrm{l}}(R) \quad |\partial_u \boldsymbol{\lambda}^j_{\mathbf{x}}| \le \epsilon_0^{\frac{\tilde{\mathrm{l}}(R)}{2L_0}}, \quad \|\partial_u \boldsymbol{\varphi}^j_{\mathbf{x}}\|_{\mathbf{x}} \le \epsilon_0^{\frac{\tilde{\mathrm{l}}(R)}{4L_0}}.$$

Proof. For all $i < \tilde{\mathrm{l}}(R)$ and $\mathbf{z}$ with $\Pi\mathbf{z} \subset \mathrm{B}_{L_i}(u)$, we have, by (K8) and (6.7),

$$\mathcal{S}[\lambda^i_{\mathbf{z}}] \cup \mathcal{S}[\boldsymbol{\varphi}^i_{\mathbf{z}}] \cup \mathcal{S}[\boldsymbol{\psi}^i_{\mathbf{z}}] \subset \mathcal{B}_{L_i}(\Pi\mathbf{z})$$

(cf. (1.1)), hence for any $0 \le i < \tilde{\mathrm{l}}(R)$ and $\mathbf{z}$ such that $u \not\in \mathcal{B}_{L_i}(\Pi\mathbf{z})$, we have

$$\partial_u \lambda^i_{\mathbf{x}},\ \partial_u \boldsymbol{\varphi}^i_{\mathbf{x}},\ \partial_u \boldsymbol{\psi}^i_{\mathbf{x}} = 0.$$

Recalling $\epsilon_i = \epsilon_0^{q^i}$ with $q > 1$, it follows that

$$|\partial_u \boldsymbol{\lambda}^j_{\mathbf{x}}| \le |\partial_u \boldsymbol{\lambda}^0_{\mathbf{x}}| + \sum_{1 \le i \le j} |\partial_u \boldsymbol{\lambda}^i_{\mathbf{x}} - \partial \boldsymbol{\lambda}^{i-1}_{\mathbf{x}}| \le \sum_{\tilde{\mathrm{l}}(R) \le i \le j} \epsilon_i^{2^-} \le \epsilon_{\tilde{\mathrm{l}}(R)} \le \epsilon_0^{\frac{\tilde{\mathrm{l}}(R)}{2L_0}}.$$

The proof of the asserted bound on $\partial_u \boldsymbol{\varphi}^j_{\mathbf{x}}$ is similar. ■

Corollary 6.4. *Fix any $j \in \mathbb{N}$, consider the AEV $\boldsymbol{\lambda}^j_{\mathbf{x}}$ as functionals of the 1-particle potential $V \in \ell^\infty(\mathbb{Z})$, and denote this dependence by $\boldsymbol{\lambda}^j_{\mathbf{x}}[V]$. Then, for any $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_{\mathbf{2}}$, the mapping $\ell^\infty(\mathbb{Z}) \ni V \mapsto \boldsymbol{\lambda}^j_{\mathbf{x}}[V] \in \mathbb{R}$ is Lipschitz continuous: for all sufficiently small $\varepsilon > 0$ in (1.5), one has*

$$\forall V, W \in \ell^\infty(\mathbb{Z}) \quad |\boldsymbol{\lambda}^i_{\mathbf{x}}[V + W] - \boldsymbol{\lambda}^i_{\mathbf{x}}[V]| \le 3\|W\|_\infty. \tag{6.8}$$

Proof. Owing to finiteness of the stochastic support $\mathcal{S}(\boldsymbol{\lambda}^j_{\mathbf{x}})$, it suffices to prove (6.8) for the functions W with $\operatorname{supp} W \subset \mathcal{S}(\boldsymbol{\lambda}^j_{\mathbf{x}})$. The transition from V to $V + W$ can be decomposed into $N = \operatorname{card} \mathcal{S}(\boldsymbol{\lambda}^j_{\mathbf{x}}))$ single-point perturbations $V_{k+1} = V_k + W(z_k)\,\mathbf{1}_{z_k}$, where $V_0 := V$, and $\{z_k, k \in [\![1, N]\!]\}$ are all the points of $\mathcal{S}(\boldsymbol{\lambda}^j_{\mathbf{x}})$ numbered in some way. Furthermore, the functions V_k and V_{k+1} can be included in a smooth parametric family $V_k(s) := V_k + sW(z_k)\,\mathbf{1}_{z_k}$, $s \in [0, 1]$. This gives rise to a function $s \mapsto \boldsymbol{\lambda}^j_{\mathbf{x}}[V_k(s)]$, and such functions, viz. the AEV depending upon a single-point perturbation of the potential, are considered in Lemmata 6.1 and 6.2.

Although $\operatorname{card} \mathcal{S}(\boldsymbol{\lambda}^j_{\mathbf{x}})$ grows as $j \to +\infty$, Lemmata 6.3 and 6.2 show that $|\partial_u \boldsymbol{\lambda}^i_{\mathbf{x}}|$ decays exponentially as $\operatorname{dist}(u, \Pi\mathbf{x}) \to +\infty$, and the principal contribution, provided by $\partial_{z_k} \boldsymbol{\lambda}^0_{\mathbf{x}}$ with $z_k \in \Pi\mathbf{x} = \{x_1, x_2\}$, is bounded by $2\|W\|_\infty$. By straightforward calculations, the higher-order terms in $\epsilon_0 = \varepsilon^{1/4}$ sum up to a quantity bounded by $\|W\|_\infty$, provided ε is small enough. This proves the claim. ■

It is plain that, for the *exact* eigenvalues, an estimate even slightly better than (6.8) would follow immediately from the min-max principle.

6.2. Proof of Lemma 6.1

In this section, $u \in \mathbb{Z}$ is an arbitrary lattice point, unless specified otherwise. To keep the notations less cumbersome, we use the shortcut $\partial \equiv \partial_u$.

By (4.35) and (4.29)–(4.30), we have, for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}$ such that $\mathbf{M}^1_{\mathbf{yx}} \neq 0$,

$$\begin{aligned} \mathbf{M}^1_{\mathbf{yx}} &= \varepsilon \sum_{\mathbf{z}:|\mathbf{z}-\mathbf{x}|=1} \mathbf{S}^0_{\mathbf{yx}} \langle \mathbf{1}_\mathbf{y} \mid \mathbf{1}_\mathbf{z} \rangle = \varepsilon\, \mathbf{S}^0_{\mathbf{yx}}\, \mathbf{1}_{\{|\mathbf{y}-\mathbf{x}|=1\}}\,, \\ \partial \mathbf{M}^1_{\mathbf{yx}} &= -\varepsilon\, \mathbf{1}_{\{|\mathbf{y}-\mathbf{x}|=1\}} (\mathbf{S}^0_{\mathbf{yx}})^2 (\partial \boldsymbol{\lambda}^0_\mathbf{x} - \partial \boldsymbol{\lambda}^0_\mathbf{y}), && (6.9) \\ |\partial \mathbf{M}^1_{\mathbf{yx}}| &\lesssim \varepsilon \delta_0^{-2} \le \epsilon_0^{1^-}. && (6.10) \end{aligned}$$

In (6.10), we have $\varepsilon = (\epsilon_0(\varepsilon))^4 \ll \epsilon_0$ for $0 < \varepsilon \ll 1$ (cf. (4.1)). Replacing ε with $\epsilon_0 = \varepsilon^{1/4}$ results in non-optimal upper bounds, but this is not crucial, and we keep here the calculations closer to those used in the next subsection, for arbitrary $j \ge 1$. Since $\boldsymbol{\Phi}^0 = \mathbf{1}$, we have $\mathbf{D}^0 = 0$. $\mathbf{Q}^0$ is a truncated version of $\widetilde{\mathbf{Q}}^0 = (\boldsymbol{\Phi}^0)^\top \boldsymbol{\Psi}^0 = \mathrm{Const}$ (cf. (4.30)), hence $\partial \mathbf{Q}^0 (\equiv \partial_u \mathbf{Q}^0) = 0$ for any u. By construction,

$$\begin{aligned} \mathbf{W}^1 &= [\mathbf{Q}^0, \mathbf{M}^1] + [\mathbf{D}^0, \mathbf{F}^0] + (\mathbf{M}^1)^2 \Lambda^0 - \mathbf{M}^1 \boldsymbol{\Lambda}^0\, \mathbf{M}^1, \quad \mathbf{D}^0 = 0, && (6.11) \\ \mathbf{F}^1_{\mathbf{yx}} &= (1 - \delta_{\mathbf{yx}}) \mathrm{W}^1_{\mathbf{yx}}, \\ \boldsymbol{\Lambda}^1_{\mathbf{xx}} &= \boldsymbol{\lambda}^1_\mathbf{x} = \boldsymbol{\lambda}^0_\mathbf{x} + \mathbf{W}^1_{\mathbf{xx}}. && (6.12) \end{aligned}$$

Assess $\partial \boldsymbol{\lambda}^1_\mathbf{x} - \partial \boldsymbol{\lambda}^0_\mathbf{x}$. By (6.11)–(6.12), and with $[\mathbf{Q}^0, \mathbf{M}^1]_{\mathbf{xx}} = 0$ by antisymmetry,

$$\partial \boldsymbol{\lambda}^1_\mathbf{x} - \partial \boldsymbol{\lambda}^0_\mathbf{x} = \partial \mathbf{W}^1_{\mathbf{xx}} = \partial((\mathbf{M}^1)^2 \boldsymbol{\Lambda}^0 - \mathbf{M}^1 \boldsymbol{\Lambda}^0\, \mathbf{M}^1)_{\mathbf{xx}}$$

where $|||\boldsymbol{\Lambda}^0|||, |||\partial \boldsymbol{\Lambda}^0||| \le \mathrm{Const}$. By assumption, $|\boldsymbol{\lambda}^0_\mathbf{x} - \boldsymbol{\lambda}^0_\mathbf{y}| \ge \delta_0 = \epsilon_0^{0^+}$ for all $\mathbf{y}$ figuring in the non-zero entries of the matrices in (6.11), hence $\delta_0^k = \epsilon_0^{0^+}$, thus we have $\epsilon_0 \delta_0^{-k} \le \epsilon_0^{1^-}$, say, for $1 \le k \le 2$, which suffices for our purposes. Therefore,

$$\begin{aligned} |\partial \boldsymbol{\lambda}^1_\mathbf{x} - \partial \boldsymbol{\lambda}^0_\mathbf{x}| &\le |\partial((\mathbf{M}^1)^2 \boldsymbol{\Lambda}^0)_{\mathbf{xx}} + \partial(\mathbf{M}^1 \boldsymbol{\Lambda}^0\, \mathbf{M}^1)_{\mathbf{xx}}| \lesssim |||\mathbf{M}^1|||\, |||\partial \mathbf{M}^1||| + |||\mathbf{M}^1|||^2 \\ &\lesssim \epsilon_0^{1^-} \cdot \epsilon_0 \delta_0^{-2} + (\epsilon_0^{1^-})^2 \le \epsilon_0^{2^-} \quad \text{(cf. (4.36) and (6.9))}\,. \end{aligned}$$

Next, assess $\tilde{\boldsymbol{\varphi}}^1_\mathbf{x} - \boldsymbol{\varphi}^0_\mathbf{x}$. By (4.37), $\tilde{\boldsymbol{\varphi}}^1_\mathbf{x} - \boldsymbol{\varphi}^0_\mathbf{x} = \sum_{|\mathbf{y}-\mathbf{x}|=1} \mathbf{M}^1_{\mathbf{yx}} \boldsymbol{\varphi}^0_\mathbf{y}$, whence

$$\partial \tilde{\boldsymbol{\varphi}}^1_\mathbf{x} - \underbrace{\partial \boldsymbol{\varphi}^0_\mathbf{x}}_{=0} = \partial \tilde{\boldsymbol{\varphi}}^1_\mathbf{x} = \sum_{|y-x|=1} (\partial \mathbf{M}^1_{\mathbf{yx}}) \boldsymbol{\varphi}^0_\mathbf{y} + \sum_{|\mathbf{y}-\mathbf{x}|=1} \mathbf{M}^1_{\mathbf{yx}} \underbrace{\partial \boldsymbol{\varphi}^0_\mathbf{y}}_{=0} = -\varepsilon \sum_{|\mathbf{y}-\mathbf{x}|=1} (\mathbf{S}^0_{\mathbf{yx}})^2 (\partial \boldsymbol{\lambda}^0_\mathbf{x} - \partial \boldsymbol{\lambda}^0_\mathbf{y}) \boldsymbol{\varphi}^0_\mathbf{y}$$

with $|\partial \boldsymbol{\lambda}^0_\mathbf{x} - \partial \boldsymbol{\lambda}^0_\mathbf{y}| \le |\partial \boldsymbol{\lambda}^0_\mathbf{x}| + |\partial \boldsymbol{\lambda}^0_\mathbf{y}| \le 2$ by (6.1) and (6.2). Therefore,

$$\|\partial \tilde{\boldsymbol{\varphi}}^1_\mathbf{x} - \partial \boldsymbol{\varphi}^0_\mathbf{x}\|_\mathbf{x} = \|\partial \tilde{\boldsymbol{\varphi}}^1_\mathbf{x}\|_\mathbf{x} \le \epsilon_0^{1^-}. \qquad (6.13)$$

To estimate the effect of normalization $\tilde{\boldsymbol{\varphi}}^1_\bullet \rightsquigarrow \boldsymbol{\varphi}^1_\bullet$, one can argue as in the proof of (6.5) in the next subsection (cf. (6.21)–(6.22)), and conclude that

$$\|\partial\boldsymbol{\varphi}^1_{\mathbf{x}} - \partial\boldsymbol{\varphi}^0_{\mathbf{x}}\|_{\mathrm{x}} \leq \epsilon_0^{1^-}.$$

Now, consider the discrepancies. By construction, $\boldsymbol{\psi}^1_{\mathbf{x}} = \sum_{\mathbf{z}\neq\mathbf{x}} \mathbf{W}^1_{\mathbf{zx}}\tilde{\boldsymbol{\varphi}}^1_{\mathbf{z}}$, so

$$\partial\boldsymbol{\psi}^1_{\mathbf{x}} = \sum_{\mathbf{z}\neq\mathbf{x}}(\partial\mathbf{W}^1_{\mathbf{zx}})\tilde{\boldsymbol{\varphi}}^1_{\mathbf{z}} + \sum_{\mathbf{z}\neq\mathbf{x}}\mathbf{W}^1_{\mathbf{zx}}\partial\tilde{\boldsymbol{\varphi}}^1_{\mathbf{z}}, \tag{6.14}$$

Here, $|||\mathbf{W}^{j+1}||| \leq \epsilon_0^{2^-}$ and $\|\partial\tilde{\boldsymbol{\varphi}}^1_{\mathbf{z}}\|_{\mathrm{x}} \leq \epsilon_0^{1^-}$ (cf. (4.46),(6.13)), so we focus on the first sum in (6.14). By (6.11) with $\mathbf{D}^0 = 0$ and $\mathbf{Q}^0 = \mathrm{Const}$,

$$\begin{aligned}\partial\mathbf{W}^1 &= [\partial\mathbf{Q}^0, \mathbf{M}^1] + [\mathbf{Q}^0, \partial\mathbf{M}^1] + \partial((\mathbf{M}^1)^2\boldsymbol{\Lambda}^0 + \mathbf{M}^1\boldsymbol{\Lambda}^0\mathbf{M}^1)\\ &= [\mathbf{Q}^0, \partial\mathbf{M}^1] + \partial((\mathbf{M}^1)^2\boldsymbol{\Lambda}^0 + \mathbf{M}^1\boldsymbol{\Lambda}^0\,\mathbf{M}^1).\end{aligned}$$

Applying (6.10) and recalling that factors O[1] can be absorbed in $\epsilon_0^{c^\pm}$, $c > 0$, we get

$$|||[\mathbf{Q}^0, \partial\mathbf{M}^1]||| \lesssim \epsilon_0^{1^-}|||\mathbf{Q}^0||| \lesssim \epsilon_0^{1^-}\cdot\epsilon_0 \leq \epsilon^{2^-},$$

$$\begin{aligned}|||\partial((\mathbf{M}^1)^2\boldsymbol{\Lambda}^0 + \mathbf{M}^1\boldsymbol{\Lambda}^0\,\mathbf{M}^1)||| &\lesssim |||\boldsymbol{\Lambda}^0|||\,|||\mathbf{M}^1|||\,|||\partial\mathbf{M}^1||| + 2|||\mathbf{M}^1|||^2|||\partial\boldsymbol{\Lambda}^0|||\\ &\leq \epsilon_0^{1^-}\cdot\epsilon_0^{1^-} + C\epsilon_0^{2^-} \leq \epsilon_0^{2^-},\end{aligned}$$

thus $|||\partial\mathbf{W}^1||| \leq \epsilon_0^{2^-} + \epsilon_0^{2^-} \leq \epsilon_0^{2^-}$. Since $\|\tilde{\boldsymbol{\varphi}}^1_{\mathbf{z}}\|_{\mathrm{x}} = \mathrm{O}[1]$, we conclude that

$$\|\partial\boldsymbol{\psi}^1_{\mathbf{x}}\|_{\mathrm{x}} \leq \epsilon_0^{2^-} + \mathrm{o}[\epsilon_0^{2^-}] \leq \epsilon_0^{2^-}.$$

This completes the proof of Lemma 6.1.

6.3. Proof of Lemma 6.2

We shall need the following estimates:

$$|||\partial\mathbf{M}^{j+1}||| \leq \epsilon_j^{1^-}, \tag{6.15}$$

$$|||\partial\mathbf{W}^{j+1}||| \leq \epsilon_j^{2^-}. \tag{6.16}$$

Remark 6.2. The explicit formulae for $\boldsymbol{\varphi}^0_\bullet$, $\boldsymbol{\lambda}^0_\bullet$ and the perturbation estimates (6.4)–(6.5) with $i \in [\![1, j]\!]$, imply a uniform boundedness of all quantities $|\partial\boldsymbol{\lambda}^i_{\mathbf{x}}|$, $\|\partial\boldsymbol{\varphi}^i_{\mathbf{x}}\|_{\mathrm{x}}$, $\|\partial\boldsymbol{\psi}^i_{\mathbf{x}}\|_{\mathrm{x}}$, $0 \leq i \leq j$, $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_2$.

By construction (cf. (4.45), (4.40), (4.35), (4.29)–(4.30)), we have

$$\boldsymbol{\lambda}^{j+1}_{\mathbf{x}} = \boldsymbol{\lambda}^j_{\mathbf{x}} + \mathbf{W}^{j+1}_{\mathbf{xx}}, \tag{6.17a}$$

$$\mathbf{W}^{j+1} = [\mathbf{Q}^j, \mathbf{M}^{j+1}] + \mathbf{D}^j\mathbf{F}^j + ((\mathbf{M}^{j+1})^2\boldsymbol{\Lambda}^j) - \mathbf{M}^{j+1}\boldsymbol{\Lambda}^j\mathbf{M}^{j+1}, \tag{6.17b}$$

$$\mathbf{M}^{j+1}_{\mathbf{yx}} = (1 - \delta_{\mathbf{yx}})\,\mathbf{S}^i_{\mathbf{yx}}\mathbf{Q}^j_{\mathbf{yx}}, \tag{6.17c}$$

$$\mathbf{Q}^j_{\mathbf{yx}} = \mathbf{1}_{\{|\mathbf{y}-\mathbf{x}|\leq L_j\}}\,\langle\boldsymbol{\varphi}^j_{\mathbf{y}} \mid \boldsymbol{\psi}^j_{\mathbf{x}}\rangle. \tag{6.17d}$$

To prove (6.15), recall that $\mathbf{M}_{\mathbf{xx}}^{j+1} = 0$ (cf. (6.17)). For $\mathbf{y} \neq \mathbf{x}$ we have

$$\begin{aligned}\partial \mathbf{M}_{\mathbf{yx}}^{j+1} &= (\partial \mathbf{S}_{\mathbf{yx}}^{j}) \cdot \mathbf{Q}_{\mathbf{yx}}^{j} + \mathbf{S}_{\mathbf{yx}}^{j} \cdot \partial \mathbf{Q}_{\mathbf{yx}}^{j} \\ &= -(\mathrm{S}_{\mathbf{yx}}^{j})^2 \mathbf{Q}_{\mathbf{yx}}^{j} (\partial \boldsymbol{\lambda}_{\mathbf{x}}^{j} - \partial \boldsymbol{\lambda}_{\mathbf{y}}^{j}) + \mathrm{S}_{\mathbf{yx}}^{j} (\langle \partial \boldsymbol{\psi}_{\mathbf{y}}^{j} \mid \boldsymbol{\varphi}_{\mathbf{x}}^{j} \rangle + \langle \boldsymbol{\psi}_{\mathbf{y}}^{j} \mid \partial \boldsymbol{\varphi}_{\mathbf{x}}^{j} \rangle)\end{aligned}$$

where $\|\boldsymbol{\psi}^j\|_{\mathbf{x}} \leq \epsilon_j^{1-}$ by (K5), and $\|\boldsymbol{\varphi}^j\| = 1$ by construction, hence

$$|\mathbf{Q}_{\mathbf{yx}}^{j}| \leq |\langle \boldsymbol{\psi}_{\mathbf{y}}^{j} \mid \boldsymbol{\varphi}_{\mathbf{x}}^{j} \rangle| \leq \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} \|\boldsymbol{\psi}^j\|_{\mathbf{x}} \leq \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} \epsilon_j^{1-}.$$

Further, $|\boldsymbol{\lambda}_{\mathbf{x}}^{j} - \boldsymbol{\lambda}_{\mathbf{y}}^{j}|^k \geq \delta_j^k \geq \epsilon_j^{0+}$ for $k = 1, 2$ and ε small enough, thus

$$|\mathbf{S}_{\mathbf{yx}}^{j}|^2 |\mathbf{Q}_{\mathbf{yx}}^{j} (\partial \boldsymbol{\lambda}_{\mathbf{x}}^{j} - \partial \boldsymbol{\lambda}_{\mathbf{y}}^{j})| \lesssim \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} (\epsilon_j^{0+})^{-1} \epsilon_j^{1-} \leq \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} \epsilon_j^{1-}.$$

On account of boundedness of $\|\partial \boldsymbol{\varphi}_{\mathbf{x}}^{j}\|_{\mathbf{x}}$ (cf. Remark 6.2) and $\|\partial \boldsymbol{\psi}_{\mathbf{y}}^{j}\|_{\mathbf{y}} \leq \epsilon_{j-1}^{2-}$ (cf. (6.6)), the estimate (6.15) now follows from the inequalities

$$\begin{aligned}&|\mathbf{S}_{\mathbf{yx}}^{j} \langle \partial \boldsymbol{\varphi}_{\mathbf{y}}^{j} \mid \boldsymbol{\psi}_{\mathbf{x}}^{j} \rangle| \leq \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} (\epsilon_j^{0+})^{-1} \|\partial \boldsymbol{\varphi}_j^{j}\|_{\mathbf{x}} \|\boldsymbol{\psi}^j\|_{\mathbf{x}} \leq \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} \epsilon_j^{1-}, \\ &|\mathbf{S}_{\mathbf{yx}}^{j} \langle \boldsymbol{\varphi}_{\mathbf{y}}^{j} \mid \partial \boldsymbol{\psi}_{\mathbf{x}}^{j} \rangle| \leq \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|} \epsilon_j^{1-}.\end{aligned}$$

Now, we turn to (6.16). By (6.17), we have

$$\begin{aligned}|||\partial \mathbf{W}^{j+1}||| \leq |||[\partial [\mathbf{Q}^j, \mathbf{M}^{j+1}]||| + |||\partial [\mathbf{D}^j, \mathbf{F}^j]||| \\ + |||\partial ((\mathbf{M}^{j+1})^2 \boldsymbol{\Lambda}^j)||| + |||\partial (\mathbf{M}^{j+1} \boldsymbol{\Lambda}^j \mathbf{M}^{j+1})|||.\end{aligned}$$

By construction, $\mathbf{Q}^j = (\boldsymbol{\Phi}^j)^{-1} \boldsymbol{\Psi}^j$, and $|||(\boldsymbol{\Phi}^j)^{-1}||| < 2$ by (K2), thus

$$\begin{aligned}|||\partial \mathbf{Q}^j||| &\leq |||\partial ((\boldsymbol{\Phi}^j)^{-1})||| \, |||\Psi^j||| + |||(\boldsymbol{\Phi}^j)^{-1}||| \, |||\partial \boldsymbol{\Psi}^j|||, \\ |||\partial ((\boldsymbol{\Phi}^j)^{-1})||| &\leq |||(\boldsymbol{\Phi}^j)^{-1}|||^2 \, |||\partial \boldsymbol{\Phi}^j||| \leq 4 |||\partial \boldsymbol{\Phi}^j|||,\end{aligned}$$

yielding f

$$|||\partial \mathbf{Q}^j||| \lesssim |||\partial \boldsymbol{\Phi}^j||| \, |||\boldsymbol{\Psi}^j||| + |||\partial \boldsymbol{\Psi}^j||| \leq \epsilon_j^{1-} \quad \text{(cf. (6.6))}.$$

Observe that the uniform in j boundedness of $|||\partial \boldsymbol{\Phi}^j|||$ follows from the bound (6.5), since it is already derived for $i = j + 1$ from its counterpart for $j = i$. Therefore,

$$|||[\partial \mathbf{Q}^j, \mathbf{M}^{j+1}]||| \lesssim \epsilon_j^{1-} \epsilon_j^{1-} \leq \epsilon_j^{2-}. \tag{6.18}$$

Further,

$$|||[\mathbf{Q}^j, \partial \mathbf{M}^{j+1}]||| \leq 2 |||\mathbf{Q}^j||| \, |||\partial \mathbf{M}^{j+1}||| \leq 4 \epsilon_j |||\partial \mathbf{M}^{j+1}||| \leq \epsilon_j^{2-}$$

(cf. (6.15)). Next,

$$
\begin{aligned}
&|||\partial((\mathbf{M}^{j+1})^2\mathbf{\Lambda}^j + \mathbf{M}^{j+1}\mathbf{\Lambda}^j\mathbf{M}^{j+1})|||\\
&\quad\lesssim |||\mathbf{\Lambda}^j|||\,|||\mathbf{M}^{j+1}|||\,|||\partial\mathbf{M}^{j+1}||| + |||\partial\mathbf{\Lambda}^j|||\,|||\mathbf{M}^{j+1}|||^2\\
&\quad\le \epsilon_j^{1^-}\cdot\epsilon_j^{1^-} + \epsilon_j^{2^-} \le \epsilon_j^{2^-}.
\end{aligned}
$$

By definition, $\mathbf{D}^j = \mathbf{C}^j - \mathbf{1}$ and $\mathbf{F}^j = (\mathbf{\Phi}^j)^{-1}\Psi^j$, so applying the identity (4.19), we obtain in a similar way the bounds

$$
|||\partial\mathbf{D}^j||| \le \epsilon_j^{1^-}, \quad |||\partial(\mathbf{D}^j\mathbf{F}^j)||| \le \epsilon_j^{2^-}. \tag{6.19}
$$

It follows from (6.18)–(6.19) that

$$
|\partial\mathbf{W}^{j+1}_{\mathbf{yx}}| \le |||\partial\mathbf{W}^{j+1}||| \le \epsilon_j^{2^-}.
$$

For the proof of (6.4), notice that, by antisymmetry, $[\mathbf{F}^j, \mathbf{M}^{j+1}]_{\mathbf{xx}} = 0$, so

$$
\partial\boldsymbol{\lambda}^{j+1}_{\mathbf{x}} - \partial\boldsymbol{\lambda}^{j}_{\mathbf{x}} = \partial((\mathbf{M}^{j+1})^2\mathbf{\Lambda}^j)_{\mathbf{xx}} + \partial(\mathbf{M}^{j+1}\mathbf{\Lambda}^j\mathbf{M}^{j+1})_{\mathbf{xx}}
$$

whence

$$
\begin{aligned}
|\partial\boldsymbol{\lambda}^{j+1}_{\mathbf{x}} - \partial\boldsymbol{\lambda}^{j}_{\mathbf{x}}| &\lesssim |||\mathbf{\Lambda}^j|||\ |||\mathbf{M}^{j+1}|||\ |||\partial\mathbf{M}^{j+1}||| + 2|||\mathbf{M}^{j+1}|||^2|||\partial\mathbf{\Lambda}^j|||\\
&\le \epsilon_j^{2^-} + \epsilon_j^{2^-} \le \epsilon_j^{2^-} \quad \text{(cf. (6.15))}.
\end{aligned}
$$

Proof of the bound (6.5) *on* $\partial\boldsymbol{\varphi}^{j+1}_{\mathbf{x}} - \partial\boldsymbol{\varphi}^j$. By construction, we have

$$
\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}} = \boldsymbol{\varphi}^{j}_{\mathbf{x}} + \sum_{\mathbf{y}\ne\mathbf{x}} \mathbf{M}^{j+1}_{\mathbf{yx}}\boldsymbol{\varphi}^{j}_{\mathbf{y}},
$$

whence

$$
\begin{aligned}
|||\partial\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}} - \partial\boldsymbol{\varphi}^{j}_{\mathbf{x}}||| &\le \sum_{\mathbf{y}\ne\mathbf{x}} |\partial\mathbf{M}^{j+1}_{\mathbf{yx}}|\,\|\boldsymbol{\varphi}^{j}_{\mathbf{y}}\|_{\mathbf{x}} + \sum_{\mathbf{y}\ne\mathbf{x}} |\mathbf{M}^{j+1}_{\mathbf{yx}}|\,\|\partial\boldsymbol{\varphi}^{j}_{\mathbf{y}}\|_{\mathbf{x}}\\
&\le \sum_{\mathbf{y}\ne\mathbf{x}} (\mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|}\epsilon_j^{1^-} + C\,\mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|}\epsilon_j^{1^-}) \lesssim \mathrm{e}^{-m|\mathbf{x}-\mathbf{y}|}\epsilon_j^{1^-} \quad \text{(cf. (6.15))}.
\end{aligned} \tag{6.20}
$$

Furthermore, it follows from the definition of $\boldsymbol{\varphi}^{j+1}_{\mathbf{x}}$, viz.

$$
\boldsymbol{\varphi}^{j+1}_{\mathbf{x}} = \|\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}\|^{-1}\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}, \quad \|\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}\| = \Big(\sum_{\mathbf{y}} (\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}(\mathbf{y}))^2\Big)^{1/2} \tag{6.21}
$$

(here $\|\cdot\|$ is the norm in $\ell^2(\mathbb{Z}_2)$), that

$$
\partial\boldsymbol{\varphi}^{j+1}_{\mathbf{x}} = \partial(\|\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}\|^2)^{-\frac{1}{2}}\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}} = -\frac{1}{2}\frac{\partial(\|\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}\|^2)}{\|\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}\|^3}\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}} + \frac{1}{\|\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}\|}\partial\tilde{\boldsymbol{\varphi}}^{j+1}_{\mathbf{x}}
$$

with $\|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\| = 1 + O[\epsilon_j^{1-}] \equiv (1 + O[\epsilon_j^{1-}])\|\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}\|$ (cf. (4.52)), and

$$\partial(\|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\|^2) = (1 + O[\epsilon_j^{1-}])\partial \sum_{\mathbf{z}} (\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}(\mathbf{z}))^2 = (1 + O[\epsilon_j^{1-}])\partial(\|\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}\|^2).$$

As mentioned above, we have $\|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\| = (1 + O[\epsilon_{j+1}^{1-}])\|\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}\|$, whence

$$\|\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1}\|^{-1} = (1 + O[\epsilon_j^{1-}])\|\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}\|^{-1}. \tag{6.22}$$

Collecting (6.20)–(6.22) and the estimates of $\|\tilde{\varphi}_x^{j+1} - \varphi_x^{j+1}\|_{\mathbf{x}}$ and $\|\partial\tilde{\boldsymbol{\varphi}}_{\mathbf{x}}^{j+1} - \partial\boldsymbol{\varphi}_{\mathbf{x}}^{j+1}\|_{\mathbf{x}}$, we come to the asserted bound (6.5) with $i = j + 1$. ■

Proof of the bound (6.6) *on* $\partial\boldsymbol{\psi}_{\mathbf{x}}^{j+1}$. We have by (4.47) and (4.44)

$$\boldsymbol{\psi}_{\mathbf{x}}^{j+1} = \sum_{\mathbf{z}\neq\mathbf{x}} (1 - \delta_{\mathbf{yx}})\mathbf{W}_{\mathbf{yx}}^{j+1}\tilde{\boldsymbol{\varphi}}_{\mathbf{z}}^{j+1} + \sum_{\mathbf{z}\neq\mathbf{x}} \mathbf{Z}_{\mathbf{yx}}^{j+1}\tilde{\boldsymbol{\varphi}}_{\mathbf{z}}^{j+1},$$

$$\mathbf{W}^{j+1} = [\mathbf{Q}^j, \mathbf{M}^{j+1}] + \mathbf{D}^j(\boldsymbol{\Phi}^j)^{-1}\Psi^j + (\mathbf{M}^{j+1})^2\boldsymbol{\Lambda}^j + \mathbf{M}^{j+1}\boldsymbol{\Lambda}^j\mathbf{M}^{j+1},$$

where $(\boldsymbol{\Phi}^j)^{-1}\Psi^j = \mathbf{F}^j$ and $\mathbf{Z}^{j+1}$ is defined in (4.41). Consider first $\partial\mathbf{W}_{\mathbf{yx}}^{j+1}$. We already have the norm-bounds on $\mathbf{Q}^j$, $\mathbf{M}^{j+1}$, $\mathbf{D}^j$, $(\boldsymbol{\Phi}^j)^{-1}$, $\boldsymbol{\Psi}^j$, $\boldsymbol{\Lambda}^j$, as well as on there derivatives, and all the terms contributing to $\partial\mathbf{W}_{\mathbf{yx}}^{j+1}$ are of order of ϵ_j^{2-}. As before, all the bounded factors can be absorbed in ϵ_j^{2-}, so $|\partial\mathbf{W}_{\mathbf{yx}}^{j+1}| \le \epsilon_j^{2-}$. A bound on $\partial\mathbf{Z}^{j+1}$ can be obtained similarly, albeit the calculations are longer, and the reader can see that it is of order of ϵ_j^{3-}. Finally,

$$\begin{aligned}\|\partial\boldsymbol{\psi}_{\mathbf{x}}^{j+1}\|_{\mathbf{x}} \le{}& |||\partial\mathbf{W}^{j+1}|||\,\|\tilde{\boldsymbol{\varphi}}_{\bullet}^{j+1}\|_{\mathbf{x}} + |||\mathbf{W}^{j+1}|||\,\|\partial\tilde{\boldsymbol{\varphi}}_{\bullet}^{j+1}\|_{\mathbf{x}} \\ &+ |||\partial\mathbf{Z}^{j+1}|||\,\|\tilde{\boldsymbol{\varphi}}_{\bullet}^{j+1}\|_{\mathbf{x}} + |||\mathbf{Z}^{j+1}|||\,\|\partial\tilde{\boldsymbol{\varphi}}_{\bullet}^{j+1}\|_{\mathbf{x}} \le \epsilon_j^{2-} = \epsilon_{j+1}^{1-}.\end{aligned}$$

This completes the inductive step and concludes the proof of Lemma 6.2. ■

7. Parametric estimates on small denominators

Recall that in Section 4.3, in the course of the inductive step (cf. Step 9), we stated the bound (4.54) and postponed its proof to the present section. The Θ-probability bound (4.54) follows from Theorem 7.2 which we are going to prove now. It provides a slightly more general and detailed version of (4.54), and its proof is based upon Theorem 7.3 and Proposition 7.4. Section 6 provides the analytic tools used below.

As to Section 8, it complements the Main Theorem 2.1 on uniform localization of eigenfunctions (ULE) and shows that, due to ULE, one can develop a rigorous smooth analysis of all principal approximate and exact spectral objects, first of all of the eigenpairs, which is quite problematic in more traditional Anderson models including (but not limited to) those with IID random potentials, where only *SULE* (semi-uniform localization of eigenfunctions) holds.

In the proofs of Theorems 7.3 and 7.2, we rely on Lemma 7.1 given below. We need it for $\mathfrak{N} = 2$ particles, but, in Section 9, we comment on the extensions to any $\mathfrak{N} \geq 2$ and $d \geq 1$, so we formulate a general statement, proved in the same way for any $\mathfrak{N}$. The reader familiar with the notion of a symmetric power of a graph (cf. Section 9, Paragraph D) can see that the proof remains valid in a lattice of any dimension $d \geq 1$. To make more transparent an extension to a more general graph $\mathcal{G}$ (e.g., $\mathcal{G} = \mathbb{Z}^d$), below we avoid where possible any reference to the algebraic nature of $\mathbb{Z}^1$ and use mainly the graph structure of the latter, except for the additions/subtractions like $x \pm 1$ in (7.1) and $|y_2 - x_2|$ in (7.2). Here, $x \pm 1$ can be replaced with "the nearest neighbors of the site x of the graph $\mathcal{G}$," and $|y_2 - x_2|$ with "$\mathrm{d}_{\mathcal{G}}(y_2, x_2)$," where $\mathrm{d}_{\mathcal{G}}(\cdot,\cdot)$ stands for the graph-distance on $\mathcal{G}$.

Assuming as before that the single-particle configuration space is $\mathbb{Z}^1$, define the $\mathfrak{N}$-particle Fermionic configuration space $\boldsymbol{\mathcal{Z}}_{\mathfrak{N}}$ as follows:

$$\boldsymbol{\mathcal{Z}}_{\mathfrak{N}} := \{(x_1, \dots, x_{\mathfrak{N}}) \in \mathbb{Z}^{\mathfrak{N}} : x_1 < x_2 < \cdots < x_{\mathfrak{N}}\}.$$

It is endowed with the graph structure inherited from $\mathbb{Z}^{\mathfrak{N}}$. Specifically, in the case $\mathfrak{N} = 2$, the edge set $\boldsymbol{\mathcal{E}}_2$ of the graph with the vertex set $\boldsymbol{\mathcal{Z}}_2$ is given by

$$\boldsymbol{\mathcal{E}}_2 = \{((x-1, y), (x, y)), ((x, y), (x, y+1)) : x, y \in \mathbb{Z}, \quad x < y\}. \tag{7.1}$$

Equivalently, the vertex set $\boldsymbol{\mathcal{Z}}_2$ can be defined as the set of all subsets $\mathbf{x} = \{s, t\} \subset \mathbb{Z}$ with $\operatorname{card}\{s, t\} = 2$, which in this particular case ($\mathfrak{N} = 2$) amounts to $s \neq t$. In this realization of $\boldsymbol{\mathcal{Z}}_2$, the edges are defined as follows: $(\mathbf{x}, \mathbf{y})$ is called an *edge* if

$$\mathbf{x} = \{x_1, x_2\}, \quad \mathbf{y} = \{x_1, y_2\}, \quad \text{where} \quad y_2 \notin \mathbf{x},\ |y_2 - x_2| = 1. \tag{7.2}$$

Similar to the case $\mathfrak{N} = 2$, we set $\Pi(x_1, \dots, x_{\mathfrak{N}}) = \{x_1, \dots, x_{\mathfrak{N}}\}$ and $\mathbf{n}_{\mathbf{x}}(\cdot) := \mathbf{1}_{\Pi\mathbf{x}}(\cdot)$ for every configuration $\mathbf{x} = (x_1, \dots, x_{\mathfrak{N}}) \in \boldsymbol{\mathcal{Z}}_{\mathfrak{N}}$.

Lemma 7.1. *For any $\mathbf{x} \in \boldsymbol{\mathcal{Z}}_{\mathfrak{N}}$ and $\mathbf{y} \in \boldsymbol{\mathcal{Z}}_{\mathfrak{N}} \setminus \{\mathbf{x}\}$, there exist $u_1 \in \Pi\mathbf{x} \setminus \Pi\mathbf{y}$, $u_2 \in \Pi\mathbf{y} \setminus \Pi\mathbf{x}$, and some subsets $\mathcal{X}, \mathcal{Y} \subset \mathbb{Z}$ with $\operatorname{card}\mathcal{X} = \operatorname{card}\mathcal{Y} = \mathfrak{N} - 1$ such that*

$$\Pi\mathbf{x} = \{u_1\} \cup \mathcal{X}, \quad \Pi\mathbf{y} = \{u_2\} \cup \mathcal{Y}, \tag{7.3}$$

$$u_1 \notin \mathcal{X} \cup \{u_2\}, \quad u_2 \notin \mathcal{Y} \cup \{u_1\}. \tag{7.4}$$

Proof. Fix any $\mathbf{x} \neq \mathbf{y}$. The mapping $f : \mathbb{Z} \ni z \mapsto \mathbf{n}_{\mathbf{x}}(z) - \mathbf{n}_{\mathbf{x}}(z) \in \{-1, 0, 1\}$ is not identically zero, for $\mathbf{n}_{\mathbf{x}} \equiv \mathbf{n}_{\mathbf{y}}$ would imply $\mathbf{x} = \mathbf{y}$. Also, $\sum_z f(z) = 0$, since $\sum_z \mathbf{n}_{\mathbf{v}}(z) = \mathfrak{N}$ for any $\mathbf{v} \in \boldsymbol{\mathcal{Z}}_{\mathfrak{N}}$. Hence, there exist some lattice points $u_1 \neq u_2$ with $f(u_1) = 1$ and $f(u_2) = -1$: otherwise, f would not change sign on $\mathbb{Z}$ without being identically zero, in contradiction with $\sum_z f(z) = 0$. Further, card $\Pi\mathbf{x} = \text{card}\,\Pi\mathbf{y} = \mathfrak{N}$ implies that $\mathcal{X} := \Pi\mathbf{x} \setminus \{u_1\}$ and $\mathcal{Y} := \Pi\mathbf{y} \setminus \{u_1\}$ have cardinality $\mathfrak{N} - 1$, and $u_1 \notin \mathcal{X}$, $u_2 \notin \mathcal{Y}$. It is plain that $u_1 \in \Pi\mathbf{x} \setminus \Pi\mathbf{y}$ and $u_2 \in \Pi\mathbf{y} \setminus \Pi\mathbf{x}$. This proves (7.3). Assertion (7.4) follows from (7.3) combined with $u_1 \neq u_2$, $u_1 \notin \mathcal{X}$, $u_2 \notin \mathcal{Y}$. ∎

In the next statement, given a number $R > 0$, we work with a partition $\mathcal{C}_{\hat{n}} = \{\mathrm{C}_{\hat{n},k}, k \in [\![1, K_{\hat{n}}]\!]\}$ of the phase space $\Omega = \mathbb{T}$, with $\hat{n} = \hat{n}(R)$ defined by

$$\hat{n} = \hat{n}(R) := \min[n \geq 1 : 2^{-n} \leq \tfrac{1}{2} C_A^{-1} R^{-A}], \tag{7.5}$$

so that, for any fixed $u \in \mathbb{Z}$, the points of the set $\{T^z\omega, z \in \mathrm{B}_R(u)\} \subset \Omega$ are separated by the elements $\mathrm{C}_{\hat{n},k}$. Recall that we introduced in Section 1.2 the σ-algebras like $\mathfrak{B}_{=n}$, $\mathfrak{B}_{\leq n}$, etc., naturally injected into $\mathfrak{B}^\Theta$, and the decompositions $\vartheta = (\vartheta_{<n}, \vartheta_{=n}, \vartheta_{>n})$. Recall also that we defined the cylinder set $\widehat{\Theta}^i := \bigcap_{0 \leq l \leq i} \Theta^l$ such that the AEV $\boldsymbol{\lambda}^i_{\mathbf{z}}(\omega, \vartheta)$, $\mathbf{z} \in \mathbb{Z}$, are well defined for all $\omega \in \Omega$ and $\vartheta \in \widehat{\Theta}^i$.

Theorem 7.2. *There exist $C, C' \in (0, +\infty)$ such that, for any bounded interval $I \subset \mathbb{R}$, any $i \in \mathbb{N} \cup \{\infty\}$, and any pair $(\mathbf{x}, \mathbf{y}) \in (\boldsymbol{\mathcal{Z}}_2)^2$ with $\mathbf{x} \neq \mathbf{y}$,* diam $\Pi\mathbf{x} \cup \Pi\mathbf{y} \leq R$, *one has a uniform bound*

$$\mathbb{P}^\Theta\{\exists\omega \in \Omega \ \ \boldsymbol{\lambda}^i_{\mathbf{x}}(\omega, \vartheta) - \boldsymbol{\lambda}^i_{\mathbf{y}}(\omega, \vartheta) \in I\} \leq \mathrm{e}^{C\ln^2(R)}|I|. \tag{7.6}$$

The most important measure-theoretic step towards the bound (7.6), covering all $\omega \in \Omega$, is the next theorem where a similar bound is stated for every *fixed* $\omega \in \Omega$.

Theorem 7.3. *Fix any $\omega \in \Omega$. For any interval $I \subset \mathbb{R}$ of length $|I| < +\infty$, any $i \geq 0$ and any pair $(\mathbf{x}, \mathbf{y}) \in (\boldsymbol{\mathcal{Z}}_2)^2$ with $\mathbf{x} \neq \mathbf{y}$,* diam $\Pi\mathbf{x} \cup \Pi\mathbf{y} \leq R$, *there exists a subset $\Theta_{\mathbf{xy}}(\omega, I) \in \mathfrak{B}_{=\hat{n}}$ satisfying, with some $C > 0$ uniform in $\omega \in \Omega$,*

$$\begin{gathered}\mathbb{P}^\Theta\{\Theta_{\mathbf{xy}}(\omega, I) \,\big|\, r\mathfrak{B}_{<\hat{n}}\} \geq 1 - \mathrm{e}^{C\ln^2(R)}|I|,\\ \forall\vartheta \in \widehat{\Theta}^i \cap \Theta_{\mathbf{xy}}(\omega, I) \quad \boldsymbol{\lambda}^i_{\mathbf{x}}(\omega, \vartheta) - \boldsymbol{\lambda}^i_{\mathbf{y}}(\omega, \vartheta) \notin I.\end{gathered}$$

Proof. Note that $\Pi\mathbf{x} \cup \Pi\mathbf{y} \subset \mathrm{B}_R(x_1)$. By T-covariance of the approximate/exact eigenvalues $\boldsymbol{\lambda}^i_\bullet$, we can assume without loss of generality that $0 \in \Pi\mathbf{x}$, so $\Pi\mathbf{x} \cup \Pi\mathbf{y} \subset \mathrm{B}_R(0) \subset \mathbb{Z}$. For future use, note that $\hat{n}(R) \leq C' \ln R$, with C' determined by A and C_A from (UPA). By Lemma 7.1, we have

$$\Pi\mathbf{x} = \{u_1, x'\}, \quad \Pi\mathbf{y} = \{u_2, y'\}, \tag{7.7}$$

where u_1 and u_2 fulfill the conditions (7.3)–(7.4). Furthermore, for each $k \in \{1, 2\}$, there is a unique partition element $C_{\hat{n},l_k}$ (cf. (3.1)) with supp $\chi_{\hat{n},l_k} = C_{\hat{n},l_k} \ni T^{u_k}\omega$. The point u_2 and the set $C_{\hat{n},l_2}$ will be useful for the proof of Theorem 8.1, but for the moment, we only need u_1 and $C_{\hat{n},l_1}$.

Estimates for fixed $\omega \in \Omega$. As a first step of the proof, we make two modifications. Firstly, we will be considering truncated samples $\vartheta = (\vartheta_{<\hat{n}}, \vartheta_{=\hat{n}}, \emptyset_{>\hat{n}})$. This corresponds to the truncation $v \rightsquigarrow v_{\hat{n}}$ of the hull function v (cf. (3.2)), and the impact of the truncation on the AEV is easily upper-bounded with the help of Corollary 6.4. (For the *exact* eigenvalues, one could simply apply the min-max principle.)

Consider the sub-algebra $\mathfrak{F}_{\mathbf{u}}$ of $\mathfrak{B}_{\le\hat{n}}$ generated by all $\vartheta_{n,k}$ with $0 \le n \le \hat{n}$ except for $(\hat{n}, l_1)$. With ω fixed, we condition on $\mathfrak{F}_{\mathbf{u}}$, which amounts to "freezing" all $a_{\hat{n}}\vartheta_{n,k}$, $n \le \hat{n}$, except $\vartheta_{\hat{n},l_1}$. The only Θ-variable left "alive," viz. $a_{\hat{n}}\vartheta_{\hat{n},l_1}$, can be considered as a coordinate, denoted s, in $\mathbb{R}$ endowed with the measure $\mu := a_{\hat{n}}^{-1}\,\mathbf{1}_{[0,a_{\hat{n}}]}(\mathrm{s})d\mathrm{s}$.

To operate with the random variables $\vartheta \mapsto \boldsymbol{\lambda}^i_{\bullet}(\omega, \vartheta)$, the latter must be well defined. By induction, this is the case for all $(\omega, \vartheta) \in \Omega \times \widehat{\Theta}^j$, so we can freely vary the value of the Θ-random variable $a_{\hat{n}}\vartheta_{\hat{n},l_1}$ in its entire range $[0, a_{\hat{n}}]$.

Adopting now the same language as in Section 6, we consider the approximate eigenvalues $\boldsymbol{\lambda}^i_{\bullet}$ as functions of a single parameter $\mathrm{s} \in \mathbb{R}$ upon which they depend through the single-point perturbation of the potential $z \mapsto V(z, \omega, \vartheta) + \mathrm{s}\,\mathbf{1}_{u_1}(z)$. It follows from the results of Section 6 that the mapping $\mathfrak{f}\colon \mathrm{s} \mapsto (\boldsymbol{\lambda}_{\mathbf{x}}(\mathrm{s}), \boldsymbol{\lambda}_{\mathbf{y}}(\mathrm{s}))$ is differentiable. Specifically, writing, for any $\mathbf{v} \in \boldsymbol{\mathcal{Z}}_2$,

$$\boldsymbol{\lambda}^i_{\mathbf{v}} = \boldsymbol{\lambda}^0_{\mathbf{v}} + (\boldsymbol{\lambda}^i_{\mathbf{v}} - \boldsymbol{\lambda}^0_{\mathbf{v}}) = \mathfrak{u}(T^{v_1}\omega, \vartheta) + \mathfrak{u}(T^{v_2}\omega, \vartheta) + \sum_{1\le j\le i}(\boldsymbol{\lambda}^j_{\mathbf{v}} - \boldsymbol{\lambda}^{j-1}_{\mathbf{v}}),$$

and recalling that $\partial_z\mathfrak{u}(T^v\omega, \vartheta) = \delta_{zv}$ for all $z, v \in \mathbb{Z}$, we have, with $\partial_{\mathrm{s}} \equiv \partial_{u_1}$:

$$\begin{aligned}&\partial_{\mathrm{s}}\mathfrak{u}(T^{u_1}\omega, \vartheta) = \delta_{u_1u_1} = 1,\\ &\partial_{\mathrm{s}}\mathfrak{u}(T^{x'}\omega, \vartheta) = \partial_{\mathrm{s}}\mathfrak{u}(T^{u_2}\omega, \vartheta) = \partial_{\mathrm{s}}\mathfrak{u}(T^{y'}\omega, \vartheta) = 0 \quad \text{(by (7.4) and (7.7))},\end{aligned}$$

while, for any $j \ge 1$ and $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}$ (indeed for all $\mathbf{z} \in \boldsymbol{\mathcal{Z}}_2$),

$$\sum_{j\ge1}|\partial_{\mathrm{s}}(\boldsymbol{\lambda}^j_{\mathbf{z}} - \boldsymbol{\lambda}^{j-1}_{\mathbf{z}})| \le \sum_{j\ge1}\epsilon_j^{1^+} \le \epsilon_1^{1^+}$$

(once again, bounded factors are absorbed in the notation $\epsilon_1^{1^+}$). We conclude that

$$\partial_{\mathrm{s}}(\boldsymbol{\lambda}^i_{\mathbf{x}} - \boldsymbol{\lambda}^i_{\mathbf{y}}) \ge 1 + \mathrm{o}[\epsilon_1] \ge 1 + \mathrm{o}[\varepsilon^{1/4}] \ge \frac{1}{2}.$$

Therefore, the probability measure $\nu^{(i)}_{\mathbf{x},\mathbf{y}}$ of the Θ-random variable $(\boldsymbol{\lambda}^i_{\mathbf{x}} - \boldsymbol{\lambda}^i_{\mathbf{y}})$ is absolutely continuous with respect to μ, and its Radon–Nikodym derivative $d\nu^{(i)}_{\mathbf{x},\mathbf{y}}/d\mu$ is upper-bounded by $(1 + \mathrm{o}[\varepsilon^{1/4}])^{-1} \le 2$.

Now, the claim follows from $a_{\hat{n}}^{-1} = \mathrm{e}^{\hat{n}^2}$, $\hat{n} \le C'\ln(R)$. ∎

The proof of Theorem 7.2 from Theorem 7.3, quite similar that of [11, Lemma 2.2] from [11, Lemma 2.1], is based upon the next proposition which is a mere reformulation of [11, Corollary 2.1]. The proof of the latter quite simple, and it explains the crucial role of the choice of the *piecewise-constant* functions $\chi_{n,k}\colon \mathbb{T} \to \mathbb{R}$ in the construction of the hull functions $\omega \mapsto v_{\vartheta}(\omega) = v(\omega, \vartheta)$ (cf. (1.3)). In essence, it allows one to replace a *continuum* family of conditions imposed on $\vartheta \in \Theta$ in Theorem 7.3 with a *finite* one.

Proposition 7.4. *For any $j \in \mathbb{N}$, there exist constants $C, C' \in (0, +\infty)$ and a finite partition $\mathcal{P}_j = \{\mathrm{P}_{j,l},\ 1 \le l \le \mathcal{L}_j \le L_j^{C'}\}$ of the phase space Ω such that for each element $\mathrm{P}_{j,l}$, the family of Θ-random variables $\left\{v_{\widetilde{N}}(T^z\omega, \vartheta), z \in \mathrm{B}^2_{L_j}(0)\right\}$ parameterized by $\omega \in \Omega$ is constant on $\mathrm{P}_{j,l}$ as a function of the parameter ω.*

By Proposition 7.4, to eliminate the set of $\vartheta \in \Theta$ for which one has the inclusion $\boldsymbol{\lambda}^i_{\mathbf{x}}(\omega, \vartheta) - \boldsymbol{\lambda}^i_{\mathbf{y}}(\omega, \vartheta) \in I$ for at least one $\omega \in \Omega$, one can pick some points $\tau_{j,l} \in \mathrm{P}_{j,l}$, and then eliminate $\mathcal{L}_j$ subsets of Θ on which $\boldsymbol{\lambda}^i_{\mathbf{x}}(\omega, \vartheta) - \boldsymbol{\lambda}^i_{\mathbf{y}}(\tau_{j,l}, \vartheta) \in I$.

Remark 7.1. A direct inspection of the above proof evidences that the probabilistic bounds on the spectral spacings $|\boldsymbol{\lambda}_{\mathbf{x}} - \boldsymbol{\lambda}_{\mathbf{y}}|$ can be easily derived from the bounds on the approximate spectral spacings $|\boldsymbol{\lambda}^i_{\mathbf{x}} - \boldsymbol{\lambda}^i_{\mathbf{y}}|$. Naturally, such bounds deteriorate as $|\mathbf{x} - \mathbf{y}| \to +\infty$.

8. Non-local Minami-type estimates

The original Minami estimate proved in the pioneering paper [37] for the 1-particle lattice Anderson model with IID random potential $V(x, \omega)$, refers to the restrictions $H_Q(\omega)$ of the Anderson Hamiltonian $H(\omega)$ to finite subsets (e.g., cubes) $Q \subset \mathbb{Z}^d$. Under the assumption that the common probability measure of the random variables $\omega \mapsto V(x, \omega)$ has a bounded density, it states that the probability analogous to the one in the LHS of (8.1) (relative to the probability space Ω) is bounded by $C|\Lambda|^2|I_1||I_2|$, but only in the particular case $I_1 = I_2$. The situation with arbitrarily placed intervals I_1 and I_2, with no restriction on their relative positions, is known to be more challenging. Certain important particular cases were treated, e.g., in [15, 32]. The model considered in the present paper has a number of distinctive features: it is 2-particle (adaptable to the case of $\mathfrak{N} \ge 2$ particles), deterministic (viz. quasi-periodic but adaptable to a richer class of dynamical systems), and features ULE which… *fails in many models* (cf. [18]) including the ones with random IID or cosine-like quasi-periodic potential (cf. [20, 38]). Yet, like the Hamiltonian (1.5) itself, it might provide some useful insight into more general Anderson models.

As was indicated in the preamble of Section 6, the proof of Theorem 8.1 relies on the results of that section.

Theorem 8.1. *Let* $\mathbf{x}, \mathbf{y} \in \mathcal{Z}_2$, $\mathbf{x} \neq \mathbf{y}$, $\operatorname{diam} \Pi\mathbf{x} \cup \Pi\mathbf{y} \leq R$. *For any bounded intervals* $I_1, I_2 \subset \mathbb{R}$, *the approximate/exact eigenvalues* $\boldsymbol{\lambda}^i_{\mathbf{x}}, \boldsymbol{\lambda}^i_{\mathbf{y}}$, $i \in \mathbb{N} \cup \{\infty\}$, *constructed in Section* 4 *admit the following bound:*

$$\mathbb{P}^{\Theta}\{\boldsymbol{\lambda}^i_{\mathbf{x}} \in I_1,\ \boldsymbol{\lambda}^i_{\mathbf{y}} \in I_2\} \leq \mathrm{e}^{C\ln^2(R)}|I_1||I_2|,$$

where $C > 0$ *is independent of* $\mathbf{x}$ *and* $\mathbf{y}$, *under the above conditions.*

For the proof, we need a variant of the inverse function theorem (cf. [12, 13]).

Proposition 8.2. *Consider real normed spaces* $(\mathbb{X}, \|\cdot\|_{\mathbb{X}}), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}}) \cong \mathbb{R}^K$, $K \in \mathbb{N}^*$, *and a mapping* $\mathfrak{f}\colon DX \to \mathbb{Y}$ *differentiable in a ball* $\mathcal{B}_\ell(0) \subset \mathbb{X}$, $\ell > 0$. *Assume that there is an invertible linear mapping* $\mathcal{A}\colon \mathbb{X} \to \mathbb{Y}$ *such that*

$$\sup_{\mathrm{x}\in\mathcal{B}_\ell(0)} \|\mathfrak{f}'(\mathrm{x}) - \mathcal{A}\| \leq \eta \leq \frac{\kappa}{\|\mathcal{A}^{-1}\|}, \quad \kappa \in \left(0, \frac{1}{2}\right).$$

Denote $\mathcal{B}^{\mathcal{A}}_R(0) := \{\mathrm{y} \in \mathbb{Y} : \|\mathcal{A}^{-1}\mathrm{y}\|_{\mathbb{X}} \leq R\}$, $R \geq 0$. *Then* $\mathfrak{f}$ *admits a differentiable inverse* $\mathfrak{f}^{-1}\colon \mathcal{B}^{\mathcal{A}}_{\kappa\ell}(0) \to \mathcal{B}_\ell(0)$, *and for all* $\mathrm{y} \in \mathcal{B}^{\mathcal{A}}_{\kappa\ell}(0)$ *one has*

$$\mathfrak{f}^{-1}(\mathrm{y}) = \mathcal{A}^{-1}\mathrm{y} + \delta(\mathrm{y}), \quad \|\delta(\mathrm{y})\|_{\mathbb{X}} \leq 2\eta\|\mathcal{A}^{-1}\|\ \|\mathcal{A}^{-1}\mathrm{y}\|_{\mathbb{X}}.$$

Furthermore, for any rectangle of the form $\mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) = I_1 \times \cdots \times I_K \subset \mathrm{B}_{\ell/4}(0)$, *with* $I_k = [\alpha_k - \epsilon_k, \alpha_k + \epsilon_k]$, $0 < \epsilon_k \leq \frac{1}{4}\ell$, *one has*

$$\mathfrak{f}^{-1}(\mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\epsilon})) \subset \bigtimes_{1\leq k\leq K} [\alpha'_k - (1 + \mathrm{O}[\epsilon_k])\epsilon_k, \alpha'_k + (1 + \mathrm{O}[\epsilon_k])\epsilon_k]$$

Proof of Theorem 8.1. As in the proof of Theorem 7.3, we apply Lemma 7.1 and fix some lattice points $u_1 \in \Pi\mathbf{x} \setminus \Pi\mathbf{y}$, $u_2 \in \Pi\mathbf{y} \setminus \Pi\mathbf{x}$ satisfying the conditions (7.3)–(7.4) (cf. also (7.7)). Letting $\hat{\mathrm{n}} \in \mathbb{N}$ be defined as in (7.5), for each $k \in \{1, 2\}$, there is a unique element $C_{\hat{\mathrm{n}},l_k}$ of the partition $\mathcal{C}_{\hat{\mathrm{n}}}$ (cf. (3.1)) with $\operatorname{supp} \chi_{\hat{\mathrm{n}},l_k} = C_{\hat{\mathrm{n}},l_k} \ni T^{u_k}\omega$.

This time, we “freeze” all $\vartheta_{n,l}$ such that either $n \neq \hat{\mathrm{n}}$, or $n = \hat{\mathrm{n}}$ but $l \notin \{l_1, l_2\}$. The Θ-random vector $(\vartheta_{\hat{\mathrm{n}},l_1}, \vartheta_{\hat{\mathrm{n}},l_2})$ is stochastically equivalent to the vector $\mathbf{s} = (\mathrm{s}_1, \mathrm{s}_2)$ in $\mathbb{R}^2$ on the probability space $(\mathbb{R}^2, \mathfrak{F}^{(2)}_{\mathrm{Leb}}, \mu)$ where $\mathfrak{F}^{(2)}_{\mathrm{Leb}}$ is the Lebesgue σ-algebra in $\mathbb{R}^2$, and μ is the measure $a^{-2}_{\hat{\mathrm{n}}}\,\mathbf{1}_{[0,a_{\hat{\mathrm{n}}}]^2}(\mathbf{s})d\mathbf{s}$. By the results of Section 6, the mapping $\mathfrak{f}\colon \mathbf{s} \mapsto (\boldsymbol{\lambda}_{\mathbf{x}}(\mathbf{s}), \boldsymbol{\lambda}_{\mathbf{y}}(\mathbf{s}))$ is differentiable, and, writing $\boldsymbol{\lambda}^i_{\mathbf{z}} = \lambda^0_{z_1} + \lambda^0_{z_2} + (\boldsymbol{\lambda}^i_{\mathbf{z}} - \boldsymbol{\lambda}^0_{\mathbf{z}})$, $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}$, $\partial_{\mathrm{s}_k} \equiv \partial_{u_k}$, $k = 1, 2$, we have (cf. (7.7))

$$\begin{pmatrix} \partial_{s_1}(\lambda^0_{u_1} + \lambda^0_{x'}) & \partial_{s_2}(\lambda^0_{u_1} + \lambda^0_{x'}) \\ \partial_{s_1}(\lambda^0_{u_2} + \lambda^0_{y'}) & \partial_{s_2}(\lambda^0_{u_2} + \lambda^0_{y'}) \end{pmatrix} = \mathbf{1} + \begin{pmatrix} \partial_{s_1}\lambda^0_{x'} & \partial_{s_2}\lambda^0_{x'} \\ \partial_{s_1}\lambda^0_{y'} & \partial_{s_2}\lambda^0_{y'} \end{pmatrix} = \mathbf{1}\,.$$

Further, by Lemma 6.2 (cf. (6.4)),

$$\forall\, j \geq 1 \quad |\partial\boldsymbol{\lambda}^j_{\mathbf{z}} - \partial\boldsymbol{\lambda}^{j-1}_{\mathbf{z}}| \leq \epsilon^{2-}_{j-1}, \quad \mathbf{z} \in \{\mathbf{x}, \mathbf{y}\},$$

whence

$$\mathfrak{f}'(\mathbf{s}) = \mathbf{1} + \mathfrak{D}(\mathbf{s}), \quad \|\mathfrak{D}(\mathbf{s})\| \leq \eta := \epsilon_0^{\tilde{\mathfrak{l}}-} = \varepsilon^{\frac{1}{2}^+}, \quad \tilde{\mathfrak{l}} \geq 2.$$

With ε assumed to be small, $\mathfrak{f}'(\mathbf{s})$ is invertible, and $\|(\mathfrak{f}'(\mathbf{t}))^{-1}\| \leq 1 + \mathrm{O}[\epsilon_0] \leq 3/2$. Therefore, Proposition 8.2 applies, with $\mathcal{A} = \mathfrak{f}'(0)$, and yields

$$\|\mathfrak{f}'(\mathbf{s}) - \mathbf{1}\| \leq \eta \leq \kappa/\|\mathcal{A}^{-1}\|, \quad \kappa = \frac{1}{4}.$$

With $\eta < \frac{1}{16}$, we get $\|\mathfrak{f}^{-1}(\mathbf{s}) - \mathcal{A}^{-1}\mathbf{s}\| \leq \frac{1}{2}\|\mathbf{s}\|$, hence the inverse image $\mathfrak{f}^{-1}(I_1' \times I_2')$ is covered by a rectangle $I_1'' \times I_2''$ with $|I_k''| \leq 2|I_k'| \leq 4|I_k|$, $k \in [\![1, 2]\!]$. Concluding,

$$\mathbb{P}^{\Theta}\{\boldsymbol{\lambda}_{\mathbf{x}}^i \in I_1,\ \boldsymbol{\lambda}_{\mathbf{y}}^i \in I_2\} \leq \mathrm{Const}\ a_{\hat{\mathrm{n}}(R)}^{-2}|I_1||I_2|$$

with $a_{\hat{\mathrm{n}}(R)}^{-1} \leq \mathrm{e}^{C' \ln^2(R)}$, so the claim follows. ■

9. Concluding remarks

A. Eigenvalue correlation estimates in the phase space. Theorem 7.2 operates with the AEV $\boldsymbol{\lambda}_{\bullet}^i$ considered as functions of the parameters $\vartheta_{n,k} \in [0, 1]$, with $(n, k) \in \bigcup_{n' \in \mathbb{N}}(\{n'\} \times [\![1, K_{n'}]\!])$. These parameters are unrelated to the main probability space (phase space) $(\Omega, \mathfrak{B}^{\Omega}, \mathbb{P}^{\Omega})$, but the estimates stated in Theorem 7.2 suit the needs of the proof of a uniform localization, for every $\omega \in \Omega$ and not just with probability one.

Taking the limit $i \to +\infty$ and making use of the perturbation formulae (4.14), one can obtain similar estimates for the exact eigenvalues $\boldsymbol{\lambda}_{\bullet}$.

However, it is worth mentioning that the parametric estimates provided by Theorem 7.2 can be transformed into probability estimates in the phase space Ω, i.e., eigenvalue correlation inequalities, with the help of a simple application of Fubini's theorem combined with Chebyshev's inequality in the disorder-parameter space $\Omega \times \Theta$, in the same way as in [9, Appendix A] (cf. also [36]).

B. Extension to any number of particles $\mathfrak{N} > 2$. The main analytic, inductive procedure carried out in Section 4 does not present any particular problem with extending it to $\mathfrak{N} > 2$, for it is quite general and robust. A more delicate point is the parametric and measure-theoretic analysis of the small denominators, as usual in the framework of the KAM-type approaches. We make, however, a step in this direction and formulate a simple but important technical Lemma 7.1 for any $\mathfrak{N} \geq 2$.

C. Particle interaction potentials of infinite range. The present text is already quite technical, and for this reason, several simplifying assumptions have been made regarding the structure and parameters of the model at hand. One of them concerns the decay of the inter-particle, two-body potential $\mathfrak{u}(\cdot)$: it is assumed to have a compact support, as in a certain number of mathematical papers on N-particle Anderson localization. However, infinite-range potentials have also been considered earlier, and it is natural to wonder if, for example, exponentially decaying potentials $\mathfrak{u}(\cdot)$ can be tolerated by the KAM-type inductive approximation scheme presented here. To avoid unproven statements, we do not formulate in this paper any formal results in this direction, but a thorough inspection of the proofs shows that extending the main results to exponentially decaying potentials is possible; it requires several technical adaptations.

A more challenging question concerns the minimal requirements on the rate of decay of $r \mapsto \mathfrak{u}(r)$ under which a uniform exponential localization can still be proved, along with a fairly explicit parametric analysis of the eigenvalues.

D. Symmetric powers of graphs. In the case where the physical, single-particle configuration space is the one-dimensional lattice, the configuration space of $\mathfrak{N} \geq 2$ fermions can be constructed similarly to $\boldsymbol{\mathcal{Z}}_{\mathbf{2}}$,

$$\boldsymbol{\mathcal{Z}}_{\mathfrak{N}} := \{(x_1, \dots, x_{\mathfrak{N}}) \in \mathbb{Z}^{\mathfrak{N}} : x_1 < x_2 < \dots < x_{\mathfrak{N}}\},$$

and it is endowed with the graph structure inherited from $\mathbb{Z}^{\mathfrak{N}}$.

However, starting with $d = 2$, the absence of a natural order in $\mathbb{Z}^d$ compels one to find a different construction. One of them relies on the notion of symmetric power of a graph. Below we present an equivalent construction making use of the "occupation numbers" functions $\mathbf{n}$ similar to those we used earlier in the case $\mathfrak{N} = 2$, $d = 1$.

Call a (Fermionic) $\mathfrak{N}$-particle *occupation number function* on $\mathbb{Z}^d$ (the latter can actually be any countable graph with uniformly bounded coordination numbers) any mapping $\mathbf{n}: \mathbb{Z}^d \to \{0, 1\}$ obeying

$$\sum_{z \in \mathbb{Z}^d} \mathbf{n}(z) = \mathfrak{N}.$$

Denote $\boldsymbol{\mathcal{Z}}_{\mathfrak{N}}$ the set of all such mappings. We endow it with the following structure of unordered graph.

Call a pair of distinct elements $\mathbf{n}', \mathbf{n}'' \in \boldsymbol{\mathcal{Z}}_{\mathfrak{N}}$ an *edge* if the symmetric difference $(\operatorname{supp} \mathbf{n}')\Delta(\operatorname{supp} \mathbf{n}'')$ is an edge $\{z', z''\}$ of the unordered graph $\mathbb{Z}^d$.

The above formal definition of the graph $\boldsymbol{\mathcal{Z}}_{\mathfrak{N}}$ has a rather obvious interpretation. A formal "occupation numbers" function $\mathbf{n}: \mathbb{Z}^d \to \{0, 1\}$ is uniquely determined by its support where it takes the constant value 1, so there is a bijection between $\boldsymbol{\mathcal{Z}}_{\mathfrak{N}}$ and the set of $\mathfrak{N}$-point lattice subsets $\mathbf{x} = \{x_1, \dots, x_{\mathfrak{N}}\}$ with $\operatorname{card} \mathbf{x} = \mathfrak{N}$: $\mathbf{x} \leftrightarrow \mathbf{n}_{\mathbf{x}} := \mathbf{1}_{\mathbf{x}}$. Naturally, the elements x_j of $\mathbf{x}$ describe $\mathfrak{N}$ distinct particle positions. Two particle confugurations $\mathbf{x}', \mathbf{x}''$ form an edge if and only if $\mathbf{x}''$ is obtained from $\mathbf{x}'$ by moving exactly one particle, positioned at some $z' \in \mathbf{x}'$, to an adjacent, unoccupied point $z'' \in \mathbb{Z}^d \setminus \mathbf{x}'$.

With such a construction of $\mathcal{Z}_{\mathfrak{N}}$, the notation $\Pi\mathbf{x}$ we had used in the main text of the paper becomes redundant: an unordered set of exactly $\mathfrak{N}$ particle positions now *is* the configuration with these positions.

Keeping the notation $\mathbf{n}_{\mathbf{x}}$ which we employed earlier has some advantages; we have seen some situations where it proves to be useful. It would become even more useful in the case (which we do not consider here) of bosonic particle configurations; clearly, in that case $\mathbf{n}_{\mathbf{x}}$ may take values in $[\![0, \mathfrak{N}]\!]$.

From the perspective of the inductive procedure presented in Section 4, replacing the subset $\mathcal{Z}_{\mathbf{2}} \subset \mathbb{Z}^2$ with a more general graph does not pose any particular problem. All we need to carry it out in a more general context is an adjacency structure, hence the canonical graph distance and graph Laplacian, and a tempered rate of growth of balls of radius R as $R \to +\infty$. The only subtle point, crucial for the proof of *uniform*, and not semi-uniform, decay of all eigenfunctions, is the possibility to construct a deterministic random potential in such a way that, by excluding a subset of small measure in the parameter (or probability) space, one could avoid abnormally small denominators at a given spatial scale on the entire, possibly infinite configuration space. This is where the algebraic (viz. periodic lattice) nature of $\mathbb{Z}^d$ provides various natural examples of deterministic operator ensembles.

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