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Critical points of the Eisenstein series E_4 and application to the spectrum of the Lamé operator

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Abstract. We give a complete description of the distribution of the critical points of the classical Eisenstein series $E_4(\tau)$. An application to the spectrum of the Lamé operator is also given.

1. Introduction

The Eisenstein series are well known as the first nontrivial examples of modular forms. Since their discovery in the early 19th century, the Eisenstein series have always played fundamental roles in the theory of modular forms and elliptic functions. On the other hand, besides their numerous applications, the Eisenstein series are rather deep objects by themselves.

This paper is the second in our project of understanding the critical points of the classical Eisenstein series. We studied the Eisenstein series $E_2(\tau)$ in [8]. By developing further the idea from [8], the aim of this paper is to completely determine all the critical points of the Eisenstein series $E_4(\tau)$ of weight 4, or equivalently the well-known invariant $g_2(\tau)$ in the theory of elliptic curves. We will see that this result has interesting applications to the spectrum of the Lamé operator.

Throughout the paper, we use the notations $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$ and $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H} = \{\tau \mid \operatorname{Im}\tau > 0\}$. Let $\wp(z) = \wp(z|\tau)$ be the Weierstrass $\wp$-function with periods Λ_τ, defined by

$$\wp(z) = \wp(z|\tau) := \frac{1}{z^2} + \sum_{\omega\in\Lambda_\tau\setminus\{0\}} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big).$$

It is well known that $\wp(z|\tau)$ satisfies the following cubic equation:

$$\wp'(z|\tau)^2 = 4\wp(z|\tau)^3 - g_2(\tau)\wp(z|\tau) - g_3(\tau).$$

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Here $g_2(\tau), g_3(\tau)$ are invariants of the elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$, which are just multiples of the basic two Eisenstein series $G_4(\tau), G_6(\tau)$, or equivalently the normalized $E_4(\tau)$, $E_6(\tau)$, respectively:

$$g_2(\tau) = \frac{4\pi^4}{3}E_4(\tau) = 60G_4(\tau) := 60\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m\tau+n)^4},$$
$$g_3(\tau) = \frac{8\pi^6}{27}E_6(\tau) = 140G_6(\tau) := 140\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m\tau+n)^6},$$

where $\sum'$ means to sum over $(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ conventionally. The derivative of $E_4(\tau)$ is given by Ramanujan's formula (see e.g., [25, p.1786])

$$E_4'(\tau) = \frac{2\pi i}{3}(E_2(\tau)E_4(\tau) - E_6(\tau)), \tag{1.1}$$

where $E_2(\tau)$ is the Eisenstein series of weight 2 defined by

$$E_2(\tau) := \frac{3}{\pi^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}' \frac{1}{(m\tau+n)^2}.$$

It is well known that $E_4(\tau), E_6(\tau)$ are both modular forms for $\mathrm{SL}(2,\mathbb{Z})$ of weight 4, 6 respectively, while $E_2(\tau)$ is not a modular form but only a *quasimodular form* (cf. [19]), and so is $E_4'(\tau)$. This infers that the set of the zeros of $E_4'(\tau)$ is *not stable* under the modular group $\mathrm{SL}(2,\mathbb{Z})$. Recently, there are many works studying the zeros of quasimodular forms including critical points of modular forms; see e.g., [3, 12, 19, 25, 26] and references therein. In particular, Saber and Sebbar [25] proved that *for each modular form f for a subgroup of* $\mathrm{SL}(2,\mathbb{Z})$*, its derivative f' has infinitely many inequivalent zeros and all, but a finite number, are simple*. As an example, they proved that $E_4'(\tau)$ has infinitely many inequivalent zeros which are all simple. However, the distribution of the zeros of $E_4'(\tau)$ is still far from being understood.

In this paper we develop our own approach to completely determine its critical points. Since the basic fundamental domain F_0 of $\Gamma_0(2)$ plays a crucial role in our approach, we would like to state our results in fundamental domains of $\Gamma_0(2)$ first. The corresponding statements Theorems 1.5–1.6 in fundamental domains of the modular group $\mathrm{SL}(2,\mathbb{Z})$ will be given as consequences.

1.1. Distribution in fundamental domains of $\Gamma_0(2)$

Recall that $\Gamma_0(2)$ is the congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ defined by

$$\Gamma_0(2) := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, c \equiv 0 \mod 2\right\},$$

and F_0 is the basic fundamental domain of $\Gamma_0(2)$ given by

$$F_0 := \left\{\tau \in \mathbb{H} \,\middle|\, 0 \leqslant \operatorname{Re}\tau \leqslant 1 \text{ and } \left|\tau - \frac{1}{2}\right| \geqslant \frac{1}{2}\right\}.$$

Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ (i.e., *consider* γ *and* $-\gamma$ *to be the same*),

$$\gamma(F_0) := \left\{\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d} \,\middle|\, \tau \in F_0\right\} = (-\gamma)(F_0)$$

is also a fundamental domain of $\Gamma_0(2)$. Note that $\gamma(F_0) = F_0 + m$ for some $m \in \mathbb{Z}$ if and only if $c = 0$. Our first result shows that the critical points of $E_4(\tau)$ satisfy the following distribution.

Theorem 1.1. *Let* $\gamma(F_0)$ *be a fundamental domain of* $\Gamma_0(2)$ *with*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)/\{\pm I_2\}.$$

Then the following hold:

(1) $E_4(\tau)$ *has no critical points in* $\gamma(F_0)$ *if* $c = 0$*;*

(2) $E_4(\tau)$ *has exactly one critical point in* $\gamma(F_0)$ *if* $c \neq 0$.

In view of Theorem 1.1, we can transform every critical point of $E_4(\tau)$ via the Möbius transformation of $\Gamma_0(2)$ action to locate it in F_0. Denote *the collection of such corresponding points in* F_0 *by* $\mathcal{D}_0$, which consists of countable many points. A fundamental question related to the distribution of the critical points is: *What is the geometry of the set* $\mathcal{D}_0$*?*

Surprisingly, it turns out that $\mathcal{D}_0$ locates on the union of *three disjoint smooth curves* $\tau(C)$'s in F_0, which are parameterized by $C \in \mathbb{R} \setminus \{0, 1\}$ via the following identity:

$$C = \tau - \frac{4\pi i g_2(\tau)}{2\eta_1(\tau)g_2(\tau) - 3g_3(\tau)}, \quad \tau \in F_0. \tag{1.2}$$

Here $\eta_1(\tau) := 2\zeta(\frac{\omega_1}{2}|\tau)$ is a quasi-period of the Weierstrass zeta function $\zeta(z) = \zeta(z|\tau) := -\int^z \wp(\xi|\tau)d\xi$. Indeed, it is a multiple of the Eisenstein series $E_2(\tau)$: $\eta_1(\tau) = \frac{\pi^2}{3}E_2(\tau)$. The relation between (1.2) and $E_4'(\tau)$ comes from Ramanujan's formula (1.1), which can be written in terms of g_2, g_3, η_1 as follows (see also [5, p.704]):

$$g_2'(\tau) = \frac{i}{\pi}\big(2\eta_1(\tau)g_2(\tau) - 3g_3(\tau)\big). \tag{1.3}$$

We will prove in Section 3 that for each $C \in \mathbb{R} \setminus \{0, 1\}$, there is *a unique point* $\tau(C) \in F_0$ such that (1.2) holds. Consequently, the parametrization (1.2) will give *three disjoint curves*

$$\mathcal{C}_0 := \{\tau(C) \mid C \in (0,1)\},$$
$$\mathcal{C}_- := \{\tau(C) \mid C \in (-\infty,0)\}, \quad \mathcal{C}_+ := \{\tau(C) \mid C \in (1,+\infty)\},$$

the limit points of which are exactly the cusps of F_0:

$$\partial\mathcal{C}_0 = \{0,1\}, \quad \partial\mathcal{C}_- = \left\{0, \frac{1}{4} + i\infty\right\}, \quad \partial\mathcal{C}_+ = \left\{1, \frac{3}{4} + i\infty\right\}.$$

Theorem 1.2. *Let $\tau(C)$ be defined by (1.2) for $C \in \mathbb{R} \setminus \{0,1\}$. Then*

$$\mathcal{D}_0 = \left\{\tau\left(\frac{-d}{c}\right) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)/\{\pm I_2\},\ c \neq 0\right\} \subset \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+. \tag{1.4}$$

Furthermore, the closure of $\mathcal{D}_0$ in F_0 is precisely the union of the three disjoint smooth curves:

$$\overline{\mathcal{D}_0} \cap F_0 = \overline{\mathcal{D}_0} \setminus \{0,1,\infty\} = \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+. \tag{1.5}$$

Remark 1.3. In fact, we will prove $\tau(C) \in \mathring{F}_0$, where $\mathring{F}_0 = F_0 \setminus \partial F_0$ denotes the set of interior points of F_0. Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$, we will prove in Theorem 4.1 that the unique critical point of $E_4(\tau)$ in $\gamma(F_0)$ is precisely

$$\frac{a\tau\left(\frac{-d}{c}\right) + b}{c\tau\left(\frac{-d}{c}\right) + d} \in \gamma(\mathring{F}_0).$$

For $\gamma_j \in \Gamma_0(2)/\{\pm I_2\}$ with $\gamma_1 \neq \pm\gamma_2$, we have

$$\gamma_1(\mathring{F}_0) \cap \gamma_2(\mathring{F}_0) = \emptyset$$

(though $\gamma_1(\partial F_0) \cap \gamma_2(\partial F_0) \neq \emptyset$ may happen). Thus, there is a one-to-one correspondence between $\mathcal{D}_0$ and the set of critical points of $E_4(\tau)$.

Remark 1.4. As mentioned before, $E_4(\tau)$ is a modular form, but $E_2(\tau)$ is not. It is interesting to compare the distribution of the critical points of $E_4(\tau)$ with that of $E_2(\tau)$. We proved in [8] that, under the Möbius transformations of $\Gamma_0(2)$ action, the images of all critical points of $E_2(\tau)$ in F_0 form a dense subset of the union of three smooth curves in F_0. However, this union is *path-connected* for the E_2 case, which is different from the situation of Theorem 1.2. See [8] for the complete theory concerning the critical points of $E_2(\tau)$. We will study the critical points of $E_6(\tau)$ in a coming work.

We will see in Section 6 that the three curves have interesting geometric meanings from the viewpoint of the multiple Green function on the elliptic curve and also monodromy meanings from the classical Lamé equation.

1.2. Distribution in fundamental domains of SL$(2, \mathbb{Z})$

Since $E_4(\tau)$ is a modular form for the modular group SL$(2,\mathbb{Z})$, it is natural to consider the distribution of the critical points of $E_4(\tau)$ in fundamental domains of SL$(2,\mathbb{Z})$. In this paper, it is convenient for us to choose

$$F := \{\tau \in \mathbb{H} \mid 0 \leqslant \operatorname{Re}\tau \leqslant 1, |\tau| \geqslant 1, |\tau - 1| \geqslant 1\} \tag{1.6}$$

to be the basic fundamental domain of SL$(2,\mathbb{Z})$, because $F \subset F_0$ and (see Figure 2 in Section 2)

$$F_0 = F \cup \gamma_1(F) \cup \gamma_2(F), \quad \gamma_1 := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \gamma_2 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, we can restate Theorem 1.1 as follows, which gives the distribution of the critical points of $E_4(\tau)$ in fundamental domains $\gamma(F)$'s of SL$(2,\mathbb{Z})$.

Theorem 1.5. *Let $\gamma(F)$ be a fundamental domain of* SL$(2,\mathbb{Z})$ *with* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in$ SL$(2,\mathbb{Z})/\{\pm I_2\}$ *and $\tau(C)$ be defined by* (1.2) *for $C \in \mathbb{R} \setminus \{0,1\}$. Then*

(1) *$E_4(\tau)$ has no critical points in $\gamma(F)$ if $\frac{-d}{c} \in (-1,2) \cup \{\infty\}$;*

(2) *$E_4(\tau)$ has exactly one critical point*

$$\frac{a\tau\left(\frac{-d}{c}\right) + b}{c\tau\left(\frac{-d}{c}\right) + d}$$

in $\gamma(F)$ if $\frac{-d}{c} \in (-\infty, -1] \cup [2, +\infty)$. Moreover, the unique critical point lies on the boundary $\partial\gamma(F) = \gamma(\partial F)$ if and only if $\frac{-d}{c} \in \{-1, 2\}$.

Similarly, we can transform every critical point of $E_4(\tau)$ via the Möbius transformation of SL$(2,\mathbb{Z})$ action to locate it in F. Denote *the collection of such corresponding points in F by $\mathcal{D}$*. Recalling the smooth curves $\mathcal{C}_-$, $\mathcal{C}_+$ in Theorem 1.2, we define two disjoint sets

$$\mathcal{C}_1 := \mathcal{C}_- \cap F, \quad \mathcal{C}_2 := \mathcal{C}_+ \cap F. \tag{1.7}$$

Theorem 1.6. *Under the above notations, the following hold.*

(1) *Each of $\mathcal{C}_1 \setminus \partial\mathcal{C}_1$ and $\mathcal{C}_2 \setminus \partial\mathcal{C}_2$ is a smooth curve in F with*

$$\partial\mathcal{C}_1 = \left\{\tau(-1), \frac{1}{4} + i\infty\right\}, \quad \partial\mathcal{C}_2 = \left\{\tau(2), \frac{3}{4} + i\infty\right\}$$

with $\tau(-1)$, $\tau(2) \in \partial F$;

(2) *$\mathcal{D}$ is a dense subset of the union of the two disjoint curves $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e.,*

$$\mathcal{D} \subset \mathcal{C}_1 \cup \mathcal{C}_2 = \overline{\mathcal{D}} \cap F = \overline{\mathcal{D}} \setminus \{\infty\}.$$

The above results give a complete description of the distribution of all critical points of $E_4(\tau)$ or equivalently $g_2(\tau)$. We believe that they will have important applications. Here we give an application to the spectrum of the $n = 3$ Lamé operator.

1.3. Application to the spectrum of the Lamé operator

As an application, we study the spectrum $\sigma(L) = \sigma(L; \tau)$ of the $n = 3$ Lamé operator

$$L := \frac{d^2}{dx^2} - 12\wp(x + z_0; \tau), \quad x \in \mathbb{R} \tag{1.8}$$

in $L^2(\mathbb{R}, \mathbb{C})$, where $z_0 \in \mathbb{C}$ is chosen such that $\wp(x + z_0; \tau)$ has no singularities on $\mathbb{R}$. The spectrum of the general Lamé operator

$$L_n := \frac{d^2}{dx^2} - n(n+1)\wp(x + z_0; \tau), \quad x \in \mathbb{R}$$

with $n \in \mathbb{N}$ has been widely studied in the literature; see e.g., [4, 7, 9, 14–17] and references therein. It is well known that the spectrum does not depend on the choice of z_0. For the $n = 3$ Lamé operator (1.8), let us recall the so-called spectral polynomial (see e.g., [14])

$$Q(B) = Q(B; \tau) = B \prod_{j=1}^{3} (B^2 - 6e_j B + 15(3e_j^2 - g_2)), \tag{1.9}$$

where $e_j = \wp(\frac{\omega_j}{2}|\tau)$ for $j = 1, 2, 3$. Since $3e_j^2 - g_2 \neq 0$ for any j, $B = 0$ is always a simple zero of $Q(B)$. Then it is already known the following.

- The spectrum $\sigma(L)$ consists of *one* semi-infinite simple analytic arc tending to $-\infty$ and *at most* 3 *bounded* analytic arcs. Furthermore, the *finite endpoints* of $\sigma(L)$ coincide with those zeros of the spectral polynomial $Q(B)$ with *odd order*. This result follows from Gesztesy and Weikard's remarkable result [16, Theorem 4.1]. Consequently, 0 is always an endpoint of $\sigma(L)$.
- When τ varies, different spectral arcs of $\sigma(L; \tau)$ might intersect with each other; see [14, 15]. There are two kinds of intersection points in general. One is that the intersection point B is not an endpoint (i.e., $Q(B) \neq 0$), so it is met by $2k$ semi-arcs for some $k \geq 2$. It was proved in [14] that such inner intersection point B must satisfy the following cubic polynomial

$$\frac{4}{15}B^3 + \frac{8}{5}\eta_1 B^2 - 3g_2 B + 9g_3 - 6\eta_1 g_2 = 0.$$

The other one is that the intersection point B is also an endpoint (i.e., $Q(B) = 0$), so it is met by $2k + 1$ semi-arcs for some $k \geq 1$. Such intersection point was called *a cusp* in [9], where it was proved that for any τ, $\sigma(L; \tau)$ has at most one cusp.

In view of the above theory and since 0 is always an endpoint of $\sigma(L;\tau)$, we ask a natural question: *When is* 0 *an intersection point of different spectral arcs, i.e., when is* 0 *is a cusp*? Here we can answer this question as follows.

Theorem 1.7. 0 *is a cusp of the spectrum* $\sigma(L;\tau)$ *if and only if* τ *is a critical point of* E_4. *In other words, using the same notations as Theorem 1.5, the following holds.*

(1) *If* $\frac{-d}{c} \in (-1,2) \cup \{\infty\}$ *and* $\tau \in \gamma(F)$, *then* 0 *is not a cusp of the spectrum* $\sigma(L;\tau)$.

(2) *If* $\frac{-d}{c} \in (-\infty,-1] \cup [2,+\infty)$ *and* $\tau \in \gamma(F)$, *then* 0 *is a cusp of the spectrum* $\sigma(L;\tau)$ *if and only if* $\tau = \frac{a\tau(\frac{-d}{c})+b}{c\tau(\frac{-d}{c})+d}$.

The rest of this paper is organized as follows. In Section 2, we introduce an auxiliary pre-modular form $Z_{r,s}^{(3)}(\tau)$ from the study of the Lamé equation in [13, 23], and recall the theorem concerning the zero structure of $Z_{r,s}^{(3)}(\tau)$ from our previous work [11]; see Theorem 2.1, which plays a fundamental role in this paper. In Section 3, we apply Theorem 2.1 to prove the existence and uniqueness of $\tau(C)$. See Theorem 3.1. In Section 4, we give the detailed proofs of Theorems 1.1–1.2. Some precise geometry of the three curves (see Theorem 4.2) will also be described. In Section 5, we apply Theorems 1.1–1.2 to prove Theorems 1.5–1.6. In Section 6, we introduce the geometric meaning of the three curves from the Green function on the elliptic curve, and also the monodromy meaning from the Lamé equation. Theorem 1.7 can be proved as a consequence.

2. An auxiliary pre-modular form

Our basic strategy is similar to our previous work [8] concerning E_2; the key point is to establish the existence and uniqueness of $\tau(C)$. For E_2, we used an auxiliary pre-modular form $Z_{r,s}^{(2)}(\tau)$ of weight 3 in [8]. Differently, here we need to study *a new auxiliary pre-modular form* $Z_{r,s}^{(3)}(\tau)$ of weight 6 introduced in [23, Example 5.9] (see also [13]). For each pair $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, we define a holomorphic function $Z_{r,s}^{(3)}(\tau)$ by

$$
\begin{aligned}
Z_{r,s}^{(3)}(\tau) := Z^6 - 15\wp Z^4 - 20\wp' Z^3 + \left(\frac{27}{4}g_2 - 45\wp^2\right)Z^2 \\
- 12\wp\wp' Z - \frac{5}{4}(\wp')^2, \qquad (2.1)
\end{aligned}
$$

where we write

$$\wp = \wp(r+s\tau|\tau), \quad \wp' = \wp'(r+s\tau|\tau)$$

for convenience and $Z = Z_{r,s}(\tau)$ is the Hecke form [18]:

$$Z = Z_{r,s}(\tau) := \zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau)$$
$$= \zeta(r + s\tau|\tau) - (r + s\tau)\eta_1(\tau) + 2\pi i s. \tag{2.2}$$

Here $\eta_k(\tau) := 2\zeta\left(\frac{\omega_k}{2}\middle|\tau\right)$, $k = 1, 2$, are the two quasi-periods of $\zeta(z|\tau)$:

$$\eta_1(\tau) = \zeta(z + 1|\tau) - \zeta(z|\tau), \quad \eta_2(\tau) = \zeta(z + \tau|\tau) - \zeta(z|\tau), \tag{2.3}$$

which satisfy the well-known Legendre relation $\eta_2(\tau) = \tau\eta_1(\tau) - 2\pi i$.

Note that $Z^{(3)}_{r,s}(\tau)$ is not well defined for $(r, s) = (0, 0)$ since $Z_{0,0} \equiv \infty$ and so do $\wp(0)$, $\wp'(0)$. To prove Theorems 1.1–1.2, in Section 3 we will "blow up" $Z^{(3)}_{r,s}(\tau)$ by considering $\lim_{s\to 0} Z^{(3)}_{-Cs,s}(\tau)$, $C \in \mathbb{R}$, and the existence and uniqueness of $\tau(C)$ will follow from that of the zero of $Z^{(3)}_{-Cs,s}(\tau)$ as $s \to 0$.

To this goal, we need to recall some basic properties of $Z^{(3)}_{r,s}(\tau)$. First, we recall the modularity of $g_2(\tau)$, $g_3(\tau)$, $\wp(z|\tau)$ and $\zeta(z|\tau)$. Given any $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2, \mathbb{Z})$, it is well known that

$$g_2\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^4 g_2(\tau), \quad g_3\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^6 g_3(\tau), \tag{2.4a}$$

$$\wp\Big(\frac{z}{c\tau + d}\ \Big|\ \frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^2\wp(z|\tau), \tag{2.4b}$$

$$\zeta\Big(\frac{z}{c\tau + d}\ \Big|\ \frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)\zeta(z|\tau). \tag{2.4c}$$

From here and $\eta_k(\tau) = 2\zeta\left(\frac{\omega_k}{2}\middle|\tau\right)$, we easily obtain

$$\begin{pmatrix} \eta_2\left(\frac{a\tau+b}{c\tau+d}\right) \\ \eta_1\left(\frac{a\tau+b}{c\tau+d}\right)\end{pmatrix} = (c\tau + d)\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\eta_2(\tau) \\ \eta_1(\tau)\end{pmatrix}. \tag{2.5}$$

In the rest of this paper, we will freely use the formulas (2.4)–(2.5).

As mentioned before, $Z^{(3)}_{r,s}(\tau)$ is not well defined for $(r, s) \in \mathbb{Z}^2$. If we take $(r, s) \in \frac{1}{2}\mathbb{Z}^2 \setminus \mathbb{Z}^2$, where

$$\frac{1}{2}\mathbb{Z}^2 := \left\{\Big(\frac{m}{2}, \frac{n}{2}\Big)\ \Big|\ m, n \in \mathbb{Z}\right\},$$

then (2.3) and the oddness of $\zeta(z|\tau)$ imply $Z_{r,s}(\tau) \equiv 0$ and so $Z^{(3)}_{r,s}(\tau) \equiv 0$, where we used $\wp'\left(\frac{\omega_k}{2}\right) = 0$. Thus, we only consider $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Then both $Z_{r,s}(\tau)$ and $Z^{(3)}_{r,s}(\tau)$ are holomorphic in $\mathbb{H}$, and it is easy to see that the following properties hold:

(P1) $Z_{r,s}(\tau) = \pm Z_{m\pm r,n\pm s}(\tau)$ and hence we get $Z^{(3)}_{r,s}(\tau) = Z^{(3)}_{m\pm r,n\pm s}(\tau)$ for any $(m, n) \in \mathbb{Z}^2$;

(P2) $Z_{r',s'}(\tau') = (c\tau + d)Z_{r,s}(\tau)$ and hence $Z^{(3)}_{r',s'}(\tau') = (c\tau + d)^6 Z^{(3)}_{r,s}(\tau)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, where $\tau' = \gamma \cdot \tau := \frac{a\tau+b}{c\tau+d}$ and $(s', r') = (s, r) \cdot \gamma^{-1} = (ds - cr, ar - bs)$.

In particular, when $(r, s) \in Q_N$ is an N-torsion point for some $N \in \mathbb{N}_{\geq 3}$, where

$$Q_N := \left\{ \left(\frac{k_1}{N}, \frac{k_2}{N} \right) \,\middle|\, \gcd(k_1, k_2, N) = 1,\ 0 \leq k_1, k_2 \leq N - 1 \right\},$$

then for any $\gamma \in \Gamma(N) := \{\gamma \in \mathrm{SL}(2, \mathbb{Z}) \mid f\gamma \equiv I_2 \bmod N\}$, we have $(r', s') \equiv (r, s) \bmod \mathbb{Z}^2$ and so properties (P1)–(P2) imply

$$Z_{r,s}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)Z_{r,s}(\tau), \quad Z^{(3)}_{r,s}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^6 Z^{(3)}_{r,s}(\tau),$$

namely $Z_{r,s}(\tau)$ and $Z^{(3)}_{r,s}(\tau)$ are *modular forms* of weight 1 and 6, respectively, with respect to $\Gamma(N)$. Due to this reason, $Z_{r,s}(\tau)$ and $Z^{(3)}_{r,s}(\tau)$ are called *pre-modular forms* in this paper as in [23]. It was proved in [23] that, given $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and τ_0, the monodromy group of the classical Lamé equation

$$y''(z) = [n(n + 1)\wp(z|\tau_0) + B]y(z), \quad n = 3 \tag{2.6}$$

for some $B \in \mathbb{C}$ is generated by

$$\begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}, \quad \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix}$$

if and only if $Z^{(3)}_{r,s}(\tau_0) = 0$.

We are interested in the zero structure of $Z^{(3)}_{r,s}(\tau)$ for $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. By property (P2), we can restrict τ in the basic fundamental domain F_0 of $\Gamma_0(2)$:

$$F_0 = \left\{ \tau \in \mathbb{H} \,\middle|\, 0 \leqslant \operatorname{Re} \tau \leqslant 1 \text{ and } \left| \tau - \frac{1}{2} \right| \geqslant \frac{1}{2} \right\},$$

and by (P1), we only need to consider $(r, s) \in [0, 1] \times \left[0, \frac{1}{2}\right] \setminus \frac{1}{2}\mathbb{Z}^2$. Define four open triangles (see Figure 1):

$$\triangle_0 := \left\{ (r, s) \,\middle|\, 0 < r, s < \frac{1}{2},\ r + s > \frac{1}{2} \right\}, \tag{2.7a}$$

$$\triangle_1 := \left\{ (r, s) \,\middle|\, \frac{1}{2} < r < 1,\ 0 < s < \frac{1}{2},\ r + s > 1 \right\}, \tag{2.7b}$$

$$\triangle_2 := \left\{ (r, s) \,\middle|\, \frac{1}{2} < r < 1,\ 0 < s < \frac{1}{2},\ r + s < 1 \right\}, \tag{2.7c}$$

$$\triangle_3 := \left\{ (r, s) \,\middle|\, r > 0,\ s > 0,\ r + s < \frac{1}{2} \right\}. \tag{2.7d}$$

Then $[0, 1] \times \left[0, \frac{1}{2}\right] = \bigcup_{k=0}^{3} \overline{\triangle_k}$. We recall the following result from our previous work [11].

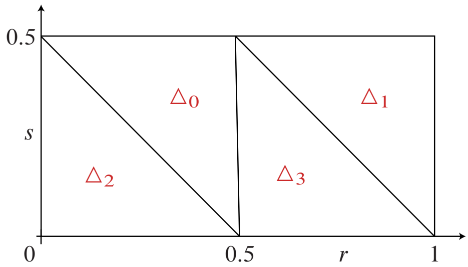

Figure 1. The four open triangles $\triangle_k$.

Theorem 2.1 ([11]). *Let* $(r,s) \in [0,1] \times \left[0, \frac{1}{2}\right] \setminus \frac{1}{2}\mathbb{Z}^2$.

(1) *For* $(r,s) \in \triangle_0$, $Z_{r,s}^{(3)}(\tau)$ *has exactly three different zeros in* F_0*, which are all simple and belong to the interior* $\mathring{F}_0$.

(2) *For* $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$, $Z_{r,s}^{(3)}(\tau)$ *has a unique zero in* F_0*, which is simple and also belongs to* $\mathring{F}_0$.

(3) *For* $(r,s) \in \bigcup_{k=0}^{3} \partial\triangle_k \setminus \frac{1}{2}\mathbb{Z}^2$, $Z_{r,s}^{(3)}(\tau)$ *has no zeros in* F_0.

In later sections, we will apply Theorem 2.1 to study the critical points of $E_4(\tau)$.

3. Existence and uniqueness of $\tau(C)$

The purpose of this section is to prove the existence and uniqueness of $\tau(C)$ for $C \in \mathbb{R} \setminus \{0,1\}$ by applying Theorem 2.1. This will give the parametrization of the three curves as mentioned in Section 1. Given $C \in \mathbb{R}$, we define a holomorphic function $f_C(\tau)$ on $\mathbb{H}$ by

$$f_C(\tau) := 2g_2(\tau)\big(C\eta_1(\tau) - \eta_2(\tau)\big) - 3g_3(\tau)(C - \tau). \tag{3.1}$$

By $\eta_2 = \tau\eta_1 - 2\pi i$, we see that $f_C(\tau) = 0$ if and only if (1.2) holds. Recall the fundamental domain F_0 of $\Gamma_0(2)$:

$$F_0 = \left\{\tau \in \mathbb{H} \;\middle|\; 0 \leqslant \operatorname{Re}\tau \leqslant 1 \text{ and } \left|\tau - \frac{1}{2}\right| \geqslant \frac{1}{2}\right\}.$$

The following result proves the existence and uniqueness of $\tau(C)$ as zeros of $f_C(\tau)$.

Theorem 3.1 (Zero structure of $f_C(\tau)$ in F_0). *The following facts hold.*

(1) *For any* $C \in \mathbb{R} \setminus \{0,1\}$, $f_C(\tau)$ *has a unique zero* $\tau(C)$ *in* F_0*. Furthermore,* $\tau(C) \in \mathring{F}_0$ *and is simple.*

(2) *For* $C \in \{0,1\}$, $f_C(\tau)$ *has no zeros in* F_0.

To prove Theorem 3.1, first we need the following lemma.

Lemma 3.2. *If $\tau = ib$ with $b > 0$, then $g_2'(\tau) \neq 0$ and*

$$2g_2(\tau)\eta_1(\tau) - 3g_3(\tau) > 0, \tag{3.2}$$

$$\tau(2\eta_2(\tau)g_2(\tau) - 3\tau g_3(\tau)) > 0. \tag{3.3}$$

Proof. Denote $q = e^{2\pi i\tau}$ and recall the q-expansion of $g_2(\tau)$ (cf. [20, p.44]):

$$g_2(\tau) = \frac{4}{3}\pi^4\Big(1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^k\Big), \quad \text{where } \sigma_3(k) = \sum_{1\leq d|k} d^3. \tag{3.4}$$

Let $\tau = ib$ with $b > 0$. Then $q = e^{-2\pi b}$ and hence $\frac{d}{db}g_2(ib) < 0$ for $b > 0$. So, $g_2'(\tau) \neq 0$ and (3.2) follows from (1.3). To prove (3.3), we use the following modular property (see (2.5)):

$$\eta_1\Big(\frac{-1}{\tau}\Big) = \tau\eta_2(\tau), \quad g_2\Big(\frac{-1}{\tau}\Big) = \tau^4 g_2(\tau), \quad g_3\Big(\frac{-1}{\tau}\Big) = \tau^6 g_3(\tau). \tag{3.5}$$

It follows that

$$\tau(2\eta_2(\tau)g_2(\tau) - 3\tau g_3(\tau)) = \frac{1}{\tau^4}\Big[2\eta_1\Big(\frac{-1}{\tau}\Big)g_2\Big(\frac{-1}{\tau}\Big) - 3g_3\Big(\frac{-1}{\tau}\Big)\Big] > 0,$$

i.e., (3.3) holds. ∎

Lemma 3.3. *For any $C \in \mathbb{R}\setminus\{0,1\}$, $f_C(\tau) \neq 0$ for $\tau \in \partial F_0 \cap \mathbb{H}$.*

Proof. Suppose $f_C(\tau) = 0$ for some $\tau \in \partial F_0 \cap \mathbb{H}$.

Case 1. $\tau \in i\mathbb{R}_{>0}$. It is known that $g_2(\tau) > 0$ and $\eta_1(\tau), g_3(\tau) \in \mathbb{R}$. It follows from $f_C(\tau) = 0$, (3.1) and (3.2) that

$$C = \tau - \frac{4\pi i g_2(\tau)}{2\eta_1(\tau)g_2(\tau) - 3g_3(\tau)} \in i\mathbb{R},$$

a contradiction with our assumption $C \in \mathbb{R}\setminus\{0\}$.

Case 2. $\left|\tau - \frac{1}{2}\right| = \frac{1}{2}$. One has $\tau' = \frac{\tau}{1-\tau} \in i\mathbb{R}_{>0}$. Define $C' := \frac{C}{1-C} \in \mathbb{R}\setminus\{0\}$. By $g_2(\tau') = (1-\tau)^4 g_2(\tau)$, $g_3(\tau') = (1-\tau)^6 g_3(\tau)$ and

$$\eta_2(\tau') = (1-\tau)\eta_2(\tau), \quad \eta_1(\tau') = (1-\tau)(\eta_1(\tau) - \eta_2(\tau)), \tag{3.6}$$

a straightforward computation leads to

$$f_{C'}(\tau') = \frac{(1-\tau)^5}{1-C} f_C(\tau) = 0.$$

Then we obtain a contradiction as in Case 1.

Case 3. $\tau \in 1 + i\mathbb{R}_{>0}$. One has $\tau' = \tau - 1 \in i\mathbb{R}_{>0}$. Define $C' := C - 1 \in \mathbb{R} \setminus \{0\}$. By using $g_2(\tau') = g_2(\tau)$, $g_3(\tau') = g_3(\tau)$ and

$$\eta_1(\tau') = \eta_1(\tau),\ \eta_2(\tau') = \eta_2(\tau) - \eta_1(\tau),$$

we easily obtain $f_{C'}(\tau') = f_C(\tau) = 0$, again a contradiction as in Case 1.

The proof is complete. ∎

Recall the pre-modular form $Z_{r,s}^{(3)}(\tau)$ in Section 2. Now, we study the precise relation between $Z_{r,s}^{(3)}(\tau)$ and $f_C(\tau)$. This is the key point of our whole idea. Fix any $C \in \mathbb{R}$, and for $s \in \left(0, \frac{1}{4(1+|C|)^2}\right)$ we define

$$F_{C,s}(\tau) := \frac{-4(\tau - C)}{9} Z_{-Cs,s}^{(3)}(\tau).$$

Lemma 3.4. *Letting $s \to 0$, $F_{C,s}(\tau)$ converges to $f_C(\tau)$ uniformly in any compact subset of $F_0 = \overline{F}_0 \cap \mathbb{H}$.*

Proof. Denote $u = -Cs + s\tau = s(\tau - C)$ for convenience. Then $u \to 0$ as $s \to 0$. Let $\tau \in K$ where K is any compact subset of F_0. Then $g_2 = g_2(\tau)$ and $g_3 = g_3(\tau)$ are uniformly bounded for $\tau \in K$. So, it follows from the Laurent series of $\zeta(\cdot|\tau)$ and $\wp(\cdot|\tau)$ (see e.g., [1,2]) that

$$\zeta(u|\tau) = \frac{1}{u} - \frac{g_2}{60}u^3 - \frac{g_3}{140}u^5 + O(|u|^7), \tag{3.7a}$$

$$\wp(u|\tau) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + O(|u|^6), \tag{3.7b}$$

$$\wp'(u|\tau) = \frac{-2}{u^3} + \frac{g_2}{10}u + \frac{g_3}{7}u^3 + O(|u|^5) \tag{3.7c}$$

hold uniformly for $\tau \in K$ as $s \to 0$. From (3.7) and (2.2), we see that

$$Z_{-Cs,s}(\tau) = \frac{1}{u} + au - \frac{g_2}{60}u^3 - \frac{g_3}{140}u^5 + O(|u|^7),$$

where

$$a := \frac{2\pi i}{\tau - C} - \eta_1 = \frac{C\eta_1 - \eta_2}{\tau - C}.$$

Inserting these into the expression (2.1) of $Z_{-Cs,s}^{(3)}(\tau)$, a lengthy yet straightforward calculation (it is much easier if using Mathematica) implies that

$$\begin{aligned} Z_{-Cs,s}^{(3)}(\tau) &= -\frac{9}{4}(2ag_2 + 3g_3) + O(|u|^2) \\ &= -\frac{9}{4}\left(2g_2\left(\frac{2\pi i}{\tau - C} - \eta_1\right) + 3g_3\right) + O(|u|^2) \\ &= -\frac{9}{4(\tau - C)} f_C(\tau) + O(|u|^2) \end{aligned} \tag{3.8}$$

uniformly for $\tau \in K$ as $s \to 0$. The proof is complete. ∎

Lemma 3.5. *Let $s > 0$. Then as $s \to 0$, any zero $\tau(s) \in \{\tau \in \mathbb{H} | \operatorname{Re}\tau \in [-1, 1]\}$ of $Z^{(3)}_{-Cs,s}(\tau)$ (if exist) is uniformly bounded.*

Proof. Assume by contradiction that up to a subsequence of $s \to 0$, $Z^{(3)}_{-Cs,s}(\tau)$ has a zero $\tau(s) \in \{\tau \in \mathbb{H} | \operatorname{Re}\tau \in [-1, 1]\}$ such that $\tau(s) \to \infty$ as $s \to 0$. Write $\tau = \tau(s) = a(s) + ib(s)$, then $a(s) \in [-1, 1]$ and $b(s) \to +\infty$.

As before, we denote $u := -Cs + s\tau = s(a(s) - C + ib(s))$ and $q = e^{2\pi i\tau}$. We also denote $x = e^{2\pi iu}$ for convenience. Then

$$e^{2\pi b(s)} = |q|^{-1} > 1 > |x| = e^{-2\pi sb(s)} > |q| = e^{-2\pi b(s)}. \tag{3.9}$$

We note that if $x \to 1$, then $sb(s) \to 0$ and

$$1 - x = 1 - e^{2\pi is(a(s)-C+ib(s))} = 2\pi sb(s) + o(sb(s)).$$

Together with $b(s) \to +\infty$ as $s \to 0$, we always have

$$s = o(|x - 1|). \tag{3.10}$$

Since $|x^{-1}q| = e^{-2\pi(1-s)b(s)} \to 0$ as $s \to 0$, there are two cases.

Case 1. *Up to a subsequence, $|x^{-1}q| \geq ds^3|x - 1|^3|x|$ for some constant $d > 0$.* It follows from (3.10) that

$$e^{-2\pi(1-2s)b(s)} = |x^{-2}q| \geq ds^3|x - 1|^3 \geq s^6$$

and so $b(s) \leq \ln\frac{1}{s}$ for $s > 0$ small. Then $u = s(\tau - C) = s(a(s) - C + ib(s)) \to 0$ as $s \to 0$ and

$$u^2 = \frac{s^2(a(s) - C + ib(s))^3}{\tau - C} = o(|\tau - C|^{-1}). \tag{3.11}$$

Recall the q-expansions of $\eta_1(\tau)$ and $g_3(\tau)$ (see e.g., [20, p.44]):

$$\eta_1(\tau) = \frac{\pi^2}{3}\Big(1 - 24\sum_{k=1}^{\infty}\sigma_1(k)q^k\Big), \qquad \text{where } \sigma_1(k) = \sum_{1\leq d|k} d. \tag{3.12}$$

$$g_3(\tau) = \frac{8}{27}\pi^6\Big(1 - 504\sum_{k=1}^{\infty}\sigma_5(k)q^k\Big), \quad \text{where } \sigma_5(k) = \sum_{1\leq d|k} d^5. \tag{3.13}$$

Since $b(s) \to +\infty$, (3.4) and (3.12)–(3.13) show that

$$g_2 = \frac{4}{3}\pi^4 + O(|q|), \quad g_3 = \frac{8}{27}\pi^6 + O(|q|)$$

and

$$\eta_1 = \frac{1}{3}\pi^2 + O(|q|)$$

are uniformly bounded, so (3.7)–(3.8) still hold, namely

$$0 = Z^{(3)}_{-Cs,s}(\tau(s)) = -\frac{9}{4}\Big(2g_2\Big(\frac{2\pi i}{\tau - C} - \eta_1\Big) + 3g_3\Big) + O(|u|^2). \tag{3.14}$$

Since

$$2g_2\eta_1 - 3g_3 = O(|q|) = O(e^{-2\pi b(s)}) = o(|\tau - C|^{-1}),$$

we easily obtain from (3.14) and (3.11) that

$$\begin{aligned} 0 &= -\frac{9}{4}\Big(2g_2\Big(\frac{2\pi i}{\tau - C} - \eta_1\Big) + 3g_3\Big) + O(|u|^2) \\ &= \frac{-12\pi^5 i}{\tau - C} + o(|\tau - C|^{-1}), \end{aligned} \tag{3.15}$$

which is a contradiction.

Case 2. *Up to a subsequence,* $|x^{-1}q| = o(s^3|x-1|^3|x|)$. We need to treat this case in a different way since $u = s(\tau - C)$ does not necessarily converge to 0 as $s \to 0$. We recall the q-expansions (see e.g. [20, p.46] for $\wp$ and [8, (3.13)] for $Z_{r,s}$): for $|q| < |e^{2\pi i z}| < |q|^{-1}$,

$$\begin{aligned} \frac{\wp(z|\tau)}{-4\pi^2} &= \frac{1}{12} + \frac{e^{2\pi i z}}{(1-e^{2\pi i z})^2} \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n q^{nm}(e^{2\pi i n z} + e^{-2\pi i n z} - 2), \end{aligned} \tag{3.16a}$$

$$\begin{aligned} \frac{\wp'(z|\tau)}{-4\pi^2} &= \frac{2\pi i e^{2\pi i z}}{(1-e^{2\pi i z})^2} + \frac{4\pi i e^{4\pi i z}}{(1-e^{2\pi i z})^3} \\ &\quad + 2\pi i \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^2 q^{nm}(e^{2\pi i n z} - e^{-2\pi i n z}), \end{aligned} \tag{3.16b}$$

and

$$\begin{aligned} Z_{r,s}(\tau) &= 2\pi i s - \pi i \frac{1+e^{2\pi i z}}{1-e^{2\pi i z}} \\ &\quad - 2\pi i \sum_{n=1}^{\infty}\Big(\frac{e^{2\pi i z}q^n}{1-e^{2\pi i z}q^n} - \frac{e^{-2\pi i z}q^n}{1-e^{-2\pi i z}q^n}\Big), \end{aligned} \tag{3.17}$$

where $z = r + s\tau$ in (3.17).

Now, we let $z = u = s(a(s) - C + ib(s))$, then $e^{2\pi i z} = x$ and it follows from (3.9) that we can apply the above q-expansions. It is easy to see that

$$2 > |x+1| > \frac{1}{2}, \quad 12 > |x^2 + 10x + 1| > \frac{1}{2} \quad \text{for } s \text{ small.}$$

Recalling $|x| < 1$ and our assumption $|x^{-1}q| = o(s^3|x-1|^3|x|) = o(s^3|x-1|^3)$, we derive from (3.4) and (3.16)–(3.17) that

$$g_2 = \frac{4}{3}\pi^4 + o(s^3|x-1|^3), \tag{3.18}$$

$$\wp := \wp(z|\tau) = -\frac{\pi^2}{3}\frac{x^2+10x+1}{(x-1)^2} + o(s^3|x-1|^3),$$

$$\wp' := \wp'(z|\tau) = 8\pi^3 i\frac{x(x+1)}{(x-1)^3} + o(s^3|x-1|^3|x|),$$

$$Z := Z_{-Cs,s}(\tau) = \pi i\Big(\frac{x+1}{x-1} + 2s\Big) + o(s^3|x-1|^3). \tag{3.19}$$

Then by (3.18)–(3.19) and (3.10) that $s = o(|x-1|)$, we easily obtain

$$\begin{aligned}
Z^6 =& -\pi^6\Big[\frac{(x+1)^6}{(x-1)^6} + 12s\frac{(x+1)^5}{(x-1)^5} + 60s^2\frac{(x+1)^4}{(x-1)^4}\Big] \\
&+ O\Big(\frac{s^3}{|x-1|^3}\Big),\\
-15\wp Z^4 =& \pi^6\Big[\frac{5(x+1)^4(x^2+10x+1)}{(x-1)^6} \\
&+ 40s\frac{(x+1)^3(x^2+10x+1)}{(x-1)^5} \\
&+ 120s^2\frac{(x+1)^2(x^2+10x+1)}{(x-1)^4}\Big] + O\Big(\frac{s^3}{|x-1|^3}\Big),\\
-20\wp' Z^3 =& -160\pi^6\Big[\frac{x(x+1)^4}{(x-1)^6} + 6s\frac{x(x+1)^3}{(x-1)^5} + 12s^2\frac{x(x+1)^2}{(x-1)^4}\Big] \\
&+ O\Big(\frac{s^3|x|}{|x-1|^3}\Big),\\
\Big(\frac{27}{4}g_2 - 45\wp^2\Big)Z^2 =& -4\pi^6(x^4-34x^3-114x^2-34x+1) \\
&\cdot\Big[\frac{(x+1)^2}{(x-1)^6} + 4s\frac{(x+1)}{(x-1)^5} + 4s^2\frac{1}{(x-1)^4}\Big] + o\Big(\frac{s^3}{|x-1|^2}\Big),\\
-12\wp\wp' Z =& -32\pi^6\Big[\frac{x(x+1)^2(x^2+10x+1)}{(x-1)^6} \\
&+ 2s\frac{x(x+1)(x^2+10x+1)}{(x-1)^5}\Big] + o\Big(\frac{s^3|x|}{|x-1|^2}\Big),\\
-\frac{5}{4}(\wp')^2 =& 80\pi^6\frac{x^2(x+1)^2}{(x-1)^6} + o(s^3|x|^2).
\end{aligned}$$

Therefore, inserting these formulas into (2.1) leads to

$$0 = Z^{(3)}_{-Cs,s}(\tau(s)) = \pi^6\Big[12s\frac{(x+1)}{(x-1)} + 44s^2 + O\Big(\frac{s^3}{|x-1|^3}\Big)\Big].$$

Since $|x-1|<2$, $|x+1|>\frac{1}{2}$ and $s=o(|x-1|)$ imply

$$44s^2 + O\Big(\frac{s^3}{|x-1|^3}\Big) = o\Big(s\Big|\frac{x+1}{x-1}\Big|\Big),$$

we finally obtain

$$0 = 12\pi^6 s\frac{(x+1)}{(x-1)} + o\Big(s\Big|\frac{x+1}{x-1}\Big|\Big),$$

a contradiction. Remark that, when $u = s(\tau - C) \to 0$ as $s \to 0$, we have $x \to 1$ and then $12\pi^6 s\frac{(x+1)}{(x-1)} = \frac{-12\pi^5 i}{\tau - C} + o(|\tau - C|^{-1})$, the same as (3.15).

The proof is complete. ∎

Recall $\triangle_k, k = 0,1,2,3$, defined in (2.7). We define

$$\begin{aligned}\tilde{\triangle}_1 &:= \{(r,s) \mid (r+1,s) \in \triangle_1\} \\ &= \Big\{(r,s) \;\Big|\; 0<s<\frac{1}{2}, \frac{-1}{2}<r<0, r+s>0\Big\}, \\ \tilde{\triangle}_2 &:= \{(r,s) \mid (r+1,s) \in \triangle_2\} \\ &= \Big\{(r,s) \;\Big|\; 0<s<\frac{1}{2}, \frac{-1}{2}<r<0, r+s<0\Big\}.\end{aligned}$$

Recalling property (P1) in Section 2 that $Z^{(3)}_{r,s}(\tau) = Z^{(3)}_{r+1,s}(\tau)$, we see from Theorem 2.1 (2) that

for $(r,s) \in \tilde{\triangle}_1 \cup \tilde{\triangle}_2 \cup \triangle_3$, $Z^{(3)}_{r,s}(\cdot)$ *has a unique and simple zero in* F_0. (3.20)

Now, we fix $C \in (-\infty,0)\cup(0,1)\cup(1,\infty)$. Then $s \in \big(0, \frac{1}{4(1+|C|)^2}\big)$ implies $(-Cs,s) \in \tilde{\triangle}_1 \cup \tilde{\triangle}_2 \cup \triangle_3$, so (3.20) implies that $Z^{(3)}_{-Cs,s}(\tau)$ *has a unique and simple zero* $\tau(s) \in F_0$. By the definition of F_0, we easily see that

$$\frac{-1}{\tau(s)}, \frac{\tau(s)}{1-\tau(s)} \in \{\tau \in \mathbb{H} \mid \operatorname{Re}\tau \in [-1,1]\}.$$

Lemma 3.6. *As* $s \to 0$, *the unique zero* $\tau(s) \in F_0$ *of* $Z^{(3)}_{-Cs,s}(\tau)$ *cannot converge to any of* $\{0,1,\infty\}$.

Proof. Lemma 3.5 shows $\tau(s) \not\to \infty$. To prove $\tau(s) \not\to \{0,1\}$, we use the properties (P1)-(P2) of $Z^{(3)}_{r,s}(\tau)$ in Section 2:

$$Z^{(3)}_{m\pm r, n\pm s}(\tau) = Z^{(3)}_{r,s}(\tau), \quad \text{for all } m,n \in \mathbb{Z},$$

and

$$Z^{(3)}_{r',s'}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^6 Z^{(3)}_{r,s}(\tau),$$

where

$$(s',r') = (s,r)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Letting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we obtain

$$Z^{(3)}_{s,-r}\Big(\frac{-1}{\tau}\Big) = \tau^6 Z^{(3)}_{r,s}(\tau).$$

Recall $C \in (-\infty,0)\cup(0,1)\cup(1,+\infty)$ and $s \in \big(0,\frac{1}{4(1+|C|)^2}\big)$.

Case 1. $C \in (-\infty,0)$. By defining

$$\widetilde{C} := \frac{-1}{C}, \quad \tilde{s} := -Cs > 0,$$

we have $\tilde{s} \in \big(0,\frac{1}{4(1+|\widetilde{C}|)^2}\big)$ for s small and

$$\tau^6 Z^{(3)}_{-Cs,s}(\tau) = Z^{(3)}_{s,Cs}\Big(\frac{-1}{\tau}\Big) = Z^{(3)}_{-s,-Cs}\Big(\frac{-1}{\tau}\Big) = Z^{(3)}_{-\widetilde{C}\tilde{s},\tilde{s}}\Big(\frac{-1}{\tau}\Big).$$

Therefore, $Z^{(3)}_{-\widetilde{C}\tilde{s},\tilde{s}}(\tau)$ has zero $\frac{-1}{\tau(s)} \in \{\tau \in \mathbb{H} \mid \operatorname{Re}\tau \in [-1,1]\}$. Since $\tilde{s} \to 0$ as $s \to 0$, Lemma 3.5 implies $\frac{-1}{\tau(s)} \not\to \infty$, i.e., $\tau(s) \not\to 0$ as $s \to 0$.

Case 2. $C \in (0,1)\cup(1,+\infty)$ By defining

$$\widetilde{C} := \frac{-1}{C}, \quad \tilde{s} := Cs > 0,$$

we have $\tilde{s} \in \big(0,\frac{1}{4(1+|\widetilde{C}|)^2}\big)$ for s small and

$$\tau^6 Z^{(3)}_{-Cs,s}(\tau) = Z^{(3)}_{s,Cs}\Big(\frac{-1}{\tau}\Big) = Z^{(3)}_{-\widetilde{C}\tilde{s},\tilde{s}}\Big(\frac{-1}{\tau}\Big).$$

Again, we obtain $\tau(s) \not\to 0$ as $s \to 0$.

To prove $\tau(s) \not\to 1$ as $s \to 0$, we let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and obtain

$$Z^{(3)}_{r,r+s}\Big(\frac{\tau}{1-\tau}\Big) = (1-\tau)^6 Z^{(3)}_{r,s}(\tau).$$

Case 3. $C \in (-\infty,0)\cup(0,1)$. By defining

$$\widetilde{C} := \frac{C}{1-C}, \quad \tilde{s} := (1-C)s > 0,$$

we have $\tilde{s} \in \big(0,\frac{1}{4(1+|\widetilde{C}|)^2}\big)$ for s small and

$$(1-\tau)^6 Z^{(3)}_{-Cs,s}(\tau) = Z^{(3)}_{-Cs,(1-C)s}\Big(\frac{\tau}{1-\tau}\Big) = Z^{(3)}_{-\widetilde{C}\tilde{s},\tilde{s}}\Big(\frac{\tau}{1-\tau}\Big).$$

So, $Z^{(3)}_{-\tilde{C}\tilde{s},\tilde{s}}(\tau)$ has zero $\frac{\tau(s)}{1-\tau(s)} \in \{\tau \in \mathbb{H} \mid \operatorname{Re}\tau \in [-1,1]\}$, and Lemma 3.5 implies $\frac{\tau(s)}{1-\tau(s)} \not\to \infty$, i.e., $\tau(s) \not\to 1$ as $s \to 0$.

Case 4. $C \in (1, +\infty)$. By defining

$$\tilde{C} := \frac{C}{1-C}, \quad \tilde{s} := -(1-C)s > 0,$$

we have $\tilde{s} \in \left(0, \frac{1}{4(1+|\tilde{C}|)^2}\right)$ for s small and similarly,

$$(1-\tau)^6 Z^{(3)}_{-Cs,s}(\tau) = Z^{(3)}_{Cs,-(1-C)s}\Big(\frac{\tau}{1-\tau}\Big) = Z^{(3)}_{-\tilde{C}\tilde{s},\tilde{s}}\Big(\frac{\tau}{1-\tau}\Big).$$

Again, we obtain $\tau(s) \not\to 1$ as $s \to 0$. The proof is complete. ∎

Now, we are in a position to prove Theorem 3.1 (1).

Proof of Theorem 3.1 (1). Fix $C \in \mathbb{R} \setminus \{0,1\}$ and let $s \in \left(0, \frac{1}{4(1+|C|)^2}\right)$. Recall that $\tau(s)$ is the unique and simple zero of $Z^{(3)}_{-Cs,s}(\tau)$ in F_0. By Lemma 3.6, up to a subsequence of $s \to 0$, we have

$$\tau(C) := \lim_{s\to 0} \tau(s) \in \overline{F}_0 \cap \mathbb{H} = F_0. \tag{3.21}$$

Recalling

$$F_{C,s}(\tau) = \frac{-4(\tau - C)}{9} Z^{(3)}_{-Cs,s}(\tau),$$

we have $F_{C,s}(\tau(s)) = 0$. Then Lemma 3.4 implies $f_C(\tau(C)) = 0$, namely $\tau(C) \in F_0$ is a zero of $f_C(\tau)$. Applying Lemma 3.3, we have $\tau(C) \in \mathring{F}_0$. Suppose $f_C(\tau)$ has another zero $\tau_1 \neq \tau(C)$ in $\mathring{F}_0$. Since $F_{C,s}(\tau)$ and $f_C(\tau)$ are all holomorphic functions, it follows from Lemma 3.4 and Rouché's theorem that $F_{C,s}(\tau)$ has a zero $\tau_1(s)$ satisfying $\tau_1(s) \to \tau_1$ as $s \to 0$, namely $Z^{(3)}_{-Cs,s}(\tau)$ has two different zeros $\tau(s)$ and $\tau_1(s)$ in F_0 for $s > 0$ small, a contradiction with (3.20). Therefore, $\tau(C)$ is the unique zero of $f_C(\tau)$ in F_0. The same argument also implies that $\tau(C)$ is a simple zero and (3.21) actually holds for $s \to 0$ (i.e., not only for a subsequence).

The proof is complete. ∎

The proof of Theorem 3.1 (2) will be postponed to the next section. As in Theorem 3.1 (1), we always denote by $\tau(C)$ the unique zero of $f_C(\tau)$ in F_0. Since $\tau(C)$ is simple, the implicit function theorem infers that $\tau(C)$ *is a smooth function of* $C \in \mathbb{R} \setminus \{0,1\}$. Here we study some basic properties of $\tau(C)$ for later usage.

Lemma 3.7. *The smooth function* $\tau(C)$ *satisfies*

$$\tau\Big(\frac{1}{1-C}\Big) = \frac{1}{1-\tau(C)}, \quad \textit{for all } C \in \mathbb{R} \setminus \{0,1\}. \tag{3.22}$$

Proof. Let $\tau' = \frac{1}{1-\tau}$ and $C' = \frac{1}{1-C}$, then we have

$$\tau' \in F_0 \iff \tau \in F_0 \quad \text{and} \quad C' \in \mathbb{R} \setminus \{0,1\} \iff C \in \mathbb{R} \setminus \{0,1\}.$$

By using $g_2(\tau') = (1-\tau)^4 g_2(\tau)$, $g_3(\tau') = (1-\tau)^6 g_3(\tau)$ and

$$\eta_2(\tau') = (1-\tau)\eta_1(\tau),\ \eta_1(\tau') = (1-\tau)(\eta_1(\tau) - \eta_2(\tau)),$$

a straightforward computation leads to

$$f_{C'}(\tau') = \frac{(1-\tau)^5}{1-C} f_C(\tau).$$

So, $f_C(\tau(C)) = 0$ gives $f_{C'}\left(\frac{1}{1-\tau(C)}\right) = 0$. Applying Theorem 3.1 (1), we obtain (3.22). This completes the proof. ■

Lemma 3.8. *Write $\tau(C) = a(C) + b(C)i$ with $a(C), b(C) \in \mathbb{R}$. Then*

$$b(C) \to +\infty,\ a(C) \to \begin{cases} 1/4- & \text{if } C \to -\infty, \\ 3/4+ & \text{if } C \to +\infty, \end{cases} \tag{3.23}$$

$$\tau(C) \to 0 \text{ as } C \to 0 \quad \text{and} \quad \tau(C) \to 1 \text{ as } C \to 1. \tag{3.24}$$

Proof. Recalling (1.2), we define

$$\phi(\tau) := \tau - \frac{4\pi i g_2(\tau)}{2\eta_1(\tau) g_2(\tau) - 3 g_3(\tau)}, \quad \tau \in F_0. \tag{3.25}$$

Write $\tau = a + bi$ and $q = e^{2\pi i \tau}$ as before. Recall from the q-expansions (3.4) and (3.12)–(3.13) that

$$\eta_1(\tau) = \frac{1}{3}\pi^2(1 - 24q - 72q^2) + O(|q|^3),$$
$$g_2(\tau) = \frac{4}{3}\pi^4(1 + 240(q + 9q^2)) + O(|q|^3),$$
$$g_3(\tau) = \frac{8}{27}\pi^6(1 - 504(q + 33q^2)) + O(|q|^3).$$

Inserting these into (3.25) leads to

$$\begin{aligned}\phi(\tau) &= \tau - \frac{i}{120\pi} q^{-1} - \frac{37i}{20\pi} + O(|q|) \\ &= a - \frac{\sin 2\pi a}{120\pi} e^{2\pi b} + i\left(b - \frac{\cos 2\pi a}{120\pi} e^{2\pi b} - \frac{37}{20\pi}\right) + O(|q|).\end{aligned}$$

Therefore, when $C \in \mathbb{R}$ and $|C| \to +\infty$, it is easy to prove the existence of $\tau_1(C) = a_1(C) + ib_1(C) \in \mathring{F}_0$ such that $C = \phi(\tau_1(C))$ and

$$b_1(C) \to +\infty, \quad a_1(C) \to \begin{cases} 1/4- & \text{if } C \to -\infty, \\ 3/4+ & \text{if } C \to +\infty. \end{cases}$$

As mentioned before, $C = \phi(\tau_1(C))$ implies $f_C(\tau_1(C)) = 0$. Since $\tau(C)$ is the unique zero of f_C in F_0, we conclude $\tau(C) = \tau_1(C)$. This proves (3.23), and then (3.24) follows from (3.23) and (3.22). ∎

4. Distribution in fundamental domains of $\Gamma_0(2)$

This section is devoted to the proof of Theorems 1.1–1.2. First, we prove the following result, which implies Theorem 1.1 as a consequence.

Theorem 4.1. *Let $\tau(C)$ be the unique zero of $f_C(\tau)$ for $C \in \mathbb{R} \setminus \{0, 1\}$ given in Theorem 3.1 (1). Then the following holds.*

(1) *For any $m \in \mathbb{Z}$, there holds $g_2'(\tau) \neq 0$ in $F_0 + m$. Consequently, $g_2'(\tau) \neq 0$ whenever $\operatorname{Im} \tau \geq \frac{1}{2}$.*

(2) *Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$. Then $\frac{a\tau(-d/c)+b}{c\tau(-d/c)+d}$ is the unique zero of $g_2'(\tau)$ in the fundamental domain $\gamma(F_0)$ of $\Gamma_0(2)$. In particular,*

$$\Theta := \left\{ \frac{a\tau\left(\frac{-d}{c}\right) + b}{c\tau\left(\frac{-d}{c}\right) + d} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)/\{\pm I_2\} \text{ with } c \neq 0 \right\} \tag{4.1}$$

gives rise to all the zeros of $g_2'(\tau)$ in $\mathbb{H}$.

Proof. (1) First, we claim that

$$2\eta_1(\tau)g_2(\tau) - 3g_3(\tau) \neq 0 \quad \text{for } \tau \in \partial F_0 \cap \mathbb{H}. \tag{4.2}$$

If $\tau \in i\mathbb{R}_{>0}$, Lemma 3.2 shows $2\eta_1(\tau)g_2(\tau) - 3g_3(\tau) > 0$. If $\tau \in i\mathbb{R}_{>0} + 1$, then

$$2\eta_1(\tau)g_2(\tau) - 3g_3(\tau) = 2\eta_1(\tau - 1)g_2(\tau - 1) - 3g_3(\tau - 1) > 0.$$

If $\left|\tau - \frac{1}{2}\right| = \frac{1}{2}$, then $\tau' = \frac{\tau}{1-\tau} \in i\mathbb{R}_{>0}$. By using (3.6), we see from (3.1) with $C = -1$ that

$$\begin{aligned} f_{-1}(\tau') &= 2g_2(\tau')(-\eta_1(\tau') - \eta_2(\tau')) - 3g_3(\tau')(-1 - \tau') \\ &= -(1 - \tau)^5[2\eta_1(\tau)g_2(\tau) - 3g_3(\tau)]. \end{aligned}$$

Since Lemma 3.3 shows $f_{-1}(\tau') \neq 0$, we obtain $2\eta_1(\tau)g_2(\tau) - 3g_3(\tau) \neq 0$. This proves (4.2).

Suppose by contradiction that $2\eta_1(\tau)g_2(\tau) - 3g_3(\tau)$ has a zero τ_0 in $\mathring{F}_0$. Then $g_2(\tau_0) \neq 0$ because $g_2^3 - 27g_3^2 \neq 0$ for any τ. Recalling $\phi(\tau)$ in (3.25), it follows that $\phi(\tau)$ is meromorphic at τ_0 with τ_0 being a pole and so maps a small neighborhood $U \subset \mathring{F}_0$ of τ_0 onto a neighborhood of ∞. Then for $C > 0$ large enough, there exists $\tau_1(C) \in U$ such that $C = \phi(\tau_1(C))$, which implies $f_C(\tau_1(C)) = 0$. Applying Theorem 3.1 (1) and Lemma 3.8, we obtain $\tau_1(C) = \tau(C) \to \infty$ as $C \to +\infty$, which contradicts with $\tau_1(C) \in U$.

Therefore, we have proved that

$$2\eta_1(\tau)g_2(\tau) - 3g_3(\tau) \neq 0 \quad \text{for any } \tau \in F_0. \tag{4.3}$$

Since (1.3) says

$$g_2'(\tau) = \frac{i}{\pi}\big(2\eta_1(\tau)g_2(\tau) - 3g_3(\tau)\big)$$

and $g_2(\tau+1) = g_2(\tau)$, we conclude that $g_2'(\tau) \neq 0$ for any $\tau \in F_0 + m$ and $m \in \mathbb{Z}$. This proves (1).

(2) Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$. Write $\tau' = \gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$ with $\tau \in F_0$. By using $g_3(\tau') = (c\tau+d)^6 g_3(\tau)$ and

$$\eta_1(\tau') = (c\tau+d)(c\eta_2(\tau) + d\eta_1(\tau)), \quad g_2(\tau') = (c\tau+d)^4 g_2(\tau),$$

we have

$$\begin{aligned} &2\eta_1(\tau')g_2(\tau') - 3g_3(\tau') \\ &\quad = -c(c\tau+d)^5\Big[2g_2(\tau)\Big(-\frac{d}{c}\eta_1(\tau) - \eta_2(\tau)\Big) - 3g_3(\tau)\Big(-\frac{d}{c} - \tau\Big)\Big] \\ &\quad = -c(c\tau+d)^5 f_{\frac{-d}{c}}(\tau). \end{aligned}$$

Clearly, $\frac{-d}{c} \in \mathbb{Q} \setminus \mathbb{Z}$, so Theorem 3.1 (1) shows that $\tau\big(\frac{-d}{c}\big) \in \mathring{F}_0$ is the unique zero of $f_{\frac{-d}{c}}(\tau)$ in F_0. Consequently,

$$\gamma \cdot \tau\Big(\frac{-d}{c}\Big) = \frac{a\tau\big(\frac{-d}{c}\big) + b}{c\tau\big(\frac{-d}{c}\big) + d} \in \gamma(\mathring{F}_0)$$

is the unique zero of $2\eta_1 g_2 - 3g_3$ in $\gamma(F_0)$. Since

$$\mathbb{H} = \bigcup_{\gamma \in \Gamma_0(2)/\{\pm I_2\}} \gamma(F_0),$$

we conclude that the set Θ defined in (4.1) gives all the zeros of $2\eta_1 g_2 - 3g_3$ and so g_2'. This proves (2). The proof is complete. ∎

Now, we can finish the proof of Theorem 3.1.

Proof of Theorem 3.1 (2). First we consider $C = 0$, i.e.,

$$f_0(\tau) = -2g_2(\tau)\eta_2(\tau) + 3\tau g_3(\tau).$$

Suppose $f_0(\tau) = 0$ for some $\tau \in F_0$. Then it is easy to see that $\tau' := \frac{\tau-1}{\tau} \in F_0$. By

$$\eta_1(\tau') = \tau\eta_2(\tau), \quad g_2(\tau') = \tau^4 g_2(\tau), \quad g_3(\tau') = \tau^6 g_3(\tau),$$

we obtain

$$2\eta_1(\tau')g_2(\tau') - 3g_3(\tau') = -\tau^5 f_0(\tau) = 0,$$

a contradiction with (4.3).

Now, we consider $C = 1$, i.e.,

$$f_1(\tau) = 2g_2(\tau)(\eta_1(\tau) - \eta_2(\tau)) - 3g_3(\tau)(1-\tau).$$

Suppose $f_1(\tau) = 0$ for some $\tau \in F_0$. Then it is easy to see that $\tau' := \frac{1}{1-\tau} \in F_0$. By $g_3(\tau') = (1-\tau)^6 g_3(\tau)$ and

$$\eta_1(\tau') = (1-\tau)(\eta_1(\tau) - \eta_2(\tau)), \quad g_2(\tau') = (1-\tau)^4 g_2(\tau),$$

we obtain

$$2\eta_1(\tau')g_2(\tau') - 3g_3(\tau') = (1-\tau)^5 f_1(\tau) = 0,$$

again a contradiction with (4.3). The proof is complete. ∎

Recall the curves defined in Section 1:

$$\mathcal{C}_0 = \{\tau(C) \mid C \in (0,1)\},$$
$$\mathcal{C}_- = \{\tau(C) \mid C \in (-\infty, 0)\}, \quad \mathcal{C}_+ = \{\tau(C) \mid C \in (1, +\infty)\}.$$

Clearly, (3.22) implies

$$\mathcal{C}_0 = \left\{\frac{1}{1-\tau} \,\middle|\, \tau \in \mathcal{C}_-\right\}, \quad \mathcal{C}_- = \left\{\frac{1}{1-\tau} \,\middle|\, \tau \in \mathcal{C}_+\right\}, \quad \mathcal{C}_+ = \left\{\frac{1}{1-\tau} \,\middle|\, \tau \in \mathcal{C}_0\right\}. \tag{4.4}$$

Before going to the proof of Theorem 1.2, we want to describe some geometry about these three curves.

Theorem 4.2. *The following holds.*

(1) *The function $C \mapsto \tau(C)$ is one-to-one for $C \in \mathbb{R} \setminus \{0,1\}$, i.e., any one of the curves $\mathcal{C}_-$, $\mathcal{C}_0$, $\mathcal{C}_+$ has no self-intersection, and any two of them are disjoint. Furthermore,*

$$\partial\mathcal{C}_0 = \{0,1\}, \quad \partial\mathcal{C}_- = \left\{0, \frac{1}{4} + i\infty\right\}, \quad \partial\mathcal{C}_+ = \left\{1, \frac{3}{4} + i\infty\right\}. \tag{4.5}$$

(2) *The curve* $\mathcal{C}_0$ *is symmetric with respect to the line* $\operatorname{Re}\tau = \frac{1}{2}$*;* $\mathcal{C}_-$ *and* $\mathcal{C}_+$ *are symmetric with respect to the line* $\operatorname{Re}\tau = \frac{1}{2}$.

(3) $\tau\left(\frac{1}{2}\right)$ *is the unique intersection point of the curve* $\mathcal{C}_0$ *with the line* $\operatorname{Re}\tau = \frac{1}{2}$. *Furthermore,* $\operatorname{Im}\tau\left(\frac{1}{2}\right) \in \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

(4) *Both* $\mathcal{C}_-$ *and* $\mathcal{C}_+$ *have no intersection with the line* $\operatorname{Re}\tau = \frac{1}{2}$.

Proof. (i) As pointed out before, $f_C(\tau(C)) = 0$ is equivalent to

$$C = \phi(\tau(C)) = \tau(C) - \frac{4\pi i g_2(\tau(C))}{2\eta_1(\tau(C))g_2(\tau(C)) - 3g_3(\tau(C))}. \tag{4.6}$$

That is, $C = \phi(\tau(C))$ for any $C \in \mathbb{R} \setminus \{0\}$. Thus, $C \mapsto \tau(C)$ is one-to-one for $C \in \mathbb{R} \setminus \{0, 1\}$. Note that (4.5) is just Lemma 3.8. This proves (i).

(ii) By the q-expansions (3.4) and (3.12)–(3.13), we easily obtain

$$\eta_1(1-\bar{\tau}) = \overline{\eta_1(\tau)}$$

and

$$g_k(1-\bar{\tau}) = \overline{g_k(\tau)}, \quad k = 2, 3.$$

We also have

$$\begin{aligned}\overline{\eta_2(\tau)} &= 2\overline{\zeta(\tau/2|\tau)} = 2\zeta(\bar{\tau}/2|1-\bar{\tau})\\ &= 2\zeta(1/2|1-\bar{\tau}) - 2\zeta((1-\bar{\tau})/2|1-\bar{\tau})\\ &= \eta_1(1-\bar{\tau}) - \eta_2(1-\bar{\tau}),\end{aligned}$$

i.e., $\eta_2(1-\bar{\tau}) = \overline{\eta_1(\tau)} - \overline{\eta_2(\tau)}$. Since $C \in \mathbb{R} \setminus \{0, 1\}$, we easily obtain

$$\begin{aligned}f_{1-C}(1-\bar{\tau}) &= 2g_2(1-\bar{\tau})\big((1-C)\eta_1(1-\bar{\tau}) - \eta_2(1-\bar{\tau})\big) - 3g_3(1-\bar{\tau})(\bar{\tau}-C)\\ &= -2\overline{g_2(\tau)}(C\overline{\eta_1(\tau)} - \overline{\eta_2(\tau)}) + 3\overline{g_3(\tau)}(C-\bar{\tau}) = -\overline{f_C(\tau)}.\end{aligned}$$

Therefore, it follows from Theorem 3.1 (1) that

$$\tau(1-C) = 1 - \overline{\tau(C)}. \tag{4.7}$$

Since τ and $1-\bar{\tau}$ is symmetric with respect to the line $\operatorname{Re}\tau = \frac{1}{2}$, we see that assertion (ii) holds.

(iii) By (ii), $\mathcal{C}_0$ has intersections with the line $\operatorname{Re}\tau = \frac{1}{2}$. Let $\tau_0 = \frac{1}{2} + ib_0$ be such an intersection point. Then $\tau_0 = \tau(C)$ for a unique $C \in (0, 1)$. Applying (4.7), we have $\tau(1-C) = \tau_0 = \tau(C)$, so assertion (i) gives $1 - C = C$, i.e., $C = \frac{1}{2}$. This proves that $\tau_0 = \tau\left(\frac{1}{2}\right)$ is the unique intersection point of the curve $\mathcal{C}_0$ with the line

$\operatorname{Re}\tau = \frac{1}{2}$. By (4.6),

$$\frac{1}{2} = \phi(\tau_0) = \tau_0 - \frac{4\pi i g_2(\tau_0)}{2\eta_1(\tau_0)g_2(\tau_0) - 3g_3(\tau_0)}$$
$$= \frac{1}{2} + ib_0 - \frac{4\pi i g_2(\tau_0)}{2\eta_1(\tau_0)g_2(\tau_0) - 3g_3(\tau_0)}. \tag{4.8}$$

By the q-expansions (3.4) and (3.12)–(3.13), we know that η_1, g_2, g_3 are all real-valued for $\operatorname{Re}\tau = \frac{1}{2}$. Therefore, (4.8) is equivalent to say that b_0 is the *unique zero* of the function

$$\varphi(b) := b - \frac{4\pi g_2(\frac{1}{2} + ib)}{2\eta_1(\frac{1}{2} + ib)g_2(\frac{1}{2} + ib) - 3g_3(\frac{1}{2} + ib)}, \quad b \geq \frac{1}{2},$$

because if $\varphi(b) = 0$, then $\frac{1}{2} = \phi\big(\frac{1}{2} + ib\big)$ and so $\frac{1}{2} + ib = \tau\big(\frac{1}{2}\big) = \frac{1}{2} + ib_0$.

It is well known that $g_2\big(\frac{1}{2} + \frac{\sqrt{3}}{2}i\big) = 0$, $g_3\big(\frac{1}{2} + \frac{1}{2}i\big) = 0$, and we proved in [8, p.32] that $\eta_1\big(\frac{1}{2} + \frac{1}{2}i\big) = 2\pi$. From here, we immediately obtain $\varphi\big(\frac{\sqrt{3}}{2}\big) = \frac{\sqrt{3}}{2}$ and $\varphi\big(\frac{1}{2}\big) = -\frac{1}{2}$. Therefore, $b_0 \in \big(\frac{1}{2}, \frac{\sqrt{3}}{2}\big)$. This proves (iii).

(iv) This assertion follows directly from (i)–(ii): $\mathcal{C}_-$ has no intersection with $\mathcal{C}_+$, and they are symmetric with respect to the line $\operatorname{Re}\tau = \frac{1}{2}$.

The proof is complete. ∎

Proposition 4.3. *The three curves $\mathcal{C}_-$, $\mathcal{C}_0$, $\mathcal{C}_+$ are all smooth curves in F_0.*

Proof. Recall $\phi(\tau)$ defined in (3.25). It follows from (4.3) that $\phi(\tau)$ is holomorphic in F_0. First we prove that

$$\phi'(\tau) \neq 0 \quad \text{for all } \tau \in \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+. \tag{4.9}$$

Fix any $\tau_0 \in \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+$. Then $\tau_0 = \tau(C)$ for some $C \in \mathbb{R} \setminus \{0, 1\}$, i.e., τ_0 is the unique and simple zero of $f_C(\tau)$ in F_0. In particular, $f_C'(\tau_0) \neq 0$. As pointed out before, $f_C(\tau_0) = 0$ implies $\phi(\tau_0) = C$, namely

$$\tau_0 - C = \frac{4\pi i g_2(\tau_0)}{(2\eta_1 g_2 - 3g_3)(\tau_0)}.$$

Consequently,

$$\phi'(\tau_0) = 1 - \frac{4\pi i g_2'}{2\eta_1 g_2 - 3g_3}(\tau_0) + \frac{4\pi i g_2(2\eta_1 g_2 - 3g_3)'}{(2\eta_1 g_2 - 3g_3)^2}(\tau_0)$$
$$= 1 - \frac{4\pi i g_2'}{2\eta_1 g_2 - 3g_3}(\tau_0) + (\tau_0 - C)\frac{(2\eta_1 g_2 - 3g_3)'}{2\eta_1 g_2 - 3g_3}(\tau_0)$$
$$= \frac{-f_C'(\tau_0)}{(2\eta_1 g_2 - 3g_3)(\tau_0)} \neq 0.$$

This proves (4.9).

On the other hand,

$$\mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+ = \{\tau \in F_0 \mid \operatorname{Im} \phi(\tau) = 0\}. \tag{4.10}$$

Write $\tau = a + bi$ with $a, b \in \mathbb{R}$. Since

$$\frac{\partial \operatorname{Im} \phi}{\partial a} = \operatorname{Im} \phi', \quad \frac{\partial \operatorname{Im} \phi}{\partial b} = \operatorname{Re} \phi',$$

we see from (4.9) that $\mathcal{C}_-$ (resp. $\mathcal{C}_0$, $\mathcal{C}_+$) is smooth at any $\tau \in \mathcal{C}_-$ (resp. $\tau \in \mathcal{C}_0$, $\tau \in \mathcal{C}_+$). The proof is complete. ■

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. The smoothness of the three curves is just Proposition 4.3. The assertion that, under Möbius transformations of the action $\Gamma_0(2)$, the collection of all critical points of $g_2(\tau)$ is precisely the set $\mathcal{D}_0$ given by (1.4), is a direct consequence of the expression (4.1) of the critical point set Θ. Recall that $\tau(C)$ is smooth as a function of $C \in \mathbb{R} \setminus \{0, 1\}$. Therefore, the denseness, i.e., the identity (1.5), follows from the fact that

$$\left\{ \frac{-d}{c} \;\middle|\; d \in \mathbb{Z},\ c \in 2\mathbb{Z} \setminus \{0\},\ (c, d) = 1 \right\}$$

is dense in $\mathbb{Q}$ and hence dense in $\mathbb{R}$. This completes the proof. ■

Recall that $g_2(\tau) \in \mathbb{R}$ for $\operatorname{Re} \tau = \frac{1}{2}$. We conclude this section by a simple observation.

Corollary 4.4. *There exists $\tilde{b} \in \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)$ such that $g_2\left(\frac{1}{2} + ib\right)$ is strictly decreasing for $b \in (0, \tilde{b})$ and strictly increasing for $b \in (\tilde{b}, +\infty)$.*

Proof. Let $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right) \in \Gamma_0(2)$. Then it is easy to prove that

$$\gamma(F_0) = \left\{ \tau \in \mathbb{H} \;\middle|\; \left|\tau - \frac{1}{2}\right| \leqslant \frac{1}{2}, \left|\tau - \frac{1}{4}\right| \geqslant \frac{1}{4}, \left|\tau - \frac{3}{4}\right| \geqslant \frac{1}{4} \right\},$$

and so

$$\left\{ \tau \in \mathbb{H} \;\middle|\; \operatorname{Re} \tau = \frac{1}{2} \right\} \subset F_0 \cup \gamma(F_0).$$

Applying Theorem 4.1 (2), we see that $\tilde{\tau} := \frac{\tau(1/2) - 1}{2\tau(1/2) - 1}$ is the unique zero of g_2' in $\gamma(F_0)$. Recall Theorem 4.2 (iii) that $\tau\left(\frac{1}{2}\right) = \frac{1}{2} + ib_0$ for some $b_0 \in \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Then $\tilde{\tau} = \frac{1}{2} + \frac{i}{4b_0}$, namely $\tilde{\tau}$ is the unique zero of g_2' on the line $\left\{\tau \in \mathbb{H} \;\middle|\; \operatorname{Re} \tau = \frac{1}{2}\right\}$ with

$\tilde{b} := \operatorname{Im} \tilde{\tau} \in \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)$. Note that $g_2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 0 < \frac{4\pi^4}{3} = \lim_{b \to +\infty} g_2\left(\frac{1}{2} + ib\right)$. Moreover, for $\tau = \frac{1}{2} + ib$,

$$g_2\left(\frac{1}{2} + \frac{i}{4b}\right) = g_2\left(\frac{\tau - 1}{2\tau - 1}\right) = 16b^4 g_2(\tau) \to +\infty \quad \text{as } b \to +\infty.$$

Thus, $g_2\left(\frac{1}{2} + ib\right)$ is strictly decreasing for $b \in (0, \tilde{b})$ and strictly increasing for $b \in (\tilde{b}, +\infty)$. The proof is complete. ∎

5. Distribution in fundamental domains of $\mathrm{SL}(2, \mathbb{Z})$

The purpose of this section is to prove Theorems 1.5–1.6 concerning the distribution of critical points of $E_4(\tau)$ in fundamental domains of $\mathrm{SL}(2, \mathbb{Z})$. Recall the basic fundamental domain F of $\mathrm{SL}(2, \mathbb{Z})$ in (1.6):

$$F = \{\tau \in \mathbb{H} \mid 0 \leqslant \operatorname{Re} \tau \leqslant 1, |\tau| \geqslant 1, |\tau - 1| \geqslant 1\}.$$

Recalling the curves $\mathcal{C}_-, \mathcal{C}_0, \mathcal{C}_+ \subset \mathring{F}_0$ and also $\mathcal{C}_1, \mathcal{C}_2$ in (1.7), we have the following important observation.

Lemma 5.1. *The following holds:*

$$\mathcal{C}_1 = \mathcal{C}_- \cap F = \left\{\tau(C) \;\middle|\; C \in (-\infty, -1]\right\} \subset F \cap \left\{\tau \;\middle|\; \operatorname{Re} \tau \in \left(0, \frac{1}{2}\right)\right\} \tag{5.1}$$

with $\partial \mathcal{C}_1 = \left\{\tau(-1), \frac{1}{4} + i\infty\right\}$;

$$\mathcal{C}_2 = \mathcal{C}_+ \cap F = \left\{\tau(C) \;\middle|\; C \in [2, +\infty)\right\} \subset F \cap \left\{\tau \;\middle|\; \operatorname{Re} \tau \in \left(\frac{1}{2}, 1\right)\right\} \tag{5.2}$$

with $\partial \mathcal{C}_2 = \left\{\tau(2), \frac{3}{4} + i\infty\right\}$;

$$\mathcal{C}_0 \cap F = \emptyset. \tag{5.3}$$

In particular, $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are symmetric with respect to the line* $\operatorname{Re} \tau = \frac{1}{2}$.

Proof. Since Theorem 4.2 (iv) implies $\mathcal{C}_- \subset \left\{\tau \;\middle|\; \operatorname{Re} \tau \in \left(0, \frac{1}{2}\right)\right\}$, we have $\mathcal{C}_- \cap \partial F \subset \{\tau \mid |\tau| = 1\}$. Suppose $\tau(C) \in \mathcal{C}_- \cap \partial F$, then $|\tau(C)| = 1$ and so $\operatorname{Re} \frac{1}{1 - \tau(C)} = \frac{1}{2}$. On the other hand, Theorem 4.2 (iii) implies

$$\mathcal{C}_0 \cap \left\{\tau \;\middle|\; \operatorname{Re} \tau = \frac{1}{2}\right\} = \left\{\tau\left(\frac{1}{2}\right)\right\}.$$

It follows from (4.4) that $\frac{1}{1 - \tau(C)} = \tau\left(\frac{1}{2}\right)$, i.e., $C = -1$ by applying (3.22). This proves

$$\mathcal{C}_- \cap \partial F = \{\tau(-1)\} \subset \{\tau \mid |\tau| = 1\}$$

and so (5.1) holds. By (5.1) and Theorem 4.2 (ii), we easily obtain (5.2), the symmetry of $\mathcal{C}_1$ and $\mathcal{C}_2$ with respect to the line $\operatorname{Re}\tau = \frac{1}{2}$, and

$$\mathcal{C}_+ \cap \partial F = \{\tau(2)\} \subset \{\tau \mid |\tau - 1| = 1\}.$$

Finally, suppose $\mathcal{C}_0 \cap F \neq \emptyset$, then $\mathcal{C}_0 \cap (\partial F \setminus \{\infty\}) \neq \emptyset$. Since Theorem 4.2 (ii) says that $\mathcal{C}_0$ is symmetric with respect to $\operatorname{Re}\tau = \frac{1}{2}$, there is $\tau \in \mathcal{C}_0$ such that $|\tau - 1| = 1$. It follows from (4.4) that $\tilde{\tau} := \frac{\tau-1}{\tau} \in \mathcal{C}_-$ and $\operatorname{Re}\tilde{\tau} = \frac{1}{2}$, a contradiction with Theorem 4.2 (iv). Therefore, (5.3) holds. ■

As a consequence of Lemma 5.1, we can restate Theorem 3.1 as follows.

Theorem 5.2. *Recall $f_C(\tau)$ defined in (3.1). Then the following holds.*

(1) *$f_C(\tau)$ has no zeros in F for any $C \in (-1, 2)$.*

(2) *For any $C \in (-\infty, -1] \cup [2, +\infty)$, $f_C(\tau)$ has a unique zero $\tau(C)$ in F. Moreover, $\tau(C) \in \mathring{F}$ for $C \notin \{-1, 2\}$ and $|\tau(-1)| = 1$, $|\tau(2) - 1| = 1$.*

Now we are in the position to prove Theorems 1.5–1.6.

Proof of Theorem 1.5. Let $\gamma(F)$ be a fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})/\{\pm I_2\}$. If $c = 0$, then $\gamma(F) = F + m \subset F_0 + m$ for some $m \in \mathbb{Z}$, and it follows from Theorem 4.1 that $E_4'(\tau)$ has no zeros in $\gamma(F)$. So, it suffices to consider $c \neq 0$, and the proof is similar to that of Theorem 4.1 (2); we omit the details here. ■

Proof of Theorem 1.6. It follows from Theorem 1.5 and Lemma 5.1 that

$$\begin{aligned}\mathcal{D} &= \left\{ \tau\left(\frac{-d}{c}\right) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})/\{\pm I_2\},\ \frac{-d}{c} \in (-\infty, -1] \cup [2, +\infty) \right\} \\ &\subset \mathcal{C}_1 \cup \mathcal{C}_2 = \overline{\mathcal{D}} \cap F = \overline{\mathcal{D}} \setminus \{\infty\}.\end{aligned}$$

The proof is complete. ■

Remark that, if we use the more standard fundamental domain

$$\mathcal{F} := \{\tau \in \mathbb{H} \mid 0 \leqslant \operatorname{Re}\tau < 1, |\tau| \geqslant 1, |\tau - 1| > 1\} \cup \{e^{\pi i/3}\}$$

of $\mathrm{SL}(2,\mathbb{Z})$ instead of F given by (1.6), then the only different thing is $\tau(2) \notin \mathcal{F}$, so Theorem 5.2 and Theorems 1.5–1.6 can be modified accordingly. We leave the details to the interested reader.

6. Geometric and monodromy interpretations of the curves

The purpose of this section is to give (i) the geometric meaning of the three curves from the Green function on flat tori $E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, and (ii) the monodromy meaning from the Lamé equation. Theorem 1.7 will be proved as a consequence.

Let $G(z) = G(z;\tau)$ be the Green function on the torus E_τ:

$$-\Delta G(z;\tau) = \delta_0 - \frac{1}{|E_\tau|} \text{ on } E_\tau, \quad \int_{E_\tau} G(z;\tau) = 0,$$

where δ_0 is the Dirac measure at 0 and $|E_\tau|$ is the area of the torus E_τ. See [22] for a detailed study of $G(z;\tau)$. In [6,21,23,24], Chai, Wang, and Lin introduced a multiple Green function G_n, $n \in \mathbb{N}$. Geometrically, any critical point of G_n is closely related to bubbling phenomena of nonlinear partial differential equations with exponential nonlinearities in two dimension; see [6,21,24] for typical examples. Thus, understanding the critical points of G_n is important for applications.

For the case $n = 3$, the multiple Green function G_3 is defined by

$$G_3(z_1,z_2,z_3;\tau) := \sum_{i<j} G(z_i - z_j;\tau) - 3\sum_{i=1}^{3} G(z_i;\tau),$$

A critical point (a_1,a_2,a_3) of G_3 satisfies

$$3\nabla G(a_i;\tau) = \sum_{j\neq i} \nabla G(a_i - a_j;\tau), \quad i = 1,2,3.$$

Clearly, if (a_1,a_2,a_3) is a critical point then so is $(a_{j_1},a_{j_2},a_{j_3})$, where (j_1,j_2,j_3) is any permutation of $(1,2,3)$, and we consider such critical points to be *the same one*. A critical point (a_1,a_2,a_3) is called a *trivial critical point* if

$$\{a_1,a_2,a_3\} = \{-a_1,-a_2,-a_3\} \quad \text{in } E_\tau.$$

Recall $\omega_1 = 1, \omega_2 = \tau$ and $\omega_3 = 1+\tau$. Since $G(z)$ is even and doubly periodic, we have $\nabla G(\frac{\omega_k}{2}) = 0$ for $k = 1,2,3$. Then $\boldsymbol{a} = (\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2})$ is a trivial critical point of G_3. Geometrically, we want to determine those τ's such that $\boldsymbol{a}$ is *degenerate* (i.e., the Hessian of G_3 at $\boldsymbol{a}$ vanishes), because bifurcation phenomena should happen and so *nontrivial* critical points of G_3 should appear near such τ's. This motivates us to define the degeneracy curve of G_3 in F_0 related to $(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2})$:

$$L := \left\{\tau \in F_0 \;\middle|\; \det D^2 G_3\left(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2};\tau\right) = 0\right\}. \tag{6.1}$$

On the other hand, we calculated in [10, Example 3.4] that the Hessian of G_3 at $\boldsymbol{a} = \left(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}\right)$ is given by

$$\det D^2 G_3(\boldsymbol{a};\tau) = -P(\tau)|2\eta_1 g_2 - 3g_3|^2 \operatorname{Im}\Big(\tau - \frac{4\pi i g_2}{2\eta_1 g_2 - 3g_3}\Big), \tag{6.2}$$

where $P(\tau) \in (0, +\infty)$ for all τ. Recalling (4.3), i.e., that $2\eta_1 g_2 - 3g_3 \neq 0$ for all $\tau \in F_0$, we see from (6.1)–(6.2) and (4.10) that

$$L = \{\tau \in F_0 \mid \operatorname{Im}\phi(\tau) = 0\} = \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+.$$

Therefore, the three curves coincide with the degeneracy curve L.

Theorem 6.1. *The degeneracy curve satisfies* $L = \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+$ *and* $L \cap F = \mathcal{C}_1 \cup \mathcal{C}_2$.

Finally, we introduce the monodromy interpretation of the curves from the Lamé equation. In a series of papers [6, 23, 24], Chai, Wang, and Lin established a theory that connects the multiple Green function G_n with the Lamé equation

$$y''(z) = [n(n+1)\wp(z|\tau) + B]y(z). \tag{6.3}$$

By applying this theory, we proved in [10] that the Hessian of G_n at a trivial critical point (i.e., a determinant of a $2n \times 2n$ real matrix which is too difficult to compute directly) can be expressed in terms of the monodromy data of (6.3). Let us take G_3 and $\boldsymbol{a} = \left(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}\right)$ for example. Let $\sigma(z) = \sigma(z|\tau)$ be the Weierstrass sigma function defined by $\sigma(z) := \exp \int^z \zeta(\xi) d\xi$. Then a direct computation shows that

$$y_1(z) := \frac{\sigma\left(z - \frac{\omega_1}{2}\right)\sigma\left(z - \frac{\omega_2}{2}\right)\sigma\left(z + \frac{\omega_3}{2}\right)}{\sigma(z)^3}$$

is an elliptic function and solves the Lamé equation (6.3) with $n = 3$ and $B = 0$:

$$y''(z) = 12\wp(z|\tau)y(z). \tag{6.4}$$

Up to a constant, we see that

$$y_1(z)^{-2} = \frac{1}{\prod_{k=1}^3 (\wp(z) - \wp(\frac{\omega_k}{2}))}$$

is even elliptic and so has no residues at $\frac{\omega_k}{2}$'s. Then $\chi(z) := \int^z y_1(\xi)^{-2} d\xi$ is meromorphic and has two quasi-periods:

$$\chi_j := \chi(z + \omega_j) - \chi(z), \quad j = 1, 2.$$

Since $y_2(z) := \chi(z)y_1(z)$ is a linearly independent solution of (6.4) with respect to $y_1(z)$, we easily obtain

$$\begin{pmatrix} \chi_1 y_1(z+\omega_1) \\ y_2(z+\omega_1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix},$$
$$\begin{pmatrix} \chi_1 y_1(z+\omega_2) \\ y_2(z+\omega_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D & 1 \end{pmatrix} \begin{pmatrix} \chi_1 y_1(z) \\ y_2(z) \end{pmatrix}, \quad D := \frac{\chi_2}{\chi_1}.$$

That is, the monodromy group of (6.4) is generated by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ D & 1 \end{pmatrix}, \tag{6.5}$$

and we refer this $D = \chi_2/\chi_1$ as *the monodromy data* of (6.4). Consequently, our result in [10] implies

$$\det D^2 G_3\Big(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}; \tau\Big) = -P_1(\tau)|\chi_1|^2 \operatorname{Im} D,$$

where $P_1(\tau) \in (0, +\infty)$. Since we calculated in [10, Example 3.4] that

$$\chi_1 = \frac{-12(2\eta_1 g_2 - 3g_3)}{g_2^3 - 27g_3^2}, \quad \frac{\chi_2}{\chi_1} = \tau - \frac{4\pi i g_2}{2\eta_1 g_2 - 3g_3} = \phi(\tau), \tag{6.6}$$

we immediately obtain (6.2).

Now, we consider the set of those τ's in F_0 such that the monodromy data $D = \chi_2/\chi_1$ of the Lamé equation (6.4) is real-valued:

$$\widetilde{L} := \{\tau \in F_0 \mid \text{the monodromy data of (6.4) is real-valued}\}.$$

Then the above argument implies that the three curves coincide with $\widetilde{L}$.

Theorem 6.2. $\widetilde{L} = \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+$ *and* $\widetilde{L} \cap F = \mathcal{C}_1 \cup \mathcal{C}_2$.

Remark 6.3. Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ with $c \neq 0$, we let $\gamma \cdot \tau_1 = \frac{a\tau_1 + b}{c\tau_1 + d}$ with $\tau_1 \in F_0$ be the unique zero of $E_4'(\tau)$ in $\gamma(F_0)$. Then the above argument indicates that the monodromy data D of the Lamé equation (6.4) with $\tau = \tau_1$ is precisely $\frac{-d}{c}$.

Proof of Theorem 1.7. In the proof of [9, Theorem 1.3], it was proved that $B = 0$ is a cusp of the spectrum $\sigma(L;\tau)$ if and only if the corresponding monodromy data D (see (6.5)) of the corresponding Lamé equation (6.4) satisfies $D = \infty$. From here and the expression (6.6) of $D = \chi_2/\chi_1$, we conclude that $B = 0$ is a cusp if and only if $2\eta_1 g_2 - 3g_3 = 0$, i.e., $g_2'(\tau) = 0$ or, equivalently, $E_4'(\tau) = 0$. ■

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