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Universal inequalities for Dirichlet eigenvalues on discrete groups

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Abstract. We prove universal inequalities for Laplacian eigenvalues with Dirichlet boundary condition on subsets of certain discrete groups. The study of universal inequalities on Riemannian manifolds was initiated by Weyl, Pólya, Yau, and others. Here we focus on a version by Cheng and Yang. Specifically, we prove Yang-type universal inequalities for Cayley graphs of finitely generated amenable groups, as well as for the d-regular tree (simple random walk on the free group).

1. Introduction

The spectral theory of Laplace–Beltrami operators on Riemannian manifolds was extensively studied in the literature, see e.g., [15,26,43,51]. For a bounded domain Ω in a Riemannian manifold, we denote by

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$$

the spectrum of the Laplace–Beltrami operator with Dirichlet boundary condition on Ω, counting the multiplicity of eigenvalues.

For the Euclidean space, Weyl [54] proved the asymptotic behavior of eigenvalues that

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \mathrm{vol}(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $\mathrm{vol}(\Omega)$ is the volume of Ω. It was conjectured by Pólya [49] that

$$\lambda_k \ge \frac{4\pi^2}{(\omega_n \mathrm{vol}(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, 3, \ldots.$$

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Li and Yau [44] proved that

$$\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, 3, \ldots.$$

Payne, Pólya, and Weinberger [47] proved the gap estimate of consecutive eigenvalues for a bounded domain in $\mathbb{R}^2$, generalized to $\mathbb{R}^n$ by Thompson [53]:

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i \quad \text{for any } k \geq 1.$$

This was improved by Hile and Protter [34]. A sharp inequality was proved by Yang [22, 55]:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} \lambda_i (\lambda_{k+1} - \lambda_i). \tag{1}$$

The above is called *Yang's first inequality* (Yang 1); it implies the Payne–Pólya–Weinberger inequality, etc. [1]. It was Mark. S. Ashbaugh who first emphasized in his papers [1, 2] the importance of the 1991 preprint of Yang. In fact, the notions of "Yang-type" inequalities were introduced Ashbaugh ("Yang 1", and "Yang 2" are Ashbaugh's designations and his take on the work of Yang.) The use of "optimal Cauchy–Schwarz" was laid out here, and further developed in [9] where a general framework, including the connections between the Payne–Pólya–Weinberger, Hile–Protter, Yang 1 and Yang 2 inequalities, and the use of the Chebyshev inequalities was first established. These are called *universal inequalities for eigenvalues* since they are independent of the domain Ω. See [4–8, 20, 33] for more results regarding Euclidean spaces.

Universal inequalities have been generalized to eigenvalues of Laplace–Beltrami operators on Riemannian manifolds. In particular, Yang's inequality has been proved for space forms. For the unit n-sphere, Cheng and Yang [20] proved that

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \Big(\lambda_i + \frac{n^2}{4} \Big).$$

For $\mathbb{H}^n$, the n-dimensional hyperbolic space of sectional curvature -1, Cheng and Yang [23] proved that

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \Big(\lambda_i - \frac{(n-1)^2}{4} \Big). \tag{2}$$

Note that $\frac{(n-1)^2}{4}$ is the bottom of the spectrum of $\mathbb{H}^n$. For a general Riemannian manifold, Chen and Cheng [16] proved a variant of Yang's inequality using related geometric quantities via isometric embedding into the Euclidean space. For universal inequalities on manifolds, we refer the readers to [17–19, 21, 23, 28, 30–32, 40, 42, 52, 56].

In this paper, we study universal inequalities for eigenvalues on graphs, in particular Cayley graphs of discrete groups. We recall the setting of general networks. A *network* is a pair (V, c) where V is a countable set and $c\colon V \times V \to [0, \infty)$ is called the *conductance*. The conductance must satisfy $0 \le c(x, y) = c(y, x) < \infty$ (symmetric) and $\pi(x) := \sum_y c(x, y) < \infty$ for every x. We write $x \sim y$ to indicate $c(x, y) > 0$ (in which case we say that $x \sim y$ is an *edge* in the network). A network naturally provides a *reversible Markov chain*, whose transition matrix is given by $P(x, y) = \frac{c(x,y)}{\pi(x)}$. The (normalized) *Laplacian* is the operator $\Delta = I - P$, where I denotes the identity operator, i.e.,

$$\Delta f(x) = \sum_y P(x, y)(f(x) - f(y)).$$

We denote by $L^2(V, \pi)$ the Hilbert space of L^2 summable functions on V, equipped with the inner product

$$\langle f, g\rangle = \langle f, g\rangle_\pi := \sum_x \pi(x) f(x)\overline{g(x)}.$$

It is well known that the Laplacian Δ is a bounded self-adjoint operator on $L^2(V, \pi)$, whose spectrum is contained in $[0, 2]$. We write $\lambda_{\min}$ for the *bottom* of the spectrum of Δ.

The Laplacian with Dirichlet boundary condition on finite subsets of networks has been investigated in the literature, see e.g., [14, 24, 25, 27, 29]. For finite $\Omega \subset V$, the Laplacian with Dirichlet boundary condition on Ω, denote by Δ_Ω, is defined as the Laplacian Δ restricted to the subspace

$$L^2(\Omega) := \{f \in L^2(V, \pi) : f|_{V\setminus\Omega} \equiv 0\}.$$

The eigenvalues of Δ_Ω, called *Dirichlet eigenvalues* on Ω, are ordered by

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{|\Omega|},$$

where $|\cdot|$ denotes the cardinality of the subset. We are interested in proving universal inequalities on graphs, in particular Yang-type inequalities (1) and (2). Due to the discrete nature of graphs, some modification is required.

Definition 1. We say that the network (V, c) satisfies *Yang's inequality* (resp. the *Yang-type inequality*) with constant C_Y (resp. C_{YT}) if the following holds for any finite subset $\Omega \subset G$. Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{|\Omega|}$ be the Dirichlet eigenvalues of Ω.

Then, for any $k < |\Omega|$,

$$\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 \le C_Y \cdot \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_{\min}), \tag{3a}$$

(resp.,

$$\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 (1 - \lambda_i) \le C_{YT} \cdot \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_{\min})). \tag{3b}$$

Since $\lambda_i \le 2$, for any $i \ge 1$, one easily sees that in case of $\lambda_{\min} = 0$, the Yang-type inequality implies Yang's inequality with $C_Y = C_{YT} + 2$. Following the arguments in [1, 22, 55], Hua, Lin, and Su [35] proved that the integer lattice $\mathbb{Z}^n$, a discrete analog of $\mathbb{R}^n$, satisfies a Yang-type inequality, with constant $C_{YT} = \frac{4}{n}$. Recently, Kobayashi [37] proved a Yang-type inequality for the eigenvalues of the Laplacian (not Dirichlet eigenvalues) of a finite edge-transitive graph.

Note that $\mathbb{Z}^n$ can be regarded as a Cayley graph of a free Abelian group. In this paper, we prove Yang-type inequalities for more general Cayley graphs of finitely generated infinite groups.

1.1. Amenable groups

Our first result regards amenable groups.

Definition 2. Let G be a finitely generated group, and let $\Gamma = (V, E)$ be a Cayley graph of G with respect to some finite symmetric generating set. Define the *Cheeger constant* of Γ to be

$$\Phi_\Gamma = \inf_{\substack{A \subset V \\ |A| < \infty}} \frac{|\partial A|}{|A|} \qquad \partial A = \{\{x, y\} \in E : x \in A,\ y \notin A\}.$$

A group G is called *amenable* if there exists a Cayley graph Γ such that $\Phi_\Gamma = 0$. Otherwise, it is called *non-amenable.*

It is a simple exercise to show that the definition of *amenable* does not depend on the specific choice of Cayley graph. That is, for any two Cayley graphs Γ, Γ' of the same group, $\Phi_\Gamma = 0$ if and only if $\Phi_{\Gamma'} = 0$.

Examples of amenable groups include finite groups, $\mathbb{Z}^d$, and Abelian groups. The free group on $d \ge 2$ generators is a non-amenable group. See [48, Chapter 7] and also below in Section 4 for more on amenable groups.

Let G be a finitely generated amenable group. Consider some probability measure μ on G (which we think of as a non-negative function $\mu: G \to [0, 1]$ such that $\sum_x \mu(x) = 1$). Assume that μ is symmetric, i.e., $\mu(x) = \mu(x^{-1})$ for all $x \in G$. Then

μ induces a corresponding Cayley graph (or network) by setting the conductances $c(x, y) = \mu(x^{-1}y)$. This network corresponds to the μ-random walk on G. This network is denoted by (G, μ).

Theorem 3. *Let G be a finitely generated infinite amenable group. Let μ be a symmetric probability measure on G, and consider the Cayley network (G, μ) of G with respect to μ. Set $\mu_* := \inf_{1 \neq y \in \operatorname{supp}(\mu)} \mu(y)$.*

Then, the network (G, μ) satisfies Yang's inequality, with constant $C_Y = \frac{6}{\mu_*}$.

For finitely generated groups with Abelian quotients, i.e., those groups which admit homomorphisms onto $\mathbb{Z}^n$ for some n, we prove the Yang-type inequality with $C_{\mathrm{YT}} = \frac{4}{n}$ for specific μ-random walks, see Theorem 7. This extends the result for $\mathbb{Z}^n$ from [35].

1.2. Free groups

Next, we consider Yang-type inequalities on regular trees, which can be regarded as Cayley graphs of free groups. Let $\mathbb{T}_d$, $d \geq 3$, be a d-regular tree with the conductances of the edges $c(x, y) = \mathbf{1}_{\{x \sim y\}} \frac{1}{d}$, which is a discrete analog of hyperbolic space $\mathbb{H}^d$. The Laplacian corresponds to the generator of the *simple random walk* on $\mathbb{T}_d$. As is well known, the bottom of the spectrum of $\mathbb{T}_d$ is $\lambda_{\min} := 1 - \frac{2\sqrt{d-1}}{d}$. Following the arguments in [23], we prove the following result.

Theorem 4. *The network given by the simple random walk on the d-regular tree $\mathbb{T}_d$ (where $d > 2$) satisfies the Yang-type inequality with constant $C_{\mathrm{YT}} = \frac{8\sqrt{d-1}}{d}$.*

We sketch the proof strategies of Theorem 3 and Theorem 4. By the variational principle, for an upper bound estimate of eigenvalues, it suffices to construct appropriate test functions. Following the arguments in [21,55], for any network and any test function $\alpha\colon V \to \mathbb{R}$, we prove the Dirichlet eigenvalues satisfy some crucial estimate involving α, see Lemma 5, a discrete analog of [21, Proposition 1]. This enables us to derive the Yang-type inequality with choice of α with nice properties for $\Delta\alpha$ and the gradient of α. For $\mathbb{R}^n$ or $\mathbb{Z}^n$, as in [22, 35, 55], linear functions are good candidates for test functions.

In order to generalize the result to Cayley graphs of amenable groups, i.e., Theorem 3, we use *harmonic cocycles* as test functions. The existence of harmonic cocycles for amenable groups was proved by [38, 45]. Harmonic cocycles are nontrivial and deep objects. They are related to notions from homology (or cohomology) and dynamics, outside the scope of this paper. We use these as a tool here, only scratching the surface. Perhaps one point of view which may be useful here is the following. A *cocycle* is a map $c\colon G \to \mathcal{H}$ from a group G to a Hilbert space $\mathcal{H}$ such that G acts on $\mathcal{H}$ by unitary operators and the map c satisfies the cocycle equation

$c(xy) = c(x) + x.c(y)$. If the action of G happened to be the trivial action (i.e., all elements of G act trivially), then c is just a group homomorphism from G to the additive group $\mathcal{H}$. For more general representations of G, what we get is that a cocycle is a kind of "twisted homomorphism" since the group action "twists" the homomorphism-defining equation. Sometimes, as is the case here, cocycles can be used to replace homomorphisms into Abelian groups to obtain similar behavior.

As an example, if $c\colon G \to \mathcal{H}$ is a homomorphism, then one can just define the action of G to be the trivial action on $\mathcal{H}$, and c will automatically be a harmonic cocycle. Finite groups do not admit non-trivial harmonic cocycles, because the projection of these onto a fixed vector would provide a bounded harmonic function, which must be constant on a finite group by the maximum principle. However, for infinite finitely generated groups, harmonic cocycles are extremely useful. They are guaranteed to exist in the case that the group does not have property (T) (originally proved by Mok [45]). Harmonic cocycles have been used for many applications, perhaps most notably in proofs of Gromov's theorem regarding groups of polynomial growth, but also in other contexts.

For $\mathbb{H}^n$, Cheng and Yang [23] used Busemann functions of geodesic rays to prove Yang-type inequality (2). To extend the result to $\mathbb{T}_d$, i.e., Theorem 4, we use the discrete analogs of Busemann functions as test functions.

For trees, Leydold characterized the subtree with Faber–Krahn property for the first Dirichlet eigenvalue in a d-regular tree $\mathbb{T}_d$, which is close to the "ball" [41]. The optimal Faber–Krahn inequality for Dirichlet eigenvalues of subtrees was proved using discrete rearrangement in Pruss' work [50]. These results gave the minimization of the first Dirichlet eigenvalue on a subtree with fixed "volume." The general thrust of papers dealing with the Payne–Pólya–Weinberger inequality and Yang's inequalities is to push for optimal results for the $\frac{\lambda_2}{\lambda_1}$ ratio [5]. By our result, we get the upper bounds in Corollary 10 for the $\frac{\lambda_2-\lambda_{\min}}{\lambda_1-\lambda_{\min}}$ ratio and in Corollary 13 for higher order ratios, where $\lambda_{\min} = 1 - \frac{2\sqrt{d-1}}{d}$ is the bottom of the spectrum of $\mathbb{T}_d$.

Note that all eigenvalue estimates in this paper are quantitative. But we do not know the sharpness of these estimates, which definitely will be interesting for further studies.

The paper is organized as follows. In next section, we introduce some basic facts on networks. In Section 3, we prove the useful estimate of eigenvalues for general networks, Lemma 5. Section 4 is devoted to the proofs of main results, Theorem 3 and Theorem 4. In the last section, we derive some applications of the Yang-type inequality, such as the Payne–Pólya–Weinberger inequality and the Hile–Protter inequality, etc.

2. Notation and basic operators

2.1. Γ calculus

Let (V, c) be a network on the set of vertices V with the conductance c. We allow $c(x, x) > 0$, which corresponds to a self-edge at $x \in V$.

Recall the inner product on functions defined in the introduction:

$$\langle f, g\rangle = \sum_x \pi(x) f(x)\overline{g(x)}.$$

Accordingly, we write $\|f\|^2 = \|f\|_\pi^2 := \langle f, f\rangle$, and the space of L^2 summable functions is given by $L^2(V, \pi) := \{f : V \to \mathbb{C} : \|f\| < \infty\}$.

The *Dirichlet energy* is defined to be

$$\mathcal{E}(f, g) := \sum_{x,y} c(x, y)(f(x) - f(y))\overline{(g(x) - g(y))},$$

and $\mathcal{E}(f) := \mathcal{E}(f, f)$. If $f, g \in L^2(V, \pi)$, then it is not difficult to prove the "integration by parts" formula,

$$\mathcal{E}(f, g) = 2\langle \Delta f, g\rangle = 2\langle f, \Delta g\rangle.$$

Define the so-called *carré du champ* operator (at $x \in V$) as follows:

$$\begin{aligned} 2\Gamma(f, g)(x) &:= \big(f\Delta\bar{g} + \bar{g}\Delta f - \Delta(f\bar{g})\big)(x) \\ &= \sum_y P(x, y)(f(x) - f(y))\overline{(g(x) - g(y))}, \end{aligned}$$

and $\Gamma(f) := \Gamma(f, f)$. Note that Γ is symmetric and bi-linear. The notion of carré du champ was introduced by Bakry and Émery in 1984 in [11] in the context of hypercontractivity, and developed further in [10, 12]; see the comprehensive reference by Bakry, Gentil, and Ledoux [13].

Finally, we define the scalar-valued (non-linear) functional

$$\Lambda(f, g) = \frac{1}{4}\sum_{x,y} c(x, y)|f(x) - f(y)|^2 \cdot |g(x) - g(y)|^2.$$

2.2. Identities

In this section we summarize a few identities which we will require in the analysis below. All are straightforward and easy to prove, and hold for all $f, g \in L^2(V, \pi)$,

$$\mathcal{E}(f, g) = 2\sum_x \pi(x)\Gamma(f, g)(x) = 2\langle \Gamma(f, g), 1\rangle.$$

Also, note that

$$\langle \Gamma(f,g), g\rangle = \frac{1}{2}\sum_{x,y} P(x,y)(f(x)-f(y))\overline{(g(x)-g(y))g(x)}\pi(x)$$

Since $\pi(x)P(x,y) = c(x,y) = \pi(y)P(y,x)$,

$$\begin{aligned}\mathcal{E}(f,g^2) &= \sum_{x,y} c(x,y)(f(x)-f(y))\overline{(g(x)^2-g(y)^2)} \\ &= \sum_{x,y} P(x,y)(f(x)-f(y))\overline{(g(x)-g(y))g(x)}\pi(x) \\ &\quad + \sum_{x,y} P(x,y)(f(x)-f(y))\overline{(g(x)-g(y))g(y)}\pi(x) \\ &= 4\langle \Gamma(f,g), g\rangle.\end{aligned}$$

So, in conclusion,

$$\langle 2\Gamma(f,g), g\rangle = \langle \Delta f, g^2\rangle.$$

We also may compute, for real functions f and g,

$$\begin{aligned}\langle 2\Gamma(f,g), f\cdot g\rangle &= \sum_{x,y} c(x,y)(f(x)-f(y))(g(x)-g(y))f(x)g(x) \\ &= \sum_{x,y} c(x,y)(f(x)-f(y))(g(x)-g(y)) \\ &\quad \cdot \frac{f(x)g(x)+f(y)g(y)}{2} \\ &= \sum_{x,y} c(x,y)(f(x)-f(y))(g(x)-g(y)) \\ &\quad \cdot \frac{(f(x)+f(y))(g(x)+g(y))+(f(x)-f(y))(g(x)-g(y))}{4} \\ &= \frac{1}{4}\sum_{x,y} c(x,y)(f(x)^2-f(y)^2)(g(x)^2-g(y)^2) \\ &\quad + \frac{1}{4}\sum_{x,y} c(x,y)|f(x)-f(y)|^2\cdot|g(x)-g(y)|^2,\end{aligned}$$

which culminates in

$$\langle 2\Gamma(f,g), f\cdot g\rangle = \frac{1}{4}\mathcal{E}(f^2,g^2) + \Lambda(f,g). \tag{4}$$

3. Universal inequality

The following is an analog of [21, Proposition 1]. It is the main estimate which will imply our results.

Let (V, c) be a network. Let $\Omega \subset V$ be a finite subset of size $n = |\Omega|$. Let $u_1, \dots, u_n$ be an orthonormal basis of eigenvectors for Δ_Ω defined on the subspace $L^2(\Omega)$ of $L^2(V, \pi)$; that is,

- $\lambda_{\min} \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$,
- $\Delta u_i = \lambda_i u_i$,
- $u_i|_{V \setminus \Omega} \equiv 0$,
- $\langle u_i, u_j \rangle = \mathbf{1}_{\{i=j\}}$.

Since the Laplacian is self-adjoint, such an orthonormal basis exists, $\lambda_i \in \mathbb{R}$ and u_i are real valued. We call such a collection $(\lambda_i, u_i)_{i=1}^n$ the *Dirichlet system* for Ω.

The next lemma is similar to Kobayashi's [37, Lemma 2.4], both following the proof of [21, Proposition 1]. Kobayashi proved the result for eigenvalues of Laplacian of a finite graph without boundary, while we proved it for eigenvalues of Laplacian on a finite subset with Dirichlet boundary condition.

Lemma 5. *Let (V, c) be a network. Let $\Omega \subset V$ be a finite subset of size $n = |\Omega|$. Let $(\lambda_i, u_i)_{i=1}^n$ be the Dirichlet system for Ω. For any $k < n$ and any $\alpha\colon V \to \mathbb{R}$ we have*

$$\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 \big(\langle \Gamma(\alpha), u_i^2 \rangle - \Lambda(\alpha, u_i)\big) \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \|u_i \cdot \Delta\alpha - 2\Gamma(\alpha, u_i)\|^2.$$

Proof. Let $\alpha\colon V \to \mathbb{R}$. Fix some $1 \leq k < n$. Set

$$a_{ij} = \langle u_i \cdot \alpha, u_j \rangle, \qquad \varphi_i = u_i \cdot \alpha - \sum_{j=1}^{k} a_{ij} \cdot u_j,$$
$$\alpha_i = u_i \cdot \Delta\alpha - 2\Gamma(u_i, \alpha), \quad b_{ij} = \langle \alpha_i, u_j \rangle,$$
$$w_i = \langle \alpha_i, \varphi_i \rangle, \qquad y_i = \Lambda(\alpha, u_i),$$
$$z_i = \langle \alpha_i, u_i \cdot \alpha \rangle.$$

We collect a few observations regarding these quantities. For all $1 \leq i, j \leq k$,

$$\langle \varphi_i, u_j \rangle = \langle u_i \cdot \alpha, u_j \rangle - \sum_{\ell=1}^{k} \langle u_\ell, u_j \rangle a_{i\ell} = a_{ij} - a_{ij} = 0. \tag{5}$$

Also, $a_{ij} = a_{ji}$ and since the Laplacian is self-adjoint,

$$\begin{aligned}\lambda_j \cdot a_{ij} &= \langle u_i \cdot \alpha, \Delta u_j\rangle = \langle \Delta(u_i \cdot \alpha), u_j\rangle \\ &= \langle \Delta u_i \cdot \alpha + u_i \cdot \Delta\alpha - 2\Gamma(u_i, \alpha), u_j\rangle \\ &= \lambda_i \cdot a_{ij} + \langle \alpha_i, u_j\rangle = \lambda_i \cdot a_{ij} + b_{ij},\end{aligned}$$

which proves that for all $1 \le i, j \le k$,

$$b_{ij} = -b_{ji} = (\lambda_j - \lambda_i) \cdot a_{ij} \tag{6}$$

$$\Delta\varphi_i = \Delta(u_i \cdot \alpha) - \sum_{j=1}^{k} \Delta u_j \cdot a_{ij} = \lambda_i u_i \cdot \alpha + \alpha_i - \sum_{j=1}^{k} \lambda_j u_j \cdot a_{ij}. \tag{7}$$

Since $\langle u_i, u_j\rangle = \mathbf{1}_{\{i=j\}}$,

$$\begin{aligned}\Big\|\alpha_i - \sum_{j=1}^{k} b_{ij} \cdot u_j\Big\|^2 &= \|\alpha_i\|^2 + \sum_{j=1}^{k} \|b_{ij} \cdot u_j\|^2 - 2\sum_{j=1}^{k} b_{ij} \cdot \langle \alpha_i, u_j\rangle \\ &= \|\alpha_i\|^2 - \sum_{j=1}^{k} |b_{ij}|^2.\end{aligned} \tag{8}$$

By (6), we know that $-\langle \alpha_i, u_j\rangle = -b_{ij} = (\lambda_i - \lambda_j)a_{ij}$, so

$$w_i = z_i - \sum_{j=1}^{k} \langle \alpha_i, a_{ij} \cdot u_j\rangle = z_i + \sum_{j=1}^{k} (\lambda_i - \lambda_j)|a_{ij}|^2. \tag{9}$$

By (4), we have that

$$\langle 2\Gamma(u_i, \alpha), u_i \cdot \alpha\rangle = \frac{1}{2}\langle \Delta(\alpha^2), u_i^2\rangle + \Lambda(\alpha, u_i).$$

Thus,

$$\begin{aligned}z_i + y_i &= \langle u_i \cdot \Delta\alpha - 2\Gamma(u_i, \alpha), u_i \cdot \alpha\rangle + \Lambda(\alpha, u_i) \\ &= \langle u_i \cdot \Delta\alpha, u_i \cdot \alpha\rangle - \frac{1}{2}\langle \Delta(\alpha^2), u_i^2\rangle \\ &= \Big\langle \Delta\alpha \cdot \alpha - \frac{1}{2}\Delta(\alpha^2), u_i^2\Big\rangle = \langle \Gamma(\alpha), u_i^2\rangle.\end{aligned} \tag{10}$$

By (5), we get that $\langle \varphi_i, u_i \cdot \alpha\rangle = \|\varphi_i\|^2$. Also, since φ_i is orthogonal to $\{u_1, \dots, u_k\}$, using (7),

$$\begin{aligned}\lambda_{k+1}\|\varphi_i\|^2 \le \langle \Delta\varphi_i, \varphi_i\rangle &= \Big\langle \lambda_i u_i \cdot \alpha + \alpha_i - \sum_{j=1}^{k} \lambda_j u_j \cdot a_{ij}, \varphi_i\Big\rangle \\ &= w_i + \lambda_i\langle u_i \cdot \alpha, \varphi_i\rangle = w_i + \lambda_i\|\varphi_i\|^2.\end{aligned}$$

Using the Cauchy–Schwarz inequality and (8),

$$\begin{aligned}(\lambda_{k+1}-\lambda_i)|w_i|^2 &= (\lambda_{k+1}-\lambda_i)\Big|\Big\langle \alpha_i - \sum_{j=1}^{k} b_{ij}\cdot u_j, \varphi_i\Big\rangle\Big|^2 \\ &\le (\lambda_{k+1}-\lambda_i)\|\varphi_i\|^2 \cdot \Big(\|\alpha_i\|^2 - \sum_{j=1}^{k}|b_{ij}|^2\Big) \\ &\le w_i \cdot \Big(\|\alpha_i\|^2 - \sum_{j=1}^{k}|b_{ij}|^2\Big).\end{aligned}$$

Thus,

$$(\lambda_{k+1}-\lambda_i)w_i \le \|\alpha_i\|^2 - \sum_{j=1}^{k}|\lambda_i-\lambda_j|^2\cdot|a_{ij}|^2. \tag{11}$$

By (9),

$$\begin{aligned}&\sum_{i=1}^{k}|\lambda_{k+1}-\lambda_i|^2 w_i \\ &\quad= \sum_{i=1}^{k}|\lambda_{k+1}-\lambda_i|^2 z_i + \sum_{i,j=1}^{k}|\lambda_{k+1}-\lambda_i|^2(\lambda_i-\lambda_j)|a_{ij}|^2 \\ &\quad= \sum_{i=1}^{k}|\lambda_{k+1}-\lambda_i|^2 z_i + \frac{1}{2}\sum_{i,j=1}^{k}\big(|\lambda_{k+1}-\lambda_i|^2-|\lambda_{k+1}-\lambda_j|^2\big)(\lambda_i-\lambda_j)|a_{ij}|^2 \\ &\quad= \sum_{i=1}^{k}|\lambda_{k+1}-\lambda_i|^2 z_i - \sum_{i,j=1}^{k}\Big(\lambda_{k+1}-\frac{\lambda_i+\lambda_j}{2}\Big)|\lambda_i-\lambda_j|^2|a_{ij}|^2 \\ &\quad= \sum_{i=1}^{k}|\lambda_{k+1}-\lambda_i|^2 z_i - \sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)|\lambda_i-\lambda_j|^2|a_{ij}|^2.\end{aligned}$$

Multiplying (11) by $\lambda_{k+1}-\lambda_i$ and summing over i, we obtain

$$\sum_{i=1}^{k}|\lambda_{k+1}-\lambda_i|^2 z_i \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|\alpha_i\|^2.$$

The proof is now complete using $z_i = \langle \Gamma(\alpha), u_i^2\rangle - \Lambda(\alpha, u_i)$ by (10). ∎

Let $\mathcal{H}$ be a Hilbert space and $\alpha\colon V\to\mathcal{H}$. The Laplacian is defined as

$$\Delta\alpha(x) = \sum_{y} P(x,y)(\alpha(x)-\alpha(y)). \tag{12}$$

We extend the definitions of the inner product and of Γ, Λ by defining

$$\begin{aligned}
2\Gamma(\alpha,u) &= \sum_y P(x,y)(u(x)-u(y))\cdot(\alpha(x)-\alpha(y)),\\
2\Gamma(\alpha)(x) &= \sum_y P(x,y)\|\alpha(x)-\alpha(y)\|_{\mathcal{H}}^2,\\
\langle\alpha,u\rangle &= \sum_x \pi(x)u(x)\cdot\alpha(x),\\
\|\alpha\|^2 = \langle\alpha,\alpha\rangle &= \sum_x \pi(x)\|\alpha(x)\|_{\mathcal{H}}^2,\\
\Lambda(\alpha,u) &= \frac{1}{4}\sum_{x,y} c(x,y)|u(x)-u(y)|^2\cdot\|\alpha(x)-\alpha(y)\|_{\mathcal{H}}^2.
\end{aligned}$$

Here $u\colon V\to\mathbb{R}$ is any (finitely supported) real valued function. With this notation, we have the following theorem generalizing Lemma 5.

Theorem 6. *Let (V,c) be a network. Let $\Omega\subset V$ be a finite subset of size $n=|\Omega|$. Let $(\lambda_i,u_i)_{i=1}^n$ be the Dirichlet system for Ω. Let $\mathcal{H}$ be a Hilbert space and let $\alpha\colon V\to\mathcal{H}$.*

Then for any $k<n$,

$$\begin{aligned}
&\sum_{i=1}^k |\lambda_{k+1}-\lambda_i|^2\cdot\big(\langle\Gamma(\alpha),u_i^2\rangle-\Lambda(\alpha,u_i)\big)\\
&\qquad\le\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\cdot\|u_i\cdot\Delta\alpha-2\Gamma(\alpha,u_i)\|^2.
\end{aligned}$$

Note that when $\mathcal{H}=\mathbb{R}$ this is exactly Lemma 5.

Proof. Let $h\in\mathcal{H}$ be any non-zero vector. Define the function $\alpha'\colon V\to\mathbb{R}$ by $\alpha'(x)=\langle\alpha(x),h\rangle_{\mathcal{H}}$. Plugging this into Lemma 5, we see that we only need to compute $\Gamma(\alpha')$, $\Lambda(\alpha',u_i)$, $\Gamma(\alpha',u_i)$, $\Delta\alpha'$. It is simple to verify that

$$\begin{aligned}
\Delta\alpha' &= \langle\Delta\alpha,h\rangle_{\mathcal{H}},\\
\Lambda(\alpha',u_i) &= \frac{1}{4}\sum_{x,y}c(x,y)|u_i(x)-u_i(y)|^2\cdot|\langle\alpha(x)-\alpha(y),h\rangle_{\mathcal{H}}|^2,\\
2\Gamma(\alpha')(x) &= \sum_y P(x,y)|\langle\alpha(x)-\alpha(y),h\rangle_{\mathcal{H}}|^2,\\
2\Gamma(\alpha',u_i)(x) &= \sum_y P(x,y)(u_i(x)-u_i(y))\cdot\langle\alpha(x)-\alpha(y),h\rangle_{\mathcal{H}}.
\end{aligned}$$

Summing this over h in an orthonormal basis for $\mathcal{H}$, we have the theorem. ∎

4. The proof of main results

4.1. Amenable groups

One application of Theorem 6 is for the case of amenable groups. Given a finitely generated group, there is a natural network one may define. Actually, the initial data are a finitely generated group G and a probability measure μ on G, which is assumed to be *symmetric*, i.e., $\mu(x) = \mu(x^{-1})$. This measure is used to construct the *random walk* on G, which is just the Markov chain with transition matrix $P(x, y) = \mu(x^{-1}y)$. This Markov chain is precisely the reversible Markov chain associated to the network on G given by conductances $c(x, y) = \mu(x^{-1}y)$. We denote this network by (G, μ), and call it the *Cayley network* of G with respect to μ. (Since μ is a probability measure, in this case $\pi(x) = 1$ for all x.)

For a probability measure μ on G, define

$$\mu_* := \inf_{1 \neq y \in \operatorname{supp}(\mu)} \mu(y).$$

Note that μ has finite support if and only if $\mu_* > 0$.

Recall that Kesten's amenability criterion [36] states that the bottom of the spectrum of Δ is 0 if and only if G is an amenable group.

We are now ready to prove Theorem 3.

Proof of Theorem 3. Since G is amenable and infinite, it does not have Kazhdan property (T). (This is very well known, and an easy exercise following the definitions of property (T) and amenability. We say that a group G has *Kazhdan property* (T) if, for any unitary representation $\rho: G \to U(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ without non-zero invariant vectors, fixed by all $g \in G$, there exists a $c > 0$ and a finite subset $K \subset G$ such that for every nonzero $v \in \mathcal{H}$ there exists $k \in K$ such that

$$\|\rho(k)v - v\| \geq c\|v\|.$$

See e.g., [48, Chapter 7].) It follows from [38, 45] that there exists a Hilbert space $\mathcal{H}$ on which the group G acts by unitary operators, with a *harmonic cocycle* $\alpha: G \to \mathcal{H}$. That is, $\alpha(xy) = \alpha(x) + x.\alpha(y)$ for all $x, y \in G$ and $\Delta\alpha \equiv 0$, see (12) for the definition. (For a short proof see, e.g., [46]. Alternatively, for a proof that uses only amenability of the group, see [39].)

Since the G-action is unitary, we may compute that

$$\|\alpha(x) - \alpha(xy)\|^2_{\mathcal{H}} = \|\alpha(y)\|^2_{\mathcal{H}},$$

so

$$2\Gamma(\alpha)(x) = \sum_y \mu(y)\|\alpha(y)\|^2_{\mathcal{H}},$$

is a constant function. We will also write $\Gamma(\alpha)$ as a constant.

Now, if u is an eigenfunction of unit length, with $\Delta u = \lambda u$, then

$$\langle \Gamma(\alpha), u^2 \rangle = \Gamma(\alpha) \cdot \sum_x \pi(x) u(x)^2 = \Gamma(\alpha).$$

Also,

$$\begin{aligned} 4\Lambda(\alpha, u) &= \sum_{x,y} c(x,y) |u(x) - u(y)|^2 \cdot \|\alpha(x) - \alpha(y)\|_{\mathcal{H}}^2 \\ &= \sum_{x,y} \mu(y) |u(x) - u(xy)|^2 \cdot \|\alpha(y)\|_{\mathcal{H}}^2. \end{aligned}$$

Since, for any $1 \neq y \in \text{supp}(\mu)$,

$$\|\alpha(y)\|_{\mathcal{H}}^2 \leq \frac{1}{\mu_*} \sum_y \mu(y) \|\alpha(y)\|_{\mathcal{H}}^2 \leq \frac{1}{\mu_*} \cdot 2\Gamma(\alpha),$$

we get that

$$4\Lambda(\alpha, u) \leq \frac{1}{\mu_*} \cdot 2\Gamma(\alpha) \cdot \sum_{x,y} \mu(y) |u(x) - u(xy)|^2 = \frac{4}{\mu_*} \Gamma(\alpha) \cdot \lambda.$$

Finally,

$$\begin{aligned} 2\Gamma(\alpha, u)(x) &= \sum_y \mu(y)(u(x) - u(xy)) \cdot (\alpha(x) - \alpha(xy)) \\ &= -\sum_y \mu(y)(u(x) - u(xy)) \cdot x.\alpha(y). \end{aligned}$$

Since G acts unitarily on $\mathcal{H}$, we have, by Jensen's inequality,

$$\begin{aligned} \|2\Gamma(\alpha, u)\|^2 &= \sum_x \left\| \sum_y \mu(y)(u(x) - u(xy)) \cdot \alpha(y) \right\|_{\mathcal{H}}^2 \\ &\leq \sum_{x,y} \mu(y) |u(x) - u(xy)|^2 \cdot \|\alpha(y)\|_{\mathcal{H}}^2 = 4\Lambda(\alpha, u). \end{aligned}$$

Plugging all the above into Theorem 6 we arrive at

$$\begin{aligned} \sum_{i=1}^k |\lambda_{k+1} - \lambda_i|^2 \cdot \Gamma(\alpha) &\leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \cdot \Lambda(\alpha, u_i) \cdot (4 + \lambda_{k+1} - \lambda_i) \\ &\leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i \cdot \frac{6}{\mu_*} \cdot \Gamma(\alpha), \end{aligned}$$

where we have used that $\lambda_{k+1} - \lambda_i \leq 2$. This completes the proof. ∎

4.2. Groups with Abelian quotients

For general groups with Abelian quotients, we can prove the Yang-type inequality, analogous to the result in [35].

Theorem 7. *Let G be a finitely generated group. Let $\alpha\colon G \to \mathbb{Z}^n$ be a surjective homomorphism. Let $S = \{s_1, \dots, s_n, k_1, \dots, k_m\}$ be a generating set for G so that $(\alpha(s_j))_{j=1}^n$ is the standard basis of $\mathbb{Z}^n$, and such that $\alpha(k_j) = 0$ for all $j = 1, \dots, m$. Let μ be a symmetric measure supported on $S \cup S^{-1}$. Let $\varepsilon = 1 - \sum_{j=1}^n (\mu(s_j) + \mu(s_j^{-1}))$. (e.g., one may take $\mu(k_j) = \mu(k_j^{-1}) = \frac{\varepsilon}{2n}$ and $\mu(s_j) = \mu(s_j^{-1}) = \frac{1-\varepsilon}{2n}$.)*

Then, the network (G, μ) satisfies the following: For any finite $\Omega \subset G$ and $k < |\Omega|$,

$$\sum_{i=1}^{k} |\lambda_{k+1} - \lambda_i|^2 \cdot (1 - \varepsilon - \lambda_i) \leq 8 \max_j \mu(s_j) \cdot \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \cdot \lambda_i .$$

Remark 8. When we choose $\mu(k_j) = \mu(k_j^{-1}) = \frac{\varepsilon}{2n}$ and $\mu(s_j) = \mu(s_j^{-1}) = \frac{1-\varepsilon}{2n}$, we get the Yang-type inequality up to an ε-defect, with constant at most $\frac{4}{n}$.

Remark 9. The case $G \cong \mathbb{Z}^n$ was already treated in [35], where the same result was shown, using similar methods. This is the case $\varepsilon = 0$ and $\mu(s_j) = \mu(s_j^{-1}) = \frac{1}{2n}$ in the above theorem.

Proof. The main advantage of α being a homomorphism is that

$$\mu(y)\alpha(y) = \begin{cases} \pm\mu(s_j)e_j, & y = (s_j)^{\pm 1}, \\ 0, & \text{otherwise.} \end{cases}$$

where $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{Z}^n$. Thus, for the Euclidean Hilbert space $\mathcal{H} = \mathbb{R}^n$,

$$2\Gamma(\alpha)(x) = \sum_y \mu(y) \|\alpha(x) - \alpha(xy)\|_{\mathcal{H}}^2 = \sum_{j=1}^{n} \big(\mu(s_j) + \mu(s_j^{-1})\big) = 1 - \varepsilon,$$

for any $x \in G$. Also, $\Delta\alpha \equiv 0$. Now, if u is an eigenfunction of unit length, with $\Delta u = \lambda u$, then

$$\langle \Gamma(\alpha), u^2 \rangle = \Gamma(\alpha) = \frac{1}{2}(1 - \varepsilon).$$

We may bound

$$\begin{aligned} 4\Lambda(\alpha, u) &= \sum_{x,y} \mu(y) |u(x) - u(xy)|^2 \cdot \|\alpha(y)\|_{\mathcal{H}}^2 \\ &= \sum_x \sum_{j=1}^{n} \mu(s_j) \big(|u(x) - u(xs_j)|^2 + |u(x) - u(xs_j^{-1})|^2 \big) \\ &\leq \sum_{x,y} \mu(y) |u(x) - u(xy)|^2 = 2\lambda . \end{aligned}$$

As in the proof of Theorem 3,

$$2\Gamma(\alpha,u)(x) = -\sum_{j=1}^{n} \mu(s_j)(u(x)-u(xs_j)-u(x)+u(xs_j^{-1}))\cdot\alpha(s_j),$$

$$\begin{aligned}\|2\Gamma(\alpha,u)\|^2 &= \sum_x\sum_{j=1}^{n}\mu(s_j)^2|u(xs_j^{-1})-u(xs_j)|^2\\ &\le 2\sum_x\sum_{j=1}^{n}\mu(s_j)^2(|u(x)-u(xs_j)|^2+|u(x)-u(xs_j^{-1})|^2)\\ &\le 2\max_j\mu(s_j)\sum_{x,y}\mu(y)|u(x)-u(xy)|^2 = \max_j\mu(s_j)\cdot 4\lambda.\end{aligned}$$

Plugging all of this into Theorem 6, we arrive at

$$\sum_{i=1}^{k}|\lambda_{k+1}-\lambda_i|^2\cdot(1-\varepsilon-\lambda_i) \le 8\max_j\mu(s_j)\cdot\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\cdot\lambda_i. \qquad \blacksquare$$

4.3. Trees

In this section, we prove the Yang-type inequality for d-regular tree $\mathbb{T}_d$, $d\ge 3$, with the conductances of the edges $c(x,y) = \mathbf{1}_{\{x\sim y\}}\frac{1}{d}$.

Proof of Theorem 4. Fix a ray to infinity, and an origin o. Let b be the *Busemann function* corresponding to the ray with $b(o)=0$. That is, let $o = x_0 \sim x_1 \sim\cdots\sim x_n\sim x_{n+1}\sim\cdots$ be an infinite *simple* path, so $x_i\ne x_j$ for all $i\ne j$. Because $\mathbb{T}_d$ is a tree, this path is necessarily a geodesic: the distance between x_j, x_i in the graph is always $|j-i|$. This path is the *ray* mentioned above. Now, for any $j\ge 0$ set $b(x_j) := -j$. Furthermore, for any vertex z, let z_* be the closest vertex to z from the above path. Set $b(z) = b(z_*) + \mathrm{dist}(z,z_*)$.

The important properties of b are thus: $b\colon\mathbb{T}_d\to\mathbb{Z}$ is a function such that $b(o)=0$ and such that every vertex x has $d-1$ neighbors $y\sim x$ with $b(y) = b(x)+1$, and exactly one neighbor $\vec{x}\sim x$ with $b(\vec{x}) = b(x)-1$. One easily sees that

$$2\Gamma(b)(x) = 1 \quad \text{for all } x\in\mathbb{T}_d.$$

It is also simple to check that the function $f(x) = \left(\frac{\xi}{\sqrt{d-1}}\right)^{b(x)}$ satisfies

$$\Delta f(x) = f(x)\cdot\Big(1-\frac{\sqrt{d-1}}{d}\cdot(\xi+\xi^{-1})\Big).$$

Hence, if $\lambda = 1-\frac{2\sqrt{d-1}}{d}$ (which corresponds to choosing $\xi=1$, maximizing the above expression) then $\Delta f = \lambda f$. Coincidentally, this is the bottom of the L^2 spectrum of Δ, i.e., $\lambda_{\min} = 1-\frac{2\sqrt{d-1}}{d}$.

For any x let $\vec{x}$ be the unique vertex with $b(\vec{x}) = b(x) - 1$. For a function f let $\vec{f}(x) := f(\vec{x})$. Note that as x ranges over the whole graph, the pair $(x, \vec{x})$ ranges over all edges in the graph, each edge counted exactly once in the direction of decreasing the Busemann function b. Thus,

$$\begin{aligned}\|f - \vec{f}\|^2 &= \sum_x |f(x) - f(\vec{x})|^2 = \frac{1}{2}\sum_{x\sim y} |f(x) - f(y)|^2 \\ &= \frac{d}{2}\sum_{x,y} c(x,y)|f(x) - f(y)|^2 = d\langle \Delta f, f\rangle.\end{aligned}$$

Also, the map $x \mapsto \vec{x}$ is a $(d-1)$-to-1 map. So,

$$\|\vec{f}\|^2 = \sum_x |f(\vec{x})|^2 = \sum_y \sum_{x:\vec{x}=y} |f(y)|^2 = (d-1)\|f\|^2. \tag{13}$$

Thus,

$$d\langle \Delta f, f\rangle = \|f - \vec{f}\|^2 = d \cdot \|f\|^2 - 2\langle f, \vec{f}\rangle. \tag{14}$$

Note that the Busemann function satisfies

$$\Delta b(x) = \sum_y P(x,y)(b(x) - b(y)) = -\frac{d-2}{d} =: -\gamma,$$

and also $|b(x) - b(y)| = 1$ for any $x \sim y$.

Let u be an eigenfunction $\Delta u = \lambda u$. Note that

$$\langle 2\Gamma(b,u), u\rangle = \frac{1}{2}\mathcal{E}(b, u^2) = \langle \Delta b, u^2\rangle = -\gamma\|u\|^2.$$

Thus,

$$\begin{aligned}\|2\Gamma(b,u) - u\Delta b\|^2 &= 4\|\Gamma(b,u)\|^2 + \gamma^2 \cdot \|u\|^2 + 2\gamma\langle 2\Gamma(b,u), u\rangle \\ &= 4\|\Gamma(b,u)\|^2 - \gamma^2 \cdot \|u\|^2.\end{aligned} \tag{15}$$

Also,

$$\begin{aligned}2\Gamma(b,u)(x) &= \sum_y c(x,y)(b(x) - b(y))(u(x) - u(y)) \\ &= -\sum_{y\neq\vec{x}} c(x,y)(u(x) - u(y)) + c(x,\vec{x})(u(x) - u(\vec{x})) \\ &= -\Delta u(x) + \frac{2}{d}(u(x) - u(\vec{x})) = \Big(\frac{2}{d} - \lambda\Big)u(x) - \frac{2}{d}\vec{u}(x),\end{aligned}$$

so using (13) and (14), assuming that $\|u\| = 1$,

$$\begin{aligned}\|2\Gamma(b,u)\|^2 &= (1-\lambda-\gamma)^2\|u\|^2 + \frac{4}{d^2}\|\vec{u}\|^2 - \frac{4}{d}(1-\lambda-\gamma)\langle u,\vec{u}\rangle \\ &= (1-\lambda)^2 + \gamma^2 - 2\gamma(1-\lambda) + \frac{4}{d^2}(d-1) - 2(1-\lambda-\gamma)(1-\lambda) \\ &= \gamma^2 + (1-\lambda_{\min})^2 - (1-\lambda)^2. \end{aligned} \tag{16}$$

Finally,

$$4\Lambda(b,u) = \sum_{x,y} c(x,y)|b(x)-b(y)|^2 \cdot |u(x)-u(y)|^2 = 2\lambda.$$

Combining this with (15), (16), and plugging into Lemma 5, we have that

$$\begin{aligned}\sum_{i=1}^{k} |\lambda_{k+1}-\lambda_i|^2 \cdot (1-\lambda_i) &\le 2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\cdot(\lambda_i-\lambda_{\min})\cdot(1-\lambda_i+1-\lambda_{\min}) \\ &\le \frac{8\sqrt{d-1}}{d} \cdot \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\cdot(\lambda_i-\lambda_{\min}),\end{aligned}$$

where we used $\lambda_i \ge \lambda_{\min} = 1 - \frac{2\sqrt{d-1}}{d}$. ∎

5. Applications of Yang-type inequalities

In this section, we derive some applications of the Yang-type inequality on graphs. Let (V,c) be the network with the bottom of the spectrum $\lambda_{\min}$. For any finite subset Ω, let $\{\lambda_i\}_{i=1}^{|\Omega|}$ be the Dirichlet eigenvalues of the Laplace on Ω. Set

$$\mu_i := \lambda_i - \lambda_{\min} \ge 0, \quad 1 \le i \le |\Omega|. \tag{17}$$

By the trace of the Laplacian,

$$\sum_{i=1}^{|\Omega|} \lambda_i \le |\Omega|.$$

Hence, for any $1 \le k \le |\Omega|$,

$$\sum_{i=1}^{k}(1-\lambda_i) \ge 0.$$

Corollary 10. *Suppose that the network (V,c) satisfies the Yang-type inequality (3). Then for any finite subset Ω,*

$$\lambda_2 - \lambda_{\min} \le \left(\frac{C_{\mathrm{YT}}}{1-\lambda_1} + 1\right)(\lambda_1 - \lambda_{\min}).$$

Proof. This follows from the Yang-type inequality (3) for $k = 1$. ∎

We prove the following consequences for the Yang-type inequality.

Corollary 11. *Suppose that the network (V, c) satisfies the Yang-type inequality* (3). *Then for any finite subset Ω, if $\lambda_k \le 1 + C_{\mathrm{YT}}$ for some $1 \le k < |\Omega|$, then we have the following:*

(i)
$$\lambda_{k+1} - \lambda_{\min} \le \frac{\sum_{i=1}^{k}(\lambda_i - \lambda_{\min})(1 + C_{\mathrm{YT}} - \lambda_i)}{\sum_{i=1}^{k}(1 - \lambda_i)};$$

(ii)
$$\sum_{i=1}^{k} \frac{\lambda_i - \lambda_{\min}}{\lambda_{k+1} - \lambda_i} \ge \frac{1}{C_{\mathrm{YT}}} \sum_{i=1}^{k}(1 - \lambda_i);$$

(iii)
$$\lambda_{k+1} - \lambda_k \le C_{\mathrm{YT}} \frac{\sum_{i=1}^{k}(\lambda_i - \lambda_{\min})}{\sum_{i=1}^{k}(1 - \lambda_i)}.$$

We remark that (i), (ii), and (iii) are discrete analogs of Yang's second inequality, the Hile–Protter inequality, and the Payne–Pólya–Weinberger inequality, respectively.

Proof. Let $C = C_{\mathrm{YT}}$. For assertion (i), without loss of generality, we may assume that $\lambda_{k+1} > \lambda_1$, otherwise the result is trivial. By the Yang-type inequality (3),

$$\frac{1}{k} \sum_i (\mu_{k+1} - \mu_i)[(\mu_{k+1} - \mu_i)(1 - \mu_i - \lambda_{\min}) - C\mu_i] \le 0,$$

where $\{\mu_i\}_i$ is defined in (17) and $C = C_{\mathrm{YT}}$. Set $a_i := \mu_{k+1} - \mu_i$ and

$$b_i := (\mu_{k+1} - \mu_i)(1 - \mu_i - \lambda_{\min}) - C\mu_i.$$

Note that the function

$$f(x) := (\mu_{k+1} - x)(1 - x - \lambda_{\min}) - Cx$$

is non-increasing in $\left(-\infty, \frac{1}{2}(1 + C + \mu_{k+1} - \lambda_{\min})\right]$. Moreover, the assumption $\lambda_k \le 1 + C$ yields that

$$\mu_i \le \frac{1}{2}(1 + C + \mu_{k+1} - \lambda_{\min}),$$

which implies that b_i is non-increasing. Using Chebyshev's inequality, i.e.,

$$\sum_i a_i b_i \ge \frac{1}{k} \sum_i a_i \sum_i b_i,$$

we have

$$\left(\mu_{k+1} - \frac{1}{k}\sum_{i=1}^{k}\mu_i\right)\left[\mu_{k+1} \cdot \frac{1}{k}\sum_{i=1}^{k}(1 - \lambda_i) - \frac{1}{k}\sum_{i=1}^{k}\mu_i(1 + C - \lambda_i)\right] \le 0.$$

Note that by $\lambda_{k+1} > \lambda_1$,

$$\lambda_{k+1} > \frac{1}{k}\sum_{i=1}^{k}\lambda_i.$$

Thus,

$$\mu_{k+1} \leq \frac{\sum_{i=1}^{k}\mu_i(1+C-\lambda_i)}{\sum_{i=1}^{k}(1-\lambda_i)},$$

which proves the result (i).

For assertion (ii), without loss of generality, we assume that $\lambda_k < \lambda_{k+1}$. Let $C = C_{\mathrm{YT}}$. Set $g(x) := \frac{x}{\mu_{k+1}-x}$, which is convex in $x \in (-\infty, \mu_{k+1})$. Hence,

$$\begin{aligned}\frac{1}{k}\sum_{i=1}^{k}\frac{\lambda_i-\lambda_{\min}}{\lambda_{k+1}-\lambda_i} &= \frac{1}{k}\sum_{i=1}^{k}\frac{\mu_i}{\mu_{k+1}-\mu_i} = \frac{1}{k}\sum_i g(\mu_i)\\ &\geq g\Big(\frac{1}{k}\sum_i \mu_i\Big) = \frac{\frac{1}{k}\sum_i\mu_i}{\mu_{k+1}-\frac{1}{k}\sum_i\mu_i},\end{aligned} \tag{18}$$

where we used Jensen's inequality for $g(x)$. By assertion (i),

$$\begin{aligned}\mu_{k+1} &\leq \frac{\sum_{i=1}^{k}\mu_i(1+C-\lambda_i)}{\sum_{i=1}^{k}(1-\lambda_i)}\\ &= \frac{C\sum_{i=1}^{k}\mu_i}{\sum_{i=1}^{k}(1-\lambda_i)} + \frac{\sum_{i=1}^{k}\mu_i(1-\lambda_i)}{\sum_{i=1}^{k}(1-\lambda_i)}\\ &\leq \frac{C\sum_{i=1}^{k}\mu_i}{\sum_{i=1}^{k}(1-\lambda_i)} + \frac{1}{k}\sum_{i=1}^{k}\mu_i,\end{aligned}$$

where we used Chebyshev's inequality in the last line. By plugging it into (18), we prove the result (ii).

For assertion (iii), we assume that $\lambda_k < \lambda_{k+1}$. By assertion (ii),

$$\frac{\sum_{i=1}^{k}(\lambda_i-\lambda_{\min})}{\lambda_{k+1}-\lambda_k} \geq \sum_{i=1}^{k}\frac{\lambda_i-\lambda_{\min}}{\lambda_{k+1}-\lambda_i} \geq \frac{1}{C_{\mathrm{YT}}}\sum_{i=1}^{k}(1-\lambda_i),$$

which yields the result. ■

We remark that for amenable groups, groups with Abelian quotients, and d-trees, the discrete analogs of the Payne–Pólya–Weinberger inequality and the Hile–Protter inequality as in Corollary 11 without the assumption that $\lambda_k \leq 1 + C_{\mathrm{YT}}$ for some $1 \leq k < |\Omega|$, can be derived using same arguments in [35, Theorem 1.1 and Theorem 1.3].

We recall a recursion formula proved by Cheng and Yang [22], see also [35, Theorem 4.2] and [3, Lemma 8.9].

Proposition 12. *Let* $a_1 \le a_2 \le \cdots \le a_{k+1}$ *be any positive numbers and* $\theta > 0$ *such that*

$$\sum_{i=1}^{k}(a_{k+1}-a_i)^2 \le \theta \sum_{i=1}^{k} a_i(a_{k+1}-a_i).$$

Define

$$F_k = \Big(1+\frac{\theta}{2}\Big)\Big(\frac{1}{k}\sum_{i=1}^{k} a_i\Big)^2 - \frac{1}{k}\sum_{i=1}^{k} a_i^2.$$

Then we have

$$F_{k+1} \le \Big(\frac{k+1}{k}\Big)^{\theta} F_k.$$

Now, we prove an upper bound estimate for λ_k.

Corollary 13. *Suppose that the network* (V, c) *satisfies the Yang-type inequality* (3). *Then for any finite subset* Ω, *if* $\lambda_k \le 1-\delta$ *for some* $\delta > 0$, *then*

$$\lambda_{k+1} - \lambda_{\min} \le (1+\theta)k^{\frac{\theta}{2}}(\lambda_1 - \lambda_{\min}),$$

where $\theta = \frac{1}{\delta}C_{\mathrm{YT}}$.

Proof. Let $\mu_i := \lambda_i - \lambda_{\min}$. By the Yang-type inequality (3), we have

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2(1-\lambda_i) \le C \cdot \sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\mu_i,$$

where $C = C_{\mathrm{YT}}$. Since $\lambda_k \le 1-\delta$, $1-\lambda_i \ge \delta$ for any $1 \le i \le k$. This yields that

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \le \theta \cdot \sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\mu_i, \tag{19}$$

where $\theta = \frac{C}{\delta}$. By the recursion formula in Proposition 12, setting $a_i = \mu_i$,

$$F_{k+1} \le \Big(\frac{k+1}{k}\Big)^{\theta} F_k.$$

Since the above result holds for all small k, we have

$$\frac{F_{k+1}}{(k+1)^{\theta}} \le \frac{F_k}{k^{\theta}} \le \cdots \le F_1 = \frac{\theta}{2}a_1^2.$$

By (19), for $A_k = \frac{1}{k}\sum_i a_i$, $B_k = \frac{1}{k}\sum_i a_i^2$,

$$\begin{aligned}\Big(a_{k+1} - \Big(1+\frac{\theta}{2}\Big)A_k\Big)^2 &\le \Big(1+\frac{\theta}{2}\Big)^2 A_k^2 - (1+\theta)B_k \\ &= (1+\theta)F_k - \frac{\theta}{2}\Big(1+\frac{\theta}{2}\Big)A_k^2.\end{aligned}$$

Hence,

$$a_{k+1}^2 \leq \frac{2(1+\theta)^2}{\theta} F_k \leq (1+\theta)^2 k^\theta a_1^2.$$

This proves the result. ■

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