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Sharp Hardy's inequalities in Hilbert spaces

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Abstract. We study the behavior of the smallest possible constants $d(a,b)$ and d_n in Hardy's inequalities

$$\int_a^b \Big(\frac{1}{x}\int_a^x f(t)dt\Big)^2 dx \le d(a,b)\int_a^b [f(x)]^2\,dx$$

and

$$\sum_{k=1}^n \Big(\frac{1}{k}\sum_{j=1}^k a_j\Big)^2 \le d_n \sum_{k=1}^n a_k^2.$$

The exact constant $d(a,b)$ and the precise rate of convergence of d_n are established and the extremal function and the "almost extremal" sequence are found.

1. Introduction and statement of the results

In the series of papers [3–5], Hardy proved the following two inequalities. Let $p > 1$. If $f(x) \ge 0$ and f^p is integrable over $(0,\infty)$, then the inequality

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty [f(x)]^p\,dx. \tag{1.1}$$

holds. This is the original Hardy's integral inequality. The discrete version of Hardy's inequality reads as

$$\sum_{k=1}^\infty \Big(\frac{1}{k}\sum_{j=1}^k a_j\Big)^p \le \Big(\frac{p}{p-1}\Big)^p \sum_{k=1}^\infty a_k^p, \quad a_k \ge 0, \quad k \in \mathbb{N} \tag{1.2}$$

Initially, Hardy proved (1.2) with the constant $(p^2/(p-1))^p$. Later, Landau, in the letter [9], which was officially published in [10], established the exact constant $(p/(p-1))^p$, in the sense that there is no smaller one for which (1.2) holds for

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every sequence of non-negative numbers a_k. Moreover, Landau observed that equality in (1.2) occurs only for the trivial sequence, that is, when $a_k = 0$ for every $k \in \mathbb{N}$. Similarly, the equality in (1.1) occurs if and only if $f(x) \equiv 0$ almost everywhere.

Inequality (1.1) has been extended to what is nowadays called the *general Hardy integral inequality*:

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t)dt \right)^p dx \leq d_p(a,b) \int_a^b [f(x)]^p \, dx, \quad f(x) \geq 0. \tag{1.3}$$

For a fixed $n \in \mathbb{N}$, it is natural to study the inequality

$$\sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{k} a_j \right)^p \leq d_{n,p} \sum_{k=1}^{n} a_k^p, \quad a_k \geq 0, k = 1, 2, \ldots, n. \tag{1.4}$$

and ask for the smallest possible $d_{n,p}$ for which it holds.

When p is a positive even integer, the assumption for non-negativity of $f(x)$ and $\{a_k\}$ can be dropped. In particular, when $p = 2$ inequalities (1.3) and (1.4) become

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t)dt \right)^2 dx \leq d(a,b) \int_a^b [f(x)]^2 \, dx \tag{1.5}$$

and

$$\sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{k} a_j \right)^2 \leq d_n \sum_{k=1}^{n} a_k^2. \tag{1.6}$$

There are many papers investigating different generalizations and applications of Hardy's inequality; see for instance [7] and the bibliography in the book [8].

In the present paper we establish the sharp inequality (1.5) in the sense that we determine the exact constant $d(a,b)$ in (1.5) as well as an extremal function f for which equality is attained. Our main result reads as follows.

Theorem 1.1. *Let a and b be any fixed numbers with $0 < a < b < \infty$. Then the inequality*

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t)dt \right)^2 dx \leq \frac{4}{1 + 4\alpha^2} \int_a^b [f(x)]^2 \, dx, \tag{1.7}$$

where α is the only solution of the equation

$$\tan\left(\alpha \log \frac{b}{a}\right) + 2\alpha = 0 \quad \textit{in the interval} \left(\frac{\pi}{2 \log \frac{b}{a}}, \frac{\pi}{\log \frac{b}{a}} \right),$$

holds for every $f \in L^2[a,b]$. *Moreover, the equality in* (1.7) *is attained for*

$$f_{a,b}(x) = x^{-1/2}\Big(2\alpha\cos\Big(\alpha\log\frac{x}{a}\Big) + \sin\Big(\alpha\log\frac{x}{a}\Big)\Big).$$

Corollary 1.2. *When either of the limits relations* $a \to 0$, $b \to \infty$, *or both hold, i.e.,* $\log\frac{b}{a} \to \infty$, *then*

$$d(a,b) \sim 4 - \frac{c}{[\log\frac{b}{a}]^2}.$$

In other words, there exist absolute constants $c_1 > 0$ *and* $c_2 > 0$, *such that*

$$4 - \frac{c_1}{[\log\frac{b}{a}]^2} \le d(a,b) \le 4 - \frac{c_2}{[\log\frac{b}{a}]^2}.$$

Since the function $f_{a,b}$ defined above obeys

$$0 < f_{a,b}(x) < x^{-1/2}\alpha\Big(2 + \Big|\log\frac{x}{a}\Big|\Big) < \frac{\pi}{\log\frac{b}{a}}x^{-1/2}\Big(2 + \Big|\log\frac{x}{a}\Big|\Big),$$

it is obvious that $f_{a,b}$ converges uniformly to $g(x) \equiv 0$ when $b \to \infty$.

We remark that an important well-known fact is that (1.1) is the prototype of the Hardy–Littlewood inequality for the maximal function. Therefore, it is natural to consider the inequality

$$\int_a^b \Big(\frac{1}{x-a}\int_a^x f(t)dt\Big)^2 dx \le d(a,b)\int_a^b [f(x)]^2\,dx, \quad f \in L^2[a,b]$$

which is equivalent, by change of variables, to

$$\int_0^b \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^2 dx \le d(b)\int_0^b [f(x)]^2\,dx, \quad f \in L^2[0,b].$$

Then, Corollary 1.2 implies that $d(b) = 4$ and the only function f for which the equality is attained is the one which vanishes almost everywhere.

Inequality (1.6) has been studied by many authors. Based on ideas of Widom [13] and his previous joint work with Widom himself [14], and with N. de Bruijn [1], on Hilbert's inequality, Herbert Wilf [15] established the following asymptotic expression for d_n:

$$d_n = 4 - \frac{16\pi^2}{\log^2 n} + O\Big(\frac{\log\log n}{\log^3 n}\Big), \quad n \to \infty. \tag{1.8}$$

The ideas developed in [2] and an important observation of the third-named author of the present note, yielded an explicit representation of d_n in terms of the smallest zero of a continuous dual Hahn polynomial of degree n, for a specific choice of the parameters, in terms of which these polynomials are defined; see Theorem A below. More recently, F. Stampach [12] studied in details the asymptotics of d_n, again employing its relation to these zeros and proved that

$$d_n = 4 - \frac{16\pi^2}{\log^2 n} + \frac{32\pi^2(\gamma + 6\log 2)}{\log^3 n} + O\Big(\frac{\log\log n}{\log^4 n}\Big), \quad n \to \infty, \tag{1.9}$$

where γ is the Euler constant. Another formula involving few more terms and an algorithm for calculating the further ones in the asymptotic expansion of d_n in terms of negative powers of $\log n$ was suggested in [12].

Here, we combine an idea similar to the one we use for the proof of Theorem 1.1 with the result in [2, Theorem 1.1] concerning the relation between d_n and the zeros of the continuous dual Hahn polynomial, and a recent one, due to W-G. Long, D. Dai, Y-T. Li, and X-S. Wang [11], about the asymptotic behaviour of those zeros. We establish sharp lower and upper bounds for d_n and obtain its full asymptotic expansion, thus extending the results of Wilf (1.8) and Stampach (1.9) to the largest possible generality.

Theorem 1.3. *Let*

$$a_k = \int_k^{k+1} h(x)\, dx,$$

where

$$h(x) = x^{-1/2}\big(2\alpha\cos(\alpha\log x) + \sin(\alpha\log x)\big), \quad 1 \le x \le n+1, \tag{1.10}$$

and α is the only solution of the equation

$$\tan(\alpha\log(n+1)) + 2\alpha = 0 \quad \textit{in the interval } \Big(\frac{\pi}{2\log(n+1)}, \frac{\pi}{\log(n+1)}\Big).$$

Then

$$\sum_{k=1}^{n}\Big(\frac{1}{k}\sum_{j=1}^{k} a_j\Big)^2 \ge \frac{4}{1+4\alpha^2}\sum_{k=1}^{n} a_k^2. \tag{1.11}$$

Since

$$\frac{4}{1+4\alpha^2} \ge 4 - 16\alpha^2 > 4 - \frac{16\pi^2}{\log^2(n+1)},$$

we obtain

$$d_n \ge 4 - \frac{16\pi^2}{\log^2(n+1)}. \tag{1.12}$$

Combining the above mentioned results with the latter, we obtain the exact rate of convergence of $\{d_n\}$ as well as very sharp estimates for d_n for every fixed n, we obtain the following result.

Theorem 1.4. *The inequalities*

$$4-\frac{16\pi^2}{[\log(n+1)]^2}\le d_n\le 4-\frac{32}{[\log n+4]^2} \tag{1.13}$$

hold for every natural $n\ge 3$.

Moreover,

$$d_n\sim 4-\frac{16\pi^2}{4\pi^2+(\gamma+6\log 2+\log n)^2}\quad \text{as } n\to\infty, \tag{1.14}$$

where γ *is the Euler constant, and the following full asymptotic expansion of* d_n *in terms of the negative powers of* $\log n$ *holds. For every fixed* $m\in\mathbb{N}$, $m\ge 2$,

$$d_n=4-\sum_{k=2}^{m} c_k\frac{1}{[\log n]^k}+O((\log n)^{-m-1}),\quad n\to\infty, \tag{1.15}$$

with

$$c_k=16\pi^2\big(4\pi^2+(\gamma+6\log 2)^2\big)^{(k-2)/2}\,U_{k-2}\Big(-\frac{\gamma+6\log 2}{\sqrt{4\pi^2+(\gamma+6\log 2)^2}}\Big), \tag{1.16}$$

where U_{k-2} *denotes the Chebyshev polynomial of the second kind of degree* $k-2$.

Needless to say, the first few coefficients c_k, $k=2,3,4,5$, in (1.16) coincide with those obtained by Wilf and Stampach.

The lower bound in (1.13), obtained as a consequence of Theorem 1.3, is amazingly close to the asymptotic value for d_n in (1.14), which is derived via [2, Theorem 1.1] and [11]. Indeed, it is not difficult to verify that for their difference one has

$$\Big(4-\frac{16\pi^2}{4\pi^2+(\gamma+\log(64n))^2}\Big)-\Big(4-\frac{16\pi^2}{\log^2(n+1)}\Big)\sim\frac{16\pi^2(2\gamma+\log 128)}{\log^3 n},\quad n\to\infty.$$

The function $h(x)$, defined in (1.10), obeys

$$0<h(x)<x^{-1/2}(2\alpha+\alpha\log x)<\frac{\pi}{\log(n+1)}x^{-1/2}(2+\log x).$$

Then it is obvious, as Landau pointed in his letter to Hardy, that if we let $n\to\infty$, then the almost extremal sequence a_k, $k=1,2,\dots$, defined in the Theorem 1.3, goes to the zero sequence, i.e., to the sequence $a_k=0$ for all k.

2. Proof of Theorem 1.1

By simple changes of variables $t = au, x = av$ in the left-hand side and $x = au$ in the right-hand side, we write inequality (1.7) in the following way:

$$\int_1^{b/a}\left(\frac{1}{v}\int_1^{v} f(au)\,du\right)^2 dv \leq \frac{4}{1+4\alpha^2}\int_1^{b/a}[f(au)]^2\,du.$$

It is obvious (by changing the notations), that it suffices to prove Theorem 1.1 for the interval $(1, b)$.

Cauchy's inequality yields

$$\left(\int_1^{x} f(t)\,dt\right)^2 \leq \left(\int_1^{x}[g(t)]^2\,dt\right)\left(\int_1^{x}\frac{[f(t)]^2}{[g(t)]^2}\,dt\right)$$

for every pair of functions $f, g \in L^2[1, b]$, such that $g(x) \neq 0$ for every $x \in (1, b)$. By multiplying both sides of the latter inequality by x^{-2} and integrating from 1 to b, we obtain

$$\int_1^{b}\left(\frac{1}{x}\int_1^{x} f(t)\,dt\right)^2 dx \leq \int_1^{b}\left(\frac{1}{x^2}\int_1^{x}[g(t)]^2\,dt\right)\left(\int_1^{x}\frac{[f(t)]^2}{[g(t)]^2}\,dt\right)dx$$

By changing the order of integration in the right-hand side, we obtain

$$\int_1^{b}\left(\frac{1}{x}\int_1^{x} f(t)\,dt\right)^2 dx \leq \int_1^{b}\left(\frac{1}{[g(t)]^2}\int_t^{b}\left(\frac{1}{x^2}\int_1^{x}[g(u)]^2\,du\right)dx\right)[f(t)]^2\,dt.$$

Let us denote for brevity

$$M(g, t) = \frac{1}{g^2(t)}\int_t^{b}\left(\frac{1}{x^2}\int_1^{x}[g(u)]^2\,du\right)dx.$$

Then the inequality

$$\int_1^{b}\left(\frac{1}{x}\int_1^{x} f(t)\,dt\right)^2 dx \leq \max_{1<t<b} M(g, t)\int_1^{b}[f(t)]^2\,dt$$

holds for every two functions $f, g \in L^2[1, b]$, with $g(x) \neq 0$ for $1 < x < b$, and consequently

$$d(1, b) \leq \max_{1<t<b} M(g, t)$$

for every non-vanishing function $g \in L^2(1, b)$.

Now, we minimize

$$\max_{1<t<b} M(g,t)$$

over all such functions g, that is, determine

$$\min_{g(x)\neq 0}\ \max_{1<t<b} \frac{1}{g^2(t)} \int_t^b \left(\frac{1}{x^2} \int_1^x [g(u)]^2\, du \right) dx.$$

Let us consider the function

$$h(x) = x^{-1/2}\big(2\alpha \cos(\alpha \log x) + \sin(\alpha \log x)\big), \quad x \in [1,b],$$

where α is the only solution of the equation

$$\tan(\alpha \log b) + 2\alpha = 0$$

in the interval $(\pi/(2\log b), \pi/\log b)$. We have

$$h'(x) = -\frac{1+4\alpha^2}{2x^{3/2}} \sin(\alpha \log x).$$

Obviously, $h'(x) < 0$ for $1 < x < b$ and consequently $h(x)$ is decreasing. Since $h(b) = 0$, it follows that $h(x) > 0$ for $1 \le x < b$. Then, for the function $g(x) = \sqrt{h(x)}$, we have

$$\begin{aligned}\int_1^x [g(u)]^2\, du &= \int_1^x u^{-1/2}\big(2\alpha \cos(\alpha \log u) + \sin(\alpha \log u)\big)\, du \\ &= 2\sqrt{x} \sin(\alpha \log x),\end{aligned}$$

and

$$\begin{aligned}\int_t^b \left(\frac{1}{x^2} \int_1^x [g(u)]^2\, du \right) dx &= 2 \int_t^b \frac{\sin(\alpha \log x)}{x^{3/2}}\, dx \\ &= -\frac{4}{1+4\alpha^2} \int_t^b h'(x)\, dx = \frac{4[g(t)]^2}{1+4\alpha^2}.\end{aligned}$$

Therefore,

$$M(g,t) = \frac{4}{1+4\alpha^2} \quad \text{and} \quad d(1,b) \le \frac{4}{1+4\alpha^2}.$$

On the other hand, by changing the order of integration, we can rewrite the left-hand side of (1.7), with $a = 1$, in the following way:

$$\int_1^b \left(\frac{1}{x}\int_1^x f(t)\,dt\right)^2 dx = \int_1^b \left(\frac{1}{f(t)}\int_t^b \left(\frac{1}{x^2}\int_1^x f(u)\,du\right) dx\right)[f(t)]^2\,dt.$$

Consequently, for the function $f_{a,b}(x) = h(x)$ we have

$$\int_1^b \left(\frac{1}{x}\int_1^x f_{a,b}(t)\,dt\right)^2 dx = \frac{4}{1+4\alpha^2}\int_1^b [f_{a,b}(x)]^2\,dx.$$

The proof of Theorem 1.1 is complete.

3. Proof of Theorem 1.3

By changing the order of summation in the left-hand side of (1.11), we obtain

$$\sum_{k=1}^n \Big(\frac{1}{k}\sum_{j=1}^k a_j\Big)^2 = \sum_{i=1}^n \Big[\frac{1}{a_i}\Big(\sum_{k=i}^n \frac{1}{k^2}\sum_{j=1}^k a_j\Big)\Big] a_i^2 = \sum_{i=1}^n M_i a_i^2$$

where

$$M_i = \frac{1}{a_i}M_i^* \quad \text{and} \quad M_i^* = \sum_{k=i}^n \Big(\frac{1}{k^2}\sum_{j=1}^k a_j\Big).$$

Then

$$\sum_{k=1}^n \Big(\frac{1}{k}\sum_{j=1}^k a_j\Big)^2 \geq \min_{1\leq i\leq n} M_i \sum_{k=1}^n a_i^2,$$

and consequently

$$d_n \geq \min_{1\leq i\leq n} M_i.$$

For the function $h(x)$ defined in (1.10), we have

$$h'(x) = -\frac{1+4\alpha^2}{2x^{3/2}}\sin(\alpha \log x).$$

Obviously, $h'(x) < 0$ for $1 < x \leq (n+1)$, that is, $h(x)$ is decreasing. Since one has $h(n+1) = 0$, it follows that $h(x) > 0$.

We shall prove that the sequence

$$a_k = \int_k^{k+1} h(x)\,dx$$

is an "almost extremal" sequence for Hardy's inequality (1.6), i.e., that inequality (1.11) holds. Since the function $h(x)$ is continuous there exists a point $\eta_i \in [i, i+1]$ such that $a_i = h(\eta_i)$.

We have

$$\sum_{j=1}^{k} a_j = \int_1^{k+1} h(x)dx = 2\sqrt{k+1}\sin(\alpha\log(k+1)).$$

The function $\sqrt{x}\sin(\alpha\log x)$ is increasing for $1 \le x \le n+1$ because

$$(2\sqrt{x}\sin(\alpha\log x))' = h(x) \ge 0$$

and consequently

$$\frac{\sqrt{k+1}}{k^2}\sin(\alpha\log(k+1)) \ge \int_k^{k+1} \frac{\sin(\alpha\log x)}{x^{3/2}}dx.$$

Then,

$$\begin{aligned} M_i^* &= 2\sum_{k=i}^{n} \frac{\sqrt{k+1}}{k^2}\sin(\alpha\log(k+1)) \ge 2\int_i^{n+1} \frac{\sin(\alpha\log x)}{x^{3/2}}dx \\ &\ge 2\int_{\eta_i}^{n+1} \frac{\sin(\alpha\log x)}{x^{3/2}}dx = -\frac{4}{1+4\alpha^2}\int_{\eta_i}^{n+1} h'(x)dx \\ &= \frac{4}{1+4\alpha^2}h(\eta_i) = \frac{4a_i}{1+4\alpha^2}. \end{aligned}$$

Thus,

$$M_i \ge \frac{4}{1+4\alpha^2}, \quad i = 1, 2, \ldots, n,$$

and

$$d_n \ge \min_{1\le i\le n} M_i \ge \frac{4}{1+4\alpha^2}.$$

The proof of Theorem 1.3 is complete.

4. Hardy's inequality (1.6), the zeros of continuous dual Hahn polynomials and the proof of Theorem 1.4

In this section we prove Theorem 1.4. Recall that the lower bound in (1.13) is nothing but (1.12), while the upper one was proved in [2, Theorem 1.1]. In what follows, we deal with the asymptotics of d_n.

The continuous dual Hahn polynomials are defined by (see [6, Section 1.3])

$$\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n} = {}_3F_2\left(\begin{matrix}-n, a+ix, a-ix\\ a+b, a+c\end{matrix};1\right) = \sum_{\nu=0}^{n} \frac{(-n)_\nu(a+ix)_\nu(a-ix)_\nu}{\nu!(a+b)_\nu(a+c)_\nu},$$

where Pochhammer's symbol is given by $(\alpha)_\nu = \alpha(\alpha+1)\cdot\dots\cdot(\alpha+\nu-1)$, $\nu \geq 1$, and $(\alpha)_0 := 1$. It is clear that each $S_n(x^2;a,b,c)$ is a polynomial of degree $2n$ with leading coefficient $(-1)^n$. Moreover, when the parameters a, b, and c are positive real numbers, $S_n(x^2;a,b,c)$ are orthogonal with respect to an absolutely continuous Borel measure (see [6, (1.3.2) on p. 29]). Hence, the smallest positive zeros $x_{n,1}(a,b,c)$ of $S_n(x^2;a,b,c)$ converge to zero when n goes to infinity. The following is one of the statements in [2, Theorem 1.1].

Theorem A. *Let d_n be the smallest possible constant such that inequality (1.6) holds. Then,*

$$d_n = 4\left(1 - \frac{4\left[x_{n,1}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right]^2}{1+4\left[x_{n,1}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right]^2}\right), \tag{4.1}$$

where $x_{n,1}(1/2,1/2,1/2)$ is the smallest positive zero of $S_n(x^2;1/2,1/2,1/2)$.

The precise uniform asymptotics for the continuous dual Hahn polynomials was obtained very recently in [11]. Set

$$A(z) = \frac{\Gamma(a-z)\Gamma(b-z)\Gamma(c-z)}{\Gamma(1-2z)}.$$

According to [11, (80)],

$$S_n(x^2) = \frac{2\gamma_{2n+1}}{x\sqrt{w(x)}}\cos\left(x\log n + \arg A(ix) - \frac{\pi}{2}\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad n\to\infty, \tag{4.2}$$

where $w(x)$ is defined in [11] and it is always positive. Therefore, we need to calculate the argument of $A(ix)$ for small values of x when $a = b = c = \frac{1}{2}$. Obviously,

$$\arg A(ix) = \arg\frac{\left[\Gamma\left(\frac{1}{2}-ix\right)\right]^3}{\Gamma(1-2ix)} = 3\arg\Gamma\left(\frac{1}{2}-ix\right) - \arg\Gamma(1-2ix)$$

and

$$\arg\left(\Gamma\left(\frac{1}{2}-ix\right)\right) = \arcsin\frac{\mathrm{Im}\left(\Gamma\left(\frac{1}{2}-ix\right)\right)}{\left|\Gamma\left(\frac{1}{2}-ix\right)\right|} = \arcsin\frac{\int_0^\infty t^{-1/2}e^{-t}\sin(-x\log t)\,dt}{\left|\Gamma\left(\frac{1}{2}-ix\right)\right|}.$$

Since the expansion

$$\begin{aligned}\int_0^\infty t^{-1/2}e^{-t}\sin(-x\log t)\,dt &= \sum_{k=0}^{\infty}\frac{(-1)^k(-x)^{2k+1}}{(2k+1)!}\int_0^\infty t^{-1/2}e^{-t}(\log t)^{2k+1}\,dt\\ &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}\Gamma^{(2k+1)}\left(\frac{1}{2}\right)}{(2k+1)!}x^{2k+1}\end{aligned}$$

certainly holds for $x \in (-1/2, 1/2)$, because the radius of convergence of the series is exactly $1/2$, then

$$\frac{\operatorname{Im}\left(\Gamma\left(\frac{1}{2}-ix\right)\right)}{\left|\Gamma\left(\frac{1}{2}-ix\right)\right|} \sim \frac{-\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}x = (\gamma+\log 4)\,x$$

for all sufficiently small x. Thus,

$$\arg\left(\Gamma\left(\frac{1}{2}-ix\right)\right) \sim \arcsin((\gamma+\log 4)x) \sim (\gamma+\log 4)x \quad \text{as } x \to 0.$$

Similar reasonings show that

$$\arg\left(\Gamma(1-2ix)\right) = \arcsin\frac{\operatorname{Im}\left(\Gamma(1-2ix)\right)}{|\Gamma(1-2ix)|} = \arcsin\frac{\int_0^\infty e^{-t}\sin(-2x\log t)\,dt}{|\Gamma(1-2ix)|},$$

$$\int_0^\infty e^{-t}\sin(-2x\log t)\,dt = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}\Gamma^{(2k+1)}(1)}{(2k+1)!}(2x)^{2k+1},$$

and

$$\arg\left(\Gamma(1-2ix)\right) \sim \frac{-\Gamma'(1)}{\Gamma(1)}(2x) = 2\gamma x \quad \text{as } x \to 0.$$

Hence,

$$\arg A(ix) \sim 3(\gamma+\log 4)x - 2\gamma x = (\gamma+\log 64)x, \quad x \to 0.$$

Since we are interested in the smallest zero, then indeed $x_{n,1}(1/2, 1/2, 1/2)$ converges to zero when n goes to infinity. Therefore, we use the latter approximation of $\arg A(ix)$. Thus, for the argument $u(x)$ of the cosine in (4.2), we obtain

$$u(x) = x\log n + (\gamma+\log 64)x - \frac{\pi}{2},$$

so we see that $x_{n,1}(1/2, 1/2, 1/2)$ is asymptotically equal to the smallest positive zero of $\cos u(x) = 0$, that is, to the solution of $u(x) = \pi/2$. In other words,

$$x_{n,1}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \sim \frac{\pi}{\gamma+\log 64+\log n}, \quad \text{as } n \to \infty.$$

The latter, together with (4.1), yields

$$d_n \sim 4 - \frac{16\pi^2}{4\pi^2 + (\gamma + \log 64 + \log n)^2}, \quad \text{as } n \to \infty,$$

which is exactly (1.14).

The explicit form of the coefficients c_k in (1.16) follows immediately after straightforward manipulations with the error term on the right-hand side of the latter expression and the generating function of the Chebyshev polynomials of the second kind

$$\frac{1}{1 - 2ty + y^2} = \sum_{j=0}^{\infty} U_j(t) y^j.$$

Indeed, setting

$$y_n = \frac{(4\pi^2 + (\gamma + 6\log 2)^2)^{1/2}}{\log n} \quad \text{and} \quad \tilde{t} = -\frac{\gamma + 6\log 2}{(4\pi^2 + (\gamma + 6\log 2)^2)^{1/2}},$$

the error term of (1.14) becomes

$$\frac{16\pi^2}{4\pi^2 + (\gamma + \log 64 + \log n)^2} = \frac{16\pi^2}{[\log n]^2} \frac{1}{1 - 2\tilde{t} y_n + y_n^2} = \frac{16\pi^2}{[\log n]^2} \sum_{j=0}^{\infty} U_j(\tilde{t}) y_n^j.$$

Since $|\tilde{t}| < 1$ and all Chebyshev polynomials of the second kind obey the inequality $|U_j(t)| \le (1 - t^2)^{-1/2}$ for $t \in (-1, 1)$, the latter series is absolutely convergent for $|y_n| < 1$. The same argument shows that the error term in (1.15) is indeed $O((\log n)^{-m-1})$.

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