

Some results on Quillen’s conjecture via equivalent-poset techniques

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Abstract. We extend the main theorem of Aschbacher and Smith on the Quillen conjecture from $p > 5$ to the remaining odd primes $p = 3, 5$. In the process, we develop further combinatorial and homotopical methods for studying the poset of non-trivial elementary abelian p -subgroups of a finite group. The techniques lead to a number of further results on the conjecture, often reducing dependence on the CFSG; in particular, we also provide some partial results toward the case of $p = 2$.

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1. Introduction: Background and statement of principal results

The original version of this paper was designed to prove an extension to $p = 3, 5$ of [4, Main theorem], which we state as Theorem 1.1 below. But our methods also

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led to other useful results, notably a closely related Theorem 1.4—which provides simplifications both in the hypothesis for the extension and in certain aspects of the inductive proof (including a “milder” use of the CFSG, as we will indicate in our discussion of Theorem 1.4 below and Remark 1.5). Other results contribute to the conjecture for the remaining prime $p = 2$.

This introduction will provide some general background for all these results, beginning with a discussion of these two versions of the extension to all odd p , highlighting their similarities and differences.

Throughout this paper, we only consider finite groups. Recall for a prime p that $\mathcal{A}_p(G)$ is the poset of non-trivial elementary abelian p -subgroups of a finite group G with ordering given by inclusion. Note that G acts on this poset by conjugation. We regard $\mathcal{A}_p(G)$ as a topological space with the topology of its order complex.

Quillen showed that if $O_p(G)$, the largest normal p -subgroup of G , is not the trivial subgroup, then $\mathcal{A}_p(G)$ is contractible [19, Proposition 2.4]. The converse is the original statement of Quillen’s conjecture (see [19, Conjecture 2.9]): that if $O_p(G) = 1$, then $\mathcal{A}_p(G)$ should be non-contractible. As in much of the subsequent literature, we instead work with a stronger “homology” form of the conjecture, replacing non-contractibility with non-vanishing of rational (reduced) homology.

(H-QC) If $O_p(G) = 1$, then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.

We will recall some of the general literature on the conjecture, starting at later Theorem 2.19.

Extending the Aschbacher–Smith result and its overall strategy

But next, we will begin our introduction by instead focusing on one particular contribution to that literature, namely, the main theorem of Aschbacher–Smith in [4]. This somewhat technical discussion will indicate some motivation for our approach and results, and the details may be appropriate mainly for the more expert reader; other readers may prefer to skip over the details and return to this discussion at the later indicated points.

Indeed, one of our principal results in this paper is Theorem 1.1 below, in which we extend their original statement, by adding the cases $p = 3, 5$ to their original treatment for $p > 5$. We mention that the original extension to $p = 5$ was given in [16]—see later Corollary 2.23.

Recall that a group G satisfies $(\mathcal{QD})_p$ if $\mathcal{A}_p(G)$ has non-zero homology in the largest possible degree, namely, $m_p(G) - 1$, where $m_p(G)$ denotes the p -rank of G . Also, a p -extension of a group L is a split extension LB of L by an elementary abelian p -group B of *outer* automorphisms (we will call such a B a p -*outer* of L). See Definition 2.15. The term “(H2u)” below is a name we have invented for the

restriction on unitary groups in the hypothesis in [4]. For *components* of G , see later Remark 2.1—these quasisimple subnormal subgroups are crucial to building up the structure of G .

We extend [4, Main theorem], from $p > 5$ there, to all odd p here.

Theorem 1.1. *Suppose that G is a finite group, p is an odd prime, and the following condition holds.*

(H2u) *If $L \cong U_n(q)$ is a component of G , with q odd and p dividing $q + 1$, then every p -extension of $U_m(q^{p^e})$ satisfies $(\mathcal{QD})_p$ for all $m \leq n$ and $e \in \mathbb{Z}$.*

Then, G satisfies (H-QC).

We will prove this extension Theorem 1.1 as later Theorem 9.2. Our proof there mainly follows the path of the proof in [4].

In fact, the possibility of such an extension was already suggested in [4], just after the statement of the main theorem there, using the language “by excluding certain further groups as components of G ”. We implement such an approach here—but now using the language of *elimination results* in our recent work [18]: that is, results which state that a particular quasisimple L cannot appear as a component in a counterexample G to (H-QC). It will be convenient to describe the role of these eliminations within the following general context.

Remark 1.2 (Adapting the overall strategy of Aschbacher–Smith). Here is our (oversimplified) viewpoint on the basic structure of the proof in [4]. The first main step can be phrased as follows.

(ElimQD). Apply [4, Proposition 1.7] to eliminate any component L of G that satisfies $(\mathcal{QD})_p$.

The remaining main step, carried out in [4, Theorem 5.3] amounts to the following.

(Rob-nonQD). Show that any remaining component L (i.e., failing $(\mathcal{QD})_p$) has a “Robinson subgroup”.

To be more precise, the final argument at [4, pp. 490–492] shows that such Robinson subgroups lead to $\tilde{\chi}(\mathcal{A}_p(G)^g) \neq 0$ for some $g \in G$. Here, $\mathcal{A}_p(G)^g$ denotes the subposet of $\mathcal{A}_p(G)$ of elements fixed by g under the conjugation action. The non-vanishing of the Euler characteristic of the fixed points $\mathcal{A}_p(G)^g$ therefore implies (H-QC) (see Section 8 for more details on this argument). Of course, this strategy is idealized: since (only) certain unitary groups might¹ fail both steps, and hence they must be excluded via the hypothesis (H2u).

¹[4, Conjecture 4.1], that unitary groups should have $(\mathcal{QD})_p$, remains open.

Now, in proving our extension as Theorem 9.2, it turns out that we will need some further elimination results—which we will establish along the way. For example, for $p = 5$ we can mostly use the original strategy above, but we will need Theorem 9.1 to eliminate three particular components, which would otherwise prevent the application of the intermediate result [4, Theorem 2.3]. And for $p = 3$, we will need Theorem 6.4 to eliminate Ree-group components—since Ree groups do not have Robinson subgroups in the argument for [4, Theorem 5.3] and also fail $(\mathcal{QD})_3$.

As a result, we are in effect implementing the following obvious adaptation of the original Aschbacher–Smith strategy.

(Elim). Apply *various* elimination results to further reduce the possible components L .

(Rob-nonelim). Show that Robinson subgroups occur for any component L in the smaller non-(Elim) list.

As we will indicate later, we will adopt this strategy and extend the approach of [4] to the final prime $p = 2$ in Section 10. We mention that because of difficulties in extending to $p = 2$ the $(\mathcal{QD})$ -List of [4, Theorem 3.1] (which we reproduce as later Theorem 2.16), it seems likely that significant further elimination results will be needed in treating $p = 2$. ■

Variant hypotheses such as (H1), related to inductive approaches

Our methods for Theorem 1.1 above also lead to another closely related principal result, namely, Theorem 1.4 below. This further variant continues the above theme of adding the cases $p = 3, 5$; but it also involves simplifications both in the hypotheses, and in certain corresponding aspects of the essentially inductive proof (including a “milder” use of the CFSG).

In order to cast some light on these simplifications, we will now continue our introduction by developing some of the rationale for the variant hypotheses—which will lead up to our statement of Theorem 1.4.

In proving [4, Main theorem], it is assumed that G is a counterexample to (H-QC), of minimal order subject to satisfying (H2u). Now, mimicking the language of “(H1)” used in the recent paper [16], we give this condition the name (H1u); that is,

(H1u) if H satisfies (H2u) and further $|H| < |G|$, then H satisfies (H-QC).

Therefore, in the proof of [4, Main theorem], those arguments which rely on the minimality of G to deduce that (H-QC) holds for certain groups of order smaller than $|G|$ can be *replaced*² by applications of (H1u)—regarded as a further formal hypothesis

²This roughly avoids the mechanics of re-verifying the various hypotheses, at each application of induction.

in Theorem 1.1. Thus, in this viewpoint, we regard the original proof in [4] as roughly showing the following:

If $p > 5$, and G satisfies (H1u) and (H2u), then G satisfies (H-QC).

In fact, our variant Theorem 1.4 takes this viewpoint even further. Namely, in place of the hypotheses (H1u), (H2u) above, it uses the simplified versions (H1), (H2) indicated below. These will lead to the promised simplifications in the proof of Theorem 1.4.

We first lead up to the variant hypothesis (H1) used in [16]. Note that (H1u) above includes mention of (H2u)—since it is designed for a minimal-order counterexample to Theorem 1.1. However, if we instead want to study the context of a minimal-order counterexample just to (H-QC) itself, we could at least in principle consider the somewhat simpler replacement hypothesis given by the following.

(H1-) Any H , with $|H| < |G|$, must satisfy (H-QC).

And then, (H2u) would be just one possible “second condition (H2)”, for our more general counterexample G . But we can also further observe that, in [4], (H1u) is in fact applied to handle just subgroups and quotients by a central p' -group. So (as in [16]), we will prefer to work instead with the further-simplified hypothesis:

(H1) Proper subgroups and proper p' -central quotients of G satisfy (H-QC).

In fact, [16] shows that it can be advantageous in proof to focus our arguments on classes of groups closed under subgroups and central quotients. In the following remark, we give further details from that paper.

Remark 1.3 (Some context for results under (H1)). First, results proved under (H1) are not really “standalone” results—in the sense that we usually would not expect to verify (H1), within some general argument toward (H-QC); instead, such results might be regarded as “segments of logic”, applicable when we already have (H1)—typically because we are in the situation of a minimal-order counterexample to (H-QC). So, in particular, as we had essentially observed in the discussion above,

(MOC) (H-QC)-results under (H1) apply to a minimal-order counterexample to (H-QC).

For example, [18, Corollary 1.2] assumes a counterexample as in (MOC)³, which gives (H1)—as needed, for quoting the more technical result Theorem 1.6 there.

But we also want to know when proposed results with hypotheses “(H2x)” other than (H1) in fact automatically satisfy (H1). Such results are important for the overall

³By a standard argument (e.g., [4, p. 487]), it suffices to take a minimal (under inclusion) counterexample.

context of the growing literature of results on the possible structure of a counterexample to (H-QC). So, we mention the following statement.

- (Inh) For an (H-QC)-result, if hypotheses (H2x) other than (H1) are inherited by subgroups and p' -central quotients, then (H1) is automatic for a minimal-order counterexample to that result—so that in fact (H-QC) holds, without *assuming* (H1).

This observation is of course just a version of the effect of induction on $|G|$ in such a result. For example, [16, Theorem 4] has “(H2x)” given by $m_p(G) \leq 4$ —which has (Inh); this is essentially why its proof can be given under (H1), so that (H1) need not be in its statement.

By contrast, Theorems 4.1, 5.1, and 6.1 there are proved under (H1)—since the relevant (H2x) in those cases does *not* satisfy (Inh), so that those results do not automatically hold without assuming (H1). ■

Considerations such as the above provide some initial motivation for stating our particular variant Theorem 1.4 below under (H1). We will indicate further benefits, as we go on.

Now, we introduce (H2), our specific choice of “(H2x)”, to use in place of (H2u) in our statement of Theorem 1.4 under (H1). Here, it turns out that a further advantage of the form (H1) is that we will be able to avoid the restrictions on $m \leq n$ and $e \in \mathbb{Z}$ that appear in (H2u) of Theorem 1.1 above. Namely, we can just use the following.

- (H2) If $L \cong U_n(q)$ is a component of G with q odd and $p \mid q + 1$, then the p -extensions of $U_n(q)$ occurring in G satisfy the $(\mathcal{QD})_p$ property.

This version allows us to focus just on the single component $U_n(q)$, and not on other unitary extensions for different n, q . This will simplify proofs. However, as mentioned above, the application of this version is for situations where, for instance, we have a minimal-order counterexample just to (H-QC), and hence (H1) automatically holds.

So, we will be proving the following variant-form of Theorem 1.1, whose appears in the latter part of Section 9.

Theorem 1.4. *Let p be an odd prime, and let G be a finite group satisfying (H1) and (H2) above. Then, G satisfies (H-QC).*

We had mentioned earlier that the proof of this theorem makes a milder use of the CFSG than the proof of Theorem 1.1. To be more precise, this amounts to avoiding the use of [4, Theorem 2.3], as we will see via our Theorem 1.4, whose proof is discussed towards the end of Section 9. Below, we briefly preview some aspects of that simplification.

Remark 1.5 (Reducing use of the CFSG in Theorem 1.4 under (H1)). The proof of our Theorem 1.4 stated under (H1) employs the classification of the finite simple

groups to a much lesser extent than in [4] (and hence in Theorem 1.1). This is thanks to our corresponding (H1)-variants of [4, Propositions 1.6 and 1.7], which are given by [16, Theorem 4.1] and Theorem 7.2 here—as we summarize below.

The original forms of those two propositions in [4] state in effect that under (H1u), we can suppose that $O_{p'}(G)$, the largest normal subgroup of G of order prime to p , is trivial, and that any simple component of G , where the indicated p -extensions satisfy $(\mathcal{QD})_p$ leads to (H-QC). We contrast that situation with our (H1)-variants. In [16, Theorem 4.1], it was shown that if G satisfies (H1), and $O_{p'}(G) \neq 1$, then (H-QC) holds for G . And in Theorem 7.2, we show that, assuming (H1), if for a given component L of G , $(\mathcal{QD})_p$ holds for all the p -extensions of L , then G satisfies (H-QC). Thus, our variants proved under (H1) give us essentially the same reductions as in [4] with Propositions 1.6 and 1.7 there.

Furthermore, the proof of [16, Theorem 4.1], leads to the reduction $O_{p'}(G) = 1$, only uses the CFSG to invoke the p -solvable case of the conjecture; and the proof of Theorem 7.2 does not use the CFSG at all. Moreover, both [16, Theorem 4.1] and Theorem 7.2 hold for any prime $p \geq 2$. In contrast, the proofs of [4, Propositions 1.6 and 1.7] invoke [4, Theorem 2.3], which is stated for odd p , and relies deeply on the CFSG. ■

Remark 1.6 (Equivalent posets and simplification of boundary-calculations). In addition, the proofs of both our above extensions of [4] to all odd p will in fact involve focusing on some more intrinsic combinatorial properties of the $\mathcal{A}_p$ -posets—as indicated in the title of this paper. See especially Sections 3 and 4. For example, results such as Proposition 3.11 allow us to “remove unnecessary points” from the posets, and so, we can work toward (H-QC) using smaller and more convenient posets homotopy equivalent to $\mathcal{A}_p(G)$. Other equivalences such as Theorem 4.10 (building on the viewpoint of Theorem 3.6) provide a convenient visual-format for poset-chains based on the Cartesian product and resulting in simplified boundary-calculations in technical results on homology propagation. This last term means, roughly, the process of showing for some $H < G$ that non-zero reduced homology for $\mathcal{A}_p(H)$ “propagates” to non-zero homology also for $\mathcal{A}_p(G)$ —to give (H-QC). We will begin a more detailed discussion of the fundamental technique of propagation at later Remark 3.1. ■

Some sample results—toward the case $p = 2$

To continue an (H1)-theme from Remark 1.5 just above, we can in fact use *any* prime p , including $p = 2$, in the reduction to $O_{p'}(G) = 1$ in [16, Theorem 4.1] (which does not require the CFSG for this prime), and in the elimination of $(\mathcal{QD})_p$ -components in Theorem 7.2. These features provide strong motivation for working under (H1), in future contributions toward (H-QC) for $p = 2$.

In particular, we could at least in principle follow essentially the Aschbacher–Smith strategy of (ElimQD), (Rob-nonQD) that we indicated earlier in Remark 1.2. However, as we had also mentioned in that remark, there seem to be significant difficulties in extending to $p = 2$ the $(\mathcal{QD})$ -List of [4, Theorem 3.1]—namely, the components which would ideally be eliminated by Theorem 7.2 in the (ElimQD) step. For this reason, it seems likely that many such components will need to instead be handled via new elimination results, following then the modified strategy of (Elim), (Rob-nonelim) in Remark 1.2.

So, we now state some partial elimination results for $p = 2$. Denote by $Z(G)$ the center of the group G .

Theorem 1.7. *Assume $p = 2$, and let G be a group satisfying (H1) and L a component of G . Suppose one of the following holds:*

- (1) $L/Z(L)$ is isomorphic to one of the following groups:

$$\begin{aligned} &\text{Alt}_5, \text{Alt}_6, \text{Sz}(2^{2n+1}), {}^2F_4(2^{2n+1})', {}^2G_2(3^{2n+1}), \\ &\text{M}_{11}, \text{M}_{12}, \text{M}_{22}, \text{M}_{23}, \text{M}_{24}, \\ &\text{J}_1, \text{J}_4, \cdot 1, \cdot 2, \cdot 3, \text{M}(23), \text{HS}, \text{Ly}, \text{Ru}, F_3, F_2, F_1; \end{aligned}$$

- (2) L is simple and every 2-extension $LB \leq G$ of L satisfies $(\mathcal{QD})_2$;

- (3) $L/Z(L)$ is a finite group of Lie type in characteristic 2 other than the types: $\text{PSL}_n(4^m)$ ($n \geq 3$); $D_n(4^m)$ ($n \geq 4$); $E_6(4^m)$;

- (4) if K is a component of G such that $K/Z(K)$ is a group of Lie type, then K is either as described in (1)–(3), or else $K/Z(K)$ is defined in characteristic 3.

Then, G satisfies (H-QC).

We remark that conclusion (4) is not really a “pure” elimination result—but instead depends on some of our later results about Robinson subgroups. The proof of this theorem can be found at the end of Section 10.

Recall that $C_G(H)$ denotes the centralizer of a subgroup H of G . Write

$$\text{O}_G(H) := \{E \in \mathcal{A}_p(G) : E \cap (HC_G(H)) = 1\}.$$

Using the above theorem in conjunction with the results of [16, 18], we get the following corollary.

Corollary 1.8. *Let $p = 2$. If G satisfies (H1) but fails (H-QC), then the following hold:*

- (1) $\text{O}_2(G) = 1 = \text{O}_{2'}(G)$;
- (2) $\mathcal{A}_2(G)$ is simply connected;
- (3) G has 2-rank at least 5;

- (4) for every component L of G , $O_G(L) \neq \emptyset$;
- (5) G contains a component L of Lie type in characteristic $\neq 3$, and some p -extension $LB \leq G$ of L fails $(\mathcal{QD})_p$;
- (6) G does not contain components of the following types:

of Lie type and Lie rank 1 in characteristic 2,

Alt_6 , $\text{Sz}(2^{2n+1})$, ${}^2F_4(2^{2n+1})'$, ${}^2G_2(3^{2n+1})$, M_{11} , M_{12} ,

M_{22} , M_{23} , M_{24} , J_1 , J_4 , $\cdot 1$, $\cdot 2$, $\cdot 3$, $M(23)$, HS , Ly , Ru , F_3 , F_2 , F_1 .

In view of the length of the paper, we provide the reader with a brief explanatory road map.

Remark 1.9 (Overview: Organization of the paper). The preliminary Section 2 establishes our overall context, collecting standard definitions and various quoted results.

Then, Sections 3–8 lead up to the main proofs in Section 9 of our two variants extending [4] to $p = 3, 5$ (namely, Theorems 1.1 and 1.4). The logical sequence is perhaps best explained in *reverse* order—regarded roughly as a sequence of successive reductions.

The proofs of the main results above follow the basic strategy indicated in Remark 1.2. The (Rob-nonelim) step requires in Section 8 some extensions of earlier results on Robinson subgroups. The (Elim) step requires some new elimination results: in Section 7, a variant Theorem 7.2 of the (ElimQD)-step eliminating $(\mathcal{QD})_p$ -components in [4]; and in Section 6, the elimination via Theorem 6.4 of some Lie-type groups in the same characteristic p .

The required results above in turn depend on some more technical results, which appear in still earlier sections. Notably, the generalized homology propagation result Theorem 5.10 established in Section 5. This result is in fact stated in the viewpoint of the pre-join (rather than the usual poset-join), which is developed in Section 4: the Cartesian product structure of the pre-join can be much more convenient for our various calculations than the standard poset join. The propagation uses a replacement poset homotopy equivalent to $\mathcal{A}_p(G)$; such a replacement is a suitable adjustment, for the pre-join, of a more standard replacement construction indicated in Definition 3.4 of Section 3.

Finally, Section A of the appendix recalls a generalized version of the Quillen fiber Theorem 2.10. Further appendix sections provide various technical details—which we moved to there, in order to not slow down the more fundamental parts of the logical flow in the main text. ■

This article is intended to be primarily self-contained. In particular, to avoid the need for frequent external look-ups, statements will be given for many quoted results.

Results concerning the Quillen conjecture will often be expressed with several alternative hypotheses, with the aim of finding different approaches to proving (H-QC), particularly those that avoid much of the use of the CFSG.

2. Preliminaries

In this section, we recall various elementary facts that we will need on finite groups, posets, and simplicial complexes. We also establish the notation that we will use throughout the article and review some of the literature on Quillen's conjecture.

Finite groups

Our main reference for the properties on finite groups is Aschbacher's book [2]. We will follow the conventions and notation on simple groups of [4, 11, 12]. We use [12, Definition 2.5.13] for the different types of automorphisms of simple groups (see [12, pp. 57–62] for more details on the automorphism group structure). Since we will sometimes refer to [11], the reader should take into account that some notation and definitions on simple groups and their automorphism types occasionally differ between [11, 12]. We will always try to indicate such differences.

We always work with finite groups. Denote by C_n , D_n , Sym_n and Alt_n the cyclic group of order n , the dihedral group of order n , the symmetric group on n letters and the alternating groups on n letters, respectively. By a simple group we always mean a *non-abelian* simple group.

Let G be a finite group. For subgroups $K, H \leq G$, let $N_K(H)$ be the normalizer of H in K , and $C_K(H)$ the centralizer of H in K . Recall that H is self-centralizing if $C_G(H) \leq H$. Denote by $N_G(H_1, \dots, H_n)$ the intersection of normalizers $N_G(H_1) \cap \dots \cap N_G(H_n)$ for subgroups $H_i \leq G$. Write $[H, K]$ for the subgroup generated by the commutators between elements of H and K . For a fixed prime p , $\Omega_1(G)$ is the subgroup of G generated by the elements of order p . The p -rank of G , denoted by $m_p(G)$, is the maximal dimension of an elementary abelian p -subgroup of G .

Denote by $Z(G)$, $F(G)$, $O_p(G)$, $O_{p'}(G)$ the center, the fitting subgroup, the largest normal p -subgroup, and the largest normal p' -subgroup of G , respectively. Recall that the fitting subgroup $F(G)$ is the direct product of all the subgroups $O_p(G)$, for p prime dividing the order of G . For solvable groups G , we have that $F(G)$ is self-centralizing: $C_G(F(G)) \leq F(G)$. However, this property does not hold for arbitrary groups G , and so, in general we instead work with the *generalized fitting subgroup* $F^*(G)$ of G ; it is fundamental for the structure of finite groups.

Remark 2.1 (Components and the generalized fitting subgroup). Recall that $F^*(G)$ is the central product of the subgroups $F(G)$ and $E(G)$; here $E(G)$ is the layer of G ,

that is, the central product of the *components* of G —namely, the quasisimple subnormal subgroups of G . The generalized fitting subgroup is self-centralizing:

$$C_G(F^*(G)) \leq F^*(G).$$

When G is solvable, $F^*(G) = F(G)$. Also, $Z(F^*(G)) = Z(F(G))$ and $Z(E(G)) \leq Z(F(G))$. ■

We sometimes use the following (non-standard) notation for an orbit of components of G .

Definition 2.2. Let L be a component of a finite group G . We denote by $\hat{L}$ the (central) product of the G -conjugates of L , and by $\hat{N}_G(L)$ the kernel of this action. That is, if $L_1, \dots, L_r$ is the G -orbit of L . Then,

$$\hat{L} = L_1 \cdots L_r \quad \text{and} \quad \hat{N}_G(L) = N_G(L_1, \dots, L_r) = \bigcap_i N_G(L_i). \quad \blacksquare$$

Note that for any prime p we have $F(G) \leq O_p(G)O_{p'}(G)$. For the Quillen conjecture, we will often work under the assumption that $O_p(G) = 1 = O_{p'}(G)$, so that $F(G) = 1$, and hence, $Z(E(G)) = 1$. Since $Z(E(G))$ is the product of the centers of the components of G , we see that the components of G are simple groups in this case. We include this in the following frequently used lemma.

Lemma 2.3. *Let G be a finite group and p a prime number dividing its order.*

- (1) *If $F(G) = 1$, then $F^*(G) = E(G) = L_1 \cdots L_t$ is the direct product of the components L_i of G , which are all simple. Since $C_G(F^*(G)) = Z(F(G)) = 1$, we have a natural inclusion*

$$F^*(G) \leq G \leq \text{Aut}(F^*(G)).$$

This holds when $O_p(G) = 1 = O_{p'}(G)$; and then p divides the order of the components.

- (2) *$E(C_G(E(G))) = 1$; and hence; $F^*(C_G(E(G))) = F(G)$.*
- (3) *Let $L_1, \dots, L_s$ be distinct components of G , then the components of $C_G(L_1 \cdots L_s)$ are the components of G other than the L_i for $1 \leq i \leq s$.*
- (4) *Under the hypothesis of (3), $F(C_G(L_1 \cdots L_s)) = F(G)$. In particular, if $O_p(G) = 1$, then $O_p(C_G(L_1 \cdots L_s)) = 1$.*

Proof. Part (1) is immediate from the previous discussion.

We prove part (2). Suppose that L is a component of $C_G(E(G))$. Since $E(G)$ is normal in G , so is $C_G(E(G))$, and therefore, L is a component of G (see [2, (31.3)]). So, L is a subgroup of $E(G)$ as well as of $C_G(E(G))$, which means that L is abelian

and not quasisimple, contrary to our assumption. This proves that $E(C_G(E(G))) = 1$, and so, $F^*(C_G(E(G))) = F(C_G(E(G)))$. But the latter is a normal nilpotent subgroup of G , which contains $F(G)$ since $[F(G), E(G)] = 1$. Hence, $F(C_G(E(G))) = F(G)$.

We prove part (3). Let $\{M_1, \dots, M_r\}$ be the set of components of G distinct from the L_i , so that $E(G) = L_1 \cdots L_s M_1 \cdots M_r$. Write $K = C_G(L_1 \cdots L_s)$. Then, $M_j \leq K$ for all $1 \leq j \leq r$ and then the M_j are components of K . That is, $M_1 \cdots M_r \leq E(K)$.

For the reverse inclusion, let M denote a component of K , hence of $E(K)$. Assume by way of contradiction, that $M \neq M_j$ for all j . By [2, (31.5)] applied in K , we see that M centralizes each M_j . Since $M \leq K = C_G(L_1 \cdots L_s)$, we get $M \leq C_G(E(G)) \leq K$. This shows that M is also a component of $C_G(E(G))$. But this contradicts item (2). In consequence, $M = M_j$ for some j . This concludes the proof of part (3).

Finally, we prove part (4). Let $M = F(C_G(L_1 \cdots L_s))$. By part (3) we see that $[M, E(G)] = 1$. Hence, $M \leq C_G(E(G)) \leq C_G(L_1 \cdots L_s) \leq G$. Then, M is a normal nilpotent subgroup of $C_G(E(G))$ using (3). Therefore, we have $M \leq F(C_G(E(G))) = F(G)$. Since $F(G) \leq C_G(E(G)) \leq C_G(L_1 \cdots L_s)$, we see that $F(G)$ is a normal nilpotent subgroup of $C_G(L_1 \cdots L_s)$, and so, $F(G) \leq M$. So, we have $M = F(G)$; hence, $O_p(C_G(L_1 \cdots L_s)) = O_p(F(C_G(L_1 \cdots L_s))) = O_p(F(G)) = O_p(G)$. ■

We will also often use (sometimes possibly implicitly) the following elementary lemma.

Lemma 2.4. *If $K, L \leq G$, where L is a component of G and K normalizes a non-central subset S of L , then K normalizes L .*

Proof. Let $k \in K$, and suppose $L^k \neq L$. Then, $S^k = S \subseteq L \cap L^k \leq Z(L)$, contrary to the hypothesis that $S \not\subseteq Z(L)$. Hence, $K \leq N_G(L)$. ■

Lemma 2.5 below is for more specialized situations: it provides a downward-inheritance property for trivial p -core, in centralizers of extensions of components by p -groups—such as the p -extensions of Definition 2.15. The $O_p(-) = 1$ condition for the centralizers in the lemma will correspond to the “non-conical”-centralizer situation in homology-propagation literature; see, for example, the hypothesis “(someNC)” in our discussion in later Remark 3.10. We will apply the non-conical conclusion of the lemma to provide the particular version of (someNC) which is appropriate for the propagation used in proving later Theorem 7.2; and also in the proof of Proposition 4.12.

Lemma 2.5. *Suppose that $L \leq G$ is a subgroup, and let $B, E \leq N_G(L)$ be abelian p -subgroups. Suppose in addition that $LB \leq LE$ and that $O_p(C_G(LE)) = 1$. Then, $O_p(C_G(LB)) = 1$.*

If further L is a component of G , then also $O_p(LB) = 1$.

Proof. Let $N := O_p(C_G(LB))$. We show that $N = 1$.

Note that LB is normal in LE since $LB/L \leq LE/L \cong E/E \cap L$ is abelian. Therefore, E acts on $C_G(LB)$, and hence on the characteristic subgroup N . Since N and E are p -groups, from $N > 1$ we would get $C_N(E) > 1$. On the other hand, $C_N(E)$ is a normal subgroup of $C_G(LE)$: since for $x \in C_G(LE)$, we see $C_N(E)^x = C_{N^x}(E^x) = C_N(E)$ using $C_G(LE) \leq C_G(LB)$. So, we get that

$$C_N(E) \leq O_p(C_G(LE)) = 1$$

using the hypothesis, which forces $N = 1$, as desired.

Finally, assume L is a component of G . Note then that $O_p(LB)$ is centralized by L and normalized by B . Thus, $C_{O_p(LB)}(B) \leq O_p(Z(LB)) \leq O_p(C_G(LB)) = N = 1$, which forces $O_p(LB) = 1$. ■

Now, we establish some notation on automorphism groups. Denote by $\text{Aut}(G)$ the group of automorphisms of G , by $\text{Inn}(G)$ the group of inner automorphisms of G (which is $G/Z(G)$), and by $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ the group of outer automorphisms of G . If $x \in \text{Aut}(G) \setminus \text{Inn}(G)$, we say that x *induces an outer automorphism* on G .

If $H \leq G$, then $\text{Aut}_G(H) = N_G(H)/C_G(H)$ is the group of automorphisms of H induced by G . Similarly, $\text{Out}_G(H) = N_G(H)/(HC_G(H))$ is the group of outer automorphisms of H induced by G . The subgroup $HC_G(H)$ can be regarded as the subgroup of G whose elements induce inner automorphisms on H . We say that a subgroup $K \leq G$ induces outer automorphisms on H if K normalizes H and K contains no non-trivial inner automorphism of H . That is, K induces outer automorphisms on H if and only if $K \cap (HC_G(H)) = 1$.

We recall now some basic facts on outer automorphisms of *simple* groups. Recall that by simple we mean non-abelian simple.

Remark 2.6 (Summary: Outer automorphisms of the finite simple groups). We will usually abbreviate the Classification of Finite Simple Groups by CFSG. The CFSG states in overview that a simple group is one of the following:

- (a) an alternating group;
- (b) a simple group of Lie type;
- (c) one of the 26 sporadic groups.

One celebrated consequence of the CFSG is the Schreier conjecture, which states that for a simple group L , $\text{Out}(L)$ is solvable. But we get fuller details from the study of the cases for L (using the CFSG to know that our list of simple groups is complete): if L is a simple group, then we have a detailed description of the structure of $\text{Out}(L)$.

These specific descriptions are important throughout the literature; but especially for the study of p -extensions: for example, in establishing the Aschbacher–Smith $(\mathcal{QD})_p$ -List (which we quote below as Theorem 2.16), and in many other results on the Quillen conjecture—including our results in Section 6. (Indeed, Theorem 2.24 shows that we should *expect* p -extensions in pursuing (H-QC).)

Here are some of those fundamental details.

- For L an alternating group, $\text{Out}(L) = C_2$, except for $L = \text{Alt}_6$, where

$$\text{Out}(\text{Alt}_6) = C_2 \times C_2.$$

See, e.g., [12, Theorem 5.2.1].

- For L a sporadic group, $\text{Out}(L) = 1$ or C_2 . See, e.g., [12, Section 5.3] for the specific cases.
- Now, consider L of Lie type—defined over a field of characteristic r , where r is a prime. Below, we give a fairly informal summary of the structure of $\text{Out}(L)$ ⁴, based on the beginning of [11, Section 7 (Part I)]. For our later results, we will require a more detailed description based on the treatment in [12, Section 5.2] (see our “first aspect” discussion after the statement of Theorem 6.4, and the underlying structures in Appendix B.) In the standard overview:

Every element of $\text{Aut}(L)$ can be written as a product $idfg$,

where i is an inner automorphism, d a diagonal automorphism, f a field automorphism, and g a graph automorphism [12, Theorem 2.5.1]. For present purposes, it will suffice to describe these automorphism types fairly informally.

- (d) Diagonal automorphisms correspond to suitable diagonal matrices, if L is given in adjoint form—namely, as a group of matrices group over the field $\mathbb{F}_{r^a}$ for suitable a . In particular, diagonal automorphisms have order dividing $r^a - 1$ and hence coprime to r .
- (e) Field automorphisms of that adjoint matrix group are induced by elements of the Galois group of $\mathbb{F}_{r^a}$ over $\mathbb{F}_r$. In particular, the subgroup of field automorphisms is cyclic⁵.

⁴In the case of the Lie-type groups $L^+ \cong \text{Sp}_4(2), {}^2F_4(2), G_2(2), {}^2G_2(3)$ for $p = 2, 2, 2, 3$, this description applies to $\text{Out}(L^+)$; whereas the simple group L is the commutator subgroup of index p —which has further automorphisms, that we must treat separately. For example, this distinction arises explicitly in our later Definition 6.1 (2). See also the discussion *Lie_{exc}* in [12, Definition 2.2.8].

⁵Of order a —or at least dividing a , depending on naming-conventions we describe in later Definition B.1.

(f) Graph automorphisms arise⁶ from elements of the symmetry group Δ of the underlying Dynkin diagram for the Lie type of L . The following are cases where $\Delta > 1$, listed by diagram type.

- (i) $\Delta = C_2$ for A_n ($n \geq 2$), B_2, C_2, D_n ($n > 4$), E_6, F_4, G_2 ; and $\Delta = \text{Sym}_3$ for D_4 .

In particular, the only possible primes s dividing the order of Δ are $s = 2, 3$. Furthermore, (see [12, Theorem 1.15.4 (b)]) for a diagram-symmetry to actually lead to a graph automorphism (in the original sense of Steinberg) of an untwisted group L , at the level of an overlying algebraic group $\bar{L}$, some of the above Lie types give a further restriction on the characteristic r .

- (ii) A graph automorphism requires $r = 2$ in $B_2 \cong C_2$ or F_4 ; and $r = 3$ in G_2 .

Here, $B_2 \cong C_2$ recalls a standard isomorphism of the finite groups of those types.

Finally, for the single-bond diagrams in (i) above (Lie types A_n, D_n, E_6), the graph automorphisms from Δ always lead to corresponding *twisted* simple groups; whereas for the Lie types in (ii) above, there is a further restriction.

- (iii) The cases in (ii) give a twisted group only over a field with order a power of r with *odd* exponent. ■

Posets and simplicial complexes. We establish the main notation and definitions on posets and simplicial complexes.

We will always consider finite posets and simplicial complexes. We typically denote the order-relation of some unspecified poset X by $\leq$. In posets of subgroups of a finite group G , the relation will usually coincide with group-inclusion $\leq$ —but we will also consider some other relations, for which we will normally specify some different notation.

We write $x < y$ if $x \leq y$ and $x \neq y$. If X is a finite poset, denote by $\mathcal{K}(X)$ its *order complex*: recall that the simplices of $\mathcal{K}(X)$ are the non-empty $<$ -chains of X . We will study the homotopy properties of the poset X —meaning the homotopy properties of the complex $\mathcal{K}(X)$ ⁷. In particular, we write $X \simeq Y$ if the geometric realisations of the posets X, Y are homotopy equivalent.

⁶Our wording here is deliberately vague, for the moment. We will be more precise about the meaning of “graph automorphism” (and various other points) in the already-mentioned later discussions beginning after Theorem 6.4.

⁷Thus, in topological situations, we in effect identify X with $\mathcal{K}(X)$: this should be implicitly clear from the context—but usually we try to make the complex $\mathcal{K}(X)$ explicit.

If $f : X \rightarrow Y$ is an order-preserving map between finite posets, then f induces a simplicial map $f : \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ between the order complexes.

The basic homotopy property: Assume $f, g : X \rightarrow Y$ are two order-preserving maps between finite posets. We say that f and g are *<-comparable*, if either $f \leq g$ (which means $f(x) \leq g(x)$ for all $x \in X$), or analogously $f \geq g$. In that case, we have a homotopy of the maps $f, g : \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ induced on the order complexes.

If G is a group acting on X by poset automorphisms, we say that X is a G -poset. In that case, it is clear that $\mathcal{K}(X)$ is a G -complex and hence its geometric realization is a G -space. If X, Y are G -posets and $f, g : X \rightarrow Y$ are G -equivariant maps such that $f \leq g$, then f, g are G -equivariant homotopy equivalent. We write $X \simeq_G Y$ if the geometric realizations of X and Y are G -homotopy equivalent as G -spaces.

We often use the following special case where $X = Y$, so that f is a poset *endomorphism*. If X is a G -poset and f is further G -equivariant, we say that f is a G -endomorphism of X .

Lemma 2.7 (Homotopy from endomorphisms). *Let X be a finite G -poset, for a group G .*

(1) *Assume $f_0, f_1, \dots, f_t$ are G -endomorphisms of X , with $f_0 = \text{Id}_X$ and such that the following statements hold:*

- (i) *for all i we have $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$;*
- (ii) *the final map f_t is constant.*

Then, $\text{Id}_X = f_0$ is G -homotopy equivalent to the constant map f_t ; so X is G -contractible.

(2) *Assume that f is a monotone G -endomorphism of X : that is, $f \leq \text{Id}_X$ or $f \geq \text{Id}_X$. Then, f induces a G -homotopy equivalence $X \simeq_G f(X)$ with its image; indeed, we get $X \simeq_G Y$ for any intermediate G -invariant subposet Y , namely, where $f(X) \subseteq Y \subseteq X$.*

If further f is the identity on $f(X)$, then f induces a poset-strong deformation retraction.

Proof. Standard; e.g., [19, Subsection 1.5]. ■

Here are a few comments about our usage of these results.

In applications of (1), the overall setup is summarized via a “zigzag” or “fence”; where for $x \in X$, we record just the following information, roughly in the form:

$$(i') \quad x(= f_0(x)) \leq f_1(x) \geq f_2(x) \leq \dots \geq (f_t(x)=)x_t.$$

In particular, this includes the definition of the maps f_i , and their *<-comparability* relations for (1) (i)—in this case $f_0 \leq f_1 \geq f_2 \leq \dots \geq f_t$. We emphasize that we must still prove that the set-maps f_i thus defined really do afford *poset* endomorphisms.

Moreover, we frequently want to use (1) to show equivariant contractibility of some subposet $Z \subseteq X$. And then, in order to have the endomorphism-condition for the restrictions $f_i|_Z$, we must *further* show that $f_i(Z) \subseteq Z$. In fact, showing this can often be the trickiest part of the proof.

We will illustrate these zigzag features, within our standard p -subgroup poset context, in the upcoming Example 2.14.

In (2), our strong deformation retraction has a further “very-strong” property, namely, the map f is even poset-homotopic to Id_Z ; that is, here we have the length-1 case of a sequence of comparable maps as in (1)—where we replace the constant map f_t with the retraction f . We will normally reserve the term “deformation retraction” for this “poset-strong” situation, where f has such a poset-homotopy sequence.

Assume $Y \subseteq X$ and $x \in X$. Set $Y_{\geq x} := \{y \in Y : y \geq x\}$. Define analogously $Y_{> x}$, $Y_{\leq x}$, $Y_{< x}$. The *link* $\text{Lk}_Y(x)$ in Y of some $x \in X$ is given by the subposet $Y_{< x} \cup Y_{> x}$. We will usually call $Y_{< x}$ the *lower link* in Y of x , and $Y_{> x}$ the *upper link* in Y of x .

The following standard condition is relevant in later Proposition 3.11.

Definition 2.8 (Upward-closed subposets). A subposet $Y \subseteq X$ is said to be *upward-closed* in X , if for all $y \in Y$, we have $X_{> y} \subseteq Y$. ■

For a poset X , a *chain* is an “inclusion-chain”—a subset $a \subseteq X$ such that the elements of a are pairwise comparable.

Write $\text{sd}(X)$ for the poset of non-empty chains of X .

This is just the *face poset* of the order complex $\mathcal{K}(X)$. An n -chain of X is a chain $a \subseteq X$ with size $|a| = n + 1$. Recall that X and $\text{sd}(X)$ are homotopy equivalent. If we wish to emphasize the order of the elements in the n -chain $a \in \text{sd}(X)$, we may write $a = (x_0 < x_1 < \cdots < x_n)$. We will denote by $\max(a)$ the maximal element of the chain a .

Definition 2.9 (The join and Cartesian product of posets). We denote by $X * Y$ the *join* of the posets X and Y . This is a poset whose underlying set is the disjoint union of X and Y , and the order is given as follows. We keep the given order in X and Y , and we put $x < y$ for $x \in X$ and $y \in Y$. It can be shown that $\mathcal{K}(X * Y) = \mathcal{K}(X) * \mathcal{K}(Y)$, where the latter join is the join of simplicial complexes. Moreover, after taking geometric realizations, it coincides with the classical join of topological spaces. That is, if $|K|$ denotes the geometric realization of the simplicial complex K , then we have a homeomorphism

$$|K * L| \cong |K| * |L|.$$

For more details see [19].

Recall also that the *Cartesian product* $X \times Y$ of two posets is a poset with the order given by $(x, y) \leq (z, w)$ if $x \leq z$ and $y \leq w$. ■

We will explore these constructions more carefully in Section 4.

We will often use Quillen's fiber theorem to prove that certain maps between posets are homotopy equivalences (see [19, Proposition 1.6]), and hence that certain posets have the same homotopy type. We include an equivariant version (see [26]) in Theorem 2.10 below, and an extended version in the appendix as Proposition A.1.

Theorem 2.10 (Quillen's fiber theorem). *Suppose $f : X \rightarrow Y$ is a map between G -posets X and Y such that for all $y \in Y$, we have that $f^{-1}(Y_{\leq y}) * Y_{> y}$ (resp., $f^{-1}(Y_{\geq y}) * Y_{< y}$) is contractible. Then, f is a homotopy equivalence. If in addition f is G -equivariant, and we have G_y -contractibility for each y , then f is a G -homotopy equivalence.*

*In particular, if $X \subseteq Y$, f is the inclusion map, and for all $y \in Y \setminus X$, we have $X_{< y} * Y_{> y}$ (respectively, $X_{> y} * Y_{< y}$) contractible, then f is a homotopy equivalence; and indeed a G -homotopy equivalence, when we have G_y -contractibility for those links. ■*

Let X be a finite poset. We denote by $\tilde{H}_*(X, R)$ the reduced homology of X with coefficients in the ring R , which is the reduced homology of its order complex $\mathcal{K}(X)$. In general, we will work with $R = \mathbb{Q}$ and we will just write $\tilde{H}_*(X)$. Finally, $\tilde{\chi}(X)$ denotes the reduced Euler characteristic of a topological space or finite poset X .

Remark 2.11 (Product homology for joins). If R is a field, the homology of a join of spaces is the tensor product of homologies. That is, we have that

$$\begin{aligned}\tilde{H}_*(X * Y, R) &= \tilde{H}_*(X, R) \otimes_R \tilde{H}_*(Y, R), \\ \tilde{H}_n(X * Y, R) &= \bigoplus_{i+j=n-1} \tilde{H}_i(X, R) \otimes_R \tilde{H}_j(Y, R).\end{aligned}$$

For more details on this isomorphism, see [14].

By the above isomorphism between the homology of a join of spaces and the tensor product of homologies, we get

$$\tilde{\chi}(X_1 * \cdots * X_n) = (-1)^{n-1} \prod_{i=1}^n \tilde{\chi}(X_i).$$

Product homology for joins is fundamental for the “classical” approach to homology propagation. See the discussion at Remark 4.5. ■

The p -subgroup posets and results on Quillen's conjecture

We close the preliminaries section with the notation of the different p -subgroup posets that we will work with, along with some background results from the literature on

Quillen's conjecture. The main posets are

$$\begin{aligned}\mathcal{S}_p(G) &= \text{Brown poset} = \text{poset of all non-trivial } p\text{-subgroups of } G, \\ \mathcal{A}_p(G) &= \text{Quillen poset} = \{A \in \mathcal{S}_p(G) : A \text{ is elementary abelian}\}, \\ \mathcal{B}_p(G) &= \text{Bouc poset} = \{P \in \mathcal{S}_p(G) : P = O_p(N_G(P))\}.\end{aligned}$$

The elements of the Bouc poset are usually called (non-trivial) radical p -subgroups or p -radical.

We will also use the following definition.

Definition 2.12 (The poset $i(X)$). Suppose next that X is finite poset such that every non-empty and lower-bounded subset $Y \subseteq X$ has an infimum in X . Denote by $i(X) \subseteq X$ the subposet whose elements are the infimum of the maximal elements above them. Note that $i(X)$ is a poset-strong deformation retract of X , via the retraction sending $x \in X$ to the infimum of the set of maximal elements of X above x . We denote this retraction by r_i .

Since the posets $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ satisfy this property (that is, every non-empty lower bounded subset has an infimum given by the intersection of that set), we will also consider the posets $i(\mathcal{S}_p(G))$ and $i(\mathcal{A}_p(G))$. Note that $i(\mathcal{S}_p(G))$ consists of the non-trivial intersections of Sylow p -subgroups of G . On the other hand, it can be shown (see [15, Remark 4.5]) that $i(\mathcal{A}_p(G))$ is the poset of elements $E \in \mathcal{A}_p(G)$ such that $E = \Omega_1(Z(\Omega_1(C_G(E))))$.

By the results of [19, 26], we have the following proposition.

Proposition 2.13. *Let G be a finite group and p a prime. Then, the posets $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, $i(\mathcal{S}_p(G))$ and $i(\mathcal{A}_p(G))$ are all G -homotopy equivalent.* ■

For example, if G is a p -group P , then $\mathcal{S}_p(P)$ is contractible to its maximal element P . Hence, the other posets above such as $\mathcal{A}_p(P)$ are also contractible.

In view of the above equivalences, we could equally well use these other posets in the statement of Quillen's conjecture, though of course it is customary to use $\mathcal{A}_p(G)$ there.

At the start of this paper, we had mentioned Quillen's result [19, Proposition 2.4] on "conical contractibility". Here, we state it in the following form.

If $O_p(G) > 1$, then $\mathcal{S}_p(G)$ is contractible.

In order to illustrate how we will later be using Lemma 2.7(1) in contractibility proofs, we now use that lemma to prove Quillen's proposition—expanding on his homotopy of maps in [19, Subsection 1.5].

Example 2.14 (Quillen's conical contractibility via the endomorphism Lemma 2.7). Recall the notation of Lemma 2.7(1), including (i') in our discussion thereafter.

Let us begin in a somewhat more general context than the p -subgroup posets above. Namely, as our overall-context poset “ X ”, we take $S(G)$ —the poset of *all* subgroups $H \leq G$ (including the identity $1 = 1_G$). And let us *assume* that N denotes some normal subgroup of G (possibly $N = 1$). Then, we have a zigzag of group inclusions determined by any $H \in S(G)$:

$$H \leq HN \leq N.$$

In the viewpoint of (i’), this defines maps

$$f_0(H) = H, \quad f_1(H) = HN, \quad f_2(H) = N;$$

and we have the $<$ -comparability relations $f_0 \leq f_1 \geq f_2$. Furthermore, the identity map $f_0 = \text{Id}_{S(G)}$ and the constant map f_2 to $\{N\}$, are easily seen to be poset endomorphisms. However, for f_1 , we further need to make a few observations. First, normality of N has the group-theoretic consequence that HN is also a subgroup—that is, $f_1(H) = HN \in S(G)$, so that f_1 is at least a set-endomorphism. Second, properties of group-inclusion show that $H \leq K$ implies $HN \leq KN$ —so that f_2 is in fact a poset map. We have now completed the verification of the hypotheses of Lemma 2.7(1), and we conclude by that result that $S(G)$ is (conically) contractible to $\{N\}$ —for any normal subgroup N . Moreover, since these maps are G -equivariant, we conclude that $S(G)$ is even G -contractible.

Now, we turn to the actual result of Quillen on p -subgroups. For our subposet “ Z ”, we take the Brown poset $\mathcal{S}_p(G)$ of non-trivial p -subgroups, and we now further assume that our normal subgroup N is a non-trivial p -group (so that $N \in \mathcal{S}_p(G)$). This time we take $H \in \mathcal{S}_p(G)$. We still get the inclusion-zigzag as above, and also the indicated properties for the restrictions f'_i of the f_i to $\mathcal{S}_p(G)$ —*except* possibly for the endomorphism-property on $\mathcal{S}_p(G)$. That is, here we must further show that each $f'_i(H)$ actually lies in $\mathcal{S}_p(G)$. For the identity map $\text{Id}_{\mathcal{S}_p(G)}$, we have $f'_0(H) = H \in \mathcal{S}_p(G)$ by our original choice of H . For the constant map to $\{N\}$, we saw $f'_2(H) = N \in \mathcal{S}_p(G)$ above—crucially using our hypothesis that N is a *non-trivial* p -group. Finally, we see $f'_1(H) = HN$ is a non-trivial p -group, using the group-theoretic product-order formula. Since the f'_i are also G -equivariant, this completes the endomorphism-proof, and hence the hypotheses of Lemma 2.7(1), applied to $\mathcal{S}_p(G)$. So, by that result, $\mathcal{S}_p(G)$ is conically G -contractible to $\{N\}$ (and indeed more generally to $\{O_p(G)\}$).

Notice this proof does not work, if for Z we take the Quillen poset $\mathcal{A}_p(G)$: for N might not be elementary abelian; and even if so (e.g., using $\Omega_1(Z(N))$), $f_1(H) = HN$ might not be elementary abelian (though both H and N are, they may not commute). Thus, to conclude that $\mathcal{A}_p(G)$ is contractible when $O_p(G) > 1$, we have to first quote the result for $\mathcal{S}_p(G)$ —and then apply the homotopy equivalence of $\mathcal{S}_p(G)$ with $\mathcal{A}_p(G)$ for general G . ■

Note that $m_p(G) - 1$ equals the dimension of the order complex of $\mathcal{A}_p(G)$. This dimension is the focus of the following definition.

Definition 2.15 (Quillen dimension and p -extensions). We say G satisfies the *Quillen dimension property* at p if

$$(\mathcal{QD})_p \text{ If } O_p(G) = 1, \text{ then } \tilde{H}_{m_p(G)-1}(\mathcal{A}_p(G)) \neq 0.$$

In particular, $(\mathcal{QD})_p$ implies (H-QC).

The condition $(\mathcal{QD})_p$ is often studied for a p -extension LB of a component L of G : following [4, p. 474], this means a split extension of L by an elementary abelian p -group B of *outer* automorphisms. (From now on, we usually reserve the notation “ LB ” for this p -extension situation.) ■

For solvable G with $O_p(G) = 1$, Quillen obtained (H-QC) using the above property (see [19, Corollary 12.2]). The term “Quillen dimension” for the property was introduced by Aschbacher–Smith in [4] they showed in their Proposition 1.7 (roughly) that for a simple component L of a general G with $O_p(G) = 1$, having the $(\mathcal{QD})_p$ -property for p -extensions LB guarantees that non-zero reduced homology for $\mathcal{A}_p(LB)$ (indeed extended to $LB C_G(LB)$, in the “non-conical” case $O_p(C_G(LB)) = 1$) propagates to $\mathcal{A}_p(G)$, and hence gives (H-QC). The proof of this result uses their original homology propagation Lemma 0.27 (we provide a full statement later, as Lemma 5.11) which exploits the interaction of the $(\mathcal{QD})_p$ -property with the calculation of the boundary operator on poset-chains. So, in order to extend their ideas, we will be studying that boundary-calculation in more detail—e.g., at later Remark 4.5.

To maximize the applicability of [4, Proposition 1.7] in their main proof, Aschbacher and Smith were able to show, by using the CFSG, that p -extensions of “most” simple groups L do indeed have the $(\mathcal{QD})_p$ -property. That result also will be relevant to our extensions, so for convenience, we recall it in Theorem 2.16 below.

If $p \nmid q$, denote by $|q|$ the multiplicative order of $q \bmod p$.

Theorem 2.16 ($(\mathcal{QD})_p$ -List, [4, Theorem 3.1]). *Assume that p is odd, and L is simple. Then, the p -extensions LB satisfy $(\mathcal{QD})_p$, except possibly when L is the following:*

- (1) of Lie type in the same characteristic p ;
- (2) $U_n(q)$ with $q \equiv -1 \pmod{p}$; and either $n \geq q(q-1)$ or $n \geq q^{1/p}(q^{1/p}-1)$, with field automorphisms $B > 1$;
- (3) for $q = r^p$, either ${}^3D_4(q)$ for $|q| = 3, 6$, or $E_8(q)$ for $|q| = 8, 12$, with field automorphisms $B > 1$;
- (4) F_1 with $p = 7$, and J_4 with $p = 11$.

(Cases with $p = 5$)

- (5) $E_8(q)$ with $|q| = 4$;
- (6) ${}^2F_4(2)'$; $\text{Sz}(32)$ or ${}^2F_4(32)$, with field automorphisms $B > 1$;
- (7) $F_5, F_2, F_1, \text{McL}, \text{Ly}$.

(Cases with $p = 3$)

- (8) alternating Alt_6 or Alt_n ($n \geq 9$);
- (9) of non-linear type over $\mathbb{F}_2$;
- (10) of non-linear type over $\mathbb{F}_8$, with field automorphisms $B > 1$;
- (11) $L_3(4)$; $L_2(8)$, ${}^2F_4(r^3)$ for $|q| = 2$, with field automorphisms $B > 1$;
- (12) $M_{11}, J_3, \cdot 1, \cdot 2, \cdot 3, M(22), M(23), M(24)', F_5, F_3, F_2, F_1, \text{McL}, \text{Suz}, \text{Ly}, \text{O'N}$. ■

In view of the above list, and the role of p -extensions in many results on Quillen's conjecture, we will be interested in understanding, for a given subgroup L of G (usually a component), the outer automorphisms of order p of L which in fact arise in $N_G(L)$. Correspondingly, we define two related posets (cf. [18, Definitions 4.2 and 6.1]).

Definition 2.17. Let $L \leq G$ and p a fixed prime. Define the p -outer poset of L to be the poset

$$\text{O}_G(L) := \{B \in \mathcal{A}_p(N_G(L)) : B \cap (LC_G(L)) = 1\}.$$

We usually call an element B of $\text{O}_G(L)$ a p -outer of L in G , since it induces outer automorphisms (and hence LB is a p -extension as in Definition 2.15). We also write

$$\hat{\text{O}}_G(L) := \text{O}_G(L) \cup \{1\}.$$

The image poset of L in G is

$$\mathcal{A}_{G,L} := \{BC_G(L)/C_G(L) : B \in \mathcal{A}_p(N_G(L)), C_B(L) = 1\}. \quad \blacksquare$$

Remark 2.18. Note that $\mathcal{A}_p(L/Z(L)) \subseteq \mathcal{A}_{G,L} \subseteq \mathcal{A}_p(\text{Aut}_G(L))$. In particular, if $Z(L) = 1$, then $\mathcal{A}_p(L) \subseteq \mathcal{A}_{G,L}$ and

$$\mathcal{A}_{G,L} \setminus \mathcal{A}_p(L) = \{(BC_G(L)/C_G(L))A : A \in \mathcal{A}_p(L) \cup \{1\}, B \in \text{O}_G(L)\}. \quad \blacksquare$$

To close this section, we summarize some main results on (H-QC) in the earlier literature.

Theorem 2.19. If G is a p -solvable group, then it satisfies $(\mathcal{QD})_p$ and, in particular, (H-QC). ■

The p -solvable case was established by various authors, based on the ideas of Quillen on the solvable case [19]. See [24, Chapter 8] for further details.

Theorem 2.20 (Aschbacher–Kleidman [3]). *An almost simple group G satisfies (H-QC).* ■

Here is the original Aschbacher–Smith result for $p > 5$, motivating our extension in Theorem 1.1.

Theorem 2.21 (Aschbacher–Smith [4]). *Let G be a finite group and p a prime. Assume that*

- (i) $p > 5$;
- (ii) *whenever G has a unitary component $U_n(q)$ with $q \equiv -1 \pmod{p}$, then $(\mathcal{QD})_p$ holds for all p -extensions $U_m(q^{p^e})$ with $m \leq n$ and $e \in \mathbb{Z}$.*

Then, G satisfies (H-QC) for p . ■

Recall that in our discussion in the introduction leading up through Remark 1.3, we indicated in (MOC) that a minimal order counterexample to (H-QC) satisfies the following induction replacement hypothesis.

(H1) Proper subgroups and proper p' -central quotients of G satisfy (H-QC).

Below, we summarize various results obtained under (H1) in the articles [16, 18].

Theorem 2.22 ([16]). *Let G be a group and p a prime. Suppose that G satisfies (H1), and that one of the following holds:*

- (1) $Z(G) \neq 1$ or $\Omega_1(G) < G$;
- (2) $O_{p'}(G) \neq 1$;
- (3) $\mathcal{A}_p(G)$ is not simply connected;
- (4) G has p -rank at most 4;
- (5) G has a component L such that $L/Z(L)$ has p -rank 1;
- (6) $p = 3$, and G has a component L such that $L/Z(L) \cong U_3(8)$.

Then, G satisfies (H-QC). ■

The result [16, Corollary 3] first showed that Theorem 2.21 above extends to $p = 5$. (So, our extension Theorem 1.1, which we prove as later Theorem 9.2, in effect adds just the “new” prime $p = 3$ in the proof.)

Corollary 2.23. *Theorem 2.21 extends to $p = 5$.* ■

However, there is a small gap in the explanation given in [16]; it can be fixed without requiring the results of this paper—and we indicate that adjustment during the proof of Theorem 9.2.

Next, we recall a recent result that establishes (H-QC) for groups G containing components L such that $\text{Out}_G(L)$ is a p' -group. The proof of the following theorem does not depend on the CFSG. Moreover, in the language of [18] (which we had described in earlier Remark 1.2), this result is an example of an “elimination result”: it allows us to eliminate possible components from a potential counterexample to (H-QC) subject to certain minimality conditions.

Theorem 2.24 (cf. [18, Corollary 5.1]). *Let p be a prime and L a component of a group G . Let $H = \hat{N}_G(L)$. Suppose the following.*

- (1) *p divides the order of L , and $C_G(\hat{L})$ satisfies (H-QC).*
- (2) *The induced map $\mathcal{A}_p(L) \rightarrow \mathcal{A}$ is not the zero map in homology, where $\mathcal{A}$ is one of the following posets:*

$$\mathcal{A}_{H,L}, \mathcal{A}_{G,L}, \mathcal{A}_p(\text{Aut}_H(L)), \mathcal{A}_p(\text{Aut}_G(L)), \mathcal{A}_p(\text{Aut}(L)).$$

Then, G satisfies (H-QC).

In particular, if G is a counterexample of minimal order to (H-QC), then $\text{Aut}_G(L)$ must contain p -outers; for odd p , this eliminates alternating and sporadic components. ■

The final statement about odd primes p uses details about outer automorphisms in Remark 2.6.

Using the above results, if G is a counterexample of minimal order to (H-QC), then by (MOC) in Remark 1.2, G satisfies (H1)—so it fails conditions (1)–(6) of Theorem 2.22. Therefore, every component L of G has order divisible by p (by Lemma 2.3); the map $\mathcal{A}_p(L) \rightarrow \mathcal{A}_{\hat{N}_G(L),L}$ is the zero map in homology (by Theorem 2.24); G is not p -solvable nor almost simple (by Theorems 2.19 and 2.20); and if $p \geq 5$, then it contains some unitary component for which some p -extension does not satisfy $(\mathcal{QD})_p$ (by Theorem 2.21).

3. Overview: Using replacement-posets homotopy equivalent to $\mathcal{A}_p(G)$

In this section, we implement a theme we had indicated in earlier Remark 1.6, namely, we recall from the literature—and extend—some methods for replacing $\mathcal{A}_p(G)$ with more convenient (and typically smaller) homotopy equivalent posets.

First, in Definition 3.4, we will review (and slightly generalize) an earlier construction of an equivalent poset $X_G(H)$, obtained with respect to a subgroup $H \leq G$. We will indicate a variant in Proposition 3.7, namely, a further-reduced equivalent subposet $X_G(i(H))$.

Finally, in the latter part of the section, we will review in Remark 3.10 a different notion of reducing $\mathcal{A}_p(G)$ to an equivalent poset—by “removing points with conical centralizer”. Then, in Proposition 3.11, we present a result for implementing these removals in some relevant situations.

As motivation for our main work on equivalences, we first give some relevant general background. Recall first that for (H-QC), we of course need to show under $O_p(G) = 1$ that $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$.

Indeed, in some situations, we can establish the non-zero homology “immediately” for $\mathcal{A}_p(G)$ itself: sometimes this proceeds by direct construction of reduced homology. See, for example, the methods in [24, Section 8.1], which, in particular, are used in obtaining the Aschbacher–Smith ($\mathcal{QD}$)-List in Theorem 2.16. But the procedure can also be indirect, namely, just establishing *existence* of non-zero homology. This is the case, for example, in the Aschbacher–Kleidman result Theorem 2.20 for almost simple G . We will examine the underlying Robinson subgroup method further in our later Section 8.

But in studying (H-QC) for more general situations, we more typically proceed via some intermediate proper subgroup $H < G$.

Remark 3.1 (Equivalences in the broad context of homology propagation). That is, often we begin just with knowledge of non-zero reduced homology of $\mathcal{A}_p(H)$ for some proper subgroup $H < G$. Sometimes via direct construction, or also possibly via induction—or induction-replacement, e.g., as in (H1) in earlier Remark 1.3.

And then (as we had briefly mentioned after Remark 1.5), we wish to show that the non-zero homology for $\mathcal{A}_p(H)$ leads to non-zero homology also for $\mathcal{A}_p(G)$. This rough notion of “homology propagation” is probably broader than the more conventional meaning in the literature, but it seems appropriate for a preliminary exposition. Stated more precisely, we mean here that the natural poset inclusion $i : \mathcal{A}_p(H) \rightarrow \mathcal{A}_p(G)$ should induce a non-zero map i_* in homology.

Of course, the simplest case of such a non-zero map arises when i is a homotopy equivalence—so that the induced map i_* is actually a homology isomorphism. This simplest situation arises for example in Claim 4 in the proof of [16, Theorem 4.1], as well as in Lemma 3.3(3) in this paper. Such results already begin to motivate our study of equivalences in this section.

And this equivalence-theme continues, when we consider the extension in [16, Lemma 3.14] of the naive propagation above—for convenience, we have given a full statement of this result as later Lemma 5.14. There, the propagation proceeds from $\mathcal{A}_p(H)$, not to $\mathcal{A}_p(G)$ but instead to some possibly-proper subset $X \subseteq \mathcal{A}_p(H)$; and then it remains to show independently that this X is homotopy equivalent to $\mathcal{A}_p(G)$. This type of equivalence arises for example in Claim 5 in the proof of [16, Theorem 4.1], as well as in various cases in this paper, such as Theorem 4.10—where X is given by a replacement-poset such as $W_G^{\mathcal{A}}(H, K)$ defined there. ■

Equivalences via shrinking a subposet inside a poset union

We now begin the main work of the section, with our review of the construction of the “basic” replacement-poset $X_G(H)$.

As a rough preliminary overview.

We want to understand how $\mathcal{A}_p(G)$ is built up from suitable subposets of $\mathcal{A}_p(H)$. To that end, we first consider the “inflation” $\mathcal{N}_G(H)$, namely, a certain larger subposet, which retracts to H ; and then we recover $\mathcal{A}_p(G)$, by adding the remaining elements $E \in \mathcal{A}_p(G) \setminus \mathcal{N}_G(H)$ —which “glue” to the poset $\mathcal{N}_G(H)$, via connections within the subposets $\mathcal{A}_p(C_G(E))$. In particular, we are viewing $\mathcal{A}_p(G)$ as the poset union $\mathcal{N}_G(H) \cup (\mathcal{A}_p(G) \setminus \mathcal{N}_G(H))$.

The smaller replacement-poset $X_G(H)$ (equivalent to $\mathcal{A}_p(G)$) is then obtained by “deflating” the inflation term $\mathcal{N}_G(H)$ inside that union: basically, collapsing it down to $\mathcal{A}_p(H)$ —so that the added- E terms indicated above now remain glued just via the corresponding smaller centralizer-posets $\mathcal{A}_p(C_H(E))$. But this process requires replacing the original inclusion-ordering $<$ on $X_G(H)$ by a new ordering $<$. As a result, $X_G(H)$ is just a subset, but not a $<$ -subposet, of $\mathcal{A}_p(G)$.

Thus, we start by recalling from the literature the definition of the underlying poset $\mathcal{N}_G(H)$, which is the inflation of $\mathcal{A}_p(H)$ (indeed slightly generalizing its context from $\mathcal{A}_p(G)$ to a suitable subposet $\mathcal{B}$).

Definition 3.2 (The $\mathcal{N} \cup \mathcal{F}$ -decomposition). For H a subgroup of G , and a subposet $\mathcal{B}$ such that $\mathcal{A}_p(H) \subseteq \mathcal{B} \subseteq \mathcal{A}_p(G)$, set

$$\mathcal{N}_{\mathcal{B}}(H) := \{E \in \mathcal{B} : E \cap H \neq 1\},$$

$$\mathcal{F}_{\mathcal{B}}(H) := \{E \in \mathcal{B} : E \cap H = 1\}.$$

Note that $\mathcal{B}$ is the *disjoint* union of $\mathcal{N}_{\mathcal{B}}(H)$ and $\mathcal{F}_{\mathcal{B}}(H)$. We focus on one feature of this union, with respect to the $<$ -ordering: note that if $F \in \mathcal{F}_{\mathcal{B}}(H)$ and $A \in \mathcal{N}_{\mathcal{B}}(H)$, then we cannot have $F > A$ —as that would violate $F \cap H = 1$. That is,

$$\text{for } A \in \mathcal{N}_{\mathcal{B}}(H), \text{ we have } \mathcal{B}_{>A} \cap \mathcal{F}_{\mathcal{B}}(H) = \emptyset; \text{ so } \mathcal{B}_{>A} = \mathcal{N}_{\mathcal{B}}(H)_{>A}.$$

Thus, members of $\mathcal{F}_{\mathcal{B}}(H)$ only appear as the *left* segment of inclusion-chains:

- (i) A $<$ -chain begins with any terms from $\mathcal{F}_{\mathcal{B}}(H)$, followed by any terms from $\mathcal{N}_{\mathcal{B}}(H)$.

This type of property will be relevant frequently, as our development continues. For example, in Remark 3.5, we will at least begin to see its significance for our viewpoint on boundary calculations, for homology propagation.

When $\mathcal{B} = \mathcal{A}_p(K)$ for some $K \leq G$, write $\mathcal{N}_K(H) := \mathcal{N}_{\mathcal{A}_p(K)}(H)$ and $\mathcal{F}_K(H) := \mathcal{F}_{\mathcal{A}_p(K)}(H)$. In particular, for $K = G$, we get

$$\mathcal{A}_p(G) = \mathcal{N}_G(H) \cup \mathcal{F}_G(H). \quad \blacksquare$$

We have the following immediate results (cf. [16, 17, 21]).

Lemma 3.3. *Assume $H \leq G$, with $\mathcal{A}_p(H) \subseteq \mathcal{B} \subseteq \mathcal{A}_p(G)$ as in Definition 3.2, and let $K \leq G$ be such that $\mathcal{A}_p(H) \subseteq \mathcal{A}_p(K)$. Then, we get the following.*

- (1) $\mathcal{N}_{\mathcal{B}}(H)$ poset-strong deformation-retracts onto $\mathcal{A}_p(H)$ —via $r : \mathcal{N}_{\mathcal{B}}(H) \rightarrow \mathcal{A}_p(H)$ defined by $r(E) := E \cap H$ (we may call this the “deflation” map).
- (2) If $E \in \mathcal{F}_K(H)$, then $\mathcal{N}_K(H)_{>E} \simeq_{N_G(H,K,E)} \mathcal{A}_p(C_H(E))$.
- (3) If $O_p(C_H(E)) > 1$ for all $E \in \mathcal{F}_K(H)$, then $\mathcal{A}_p(H) \hookrightarrow \mathcal{A}_p(K)$ is a $N_G(H, K)$ -homotopy equivalence. So, if this further condition holds for $K = G$, we get $\mathcal{A}_p(H) \simeq_{N_G(H)} \mathcal{A}_p(G)$.

Proof. For part (1), we apply Lemma 2.7 (2), with $r, \mathcal{N}_{\mathcal{B}}(H)$ in the roles of “ f, X ”: Notice that we have $r(E) \leq E$, with r the identity when restricted to $r(\mathcal{N}_{\mathcal{B}}(H)) = \mathcal{A}_p(H)$. We emphasize that for the endomorphism-hypothesis in that Lemma, we further need to check that $r(E) = E \cap H$ lies in $\mathcal{N}_{\mathcal{B}}(H)$: this holds here, since we have $E \cap H \in \mathcal{A}_p(H) \subseteq \mathcal{B}$, while $(E \cap H) \cap H = E \cap H > 1$ —so that $E \cap H \in \mathcal{N}_{\mathcal{B}}(H)$. For part (2), we use the poset map $f : \mathcal{N}_K(H)_{>E} \rightarrow \mathcal{A}_p(C_H(E))$ defined by $f(B) := B \cap H$, with homotopy inverse $g(C) := EC$; for the homotopies, we observe that $gf(B) = E(B \cap H) \leq B$, along with $fg(C) = EC \cap H \geq C$ —that is, $gf \leq \text{Id}_{\mathcal{N}_K(H)_{>E}}$, and $fg \geq \text{Id}_{\mathcal{A}_p(C_H(E))}$. Part (3) is a consequence of (2), via the Quillen fiber Theorem 2.10. Finally, the equivariant assertions are easy to verify from the given endomorphisms. ■

In Definition 3.4, we introduce $X_G(H)$ —the first of our replacement-posets for $\mathcal{A}_p(G)$. Similar constructions had been considered earlier in [21], and also at [25, p. 134]; compare also with [1, (6.8)]. The related construction in [16] shrank $\mathcal{A}_p(G)$ to a smaller equivalent poset but did not change the order relation. In this paper, we change the ordering—obtaining more general equivalent posets, which will help to make the later homology propagation clearer and more explicit.

In fact, we will see in Definition 3.4 how we can use the deflation map r above to shrink $\mathcal{N}_{\mathcal{B}}(H)$ to $\mathcal{A}_p(H)$ —within the decomposition given by the poset union in Definition 3.2; though this comes at the cost of defining a new poset-ordering on the resulting subset. We will denote the new order-relation by $\prec$: since it agrees with inclusion $<$, except at certain specified cross-terms between $\mathcal{F}$ and $\mathcal{N}$, in the language of the decomposition in Definition 3.2. The result is a “replacement-poset” $X_{\mathcal{B}}(H)$. That is, we will see in Theorem 3.6 that it will be equivalent to $\mathcal{B}$ in our standard situations, where it will often be more convenient for us to work with, for our later applications on homology propagation.

Definition 3.4 (The replacement-poset $X_{\mathcal{B}}(H)$, with its ordering $\prec$). Let $H \leq G$ and take an intermediate poset $\mathcal{B}$ with $\mathcal{A}_p(H) \subseteq \mathcal{B} \subseteq \mathcal{A}_p(G)$, as in Definition 3.2. Define

$X_{\mathcal{B}}(H)$ to be the poset which—just as a set, rather than a poset—is the disjoint union:

$$X_{\mathcal{B}}(H) = \mathcal{A}_p(H) \cup \mathcal{F}_{\mathcal{B}}(H)$$

and with a new poset order-relation $\prec := \prec_{X_{\mathcal{B}}(H)}$ given as follows.

- (1) Inside $\mathcal{A}_p(H)$, and also inside $\mathcal{F}_{\mathcal{B}}(H)$, as $\prec$ we keep the group-inclusion ordering $\leq$.
- (2) When $F \in \mathcal{F}_{\mathcal{B}}(H)$, and $A \in \mathcal{A}_p(H)$ with $C_A(F) > 1$, we define $F \prec A$.

We need to check transitivity for this new relation $\prec$. Since $\prec$ agrees with the old ordering $\leq$ inside each of the two factors in the union for $X_{\mathcal{B}}(H)$, we only need to check transitivity of extensions at either end of cross-terms⁸ between the two factors; and by the $\mathcal{F}$ -left property of $\prec$ in (2) above (see also (3) below), these have form $F \prec A$, for $F \in \mathcal{F}_{\mathcal{B}}(H)$ and $A \in \mathcal{A}_p(H)$ —where we have $C_A(F) > 1$. And such extensions must then have the forms $E < F$ for $E \in \mathcal{F}_{\mathcal{B}}(H)$, or $A < B$ for $B \in \mathcal{A}_p(H)$. When $E < F$, we get $1 < C_A(F) \leq C_A(E)$, so that also $E \prec A$. And when $A < B$, we get $1 < C_A(F) \leq C_B(F)$, so that also $F \prec B$. This completes the transitivity proof. Hence, $X_{\mathcal{B}}(H)$ is indeed a poset.

Note that (2) gives the analog, for $\prec$, of the $\mathcal{F}$ -left property (i) in Definition 3.2.

- (3) A $\prec$ -chain begins with terms from $\mathcal{F}_{\mathcal{B}}(H)$, followed by terms from $\mathcal{A}_p(H)$.

We typically write out such a $\prec$ -chain a in the form

$$a = (F_1 < \cdots < F_r < A_1 < \cdots < A_s) \quad \text{for } F_i \in \mathcal{F}_{\mathcal{B}}(H) \text{ and } A_j \in \mathcal{A}_p(H).$$

Here, we have in fact written the new notation $\prec$ only at the cross-term case (where it is definitely required), but used the old notation $<$ at all the other adjacencies, where $\prec$ in fact agrees with $<$. This viewpoint has another alternative form, again analogous to a property in Definition 3.2: if $A \in \mathcal{A}_p(H)$, then $X_{\mathcal{B}}(H)_{>A} \cap \mathcal{F}_{\mathcal{B}}(H) = \emptyset$. That is, $\mathcal{A}_p(H) \supseteq X_{\mathcal{B}}(H)_{>A} = \mathcal{A}_p(H)_{>A}$.

If $\mathcal{B} = \mathcal{A}_p(K) \supseteq \mathcal{A}_p(H)$, for $K \leq G$, we write just $X_K(H)$ for $X_{\mathcal{B}}(H)$. When $K = G$, we get

$$X_G(H) = \mathcal{A}_p(H) \cup \mathcal{F}_G(H)$$

as a subset of $\mathcal{A}_p(G)$ (though not a subposet under group-inclusion $\leq$). ■

Remark 3.5 (A left-focused format for poset-chains involved in boundary calculations). A new feature of our use of $X_G(H)$ in this paper is that (as we had briefly suggested in earlier Remark 1.6) we can exploit the $\mathcal{F}$ -left condition in Definition 3.4 (3),

⁸Notational comment: We will typically reserve the letter-pairs F, A and F, B for such cross-terms.

to get a more “visual” format for its poset-chains—which makes analysis of their boundaries easier than in $\mathcal{A}_p(G)$.

Very roughly, for a suitable cycle from $\mathcal{A}_p(H)$ involving a $\prec$ -chain a , the main preliminary calculation in homology propagation shows that chains from $\mathcal{A}_p(G)$ properly containing a , which are built just from further members of $\mathcal{N}_G(H)$, do not contribute to the boundary of a in $\mathcal{A}_p(G)$; so that the only problematic members must come from the $\mathcal{F}_G(H)$ factor in the union.

And after applying the homotopy equivalence of $X_G(H)$ with $\mathcal{A}_p(G)$ in Theorem 3.6 below, it will be an advantage of the $X_G(H)$ -formulation for the boundary calculation that any such problematic members of $\mathcal{F}_G(H)$ appearing in $\prec$ -chains containing a are concentrated together at the *left* end of those chains. We will flesh out this vague description later, notably in Remark 5.5.

This left-focused format for the chains in $X_G(H)$ (and in other posets later in the paper) will help simplify the proofs of some new homology propagation results, such as Theorem 6.6. ■

As promised earlier, we next get an explicit equivariant homotopy equivalence between $\mathcal{B}$ and $X_{\mathcal{B}}(H)$ under suitable extra conditions on $\mathcal{B}$ which hold, for example, when $\mathcal{B}$ is of the usual form $\mathcal{A}_p(K)$.

Theorem 3.6. *Assume $H \leq G$, with $\mathcal{A}_p(H) \subseteq \mathcal{B} \subseteq \mathcal{A}_p(G)$ as in Definition 3.2; and write $N_G(H, \mathcal{B})$ for the largest subgroup of $N_G(H)$ acting on $\mathcal{B}$. Recall the poset $X_{\mathcal{B}}(H) = \mathcal{A}_p(H) \cup \mathcal{F}_{\mathcal{B}}(H)$ under $\prec$ from Definition 3.4. Assume also the further condition.*

- (i) *When $F \in \mathcal{F}_{\mathcal{B}}(H)$, $B \in \mathcal{N}_{\mathcal{B}}(H)$ satisfy $F \prec (B \cap H)$ (that is, $C_{B \cap H}(F) > 1$), then also $C_B(F), C_B(F)F \in \mathcal{B}$.*

Then, we have an $N_G(H, \mathcal{B})$ -homotopy equivalence $\alpha_{\mathcal{B}, H} : \mathcal{B} \rightarrow X_{\mathcal{B}}(H)$, defined using the deflation map on the $\mathcal{N}_{\mathcal{B}}(H)$ -term:

$$\alpha_{\mathcal{B}, H}(D) = \begin{cases} D \cap H : D \in \mathcal{N}_{\mathcal{B}}(H), \\ D : D \in \mathcal{F}_{\mathcal{B}}(H). \end{cases}$$

Moreover, the following statements hold.

- (1) *The homotopy equivalence $\alpha_{\mathcal{B}, H}$ is the identity when restricted to $\mathcal{A}_p(H)$.*
- (2) *For $F \in \mathcal{F}_{\mathcal{B}}(H)$, $X_{\mathcal{B}}(H)_{>F} \setminus \mathcal{F}_{\mathcal{B}}(H) = \mathcal{A}_p(H)_{>F} = \mathcal{N}_H(C_H(F))$; this has a poset-strong deformation retraction to $\mathcal{A}_p(C_H(F))$ via the $C_H(F)$ -deflation map $B \mapsto B \cap C_H(F) = C_{B \cap H}(F) = C_B(F)$.*
- (3) *If $\mathcal{B} \subseteq \mathcal{N}_G(H)$, then $\mathcal{B} = \mathcal{N}_{\mathcal{B}}(H) \simeq \mathcal{A}_p(H)$; and $\alpha_{\mathcal{B}, H}$ is the restriction to $\mathcal{B}$ of the deflation map, i.e., the retraction $B \mapsto B \cap H$.*

In particular, if $\mathcal{B} = \mathcal{A}_p(K) \supseteq \mathcal{A}_p(H)$ for $K \leq G$, the above hypotheses (notably (i)) hold, and we write $\alpha_{K,H}$ for the above map, which is an $N_G(H, \Omega_1(K))$ -homotopy equivalence.

Thus, for $K = G$, we get an $N_G(H)$ -homotopy equivalence of $\mathcal{A}_p(G)$ with the poset $X_G(H)$ ⁹.

Proof. First, we check that $\alpha_{\mathcal{B},H}$ is order-preserving—from $<$ to $<$. Since $<$ agrees with $<$ inside the images of each of the two factors in the union giving the domain $\mathcal{B}$, we only need to check images of cross-terms between those two factors, which by the $\mathcal{F}$ -left property (i) in Definition 3.2 must be of the form $F < A$, for $F \in \mathcal{F}_{\mathcal{B}}(H)$, and $A \in \mathcal{N}_{\mathcal{B}}(H)$ (so that $A \cap H > 1$). Here, F centralizes A just since A is abelian, so that $C_{A \cap H}(F) = A \cap H > 1$ —and so, we do indeed get $F < A \cap H$ for the images, as needed.

To show $\alpha_{\mathcal{B},H}$ is an equivariant homotopy equivalence, we will use the Quillen fiber Theorem 2.10.

The first step is the equivalence (with equivariance indicated later) for $x \in X_{\mathcal{B}}(H)$, we will show below that the preimage

$$Z_x := \alpha_{\mathcal{B},H}^{-1}(X_{\mathcal{B}}(H)_{\succeq x})$$

is contractible.

Consider first the case $x = A \in \mathcal{A}_p(H)$. Then, using the $\mathcal{F}$ -left property in Definition 3.4 (3), and the hypothesis $\mathcal{A}_p(H) \subseteq \mathcal{B}$, we have $X_{\mathcal{B}}(H)_{\succeq A} = \mathcal{A}_p(H)_{\succeq A}$, so that our preimage is

$$\begin{aligned} Z_A &= \alpha_{\mathcal{B},H}^{-1}(X_{\mathcal{B}}(H)_{\succeq A}) = \alpha_{\mathcal{B},H}^{-1}(\mathcal{A}_p(H)_{\succeq A}) = \{B \in \mathcal{N}_{\mathcal{B}}(H) : B \cap H \geq A\} \\ &= \mathcal{N}_{\mathcal{B}}(H)_{\succeq A}, \end{aligned}$$

which is contractible since it has minimal element $A > 1$.

Now, consider the remaining case $x = F \in \mathcal{F}_{\mathcal{B}}(H)$. This argument will be lengthier. Note that

$$X_{\mathcal{B}}(H)_{\succeq F} = \mathcal{A}_p(H)_{\succeq F} \cup \mathcal{F}_{\mathcal{B}}(H)_{\succeq F} = \mathcal{N}_H(C_H(F)) \cup \mathcal{F}_{\mathcal{B}}(H)_{\succeq F},$$

and in particular, we get the first part of (2). Furthermore, our preimage then has the following form.

$$(ii) \ Z := Z_F = \alpha_{\mathcal{B},H}^{-1}(X_{\mathcal{B}}(H)_{\succeq F}) = \mathcal{N}_{\mathcal{B}}(C_H(F)) \cup \mathcal{F}_{\mathcal{B}}(H)_{\succeq F}.$$

We will show using Lemma 2.7 (1) that Z is contractible (to $\{F\}$). As in (i') in our discussion after that lemma (compare also the worked-out details of the application

⁹Notice that we do not say “retraction” because $X_G(H)$ is just a subset, not a $\leq$ -subposet of $\mathcal{A}_p(G)$.

in earlier Example 2.14), the arguments are summarized by the following zigzag of group-inclusions for $D \in Z$:

$$D \geq C_D(F) \leq C_D(F)F \geq F. \quad (3.1)$$

This abbreviated form expands to defining maps j_0, j_1, j_2, j_3 on Z , with images at D given by $D, C_D(F), C_D(F)F, F$. In particular, $j_0 = \text{Id}_Z$, and j_3 is the constant map to $\{F\}$. We see via elementary group-theoretic properties that each j_i is a poset map with the $<$ -comparability conditions $j_0 \geq j_1 \leq j_2 \geq j_3$.

So, to complete the hypotheses of Lemma 2.7(1), it remains to check the endomorphism condition, namely, that $j_i(D) \in Z$ for all i . For $j_0 = \text{Id}_Z$, this is clear (by choice of $D = j_0(D)$ in Z); and for j_3 also, since $j_3(D) = F \in \mathcal{F}_{\mathcal{B}}(H)_{\geq F} \subseteq Z$ by (ii). Thus, it remains to consider j_1, j_2 , that is, to show that the corresponding images $C_D(F), C_D(F)F$ lie in Z .

Now, since our Z is defined in the context of a subposet $\mathcal{B}$, which is not necessarily all of $\mathcal{A}_p(G)$, as a preliminary, we first need to see that these images fall into $\mathcal{B}$.

(a) Show that the images $C_D(F)$ and $C_D(F)F$ lie in $\mathcal{B}$.

In the case D lies in the right-factor for Z in (ii) above, i.e., $D = E \in \mathcal{F}_{\mathcal{B}}(H)_{\geq F}$, we have $C_E(F) = E = C_E(F)F$, where $E \in \mathcal{F}_{\mathcal{B}}(H)_{\geq F} \subseteq \mathcal{B}$, giving (a) for both images. Indeed, since these images lie in $\mathcal{F}_{\mathcal{B}}(H)_{\geq F} \subseteq Z$ in (ii), this even gives (b) and (c) below for such $D = E$. The remaining case, that is, the left-factor for Z in (ii), has $D = B \in \mathcal{N}_{\mathcal{B}}(C_H(F))$, so that $B \cap C_H(F) > 1$. Observe that we have several ways of viewing this intersection, so that we can write this condition in the expanded form

$$B \cap C_H(F) = C_{B \cap H}(F) = C_B(F) \cap C_H(F) > 1.$$

In particular, we have $C_{B \cap H}(F) > 1$, so that we are in the situation of hypothesis (i) of the theorem. We now crucially quote that hypothesis, to get $C_B(F)$ and $C_B(F)F$ in $\mathcal{B}$, as desired. This completes (a) in both cases $D = E, B$ for a member of Z .

(b) After (a), it is routine to check that $j_1(D) = C_D(F) \in Z$.

For we covered the right-factor case $D = E \in \mathcal{F}_{\mathcal{B}}(H)_{\geq F}$ in the proof of (a) above. The left-factor case, where $D = B \in \mathcal{N}_{\mathcal{B}}(C_H(F))$, has $C_B(F) \cap C_H(F) > 1$ as in that proof; and so, now using the conclusion of (a), we have $C_B(F) \in \mathcal{N}_{\mathcal{B}}(C_H(F)) \subseteq Z$ in (ii), as desired.

(c) After (a), it is routine to check that $j_2(D) = C_D(F)F \in Z$.

For again the right-factor case $D = E \in \mathcal{F}_{\mathcal{B}}(H)_{\geq F}$ was covered in the proof of (a). In the left-factor case given by $D = B \in \mathcal{N}_{\mathcal{B}}(C_H(F))$, we have

$$C_B(F)F \cap C_H(F) \geq B \cap C_H(F) > 1$$

as in that proof. So, using the conclusion of (a), we get $C_B(F)F \in \mathcal{N}_B(C_H(F)) \subseteq Z$ in (ii), as desired.

This completes the proof that $j_1(D), j_2(D) \in Z$, and hence of the hypotheses of Lemma 2.7 (1). We conclude by that lemma that $Z = Z_F$ is contractible, in this case for $x = F$.

In conjunction with the contractibility proof for the preimage in the previous case $x = A$, we have completed the proof that $\alpha_{B,H}$ is a homotopy equivalence.

For the equivariance of the equivalence, we need to note that the homotopies we produced above preserve the action of the stabilizer of x . To be more precise, let $K := N_G(H, B)$ and let $x \in X_B(H)$. The contractibility of Z_x in the case $x = A \in \mathcal{A}_p(H)$ is given by $B \geq A$, which is $N_K(A)$ -equivariant; and in the case $x = F \in \mathcal{F}_B(H)$ the zigzag in equation (3.1) showing contractibility for Z_x is also $N_K(F)$ -equivariant. Since $\alpha_{B,H}$ is also K -equivariant, we conclude it is a K -homotopy equivalence by the equivariant version of the Quillen fiber Theorem 2.10.

Furthermore, the homotopies above are the identity when we restrict them to elements of $\mathcal{A}_p(H)$ —additionally giving (1). Finally, item (3) follows from Lemma 3.3 (1). This also gives the remaining part of (2) when we take $H, C_H(F)$ in the roles of “ G, H ”. ■

In our later equivalence-result Proposition 4.9, we will adjust the construction of $X_G(H)$ in Definition 3.4 as follows. First, the role of “ H ” will in fact be played by a p' -central product $HK = H * K$; and second, we will replace the customary corresponding poset-join $\mathcal{A}_p(H) * \mathcal{A}_p(K) \simeq \mathcal{A}_p(HK)$ by its “pre-join” variant—which we describe in Section 4. Indeed, in the subsequent equivalence-result Theorem 4.10, we will even generalize our notion of shrinking to actual replacement of the pre-join factor $\mathcal{A}_p(H)$ by related posets such as $\mathcal{S}_p(H)$ and $\mathcal{B}_p(H)$ —resulting in replacement-posets which are more general than just $\mathcal{A}_p$ -posets.

For the moment, we indicate several variations on $X_B(H)$, using just naive-shrinking.

First, inside $X_B(H)$, we can shrink $\mathcal{A}_p(H)$ to the subposet $i(\mathcal{A}_p(H))$ of Definition 2.12.

Proposition 3.7. *Under the hypotheses of Theorem 3.6, set*

$$X_B(i(H)) := i(\mathcal{A}_p(H)) \cup \mathcal{F}_B(H) \subseteq X_B(H).$$

Then, the retraction $r_1 : \mathcal{A}_p(H) \rightarrow i(\mathcal{A}_p(H))$ extends to an $N_G(H, B)$ -equivariant retraction map $\hat{r}_1 : X_B(H) \rightarrow X_B(i(H))$. In particular, we have $N_G(H, B)$ -homotopy equivalences

$$B \simeq X_B(H) \simeq X_B(i(H)) = i(\mathcal{A}_p(H)) \cup \mathcal{F}_B(H),$$

which are the identity when restricted to $i(\mathcal{A}_p(H))$.

In the subcase for $\mathcal{B} = \mathcal{A}_p(G)$, we get $N_G(H)$ -homotopy equivalences

$$\mathcal{A}_p(G) \simeq X_G(H) \simeq X_G(i(H)) = i(\mathcal{A}_p(H)) \cup \mathcal{F}_G(H).$$

Proof. Let $\hat{r}_i : X_{\mathcal{B}}(H) \rightarrow i(\mathcal{A}_p(H)) \cup \mathcal{F}_{\mathcal{B}}(H)$ be the set-map extending r_i by the identity on $\mathcal{F}_{\mathcal{B}}(H)$. We check that it is a poset map, for the corresponding extension of the ordering $<$ to $\prec$. Recall that $r_i(B) \geq B$ for all $B \in \mathcal{A}_p(H)$; and the identity is a poset map on the $\mathcal{F}_{\mathcal{B}}(H)$ -part of the union. So, it remains to consider cross-terms, which by the $\mathcal{F}$ -left property in Definition 3.4 (3) must be of form $F \prec A$, for $F \in \mathcal{F}_{\mathcal{B}}(H)$ and $A \in i(\mathcal{A}_p(H))$. Then, we have $\hat{r}_i(F) = F \prec A \leq r_i(A) = \hat{r}_i(A)$. Therefore, using transitivity, $\hat{r}_i$ is order-preserving. Further, $\hat{r}_i(A) \geq A$, for all $A \in X_{\mathcal{B}}(H)$: that is, we have $\hat{r}_i \geq \text{Id}_{X_{\mathcal{B}}(H)}$; the equivariant-retraction follows from this (cf. Lemma 2.7 (2)). ■

Here is one sample use of the above further-shrinking proposition.

Example 3.8 (Constructing $X_{\text{Sym}_5}(i(\text{Alt}_5))$). Let $G = \text{Sym}_5$ and $H = \text{Alt}_5$, with $p = 2$. The choice $\mathcal{B} = \mathcal{A}_2(G)$ satisfies the “In particular” hypotheses of Theorem 3.6, and hence also of Proposition 3.7—from which (using normality of H in G), we obtain G -homotopy equivalences

$$\mathcal{A}_2(G) \simeq X_G(i(H)) \simeq i(\mathcal{A}_2(H)) \cup \mathcal{F}_G(H).$$

We can use this equivalent form to explicitly determine the G -homotopy type of $\mathcal{A}_2(G)$. Note that $\mathcal{A}_2(G)$ is obtained from $\mathcal{A}_2(H)$ by adding $\binom{5}{2} = 10$ involutions. That is, $|\mathcal{F}_G(H)| = 10$, and the elements of this set correspond to the involutions of $G \setminus H$ (that is, the transpositions). Now, the Sylow 2-subgroups of Alt_5 are elementary, so here we get $\mathcal{A}_2(H) = \mathcal{S}_2(H)$; and recall from Definition 2.12 that $i(\mathcal{S}_2(H))$ is the poset of 2-Sylow intersections in H . So, since the distinct conjugates of the 5 Sylow 2-subgroups of Alt_5 intersect trivially, the poset $i(\mathcal{A}_2(H))$ is here just a discrete set with those 5 points (and indeed it coincides with $\mathcal{B}_2(H)$). For each $A \in \mathcal{F}_G(H)$, there are exactly three maximal elementary abelian groups containing some involution commuting with A , since $C_H(A) \cong \text{Sym}_3$. Therefore, $X_G(i(H))$ is in fact a connected bipartite graph with $10 + 5$ points and $3 \cdot 10 = 30$ edges. Hence, computing with the G -equivalent $X_G(i(H))$ in place of $\mathcal{A}_2(G)$: we see $\tilde{H}_1(\mathcal{A}_2(G))$ has dimension $30 - 15 + 1 = 16$, and $\mathcal{A}_2(G)$ is G -homotopy equivalent to a bouquet of 16 1-spheres.

We mention that this graph $X_G(i(H))$ is in fact the “triples-geometry” for Sym_5 , as described in [24, Example 2.3.5], which has been used in various places in the geometric literature.

Finally, we note that this $X_G(i(H))$ can also be realized as $W_G^{\mathcal{B}}(H, 1)$ in our later construction in Theorem 4.10. ■

Another interesting variant of $X_G(H)$ uses the context of the poset introduced at [25, p. 134]; notice it does *not* force members of $\mathcal{F}_G(H)$ to the left end of $\prec$ -chains, but instead leaves them on the right (that is, the left end of $\succ$ -chains). We get the relevant analogue of Theorem 3.6.

Proposition 3.9 (Thévenaz poset). *Let $H \leq G$. Define the poset $\hat{X}_G(H)$ as the disjoint union of $\mathcal{A}_p(H)$ and $\mathcal{F}_G(H)$, and with the following ordering $\prec := \prec_{\hat{X}_G(H)}$:*

- (1) *inside $\mathcal{A}_p(H)$, as $\prec$, we keep the subgroup-inclusion ordering $<$;*
- (2) *inside $\mathcal{F}_G(H)$, as $\prec$, we use the opposite of $<$ —that is, we use group-containment ordering $>$;*
- (3) *when $F \in \mathcal{F}_G(H)$, we set $A \prec F$ for all $A \in \mathcal{A}_p(C_H(F))$.*

Then, $\mathcal{A}_p(G) \simeq \hat{X}_G(H)$ (and indeed this is an $N_G(H)$ -equivalence).

Proof. There will be some limited analogies with arguments from the proof of Theorem 3.6. Indeed, since here we are in effect using the full $\mathcal{A}_p(G)$ in the role of “ $\mathcal{B}$ ” there, the arguments will be easier. In particular, we mentioned there that its hypothesis (i) is automatic for $\mathcal{A}_p(G)$.

The proof of transitivity of $\prec_{\hat{X}_G(H)}$ here is roughly dual to that for $\prec_{X_G(H)}$ in Definition 3.4—due to the reversal of the usual ordering in $\mathcal{F}_G(H)$. As before, we only need to check cross-terms; so suppose $A \prec F$, so that $A \leq C_H(F)$. Given an extension $B < A$, we have $B < A \leq C_H(F)$, so that also $B \prec F$. And similarly, given $F > E$, we have $A \leq C_H(F) \leq C_H(E)$, so that also $A \prec E$. By the transitivity, $\hat{X}_G(H)$ is indeed a poset with this order-relation.

Next, in order to construct a homotopy equivalence between $\mathcal{A}_p(G)$ and $\hat{X}_G(H)$, we are going to work with face posets, that is, the poset of chains from our original posets; this should help avoid confusion, since we have reversed some orders in the new poset $\hat{X}_G(H)$. Concretely, we first abbreviate our original poset by $Y := \mathcal{A}_p(G)$, with the usual ordering $<$, and set $X := \hat{X}_G(H)$, with the new ordering $\prec$ above. The set $\mathcal{F} := \mathcal{F}_G(H)$ is a subset of both Y and X —and indeed a subposet of each, though with opposite orderings $<$ and $\prec_{\mathcal{F}_G(H)}$ (namely, $>$). We now consider chains in these posets. For a $\prec$ -chain $a \in \text{sd}(Y)$, we define $a \cap H := \{A \cap H : A \in \mathcal{N}_G(H) \cap a\}$ and $a \cap \mathcal{F} := \{A \in a : A \in \mathcal{F}\}$. Recall that the ordering from a on the set $a \cap \mathcal{F}$ is given by group-inclusion $<$.

Now, for our homotopy equivalence, define the map

$$\beta : \text{sd}(Y) \rightarrow \text{sd}(X), \quad \beta(a) := (a \cap H) \cup (a \cap \mathcal{F})_{\prec},$$

where the subscript $\prec$ indicates reversing the $<$ -ordering on $a \cap \mathcal{F}$ to $>$, to get the $\prec$ -ordering. Notice that this map is essentially the deflation map, as in the equivalence map $\alpha_{\mathcal{B},H}$ in the proof of Theorem 3.6.

We see that β is an order-preserving map (from $<$ to $<$). Since $<$ is either equal, or dual, to $<$ on the two factors in the union defining the domain $\text{sd}(Y)$, we only need to check images from cross-terms $F < A$ between the two factors; here A centralizes F just by abelianness, so $\beta(A) = A \cap H \leq C_H(F)$, and hence, we get $\beta(A) < F$ for the images. It follows at the level of face posets that $\beta(a)$ is indeed a $<$ -chain in $\text{sd}(X)$.

Now, we will show that β is a homotopy equivalence. To this end, we will show that the preimage $\beta^{-1}(\text{sd}(X_{\leq x}))$ is contractible for all $x \in X$. And then in summary, in the language used by McCord in [13], the $\text{sd}(X_{\leq x})$ as x varies give a basis-like open cover, and by [13, Theorem 6] we will be able to conclude that β is a homotopy equivalence.

We split the contractibility proof into the two usual cases for x . Because we use lower links (with $\leq x$), rather than upper links (with $\geq x$) as in the proof of Theorem 3.6, the preimages here are slightly different in form.

- $x = A \in \mathcal{A}_p(H)$. In this case, we have

$$X_{\leq A} = \mathcal{A}_p(H)_{\leq A} = \mathcal{A}_p(A).$$

So that for the preimage, we get

$$\beta^{-1}(\text{sd}(X_{\leq A})) = \text{sd}(Z), \quad \text{where } Z := Z_A = \{B \in \mathcal{N}_G(H) : 1 < B \cap H \leq A\}.$$

Then, Z is conically contractible, essentially via Quillen's homotopy

$$B \geq B \cap H \leq A.$$

The proof is basically dual to that in Example 2.14. Namely, we apply Lemma 2.7 condition (1) to the corresponding maps j_0, j_1, j_2 ; they are visibly poset maps, with $j_0 \geq j_1 \leq j_2$. For the endomorphism requirement: note using the description of Z displayed above that we get $1 < B \cap H \leq A$; and then as $(B \cap H) \cap H = B \cap H$, we see again using that description of Z that we also get $j_1(B) = B \cap H \in Z$ for the middle term $B \cap H$ above. This completes the hypotheses of Lemma 2.7 (1), and by that result, we get contractibility of $Z = Z_A$.

- $x = F \in \mathcal{F}$. Here, we have

$$X_{\leq F} = \mathcal{A}_p(C_H(F)) \cup (\mathcal{F}_{<}^{\text{op}})_{\leq F} = \mathcal{A}_p(C_H(F)) \cup \mathcal{F}_{\geq F}.$$

Therefore, for the preimage we get an analogue of (ii) in the proof of Theorem 3.6; namely, we have $\beta^{-1}(\text{sd}(X_{\leq F})) = \text{sd}(Z)$, where

$$(ii') \quad Z := Z_F = \{B \in \mathcal{N}_G(H) : 1 < B \cap H \leq C_H(F)\} \cup \mathcal{F}_{\geq F}.$$

We claim Z is contractible, via the homotopy zigzag abbreviated by the group-inclusions

$$D \geq C_D(F) \leq C_D(F)F \geq F$$

as D runs over Z . Recall this is the sequence that arose in the case “ $x = F$ ” in the proof of Theorem 3.6; and indeed the proof now parallels that earlier one: we show using Lemma 2.7 (1) that Z is contractible.

Again, j_0, j_1, j_2, j_3 are clearly poset maps, with $j_0 \geq j_1 \leq j_2 \geq j_3$. And again $j_0 = \text{Id}_Z$, and the constant map j_3 to $F \in Z$, visibly satisfy the endomorphism requirement.

So, it remains to establish the endomorphism requirement for j_1, j_2 ; that is, we must show that $j_1(D) = C_D(F)$ and $j_2(D) = C_D(F)F$ lie in Z . This time, since here $\mathcal{A}_p(G)$ plays the role of “ $\mathcal{B}$ ” there, we automatically get step (a) of the earlier argument; so it remains to obtain steps (b) and (c) there. In the subcase for Z in (ii') where $D = E \in \mathcal{F}_{\geq F}$, we again have $C_E(F) = E = C_E(F)F$ —so again the earlier argument goes through. In the other subcase of (ii'), we have $D = B \in \mathcal{N}_G(H)$ with $1 < B \cap H \leq C_H(F)$. Then,

$$1 < B \cap H \leq (C_B(F) \cap H) \cap C_H(F),$$

which shows using (ii') that also $j_1(B) = C_B(F) \in Z$ —giving step (b). Step (c) follows, on replacing $C_B(F)$ in the above display by $C_B(F)F = j_2(B)$. This completes the proof of contractibility of Z in the case $x = F$.

We have now shown that for every set $U \in \mathcal{U} := \{\text{sd}(X_{\leq x}) : x \in X\}$, $\beta^{-1}(U)$ is contractible. Here, $\mathcal{U}$ is a basis-like open cover: this means that it is a basis for a topology. That is, if $U_1, U_2 \in \mathcal{U}$ and we have $x \in U_1 \cap U_2$, then there exists $U_3 \in \mathcal{U}$ such that $x \in U_3 \subseteq U_1 \cap U_2$. To illustrate this situation in our present case: if $U_1 = \text{sd}(X_{\leq B_1})$, $U_2 = \text{sd}(X_{\leq B_2})$, and $x \in U_1 \cap U_2$, then for the $\prec$ -maximal element of x , we have $\max(x) \leq B_1, B_2$, so that we get $x \in \text{sd}(X_{\leq \max(x)}) \subseteq \text{sd}(X_{\leq B_1}) \cap \text{sd}(X_{\leq B_2})$, and further that $U_3 := \text{sd}(X_{\leq \max(x)}) \in \mathcal{U}$.

So, since we have contractibility for this basis-like cover, [13, Theorem 6] applies, and we conclude that β is a homotopy equivalence.

Finally, to show that β is an $N_G(H)$ -homotopy equivalence, note first that β is an $N_G(H)$ -equivariant poset map. Second, for $K \leq N_G(H)$, we have that

$$\mathcal{U}^K := \{\text{sd}(X_{\leq x}^K) : x \in X^K\}$$

is a basis-like open cover of the K -fixed subposet $(\text{sd}(X))^K$ (notice that we always have $|\text{sd}(X)|^K = |(\text{sd}(X))^K|$). Then, the same proof as above shows indeed that the restriction $\beta^K : (\text{sd}(Y))^K \rightarrow (\text{sd}(X))^K$ verifies that $\beta^K(\text{sd}(X_{\leq x}^K))$ is contractible for all $x \in X$: since if K fixes B, D, F as above, then K fixes $B \cap H, C_D(F)$ and

$C_D(F)F$. Again, by [13, Theorem 6], we see that β^K is a homotopy equivalence. Since $K \leq N_G(H)$ is arbitrary, an application of the equivariant Whitehead's theorem allows us to conclude that β is an $N_G(H)$ -homotopy equivalence. ■

Equivalences via removal of points with conical centralizers

In the final part of the section, we turn to a different kind of replacement-poset for $\mathcal{A}_p(G)$. As in the later propagation analysis (cf. Hypothesis 4.2), our background aim is to analyze the homotopy type of $\mathcal{A}_p(G)$ with respect to a fixed component L of G . This time, however, our equivalent subposet is obtained by using the Quillen fiber Theorem 2.10 to “homotopically remove” only certain members $\mathcal{N}$ of the inflation $\mathcal{N}_G(L)$ of L (rather than applying deflation, which was the method of the previous subsection)—namely, those which are faithful on L , and are suitably associated with “conical centralizers”. Here is some further background from the literature, regarding this distinction.

Remark 3.10 (Removing points via conical centralizers). We quickly preview a particular feature in our further discussion of homology propagation in the early part of Section 4. Namely, we often have the situation in Hypothesis 4.2, of a component L of G , with a p -outer $B \in O_G(L)$; and we study the resulting p' -central product $H * K$, where $H := LB$ and $K := C_G(LB)$ as in Hypothesis 4.4. If $\mathcal{A}_p(K) = \emptyset$, then $\mathcal{A}_p(HK)$ in fact just reproduces $\mathcal{A}_p(H)$, so that we need no further constructions for HK . Thus, our development below is really for the case that $\mathcal{A}_p(K) \neq \emptyset$.

Now, in view of results like the almost simple case in Theorem 2.21, we can usually begin with knowledge of non-zero homology $\tilde{H}_*(\mathcal{A}_p(H))$ for our $H = LB$. However, we need to deal with the possibility that this homology could become zero in $\mathcal{A}_p(G)$, from calculating in cones over those cycles—arising from points of $\mathcal{A}_p(K)$ for our $K = C_G(LB)$. Instead, we would like to preserve non-zero homology, by having at least double-cones, i.e., suspensions, over those cycles. One obvious special case of this cone problem arises when $O_p(K) > 1$: so that $\mathcal{A}_p(K)$ is contractible to a point $*_K$ by Quillen's theorem, and then we cannot produce non-zero homology cycle by joining cycles of $\mathcal{A}_p(LB)$ and $\mathcal{A}_p(K)$. However, when $O_p(K) = 1$, we can hope instead for non-zero cycles for $\mathcal{A}_p(K)$ —giving non-zero product-homology cycles for $\mathcal{A}_p(HK)$, which we can then hope to propagate to non-zero homology for $\mathcal{A}_p(G)$. Correspondingly, adapting the terminology of [4], we say K is *conical* if $O_p(K) > 1$; and K is *non-conical* otherwise, i.e., if $O_p(K) = 1$.

Therefore, for classical homology propagation, we need to start (as just indicated) with $O_p(K) = 1$ and indeed $\tilde{H}_*(\mathcal{A}_p(K)) \neq 0$: see, e.g., hypothesis (2) of Lemma 5.11 (quoted from [4]), and hypothesis (v) of Lemma 5.14 (quoted from [16]); and compare also hypothesis (3) in our new pre-join propagation Theorem 5.10. Ideally, we obtain this non-zero homology essentially by induction, when we have the

non-conical situation $O_p(K) = 1$. So, we often have a natural case division, where the more propagation-friendly branch corresponds to the following.

(someNC) There is some $B \in O_G(L)$ with $O_p(C_G(LB)) = 1$.

For example, [4, Theorem 2.4] (existence of “non-conical complements”) is a crucial tool, which in effect establishes (someNC)—allowing for the homology propagation required in the proof. Furthermore, earlier Lemma 2.5 in this paper helps to extend non-conicality to further subgroups.

The other, less propagation-friendly, branch (which we instead might call ‘removal-friendly’) of the case division is as follows.

(allC) For all $B \in O_G(L)$, we have $O_p(C_G(LB)) > 1$.

Some version of (allC) arises in various results in the literature; and we could say roughly that it is more relevant to homotopy equivalence methods, typically based on using the Quillen fiber Theorem 2.10; the equivalence is with a subset of $\mathcal{A}_p(G)$, obtained by removing points corresponding to the conical centralizers. Suitable such equivalences are implemented in a number of ways in the literature—cf. [16, Theorems 4.1, 5.1, and 6.1]. And Proposition 3.11 below produces such equivalences for situations satisfying an upper-link version of (allC). ■

In Proposition 3.11 below, we provide a somewhat flexible homotopy equivalence result: where the set $\mathcal{F}_1$ corresponds to members of $\mathcal{A}_p(G)$ with the above condition (allC); and the conclusion allows for some leeway in choosing the set $\mathcal{N}$ of points to be removed. This equivalence is used in later Proposition 4.12, with $\mathcal{N} = \mathcal{N}_1$ itself; and in Theorem 7.2, with $\mathcal{N}$ a possibly proper subset of $\mathcal{N}_1$.

We mention that the first condition $E \in \mathcal{A}_p(N_G(L))$ in the definition of $\mathcal{N}_1$ below is actually redundant: for in view of Lemma 2.4, we get E normalizing the component L from the second condition $E \cap L > 1$ —once we use the third (“faithful”) condition $C_E(L) = 1$, to guarantee that $E \cap L \not\leq Z(L)$. However, we keep that condition about $N_G(L)$ in the definition, as it provides some natural context for those later conditions.

Proposition 3.11. *Let G be a group, with a component L of order divisible by the prime p . Define $\mathcal{F}_1$ as the subset of p -outers of L with $C_G(LB)$ conical*

$$\mathcal{F}_1 := \{B \in O_G(L) : O_p(C_G(LB)) > 1\}.$$

Note that $\mathcal{F}_1$ is upward-closed in $O_G(L)$, by the contrapositive of Lemma 2.5.

Define the subset $\mathcal{N}_1$ of members of the inflation $\mathcal{N}_G(L)$ acting faithfully on L , and containing some member of $\mathcal{F}_1$:

$$\begin{aligned} \mathcal{N}_1 &:= \{E \in \mathcal{A}_p(N_G(L)) : E \cap L > 1, C_E(L) = 1, \exists B \in \mathcal{F}_1 : B < E\} \\ &\subseteq \mathcal{N}_G(L) \cap \mathcal{A}_p(N_G(L)). \end{aligned}$$

Note that the condition $B < E$ guarantees that $\mathcal{N}_1 \cap \mathcal{A}_p(LC_G(L)) = \emptyset$.

Assume that $\mathcal{N}$ is an upward-closed subset of $\mathcal{N}_1$. Then, $\mathcal{A}_p(G) \setminus \mathcal{N} \hookrightarrow \mathcal{A}_p(G)$ is a homotopy equivalence. Furthermore, $\mathcal{A}_p(LC_G(L)) \subseteq \mathcal{A}_p(G) \setminus \mathcal{N}$.

In particular, for $\mathcal{N} = \mathcal{N}_1$ we see that $\mathcal{A}_p(G) \setminus \mathcal{N}_1 \hookrightarrow \mathcal{A}_p(G)$ is a homotopy equivalence.

Proof. We abbreviate $Y := \mathcal{A}_p(G) \setminus \mathcal{N}$. Note since $\mathcal{N}_1 \cap \mathcal{A}_p(LC_G(L)) = \emptyset$ that $\mathcal{A}_p(LC_G(L)) \subseteq Y$, as required.

We will obtain the desired equivalence using the Quillen fiber Theorem 2.10, namely, we will show for any $E \in \mathcal{A}_p(G) \setminus Y = \mathcal{N}$ that $Y_{>E}$ is contractible. Recall since $E \in \mathcal{N} \subseteq \mathcal{N}_1$ that we have

$$E \leq N_G(L), \quad E \cap L > 1, \quad C_E(L) = 1 \text{ and there is some } B \in \mathcal{F}_1 \text{ such that } B < E.$$

We begin by showing that $Y_{>E}$ is homotopy equivalent to $\mathcal{A}_p(C_G(LE))$. For this, we will directly construct poset maps in each direction, which are homotopy inverses. So, consider any A in the domain space $Y_{>E}$. Then, $A \cap L \geq E \cap L > 1$. But $A \cap L \not\leq Z(L)$, since $E \cap Z(L) \leq C_E(L) = 1$; so $A \leq N_G(L)$ by Lemma 2.4. Further $B < E < A$. Thus, we have for A three of the four defining properties for membership in $\mathcal{N}_1$. So, if we had the remaining faithful property $C_A(L) = 1$, we would then get $A \in \mathcal{N}_1$; but then since $E \in \mathcal{N} \subseteq \mathcal{N}_1$ with $E < A$, and $\mathcal{N}$ is upward-closed in $\mathcal{N}_1$ by hypothesis, we would get $A \in \mathcal{N}$ —contrary to its choice in $Y = \mathcal{A}_p(G) \setminus \mathcal{N}$. This contradiction shows that $C_A(L) > 1$. Now, $A = C_A(E)$ since $A > E$; so $C_A(LE) = C_A(L) \cap C_A(E) = C_A(L) > 1$. Thus, $a : A \mapsto C_A(LE)$ is a poset map from $Y_{>E}$ to $\mathcal{A}_p(C_G(LE))$. For the other direction, consider now some $C \in \mathcal{A}_p(C_G(LE))$. Since $C_E(L) = 1$, we have $CE > E$; and also $C_{CE}(L) = C > 1$, so that $CE \notin \mathcal{N}_1$ —and hence $CE \notin \mathcal{N}$; so we conclude that $CE \in Y_{>E}$. Then, $b : C \mapsto CE$ is a poset map from $\mathcal{A}_p(C_G(LE))$ to $Y_{>E}$. Note that $ba(A) = C_A(LE)E \leq A$ and $ab(C) = C_{CE}(LE) = C$; so that the two maps are homotopy inverses, giving the desired homotopy equivalence.

Using this equivalence, the contractibility of $Y_{>E}$ will follow once we show that $O_p(C_G(LE)) > 1$.

Case 1. Assume E contains an element $e \in LC_G(L) \setminus L$. Then, $O_p(C_G(LE)) > 1$.

Here, we write $e = lc$, where $l \in L$ and $c \in C_G(L)$. Since e has order p , but $e \notin L$, we see that c has non-trivial p -power order. We will show that $c \in Z(C_G(LE))$; so consider any $x \in C_G(LE)$:

$$lc = e = e^x = l^x c^x = lc^x,$$

which forces $c = c^x$. This shows $c \in Z(C_G(LE))$. So,

$$O_p(C_G(LE)) \geq O_p(Z(C_G(LE))) > 1,$$

since we saw c is a non-trivial p -element. This finishes the proof of Case 1.

Case 2. Otherwise, $E \cap (LC_G(L) \setminus L) = \emptyset$. And again, we get $O_p(C_G(LE)) > 1$.

Here, the hypothesis implies that when $e \in E$ induces an inner automorphism of L , then $e \in L$. And so, if we write $E = (E \cap (LC_G(L)))C$, where $C \in O_G(L)$, we see that $E \cap (LC_G(L)) = E \cap L$; hence, $LE = LC$. Moreover, since $B < E$, we can choose C so that $C \geq B$. Since $B \in \mathcal{F}_1 \subseteq O_G(L)$, and $C \in O_G(L)_{\geq B}$, we conclude that $O_p(C_G(LE)) = O_p(C_G(LC)) > 1$, by the upward-closed property of $\mathcal{F}_1$. This finishes the proof of Case 2.

In summary, we have shown that $Y_{>E} \simeq \mathcal{A}_p(C_G(LE)) \simeq *$, since by Cases 1 and 2 we have $O_p(C_G(LE)) > 1$. To complete the proof, apply the Quillen fiber Theorem 2.10 to the inclusion $Y \hookrightarrow \mathcal{A}_p(G)$. ■

4. The pre-join construction and some corresponding replacement-posets

In the next section, we will prove Theorem 5.10. This is essentially a variant of Lemma 5.11, namely, the original homology propagation result [4, Lemma 0.27] of Aschbacher–Smith—but now using, in place of the usual join of posets, the context of the “pre-join”, which we develop in this section.

Furthermore, Theorem 5.10 is stated in the language just of posets—which now do not have to be $\mathcal{A}_p$ -posets, in contrast to “classical” propagation of [4]. In our context of (H-QC), this allows us to use more general replacement-posets “ W ” equivalent to $\mathcal{A}_p(G)$, in the spirit of the various equivalent constructions “ X ” in the previous Section 3. So, the present section, in particular, develops some replacement-posets which use the pre-join; notably in Theorem 4.10 and Proposition 4.12.

Background: The classical poset join in homology propagation

Before introducing the pre-join, we will first review the more standard context of the usual poset join in homology propagation.

We start with a brief continuation of our earlier more general viewpoint just on propagation.

Remark 4.1 (Some further broad context on homology propagation). Recall from Remark 3.1, we typically have a suitable proper subgroup $H < G$ with $\widetilde{H}_*(\mathcal{A}_p(H)) \neq 0$; and we wish to show that this non-zero reduced homology “propagates” to non-zero homology also for $\mathcal{A}_p(G)$. More precisely, this means that the natural inclusion $i : \mathcal{A}_p(H) \rightarrow \mathcal{A}_p(G)$ should induce a non-zero map i_* in homology. In that earlier remark, we focused on the simplest case, where i is a homotopy equivalence, and i_* an isomorphism; but now we wish to examine more general situations for i_* .

Note first that i_* can be non-zero, without even needing to be injective; that is, it suffices to have just the (non-constructive) *existence* of a non-zero image i_* in homology. For example, this is the case in Theorem 2.24 quoted from [18]. (We mention also that the knowledge of non-zero homology for $\mathcal{A}_p(H)$ in that result is also non-constructive: it relies on the existential result Theorem 2.20 quoted from [3]—applied to $H = \text{Aut}_G(L)$, for L a component of G .)

However, it is in fact more common in the literature to have the propagation¹⁰ be more constructive by showing i_* is essentially a monomorphism in some particular non-zero degree.

For historical background, we illustrate this via Quillen's [19, Corollary 12.2], which in fact essentially inspired the study of homology propagation. There we have G solvable, with $H := LA$, where $L := O_{p'}(G)$ and A is of maximal rank m in $\mathcal{A}_p(H)$ (which is also the maximal rank for $\mathcal{A}_p(G)$). In particular, we get the following description of the upper link of A :

$$(i) \quad \mathcal{A}_p(G)_{>A} = \emptyset,$$

just from the particular maximality condition that we have assumed for A .

Next, the construction there of non-zero reduced homology for $\mathcal{A}_p(H)$ (basically the property $(\mathcal{QD})_p$) gives us a non-zero cycle α in the topological dimension $m - 1$ of $\mathcal{A}_p(H)$: this must involve some poset-chain a with m members, so we may take our $A := \max(a)$. So, suppose, by way of contradiction to (H-QC), that α is zero in $\tilde{H}_*(\mathcal{A}_p(G))$ (that is, in the kernel of i_*). Then, α would appear in the image under the boundary operator of some longer chain b with $(m + 1)$ members from $\mathcal{A}_p(G)$, and with $b \in \text{Lk}_{\mathcal{A}_p(G)}(a)$.

We examine this situation using the later viewpoint of [4]. Note since a has m members, with $\max(a) = A$ of rank m , those members have consecutive orders $p, p^2, \dots, p^m$. So that naively, there is “no room” to insert further members of $\mathcal{A}_p(G)$ either among, or below, those of a , but only above. We state this condition in the language of “full chain” at later Definition 5.1 (2).

$$(ii) \quad \text{Lk}_{\mathcal{A}_p(G)}(a) \subseteq \mathcal{A}_p(G)_{>A}, \text{ where } A = \max(a) \text{ (i.e., } a \text{ is a full chain in } \mathcal{A}_p(G)).$$

And now applying (i) in (ii), we see that no such longer chain b is possible. This contradiction shows that α is non-zero for $\mathcal{A}_p(G)$, giving (H-QC); and, in particular, shows that i_* is injective in this degree. (Even in all degrees, since here $\mathcal{A}_p(LA)$ is a wedge of spheres of dimension $m - 1$.)

The above interaction of the maximality condition on A from $(\mathcal{QD})_p$, giving the upper-link condition in (i), with a relevant restriction in (ii) on the structure of links of simplices, is essentially the starting point for later extensions of the notion of this

¹⁰And indeed the non-zero homology of $\mathcal{A}_p(H)$.

elementary propagation. So, our continuing exposition below will follow this interaction theme; up through the analysis in Remark 5.5, which underlies our later main propagation result Theorem 5.10. ■

The use of the usual poset join in propagation originates with Aschbacher–Smith [4], where the argument indicated above is adapted to cover the case where Quillen’s $L = O_{p'}(G)$ is replaced with a component L of G . The adjustment, in particular, replaces the single subgroup “ H ” above, with a p' -central product $H * K$; and then the corresponding poset $\mathcal{A}_p(H * K)$ is homotopy equivalent to the usual join $\mathcal{A}_p(H) * \mathcal{A}_p(K)$. So, as further background for our later pre-join variants, we now indicate some relevant details in their development.

Thus, we consider a component L of a counterexample G to (H-QC). We’ll need to consider possible p -outers of L ; so we replace “ A ” above by some $B \in \widehat{O}_G(L)$ as in Definition 2.17 (which covers the p -outers $B \in O_G(L)$, as well as the identity case $B = 1$). Hence, the basic context for propagation in [4] is the component-hypothesis.

Hypothesis 4.2. L is a component of a counterexample G to (H-QC), with $B \in \widehat{O}_G(L)$. ■

Recall that, in Remark 3.10, we had also indicated the relevance, for propagation, of the product of LB with $C_G(LB)$. In Lemma 4.3 below, we will see (postponing momentarily the easy proof) that the above component—Hypothesis 4.2, via the product of LB with $C_G(LB)$, gives our standard case of a somewhat more general $\mathcal{A}_p$ -poset Hypothesis 4.4. This latter product-hypothesis is what is actually assumed (as [4, Hypothesis 0.15]) in the propagation lemma [4, Lemma 0.27] (which we have quoted as later Lemma 5.11).

Lemma 4.3. *Assume the component-Hypothesis 4.2 for L , B . As H , choose a subgroup of LB (ideally LB or even L when $B > 1$), with $K := C_G(LB)$ for any choice. Then, we also have the following central product Hypothesis 4.4 for H , K .*

Hypothesis 4.4 (p' -central product). We have $H, K \leq G$, with $[H, K] = 1$ and $H \cap K$ a p' -group. In particular, we have a p' -central product $HK = H * K$. ■

Proof of Lemma 4.3. Notice that $[H, K] = 1$ from the hypothesis $K = C_G(LB)$ of the lemma. We have

$$H \cap K \leq LB \cap C_G(LB) = Z(LB) \leq Z(L),$$

since B is faithful (including when $B = 1$). The intersection $H \cap K$ is then a p' -group since $O_p(L) \leq O_p(G) = 1$ in our counterexample to (H-QC). ■

In fact, in the proof of [4, Proposition 1.7], we also have $O_{p'}(G) = 1$ —so as in Lemma 2.3 (1), in this case the product $HK = H \times K$ is even direct.

Since we will be using such central products within the Cartesian product context of the pre-join below, we will frequently make use of the property:

$$\text{Under Hypothesis 4.4, for } A \in \mathcal{A}_p(HK), \text{ we have } A \leq p_H(A) \times p_K(A), \quad (4.1)$$

where p_H and p_K denote the respective projections.

This central product situation of $H * K$ leads us to analyze some features of the usual join.

Remark 4.5 (Homology propagation using the classical join of $\mathcal{A}_p$ -posets). We continue to study certain aspects of the proof of [4, Proposition 1.7]—the elimination of $(\mathcal{QD})_p$ -components; under Hypothesis 4.2 (and hence, Hypothesis 4.4, in view of Lemma 4.3). Recall in that result we assume also that $H = LB$ has $(\mathcal{QD})_p$. We follow the interaction viewpoint of Remark 4.1.

Here, “Step iii” at [4, p. 489] shows that we may choose an $A \in \mathcal{A}_p(LB)$ with $LA = LB = H$, so that A exhibits $(\mathcal{QD})_p$ for H . And then with α and a as in the Quillen case in Remark 4.1, as before we get condition (ii) of that remark.

(ii') Again, $\text{Lk}_{\mathcal{A}_p(G)}(a) \subseteq \mathcal{A}_p(G)_{>A}$ (that is, a is a full chain in $\mathcal{A}_p(G)$).

However, we can no longer expect to get condition (i) of Remark 4.1. For example, consider larger products of A with any $C \in \mathcal{A}_p(K)$, where $K = C_G(LB)$. Instead, the argument at [4, p. 490] (referring back to [4, p. 488]) obtains a suitable replacement for (i).

(i') $\mathcal{A}_p(G)_{>A} \subseteq A \times K$.

We mention that in the replacement-poset language of earlier Definition 3.4, with HK in the role of “ H ”, this amounts to showing that¹¹

$$\mathcal{A}_p(G)_{>A} \cap \mathcal{F}(HK) = \emptyset \quad \text{so that } \mathcal{A}_p(G)_{>A} \subseteq \mathcal{A}_p(H * K) \simeq \mathcal{A}_p(H) * \mathcal{A}_p(K).$$

And this suffices: for the original propagation lemma [4, Lemma 0.27] (which we have quoted as Lemma 5.11) assumes (i') as its first hypothesis. That lemma shows roughly that $A * K$ -terms in (i') do not obstruct propagation. Namely, if a shuffle product¹² $\alpha \times \beta$ (with β a non-zero cycle in $\tilde{H}_*(\mathcal{A}_p(K))$) is a boundary for $\mathcal{A}_p(G)$, then the “global” boundary calculation for $\alpha \times \beta$ from $\mathcal{A}_p(G)$ reduces via the poset join

¹¹Notice the similarity of this condition with the displayed condition before Definition 3.2 (i). This starts to link our earlier equivalence methods with propagation conditions; the theme continues, e.g., at (i'') below.

¹²The classical shuffle product (see [4, Definition 0.21]) implements product homology for the join as in Remark 2.11. This amounts to recovering the Künneth formula for the join of spaces (cf. [7, 14]).

$\mathcal{A}_p(H) * \mathcal{A}_p(K)$ to a parallel calculation from $\mathcal{A}_p(K)$ —showing that β a boundary, contrary to its non-zero choice. This contradiction establishes (H-QC).

We mention that Lemma 5.14 (the quoted result [16, Lemma 3.14]) generalizes the arguments above, in several ways relevant to our exposition so far. First, the propagation from $\mathcal{A}_p(H) * \mathcal{A}_p(K)$ is not necessarily to $\mathcal{A}_p(G)$, but instead to a suitable subposet X —which must then be shown to be homotopy equivalent to $\mathcal{A}_p(G)$ (as in the X -viewpoint in Remark 3.1). Further, the sufficient condition for propagation now takes the related form:

$$(i'') \quad X_{>A} \subseteq \mathcal{N}_G(K),$$

in the language of Definition 3.2. Finally, the assumption of $(\mathcal{QD})_p$ for H is weakened essentially to its consequence of a being a full chain with $A = \max(a)$; that is, we have the relevant analogue of Remark 4.1 (ii):

$$(ii'') \quad \text{Lk}_X(a) \subseteq X_{>A}, \text{ where } A = \max(a) \text{ (} a \text{ is a full chain in } X \text{)}.$$

The proof of propagation in this situation then proceeds via a similar reduction of the boundary calculation, via the shuffle product for the link, to $\mathcal{A}_p(K)$. ■

The pre-join and some basic properties using the Cartesian product

With the above classical context in place, we now turn to the extended viewpoint of the present paper. For applications in the next section (cf. Hypothesis 5.2 there), we will start with the basic situation of Hypothesis 4.2 for a component L , along with a non-trivial p -outer $B \in \text{O}_G(L)$ —but in Lemma 4.3, we will now usually make the non-classical choice of $H := L$ and $K := C_G(LB)$. We emphasize that by that lemma, *we will still get the central product Hypothesis 4.4 for this choice of H , K* —as needed for the hypothesis of the results given later in this section.

In fact, our main new propagation result Theorem 5.10 will use the pre-join, in generalizing the classical context: we will replace $\mathcal{A}_p(G)$ with non-Quillen equivalent posets such as $W_G^{\mathcal{B}}(H, K)$ in Theorem 4.10 (a further development beyond $X_G(HK)$ in earlier Definition 3.4). This will give us more flexibility for propagating non-zero homology, in the sense that we can correspondingly relax certain hypotheses that had been required in the classical Lemma 5.11 (i.e., [4, Lemma 0.27]). We can also pursue further analogs of the interaction of (i'') and (ii'') in Remark 4.5: using a more natural visual format for simplices and their links—in which the Cartesian product form of the pre-join will be more convenient for our calculations than the usual poset join. These extensions to propagation will allow us to establish new elimination results, such as Theorem 6.4—in whose proof we apply Theorem 5.10, replacing the Quillen poset for a p -extension of a Lie type group (in the same characteristic p) with the homotopy equivalent non-Quillen poset $W := W_G^{\mathcal{B}}(H, K)$ of typically smaller topological

dimension. Indeed, the climax of the argument occurs in the proof of Lemma 6.7, where a is shown to be a full chain in W —i.e., the analogue of Remark 4.5 (ii'').

We now begin the main work of the section, primarily adapting results to the pre-join viewpoint.

Inspired by the constructions of [4, 19] using the poset join, we generalize the context by further defining the *pre-join* of two general posets X, Y —which are of course inspired by the classical $\mathcal{A}_p(H), \mathcal{A}_p(K)$; but for us, they will not always be $\mathcal{A}_p$ -posets. The pre-join turns out to be homotopy equivalent to the classical join of the posets; but the pre-join will be more convenient for our later purposes.

Definition 4.6 (The pre-join of posets). Let X and Y be finite posets. Denote by C^-X the poset obtained from X by adjoining a minimum element 1. The *pre-join* of X and Y is the subposet

$$X \underline{*} Y := C^-X \times C^-Y \setminus \{(1, 1)\}$$

of the Cartesian product poset (cf. Definition 2.9). Regard X and Y as subposets of $X \underline{*} Y$ via $x \mapsto (x, 1)$ and $y \mapsto (1, y)$, respectively. ■

Note, in particular, that in the format for poset-chains in the pre-join, any terms from X itself will appear to the *left* of “mixed” terms—that is, of (x, y) with $y \neq 1$. (This continues the theme of left-focusing that we had mentioned in the earlier Remark 3.5.)

The pre-join of posets $X \underline{*} Y$ was used by Quillen to show that the join of simplicial complexes (or posets) is homotopy equivalent to the classical join of topological spaces of their geometric realizations (see [19, p. 104]). We record the G -homotopy equivalence as follows.

Proposition 4.7. *If X, Y are G -posets, then we have a G -homotopy equivalence $\Gamma : X \underline{*} Y \rightarrow X * Y$ defined by*

$$\Gamma(x, y) := \begin{cases} x \in X & \text{when } y = 1, \\ y \in Y & \text{when } y \neq 1. \end{cases}$$

Proof. This follows by applying (to the preimages of lower links) the equivariant version of the Quillen fiber Theorem 2.10. For example, if $y \in Y$, then

$$\Gamma^{-1}((X * Y)_{\leq y}) = X \times \{1\} \cup C^-X \times Y_{\leq y} = X \underline{*} Y_{\leq y}.$$

This poset is G_y -contractible via the zigzag G_y -equivariant homotopy

$$(x, z) \leq (x, y) \geq (1, y)$$

(see Lemma 2.7). ■

Next, we examine the pre-join in the context of subgroup posets of products. When X is a poset of subgroups, regard the minimum element $1 \in C^-X$ as the trivial subgroup. (Of course, 1 lies outside our usual posets of non-trivial p -subgroups, such as $\mathcal{A}_p$ -posets.)

Then, in Quillen's application of the usual poset join to the $\mathcal{A}_p$ -poset of a product, we can now also add the pre-join.

Corollary 4.8. *Assume the p' -central product Hypothesis 4.4 for H, K . Then, we have $N_G(H, K)$ -homotopy equivalences*

$$\mathcal{A}_p(HK) \simeq \mathcal{A}_p(H) \ast \mathcal{A}_p(K) \simeq \mathcal{A}_p(H) \ast \mathcal{A}_p(K).$$

Here, $\mathcal{A}_p(HK)$ poset-strongly retracts to its subposet of members which are the product of their projections; and we may in turn identify subposet with $\mathcal{A}_p(H) \ast \mathcal{A}_p(K)$.

In particular, for a direct product $G = HK = H \times K$, normality gives G -homotopy equivalences.

Proof. The first homotopy equivalence is given by the coordinate $a : A \mapsto (p_H(A), p_K(A))$, where the projection $p_H : \mathcal{A}_p(HK) \rightarrow C^- \mathcal{A}_p(H)$ is map obtained from the quotient by K . Observe, since $H \cap K$ is a central p' -subgroup of H , that $\mathcal{A}_p(H) \cong \mathcal{A}_p(H/H \cap K)$. Define p_K analogously. The map b in the reverse direction is obtained by multiplying the two coordinates; observe that $ba(A) = p_H(A)p_K(A) \geq A$ using (4.1), and $ab(C, D) = (p_H(CD), p_K(CD)) = (C, D)$, so that the maps are homotopy inverses. Note using Lemma 2.7 (2) that from $ba \geq \text{Id}_{\mathcal{A}_p(HK)}$ above, we get the retraction conclusion.

The second equivalence is that of Quillen [19, Proposition 2.6], in earlier Proposition 4.7. The assertions about the case $G = H \times K$ are then straightforward. ■

We mention that as our development proceeds, we will be seeing a number of further variations on the projection/product equivalence above.

Some replacement-posets based on the pre-join

We just gave in Corollary 4.8 the relationship of $\mathcal{A}_p(HK)$, for the central product HK of Hypothesis 4.4, with the pre-join $\mathcal{A}_p(H) \ast \mathcal{A}_p(K)$.

We will wish to apply this to the context of the replacement-poset $X_G(HK)$, as defined in the previous section. Note that from Theorem 3.6, with HK in the role of “ H ”, we get

$$X_G(HK) = \mathcal{A}_p(HK) \cup \mathcal{F}_G(HK) \text{ is } N_G(H, K)\text{-homotopy equivalent to } \mathcal{A}_p(G);$$

where we will write $\prec_X$ for the ordering in $X_G(HK)$.

We indicate how to substitute the corresponding pre-join for the term $\mathcal{A}_p(HK)$ above, so as to get a replacement-subposet $W_G(H, K)$ of $X_G(HK)$, adapted now to our pre-join viewpoint.

Proposition 4.9. *Assume the p' -central product Hypothesis 4.4 for H, K . Regard $\mathcal{A}_p(H) \ast \mathcal{A}_p(K)$ as the direct product subposet of $\mathcal{A}_p(HK)$, by the identification in Corollary 4.8.*

Then, we may similarly regard as a subposet of $X_G(HK)$ the following poset:

$$W_G(H, K) := (\mathcal{A}_p(H) \ast \mathcal{A}_p(K)) \cup \mathcal{F}_G(HK),$$

whose order-relation $\prec_W$ is the obvious corresponding re-interpretation of $\prec_X$:

- (1) inside $\mathcal{A}_p(H) \ast \mathcal{A}_p(K)$ and $\mathcal{F}_G(HK)$, $\prec_W$ agrees with $\prec_X$ (i.e., with inclusion $<$),
- (2) for $F \in \mathcal{F}_G(HK)$ and $(B, C) \in \mathcal{A}_p(H) \ast \mathcal{A}_p(K)$, we set $F \prec_W (B, C)$ iff $C_{BC}(F) > 1$ ¹³.

Using the retraction in Corollary 4.8, we see $\mathcal{A}_p(G)$ and $W_G(H, K)$ are $N_G(H, K)$ -homotopy equivalent.

Proof. Here, we are essentially observing the poset retraction $\mathcal{A}_p(HK)$ to $\mathcal{A}_p(H) \ast \mathcal{A}_p(K)$ in Corollary 4.8 extends, via the identity on $\mathcal{F}_G(HK)$, to a poset-strong retraction from $X_G(HK)$ to $W_G(H, K)$. The retraction again proceeds via the projection/product equivalence. ■

We modify $W_G(H, K)$ from Proposition 4.9, in the spirit of our earlier modification Proposition 3.7 of Theorem 3.6: this time replacing the term $\mathcal{A}_p(H)$ in the pre-join by $\mathcal{S}_p(H)$ or $\mathcal{B}_p(H)$. Note that the resulting equivalent replacement-posets are typically not $\mathcal{A}_p$ -posets: since they include pre-joins such as $\mathcal{S}_p(H) \ast \mathcal{A}_p(K)$, which do not have a natural HK -interpretation like Corollary 4.8—except possibly via subposets of

$$\mathcal{S}_p(H) \ast \mathcal{S}_p(K) \simeq \mathcal{S}_p(HK).$$

In fact, the new poset $W_G^{\mathcal{B}}(H, K)$ will be the “ W ” used to fit the hypotheses of our new propagation result Theorem 5.10, in the proof of later Theorem 6.4—more specifically in the branch of that theorem given by Lemma 6.7. (We had also mentioned, in earlier Example 3.8, that the construction there via Proposition 3.7 could also be realized in the language here of $W_{\text{Sym}_5}^{\mathcal{B}}(\text{Alt}_5, 1)$.)

¹³This definition begins to illustrate how we exploit the Cartesian product format of the pre-join; we’ll be seeing more such definitions as we proceed.

Theorem 4.10. *Assume the p' -central product Hypothesis 4.4 for H, K . Consider the posets:*

$$\begin{aligned} W_G^{\mathcal{A}}(H, K) &= (\mathcal{A}_p(H) \ast \mathcal{A}_p(K)) \cup \mathcal{F}_G(HK), \\ W_G^{\mathcal{S}}(H, K) &= (\mathcal{S}_p(H) \ast \mathcal{A}_p(K)) \cup \mathcal{F}_G(HK), \\ W_G^{\mathcal{B}}(H, K) &= (\mathcal{B}_p(H) \ast \mathcal{A}_p(K)) \cup \mathcal{F}_G(HK). \end{aligned}$$

The order-relation $<$ in each of these posets is the obvious extension of $<_W$ in Proposition 4.9:

- (1) *inside $\mathcal{S}_p(H) \ast \mathcal{A}_p(K)$ and $\mathcal{F}_G(HK)$, as $<$ we keep the ordering induced by inclusion $<$,*
- (2) *for $F \in \mathcal{F}_G(HK)$ and $(B, C) \in \mathcal{S}_p(H) \ast \mathcal{A}_p(K)$, we set $F < (B, C)$ if and only if $C_{BC}(F) > 1$.*

Thus, $W_G^{\mathcal{A}}(H, K)$ and $W_G^{\mathcal{B}}(H, K)$ are subposets of $W_G^{\mathcal{S}}(H, K)$; and from Proposition 4.9, we have

$$W_G^{\mathcal{A}}(H, K) = W_G(H, K) \simeq \mathcal{A}_p(G).$$

Then, the three posets displayed above are homotopy equivalent; thus, we have

$$\mathcal{A}_p(G) \simeq W_G^{\mathcal{A}}(H, K) \simeq W_G^{\mathcal{S}}(H, K) \simeq W_G^{\mathcal{B}}(H, K).$$

Proof. Note that in our construction, we do not get $W_G^{\mathcal{S}}(H, K)$ as a natural subposet of $X_G(HK)$ —though we could in fact construct a suitable over-poset of $X_G(HK)$, which uses $\mathcal{S}_p$ -posets rather than $\mathcal{A}_p$ -posets. However, here we really just need a poset inclusion $i : W_G^{\mathcal{A}}(H, K) \rightarrow W_G^{\mathcal{S}}(H, K)$.

So, we can begin the proof by checking transitivity of our extension of the order-relation $<_W$ from $W_G^{\mathcal{A}}(H, K) = W_G(H, K)$, to $<$ for $W_G^{\mathcal{S}}(H, K)$. And here we can just follow the transitivity proof for $<_X$ in Definition 3.4—since that argument depends just on standard properties of group-inclusion, and not on elementary-abelianness in $\mathcal{A}_p$ -posets.

Now, let i above and $j : W_G^{\mathcal{B}}(H, K) \rightarrow W_G^{\mathcal{S}}(H, K)$ be the natural inclusion maps. To complete the proof of the theorem, we will show that i, j are homotopy equivalences.

Case i. The map i is a homotopy equivalence.

Proof. We will apply the Quillen fiber Theorem 2.10 to preimages for the map i . Namely, we will show that for any $x \in W_G^{\mathcal{S}}(H, K) \setminus W_G^{\mathcal{A}}(H, K)$, we get contractibility for the corresponding preimage $i^{-1}(W_G^{\mathcal{S}}(H, K)_{\leq x}) = W_G^{\mathcal{A}}(H, K)_{< x}$. Notice that this preimage is the left-hand term of the join of form “ $f^{-1}(Y_{\leq y}) \ast Y_{> y}$ ” in our statement of Theorem 2.10.

Such an x has the form (C, D) , with $C \in \mathcal{S}_p(H) \setminus \mathcal{A}_p(H)$ (so that C is not elementary abelian), and $D \in \mathcal{A}_p(K) \cup \{1\}$. Then,

$$(i) \quad W_G^{\mathcal{A}}(H, K)_{\prec(C,D)} = (\mathcal{A}_p(H)_{\leq C}) * (\mathcal{A}_p(K)_{\leq D}) \cup \mathcal{F}_G(HK)_{\prec(C,D)} \\ = (\mathcal{A}_p(C) * \mathcal{A}_p(D)) \cup \{F \in \mathcal{F}_G(HK) : C_{CD}(F) > 1\}.$$

Now, even though C is not elementary abelian, it is a standard property of the $\mathcal{A}_p$ -poset in the case of a p -group that $\mathcal{A}_p(C) \simeq \mathcal{S}_p(C) \simeq *$. On the other hand, we could get $\mathcal{A}_p(D) = \emptyset$, when we have $D = 1$. But in any case for D , we get

$$\mathcal{A}_p(C) * \mathcal{A}_p(D) \simeq *.$$

So, it suffices to show the inclusion

$$k : \mathcal{A}_p(C) * \mathcal{A}_p(D) \hookrightarrow (\mathcal{A}_p(C) * \mathcal{A}_p(D)) \cup \mathcal{F}_G(HK)_{\prec(C,D)}$$

is a homotopy equivalence—as this gives

$$W_G^{\mathcal{A}}(H, K)_{\prec(C,D)} \simeq \mathcal{A}_p(C) * \mathcal{A}_p(D) \simeq *,$$

as desired.

To establish the equivalence for k , we will again use the Quillen fiber Theorem 2.10 to remove the elements $F \in \mathcal{F}_G(HK)_{\prec(C,D)}$ from the union by showing that $(\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\succ F}$ is contractible. Recall that such an F satisfies $C_{CD}(F) > 1$; and $\mathcal{A}_p(C_{CD}(F)) \simeq \mathcal{S}_p(C_{CD}(F))$ is contractible. So, it will even suffice to show for each such F that

$$(ii) \quad (\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\succ F} \text{ is homotopy equivalent to } \mathcal{A}_p(C_{CD}(F)) \simeq *.$$

To obtain (ii), we see that for any $(A, B) \in (\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\succ F}$, we get $1 < C_{AB}(F) \leq C_{CD}(F)$. Then, our desired homotopy equivalence will be given by a variation on the projection/product equivalence in Corollary 4.8. Here, we replace the earlier product-map b with the centralizer-adjusted variant given by the map $b' : (A, B) \mapsto C_{AB}(F) \in \mathcal{A}_p(C_{CD}(F))$; while the homotopy inverse for b' will still be given by the earlier projection map $a : E \mapsto (p_C(E), p_D(E))$ for $E \in \mathcal{A}_p(C_{CD}(F))$. For observe using (4.1) that we have $1 < E \leq p_C(E)p_D(E)$, so that we get $C_{p_C(E)p_D(E)}(F) > 1$ —and hence we obtain that $F \prec (p_C(E), p_D(E))$, giving the image $a(E) = (p_C(E), p_D(E)) \in (\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\succ F}$. Finally, note that

$$ab'(A, B) = (p_C(C_{AB}(F)), p_D(C_{AB}(F))) = (p_A(C_{AB}(F)), p_B(C_{AB}(F))) \leq (A, B).$$

For the inverse correspondingly, we have

$$b'a(E) = C_{p_C(E)p_D(E)}(F) = p_C(E)p_D(E) \cap C_{CD}(F) \geq E.$$

These homotopies complete the proof of the contractibility in (ii), and hence of the contractibility of $W_G^{\mathcal{A}}(H, K)_{\prec(C,D)}$ in (i). This in turn establishes the homotopy equivalence needed for Case i . ■

Case j . The map j is a homotopy equivalence.

Proof. As in Case i , we will apply the Quillen fiber Theorem 2.10: but this time not to the left term, but instead to the right term of the join of form “ $f^{-1}(Y_{\leq y}) * Y_{> y}$ ” in the theorem. That is, we will show that if $x \in W_G^{\mathcal{S}}(H, K) \setminus W_G^{\mathcal{B}}(H, K)$, then $W_G^{\mathcal{S}}(H, K)_{>x}$ is contractible. Such an x has the form $x = (C, D)$: where we have $D \in \mathcal{A}_p(K) \cup \{1\}$, and $C \in \mathcal{S}_p(H) \setminus \mathcal{B}_p(H)$ (so that C is not a “ p -radical” subgroup of H). We get

$$\begin{aligned} W_G^{\mathcal{S}}(H, K)_{>x} &= \{(A, B) \in \mathcal{S}_p(H) * \mathcal{A}_p(K) : (A, B) > (C, D)\} \\ &= (\mathcal{S}_p(H)_{\geq C} \times \mathcal{A}_p(K)_{>D}) \cup (\mathcal{S}_p(H)_{>C} \times (\mathcal{A}_p(K) \cup \{1\})_{\geq D}) \\ &= \mathcal{S}_p(H)_{>C} * \mathcal{A}_p(K)_{>D} \simeq *; \end{aligned}$$

where the contractibility holds because $\mathcal{S}_p(H)_{>C}$ is contractible: this is a standard property of non- p -radical subgroups $C \in \mathcal{S}_p(H) \setminus \mathcal{B}_p(H)$ (see [5] and [24, p. 152]). Then, j is a homotopy equivalence by the Quillen fiber Theorem 2.10. This concludes the proof of Case j . ■

And that in turn completes the proof of the equivalences in Theorem 4.10. ■

It turns out that in proving Theorem 6.4—in the (allC)-branch covered by Lemma 6.8—we need to apply the propagation Theorem 5.10 to a variant of the replacement-poset $W = W_G^{\mathcal{B}}(H, K)$ indicated in Theorem 4.10 above. We denote this variant using a tilde, in Proposition 4.11 below.

So, to finish the section, we now describe a further modification of the posets used in Theorems 3.6 and 4.10; recall these are based on the method of “shrinking a subposet inside a union” in the first subsection of Section 3. The modification roughly allows us to take account of the method of removals in the second subsection of Section 3, namely, the (allC)-removals indicated in Proposition 3.11 there. We will not in fact make any further removals of elements from our poset union; we work instead at the level of poset-orderings, where we will adjust the earlier $\prec$ to a new $\sqsubset$, by in effect removing certain order-*relations* (i.e., pairs) from $\prec$, rather than removing actual elements from the poset¹⁴. We will thus obtain suitable new replacement-posets, now denoted using a tilde, which are homotopy equivalent to $\mathcal{A}_p(G)$.

¹⁴However, when we later consider subposets of the form $Z_{\sqsubseteq x}$, there *will* be corresponding exclusions of elements.

We will first *construct* the posets in Proposition 4.11 below, with the equivalences given afterwards in Proposition 4.12. Recall that earlier Proposition 3.11 is for the analysis of a fixed component L of G . But in these two upcoming propositions, we will not actually require G to be a counterexample to (H-QC); and correspondingly, instead of assuming the component-Hypothesis 4.2 for L (with $B = 1$).

In the remainder of the section, we assume $Z(L)$ is a p' -group, and set

$$H := LC_G(L).$$

This will in effect still provide the p' -central product Hypothesis 4.4 for H ; in particular, we still have the analogue of (4.1):

$$\text{For } A \in \mathcal{A}_p(H), \quad \text{we have } A \leq p_L(A)p_{C_G(L)}(A), \quad (4.2)$$

where p_L and $p_{C_G(L)}$ denote the respective projections. We get from Lemma 2.4 the following:

$$\text{If } A \in \mathcal{A}_p(G) \quad \text{and} \quad C_L(A) > C_{Z(L)}(A), \quad \text{then } A \leq N_G(L). \quad (4.3)$$

We will use such properties in our results in the remainder of the section, beginning with the following proposition.

Proposition 4.11. *Assume G has a component L , with $Z(L)$ a p' -group. Set $H := LC_G(L)$, so*

$$\mathcal{F}_G(H) \cap \mathcal{A}_p(N_G(L)) = \mathcal{O}_G(L).$$

Recall also from Proposition 3.11 the subposet

$$\mathcal{F}_1 := \{B \in \mathcal{O}_G(L) : O_p(C_G(LB)) > 1\},$$

which we saw that it is upward-closed in $\mathcal{O}_G(L) \subseteq \mathcal{F}_G(H)$.

Then, the definitions below give transitive order-relations—so the given set-unions afford posets.

(1) $\tilde{X}_G(H) := \mathcal{A}_p(H) \cup \mathcal{F}_G(H)$, with order $\sqsubset_X$ “mostly” given by $\prec_X$ from $X = X_G(H)$:

- *inside $\mathcal{A}_p(H)$, and inside $\mathcal{F}_G(H)$, $\sqsubset_X$ agrees with $\prec_X$ (hence with inclusion $<$),*
- *for $A \in \mathcal{A}_p(H)$ and $F \in \mathcal{F}_G(H)$, $\sqsubset_X$ agrees with $\prec_X$ —except we exclude certain pairs.*

For $F \in \mathcal{F}_1$, we require $F \not\sqsubset_X A$, when $F \prec_W A$ but $C_A(F) \leq L$. We denote this by

$$F \sqsubset_X A \text{ iff } 1 < C_A(F) \quad (\not\leq L \text{ when } F \in \mathcal{F}_1).$$

(2) $\tilde{W}_G^{\mathcal{S}}(L, C_G(L)) := \mathcal{S}_p(L) \ast \mathcal{A}_p(C_G(L)) \cup \mathcal{F}_G(H)$, with order $\sqsubset_W$ “mostly” given by $<_W$ from $W = W_G^{\mathcal{S}}(L, C_G(L))$:

- inside $\mathcal{S}_p(L) \ast \mathcal{A}_p(C_G(L))$, and inside $\mathcal{F}_G(H)$, $\sqsubset_W$ agrees with $<_W$ (hence with $<$),
- for $(A, C) \in \mathcal{S}_p(L) \ast \mathcal{A}_p(C_G(L))$ and $F \in \mathcal{F}_G(H)$, $\sqsubset$ agrees with $<_W$ —except that we exclude certain pairs.

For $F \in \mathcal{F}_1$, we require $F \not\sqsubset_W (A, C)$, when $F <_W (A, C)$ but $C_{AC}(F) \leq L$. We write

$$F \sqsubset_W (A, C) \text{ iff } 1 < C_{AC}(F) \quad (\not\sqsubset L \text{ when } F \in \mathcal{F}_1).$$

(3) We also define the following subposets of $\tilde{W}_G^{\mathcal{S}}(L, C_G(L))$:

$$\tilde{W}_G^{\mathcal{B}}(L, C_G(L)) := \mathcal{B}_p(L) \ast \mathcal{A}_p(C_G(L)) \cup \mathcal{F}_G(H),$$

$$\tilde{W}_G^{\mathcal{A}}(L, C_G(L)) := \mathcal{A}_p(L) \ast \mathcal{A}_p(C_G(L)) \cup \mathcal{F}_G(H).$$

Using the identification of $\mathcal{A}_p(L) \ast \mathcal{A}_p(C_G(L))$ as a subposet of $\mathcal{A}_p(LC_G(L))$ in Corollary 4.8, we may regard $\tilde{W}_G^{\mathcal{A}}(L, C_G(L))$ as a subposet of

$$\tilde{X}_G(LC_G(L)) = \tilde{X}_G(H).$$

Note that each $\sqsubset$ has the $\mathcal{F}$ -left property: that members of $\mathcal{F}_G(H)$ can appear only below members of the left-factor in the relevant union.

Proof. We must show that the restricted relations above for the sets $\tilde{X}_G(H)$ in (1) and $\tilde{W}_G^{\mathcal{S}}(L, C_G(L))$ in (2) are indeed transitive order-relations. Since the new $\sqsubset$ agrees with our known ordering $<$ within the factors in the union, as well as on cross-terms between the factors when the $\mathcal{F}$ -term lies outside $\mathcal{F}_1$, the work amounts to checking transitivity of extensions at either end of a cross-term relation (of the form “ $F \sqsubset A$ ” by the $\mathcal{F}$ -left property), where the extension involves at least one term from $\mathcal{F}_1$.

(1) To see transitivity, so that $\tilde{X}_G(H)$ is a poset.

Assume $A, B \in \mathcal{A}_p(H)$, with $A < B$; and $F_1, F_2 \in \mathcal{F}_G(H)$, with $F_1 < F_2$; and also,

(i) $F_2 \sqsubset A$. Hence, $1 < C_A(F_2)$ ($\not\sqsubset L$ when $F_2 \in \mathcal{F}_1$).

We must show that $F_1 \sqsubset A$ and $F_2 \sqsubset B$; we may assume at least one of F_1, F_2 lies in $\mathcal{F}_1$.

We begin by adding some easy subgroup-inclusions to (i):

(ii) $1 < C_A(F_2) \leq C_A(F_1) \cap C_B(F_2)$.

Then, we consider the cases where at least one of F_1, F_2 lies in $\mathcal{F}_1$:

- $F_2 \in \mathcal{F}_1$: Here, (i) includes also $C_A(F_2) \not\leq L$. So, from (ii) we get $1 < C_A(F_1) \not\leq L$, and similarly $1 < C_B(F_2) \not\leq L$; and hence $F_1 \sqsubset A$ and $F_2 \sqsubset B$;
- $F_2 \notin \mathcal{F}_1, F_1 \in \mathcal{F}_1$: Here, (ii) gives $C_B(F_2) > 1$, so we get $F_2 \sqsubset B$; while for $F_1 \in \mathcal{F}_1$, we similarly get $C_A(F_1) > 1$ —but we still need to show $C_A(F_1) \not\leq L$. So, assume by way of contradiction that $C_A(F_1) \leq L$. Then, using (ii), F_2 centralizes the non-trivial p -subgroup $C_A(F_2) \leq C_A(F_1) \leq L$, so that F_2 normalizes L by (4.3). So, we obtain that $F_2 \in \mathcal{F}_G(H) \cap \mathcal{A}_p(N_G(L)) = O_G(L)$. Now, $F_1 < F_2$, and we have $F_1 \in \mathcal{F}_1 \subseteq O_G(L)$; and we saw in Proposition 3.11 that $\mathcal{F}_1$ is by its construction upward-closed in $O_G(L)$. We conclude that $F_2 \in \mathcal{F}_1$ —contrary to our choice in this case of $F_2 \notin \mathcal{F}_1$. This contradiction shows that we have $C_A(F_1) \not\leq L$, and so, completes the proof that $F_1 \sqsubset A$ in this final case.

This establishes transitivity of the order-relation; so that $\tilde{X}_G(H)$ is indeed a poset.

- (2) To see transitivity, so that $\tilde{W}_G^{\mathcal{S}}(L, C_G(L))$ is a poset, we parallel the previous proof as follows. First, we replace “ $A < B$ ” above by elements $(A, C) < (B, D)$ from the pre-join $\mathcal{S}_p(L) \ast \mathcal{A}_p(C_G(L))$, where of course $C \leq D$; and with members $F_1 < F_2$ of $\mathcal{F}_G(H)$. Assume the analogue of (i):

(i') $F_2 \sqsubset_W (A, C)$. Hence, $1 < C_{AC}(F_2) (\not\leq L \text{ when } F_2 \in \mathcal{F}_1)$.

We must show $F_1 \sqsubset_W (A, C)$ and $F_2 \sqsubset_W (B, D)$; we may assume one of F_1, F_2 lies in $\mathcal{F}_1$. We get the analogue of (ii):

(ii') $1 < C_{AC}(F_2) \leq C_{AC}(F_1) \cap C_{BD}(F_2)$.

And we have a similar case division according to membership of F_1, F_2 in $\mathcal{F}_1$. In the first case, where $F_2 \in \mathcal{F}_1$, (i') includes also $C_{AC}(F_2) \not\leq L$. This is what is further needed to complete the proof, using (ii'), that $F_1 \sqsubset_W (A, C)$ and $F_2 \sqsubset_W (B, D)$.

So, the only substantial argument required is for the second case, where $F_1 \in \mathcal{F}_1, F_2 \notin \mathcal{F}_1$. Again $F_2 \sqsubset (B, D)$ follows directly from $1 < C_{BD}(F_2)$ in (ii'); but for (A, C) we only get $C_{AC}(F_1) > 1$ —so we still need to show $C_{AC}(F_1) \not\leq L$. So, suppose by way of contradiction that $C_{AC}(F_1) \leq L$. Then, F_2 centralizes the non-trivial p -subgroup $C_{AC}(F_2) \leq C_{AC}(F_1) \leq L$, so again F_2 normalizes L by (4.3), and then we get $F_2 \in \mathcal{F}_G(H) \cap \mathcal{A}_p(N_G(L)) = O_G(L)$. And since $F_1 < F_2$ with $F_1 \in \mathcal{F}_1 \subseteq O_G(L)$, the upward-closed property of $\mathcal{F}_1$ gives $F_2 \in \mathcal{F}_1$ —contrary to its choice in this case. This contradiction shows that $C_{AC}(F_1) \not\leq L$ —which gives $F_2 \sqsubset_W (A, C)$, to complete the proof for the order-relation; so that $\tilde{W}_G^{\mathcal{S}}(L, C_G(L))$ is indeed a poset, as desired.

Of course, conclusion (3) is just a definition of subsets of giving subposets, and so, does not require further proof. ■

Now, we prove our equivalence-result analogous to Theorem 4.10, showing that the new tilde-posets can be used as replacement-posets for $\mathcal{A}_p(G)$ —notably in applying our new propagation Theorem 5.10 in situations related to Proposition 3.11, as we will do in the proof of the branch of Theorem 6.4 given by Lemma 6.8.

Proposition 4.12. *Continue the hypotheses of Proposition 4.11: thus G has a component L , with $Z(L)$ a p' -group; and we set $H := LC_G(L)$ for the p' -central product of these groups, so that*

$$\mathcal{F}_G(H) \cap \mathcal{A}_p(N_G(L)) = O_G(L).$$

Recall also from Proposition 3.11 the poset

$$\mathcal{F}_1 := \{B \in O_G(L) : O_p(C_G(LB)) > 1\},$$

which we saw there is upward-closed in $O_G(L)$.

Then, we have homotopy equivalences relating the new posets in Proposition 4.11:

$$\mathcal{A}_p(G) \simeq \tilde{X}_G(H) \simeq \tilde{W}_G^{\mathcal{S}}(L, C_G(L)) \simeq \tilde{W}_G^{\mathcal{B}}(L, C_G(L)) \simeq \tilde{W}_G^{\mathcal{A}}(L, C_G(L)).$$

Proof. Recall the further notation of Proposition 3.11, including that

$$\begin{aligned} \mathcal{N}_1 &= \{E \in \mathcal{A}_p(N_G(L)) : E \cap L > 1, C_E(L) = 1, \exists D \in \mathcal{F}_1 \text{ with } D < E\} \\ &\subseteq \mathcal{N}_G(L) \subseteq \mathcal{A}_p(N_G(L)), \end{aligned}$$

where the last inclusion holds using (4.3). It will be convenient to begin by replacing $\mathcal{A}_p(G)$ with

$$Y := \mathcal{A}_p(G) \setminus \mathcal{N}_1; \quad \text{recall by Proposition 3.11 that } \mathcal{A}_p(H) \subseteq Y \simeq \mathcal{A}_p(G).$$

In fact, we will show first that $Y \simeq \tilde{X}_G(H)$. Then, we will prove that $\tilde{X}_G(H) \simeq \tilde{W}_G^{\mathcal{A}}(L, C_G(L))$; and finally that the inclusions $\tilde{W}_G^{\mathcal{A}}(L, C_G(L)) \hookrightarrow \tilde{W}_G^{\mathcal{S}}(L, C_G(L))$ and $\tilde{W}_G^{\mathcal{B}}(L, C_G(L)) \hookrightarrow \tilde{W}_G^{\mathcal{S}}(L, C_G(L))$ are homotopy equivalences.

Claim 1. $Y \simeq \tilde{X}_G(H)$.

Proof. The proof of Claim 1 is somewhat intricate and will require several pages.

We will mimic certain parts of the proof of Theorem 3.6: Let $\tilde{\alpha} : Y \rightarrow \tilde{X}_G(H)$ be the analogue of the map α there defined by using the deflation map on the $\mathcal{N}_Y(H)$ -term:

$$\begin{aligned} A \in \mathcal{N}_Y(H) &\Rightarrow \tilde{\alpha}(A) = A \cap H, \\ F \in \mathcal{F}_Y(H) &\Rightarrow \tilde{\alpha}(F) = F. \end{aligned}$$

Note since $\mathcal{N}_1 \subseteq \mathcal{N}_G(L) \subseteq \mathcal{N}_G(H)$ that $\mathcal{N}_Y(H) = \mathcal{N}_G(H) \setminus \mathcal{N}_1 \supseteq \mathcal{A}_p(H)$ and $\mathcal{F}_Y(H) = \mathcal{F}_G(H)$.

We first check that $\tilde{\alpha}$ is order-preserving—from $<$ to $\sqsubset$.

The argument begins by paralleling the corresponding segment of the proof of Theorem 3.6. On pairs $E < F$ within $\mathcal{F}_Y(H)$, for the images we get $E \sqsubset F$, since $\sqsubset$ coincides with $<$ here. On pairs $A < B$ within $\mathcal{N}_Y(H)$, for the images we get $A \cap H \sqsubset B \cap H$ since again $\sqsubset$ coincides with $<$ here—and we checked at Lemma 3.3 (1) that the deflation map preserves $<$. So, it remains to consider cross-terms—which by the $\mathcal{F}$ -left property (i) in Definition 3.2 must be of the form $F < A$, where $F \in \mathcal{F}_Y(H)$, and $A \in \mathcal{N}_Y(H)$ (so $A \cap H > 1$). Here, $[F, A] = 1$, which implies that

$$C_{A \cap H}(F) = A \cap H > 1.$$

When $F \notin \mathcal{F}_1$, this already gives $F \sqsubset A \cap H$ for the images under $\tilde{\alpha}$. So, assume $F \in \mathcal{F}_1$: where it remains to show that $C_{A \cap H}(F) \not\leq L$. (That is, now we must go beyond the earlier argument in the proof of Theorem 3.6.) Assume by way of contradiction that $C_{A \cap H}(F) \leq L$ —that is, that $A \cap H \leq L$; we will show $A \in \mathcal{N}_1$. We see since $H = LC_G(L)$ that $A \cap L = A \cap H > 1$; thus $C_L(A) \geq A \cap L$ contains non-trivial p -elements, and so, A normalizes the component L by (4.3). Furthermore,

$$C_A(L) = A \cap C_G(L) = (A \cap H) \cap C_G(L) = (A \cap L) \cap C_G(L) \leq Z(L),$$

which is a p' -group by hypothesis—so that $C_A(L) = 1$, and hence A is faithful on L . Recall $F < A$ with $F \in \mathcal{F}_1$; so we have completed the requirements for $A \in \mathcal{N}_1$ —contrary to $A \in Y = \mathcal{A}_p(G) \setminus \mathcal{N}_1$. This contradiction shows that $C_{A \cap H}(F) \not\leq L$ when $F \in \mathcal{F}_1$; so we conclude in this final case also that $\tilde{\alpha}(F) = F \sqsubset A \cap H = \tilde{\alpha}(A)$. Thus, we have shown that $\tilde{\alpha}$ is an order-preserving map.

We will now prove that $\tilde{\alpha}$ is a homotopy equivalence, by applying Quillen's fiber Theorem 2.10. This reduces the rest of the proof of Claim 1 to showing for all $x \in \tilde{X}_G(H)$ that the preimage

$$Z_x := \tilde{\alpha}^{-1}(\tilde{X}_G(H)_{\supseteq x})$$

is contractible. Recall that $\mathcal{F}_Y(H) = \mathcal{F}_G(H)$.

The case $x = A \in \mathcal{A}_p(H)$ is quickly treated: we can just parallel the case “ $x = A$ ” in proof of Theorem 3.6. Recall that $\sqsubset$ agrees with group-inclusion $<$ inside $\mathcal{A}_p(H)$. Hence, using the $\mathcal{F}$ -left property of $\sqsubset$ from the definition in Proposition 4.11, we get $\tilde{X}_G(H)_{\supseteq A} = \mathcal{A}_p(H)_{\supseteq A} = \mathcal{A}_p(H)_{\geq A}$; so that

$$Z_A = \tilde{\alpha}^{-1}(\tilde{X}_G(H)_{\supseteq A}) = \tilde{\alpha}^{-1}(\mathcal{A}_p(H)_{\geq A}) = \mathcal{N}_Y(H)_{\geq A}.$$

And now, as in the proof of Theorem 3.6, contractibility follows from the minimum A .

This reduces the proof of Claim 1 to contractibility of $Z := Z_x = Z_F$, for the remaining case $x = F \in \mathcal{F}_Y(H) = \mathcal{F}_G(H)$.

Recall that the ordering $\sqsubseteq$ agrees with $<$ also on pairs inside $\mathcal{F}_G(H)$ and $\mathcal{F}_G(H) \cap \mathcal{A}_p(N_G(L)) = O_G(L) \supseteq \mathcal{F}_1$. Here, the parallel with the case “ $x = F$ ” in the proof of Theorem 3.6 will require more substantial adaptations.

For example, here for $\sqsubseteq$, we get the exclusion-modified form

$$\begin{aligned} \tilde{X}_G(H)_{\sqsupseteq F} &= \mathcal{A}_p(H)_{\sqsupseteq F} \cup \mathcal{F}_G(H)_{\sqsupseteq F} \\ &= \{A \in \mathcal{N}_H(C_H(F)) : 1 < C_A(F) (\not\leq L \text{ when } F \in \mathcal{F}_1)\} \cup \mathcal{F}_G(H)_{\geq F}. \end{aligned}$$

Hence, for the preimage, we get a correspondingly exclusion-modified form (for which we choose the numbering (ii) parallel to that in the proof of Theorem 3.6):

$$\begin{aligned} \text{(ii) } Z &= \tilde{\alpha}^{-1}(\tilde{X}_G(H)_{\sqsupseteq F}) = Z_{\mathcal{N}} \cup \mathcal{F}_G(H)_{\geq F}, \text{ where } Z_{\mathcal{N}} := \{B \in \mathcal{N}_Y(C_H(F)) : \\ &1 < B \cap C_H(F) (\not\leq L \text{ when } F \in \mathcal{F}_1)\}. \end{aligned}$$

Thus, for this F , we have $Z_{\mathcal{N}} \subseteq \mathcal{N}_Y(C_H(F))$ —with equality except possibly when $F \in \mathcal{F}_1$.

As usual, we begin with an easier sub case—namely, $F \not\leq N_G(L)$; that is, $F \notin O_G(L)$. Then, in particular, $F \notin \mathcal{F}_1$; so for Z in (ii) above, we have that $Z_{\mathcal{N}} = \mathcal{N}_Y(C_H(F))$, i.e., there are no exclusions required. Hence, Z is the analogue, with Y in the role of “ $\mathcal{B}$ ”, of the preimage Z in the case “ $x = F$ ” in the proof of Theorem 3.6. So, to continue our theme of adapting that proof: for any $D \in Z$, we have the same group-inclusion relations

$$D \geq C_D(F) \leq C_D(F)F \geq F.$$

So here, as there, we will again show via Lemma 2.7 (1) that Z is contractible to $\{F\}$. Along the way, we will need to adapt the proof of (a) there—now with Y in the role of “ $\mathcal{B}$ ”. Again, the right-factor case, where $D = E \in \mathcal{F}_G(H)_{\geq F}$, is no problem: since as there, we get that both $C_E(F)$ and $C_E(F)F$ are equal to $E \in \mathcal{F}_G(H)_{\geq F} \subseteq Y$. But in the left-factor case, where $D = B \in \mathcal{N}_Y(C_H(F))$ so that $B \cap C_H(F) > 1$, we do *not* try to prove hypothesis (i) of Theorem 3.6, with Y in the role of “ $\mathcal{B}$ ”, for a general $F \in \mathcal{N}_Y(H)$: instead, we argue for this particular F —using $F \not\leq N_G(L)$ —that we must have $C_B(F)$ and $C_B(F)F$ in Y for (a). Consider $C_B(F)$ first: assume by way of contradiction that $C_B(F) \in \mathcal{N}_1 \subseteq N_G(L)$. Then, we have $C_B(F) \cap L > 1$; so we conclude by (4.3) that $F \leq N_G(L)$ —contrary to the choice of F in this case. This contradiction shows $C_B(F) \in Y$, as desired. Similarly, assuming $C_B(F)F \in \mathcal{N}_1$ would give $1 < C_B(F)F \cap L$ centralized by F —so that F normalizes L by (4.3), again contrary to the choice of F . This contradiction gives $C_B(F)F \in Y$, as desired—completing the proof of (a). With (a) in hand, steps (b) and (c) go through as in the proof of Theorem 3.6. This completes the proof of contractibility of the preimage Z , for this case where $F \not\leq N_G(L)$.

We have now reduced the proof of Claim 1 to contractibility of Z for $F \leq N_G(L)$.

For such F , we will show that the preimage Z (where now we might have $F \in \mathcal{F}_1$, so that we could get $Z_{\mathcal{N}} \subsetneq \mathcal{N}_Y(C_H(F))$ in (ii)), we will obtain contractibility via a somewhat more elaborate argument using the general method above, namely, Lemma 2.7(1). That is, we will use the zigzag of homotopies abbreviated by the following longer sequence of group-inclusions (for $D \in Z$):

$$\begin{aligned}
 D &\geq (D \cap F)(D \cap H) \\
 &\leq (D \cap F)p_L(D \cap H)p_{C_G(L)}(D \cap H) \\
 &\geq (D \cap F)p_L(C_{D \cap H}(F))p_{C_G(L)}(C_{D \cap H}(F)) \\
 &\leq Fp_L(C_{D \cap H}(F))p_{C_G(L)}(C_{D \cap H}(F)) \\
 &\geq F.
 \end{aligned} \tag{4.4}$$

Here, we use $j_1, \dots, j_5$ to denote the maps where the image of D is given in each successive line of (4.4), with $j_0 := \text{Id}_Z$. Note that these j_i are order-preserving, with $<$ -comparability relations given by $j_0 \geq j_1 \leq j_2 \geq j_3 \leq j_4 \geq j_5$.

Therefore, we have now reduced the proof of Claim 1 to verifying the endomorphism requirement for that lemma, namely, that each $j_i(D) \in Z$. Of course, this holds already for $j_0 = \text{Id}_Z$, and the constant map j_5 to $F \in Z$; so we only need to work with j_i for $1 \leq i \leq 4$.

Since $F \in O_G(L)$, so that F acts on L and on $C_G(L)$, we see that if further $[F, D] = 1$, then also $[F, p_L(D)] = 1 = [F, p_{C_G(L)}(D)]$.

As usual in such zigzags, the right factor for Z in (ii), where $D = E \in \mathcal{F}_G(H)_{\geq F}$, is no problem: for note that $E \cap F = F$ and $E \cap H = 1$; so we see that $j_1(E) = F$. Also, $j_i(F) = F$ for all i ; so that for all $i \geq 1$, we get $j_i(E) = F \in Z$, as desired.

Since we dealt with $D = E \in \mathcal{F}_G(H)_{\geq F}$ just above, we have in fact made yet another reduction, namely, to D lying in the left-factor for Z in (ii): that is, to $D = B \in Z_{\mathcal{N}} \subseteq Z$. So, in particular, from (ii) we have for this B the following:

$$(iii) \quad 1 < B \cap C_H(F) (\not\leq L \text{ if } F \in \mathcal{F}_1).$$

As in the proof of Theorem 3.6, the first part of (iii) can be viewed in several ways:

$$(iii') \quad 1 < B \cap C_H(F) = C_{B \cap H}(F) = C_B(F) \cap C_B(H).$$

We begin our treatment of such B with the following cases.

Case $i = 1$. We have $j_1(B) = (B \cap F)(B \cap H) \in Z$.

Proof. We will show $(B \cap F)(B \cap H) \in Z_{\mathcal{N}}$ —where of course $Z_{\mathcal{N}} \subseteq Z$ in (ii).

We begin with the case where $B \cap F = 1$. Here, $(B \cap F)(B \cap H) = B \cap H \in \mathcal{A}_p(H) \subseteq Y$; and we see using (iii/iii') that $1 < B \cap C_H(F) = (B \cap H) \cap C_H(F)$ ($\not\leq L$ when $F \in \mathcal{F}_1$)—so by (ii) we have $B \cap H \in Z_{\mathcal{N}}$ —as desired.

We now turn to the case where $B \cap F > 1$. We will show first that $(B \cap F)(B \cap H) \in Y$, so we assume by way of contradiction that we have $(B \cap F)(B \cap H) \in \mathcal{N}_1$. We will prove this leads to $B \in \mathcal{N}_1$. For we observe, using the definition of $(B \cap F)(B \cap H) \in \mathcal{N}_1$, that $1 < C_{(B \cap F)(B \cap H)}(F) \leq C_B(F)$ and $1 < ((B \cap F)(B \cap H)) \cap L = B \cap L$ (since $L \leq LC_G(L) = H$); and finally, there is $E \in \mathcal{F}_1$ with $E \leq (B \cap F)(B \cap H) \leq B$. Thus, we have $B \in \mathcal{N}_1$ —contrary to the choice of $B \in Y = \mathcal{A}_p(G) \setminus \mathcal{N}_1$. This contradiction shows that $(B \cap F)(B \cap H) \in Y$.

Using (iii/iii') as before now gives

$$1 < B \cap C_H(F) = (B \cap H) \cap C_H(F) \leq (B \cap F)(B \cap H) \cap C_H(F) \\ (\not\leq L \text{ when } F \notin \mathcal{F}_1),$$

so by (ii) we obtain $(B \cap F)(B \cap H) \in Z_{\mathcal{N}}$, as desired—completing Case $i = 1$. ■

Thus, to complete the proof of the endomorphism requirement, it now remains to show the following.

Cases $i = 2, 3, 4$. For $2 \leq i \leq 4$, we have $j_i(B) \in Z$.

Using (iii') and (4.2), we get a consequence for the H -projection factors in the $j_i(B)$:

$$(iv) \quad 1 < C_{B \cap H}(F) \leq p_L(C_{B \cap H}(F)) \cdot p_{C_G(L)}(C_{B \cap H}(F)) \leq p_L(B \cap H) \cdot p_{C_G(L)}(B \cap H).$$

Initial reduction. The endomorphism result for $2 \leq i \leq 4$ holds either if $F \in \mathcal{F}_1$, or $B \cap C_H(F) \not\leq L$.

Proof. By (iii), the first if-condition $F \in \mathcal{F}_1$ implies the second; so we just assume $B \cap C_H(F) \not\leq L$. This assumption shows via (iii') that $C_{B \cap H}(F)$ does not lie in the $p_L(C_{B \cap H}(F))$ -factor in (iv)—so the following hold.

$$(v) \quad \text{Since } B \cap C_H(F) \not\leq L \text{ here, we have } p_{C_G(L)}(C_{B \cap H}(F)) > 1.$$

In particular, p -overgroups of $p_{C_G(L)}(C_{B \cap H}(F))$ (such as the $j_i(B)$ for $i = 2, 3, 4$) are not faithful on L , and so, do not lie in $\mathcal{N}_1$ —that is, they lie in Y .

To complete the proof that $j_i(B) \in Z$, note that by (iv) and (v), for $i = 2, 3, 4$ there are inclusions:

$$j_i(B) \cap C_H(F) = (p_L(C_{B \cap H}(F))p_{C_G(L)}(C_{B \cap H}(F))) \cap C_H(F) \geq C_{B \cap H}(F) > 1;$$

so that $j_i(B) \in \mathcal{N}_Y(C_H(F))$, hence $j_i(B) \in Z_{\mathcal{N}} \subseteq Z$ by (ii), since we assume $B \cap C_H(F) \not\leq L$. ■

By the initial reduction, in the remainder of the proof of Cases $i = 2, 3, 4$, we have

$$F \in O_G(L) \setminus \mathcal{F}_1 \text{—so that } Z_{\mathcal{N}} = \mathcal{N}_Y(C_H(F)) \text{ in (ii); and } 1 < C_{B \cap H}(F) \leq L.$$

Hence, (iv) simplifies to

$$1 < C_{B \cap H}(F) = p_L(C_{B \cap H}(F)) = C_{B \cap L}(F) \leq L \quad \text{and} \quad p_{C_G(L)}(C_{B \cap H}(F)) = 1. \quad (4.5)$$

We also get the correspondingly simplified expressions

$$j_3(B) = (B \cap F)C_{B \cap L}(F) \quad \text{and} \quad j_4(B) = FC_{B \cap L}(F).$$

At this point, we can use the non-triviality in (4.5) of $p_L(C_{B \cap H}(F))$ (and hence of $p_L(B \cap H)$), as we used (v) in the argument on $j_4(B) \cap C_H(F)$ in the proof of the Initial Reduction, to conclude that $j_i(B) \in \mathcal{N}_G(C_H(F))$.

But this time, it will require some further work below to get $j_i(B) \in Y$, which will complete the proof by giving $j_i(B) \in \mathcal{N}_Y(C_H(F)) = Z_{\mathcal{N}} \subseteq Z$ via (ii), since we are assuming $F \notin \mathcal{F}_1$.

So, we treat the successive cases for $j_i(B) \in Y$ as follows.

- To see $j_2(B) = (B \cap F)p_L(B \cap H)p_{C_G(L)}(B \cap H) \in Y$: if $p_{C_G(L)}(B \cap H) > 1$, then $j_2(B)$ is not faithful on L , and so, cannot lie in $\mathcal{N}_1$ —so it lies in Y . Otherwise, $p_{C_G(L)}(B \cap H) = 1$, so that $B \cap H = B \cap L = p_L(B \cap H)$. Hence, $j_2(B) = (B \cap F)(B \cap L) = (B \cap F)(B \cap H)$. And this equals $j_1(B) \in Z$ using Case $i = 1$ (so, in particular, $j_2(B) \in Y$).
- We saw using (4.5) that $j_3(B)$ simplifies to $(B \cap F)C_{B \cap L}(F)$ —which is a subgroup of LF . Assume by way of contradiction that $(B \cap F)C_{B \cap L}(F) \in \mathcal{N}_1$. Then, there is some $E \in \mathcal{F}_1$ with $E \leq (B \cap F)C_{B \cap L}(F)$; so in particular, $LE \leq LF$. Recall since $F \in O_G(L) \setminus \mathcal{F}_1$ that $O_p(C_G(LF)) = 1$. So, by Lemma 2.5, we also get $O_p(C_G(LE)) = 1$; that is, $E \notin \mathcal{F}_1$ —contrary to its choice. This contradiction shows that $j_3(B) = (B \cap F)C_{B \cap L}(F) \in Y$.
- To see $j_4(B) = Fp_L(C_{B \cap H}(F)) \in Y$: the same proof as just above, with $(B \cap F)$ replaced by F , goes through for this case.

This completes the proof that $j_i(B) \in Y$ for $i = 2, 3, 4$; and hence (as we had mentioned) that $j_i(B) \in \mathcal{N}_Y(C_H(F)) = Z_{\mathcal{N}} \subseteq Z$.

This in turn completes the proof for all $1 \leq i \leq 5$ of the endomorphism requirement for our zigzag of equivalences in (4.4), establishing contractibility of Z in this final case where $F \leq N_G(L)$.

Now, working back through our long sequence of reductions in the proof: this case for F then completes the proof of contractibility of Z in all cases for F , and hence in all cases for $D \in Z$; and thus also completes the proof that $\tilde{\alpha}$ is a homotopy equivalence. So, we have finally completed the proof of Claim 1. ■

Claim 2. $\tilde{X}_G(H) \simeq \tilde{W}_G^{\mathcal{A}}(L, C_G(L))$.

Proof. Our equivalence arises essentially by taking the projection/product maps a , b of Corollary 4.8, retracting $\mathcal{A}_p(H)$ to its product-subposet that we identify with $\mathcal{A}_p(L) \ast \mathcal{A}_p(C_G(L))$, and extending them via the identity on $\mathcal{F}_G(H)$.

We define the forward-map $\tilde{a} : \tilde{X}_G(H) \rightarrow \tilde{W}_G^{\mathcal{A}}(L, C_G(L))$ by

$$\tilde{a}(x) = \begin{cases} (p_L(A), p_{C_G(L)}(A)) & x = A \in \mathcal{A}_p(H), \\ F & x = F \in \mathcal{F}_G(H). \end{cases}$$

To see that $\tilde{a}$ is order-preserving, as usual it suffices to check cross-terms. Assume $F \sqsubset_X A$, so that $1 < C_A(F)$ ($\not\leq L$ when $F \in \mathcal{F}_1$). Using (4.2), we get

$$1 < C_A(F) \leq C_{p_L(A)p_{C_G(L)}(A)}(F). \quad (4.6)$$

When $F \notin \mathcal{F}_1$, this completes the proof that $F \sqsubset_W (p_L(A), p_{C_G(L)}(A))$ for the images under $\tilde{a}$. So, now assume further that $F \in \mathcal{F}_1$, that is $C_A(F) \not\leq L$. Then, also $C_{p_L(A)p_{C_G(L)}(A)}(F) \not\leq L$ by (4.6), so $F \sqsubset_W (p_L(A), p_{C_G(L)}(A))$ for the images. Hence, $\tilde{a}$ is indeed order-preserving.

For the reverse direction, note that $\tilde{W}_G^{\mathcal{A}}(L, C_G(L))$ embeds into $\tilde{X}_G(H)$ via the map $\tilde{b}$ such that $\tilde{b}(A, C) = AC \in \mathcal{A}_p(H)$ for all $(A, C) \in \mathcal{A}_p(L) \ast \mathcal{A}_p(C_G(L))$, and $\tilde{b}|_{\mathcal{F}_G(H)} = \text{Id}_{\mathcal{F}_G(H)}$. To see that $\tilde{b}$ is order-preserving, assume $F \sqsubset_W (A, C)$: then $1 < C_{AC}(F)$ ($\not\leq L$ when $F \in \mathcal{F}_1$). And this is just the definition of $\sqsubset_X$, in the relation $F \sqsubset_X AC$ for the images under $\tilde{b}$, as desired.

Finally, note that $\tilde{a}\tilde{b}$ is the identity on $\tilde{W}_G^{\mathcal{A}}(L, C_G(L))$. Further $\tilde{b}\tilde{a}(A) = p_L(A)p_{C_G(L)}(A) \geq A$ using (4.2), and $\tilde{b}\tilde{a}(F) = F$ for $F \in \mathcal{F}_G(H)$; so that $\tilde{b}\tilde{a} \geq \text{Id}_{\tilde{X}_G(H)}$. Then, $\tilde{a}$ is a homotopy equivalence with homotopy inverse $\tilde{b}$. This completes the proof that $\tilde{X}_G(H) \simeq \tilde{W}_G^{\mathcal{A}}(L, C_G(L))$, that is, of Claim 2.

In particular, assuming the identification in Corollary 4.8 of the product-subposet of $\mathcal{A}_p(H)$ with $\mathcal{A}_p(L) \ast \mathcal{A}_p(C_G(L))$, we see that our maps retract $\tilde{X}_G(H)$ to a subposet we may identify with $\tilde{W}_G^{\mathcal{A}}(L, C_G(L))$. ■

Claim 3. The maps given by the following natural inclusions $\tilde{i} : \tilde{W}_G^{\mathcal{A}}(L, C_G(L)) \hookrightarrow \tilde{W}_G^{\mathcal{S}}(L, C_G(L))$ and $\tilde{j} : \tilde{W}_G^{\mathcal{B}}(L, C_G(L)) \hookrightarrow \tilde{W}_G^{\mathcal{S}}(L, C_G(L))$ are homotopy equivalences.

Proof. We proceed by making relevant adjustments to the proof of Theorem 4.10.

We first observe that $\tilde{j}$ is a homotopy equivalence, by the argument given in Case j there: For we want to similarly remove the elements $x \in \tilde{W}_G^{\mathcal{S}}(L, C_G(L)) \setminus \tilde{W}_G^{\mathcal{B}}(L, C_G(L))$; and the subposets $\tilde{W}_G^{\mathcal{S}}(L, C_G(L))_{\sqsupset x}$ do not contain members of $\mathcal{F}_G(LC_G(L))$. Hence, $\sqsupset$ agrees with $\succ$ (that is, with $>$) on that subposet, so the contractibility argument there using $<$ and $>$ goes through here also.

So, it remains to show that $\tilde{i}$ is a homotopy equivalence. We follow the general strategy for Case i in the proof of Theorem 4.10. We will again show that for

$x \in \mathcal{S}_p(L) * \mathcal{A}_p(C_G(L))$, the interval $\tilde{W}_G^{\mathcal{A}}(L, C_G(L))_{\sqsubset x}$ is contractible, and so, x can be homotopically removed from $\tilde{W}_G^{\mathcal{S}}(L, C_G(L))$. Again, such an x has the form (C, D) , with $C \in \mathcal{S}_p(L) \setminus \mathcal{A}_p(L)$ (so that C is not elementary abelian), and $D \in \mathcal{A}_p(C_G(L)) \cup \{1\}$. But this time, we must replace (i) there with an exclusion-modified version for $\sqsubset$:

(i') $\tilde{W}_G^{\mathcal{A}}(L, C_G(L))_{\sqsubset(C,D)}$ is equal to the union

$$(\mathcal{A}_p(C) * \mathcal{A}_p(D)) \cup \{F \in \mathcal{F}_G(LC_G(L)) : 1 < C_{CD}(F) (\not\leq L \text{ when } F \in \mathcal{F}_1)\}.$$

Again, it will suffice for contractibility in (i') to homotopically remove the indicated elements of $F \in \mathcal{F}_G(LC_G(L))_{\sqsubset(C,D)}$, by showing that $(\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\sqsubset F}$ is contractible for each such F .

So, assume that $F \notin \mathcal{F}_1$. Then, for this F , the exclusions specified by $C_{CD}(F) \not\leq L$ in (i') are not required; so $(\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\sqsubset F}$ agrees with $(\mathcal{A}_p(C) * \mathcal{A}_p(D))_{> F}$ as in (i), and of course $\sqsubset$ agrees with $<$ (that is, with $<$) on this set. So, the argument for contractibility from the proof of Theorem 4.10, namely, via the equivalence in (ii) there with $\mathcal{A}_p(C_{CD}(F)) \simeq *$, goes through here also—completing the case $F \notin \mathcal{F}_1$.

Finally, we assume the remaining case where $F \in \mathcal{F}_1$. Here, to describe $(\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\sqsubset F}$, we now do require the exclusions specified in (i'), namely, $1 < C_{CD}(F) \not\leq L$. In this situation, $C_{CD}(F) \neq p_L(C_{CD}(F)) = p_C(C_{CD}(F))$. Now, since F normalizes L and $C_G(L)$, if $c \in C$ and $d \in D$ are such that $cd \in C_{CD}(F)$ then, for $f \in F$ we have $cd = (cd)^f = c^f d^f$. This implies $c^{-1}c^f \in Z(L)$, which is a p' -group. In particular, $c^f = (c^{-1}c^f)c$, that is c, c^f commute, and hence $c^{-1}c^f$ is a p -element lying in $Z(L)$. This shows that $c^f = c$, and thus also $d^f = d$, that is, $d \in C_D(F)$. From this we conclude $p_D(C_{CD}(F)) \leq C_D(F)$ and

$$1 < p_{C_G(L)}(C_{CD}(F)) = p_D(C_{CD}(F)) \leq C_D(F).$$

These exclusions require a corresponding change to the codomain for the homotopy equivalence in (ii) of Theorem 4.10. This time, we will get our equivalence with the exclusion-modified poset given by the difference $\mathcal{A}_p(C_{CD}(F)) \setminus \mathcal{A}_p(C_C(F))$. (And then contractibility of the latter poset will be the corresponding variant below of the standard contractibility argument for $\mathcal{A}_p(C_{CD}(F))$.)

In particular, for the equivalence we use essentially the adjusted projection/product maps a, b' as in the analogous proof in (ii) for Theorem 4.10; the only new feature is that we must check that their images really do lie in our exclusion-modified subposets. First, take $(A, B) \in (\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\sqsubset F}$, so that we have the restriction $1 < C_{AB}(F) \not\leq L$; then $C_{AB}(F) \neq p_L(C_{AB}(F)) = p_C(C_{AB}(F))$, so that $C_{AB}(F) \not\leq C$. So, we conclude that $b'(A, B) = C_{AB}(F) \in \mathcal{A}_p(C_{CD}(F)) \setminus \mathcal{A}_p(C_C(F))$, as required. Next, consider some $E \in \mathcal{A}_p(C_{CD}(F)) \setminus \mathcal{A}_p(C_C(F))$ —here we see using (4.2) that we get $1 < E \leq p_C(E)p_D(E) = C_{p_C(E)p_D(E)}(F) \not\leq L$ since $E \not\leq C_C(F)$. Hence,

$F \sqsubset (p_C(E), p_D(E))$, so that $a(E) = (p_C(E), p_D(E)) \in (\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\sqsubset F}$, as required. We had already checked the usual poset-equivalence properties of the pair a, b' , so we get the equivalence of $(\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\sqsubset F}$ with $\mathcal{A}_p(C_{CD}(F)) \setminus \mathcal{A}_p(C_C(F))$.

The contractibility of the latter poset $\mathcal{A}_p(C_{CD}(F)) \setminus \mathcal{A}_p(C_C(F))$ will follow using Lemma 2.7(1) via the standard Quillen conical-contractibility zigzag

$$E \leq E \cdot \Omega_1 Z(C_{CD}(F)) \geq \Omega_1 Z(C_{CD}(F)),$$

once we show the endomorphism requirement, namely, that the two right-hand image-terms in fact lie in the poset $\mathcal{A}_p(C_{CD}(F)) \setminus \mathcal{A}_p(C_C(F))$. But

$$\Omega_1 Z(C_{CD}(F)) \geq \Omega_1 Z(C_C(F)) \cdot \Omega_1 Z(C_D(F));$$

so since we saw above that $C_D(F) > 1$, we conclude that the two terms indeed do not fall into $C_C(F)$, as required. This completes the proof of contractibility of $\mathcal{A}_p(C_{CD}(F)) \setminus \mathcal{A}_p(C_C(F))$, and hence also of $(\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\sqsubset F}$, for $F \in \mathcal{F}_1$.

We have now established the contractibility of $(\mathcal{A}_p(C) * \mathcal{A}_p(D))_{\sqsubset F}$, for all cases of F . So, we conclude that $\mathcal{A}_p(C) * \mathcal{A}_p(D) \hookrightarrow \tilde{W}_G^{\mathcal{A}}(L, C_G(L))_{\sqsubset (C, D)}$ is a homotopy equivalence.

Thus, $\tilde{W}_G^{\mathcal{A}}(L, C_G(L))_{\sqsubset (C, D)} \simeq \mathcal{A}_p(C) * \mathcal{A}_p(D) \simeq *$, where the contractibility follows from the standard contractibility of $\mathcal{A}_p(C)$, as in the argument after (i) in the proof of Theorem 4.10. This contractibility for all (C, D) in turn completes the proof that $\tilde{i}$ is a homotopy equivalence; and hence completes the proof of Claim 3. ■

And that in turn completes the proof of Proposition 4.12. ■

5. Using pre-join replacement-posets for homology propagation

In this section, we will study how to propagate non-zero homology using the more general context of the pre-join of posets (which might not be $\mathcal{A}_p$ -posets), namely, propagating (essentially) from a pre-join poset $X * Y$, up to a larger poset W .

In proving our new propagation result Theorem 5.10, we will in effect give a pre-join generalization of the arguments leading up to the original propagation result [4, Lemma 0.27] of Aschbacher and Smith (we give a statement of their result as Lemma 5.11). Note that their overall central product context at [4, Hypothesis 0.15] is correspondingly generalized at Hypothesis 5.4 below. So after that, we will be providing various associated definitions, which roughly generalize those starting 0.9 and especially 0.19 of [4].

First, we recall some standard features of chain complexes and homology for general posets. The following constructions work in any commutative ring with unit R ,

but for simplicity we will work with $R = \mathbb{Q}$ or $\mathbb{Z}$. We also suppress the notation of the coefficient ring R , and we implicitly assume that all the computations are always carried out in R . For a finite poset X and $n \geq -1$, let $C_n(X)$ denote the n th chain group of the augmented chain complex of X . Recall that $C_n(X)$ is the free R -module generated by the n -chains of X (these are inclusion-chains of length n —that is, with $(n + 1)$ members). And $Z_n(X)$ is the group of n -cycles, the kernel of the boundary map.

We emphasize an important notational distinction:

For an abstract X in this section, we write the order-relation as $<$ —used “unspecifically”,

meaning that in later applications, we will specify $<$ —as either standard group-inclusion $<$; or else one of our variants from Sections 3 and 4 such as $<_*$.

We follow [4, Section 0] in using the following general terminology.

Definition 5.1 (Full and a -initial chains). Let X be a finite poset, and $a \subseteq X$ an n -chain.

- (1) For an arbitrary element $\alpha \in C_n(X)$ we write $\alpha = \sum_b \alpha_b \cdot b$, where b runs over the set of all chains n -chains of X , and $\alpha_b \in R$ is trivial except for a finite number of n -chains b . We write $a \in \alpha$ if and only if $\alpha_a \neq 0$. If $\alpha \in Z_n(X)$ gives a non-zero cycle in $\tilde{H}_n(X)$ and $a \in \alpha$, we say that a (or its right-end member $\max(a)$) *exhibits* homology for X .
- (2) We say that a is a *full* chain in X , if for every $x \in X$ such that $a \cup \{x\}$ is a longer chain, we have $x > \max(a)$; that is, $\text{Lk}_X(a) \subseteq X_{>\max(a)}$. (That is, there is “no room” for extending a either below or within it.)
- (3) For $a \neq \emptyset$, a chain $b \in \text{sd}(X)$ containing a is called an *a -initial chain* if for every $x \in b \setminus a$, we have that $x > \max(a)$ (i.e., b “begins with” a .)
- (4) If $\beta \in C_m(X)$ is any chain, then $\beta_a = \sum_{b: b \text{ is } a\text{-initial}} \beta_b \cdot b$.

In particular, if a is a full chain, then every chain b containing a is a -initial.

Furthermore, a is a *maximal* chain in X if it cannot be lengthened, that is,

$$\text{Lk}_X(a) = \emptyset.$$

In particular, a maximal chain is full. ■

Our further development below will continue the theme around Remark 4.5, of propagation calculations which exploit a full chain a as in Definition 5.1 (2) above, in interaction with restrictions arising from $(\mathcal{QD})$ -type conditions.

Next, in the spirit of earlier Hypothesis 4.4, which provided the more general group-theoretic context for the propagation in Lemma 5.11—we give in Hypothesis 5.4 below a more general poset-context for our new propagation in Theorem 5.10.

Here is a brief, informal preview. We will propagate up to a poset W as in Theorem 4.10—where this W is the union of a pre-join with a related $\mathcal{F}$ -poset, for $\mathcal{F}$ in the spirit of Definition 3.2. Now, the naive analogue of the classical case would begin propagation just at that pre-join. However, we saw after Remark 4.5 that we wish to make the non-classical choice of $H := L$ rather than LB ; so in order to include treatment of the p -outer B in the p -extension LB , we will actually begin propagation at the union of the pre-join with $O_{LB}(L)$ —which is a subposet of the $\mathcal{F}$ -poset in W . (Indeed, in Hypothesis 5.2 below, we see that $O_{LB}(L)$ will be the model for the more abstract “ Z ” in our upcoming Hypothesis 5.4.)

We now flesh out the above overview more formally.

First, we present our augmented version Hypothesis 5.2 of the earlier component-Hypothesis 4.2.

Hypothesis 5.2. Assume Hypothesis 4.2 for L, B . Choose $K := C_G(LB)$ as in the classical case.

But now take $H := L$ only (even though we will sometimes have $B > 1$). Recall from Lemma 4.3 that we still get Hypothesis 4.4—so that we can apply the later results from Section 4. In particular, we can choose our replacement-poset as in Theorem 4.10 as follows:

$$W := W_G^{\mathcal{B}}(H, K) = \mathcal{B}_p(H) \ast \mathcal{A}_p(K) \cup \mathcal{F}_G(HK) = X \ast Y \cup \mathcal{F}_G(HK),$$

where we have used the abbreviations $X := \mathcal{B}_p(H) = \mathcal{B}_p(L)$ and $Y := \mathcal{A}_p(K) = \mathcal{A}_p(C_G(LB))$. Finally, set $Z := O_{LB}(L)$.

Here, we specify the abstract order-relation “ $<$ ” as the relation $\prec$ in the definition of W . ■

Next—mimicking the sequence after earlier Hypothesis 4.2, and again postponing briefly the proof—we see in part (b) of Lemma 5.3 below that the above hypothesis is a case of a somewhat more general poset-Hypothesis 5.4, which is what is actually assumed in our new propagation result Theorem 5.10. We also add a further part (a)—which we use in later Corollary 5.12, in checking that Lemma 5.11 (i.e., our quoted statement of “old” [4, Lemma 0.27]) is in fact a particular case of our new-Theorem 5.10.

Lemma 5.3. *Assume that we have either*

- (a) *Hypothesis 4.4 for H, K —with $X := \mathcal{A}_p(H)$, $Y := \mathcal{A}_p(K)$, $Z := \emptyset$, and with W given by $\mathcal{A}_p(G)$ under group-inclusion $<$, or*
- (b) *Hypothesis 5.2 for L, B (including the definitions of H, K, X, Y, Z , and W under $\prec$).*

Then, we also get Hypothesis 5.4 below for X, Y, Z, W .

Hypothesis 5.4 (The poset-context (Zleft)). For our new propagation results, we will work in the following poset-context; with the order-relation denoted unspecifically by $<$.

- (Zleft) (i) we have posets X, Y, Z (disjoint) and W , with $(X * Y) \cup Z \subseteq W$,
(ii) if $(x, y) \in X * Y$ and $z \in Z$ are comparable in W , then $z < (x, y)$,
(iii) if $(x, y) \in X * Y$ with $y \neq 1$, then $Z \subseteq W_{<(x,y)}$.

We identify X and Y as subposets of W via $X \times 1$ and $1 \times Y$, respectively. ■

For propagation in the context of the (Zleft)-Hypothesis 5.4, we will be showing how to construct homology cycles for W , from homology cycles of $X \cup Z$ and Y . But first, let us see the following proof which we had postponed.

Proof of Lemma 5.3. First, consider the posets X, Y, Z, W chosen in part (a) of the lemma. Here, we have $X * Y = \mathcal{A}_p(H) * \mathcal{A}_p(K)$, which as in Corollary 4.8 can be identified with the subposet of $\mathcal{A}_p(HK)$ whose elements are of form $C \cdot D$, where $(C, D) \in \mathcal{A}_p(H) * \mathcal{A}_p(K)$; these $\mathcal{A}_p$ -posets are disjoint, since $H \cap K$ is a p' -group by Hypothesis 4.4. And since here we also have $Z = \emptyset$, we now easily get the desired containment for (i) of (Zleft):

$$X * Y \cup Z = X * Y = \mathcal{A}_p(H) * \mathcal{A}_p(K) \subseteq \mathcal{A}_p(HK) \subseteq \mathcal{A}_p(G) = W,$$

as chosen in (a). And again, using $Z = \emptyset$, we get conditions (ii) and (iii) vacuously. This completes (a).

We turn to the posets X, Y, Z, W chosen in (b), i.e., in Hypothesis 5.2—where we recall $H = L$ and $K = C_G(LB)$. Here, we observe that

$$Z = O_{LB}(L) \subseteq \mathcal{F}_G(LC_G(LB)) = \mathcal{F}_G(HK)$$

since, given a member $C \in O_{LB}(L)$, we have $C \cap LC_G(LB) \leq C \cap LC_G(L) = 1$. Further X, Y, Z are disjoint—since we have $H \cap K$ a p' -group just as in part (a); while members of $\mathcal{F}_G(HK)$ have trivial intersection with HK . Hence, $X * Y \cup Z \subseteq X * Y \cup \mathcal{F}_G(HK) = W$, as chosen in Hypothesis 5.2, giving (i). Now, just as we had emphasized in Remark 3.5 on the earlier X_G -construction, the order-relation $<$ in the construction of our present choice of

$$W = W_G^{\mathcal{B}}(H, K) = \mathcal{B}_p(H) * \mathcal{A}_p(K) \cup \mathcal{F}_G(HK)$$

shows that elements $z \in Z \subseteq \mathcal{F}_G(HK)$ can only appear *below* elements of $\mathcal{B}_p(H) * \mathcal{A}_p(K) = X * Y$ —giving (ii). Finally, the p -outer $B \in Z = O_{LB}(L) \leq LB$ commutes with all of the members of $\mathcal{A}_p(C_G(LB)) = Y$; so we see that for any (C, D) with $D > 1$ as assumed in (iii), we in fact get $1 < D = C_D(B) \leq C_{CD}(B)$ —so that $B < (C, D)$ in the ordering on W , as desired. ■

We mention that conditions (ii) and (iii) for Z , in the poset-context in our (Zleft)-Hypothesis 5.4 above, will tie in with our various $\mathcal{F}$ -left conditions in Sections 3 and 4; and hence with the left-focusing theme of earlier Remark 3.5. This is of course the reason for the nickname “(Zleft)”. To continue with that viewpoint.

Remark 5.5 (Simplex format under Hypothesis 5.2, for boundary calculations under (Zleft)). We are now in a position to expand on our earlier preview of left-focused format in Remark 3.5; here is a brief (and still somewhat informal) overview of corresponding aspects in the upcoming arguments.

We will mainly describe applications of propagation based on LB under Hypothesis 5.2; and we had mentioned in introducing hypothesis that we expect $B > 1$. However, we will essentially avoid dealing with the full poset of elementary subgroups $\mathcal{A}_p(LB)$ of LB —by working instead with the smaller poset $\mathcal{B}_p(L) \cup O_{LB}(L) = X \cup Z$; here we are using the language of the (Zleft)-Hypothesis 5.4, as we may by Lemma 5.3. For this smaller poset, we will have a suitable maximal-dimension homology condition analogous to $(\mathcal{QD})_p$; cf. Theorem 6.6. As usual, we then study a chain a —involved in a non-zero cycle α in that appropriate maximal dimension. Recall that by definition of the ordering in $W_G^{\mathcal{B}}(L, C_G(LB)) = W$, we get the format:

(i) A chain a of $X \cup Z$ has the format $a = (B_1 < \cdots < B_r < A_1 < \cdots < A_s)$, where $B_i \in O_{LB}(L) = Z$ and $A_j \in \mathcal{B}_p(L) = X$. In particular, the p -outer terms B_i all appear at the *left* end of the sequence (prior to the occurrence of the order-relation $<$, which may differ from group-inclusion $<$).

Next, just as in the classical case in earlier Remark 4.5, we must in fact consider products, namely, of α , with β arising from $Y = \mathcal{A}_p(C_G(LB))$; see Lemma 5.9 below for the appropriate generalization $T_*(\alpha \otimes \beta)$ of the classical shuffle product. We will wish to apply propagation to these products, to see they remain non-zero in the homology of W , hence establishing (H-QC).

Note that these usual α and β appear in hypotheses (1), (2)(a), and (3) of our new propagation result Theorem 5.10. So, in order to apply that propagation, results assuming Hypothesis 5.2 will mainly focus on establishing the other hypothesis (2)(b); note that this is the relevant analogue of (i') and (ii'), in the interaction-viewpoint of Remark 4.5. The condition in (2)(b) primarily involves eliminating the possibility of any member of $\mathcal{F}_G(LC_G(LB))$ lying in $\text{Lk}_M(a)$ there; note that this analysis takes place in the left-hand part of (i) above—before A_1 , hence in the vicinity of the B_i . After that, it remains to similarly eliminate members of $(\mathcal{B}_p(L) * \mathcal{A}_p(C_G(LB)))_{<A_s}$, and establish the precise structure in (2)(b) for members above A_s . Note that this latter analysis instead takes place in the right-hand part of (i) above—first before or among the A_j , and then after A_s . These remarks illustrate the “visual” convenience (mentioned earlier in the paper) of the format in (i), arising from the definition of the ordering.

Now, the argument so far has basically reduced $\text{Lk}_W(a)$ to members of $X \ast Y$ having the special form given in (2) (b). Applying the propagation result, Theorem 5.10 will show in effect that these members are *not* an obstruction to propagation—by reducing to a boundary calculation just in Y . We had made a similar in-effect comment at Remark 4.5, in describing 0.27 of [4] (i.e., our quoted result Lemma 5.11); and indeed the new propagation proof below is essentially a pre-join version of that earlier proof. ■

Now, we begin the details for our pre-join variant of homology propagation.

The next few pages, through Lemma 5.9, require the (Zleft)-Hypothesis 5.4 only for X, Y, Z ; we will not involve W until Theorem 5.10. More precisely, recall that we regard X and Y as subposets of W , via $X \times 1$ and $1 \times Y$, respectively. So, before propagating to W , we will first in Lemma 5.9 below construct product-homology cycles for $(X \ast Y) \cup Z$, from homology cycles of $X \cup Z$ and Y .

We begin by introducing some notions which will allow us to generalize the classical shuffle product; recall that definition was given at [4, p. 483], and later extended in [16, Section 3].

Let a be a chain of $X \cup Z$. Then, $a = a_Z \cup a_X$, where $a_Z = a \cap Z$, $a_X = a \cap X$. Note by condition (Zleft) (ii) that for all $z \in a_Z$, and $x \in a_X$, we get $z < (x, 1)$, where of course we have identified $(x, 1)$ in the pre-join $X \ast Y$ with $x \in X$. That is, (Zleft) (ii) in particular, essentially reproduces the $\mathcal{F}$ -left property in the poset W of Hypothesis 5.2, so we still have the simplex format of Remark 5.5 (i):

- (i') A chain a of $X \cup Z$ has the form $a = a_Z \cup a_X = (z_1 < \cdots < z_r < x_1 < \cdots < x_s)$,

where $z_i \in Z$ and $x_j \in X$.

Definition 5.6 (The notation $a_X \ast b$, and classical shuffles). Now, take a chain $b \subseteq Y$. Following [4, Definition 0.19], we write

$a_X \ast b :=$ the chain “ $a_X < b$ ”, obtained by placing a_X to the left end of b .

More formally, $a_X \ast b$ must be a $<$ -chain in the pre-join $X \ast Y$; so our discussion here is a variant of that in [4], which takes place instead in the context of $\mathcal{A}_p(HK)$. Namely, we first “naively” regard $a_X \ast b$ just as a sequence, from the set $X \cup Y$ —and then we specify the requirements for the needed *poset*-relations, in terms of the Cartesian product coordinates of the pre-join $X \ast Y$. Note that the chain $a_X \ast b$ is a_X -initial, in the sense of Definition 5.1 (3).

As a simple example: suppose we have $a = (z_1 < z_2 < x_1 < x_2)$ and $b = (y_1 < y_2)$; then from the naive-sequence (x_1, x_2, y_1, y_2) , we get the $<$ -chain

$$a_X \ast b = ((x_1, 1) < (x_2, 1) < (x_2, y_1) < (x_2, y_2))$$

in the pre-join. In particular, the terms $y_i \in b$ in the sequence for $a_X * b$ are appearing in the pre-join element $a_X * b$ —but *not* in the form $(1, y_i)$ (from our standard identification of Y with a subset of the pre-join)—they appear instead in the two right-hand terms, in the form needed for the poset-relations in the pre-join.

This sequence-viewpoint also makes it easier to discuss the classical shuffle product.

Recall, essentially as in [4, p. 483], that a *shuffle* is a set-permutation σ of the sequence for $a_X * b$, whose image preserves the original order within the two subsets a_X and b . Thus, it can “shuffle” members of b to the left, among those of a_X (and, in particular, the result need not be a_X -initial). For instance, if we apply the shuffle $\sigma = (13)(24)$ to the sequence for $a_X * b$ in the example above, we get the shuffled sequence (y_1, y_2, x_1, x_2) —resulting in the pre-join element given by $((1, y_1) < (1, y_2) < (x_1, y_2) < (x_2, y_2))$.

We will write $(a_X \times b)_\sigma$ for the pre-join element corresponding to the shuffled-sequence $\sigma(a_X * b)$; and then the classical *shuffle product* of a_X and b is defined as the alternating sum of those elements

$$a_X \times b := \sum_{\sigma} (-1)^{\sigma} (a_X \times b)_{\sigma},$$

where σ runs through all the shuffle permutations. ■

We now examine a special feature of the shuffling process above—making crucial use of our Z -left Hypothesis 5.4. Consider some member (x, y) of $(a_X * b)$ for some σ . First, if $y = 1$, then $x \in a_X$ (with no member of b shuffled before it); and we see using (Zleft) (ii) (compare (i') above) that each $z \in a_Z$ has $z < (x, 1) = (x, y)$. Otherwise, $y > 1$, and here by (Zleft) (iii), we see that each $z \in a_Z$ has $z < (x, y)$. Thus, we obtain the fundamental property

Every element of a_Z is $<$ -below every element of $(a_X \times b)_{\sigma}$.

Therefore, we can write $a_Z \cup (a_X \times b)_{\sigma}$, for the unique $<$ -chain obtained by adding the elements of a_Z to the bottom of the chain $(a_X \times b)_{\sigma}$. (This ability to add-before illustrates another visually natural feature of the format in (i') above.) We can extend this notation by linearity and define the operation

$$(a, b) \mapsto \sum_{\sigma} (-1)^{\sigma} (a_Z \cup (a_X \times b)_{\sigma}) =: a_Z \cup (a_X \times b).$$

We emphasize a notational point here: our definition above is on *pairs* (a, b) of *chains* taken from $(X \cup Z)$ and Y ; we are *not* claiming there is any subposet of form $(X \cup Z) \ast Y$ —the important point here is that the *image* of our map is in fact a chain, in the target-poset $(X \ast Y) \cup Z$.

Definition 5.7. Let a, b be chains in $X \cup Z$ and Y as above, under the (Zleft)-Hypothesis 5.4 for X, Y, Z . Then, the *initial shuffle product* T of a and b (i.e., of the pair (a, b) in $\text{sd}(X \cup Z) \times \text{sd}(Y)$) is

$$T(a, b) := a_Z \cup (a_X \times b). \quad \blacksquare$$

In Lemma 5.9, we record pre-join versions of the basic facts (cf. [4, pp. 484–485]) about the classical shuffle product—for use with our generalized version of the product in Definition 5.7.

To this end, we first introduce the following useful notation: if a denotes a chain of a poset X , which we write as a sequence $a = (x_0 < \cdots < x_r)$, then for $0 \leq i \leq r$, we set

$$\partial_i(a) := a \setminus \{x_i\} = (x_0 < \cdots < \widehat{x_i} < \cdots < x_r).$$

We let $|a| = r + 1$ denote the size of a as a set; recall that this a has length r as a poset-chain—and defines an r -chain in the chain complex $C_*(X)$.

We will also use the following straightforward extension to complexes of the notion of full chain in earlier Definition 5.1 (2).

Definition 5.8. Let X be a finite poset. For a subcomplex $M \subseteq \mathcal{K}(X)$, and a chain $a \in \text{sd}(X)$ such that $a \in M$, we say a is a *full chain in M* if $\text{Lk}_M(a) \subseteq \mathcal{K}(X_{>\max(a)})$. $\blacksquare$

We now recollect the basic facts needed for our generalization to the initial shuffle product.

Lemma 5.9. Let X, Y, Z be as in the (Zleft)-Hypothesis 5.4, and let a, b denote chains of $X \cup Z$ and Y , respectively. Write $a_Z := a \cap Z$ and $a_X := a \cap X$.

- (1) If $a_Z = \emptyset$, then $T(a, b) = a \times b$ is the classical shuffle product.
- (2) T extends by linearity to a map of chain complexes

$$T_* : C_m(X \cup Z) \otimes C_n(Y) \rightarrow C_{m+n+1}((X \ast Y) \cup Z).$$

- (3) In particular, if α is a cycle of $X \cup Z$, and β is a cycle of Y , then $T_*(\alpha \otimes \beta)$ is a cycle of $(X \ast Y) \cup Z$.
- (4) The a -initial part of $T(a, b)$ is $T(a, b)_a = a_Z \cup (a_X \ast b)$. (Recall Definition 5.6 for $\ast$.)
- (5) If V is a finite poset, and c is a full chain of a subcomplex $M \subseteq \mathcal{K}(V)$, then when $\gamma \in C_t(M)$ for some t , we have $(\partial\gamma)_c = (\partial(\gamma_c))_c$.

Proof. Item (1) is clear from the definition of the initial shuffle product T .

We next show item (2). We need to prove that T_* commutes with the boundary operator; that is, that $T_*(\partial(a \otimes b)) = \partial(T_*(a \otimes b))$, for every m -chain a of $X \cup Z$, and n -chain b of Y .

We first compute $T_*(\partial(a \otimes b))$. Using the graded Leibniz derivation rule, we get

$$\partial(a \otimes b) = (\partial a) \otimes b + (-1)^m a \otimes (\partial b).$$

Decomposing $(\partial a) \otimes b$ into its Z - and X -sums, we get

$$\begin{aligned} T_*(\partial(a \otimes b)) = T_* \left(\sum_{i=0}^{|a_Z|-1} (-1)^i \partial_i(a_Z \cup a_X) \otimes b \right. \\ \left. + \sum_{i=|a_Z|}^m (-1)^i \partial_i(a_Z \cup a_X) \otimes b + (-1)^m a \otimes (\partial b) \right). \end{aligned}$$

Moving T_* inside the sums by linearity and re-indexing the second for the a_X -ordering we have

$$\begin{aligned} &= \sum_{i=0}^{|a_Z|-1} (-1)^i T_*((\partial_i(a_Z) \cup a_X) \otimes b) \\ &\quad + (-1)^{|a_Z|} \sum_{i=0}^{m-|a_Z|} (-1)^i T_*((a_Z \cup \partial_i(a_X)) \otimes b) \\ &\quad + (-1)^m T_*((a_Z \cup a_X) \otimes (\partial b)). \end{aligned}$$

Recall from Definition 5.7 that we defined T_* on pairs (a, b) of chains from $X \cup Z$ and Y ; so identifying $a \otimes b$ with the pair (a, b) , and applying that definition (which has the effect of shifting relevant parentheses), we have

$$\begin{aligned} &= \sum_{i=0}^{|a_Z|-1} (-1)^i (\partial_i(a_Z) \cup (a_X \times b)) + (-1)^{|a_Z|} \sum_{i=0}^{m-|a_Z|} (-1)^i (a_Z \cup (\partial_i(a_X) \times b)) \\ &\quad + (-1)^m (a_Z \cup (a_X \times (\partial b))). \end{aligned}$$

Now, combining the last two summands, while extracting the common factor a_Z , yields

$$\begin{aligned} &= \sum_{i=0}^{|a_Z|-1} (-1)^i (\partial_i(a_Z) \cup (a_X \times b)) \\ &\quad + (-1)^{|a_Z|} a_Z \cup \left(\sum_{i=0}^{m-|a_Z|} (-1)^i (\partial_i(a_X) \times b) + (-1)^{m-|a_Z|} (a_X \times (\partial b)) \right) \\ &= \sum_{i=0}^{|a_Z|-1} (-1)^i (\partial_i(a_Z) \cup (a_X \times b)) + (-1)^{|a_Z|} a_Z \cup (\partial(a_X \times b)); \end{aligned}$$

where the last equality holds by the classical shuffle product

$$\partial(a \times \beta) = \partial(a) \times \beta + (-1)^{|a|} a \times \partial(\beta).$$

See [6, Theorem 5.2].

The above expression for $T_*(\partial(a \otimes b))$ will now turn out to be equal to our computation below of $\partial(T_*(a \otimes b))$:

$$\begin{aligned} \partial(T_*(a \otimes b)) &= \partial(a_Z \cup (a_X \times b)) \\ &= \sum_{i=0}^{|a_Z|-1} (-1)^i (\partial_i(a_Z) \cup (a_X \times b)) \\ &\quad + \sum_{\sigma} (-1)^{\sigma} \sum_{i=|a_Z|}^{m+n+1} (-1)^i \partial_i(a_Z \cup (a_X \times b)_{\sigma}). \end{aligned}$$

In view of the expression obtained above for $T_*(\partial(a \otimes b))$, it remains to show that the sum over σ in the second line of the previous equation coincides with $(-1)^{|a_Z|} a_Z \cup (\partial(a_X \times b))$. We have

$$\begin{aligned} &\sum_{\sigma} (-1)^{\sigma} \sum_{i=|a_Z|}^{m+n+1} (-1)^i \partial_i(a_Z \cup (a_X \times b)_{\sigma}) \\ &= (-1)^{|a_Z|} \sum_{\sigma} (-1)^{\sigma} \sum_{i=0}^{m-|a_Z|+n+1} (-1)^i a_Z \cup \partial_i((a_X \times b)_{\sigma}) \\ &= (-1)^{|a_Z|} a_Z \cup \left(\sum_{\sigma} (-1)^{\sigma} \sum_{i=0}^{m-|a_Z|+n+1} (-1)^i (\partial_i(a_X \times b)_{\sigma}) \right) \\ &= (-1)^{|a_Z|} a_Z \cup (\partial(a_X \times b)). \end{aligned}$$

This is exactly the expression that we obtained above for $T_*(\partial(a \otimes b))$, completing the proof of item (2). Note also that item (3) follows from item (2).

Item (4) follows from the same property of the classical shuffle product since

$$T(a, b)_a = (a_Z \cup (a_X \times b))_a = a_Z \cup (a_X \times b)_{a_X} = a_Z \cup (a_X * b).$$

Item (5) holds in the context of general posets; cf. [4, Lemma 0.24]. ■

Now, we will state and prove our pre-join version of the homology propagation lemma [4, Lemma 0.27] (for comparison, we will state that earlier result afterward, as Lemma 5.11).

Theorem 5.10. *Let X, Y, Z, W be as in the (Zleft)-Hypothesis 5.4. Assume M is some intermediate simplicial complex, namely, one satisfying*

$$\mathcal{K}((X \ast Y) \cup Z) \subseteq M \subseteq \mathcal{K}(W).$$

Suppose that we further have the following.

- (1) *A non-zero homology cycle $\alpha \in Z_m(X \cup Z)$, $m \geq 0$.*
- (2) *A chain $a \in \alpha$ that the following hold:*
 - (a) *the coefficient q of a in α is invertible,*
 - (b) *$\text{Lk}_M(a) \subseteq \mathcal{K}(\{\max(a)\} \times Y)$. In particular, a is a maximal chain of $X \cup Z$.*
- (3) *A non-zero homology cycle $\beta \in Z_n(Y)$, $n \geq -1$.*

Then, $T_(\alpha \otimes \beta)$ defines a non-zero homology cycle in $Z_{m+n+1}(M)$.*

Proof. We mimic the proof of [4, Lemma 0.27]—i.e., our quoted result Lemma 5.11 below. Compare also with the proof of [16, Lemma 3.14], quoted as Lemma 5.14.

By Lemma 5.9 (3), $T_*(\alpha \otimes \beta)$ is a cycle of $(X \ast Y) \cup Z$. So, since $\mathcal{K}((X \ast Y) \cup Z) \subseteq M$ by hypothesis (which implicitly involves the containment $(X \ast Y) \cup Z \subseteq W$ in (Zleft) (i)), it follows that $T_*(\alpha \otimes \beta)$ is also a cycle of M . Suppose by way of contradiction that it is in fact a boundary

$$T_*(\alpha \otimes \beta) = \partial(\gamma) \tag{5.1}$$

for some $\gamma \in C_{m+n+2}(M)$. Note that hypothesis (2) (b) implies that a is a full chain of M . Then, we may apply Lemma 5.9 (5) to get

$$(\partial(\gamma_a))_a = (\partial(\gamma))_a = T_*(\alpha \otimes \beta)_a = q(a_Z \cup (a_X \ast \beta)), \tag{5.2}$$

where the second equality follows using (5.1), and the third using Lemma 5.9 (4).

We describe the chains involved in γ_a ; these must involve elements of $\text{Lk}_M(a)$. We saw via our format-discussion—at (i') after Remark 5.5—that we may write $a = a_Z \cup a_X$, where a_X has the form $((x_{r+1}, 1) < \cdots < (x_m, 1))$ (with all members of a_Z below $\min(a_X) = (x_{r+1}, 1)$). Let $c \in \gamma_a$ be a -initial, and pick an element $w \in c \setminus a$. By the description of $\text{Lk}_M(a)$ in hypothesis (2) (b), it follows that $w > \max(a) = (x_m, 1)$; so $w = (x_m, y)$ for some element $y \in Y$ —by the definition of the ordering in the pre-join. Hence, we see $c = a_Z \cup (a_X \ast \hat{c})$, where $\hat{c} = \{y \in Y : (x_m, y) \in c\}$. Note that $\hat{c}$ is a non-empty chain in Y since $|c| = m + n + 2 > m = |a|$. If we write

$$\gamma_a = \sum_{i \in I} q_i c_i = \sum_{i \in I} q_i (a_Z \cup (a_X \ast \hat{c}_i)),$$

then working at the i th term, we have

$$\partial(a_X * \hat{c}_i) = \sum_j (-1)^j \partial_j(a_X * \hat{c}_i);$$

and so, since the a -initial part of the boundary only removes members of the chain $\hat{c}_i$, we see that

$$\partial(a_X * \hat{c}_i)_a = \sum_{j > |a_X|} (-1)^j \partial_j(a_X * \hat{c}_i) = (a_X * (-1)^m \partial(\hat{c}_i)).$$

Thus, we obtain

$$\begin{aligned} (\partial(\gamma_a))_a &= \sum_i q_i (\partial(a_Z \cup (a_X * \hat{c}_i)))_a = \sum_i q_i (a_Z \cup (a_X * (-1)^m \partial(\hat{c}_i))) \\ &= a_Z \cup \left(a_X * (-1)^m \partial \left(\sum_i q_i \hat{c}_i \right) \right). \end{aligned} \quad (5.3)$$

Equating the final values for $(\partial(\gamma_a))_a$ in (5.2) and (5.3), we get

$$q(a_Z \cup (a_X * \beta)) = a_Z \cup \left(a_X * (-1)^m \partial \left(\sum_i q_i \hat{c}_i \right) \right),$$

and this holds if and only if

$$q\beta = \partial \left((-1)^m \sum_i q_i \hat{c}_i \right).$$

But then, $\beta = \partial(q^{-1}(-1)^m \sum_i q_i \hat{c}_i)$ is zero in the homology $\tilde{H}_n(Y)$, contrary to hypothesis (3). This contradiction completes the proof. ■

As a natural first application, we will deduce the original propagation result [4, Lemma 0.27] from Theorem 5.10. For reference, we first provide a statement of that earlier result.

Lemma 5.11 (Original homology propagation, [4, Lemma 0.27]). *Assume the p' -central product Hypothesis 4.4 for $H, K \leq G$; and that the following further conditions hold:*

- (i) *for some A exhibiting $(\mathcal{QD})_p$ for H , $\mathcal{A}_p(G)_{>A} \subseteq A \times K$,*
- (ii) *$\tilde{H}_*(\mathcal{A}_p(K), \mathbb{Q}) \neq 0$.*

Then, also $\tilde{H}_(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.* ■

Corollary 5.12. [4, Lemma 0.27] (i.e., Lemma 5.11) is a particular case of Theorem 5.10.

Proof. Let A, H, K, G be as in the hypotheses of [4, Lemma 0.27]—i.e., Lemma 5.11 above. We saw in Lemma 5.3 (a) that we get the (Zleft) Hypothesis 5.4, when we make the indicated choices $X = \mathcal{A}_p(H)$, $Y = \mathcal{A}_p(K)$, $Z = \emptyset$, and $W = \mathcal{A}_p(G)$ under group-inclusion $<$.

We now check the conditions required for our new propagation Theorem 5.10, for the further choice $M = W = \mathcal{A}_p(G)$. By hypothesis (i) of Lemma 5.11, H has $(\mathcal{QD})_p$ exhibited by A ; giving a non-zero homology cycle $\alpha \in Z_{m_p(H)-1}(\mathcal{A}_p(H))$ and a chain $a \in \alpha$ such that $\max(a) = A$ —yielding condition (1) for Theorem 5.10. Moreover, since in [4] the authors work with rational homology, we immediately get invertibility for the coefficient q of a in α —giving condition (2) (a). The latter part of hypothesis (i) also gives

$$\mathcal{A}_p(G)_{>\max(a)} = \mathcal{A}_p(G)_{>A} \subseteq \{A\} \times \mathcal{A}_p(K).$$

And since A has maximal rank in $\mathcal{A}_p(H)$, we have

$$\text{Lk}_{\mathcal{A}_p(G)}(a) \subseteq \mathcal{A}_p(G)_{>A};$$

and this with the previous inclusion completes condition (2) (b). Finally, condition (3) holds since $\tilde{H}_*(\mathcal{A}_p(K)) \neq 0$ by hypothesis (ii).

Since all the conditions for Theorem 5.10 hold, we conclude that

$$\tilde{H}_*(\mathcal{A}_p(G)) = \tilde{H}_*(M) \neq 0. \quad \blacksquare$$

Remark 5.13. We can alternatively prove the above corollary by invoking Theorem 5.10, with the same posets $X = \mathcal{A}_p(H)$, $Y = \mathcal{A}_p(K)$, $Z = \emptyset$, but by taking $M = W$ to be the Thévenaz poset $\hat{X}_G(HK)$ of Proposition 3.9 (with its order-relation $<$)—of course we saw there that this poset is homotopy equivalent to $\mathcal{A}_p(G)$. $\blacksquare$

We mention that for the purposes of this paper, the main applications of Theorem 5.10 will come in the proof of Theorem 6.4—more precisely, in the proofs of Lemmas 6.7 and 6.8, covering the two parts of that theorem.

We close the section by giving a statement below the extended propagation result [16, Lemma 3.14]; which we will use several times later—but which does *not* follow from Theorem 5.10 above. (However, note the similarity of conditions (iv) + (ii) in the lemma with condition (2) (b) in the theorem.)

The lemma holds for homology with coefficients in any commutative ring R with unit. For the purposes of this article, we can suppose that $R = \mathbb{Q}$ or $\mathbb{Z}$.

Lemma 5.14 ([16, Lemma 3.14]). *Assume that Hypothesis 4.4 for $H, K \leq G$ have*

- (i) $[H, K] = 1$, and $H \cap K$ is a p' -group.

Further, assume there is $X \subseteq \mathcal{A}_p(G)$ with the following.

- (ii) $\mathcal{N}_G(K) \subseteq X$.
- (iii) There exist a chain $a \in \text{sd}(\mathcal{A}_p(H)) \cap \text{sd}(X)$ and a cycle $\alpha \in C_m(\mathcal{A}_p(H)) \cap C_m(X)$ such that the coefficient of a in α is invertible, and α is non-zero in $\tilde{H}_m(\mathcal{A}_p(H))$ (for some $m \geq -1$).
- (iv) In addition, a is a full chain in X , and also $X_{>\max(a)} \subseteq \mathcal{N}_G(K)$.
- (v) $\tilde{H}_*(\mathcal{A}_p(K)) \neq 0$.

Then, $\tilde{H}_*(X) \neq 0$. ■

6. Eliminating many simple components of Lie type in the same characteristic p

When we prove later Theorem 9.2 (i.e., our extension of the Aschbacher–Smith main theorem stated earlier as Theorem 1.1), we will see toward the end of that proof that in the case $p = 3$ (cf. Remark 9.3), the argument for [4, Theorem 5.3] breaks down for components L given by the simple Ree groups in characteristic 3. We will want to avoid this problem by establishing an elimination result: to guarantee (earlier in the overall proof) that such a component could not arise in our counterexample.

In fact, it turns out to be no more difficult, in our results Theorems 6.4 and 6.5 below, to eliminate a wide array of simple Lie-type components L in the “same” characteristic p (i.e., the p for which we study (H-QC)), subject to a restriction on the p -outers $O_G(L)$ arising in G . This class of possible components is significant: for example, these same-characteristic groups give the main family (1) of obstructions in the Aschbacher–Smith $(\mathcal{QD})_p$ -List (which we have quoted as Theorem 2.16).

To treat these cases, we will use propagation techniques applied to some of the replacement-posets we studied in the previous section. We will show in Theorem 6.4 that if the p -outers of a simple component L of Lie type in characteristic p are of “field” type Φ (in the sense of Definition 6.3 below), then (H-QC) holds for our group G —under suitable inductive hypotheses (notably (H1)) on $C_G(L)$.

It will be convenient to abbreviate our overall generic-context hypothesis with the following name.

Definition 6.1. We say that a group L is of type $(sLie-p)$, if it is of Lie type in characteristic p , and is simple.

We expand on several technical issues, which may not be obvious, related to this definition.

(1) Our choice of L simple—rather than just quasisimple (e.g., for a component)—is primarily for convenience. In fact, our arguments will typically apply to L

quasisimple of Lie type¹⁵, and indeed we hope, in a future version, to adjust this section to proceed under that more general quasisimple hypothesis (qLie- p). For now, we assume L is simple, and this guarantees, for example, that L is adjoint (cf. [12, Theorems 2.2.6 and 2.2.7]).

(2) Of course, those adjoint Lie type groups are almost always automatically simple, and those simple L in (sLie- p) correspondingly provide our generic case. But we also need to indicate the status, for our arguments, of the few non-generic simple groups not satisfying (sLie- p): these arise from the four non-solvable members of the set called $\mathcal{L}ie_{\text{exc}}$ in [12, Definition 2.2.8]; namely, the Lie-type groups $L^+ = B_2(2), {}^2F_4(2), G_2(2), {}^2G_2(3)$. For these, it is the commutator subgroup $(L^+)'$ (of index $p = 2, 2, 2, 3$) which is the actual simple group. These simple commutators $(L^+)'$ are considered (more or less honorarily) to be of Lie type—for the purposes of stating the CFSG. However, they are not themselves of Lie type, in the strict sense of satisfying the Lie theory (though the overgroup L^+ does). As a result, these four commutator subgroups—namely, $\text{Sp}_4(2)' \cong \text{Alt}_6$, ${}^2F_4(2)'$ (the Tits group), $G_2(2)' \cong \text{U}_3(3)$, and the smallest Ree group ${}^2G_2(3)' \cong \text{L}_2(2^3)$ —do not satisfy (sLie- p) for the relevant primes $p = 2, 2, 2, 3$. Hence, they must be excluded, when we apply results depending on the full Lie theory—notably our generic analysis of outer automorphisms in the set Φ mentioned earlier, as in Theorem 6.4. However, we are still able to treat these four non-generic cases via other arguments, particularly under the inductive hypothesis (H1)—as we will see in Theorem 6.5 below. Of course, we will need to make a specific mention of this exceptional situation later, when any of these non-generic cases arises. ■

Thus, (sLie- p) gives the main hypothesis for our generic-case Theorem 6.4 below.

Furthermore, now *assuming* characteristic- p in (sLie- p), we can also provide some initial motivation for another main hypothesis used in part (2) of Theorem 6.4, namely,

(p -cyclic) The members of $\text{O}_G(L)$ are cyclic.

That is, the p -outers arising in G should be of order exactly p . This means that $\text{O}_G(L)$ will be a poset of dimension 0, which will be important for our applications.

We can begin the motivation for frequently expecting (p -cyclic) by using just the overview description of $\text{Out}(L)$ in earlier Remark 2.6. Observe first the following.

Under (sLie- p), we have $p = r$ (namely, the characteristic prime “ r ” for L).

¹⁵But note that we avoid the further generalization to just quasisimple with $L/Z(L)$ of Lie type: for then we might get an “exceptional” Schur multiplier term (cf. [12, Table 6.1.3]) in $Z(L)$, which would prevent us from applying the Lie theory to L itself.

And then since p cannot divide the order $p^a - 1$ of the multiplicative group of the field $\mathbb{F}_{p^a}$ in the description of diagonal automorphisms in (d) there, we further see the following.

- (i) $\text{Outdiag}(L)$ is a p' -group.

It follows from (i) that each member of $O_G(L)$ maps faithfully into the quotient

$$\text{Out}(L)^* := \text{Out}(L) / \text{Outdiag}(L).$$

We observe next that each $\text{Outdiag}(L)$ -conjugacy class falls into a single coset of $\text{Outdiag}(L)$ —which might contain several such classes. So, to investigate the behavior of some particular individual p -outer $B \in O_G(L)$, i.e., defined at the group-element level in $\text{Aut}(L)$, we can view B up to such conjugacy via its image B^* in $\text{Out}(L)^*$, i.e., at the level of quotients modulo inner-diagonal elements. In summary:

- (ii) Study p -outers via their faithful images in $O_G(L)^* \subseteq \mathcal{A}_p(\text{Out}(L)^*)$.

Now, we move on to a description of $\text{Out}(L)$ more detailed than in Remark 2.6.

Caution: We now temporarily—up to the end of Remark 6.2—suspend our assumption that $p = r$ (which came from (sLie- p)); that is, this more-detailed description holds for simple L of Lie type of general characteristic r , independent of our special prime p for (H-QC).

The above quotient $\text{Out}(L)^*$ can be described via a standard normal series for $\text{Aut}(L)$, which more generally gives a natural viewpoint on the *idfg*-representation of automorphisms that we quoted in Remark 2.6. One form of the series appears in (7-1) of [11, Part I], but we will use the more extensive treatment in [12, Section 2.5]. By Theorem 2.5.12 (b) there, $\text{Out}(L)$ is a split extension of $\text{Outdiag}(L)$ ¹⁶ by the product

$$\text{Out}(L)^* = \Phi_L \Gamma_L \quad \text{with } \Phi_L \text{ cyclic and normal;} \quad (6.1)$$

where Φ_L and Γ_L are given in parts (a) (b), and (c) of Definition 2.5.10 there—for present purposes, we'll just informally single out a few overview features.

First, elements of Γ_L require elements of the group Δ of symmetries of the Dynkin diagram—and we listed the possible Dynkin diagrams for L with $\Delta > 1$, in (i) of Remark 2.6. Second, the normal subgroup Φ_L is the image of the natural field automorphisms of an overlying algebraic group $\bar{L}$: arising (cf. [12, Definition 1.15.5 (a)]) from powers of the generator $x \mapsto x^r$ of the Galois group of the algebraic closure $\bar{\mathbb{F}}_r$ of the characteristic r of the field $\mathbb{F}_q$ of definition of L . This, in particular, shows that Φ_L is cyclic.

Furthermore, the case of $\Delta > 1$ for $\Phi_L \Gamma_L$ splits into cyclic and “nearly cyclic” subcases below.

¹⁶We mention also that $\text{Outdiag}(L)$ is cyclic—except when $L \cong D_n(q)$ with $n \geq 4$ even and q odd, where $\text{Outdiag}(L) = C_2 \times C_2$.

Remark 6.2 (The cases for $\Phi_L \Gamma_L$). From [12, Theorem 2.5.12 (f),(e)], we get the following possibilities for the product $\text{Out}(L)^* = \Phi_L \Gamma_L$ in (6.1).

- (0) For twisted L , or untwisted L with diagram having $\Delta = 1$: $\Gamma_L = 1$; so $\Phi_L \Gamma_L = \Phi_L$ is cyclic.
Otherwise, L is untwisted with diagram having $\Delta > 1$ and the possibilities are:
- (1) For untwisted L of type B_2, F_4, G_2 with $r = 2, 2, 3$: $\Phi_L \Gamma_L$ is cyclic, and $\Phi_L \Gamma_L / \Phi_L \cong \Delta \cong C_2$.
- (2) For untwisted types $A_{\geq 2}, D_{\geq 4}, E_6$: $\Phi_L \Gamma_L = \Phi_L \times \Gamma_L$, and $\Gamma_L \cong \Delta \cong C_2$, or Sym_3 for D_4 . ■

Now, as case (0) above suggests, we will often have the condition $\text{Out}_G(L)^* \leq \Phi_L$; indeed, under (sLie- p) where $p = r$, we will usually get the following stronger condition for p -outers:

$$(p\text{-}\Phi) \quad \text{O}_G(L)^* \subseteq \mathcal{A}_p(\Phi_L).$$

Note that $(p\text{-}\Phi)$ implies $(p\text{-cyclic})$ since Φ_L is cyclic in (6.1).

Furthermore, the definition of Φ_L via powers of $x \mapsto x^r$ suggests that its members exhibit “field-like enough” behavior for our purposes in this section. (Of course, under (sLie- p) we will have $p = r$.) Consequently, we have the following definition.

Definition 6.3. Assume that L has type (sLie- p), and that $L \leq G$ for some group G . We say that a p -outer $B \in \text{O}_G(L)$ of order p is of type Φ if $B^* \leq \Phi_L$. Write $\text{O}_G^\Phi(L)$ for the subposet of $\text{O}_G(L)$ whose elements are of type Φ :

$$\text{O}_G^\Phi(L) = \{B \in \text{O}_G(L) : |B| = p \text{ and } B^* \leq \Phi_L\}.$$

Note that $\text{O}_G^\Phi(L)$ is a *discrete* poset of cyclic p -outers (no proper $<$ -relations). ■

In the following technical result, the two cases corresponding to Remark 3.10 will proceed under somewhat different hypotheses, for reasons to be indicated later.

So, we state our generic theorem below. We work with rational homology. We will prove the following.

Theorem 6.4 (Generic Lie-eliminations under (sLie- p)). *Assume p is a prime, and G a finite group satisfying the following:*

(sLie- p) G has a component L of type (sLie- p).
Assume further that one of the following conditions holds.

- (allC+H) For all $B \in \text{O}_G(L)$, $O_p(C_G(LB)) > 1$, and $C_G(L)$ satisfies (H-QC).
- (someNC+ Φ) For some $B_0 \in \text{O}_G^\Phi(L)$, $\tilde{H}_*(\mathcal{A}_p(C_G(LB_0))) \neq 0$, and $(p\text{-cyclic})$ holds for L .

Then, G satisfies (H-QC).

In particular, if G satisfies (H1), and also contains a component L of type (sLie- p) with $(p-\Phi)$ (that is, $O_G(L)^ \subseteq \mathcal{A}_p(\Phi_L)$), then G satisfies (H-QC). Notice that any L of Lie-rank 1 has $(p-\Phi)$ by Remark 6.2 (0).*

Furthermore, the above statement holds if “(H1)” is replaced by “(H1u) + (p odd)”.

We will postpone the proof of Theorem 6.4—while we first discuss two aspects of its hypotheses.

First aspect. The further hypotheses (allC+H) and (someNC+ Φ) of Theorem 6.4 are of course extensions of the case-division we had described earlier in Remark 3.10. Namely,

(allC)—where for all $B \in O_G(L)$ we have $O_p(C_G(LB)) > 1$;

(someNC)—where we have some B with $O_p(C_G(LB)) = 1$.

These extensions are meant (roughly) to provide purely propagation-theoretic hypotheses. Note that (someNC+ Φ) adds, to the propagation-friendly situation of (someNC), a p -outer B_0 which must specifically be of the type Φ , which behaves well in homology calculations; here we will need the assumption (p -cyclic) to guarantee propagation. By contrast, hypothesis (allC+H) adds to (allC) a propagation-friendly non-zero homology condition on $C_G(L)$; in applications, we will typically obtain this condition essentially automatically, via a suitable inductive hypothesis such as (H1)—as in the “In particular” statement in the theorem.

Second aspect. Here, we argue roughly that under the main hypothesis (sLie- p) of Theorem 6.4, the later hypothesis (p -cyclic) in (someNC+ Φ), and indeed the stronger hypothesis ($p-\Phi$), should hold “most of the time”. Furthermore, the discussion below of when those cyclic hypotheses might fail, under the generic hypothesis (sLie- p), will also lead essentially to the list of exceptional situations, that we must further treat in the non-generic Theorem 6.5—thus basically explaining the list of excluded Lie types (i.e., that we do not treat) in the statement of that theorem below.

So, during this second-aspect discussion, we continue to assume (sLie- p), and argue to isolate those situations where ($p-\Phi$)—that is, the condition that $O_G(L)^* \subseteq \mathcal{A}_p(\Phi)$ —might fail.

We saw in earlier (ii) that we may study individual members of $O_G(L)$, up to $\text{Outdiag}(L)$ -conjugacy, via the cases for $\Phi_L \Gamma_L$ in Remark 6.2. In case (0) (i.e., for twisted groups, and untwisted groups where $\Delta = 1$) we have $\text{Out}(L)^* = \Phi$, giving ($p-\Phi$). Therefore, we are reduced to the untwisted groups L in cases (1) and (2) there (namely, with $\Delta > 1$).

First, assume case (1), with multiple bonds, and characteristic $r = p$ by (sLie- p). If L has type G_2 , we have $p = r = 3$, with $|\Phi_L \Gamma_L : \Phi_L| = 2$, and, in particular,

3-outers lie in Φ_L . Here, $O_G(L)^* \subseteq \mathcal{A}_3(\Phi)$, and again, we have $(p-\Phi)$. So, consider instead types B_2 and F_4 , where we have $p = r = 2$. We first single out the odd-power field cases. When $O_2(\Phi_L) = 1$ (i.e., L is defined over some $\mathbb{F}_{2^{2a-1}}$), then we have $\text{Out}(L)^* = \Phi_L \times \Gamma_L$ with Φ_L of $2'$ -order, so $(p-\Phi)$ fails but $(p\text{-cyclic})$ does hold. Here, $\text{Sp}_4(2)$ is not simple, and so, does not arise under $(\text{sLie-}p)$. However, we treat $\text{Sp}_4(2)'$ via independent arguments in the non-generic Theorem 6.5. In fact, we are also able to treat $F_4(2)$, and indeed the higher odd-powers for types B_2 and F_4 , similarly there. Thus, these groups B_2 and F_4 with odd-power fields do not appear in the excluded list in that theorem below. Now, we turn to groups of those types with even-power fields: in this case we have $O_2(\Phi_L) > 1$; so since $\Phi_L \Gamma_L$ is cyclic, the 2-elements of $\Phi_L \Gamma_L \setminus \Phi_L$ must have order at least 4—hence $(p-\Phi)$ holds here, and the groups are treated via our generic methods.

Finally, assume case (2), with single bonds, and characteristic $r = p$ by $(\text{sLie-}p)$. Here, we always have $\text{Out}(L)^* = \Phi_L \times \Gamma_L$ with $\Gamma_L \cong \Delta \cong C_2$, or Sym_3 for D_4 . Now, it might happen that $(p-\Phi)$ holds, when $O_G(L)^*$ induces only field automorphisms on L , and then we can use our generic methods. But it may also happen that p divides the order of both Φ_L and Γ_L , and then we see that $(p-\Phi)$ and $(p\text{-cyclic})$ can definitely fail. If $O_G(L)^*$ induces only a graph- or graph-field automorphism, we could still have $(p\text{-cyclic})$. However, we do not attempt to treat such automorphisms by our methods; so these single-bond groups, in the case away from $(p-\Phi)$, appear in the excluded-list in the non-generic theorem below.

Notice that we have now reduced to exactly the cases in that excluded-list.

In fact, it is now not difficult to establish our non-generic Theorem 6.5: this time not constructively via propagation as in the generic-Theorem 6.4, but instead via the less-constructive elimination methods of [18]—which turn out to be suited to the few, and suitably-“small”, cases that we in practice must treat.

Theorem 6.5 (Lie eliminations under (H1)—including non-generic cases). *Assume that G satisfies (H1) and contains a component L which is simple of Lie type in characteristic p , in the wider CFSG-sense: either L satisfies $(\text{sLie-}p)$, or L is one of the four Lie_{exc} -commutator groups in Definition 6.1 (2).*

Then, one of the following holds:

- (1) G satisfies $(H\text{-}QC)$,
- (2) L is one of the following untwisted Lie types with the indicated characteristic $r = p$:

$$\text{PSL}_n(2^{2m}) (n \geq 3); \quad D_n(p^{pm}) \quad (n \geq 4; p = 2, 3); \quad E_6(2^{2m})$$

with $O_G^\Phi(L) \neq \emptyset$ and some $B \in O_G(L)$ inducing a graph-automorphism or graph-field automorphism.

We mention that there are further technical conditions related to B which could reduce the excluded-list above, but we have preferred to keep this simpler form of the statement. We provide in Remark 6.9 some further discussion of the excluded cases.

Proof of Theorem 6.5. Note that under (H1), we may assume by [18, Theorem 1.6] that $O_G(L)$ is non-empty.

Here, most of the work is done by the “In particular” statement for (H1) in the generic Theorem 6.4: that is, we are done in the vast majority of cases, where L has both (sLie- p) and (p -cyclic) (and indeed (p - Φ)). Thus, we only need to treat the cases where one of the hypotheses fails.

Now, the simple groups failing the strict Lie-theory condition of (sLie- p) are the four commutator subgroups in Definition 6.1 (2). And our analysis above isolated the cases where (p - Φ) and (p -cyclic) can fail: our excluded-list records the cases that we will not attempt to treat—but we will in fact cover $B_2(q)$ and $F_4(q)$ for q an odd power of $p = 2$.

We note that the condition (p -cyclic) arises in each of the separate cases just listed in the paragraph above (it is needed as hypothesis for the results in [18]).

We first consider the four non-generic commutator groups.

First, $L \cong {}^2F_4(2)'$ has $O_G(L) = \emptyset$ since outer 2-automorphisms B have order 4 as group elements in LB ; so this case is already handled by our initial remark.

Next, the smallest case $L \cong \mathrm{Sp}_4(2)'$ is handled specifically for $p = 2$ in [18, Corollary 8.1].

Furthermore, [18, Corollary 8.1] invokes Proposition 6.9 there, and indeed via an easier application of that proposition, we can in fact treat in parallel the remaining two cases, i.e., $L = (G_2(2)', {}^2G_2(3)')$ for $p = (2, 3)$. Since we may assume $O_G(L)^*$ is non-empty, we see that it is given by B^* inducing an automorphism of type (graph, graph-field), respectively, (though the first case is instead also called graph-field in [12]—see Definition 2.5.13(b)(2) there). We check then that $C_L(B)$ has structure (order 24, Sym_3) with $O_p(C_L(B)) > 1$, and it follows that the reduced homology of $\mathcal{A}_p(C_L(B))$ for any term vanishes (i.e., in all degrees). However, $\mathcal{A}_p(L)$ is a wedge of spheres of topological dimension $(1, 0)$. So, taking these values for “ k ” in [18, Proposition 6.9], we conclude that G satisfies (H-QC).

The elimination of the groups of type $(B_2(q), F_4(q))$ (for q an odd power of $p = 2$) is only marginally more complicated. Here, $O_G(L)^*$ is either empty, or given by B^* inducing an automorphism of (graph, graph) type (again with [12] using the graph-field terminology). Now, we get $C_L(B)$ of structure $({}^2B_2(q), {}^2F_4(q))$: so that $\mathcal{A}_2(C_L(B))$ is a wedge of spheres in topological dimension $(0, 1)$, with reduced homology vanishing in degrees above those values. However, $\mathcal{A}_2(L)$ is a wedge of spheres in topological dimension $(1, 3)$. So, by taking these values for “ k ” in [18, Proposition 6.9], we conclude that G satisfies (H-QC).

Finally, the structure of the groups in item (2) follows by Remark 6.9. ■

We now start the details for our earlier preview, and culminating in the proof of our generic Theorem 6.4.

Recall that the more technical aspects of our study of p -outer automorphisms are presented in Section B of the appendix—notably the conventions in Definition B.1 for the terminology of “field” and “graph” automorphisms.

First, we obtain an analogue of the Quillen dimension property of [4] (recall Definition 2.15), in the sense that, for our almost simple extensions, we obtain non-zero homology in a particular natural degree: given L of type (sLie- p) with Lie rank m , and a p -outer B of type Φ , this degree for $\mathcal{A}_p(LB)$ is in fact m . Since the behavior arises from an underlying use of the Bouc poset $\mathcal{B}_p(L)$ of L , we have nicknamed the property as “Bouc dimension” below. And this homology will be the starting point for our propagation up to $\mathcal{A}_p(G)$. We establish the following theorem.

Theorem 6.6 (“Bouc dimension”). *Assume that L is of type (sLie- p) with Lie rank m , and we have some $B \in \mathcal{O}_{\text{Aut}(L)}^\Phi(L)$. Then, $\tilde{H}_m(\mathcal{A}_p(LB)) \neq 0$.*

Proof. Fix $B \in \mathcal{O}_{\text{Aut}(L)}^\Phi(L)$. By part (3) of Proposition B.2, we may choose $D \in \mathcal{O}_{LB}^\Phi(L)$ such that $O_p(C_L(D)) = 1$. (We would only need to make this choice, if our original B happened to be in case (2)(a) of the proposition.) Since $LB = LD$, we may take D in the role of “ B ”; thus we have $O_p(C_L(B)) = 1$ —that is, we have our chosen B in case (1) or (2)(b) of the proposition.

We are going to work with the Bouc poset $\mathcal{B}_p(L)$. Recall that $\mathcal{B}_p(L)$ is homotopy equivalent to the Tits building of L : it is a poset of topological dimension $m - 1$, where m is the Lie rank of L , and it is homotopy equivalent to a wedge of spheres of dimension $m - 1$. The number of such spheres is the dimension of $\tilde{H}_{m-1}(\mathcal{A}_p(L)) = \tilde{H}_{m-1}(\mathcal{B}_p(L))$ as a vector-space, which in turn coincides with the order of a Sylow p -subgroup of L , namely, $|L|_p$. In particular, we have $\tilde{H}_*(\mathcal{A}_p(L)) = \tilde{H}_{m-1}(\mathcal{B}_p(L))$.

We will use the abbreviation $\mathcal{O} := \mathcal{O}_{LB}(L)$. By Theorem 4.10 applied to LB —with $L, 1$ in the roles of “ H, K ”—we get

$$\mathcal{A}_p(LB) \simeq W_{LB}^{\mathcal{B}}(L, 1) (= \mathcal{B}_p(L) \cup \mathcal{O}).$$

Therefore, $\mathcal{A}_p(LB)$ is homotopy equivalent to the poset $\mathcal{B}_p(L) \cup \mathcal{O}$ of topological dimension m .

We will now let D vary over $\mathcal{O}$. By (3) of Proposition B.2, we get $D^* \subseteq \Phi_L$; and since we also have $B^* = \Omega_1(O_p(\Phi_L))$ there, we see that $D^* = B^*$. Next, apply the proposition to this general D : then either case (2)(a) holds, with $O_p(C_L(D)) > 1$ —so that $\mathcal{A}_p(C_L(D))$ is contractible; or else case (1) or case (2)(b) holds, with $C_L(D)$ again a group of Lie type in characteristic p of the same Lie rank m (possibly extended by p' -diagonal automorphisms), so that $\mathcal{A}_p(C_L(D))$ is homotopy equivalent to a wedge of $|C_L(D)|_p$ spheres of topological dimension $(m - 1)$. In any case, the

reduced homology of $\mathcal{A}_p(C_L(D))$ vanishes in degrees below $(m - 1)$, just as for L itself.

At this point, we invoke the long exact sequence of [21, Main theorem] and take L, LB in the roles of “ N, G ” there. Note that “ $\mathcal{M}$ ” there is $\mathcal{O}$ here, which consists of cyclic elements of order p of type Φ . With the notation of that article (where Σ denotes suspension), we have that $\mathcal{A}_p(LB)_L \simeq \bigvee_{D \in \mathcal{O}} \Sigma \mathcal{A}_p(C_L(D))$. Now, we saw that homology from $\mathcal{A}_p(L)$ and $\mathcal{A}_p(C_L(D))$ vanishes (in any case, contractible or otherwise) in degrees below $(m - 1)$; and similarly homology from $\mathcal{A}_p(LB)$ vanishes in degrees above the topological dimension m . So, we get

$$\begin{aligned} 0 \rightarrow \tilde{H}_m(\mathcal{A}_p(LB)) &\rightarrow \bigoplus_{D \in \mathcal{O}} \tilde{H}_{m-1}(\mathcal{A}_p(C_L(D))) \rightarrow \\ &\rightarrow \tilde{H}_{m-1}(\mathcal{A}_p(L)) \rightarrow \tilde{H}_{m-1}(\mathcal{A}_p(LB)) \rightarrow 0. \end{aligned}$$

Using the above exact sequence of vector spaces, we now calculate and give a lower bound for the dimension of $\tilde{H}_m(\mathcal{A}_p(LB))$. Note we get a zero contribution from any cases where $\mathcal{A}_p(C_L(D))$ is contractible; and we will use the fact, from (1) or (2) (b) of Proposition B.2, that there is just one $\text{Inndiag}(L)$ -conjugacy class of D with the indicated centralizer structure, represented by our original choice B :

$$\begin{aligned} &\dim(\tilde{H}_m(\mathcal{A}_p(LB))) \\ &= \dim(\tilde{H}_{m-1}(\mathcal{A}_p(LB))) - \dim(\tilde{H}_{m-1}(\mathcal{A}_p(L))) \\ &\quad + \sum_{D \in \mathcal{O}} \dim(\tilde{H}_{m-1}(\mathcal{A}_p(C_L(D)))) \\ &\geq -\dim(\tilde{H}_{m-1}(\mathcal{A}_p(L))) + \sum_{D \in \mathcal{O}} \dim(\tilde{H}_{m-1}(\mathcal{A}_p(C_L(D)))) \\ &\geq -\dim(\tilde{H}_{m-1}(\mathcal{A}_p(L))) + \sum_{g N_L(B) \in L/N_L(B)} \dim(\tilde{H}_{m-1}(\mathcal{A}_p(C_L(B^g)))) \\ &= -|L|_p + |L : N_L(B)| |C_L(B)|_p \\ &= -|L|_p + \frac{|L|}{|C_L(B)|} \frac{|C_L(B)|}{|N_L(B)|} |C_L(B)|_p \\ &\geq -|L|_p + \frac{|L|}{|C_L(B)|} \frac{1}{(p-1)} |C_L(B)|_p \\ &= |L|_p \left(-1 + \frac{|L|_{p'}}{(p-1)|C_L(B)|_{p'}} \right). \end{aligned}$$

Now, our chosen B has $O_p(C_L(B)) = 1$ (that is, case (1) or (2) (b) of Proposition B.2); so Lemma B.3 gives the inequality

$$\frac{|L|_{p'}}{(p-1)|C_L(B)|_{p'}} > 1,$$

and hence the above dimension is positive. We conclude that $\tilde{H}_m(\mathcal{A}_p(LB)) \neq 0$, as desired. $\blacksquare$

We now begin the details for our main proof, of the generic Theorem 6.4, splitting it into the two cases of its conicality-hypotheses. We first deal essentially with the case (someNC+ Φ) there, corresponding to condition (someNC) in Remark 3.10.

Lemma 6.7. *Let G be a finite group such that the following hold:*

- (sLie- p) G has a component of L of type (sLie- p);
- (p -cyclic) $O_G(L)$ contains only cyclic p -outers;
- (someNC+ Φ)⁻ For some $B_0 \in O_G^\Phi(L)$, $\tilde{H}_*(\mathcal{A}_p(C_G(LB_0))) \neq 0$.

Then, G satisfies (H-QC).

Proof. As usual to show (H-QC) for G , we assume $O_p(G) = 1$, and show that

$$\tilde{H}_*(\mathcal{A}_p(G)) \neq 0.$$

Let L and B_0 be as in the hypotheses (sLie- p) and (someNC+ Φ)⁻. Then, the choices $H := L$ and $K := C_G(LB_0)$ satisfy the component-Hypothesis 4.2; and we showed in Lemma 4.3 that the less-usual choice of $H = L$ rather than LB_0 still gives the central product Hypothesis 4.4 for HK . So, by Theorem 4.10,

$$\mathcal{A}_p(G) \simeq W := W_G^{\mathcal{B}}(L, C_G(LB_0)) = \mathcal{B}_p(L) \ast \mathcal{A}_p(C_G(LB_0)) \cup \mathcal{F}_G(LC_G(LB_0)),$$

where we recall that $\prec$ denotes the order-relation in the poset W . Now, note that in the above we have $O_{LB_0}(L) \subseteq \mathcal{F}_G(LC_G(LB_0))$; and also we have an inclusion of posets

$$\mathcal{A}_p(LB_0) \simeq W_{LB_0}^{\mathcal{B}}(L, 1) = (\mathcal{B}_p(L) \times \{1\}) \cup O_{LB_0}(L) \subseteq W_G^{\mathcal{B}}(L, C_G(LB_0)).$$

We are going to invoke Theorem 5.10: with the subcomplex $M = \mathcal{K}(W)$, and making the choices that $X = \mathcal{B}_p(L)$, $Y = \mathcal{A}_p(K)$, and $Z = O_{LB_0}(L)$. Recall that these are the choices of X, Y, Z, W made in our earlier augmented-Hypothesis 5.2 (extending the component-Hypothesis 4.2); and that in Lemma 5.3 (b), we showed that these choices satisfy the (Zleft)-Hypothesis 5.4, as needed for that theorem.

Now, we check the other conditions for Theorem 5.10. By Theorem 6.6, we see that $W_{LB_0}^{\mathcal{B}}(L, 1)$ has dimension m , and that it has non-zero homology in degree m . Since $\mathcal{B}_p(L)$ has dimension $m - 1$, there is a non-zero homology cycle α of $\tilde{H}_m(W_{LB_0}^{\mathcal{B}}(L, 1)) \cong \tilde{H}_m(\mathcal{A}_p(LB_0))$ with

$$\alpha = \sum_{i \in I} q_i (B^i \prec (R_1^i, 1) < \cdots < (R_m^i, 1)),$$

where $R_j^i \in \mathcal{B}_p(L)$ for all $j = 1, \dots, m$, and $B^i \in \mathcal{O}_{LB_0}(L)$. Moreover, since $\mathcal{B}_p(L)$ has dimension $m - 1$, note that

$$(R_1^i, 1) < \dots < (R_m^i, 1) \text{ is a maximal chain in } \mathcal{B}_p(L);$$

so, e.g., $R_m^i \in \text{Syl}_p(L)$ for all i . In particular, if $x \in W$ and $x > (R_m^i, 1)$, then just from the ordering in the Cartesian product we have $x = (R_m^i, E) \in \mathcal{K}(\{\max(a)\} \times Y)$, for some $E \in \mathcal{A}_p(K)$.

Now, let $a \in \alpha$ be one of the above chains involved in α , with invertible coefficient $q = q_i \neq 0$. We immediately get conditions (1) and (2) (a) of Theorem 5.10, and the above- a part of condition (2) (b) there. By $(\text{someNC}+\Phi)^-$, we can take a non-zero cycle β of $\mathcal{A}_p(K)$, giving condition (3) of Theorem 5.10. It remains to establish the rest of condition (2) (b); at this point, we will strongly use our hypothesis (p -cyclic). Namely, for a as just chosen, we will establish the following claim.

Claim. $\text{Lk}_M(a) \subseteq \mathcal{K}(\{\max(a)\} \times Y)$. In particular, a is a maximal chain of $X \cup Z$.

Proof. Suppose that $a = (B < (R_1, 1) < \dots < (R_m, 1))$, and let $w \in M \setminus a$ be $<$ -comparable with every element of a . Note that $|B| = p$, since $B \in \mathcal{O}_{LB_0}(L)$ and $|B_0| = p$. We also have $B \in \mathcal{F}_G(HK)$, so $B < w$ by the $\mathcal{F}$ -left property in the definition of the ordering. In fact, we will show that $w > \max(a) = R_m$, where we saw earlier that such a w satisfies the claim.

First, we saw earlier that $a \setminus \{B\}$ is a maximal chain in $\mathcal{B}_p(L)$, so we cannot have $1 \leq i < m$ with $(R_i, 1) < w < (R_{i+1}, 1)$. Thus, we only need to eliminate the case that $B < w < (R_1, 1)$.

So, suppose that $B < w < (R_1, 1)$. Now, $w \notin \mathcal{B}_p(L)$, again since we saw $a \setminus \{B\}$ is a maximal chain in $\mathcal{B}_p(L)$. Hence, $w = D \in \mathcal{F}_G(HK)$; that is, $D \cap (LC_G(LB_0)) = 1$. Since $D < (R_1, 1)$ we see that $C_{R_1}(D) > 1$, so D normalizes L . Then, $D > B$ implies that D also normalizes LB . Now, since $D \leq C_G(B)$, we get

$$\begin{aligned} D \cap (LC_G(L)) &= D \cap (LC_G(L)) \cap C_G(B) \\ &= D \cap (C_L(B)C_G(LB)) \\ &\leq D \cap (LC_G(LB_0)) = 1. \end{aligned}$$

Here, we have used the fact that $LB = LB_0$ since $B \in \mathcal{O}_{LB_0}(L)$. Thus, $D \in \mathcal{O}_G(L)$, which consists only of cyclic p -outers by our hypothesis (p -cyclic). This is contrary to D of p -rank at least 2, since $D > B$. This contradiction shows that no such D exists.

We have reduced to the case that $w > (R_m, 1) = \max(a)$; which we saw earlier satisfies the claim. In particular, a is a maximal M -chain in $X \cup Z$. ■

Therefore, Theorem 5.10 applies, with these choices of $X, Y, Z, W, M = \mathcal{K}(W)$, and of a, α, β ; so we get $\tilde{H}_*(\mathcal{A}_p(G)) \cong \tilde{H}_*(W) \neq 0$, as desired. ■

Next, we prove case (allC+H) of the generic Theorem 6.4, corresponding to condition (allC) in Remark 3.10.

Lemma 6.8. *Let G be a finite group such that the following hold:*

(sLie- p) *L is a component of G of type (sLie- p);*

(allC+H) *For all $B \in \text{O}_G(L)$, $O_p(C_G(LB)) > 1$, and $C_G(L)$ satisfies (H-QC).*

Then, G satisfies (H-QC).

Proof. We assume $O_p(G) = 1$, and prove that $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$.

We begin with a slightly easier version of the setup as in the previous lemma.

Let $H := L$, $K := C_G(L)$: again these choices satisfy the component Hypothesis 4.2; and again using Lemma 4.3 we see that these usual choices give the central product Hypothesis 4.4 for HK . This time, in order to avoid centralizers in $X *_Y$ (of members of $\text{O}_G(L)$) which lie entirely in L , we instead apply Proposition 4.12 (rather than Theorem 4.10) in order to be able to make use of the replacement-poset

$$\mathcal{A}_p(G) \simeq W := \tilde{W}_G^{\mathcal{B}}(H, K) = \mathcal{B}_p(H) *_K \mathcal{A}_p(K) \cup \mathcal{F}_G(HK),$$

namely, the poset of Proposition 4.11. Recall that $\sqsubset$ denotes the order-relation in the poset W .

Note now by the initial (allC)-part of hypothesis (allC+H), we have

$$\mathcal{F}_1 = \{B \in \text{O}_G(L) : O_p(C_G(LB)) > 1\} = \text{O}_G(L);$$

so that the subset $\mathcal{F}_1$ in fact covers all of $\text{O}_G(L)$.

In view of this, we will again be applying Theorem 5.10 for propagation: again with $M = \mathcal{K}(W)$ but now for the new variant- W as above; and with $X := \mathcal{B}_p(L)$, $Y := \mathcal{A}_p(K)$ and the “trivial” choice $Z := \emptyset$. These choices of X , Y , Z , W extend the component-Hypothesis 4.2; and we can also directly mimic the proof in the simpler case (a) of Lemma 5.3. Namely, we see that these choices satisfy the (Zleft)-Hypothesis 5.4, as needed for Theorem 5.10. That is, the inclusion of $X *_Y$ in our present choice of W for (Zleft) (i) is automatic—by construction; and the choice of $Z = \emptyset$ makes (ii) and (iii) there vacuous.

So, we turn to verifying the other conditions needed for Theorem 5.10.

Since $O_p(G) = 1$, we have $O_p(C_G(L)) = 1$ by Lemma 2.3 (4). Indeed, by the latter part of hypothesis (allC+H), we see that $\tilde{H}_*(\mathcal{A}_p(K)) = \tilde{H}_*(\mathcal{A}_p(C_G(L))) \neq 0$. Thus, we can take a non-zero homology cycle $\beta \in Z_n(\mathcal{A}_p(K))$ for some $n \geq -1$. This establishes condition (3) of Theorem 5.10.

By hypothesis (sLie- p), if m is the Lie rank of L , there exists a non-zero homology cycle $\alpha \in Z_{m-1}(\mathcal{B}_p(L))$, giving condition (1) of Theorem 5.10. To finish, we need

to show condition (2) of Theorem 5.10 for such an α . Pick $a \in \alpha$, so $a = ((R_1, 1) < \dots < (R_m, 1)) \subseteq W$. First, since we work with homology with rational coefficients, we automatically get condition (2) (a) of Theorem 5.10. For condition (2) (b), we need to show that $\text{Lk}_M(a) \subseteq \mathcal{K}(\{\max(a)\} \times Y)$. Since $M = \mathcal{K}(W)$, this is equivalent to showing that a is a full chain in W , and that if $w > \max(a)$ then $w \in \{\max(a)\} \times Y$.

So, let $w \in W$ be an element comparable with every element of a . Since a is a maximal chain of $\mathcal{B}_p(L)$, we see that either $w > \max(a)$, or else $w \sqsubset \min(a)$ with $w \notin X$. In the former case, by the definition of the order in W we see that $w = (\max(a), y) \in \{\max(a)\} \times Y$, for some $y \in Y$. In the latter case, $w =: D \in \mathcal{F}_G(LC_G(L))$. Since $D \sqsubset (R_1, 1)$, we have $C_{R_1,1}(D) > 1$, and this forces $D \in \mathcal{A}_p(N_G(L))$. Now, the condition $D \cap (LC_G(L)) = 1$ implies that $D \in O_G(L) = \mathcal{F}_1$, an equality we observed earlier using the (allC)-part of our hypothesis. By definition of the order-relation in the poset $W = \tilde{W}_G^{\mathcal{B}}(L, C_G(L))$ in Proposition 4.11, we see that $D \sqsubset (E, F) \in X * Y$ implies $C_{EF}(D) \not\leq L$, so $F > 1$, contrary to $D \sqsubset (R_1, 1)$ above. This contradiction shows that no such w exists, so this case cannot hold. In conclusion, $\text{Lk}_M(a) \subseteq \{\max(a)\} \times Y$, giving condition (2) (b) of Theorem 5.10.

We have fulfilled the conditions of Theorem 5.10, so $0 \neq \tilde{H}_*(M) = \tilde{H}_*(W) \cong \tilde{H}_*(\mathcal{A}_p(G))$, completing the proof. ■

Now, we can prove our generic Theorem 6.4.

Completing the proof of Theorem 6.4. We saw that Case (someNC+ Φ) holds by Lemma 6.7; while the other Case (allC+H) follows from Lemma 6.8: since both Lemmas assume the initial hypothesis (sLie- p) of the theorem.

Thus, we had already essentially completed the proof of the main assertion of the theorem. We now turn to the final “In particular” statements.

Suppose first that G satisfies (H1). Then, that assumption gives (H-QC) for $C_G(L)$ and for every centralizer $C_G(LB)$, where $B \in O_G(L)$. Suppose further that L has type (sLie- p), and that $(p-\Phi)$ holds. We see now that either $O_p(C_G(LB)) > 1$ for all $B \in O_G(L)$ (so that (allC+H) holds, using (H1) for $C_G(L)$ as above), or else that some $B \in O_G(L) = O_G^\Phi(L)$ verifies that $O_p(C_G(LB)) = 1$ —so since $C_G(LB)$ satisfies (H-QC), we get (someNC+ Φ). Then, we have indeed reduced to the hypotheses of the main statement of the theorem. Namely, G satisfies (sLie- p) and one of (allC+H) or (someNC+ Φ). Hence, G satisfies (H-QC) by that main result.

Now, suppose finally that we replace “(H1)” above with “(H1u)+(p odd)”; in particular, we are still assuming that G also satisfies (sLie- p) and $(p-\Phi)$ (and hence $(p$ -cyclic)). As usual for (H-QC), we assume $O_p(G) = 1$, and show that $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$.

As we are assuming the inductive hypothesis (H1u) of [4], we will use the results of that article. Since $C_G(L) < G$, and the components of $C_G(L)$ are components of G (by Lemma 2.3 (3)), we see that $C_G(L)$ satisfies (H-QC) by (H1u). Hence, if either

$O_G(L) = \emptyset$, or $O_p(C_G(LB)) > 1$ for all $B \in O_G(L)$, then (allC) holds, and in fact we have reduced to the hypothesis (allC+H) for the main statement of the theorem, and we get (H-QC) for G just as before.

Thus, to finish, we may assume (allC) fails, so there is $B \in O_G(L)$ such that $O_p(C_G(LB)) = 1$. And we need to show that $\tilde{H}_*(\mathcal{A}_p(C_G(LB))) \neq 0$. In view of our still-assumed conditions, this will complete the additional hypothesis (someNC+ Φ) as earlier, again reducing us to the main statement of the theorem.

For this non-zero condition, we will adapt arguments from [4] (notably Theorem 2.4 there), to show that there is a replacement $B' \in O_G(L)$ for B satisfying the non-zero homology condition for the desired centralizer. At some points in the argument, (H1u) will directly imply (H-QC) for G , and in those cases we will not need to get non-zero homology for the centralizer.

Following [4, Propositions 1.4 and 1.5], if we have non-trivial $N := Z(G)$ or $Z(E(G))$, then the components of G/N are covered by components of G . Since $\tilde{H}_*(\mathcal{A}_p(G/N)) \subseteq \tilde{H}_*(\mathcal{A}_p(G))$ by [4, Lemma 0.12], without loss of generality, we can suppose that $Z(G) = 1 = Z(E(G))$, since otherwise we get (H-QC) for G using (H1u). By our quoted elimination result Theorem 9.1, we can also assume that G has no component of type $L_2(2^3)$, $U_3(2^3)$, or $Sz(2^5)$, for $p = 3, 3, 5$, respectively.

Now, [4, Proposition 1.6] allows us to reduce to the case $O_{p'}(G) = 1$; and in particular, $F(G) = 1$ using our Lemma 2.3. Note that the argument in [4] that we are quoting here depends on Theorem 2.4 there—and hence on Theorem 2.3 there, which requires p odd; so that we are now using that part of our current “In particular” hypothesis. We are also using the elimination result indicated above, again in order to apply [4, Theorem 2.3].

Now, we pick $B \in O_G(L)$ as above. At this point, we again invoke [4, Theorem 2.4], this time with $C_G(L)$ in the role of “ I ” there; so in (ii) below, as in the previous paragraph, we are once again using our hypothesis that p is odd, and our eliminations, in order to apply [4, Theorem 2.3]. We check conditions (i) and (ii) of [4, Theorem 2.4] for this I , as follows.

- (i) Using our Lemma 2.3 (4), $Z(I) \leq F(I) = F(C_G(L)) = F(G) = 1$.
- (ii) The components of I , which are components of G by Lemma 2.3, have non-conical complements as defined in [4, Theorem 2.3]: since we have p odd, and we eliminated components of G of type $L_2(2^3)$, $U_3(2^3)$ or $Sz(2^5)$ for $p = 3, 3, 5$, respectively.

This establishes conditions (i) and (ii) of [4, Theorem 2.4] for $I = C_G(L)$. By that result, there exists a complement $B_0 \in \mathcal{A}_p(IB)$ to I in IB such that $1 = O_p(C_I(B_0)) = O_p(C_G(LB_0))$. In particular, $B_0 \in O_G(L)$ and $O_p(C_G(LB_0)) = 1$. Also, unitary components of $C_I(B_0) = C_G(LB_0)$ satisfy the conditions of (H1u) by the conclusions of [4, Theorem 2.4]. In consequence, by (H1u), we see $C_G(LB_0)$ satisfies (H-QC). So,

B_0 fulfills the requirements of (someNC+ Φ) of Theorem 6.4. This yields, as desired, the hypotheses of the main statement there—so that G satisfies (H-QC). ■

Remark 6.9. From Theorem 6.4, if G is a counterexample of minimal order to (H-QC) and contains a simple component L of Lie type in characteristic $r = p$, then one of the following three (mutually exclusive) conditions should hold.

- (1) (p -cyclic) fails and there exists $B \in O_G(L)$ with $O_p(C_G(LB)) = 1$.
- (2) (p -cyclic) holds, $O_G(L) \neq \emptyset$ and $O_G^\Phi(L) = \emptyset$.
- (3) (p -cyclic) holds, $O_G^\Phi(L) \neq \emptyset$, every $B \in O_G^\Phi(L)$ has $O_p(C_G(LB)) > 1$, and there exists $D \in O_G(L) \setminus O_G^\Phi(L)$ with $O_p(C_G(LD)) = 1$.

In cases (1) and (3), one gets $O_G^\Phi(L) \neq \emptyset$ and $O_G(L) \setminus O_G^\Phi(L) \neq \emptyset$, so L is untwisted and it is defined over a field of p^{pm} elements.

On the other hand, case (2) can be ruled out by an application of [18, Proposition 6.9]. Namely, under the conditions of (2), if $B \in O_G(L)$, then $\mathcal{A}_p(C_L(B))$ has homological dimension less than that of $\mathcal{A}_p(L)$. This can be proved by using a more detailed description of the structure of centralizers of graph and graph-field automorphisms: since $C_L(B)$ is a group of Lie type in characteristic p but of Lie rank less than that of L . See also [12, Propositions 4.9.1 and 4.9.2]. Then, the inclusion map $\mathcal{A}_p(C_L(B)) \hookrightarrow \mathcal{A}_p(L)$ is zero in homology. This fulfils the requirement for the application of [18, Proposition 6.9], where the value “ k ” there is given by the Lie rank of L minus 1. ■

7. Eliminating QD-components under (H1)

In this section, we provide in Theorem 7.2 an alternative version of [4, Proposition 1.7] (namely, elimination of $(\mathcal{QD})$ -components)—that works for any prime p , under a somewhat different inductive hypothesis. Our proof by contrast does not invoke the CFSG. We then use Theorem 7.2 in our proof, at the end of Section 9, of the variant Theorem 1.4 of our more direct extension Theorem 1.1 to $p = 3, 5$ of the Aschbacher–Smith main theorem.

But first, we will recall the original result [4, Proposition 1.7] which roughly establishes that under (H1u) for odd p , either G satisfies (H-QC), or every component of G has a p -extension failing $(\mathcal{QD})_p$. Namely, we give in Proposition 7.1 below a slightly different alternative formulation of [4, Proposition 1.7]—which is closer to the original argument than our Theorem 7.2, and which we expect could be applied in wider contexts, beyond the present paper.

The proof of [4, Proposition 1.7] uses Theorems 2.3 and 2.4 there to obtain the non-conical complement A needed for homology propagation. In hypothesis (MaxNC)

for our version in the proposition below, we in effect isolate that non-conicality argument (and in particular, its dependence on having odd p)—by roughly encoding the maximal choices made for A in the original Aschbacher–Smith proof.

Proposition 7.1. *Suppose that G satisfies the following conditions.*

(p -extQD) *G contains a simple component L of order divisible by p for which all the p -extensions satisfy $(\mathcal{QD})_p$.*

(MaxNC) *There exists $A \in \mathcal{A}_p(N_G(L))$ faithful on L and of maximal p -rank such that $|A \cap L|$ is maximal and $\tilde{H}_*(\mathcal{A}_p(C_G(LA))) \neq 0$.*

Then, G satisfies (H-QC).

In particular, if p is odd, (p -extQD) holds, and (H1) or (H1u) holds, then G satisfies (H-QC).

Proof. The proof of this proposition essentially follows the original proof given by Aschbacher–Smith for their [4, Proposition 1.7, pp. 489–490]. But note that Step iv of that original proof is now replaced by the hypothesis (MaxNC). That is, here we in effect assume the conclusion of [4, Theorem 2.4], so we are not using their Theorem 2.3, which would require p odd. Also, we do not need to invoke the CFSG since we have (MaxNC).

For the “In particular” part under (H1), we can use the earlier reductions under (H1) that we quoted in our Theorem 2.22—first parts (1), (2), to establish [4, Propositions 1.4–1.6]; and then parts (5), (6) to eliminate the problematic components $\mathrm{PSL}_2(2^3)$, $\mathrm{PSU}_3(2^3)$ and $\mathrm{Sz}(2^5)$, $p = 3, 3, 5$, respectively. Then, we may use the remainder of the original proofs of [4, Theorems 2.3 and 2.4], to establish (MaxNC)—completing the reduction to the hypotheses of the main statement, and hence giving (H-QC).

Under (H1u), the proof is essentially given using the adjustment to [4] that we used for the corresponding “In particular” statement in Theorem 6.4 above, which we now summarize only briefly. Here, we again get (MaxNC) from [4, Theorem 2.4]—recalling that we may apply Theorem 2.3 there, by first eliminating the above components $\mathrm{PSL}_2(2^3)$, $\mathrm{PSU}_3(2^3)$ and $\mathrm{Sz}(2^5)$, for $p = 3, 3, 5$, respectively, now via [4, Proposition 1.5] and our quoted result Theorem 9.1 under (H1u), and then reducing to $O_{p'}(G) = 1$ via [4, Proposition 1.6]. ■

We now provide, as Theorem 7.2 below, a further alternative version of [4, Proposition 1.7], that works for *any* prime p . In particular, we do not invoke [4, Theorems 2.3 and 2.4] which depend on the CFSG, assume p odd and further require the exclusion of components of type $\mathrm{PSL}_2(2^3)$, $\mathrm{PSU}_3(2^3)$, and $\mathrm{Sz}(2^5)$, $p = 3, 3, 5$, respectively. The proof of our theorem instead relies on the combinatorial properties of the $\mathcal{A}_p$ -posets, and does not invoke the CFSG.

In contrast with Proposition 7.1 just above, here we will not assume that

$$\tilde{H}_*(\mathcal{A}_p(C_G(LA))) \neq 0$$

for some $A \in \mathcal{A}_p(N_G(L))$ satisfying subtle conditions of maximality. Instead, we will develop fairly elementary conditions of maximality for A , within the proof—and then we will see roughly that we can homotopically remove any such A which would have zero-homology for $\mathcal{A}_p(C_G(LA))$ (that is, guaranteeing that our A produces a non-conical complement).

As a brief summary of the proof: because of the hypothesis (p -extHQC) of Theorem 7.2 below, zero-homology for $\mathcal{A}_p(C_G(LA))$ would force $O_p(C_G(LA)) > 1$. This guarantees the contractibility of links in $\mathcal{A}_p(G)$, so that by Proposition 3.11 we can remove such points A from our poset—getting a homotopy equivalent smaller poset X , to which we can apply the earlier propagation result which we quoted as Lemma 5.14. Any such remaining A will have the non-conical requirement for hypothesis (v) of Lemma 5.14. The removals also guarantee the condition on $X_{>A}$ needed in hypothesis (iv) of Lemma 5.14.

Theorem 7.2. *Suppose that G satisfies the following conditions.*

(p -extQD) *G contains a simple component L of order divisible by p for which all of the p -extensions $LB \leq G$, $B \in \hat{O}_G(L)$, satisfy $(QD)_p$.*

(p -extHQC) *If $LB \leq G$ is a p -extension of L , then $C_G(LB)$ satisfies (H-QC).*

Then, G satisfies (H-QC).

Notice in particular that (p -extHQC) holds when assuming (H1).

Proof. As usual for (H-QC), we suppose $O_p(G) = 1$ and show that $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$ under the hypotheses of the theorem. We use the notation of Proposition 3.11, in particular, recalling the undesirable conical-subset of p -outers defined by

$$\mathcal{F}_1 := \{B \in O_G(L) : O_p(C_G(LB)) > 1\}.$$

Eventually, we will choose an A exhibiting $(QD)_p$ for a suitable LB , so this LB will need to instead be a member of the corresponding non-conical subset

$$\mathcal{E} := \{LE : E \in \hat{O}_G(L) \setminus \mathcal{F}_1\}.$$

Note that $L \in \mathcal{E}$ (the case where $E = 1$) since $O_p(C_G(L)) = 1$ by Lemma 2.3 (4); in particular, we have $\mathcal{E} \neq \emptyset$.

Now, in pursuing a suitable LB , it is natural to pick a configuration $LE \in \mathcal{E}$, with $m_p(LE)$ maximal among the p -ranks of the groups in $\mathcal{E}$. Then, we take LE of maximal order subject to this property; in particular, then E is a maximal element of $\hat{O}_G(L) \setminus \mathcal{F}_1$. Just from the definition of E , as a p -outer of the component L not lying in $\mathcal{F}_1$, we have the following claim.

Claim 1. $O_p(C_G(LE)) = 1$, and hence, $O_p(LE) = 1$ using Lemma 2.5.

Next, we develop some conditions for suitable A and B in $\mathcal{A}_p(LE)$. Notice that any $A \in \mathcal{A}_p(LE)$ of maximal p -rank intersects L non-trivially: since p divides the order of L by (p -extQD), we see the following claim.

Claim 2. We take $A \in \mathcal{A}_p(LE)$ of p -rank $m_p(LE)$ such that $|A \cap L|$ is maximal. Then, $|A \cap L| > 1$, in particular, using Claim 1, we have $O_p(LA) \leq O_p(LE) = 1$.

Now, decompose $A = (A \cap L) \times B$. Since $O_p(LE) = 1$ by Claim 1, LE embeds into $\text{Aut}_G(L)$, so we see $LA = LB \leq LE$. By Claim 1 and Lemma 2.5, we get $O_p(LB) = 1 = O_p(C_G(LB))$, and thus $LB \in \mathcal{E}$. In particular, $B \in \hat{O}_G(L)$, we get

$$|B| = |A/A \cap L| = |LA|/|L| \leq |LE|/|L| = |E|,$$

so that using our choice in Claim 2 with $m_p(A) = m_p(LE)$, we have

$$m_p(LE) = m_p(A) \leq m_p(LB) \leq m_p(LE),$$

and hence equality holds for these ranks.

Finally, we show that we can indeed take this A to exhibit $(\mathcal{QD})_p$ for LB . Since we saw above that $O_p(LB) = 1$, we know by our hypothesis (p -extQD) that LB has $(\mathcal{QD})_p$ —exhibited by some $A_0 \in \mathcal{A}_p(LB)$ of p -rank $m_p(LB) = m_p(A)$. If we write $A_0 = (A_0 \cap L) \times B_0$. Then,

$$|B_0| = |A_0/A_0 \cap L| = |LA_0|/|L| \leq |LA|/|L| = |B|,$$

and by maximality of $|A \cap L|$, we also have

$$|A_0 \cap L| \leq |A \cap L|.$$

Now, $m_p(A_0) = m_p(A)$ implies that the two inequalities above must be equalities; that is, we must have $|B_0| = |B|$ and $|A_0 \cap L| = |A \cap L|$. Therefore, on replacing our original A by this A_0 which satisfies the same maximality properties, we can suppose without loss of generality that A exhibits $(\mathcal{QD})_p$ for LB . This occurs via some non-zero homology cycle, say α involving a full chain a ending in A , of $\mathcal{A}_p(LB)$.

We now verify the hypotheses for Lemma 5.14.

Recall from Proposition 3.11 the definition of the subset $\mathcal{N}_1$ of members of the inflation $\mathcal{N}_G(L)$ acting faithfully on L , and containing some member of $\mathcal{F}_1$:

$$\mathcal{N}_1 := \{D \in \mathcal{A}_p(N_G(L)) : D \cap L > 1, C_D(L) = 1, \exists B_0 \in \mathcal{F}_1 : B_0 < D\}.$$

Set $X := \mathcal{A}_p(G) \setminus \mathcal{N}$, where

$$\mathcal{N} := \{D \in \mathcal{N}_1 : m_p(LD) > m_p(LE)\}.$$

Note that $\mathcal{N}$ is an upward-closed subposet of $\mathcal{N}_1$. Therefore, $X \hookrightarrow \mathcal{A}_p(G)$ is a homotopy equivalence by Proposition 3.11.

Set $H := LB$ and $K := C_G(LB)$. Since elements of $\mathcal{N}$ are faithful on L , we immediately get conditions (i) and (ii) of Lemma 5.14.

Furthermore, our cycle α above exhibiting $(\mathcal{QD})_p$ for $LB = H$ now extends to a cycle for X : for note that if $D \in \mathcal{A}_p(LB)$ is involved in α , then we get $m_p(LD) \leq m_p(LB) = m_p(LE)$; so by definition, D cannot be contained in $\mathcal{N}$. This proves the following claim.

Claim 3. If $D \in \mathcal{A}_p(LB)$ is involved in α , then $D \in X$. In particular, α is also a cycle of X containing a (full) chain a with $A = \max(a)$. Hence, we get condition (iii) of Lemma 5.14.

Claim 4. If $D \in X_{>A}$, then $C_D(LB) > 1$. Hence, we get (iv) of Lemma 5.14.

Proof. Suppose by way of contradiction that $D \in X_{>A}$, but $1 = C_D(LB) = C_D(LA) = C_D(L)$. Since $D \cap L \geq A \cap L > 1$, we get $D \leq N_G(L)$.

Now, suppose by way of further contradiction that we have some $d \in (D \cap LC_G(L)) \setminus L$; write $d = lc$, with $l \in L$ and $c \in C_G(L)$, $|c| = p$. Since $[d, A] = 1$, we get $[l, A] = 1 = [c, A]$. But $l \in L \leq LA$, and A is of maximal p -rank in LA , so $l \in A \leq D$. Therefore, $c \in D$, that is, $c \in C_D(L) = 1$, contrary to c of order p above. This intermediate contradiction shows that $D \cap LC_G(L) = D \cap L$.

So, we may write $D = (D \cap L) \times D_1$, where now $D_1 \in \hat{\mathcal{O}}_G(L)$. Since $D \notin \mathcal{N}$, while we have $m_p(LD) \geq m_p(D) > m_p(LE) = m_p(A)$, we must have $D \notin \mathcal{N}_1$. In particular, $D_1 \notin \mathcal{F}_1$, and hence $LD = LD_1 \in \mathcal{E}$ with $m_p(LD_1) > m_p(LE)$. But this contradicts the earlier maximality of $m_p(LE)$ in $\mathcal{E}$. This final contradiction shows that $1 < C_D(L) = C_D(LB)$, as we wanted. ■

Finally, we saw earlier that $O_p(K) = O_p(C_G(LB)) = 1$. Hence, we get condition (v) of Lemma 5.14, using our hypothesis (p -extHQC).

In conclusion, we have shown that $X \simeq \mathcal{A}_p(G)$ with $H = LB$ and $K = C_G(LB)$ satisfy the hypotheses of Lemma 5.14. Thus,

$$\tilde{H}_*(\mathcal{A}_p(G)) \cong \tilde{H}_*(X) \neq 0,$$

as required. ■

8. Robinson subgroups and the Lefschetz-module version of Quillen's conjecture

In this section, we recall how to establish the Lefschetz-module version (L-QC) of Quillen's conjecture via the construction of *Robinson subgroups*; cf. the conditions (Rob-xx) described in our discussion of strategy in Remark 1.2.

In our context, the idea is to show that for a minimal counterexample to (H-QC), once we have eliminated (e.g., via propagation methods) as many simple components as possible, we can in fact establish this stronger version (L-QC) of the conjecture, and so, obtain a final contradiction. The key point is that we can reduce the study of this conjecture to analyzing particular aspects of the components of the group. To that end, we take advantage of the very restrictive structures of any remaining components in a minimal counterexample to (H-QC). The idea goes back to Robinson [20], and it was later exploited by Aschbacher–Kleidman to establish the almost simple case of the conjecture [3] (which we have quoted as Theorem 2.20), and then by Aschbacher–Smith [4] to conclude the proof of their main theorem (compare (Rob-nonQD) in Remark 1.2).

We denote by (L-QC) the following version of Quillen’s conjecture in terms of the (reduced) Lefschetz module.

(L-QC) If $O_p(G) = 1$, then $\tilde{L}_G(\mathcal{A}_p(G)) := \sum_{n \geq -1} (-1)^n [C_n(\mathcal{A}_p(G))] \neq 0$.

Recall that $C_*(X)$ denotes the augmented chain complex of a poset X with coefficients in the rational numbers. The above alternating sum is taken in the *Grothendieck ring* of representations of G over $\mathbb{Q}$, with brackets denoting the character of a representation, and $\tilde{L}_G(\mathcal{A}_p(G))$ is the *reduced Lefschetz module* of $\mathcal{A}_p(G)$.

By the Hopf trace formula, we get a homology-version of the Lefschetz module

$$\tilde{L}_G(\mathcal{A}_p(G)) = \sum_{n \geq -1} (-1)^n [\tilde{H}_n(\mathcal{A}_p(G))].$$

In particular, we see that (L-QC) implies (H-QC). Moreover, the elements of the Grothendieck ring of G are completely determined by their (virtual) characters, so $\tilde{L}_G(\mathcal{A}_p(G)) = 0$ if and only if its virtual character is 0. By the Lefschetz fixed point formula, the value of this character in a given element $g \in G$ is $\tilde{\chi}(\mathcal{A}_p(G)^g)$, the reduced Euler characteristic of the fixed point subposet $\mathcal{A}_p(G)^g$. Therefore, (L-QC) is equivalent to the following version of the conjecture:

(L'-QC) if $O_p(G) = 1$, then $\tilde{\chi}(\mathcal{A}_p(G)^g) \neq 0$ for some $g \in G$.

Following [3, 4, 20], in some particular situations, we will establish (L'-QC), and hence (H-QC), by exhibiting an element $g \in G$ such that $\tilde{\chi}(\mathcal{A}_p(G)^g) \neq 0$. In fact, we will show that there exists a suitable p' -subgroup $Q \leq G$ of the form $Q = \langle g \rangle \times O_q(Q)$, with q a prime, and

$$\tilde{\chi}(\mathcal{A}_p(G)^g) \equiv \tilde{\chi}(\mathcal{A}_p(G)^Q) \not\equiv 0 \pmod{q}.$$

Note that Q is a q -elementary group, that is, a direct product of a cyclic group with a q -group. The point of considering q -elementary p' -subgroups Q is that the fixed point subposet $\mathcal{A}_p(G)^Q$ can be smaller and easier to control. For example, we typically look for $\mathcal{A}_p(G)^Q$ empty.

With the aim of proving (H-QC), we will try to establish (L-QC) for the groups G not satisfying the hypotheses of some of the theorems of the previous sections, such as Theorems 2.22, 2.24, and 7.2, and Theorem 6.4 (compare (Rob-nonelim) in Remark 1.2). For these groups G , we have several restrictions on the components, and possibly a concrete list such as the $(\mathcal{QD})$ -List Theorem 2.16 when p is odd. The goal is to construct q -elementary p' -subgroups Q_L for each component L of G , in such a way that we can produce a q -elementary p' -subgroup Q of G satisfying the extra property¹⁷:

$$\mathcal{A}_p(G)^Q \simeq \mathcal{A}_p(L_1)^{Q_{L_1}} * \dots * \mathcal{A}_p(L_t)^{Q_{L_t}}, \quad (8.1)$$

where $L_1, \dots, L_t$ are the components of G . Then, using product homology

$$\tilde{\chi}(\mathcal{A}_p(G)^Q) \equiv (-1)^{t-1} \prod_{i=1}^t \tilde{\chi}(\mathcal{A}_p(L_i)^{Q_{L_i}}) \pmod{q}. \quad (8.2)$$

Hence, if $\tilde{\chi}(\mathcal{A}_p(L_i)^{Q_{L_i}}) \not\equiv 0 \pmod{q}$ for all i , then $\tilde{\chi}(\mathcal{A}_p(G)^Q) \not\equiv 0 \pmod{q}$. This reduces the problem of establishing (L'-QC) for G to understanding the structural properties on the components L_i , that allow us to construct these subgroups Q_{L_i} . These q -elementary p' -subgroups Q are also known as *Robinson subgroups*, the terminology employed in [4] to indicate the origin of the method in [20].

Now, we will study the properties that these Q_{L_i} have to satisfy, in order to get such subgroup Q satisfying conditions (8.1) and (8.2). Note that each Q_i is q -elementary, for a fixed prime q common to all the i : that is, we fix the same prime $q \neq p$ for all the components. For example, when p is odd, the natural choice is $q = 2$. When $p = 2$, one fairly natural choice could be $q = 3$, since most of the simple groups have order divisible by 3.

Thus, for our extension of [4], we want to extend the context of Theorem 5.3 at the end of Section 1 there by considering the following properties.

Property $\mathcal{R}_0(p)$. L is a simple group, having a subgroup Q_L such that for some prime $q \neq p$, we have the following.

- (1) Q_L is a q -elementary p' -subgroup.
- (2) $O_q(Q_L) > 1$.
- (3) $\tilde{\chi}(\mathcal{A}_p(L)^{Q_L}) \not\equiv 0 \pmod{q}$.
- (4) $\mathcal{A}_p(C_{\text{Aut}(L)}(Q_L)) \subseteq \mathcal{A}_p(L)$.

¹⁷Below, we are using product homology as in Remark 2.11; notice this is a crucial use of the basic group-theory fact that $E(G)$ is a central product—even a direct product, in our usual situation of Lemma 2.3 (1).

Property $\mathcal{R}(p)$. Q_L satisfies (1)–(4) as in $\mathcal{R}_0(p)$ above, and further,

(5) $q = 2$ when $p \neq 2$; while $q = 3$ when $p = 2$.

We pause to state some consequences of (4) above.

(4+) If $A \in \mathcal{A}_p(\text{Aut}(L))^{Q_L}$, then $A \cap L > 1$. In particular, $\mathcal{A}_p(\text{Aut}(L))^{Q_L} \simeq \mathcal{A}_p(L)^{Q_L}$.

For suppose by way of contradiction that $A \cap L = 1$. Then, $[A, Q_L] \leq A \cap L = 1$, so $A = C_A(Q_L)$, and then (4) forces $A \in \mathcal{A}_p(L)$, contrary to our assumption $A \cap L = 1$. This contradiction shows that $A \cap L > 1$. In particular, $A \mapsto A \cap L$ is a poset endomorphism of $\mathcal{A}_p(\text{Aut}(L))^{Q_L}$ with image $\mathcal{A}_p(L)^{Q_L}$, giving the final statement via Lemma 2.7 (2).

Furthermore, we will often be able to establish the stronger condition.

(3+) $\mathcal{A}_p(\text{Aut}(L))^{Q_L} = \emptyset$; which implies (3) and (4).

For certainly (3+) gives $\mathcal{A}_p(L)^{Q_L} = \emptyset$, with the value of -1 for (3); and gives $\mathcal{A}_p(C_{\text{Aut}(L)}(Q_L)) = \emptyset$, so that (4) holds vacuously.

In the following extension of [4], we show that we can establish (L-QC) if the components of G satisfy $\mathcal{R}(p)$. It roughly states that, under those very particular Robinson-type conditions, we can propagate (L-QC): from the components of G , to G itself. An important observation here is that this propagation depends exclusively on the individual structure of each simple component L . The proof is based on the original argument given in [4]—at the end of Section 1 there.

Proposition 8.1. *Suppose that $O_p(G) = 1 = O_{p'}(G)$, with $F^*(G) = L_1 \cdots L_t$, and let q be a fixed prime distinct from p . Assume that every component L_i of G satisfies Property $\mathcal{R}_0(p)$ for some q -elementary subgroup $Q_i \leq L_i$. Then, G satisfies (L-QC).*

Proof. In overview, we construct a single group Q to satisfy (8.1) and (8.2); roughly, we need to establish for Q properties like $\mathcal{R}_0(p)$ —but simultaneously, that is, with respect to *all* the components L in the product $E(G) = F^*(G)$.

Write $Q_i := \langle g_i \rangle \times O_q(Q_i)$, and let $g := g_1 \cdots g_t$, $C := \langle g \rangle$, $R := \prod_i O_q(Q_i)$, and $Q := C \times R$. Then, Q is a q -elementary p' -subgroup of G by conditions (1), (2) of Property $\mathcal{R}_0(p)$ of each $Q_i \leq L_i$. We show now that $\tilde{\chi}(\mathcal{A}_p(G)^g) \neq 0$ by showing that this integer is not zero mod q . Recall that $\tilde{\chi}(\mathcal{A}_p(G)^g) = \tilde{\chi}(\mathcal{A}_p(G)^C) \equiv \tilde{\chi}(\mathcal{A}_p(G)^Q) \pmod{q}$ as $Q/C = R$ is a q -group. Therefore, it is enough to establish that $\tilde{\chi}(\mathcal{A}_p(G)^Q) \not\equiv 0 \pmod{q}$.

Consider any $E \in \mathcal{A}_p(G)^Q$. By coprime action, $E = C_E(Q)[E, Q]$. Moreover, we note that $[E, Q] \leq E \cap F^*(G)$. We first eliminate a potentially troublesome case for E .

Case 1. We cannot have $E \cap F^*(G) = 1$.

For in this case, we have $E = C_E(Q)$, and, in particular, $E \leq C_G(O_q(Q_i))$ for all i . Since $O_q(Q_i)$ is a non-trivial subgroup of L_i by (2) of condition $\mathcal{R}_0(p)$, we see that $E \leq N_G(L_i)$. Fix some i , and decompose E as the direct product of $E \cap (L_i C_G(L_i))$ with some complement E_0 . Then, E_0 induces outer automorphisms on L_i , and Q_i centralizes E_0 . Thus, we can embed E_0 into $\text{Aut}(L_i)$ and we get $E_0 \leq C_{\text{Aut}(L_i)}(Q_i)$. However, (4) of condition $\mathcal{R}_0(p)$ along with our case hypothesis that $E \cap F^*(G) = 1$ forces $E_0 = 1$. Therefore, $E \leq L_i C_G(L_i)$. Varying i , we see that

$$E \leq \bigcap_i L_i C_G(L_i) = F^*(G) C_G(F^*(G)) = F^*(G).$$

This is contrary to our case hypothesis that $E \cap F^*(G) = 1$, and this contradiction completes the proof that this case cannot arise.

Thus, all E must satisfy the remaining case.

Case 2. Every $E \in \mathcal{A}_p(G)^{\mathcal{Q}}$ intersects $F^*(G)$ non-trivially.

Thus, via Lemma 2.7 (2), we have a homotopy equivalence given by the poset endomorphism

$$\mathcal{A}_p(G)^{\mathcal{Q}} \rightarrow \mathcal{A}_p(F^*(G))^{\mathcal{Q}}, \quad E \mapsto E \cap F^*(G),$$

with the homotopy inverse given by the inclusion.

Finally, note that the direct product of components gives us a homotopy equivalence

$$\begin{aligned} \mathcal{A}_p(F^*(G))^{\mathcal{Q}} &= \mathcal{A}_p(L_1 \cdots L_t)^{\mathcal{Q}} \\ &\simeq (\mathcal{A}_p(L_1) * \cdots * \mathcal{A}_p(L_t))^{\mathcal{Q}} \\ &= \mathcal{A}_p(L_1)^{\mathcal{Q}_1} * \cdots * \mathcal{A}_p(L_t)^{\mathcal{Q}_t}. \end{aligned}$$

That is, we have established (8.1). For (8.2), we compute the reduced Euler characteristics of these fixed point subposets. Recall using product homology as in Remark 2.11 that the Euler characteristic of a join of spaces is the product of the Euler characteristics of the spaces involved, up to a dimension-shift sign

$$\tilde{\chi}(\mathcal{A}_p(G)^{\mathcal{Q}}) = \tilde{\chi}(\mathcal{A}_p(L_1)^{\mathcal{Q}_1} * \cdots * \mathcal{A}_p(L_t)^{\mathcal{Q}_t}) = (-1)^{t+1} \prod_i \tilde{\chi}(\mathcal{A}_p(L_i)^{\mathcal{Q}_i}).$$

The rightmost term is non-zero mod q by (3) of condition $\mathcal{R}_0(p)$ on the Q_i -subgroups. In consequence, $\tilde{\chi}(\mathcal{A}_p(G)^{\mathcal{Q}}) \neq 0$; and so, G satisfies (L-QC). ■

In [4], Robinson methods are used to treat components L which fail the $(\mathcal{QD})_p$ -property for odd primes p , as given in the $(\mathcal{QD})$ -list, which we have quoted as Theorem 2.16. In this context, simple groups of Lie type are a major exception to $(\mathcal{QD})$. For example, in that list we see that there are basically three sub-families of

groups of Lie type which might fail $(\mathcal{QD})_p$ for odd p : Lie-type groups in the same characteristic p , Lie-type groups in characteristic 2, and unitary groups $U_n(q)$ with $p \mid q + 1$. There are also some assorted particular cases (arising from exceptional or twisted groups).

The following proposition shows, in particular, that groups of Lie type in characteristic $r = 2$ satisfy Property $\mathcal{R}(p)$ for odd p . Consequently, we will then have to focus our attention on studying the remaining components of Lie type described in the previous paragraph. Most of the work for the proposition was already done in [4, Theorem 5.3] for odd p .

The proof of the proposition is based on [3, Lemma 4]. However, the proof of that lemma seems to have a small gap, which we show below how to fix. We are grateful to R. Solomon, who provided us with some references on situations related to the Borel–Tits theorem.

Proposition 8.2. *Suppose that L is a simple group of Lie type in characteristic $r \neq p$. Suppose in addition that $L \neq B_2(2)' (\cong \mathbb{A}_6)$. Then, a Sylow r -subgroup X of L satisfies conditions (1)–(4) of Property $\mathcal{R}_0(p)$, where we take r as q in that definition. Moreover, $\mathcal{A}_p(\text{Aut}(L))^X = \emptyset$. In particular, if (p is odd, and $r = 2$) or ($p = 2$, and $r = 3$), we even have Property $\mathcal{R}(p)$ with Q_L a Sylow r -subgroup of L .*

Proof. Let X be a Sylow r -subgroup of L . We show that X satisfies conditions (1)–(4) of Property $\mathcal{R}_0(p)$ with $q = r$, by using our hypothesis that $r \neq p$.

Clearly, X satisfies conditions (1), (2) of Property $\mathcal{R}_0(p)$. On the other hand, since $P := N_L(X)$ is a Borel subgroup of L , we have

$$C_{\text{Aut}(L)}(X) \leq X,$$

by [11, (13-2)], except when $L = B_2(2)' \cong \mathbb{A}_6$ ¹⁸. This shows that X satisfies condition (4) under our hypothesis that $L \neq B_2(2)'$.

It remains to establish condition (3) of Property $\mathcal{R}_0(p)$. Indeed, we will show the stronger condition (3+) mentioned earlier, namely, $\mathcal{A}_p(\text{Aut}(L))^X = \emptyset$, via the ideas of [3, Lemma 4].

We first show $\mathcal{A}_p(L)^X = \emptyset$ (which in fact is enough for (3)). Assume by way of contradiction that we have some $E \in \mathcal{A}_p(L)^X$. Let $M := EX \leq L$. At this point, [3] claims that M lies in a parabolic by the Borel–Tits theorem. However, Borel–Tits does not apply here, since $r \neq p$. So instead, we need to quote some further development, such as [23]. Since $|L : M|_r = 1$, we have $O_r(M) > 1$ by [23, Proposition C]

¹⁸In the latter case, the centralizer of a Sylow 2-subgroup of $B_2(2)'$ in $\text{Aut}(B_2(2)')$ is isomorphic to $C_2 \times C_2$, and contains non-trivial elementary abelian 2-subgroups inducing outer automorphisms on $L = B_2(2)'$.

(see also [22]). We now apply the Borel–Tits theorem to conclude that there exists a parabolic subgroup P of L such that $O_r(M) \leq O_r(P)$ and $N_L(O_r(M)) \leq P$. In particular, we see that $EX = M \leq P$ and $O_r(P)$ lies in the r -Sylow X of L . Hence, E and $O_r(P)$ normalize each other, so $[E, O_r(P)] \leq E \cap O_r(P) = 1$ since $p \neq r$. This implies that $E \leq C_L(O_r(P))$. Since $O_r(L) = 1$ and $O_r(P) > 1$, P is proper in L ; and then using [11, (13-2)], $C_{\text{Aut}(L)}(O_r(P)) \leq O_r(P)$. Again, $E \leq O_r(P)$ forces $E = 1$ since $p \neq r$, contrary to $E > 1$ in our assumption that $E \in \mathcal{A}_p(L)^X$. This contradiction shows that $\mathcal{A}_p(L)^X = \emptyset$, and (3) follows. (So, the proof is actually now complete.)

Finally, if we had some $E \in \mathcal{A}_p(\text{Aut}(L))^X$, then $E \cap L$ is invariant under X , and by the previous paragraph, $E \cap L = 1$. Therefore, $[E, X] \leq E \cap L = 1$, so $E \leq C_{\text{Aut}(L)}(X) \leq X$, again contrary to $p \neq r$. This contradiction shows $\mathcal{A}_p(\text{Aut}(L))^X = \emptyset$, so we even get (3+). As we had mentioned earlier, this gives another proof that X satisfies condition (4) of Property $\mathcal{R}_0(p)$. ■

9. Proving our two variants of Aschbacher–Smith for the primes 3 and 5

This section is devoted to proving our two extensions of [4, Main theorem] to $p = 3, 5$: first under (H1u), and then instead under (H1).

So, we begin by considering the variant under (H1u): that is, we extend [4, Main theorem] to every odd prime p . Recall from the introduction that our extension Theorem 1.1 will be established below as Theorem 9.2, where the proof will involve (H1u).

In fact, in [16], we already extended the Aschbacher–Smith theorem to the case $p = 5$. We recall that the only obstruction there was the possible presence of a component of Suzuki type $\text{Sz}(2^5)$. So, in [16], we showed that if G contains such a component, then—under (H1)— G satisfies (H-QC). This is of course an example of an elimination result.

The explanation given in [16] claiming that [4, Main theorem] extends, as it is, to $p = 5$, has a small gap: one needs to work under (H1u), instead of (H1), but the proof of such extension to $p = 5$ invokes [16, Theorem 5.1] which requires in fact (H1). In view of this discrepancy, we provide below a correspondingly adjusted statement of [16, Theorem 5.1] instead of using (H1u). We will then check that the proof of the original theorem adapts just as well to this alternative version, by only using the results of [16].

Theorem 9.1 (cf. [16, Theorem 5.1]). *Suppose that G satisfies (H1u) and contains a component L such that*

$$L \cong \text{PSL}_2(2^3)(p = 3), \quad \text{PSU}_3(2^3)(p = 3) \quad \text{or} \quad \text{Sz}(2^5)(p = 5).$$

Then, G satisfies (H-QC).

Proof. We suppose that $O_p(G) = 1$ and show $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$. The original proof in [16] begins by reducing to $O_{p'}(G) = 1$, just to claim that $Z(L) = 1$. However, here we have the property $Z(L) = 1$ directly by hypothesis, so we do not need to reduce to $O_{p'}(G) = 1$.

And the rest of the original proof in [16] invokes (H1) to claim that $\mathcal{A}_p(C_G(L))$ has non-zero homology. For that purpose, we will invoke instead (H1u). Hence, we need to show that $O_p(C_G(L)) = 1$, and that $C_G(L)$ satisfies (H-QC). By Lemma 2.3 (3), (4), the components of $C_G(L)$ are the components of G distinct from L , and $O_p(C_G(L)) = O_p(G) = 1$. Therefore, since $C_G(L)$ is proper in G , while any unitary component of $C_G(L)$ is a unitary component of G , $C_G(L)$ satisfies the conditions of (H1u). Hence, $\tilde{H}_*(\mathcal{A}_p(C_G(L))) \neq 0$, as desired. And now the original proof in [16] goes through. ■

With the above re-statement of [16, Theorem 5.1] in hand, we can now complete the extension of [4, Main theorem] to every odd prime, by also using the results of this article. While the extension to $p = 5$ can be done here by using Theorem 9.1 and the existing argument of [4], the extension of [4, Main theorem] to $p = 3$ now further requires us to deal with some more exceptional components, and also to extend certain results of [4] to $p = 3$. Concretely, although most of the hard work has already been done in [4], it remains to eliminate some further components on the $(\mathcal{QD})$ -List (recall we have quoted this list as Theorem 2.16), before constructing Robinson subgroups—to finish as in the cases for $p \geq 5$. Indeed, the only component that presents a real obstruction to constructing Robinson subgroups is the Ree group ${}^2G_2(3^a)'$; and our earlier result Theorem 6.4 shows that such a component is not a problem in our setting.

We proceed now with the details of our extension of [4, Main theorem] to every odd prime p .

Theorem 9.2. *Let p be an odd prime. Suppose that if G has a unitary component $L \cong \text{PSU}_n(q)$ such that q is odd and $p \mid q + 1$, then $(\mathcal{QD})_p$ holds for the p -extensions of $\text{PSU}_m(q^{p^e})$ for $m \leq n$ and $e \in \mathbb{Z}$.*

Then, G satisfies (H-QC).

Proof. The proof largely follows the path of the original proof of [4, Main theorem]: indeed we quote those original arguments when $p > 5$; and also, when they apply equally well to the “new” primes $p = 3, 5$ —but when not, we must indicate alternative arguments.

We take G to be a counterexample of minimal order subject to the conditions of the theorem. Hence, G satisfies (H1u). Then, we have $O_p(G) = 1$ and $\tilde{H}_*(\mathcal{A}_p(G)) = 0$. The proofs of the reductions in [4, Propositions 1.3–1.5] apply for any p (including the primes $p = 3, 5$), so we also get $Z(E(G)) = 1 = Z(G)$. In particular, $E(G)$ is the direct product of the components of G , which are simple.

At this point, we insert into the original argument of [4] an application of the elimination result Theorem 9.1 to exclude $\mathrm{PSL}_2(2^3)$ ($p = 3$), $\mathrm{PSU}_3(2^3)$ ($p = 3$), and $\mathrm{Sz}(2^5)$ ($p = 5$) as possible components L of G . We could optionally also apply further elimination results here, which would have the effect of some simplification, over the original path of [4], of later details. However, here we instead prefer to closely parallel that original path.

The three eliminations above now remove the obstruction, for the new primes $p = 3, 5$, to the proof of the non-conical complement result [4, Theorem 2.3]. Hence, also the further result [4, Theorem 2.4] can be applied in the remainder of our proof.

So, we now obtain, first, the reduction [4, Proposition 1.6] to $O_{p'}(G) = 1$; and then also [4, Proposition 1.7], namely, the elimination of $(\mathcal{QD})$ -components. This is because their proofs use [4, Theorem 2.4], as allowed by the previous paragraph. We note that this part of the proof does not depend on the particular prime p (and so, in particular, goes through for $p = 3, 5$).

In the next few paragraphs, let us describe more precisely the effect of the eliminations above. Now, every component L of G has some p -extension $LB \leq G$ such that LB fails $(\mathcal{QD})_p$. In particular, if L is a component of G , then by the hypothesis (H1u), L is not $\mathrm{PSU}_n(q)$ with q odd and $p \mid q + 1$. This gives the hypothesis for [4, Theorem 3.1]—which is stated for all odd p (hence including $p = 3, 5$). That result shows that L must appear in the $(\mathcal{QD})$ -List—which we have reproduced as Theorem 2.16 in the present paper.

Now, for $p > 5$, Aschbacher and Smith complete the proof by invoking [4, Theorem 5.3]—which establishes $\mathcal{R}(p)$ for any L in the above $(\mathcal{QD})$ -List. Indeed, since that result applies also to $p = 5$, we obtain our extension of their main result to $p = 5$ as well.

Hence, in the remainder of our proof, we may assume $p = 3$. In brief overview: we will first apply some further elimination results to avoid certain classes of groups in the $(\mathcal{QD})$ -List; and after that, finish with results on $\mathcal{R}(3)$ much as in [4, Theorem 5.3]—cf. Proposition 8.1.

So, let us briefly assess the $(\mathcal{QD})$ -sublist for $p = 3$ that we must start with.

- (1) We must deal with the Lie(3) cases in (1) of the $(\mathcal{QD})$ -List.
- (2) The unitary cases in (2) of that list are ruled out by hypothesis (H1u).
- (3) Case (3) of the $(\mathcal{QD})$ -List is eliminated: For there, we would have $q = r^3$, with the orders-mod-3 given by $|q| = |r| = 3, 6, 8, 12$; but here, 3 cannot divide r, q (otherwise, these orders are not defined)—and so, for us, those orders-mod-3 can only be 1, 2.
- (4) We do not get case (4), since there $p = 7$ or 11.

- (5) But we must still contend with the specific cases for $p = 3$ in (8)–(12) of the $(\mathcal{QD})$ -List. (We mention that the three special eliminations applied at the start of our proof have already removed $L_2(2^3)$ from (11), and $U_3(2^3)$ from (10).)

Thus, it remains to deal with the appropriate cases in (1) and (8)–(12) of the $(\mathcal{QD})$ -sublist for $p = 3$.

We next setup to apply the elimination result Theorem 2.24. Note by our early reduction to $O_{p'}(G) = 1$ that every component L has order divisible by $p = 3$. Furthermore, by Lemma 2.3 (3), the components of $C_G(\hat{L})$ are components of G and hence satisfy (H1u); so since G is a counterexample of minimal order to our theorem, we have that $C_G(\hat{L})$ satisfies (H-QC). This gives the hypotheses of Theorem 2.24, up to (1) there.

Since G is, in particular, a counterexample to (H-QC), we see that the remaining hypothesis (2) of Theorem 2.24 must fail; so that p must divide the order of $\text{Out}_G(L)$. Thus, for p odd (and hence for our present $p = 3$), this eliminates L of alternating or sporadic type. So, we eliminate cases (8) and (12) in the $(\mathcal{QD})$ -List from our list of possible components of G .

We will here apply one further elimination result, namely, Theorem 6.4. Note that we have the “in particular” hypotheses (H1u) and $p = 3$ odd there; furthermore, the previous paragraph shows that we have $B > 1$. And we saw that the remaining in-particular hypothesis of $(p\text{-}\Phi)$ holds for the Lie-type groups of Lie-rank 1, by Remark 6.2 (0). So, we eliminate the subcase of Lie-rank 1 from the Lie(3) part of case (1) of the $(\mathcal{QD})$ -list. (Note that the smallest Ree group $L \cong {}^2G_2(3)' \cong L_2(2^3)$ was already eliminated by our earlier application of Theorem 9.1; we cannot apply Theorem 6.4 to it since it is one of the exceptional commutators which are not of type (sLie- p) in Definition 6.1 (2).)

To summarize the effect of these eliminations: we have reduced our $p = 3$ sublist (1), (8)–(12) of the $(\mathcal{QD})$ -List to Case (1) for Lie-rank > 1 , and cases (9)–(11) (with $L_2(2^3)$ already eliminated from (11)).

And now, we are ready to finish by turning to results which will give $\mathcal{R}(3)$ for this latest sublist. Proposition 8.2 gives $\mathcal{R}(3)$ for the Lie(2) cases—except $\text{Sp}_4(2)' = \text{Alt}_6$, which was already eliminated above via Theorem 2.24 as the alternating group Alt_6 . This completes the treatment of the cases in (9)–(11) and reducing us to Case (1) with Lie-rank > 1 .

We now observe that the proof of [4, Theorem 5.3] for the Lie(p) case actually works for $p = 3$ and $\mathcal{R}(3)$ —except when $L \cong {}^2G_2(3^a)$ for $a > 1$ odd, which we already eliminated above via Theorem 6.4.

In conclusion, we have shown every remaining possible component L of G (that is, not ruled out by our elimination results above) satisfies $\mathcal{R}(3)$. By Proposition 8.1,

G satisfies (L-QC)—and hence also (H-QC). This completes our proof of Theorem 9.2. ■

Remark 9.3. Property $\mathcal{R}(3)$ cannot be established for the Ree groups ${}^2G_2(3^a)$ (for $a > 1$ odd), since there every field automorphism of order 3 of the Ree group commutes with some Sylow 2-subgroup. However, $\mathcal{R}(3)$ does hold for the smallest Ree group ${}^2G_2(3)'$, which has no field automorphisms. ■

Finally, we close this section by proving Theorem 1.4, the alternative version under (H1) of Theorem 1.1 (that is of Theorem 9.2 above). We will particularly emphasize the points of the proof that use the CFSG.

Proof of Theorem 1.4. We assume that G satisfies both (H1) and (H2). As usual for (H-QC), we may suppose that $O_p(G) = 1$, and we must show then $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$ —now for all odd p , including $p = 3, 5$.

By Theorem 2.22 (2), we can assume that $O_{p'}(G) = 1$ (this reduction makes use of the CFSG to invoke the p -solvable case of (H-QC)). Note that this implies that every component of G is simple and of order divisible by p (see Lemma 2.3).

We get the following constraints on the components of G , which do not invoke the CFSG.

- (1) By (5) and (6) of Theorem 2.22, we can suppose that G does not contain components isomorphic to $\mathrm{PSL}_2(2^3)$, $\mathrm{PSU}_3(2^3)$, or $\mathrm{Sz}(2^5)$ for $p = 3, 3, 5$, respectively.
- (2) By Theorem 2.24, we can suppose that G does not contain components of alternating or sporadic type.
- (3) By Theorem 7.2, we can suppose that for every component L of G there is a p -extension of L in G failing $(\mathcal{QD})_p$.
- (4) By Theorem 6.4, we can suppose that G does not contain a component of Lie type and Lie rank 1 in characteristic p .

In view of (3) above, we can invoke at this point the $(\mathcal{QD})$ -List, which we have quoted as Theorem 2.16, to eliminate possible components of G . Here, we do use the CFSG.

Note that (H2) guarantees that the unitary groups in item (2) of Theorem 2.16 are not a possibility in our context. Hence, using the further eliminations noted in (1), (2), (4) just above, we see that items (1), (3), (5), (6), (9), (10), and (11) of the $(\mathcal{QD})$ -List Theorem 2.16 contain the possible components of G . Now, all of these simple groups satisfy the Robinson property $\mathcal{R}(p)$ in view of [4, Theorem 5.3] and its proof (compare with the end of the proof of Theorem 9.2). Recall here that ${}^2G_2(3^a)'$ does not arise as a component of G when $p = 3$, in view of the elimination in item (4) above, along with (1) when $a = 1$ since $\mathrm{PSL}_2(2^3) \cong {}^2G_2(3)'$. Finally, note that this

step does not require the CFSG, since Property $\mathcal{R}(p)$ is shown for specific families of simple groups. ■

Remark 9.4. In summary, in the proof of Theorem 1.4 above, the CFSG is used just in two steps: first, to appeal to the p -solvable case of the Quillen conjecture in reducing to $O_{p'}(G) = 1$, and then, to invoke the $(\mathcal{QD})$ -List of [4]. ■

10. Some results toward the case $p = 2$ of Quillen's conjecture

In view of the previous results on odd p , in this section, we give some insights for the study of (H-QC) for $p = 2$.

First, for overall context, recall our general discussions of strategy in Remark 1.2.

In general, the theorems of [4] only work for p odd. We have been able to extend some of them to every prime p by using more general combinatorial methods in $\mathcal{A}_p$ -posets. For example, the reduction $O_{p'}(G) = 1$ was originally stated in [4, Proposition 1.6] for $p > 5$; and an extension to all primes p was given in [16] by using a more general combinatorial argument. In addition, the main reductions in Theorems 2.22 and 2.24 here also have no restrictions on p , and so, we can apply them to $p = 2$. In a similar vein, Theorem 7.2 extends [4, Proposition 1.7] to every prime p , at the cost of assuming an alternative inductive hypothesis, namely, (H1); so in this section, we will make use of the (H1)-versions of various results. Finally, the Robinson path to (L-QC) described in the previous two sections (notably Proposition 8.1) has no a priori restrictions on p , so it also admits the possibility of treating $p = 2$.

Thus, (in view of (MOC) in Remark 1.3) we see using (H1) that a minimal counterexample G to (H-QC) when $p = 2$ must satisfy $O_2(G) = 1 = O_{2'}(G)$, using [16, Theorem 4.1]. Further, for every component L of G we have that some p -extension in G fails $(\mathcal{QD})_p$ (by Theorem 7.2), it is not of Lie type in characteristic $p = 2$ except possibly for the cases listed in Theorem 6.5, and $\text{Out}_G(L)$ is not a $2'$ -group (by Theorem 2.24).

In consequence, if we adapt the strategy of [4] in the spirit of (Elim) and (Robnonelim) in Remark 1.2, we can try to exhibit a $(\mathcal{QD})$ -List for $p = 2$, and then show that the possible components in such a list satisfy the Robinson property $\mathcal{R}(2)$.

So, in this section, we give some further elimination results for $p = 2$, as consequences of our methods. We also establish the Robinson property $\mathcal{R}(2)$ for some of the simple groups. These results will allow us to conclude, for example, that a minimal counterexample G to (H-QC) for $p = 2$ has some component L which is of Lie type in some characteristic $\neq 3$. (cf. Theorem 1.7, and its proof at the end of this section.)

So, we now embark on the various results just described. The following initial elimination result is an easy consequence of Theorem 2.24.

Proposition 10.1. *Let $p = 2$, and suppose that G satisfies (H1). If a component L of G is of Suzuki type $\text{Sz}(2^{2n+1})$, or of Ree type ${}^2F_4(2^{2n+1})'$ or ${}^2G_2(3^{2n+1})'$, then G satisfies (H-QC).*

In particular, if G satisfies (H1) but fails (H-QC), then G contains no component of Suzuki type nor of Ree type.

Proof. We saw in the discussion preceding Definition B.1 that $\text{Out}(L)$ is a $2'$ -group in each case. Hence, $\mathcal{A}_2(L) = \mathcal{A}_2(\text{Aut}_G(L))$; and $\mathcal{A}_2(L) \rightarrow \mathcal{A}_2(\text{Aut}_G(L))$ is the identity map, and so, is non-zero in homology using the almost simple case of the conjecture, which we quoted earlier as Theorem 2.20. The result follows then by Theorem 2.24 (cf. [18, Corollary 1.5]). ■

Corollary 10.2. *For $p = 2$, if G satisfies (H1), then either G satisfies (H-QC), or each component of G has order divisible by 3.*

Proof. The only¹⁹ simple groups of order not divisible by 3 are the Suzuki-type groups, which as components lead to (H-QC) by the previous proposition. ■

So, it is natural in this section to use $q = 3$ in our q -elementary Robinson subgroup analysis.

In the remainder of the section, we, in particular, show that alternating or sporadic components of G are either eliminated from a minimal counterexample to (H-QC) with $p = 2$ (by using one of our homology propagation methods), or else they satisfy the Robinson property $\mathcal{R}(2)$. We summarize the results for sporadic components, in Proposition 10.4 below; and for alternating components, in later Corollary 10.8.

The following lemma will give us a quick condition for establishing Property $\mathcal{R}(2)$ on a simple group. For primes p, q , recall that the p -local q -rank of a finite group L is the largest q -rank of a p -local subgroup of L . More concretely, it is the following number:

$$m_{p,q}(L) := \max\{m_q(N_L(A)) : A \in \mathcal{A}_p(L)\}.$$

Lemma 10.3. *Let L be a simple group such that one of the following conditions hold.*

- (1) *Whenever $E \in \mathcal{A}_2(\text{Aut}(L))$, then $N_L(E)$ does not contain a Sylow 3-subgroup of L .*
- (2) $m_{2,3}(\text{Aut}(L)) < m_3(L)$.

Then, L has property $\mathcal{R}(2)$.

¹⁹This is a well-known pre-CFSG result, going back to Thompson and Glauberman (see [9, Theorem C]); cf. Theorem 4.174 and the discussion beforehand (using also [10, Theorem 4.126]).

Proof. We show first that (2) implies (1). Assuming (2), take a Sylow 3-subgroup P of L . Since $m_3(L) = m_3(P)$, and $m_{2,3}(\text{Aut}(L)) < m_3(L)$ by hypothesis, for every $E \in \mathcal{A}_2(\text{Aut}(L))$, we have

$$m_3(N_L(E)) \leq m_3(N_{\text{Aut}(L)}(E)) \leq m_{2,3}(\text{Aut}(L)) < m_3(L) = m_3(P).$$

Therefore, $N_L(E)$ cannot contain P , giving (1) as claimed.

It remains to show that condition (1) implies Property $\mathcal{R}(2)$ (with P in the role of “ Q_L ”). By (1), P does not normalize an element $E \in \mathcal{A}_2(\text{Aut}(L))$. Therefore, we have established (3+), namely, that $\mathcal{A}_2(\text{Aut}(L))^P = \emptyset$; and we had seen earlier that (3+) suffices to guarantee (3) and (4) of $\mathcal{R}_0(2)$, so we see L satisfies Property $\mathcal{R}(2)$ for P . ■

We show next how to deal with sporadic components when studying (H-QC) for $p = 2$.

Proposition 10.4. *Let L be a sporadic simple group and let $p = 2$. Then, $\mathcal{A}_2(L) \rightarrow \mathcal{A}_2(\text{Aut}(L))$ does not induce the zero map in homology if L is one of the following 17 sporadic groups:*

$$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_4, \cdot 1, \cdot 2, \cdot 3, M(23), \text{HS}, \text{Ly}, \text{Ru}, F_3, F_2, F_1. \quad (10.1)$$

In the remaining 9 sporadic cases, L satisfies Property $\mathcal{R}(2)$:

$$J_2, J_3, M(22), M(24)', \text{McL}, \text{He}, \text{Suz}, \text{O}'\text{N}, F_5. \quad (10.2)$$

Proof. For the proof of this proposition, we use the results of [12, Table 5.3].

We first consider L in the list (10.1). Then, either $\text{Out}(L) = 1$ or else $L = M_{12}, M_{22}, \text{HS}$. In the cases where $\text{Out}(L) = 1$, $\mathcal{A}_2(L) \rightarrow \mathcal{A}_2(\text{Aut}(L))$ is the identity map and hence non-zero in homology by the almost simple case of (H-QC), which we had quoted as Theorem 2.20. So, we study the remaining three cases of this list.

- If $L = M_{12}$, then for all $B \in \text{O}_{\text{Aut}(L)}(L)$, B is of type $2C$ and hence $O_2(C_L(B)) > 1$ (see [12, Table 5.3 (b)]). So, $\mathcal{A}_2(L) \rightarrow \mathcal{A}_2(\text{Aut}(L))$ is a homotopy equivalence using part (3) of Lemma 3.3, and hence is non-zero in homology using the almost simple case as above.
- If $L = M_{22}$, then for all $B \in \text{O}_{\text{Aut}(L)}(L)$, B is of type $2B$ or $2C$, and in either case we have $O_2(C_L(B)) > 1$ (see [12, Table 5.3 (c)]). Then, $\mathcal{A}_2(L) \rightarrow \mathcal{A}_2(\text{Aut}(L))$ is again a homotopy equivalence using part (3) of Lemma 3.3, and is non-zero in homology as above.
- If $L = \text{HS}$, then using [18, Theorem 7.1], we see that $\mathcal{A}_2(L) \rightarrow \mathcal{A}_2(\text{Aut}(L))$ is non-zero in homology.

This completes the treatment of the list (10.1).

It remains to consider L in the list (10.2). Then, $|\text{Out}(L)| = 2$. Note that, in each case, if $B \in \text{O}_{\text{Aut}(L)}(L)$, then $|B| = 2$, so every element there is maximal, and also $\text{Aut}(L) = LB$. We study Property $\mathcal{R}(2)$ in each case. From [12, Tables 5.6.1 and 5.6.2], we see that in fact $m_3(L) > m_{2,3}(\text{Aut}(L))$ if L is one of the following seven sporadic groups:

$$J_3, M(22), M(24)', \text{McL}, \text{Suz}, \text{O}'\text{N}, F_5;$$

and then we may apply Lemma 10.3 (2) to obtain $\mathcal{R}(2)$ for these cases. We complete the details of the proof by analyzing the two remaining cases J_2 and He .

Case $L = J_2$. Let $Q \in \text{Syl}_3(L)$. Note that Q does not centralize outer involutions. Moreover, we get $C_{\text{Aut}(L)}(Q) \leq Q$ and $\mathcal{A}_p(L)^Q = \emptyset$ —either by looking into the maximal subgroups, or by direct computation. This gives (4) and (3) of Property $\mathcal{R}_0(2)$, so we see that L satisfies property $\mathcal{R}(2)$.

Case $L = \text{He}$. We get Property $\mathcal{R}(2)$ by Lemma 10.5 below. ■

The assertions in the following lemma were checked with GAP [8]. We follow the conventions of [12, Table 5.3 (p)].

Lemma 10.5. *Let $L = \text{He}$, and let Q be a cyclic subgroup of order 21 generated by elements of type $3B$ and $7A$. Then, Q satisfies the requirements of Property $\mathcal{R}(2)$.*

Proof. Clearly, Q is q -elementary for $q = 3$, giving (1) and (2) of Property $\mathcal{R}_0(2)$. And $C_{\text{Aut}(L)}(Q) = Q$, giving $\mathcal{A}_2(C_{\text{Aut}(L)}(Q)) = \emptyset$, for (4) there. Further, $\mathcal{B}_2(L)^Q$ consists of exactly two non-conjugate elements E_1 and E_2 ²⁰. Since the equivalence of $\mathcal{A}_2(L)$ with $\mathcal{B}_2(L)$ in Proposition 2.13 is L -equivariant, we get the value of $+1$ for the reduced Euler characteristic in (3) of $\mathcal{R}_0(2)$. So, as $q = 3$, we conclude that Q satisfies the requirements of property $\mathcal{R}(2)$. ■

We now make an analysis for alternating components, similar in spirit to Proposition 10.4 above for sporadic components.

Note that the alternating components $\text{Alt}_5 \cong \text{PSL}_2(2^2)$ and $\text{Alt}_6 \cong \text{Sp}_4(2)'$ are eliminated from a minimal counterexample in view of Theorem 6.5. So, we show that Alt_n , with $n \geq 7$, satisfies the Robinson property $\mathcal{R}(2)$.

Proposition 10.6. *Let L be an alternating group Alt_n with $n \geq 7$. Then, L satisfies Property $\mathcal{R}(2)$.*

Proof. Recall that we have $\text{Aut}(\text{Alt}_n) = \text{Sym}_n$ since $n \geq 7$. In particular, $\text{O}_{\text{Sym}_n}(\text{Alt}_n)$ consists only of elements of order 2, which, in particular, are minimal. So, we see that $\text{Sym}_n = \text{Alt}_n B$ for all $B \in \text{O}_{\text{Sym}_n}(\text{Alt}_n)$.

²⁰Both are isomorphic to $C_2 \times C_2$, and are generated by involutions of type $2A$, with $E_1 \cap E_2 = 1$; and E_1 is L -conjugate to $\text{O}_2(C_L(2A))$, with $\langle E_1, E_2 \rangle \cong \text{PSL}_3(2)$.

Now, we establish Property $\mathcal{R}(2)$ for Alt_n for our value of $n \geq 7$. We let Q denote the following subgroup of Alt_n :

$$Q = \begin{cases} \langle (1\ 2\ 3), (4\ 5 \dots n) \rangle & 2 \mid n, \\ \langle (1\ 2 \dots n) \rangle & 2 \nmid n, 3 \mid n, \\ \langle (1\ 2\ 3), (4\ 5\ 6), (7 \dots n) \rangle & 2 \nmid n, 3 \nmid n. \end{cases}$$

Note that in each case Q is a 3-elementary $2'$ -subgroup of Alt_n with $O_3(Q) > 1$, so we get conditions (1), (2), and (5) of Property $\mathcal{R}(2)$ for this $Q \leq \text{Alt}_n$. So, it remains to prove conditions (3) and (4) for each case of Q .

Case 1. $2 \mid n$.

This case is done in [3, p. 212], below their Lemma 3, where it is shown that $\mathcal{A}_2(\text{Sym}_n)^Q = \emptyset$ (including for $n = 6$). Therefore, we have (3+), which we saw suffices to give conditions (3) and (4) of Property $\mathcal{R}(2)$ for this Q . This finishes the proof of Case 1.

Case 2. $2 \nmid n, 3 \mid n$.

Note that Q is transitive on $X := \{1, \dots, n\}$, which has an odd number of elements by hypothesis. If there is any $E \in \mathcal{A}_2(\text{Sym}_n)$, then E acts non-trivially on X ; hence Lemma 10.7 below would show n is even, contrary to the hypothesis of this case. This contradiction shows that $\mathcal{A}_2(\text{Sym}_n)^Q = \emptyset$. So, again we have (3+), finishing the proof of Case 2.

Case 3a. $2 \nmid n, 3 \nmid n, n > 7$ (so $n \geq 11$).

Suppose that there is some $E \in \mathcal{A}_2(\text{Sym}_n)^Q$. Write $x := (1\ 2\ 3)$, $y = (4\ 5\ 6)$, and $z := (7\ 8 \dots n)$, with $Q := \langle x, y, z \rangle$. Note that both $C_E(x)$ and $C_E(y)$ are Q -invariant.

Claim. $C_E(x) = 1 = C_E(y)$; in particular, E is not trivial on $\{1, 2, 3\}$ or $\{4, 5, 6\}$.

Note that $C_E(x) \leq C_{\text{Sym}_n}(x) = \langle x \rangle \times \text{Sym}_{\{4,5,\dots,n\}}$ by [12, Proposition 5.2.6]. Now, since E is a 2-group, we conclude that $C_E(x) \leq \text{Sym}_{\{4,5,\dots,n\}}$. But also note that $y, z \in \text{Sym}_{\{4,5,\dots,n\}}$, so $C_E(x) \in \mathcal{A}_2(\text{Sym}_{\{4,5,\dots,n\}})^{\langle y,z \rangle} \cup \{1\}$. By Case 1 (with $n - 3 \geq 8$ in the role of “ n ”), we see that we have $\mathcal{A}_2(\text{Sym}_{\{4,5,\dots,n\}})^{\langle y,z \rangle} = \emptyset$: so that we must have $C_E(x) = 1$.

Since y is also a 3-cycle, an analogous argument shows that $C_E(y) = 1$. This finishes the proof of the claim.

Now, since n is odd with E a 2-group, we see that $\text{Fix}(E) \neq \emptyset$; and Q acts on $\text{Fix}(E)$. Recall that the Q -orbits of the action of Q on $\{1, \dots, n\}$ are $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, \dots, n\}$. Further, $\text{Fix}(E)$ is a non-empty union of Q -orbits. On the other hand, the above claim shows that E cannot be trivial on $\{1, 2, 3\}$ or $\{4, 5, 6\}$. This forces $\text{Fix}(E) = \{7, 8, \dots, n\}$, and so, $E \leq \text{Sym}_6$, that is, $E \in \mathcal{A}_2(\text{Sym}_6)^{\langle x,y \rangle}$. But

this is Case 1 again, and we saw there that also for $n = 6$, we get $\mathcal{A}_2(\text{Sym}_6)^{(x,y)} = \emptyset$, contrary to our choice of E just before the claim. This contradiction shows that $\mathcal{A}_2(\text{Sym}_n)^Q = \emptyset$. In particular, we again get condition (3+), and hence Property $\mathcal{R}(2)$ for this Q . This concludes the proof of Case 3a.

Case 3b. $n = 7$.

It follows by direct computation that $\mathcal{A}_2(\text{Sym}_7)^Q = \mathcal{A}_2(\text{Alt}_7)^Q$ is discrete with two points. In particular, $\mathcal{A}_2(C_{\text{Sym}_n}(Q)) \subseteq \mathcal{A}_2(\text{Alt}_7)$ and $\tilde{\chi}(\mathcal{A}_2(\text{Alt}_7)^Q) = 1 \not\equiv 0 \pmod{3}$. Hence, Q satisfies conditions (4) and (3) of Property $\mathcal{R}(2)$ for $L = \text{Alt}_7$. This finishes the proof of Case 3b, and hence of Case 3.

We have shown in all cases that Alt_n for $n \geq 7$ satisfies Property $\mathcal{R}(2)$. ■

During the proof, we used the following easy consequence of non-trivial action under a p -group.

Lemma 10.7 (cf. [3, Lemma 3]). *Assume that $Q \leq G$ and that $E \in \mathcal{A}_p(G)^Q$ acts non-trivially on some Q -orbit $\mathcal{O}$. Then, $p \mid |\mathcal{O}|$.* ■

We combine the results above on the alternating-component case in the following corollary.

Corollary 10.8. *Let $p = 2$, and suppose that G satisfies (H1). Suppose in addition that L is a component of G , such that $L/Z(L) \cong \text{Alt}_n$.*

- (1) *If $n = 5$ or 6 , then G satisfies (H-QC).*
- (2) *If $n \geq 7$, then $L/Z(L)$ satisfies Property $\mathcal{R}(2)$.* ■

We can now prove Theorem 1.7.

Proof of Theorem 1.7. The elimination-parts (1)–(3) are established as follows. Note that from Theorem 6.5 and the list (10.1) in Proposition 10.4, we get conclusion (1) there. Then, Theorem 7.2 gives conclusion (2); and Theorem 6.5 gives conclusion (3).

In proving conclusion (4), we may as well assume we have already eliminated components L described in (1)–(3). Then, all the remaining components L of G satisfy $\mathcal{R}(2)$: the alternating components by Proposition 10.6, the sporadic components via the “remaining-9”-list (10.2) in Proposition 10.4, and the Lie-type components, which must be defined in characteristic 3 by the hypothesis of (4), by Proposition 8.2. Then, the standard Robinson-type argument on $\mathcal{R}(2)$ in Proposition 8.1 gives (L-QC), and hence (H-QC) as required. ■

Remark 10.9. The simple group $L = \text{PSL}_3(2^2)$ is a potential candidate to fail most arguments that we describe here, for the case $p = 2$.

- This simple group satisfies (sLie- p), but corresponds to one of groups in the exclusion list of Theorem 6.5: there exist 2-outers $B \in \text{O}_{\text{Aut}(L)}(L)$ of 2-rank 2,

generated by graph and field automorphisms. Hence, $O_G(L)$ may contain non-cyclic 2-outers if L is a component of G (see also the discussion in Remark 6.9.)

- We have $m_2(L) = 4$ and $\mathcal{A}_2(L) \simeq \mathcal{B}_2(L)$ has the homotopy type of a bouquet of 1-spheres. This shows that $(\mathcal{QD})_2$ fails for L . Thus, Theorem 7.2 cannot be applied for L .
- It can also be proved that $\mathcal{A}_2(\text{Aut}(L))$ has the homotopy type of a bouquet of 2-spheres, so the map $\mathcal{A}_2(L) \rightarrow \mathcal{A}_2(\text{Aut}(L))$ is the zero map in homology. This shows that we cannot apply Theorem 2.24 in general.
- Finally, we see that we cannot produce Robinson subgroups for $q = 3$. That is, $\text{PSL}_3(2^2)$ fails $\mathcal{R}(2)$. Indeed, it fails $\mathcal{R}_0(2)$ for any possible choice of q : since $|\text{PSL}_3(4)| = 20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, here q could be 3, 5, 7. In any case, for a non-trivial q -subgroup Q of L , we have that $C_L(Q)$ is a Sylow q -subgroup of L , and $C_{\text{Aut}(L)}(Q)$ contains 2-outers of L . So, $\mathcal{A}_2(C_{\text{Aut}(L)}(Q))$ is not included in $\mathcal{A}_2(L)$.

The pathological behavior of this simple group shows that, as a component of a finite group, this cannot be treated by using some of the theorems described in this article. In conclusion, further elimination results will be necessary to study the case $p = 2$ of the conjecture, if we attach to the usual strategy (Elim)/(Rob-nonelim).

This behavior of $\text{PSL}_3(2^2)$ is closely related to the fact that $\text{Aut}(\text{PSL}_3(2^2))$ satisfies the Robinson Property (R2), as defined in [3] (cf. Theorem 2 there). ■

A. An extended form of Quillen's fiber theorem

We had stated Quillen's fiber theorem in earlier Theorem 2.10. Below, we recall generalized version and outline a combinatorial proof.

Recall that an n -equivalence is a continuous function $f : X \rightarrow Y$ such that f induces isomorphisms in the homotopy groups π_i with $i < n$, and an epimorphism in π_n . By the Hurewicz theorem, an n -equivalence also induces isomorphisms in the homology groups of degree $\leq n - 1$, and an epimorphism in degree n .

Proposition A.1 (Extended quillen fiber theorem). *Let $f : X \rightarrow Y$ be a map between finite posets, and let $n \geq 0$.*

- (1) *Suppose that for all $y \in Y$, $f^{-1}(Y_{\leq y}) * Y_{> y}$ (resp., $f^{-1}(Y_{\geq y}) * Y_{< y}$) is in fact $(n - 1)$ -connected. Then, f is an n -equivalence.*
- (2) *In addition, assume that X, Y are G -posets for some group G , and that f is a G -equivariant map of posets such that for all $y \in Y$, $f^{-1}(Y_{\leq y}) * Y_{> y}$ (resp., $f^{-1}(Y_{\geq y}) * Y_{< y}$) is G_y -contractible. Then, f is a G -homotopy equivalence.*

Proof. (Sketch) We begin by showing item (1). To that end, we consider the non-Hausdorff cylinder of the map f . This is the finite poset $B(f)$ whose underlying set is the disjoint union of X and Y . It keeps the given order in X and Y ; and if $x \in X$ and $y \in Y$, set $x < y$ if $f(x) \leq y$. It can be shown that Y is a strong deformation retract of $B(f)$, via the map $r : B(f) \rightarrow Y$ defined by $r(x) = f(x)$ if $x \in X$ (and the identity in Y). Moreover, if $i : X \hookrightarrow B(f)$ is the inclusion map, then $(ri)(x) = f(x)$, i.e., $ri = f$. Since r is a homotopy equivalence, we see that f is an n -equivalence if and only if the inclusion $i : X \hookrightarrow B(f)$ is an n -equivalence.

Let $Y := \{y_1, \dots, y_s\}$ be a linear extension of Y , that is, a numbering of the elements of Y such that $y_i \leq y_j$ implies $i \leq j$. Further, set $X_i := X \cup \{y_i, \dots, y_s\}$. Hence, $X_0 = B(f)$ and $X_{s+1} = X$. Note that $X \hookrightarrow B(f)$ is the composition of all the inclusions $X_{i+1} \hookrightarrow X_i$. We show that each inclusion is an n -equivalence. Note that

$$\begin{aligned} X_i &= X \cup \{y_i, \dots, y_s\} = X_{i+1} \cup \{y_i\}, \\ X_{i>y_i} &= Y_{>y_i} \quad \text{and} \quad X_{i<y_i} = X_{<y_i} = f^{-1}(Y_{\leq y_i}). \end{aligned}$$

Therefore, the conclusion of item (1) follows from Lemma A.2 below.

Next, we prove item (2). Here, we will use the fact that if $f, g : X \rightarrow Y$ are two equivariant maps between G -posets such that $f \leq g$, then f and g are equivariantly homotopic (see also the discussion in [15, p. 285]). Observe that the non-Hausdorff cylinder $B(f)$ is naturally a G -poset; we have $X \subseteq B(f)$ and $r : B(f) \rightarrow Y$ is a G -equivariant strong deformation retract (and hence a G -homotopy equivalence) since $r(z) \geq z$ for all $z \in B(f)$. And by the equality $ri = f$, we see that it remains to show that the inclusion $i : X \hookrightarrow B(f)$ is a G -homotopy equivalence.

Now, instead of taking an arbitrary linear extension of Y , we decompose Y into a disjoint union of G -orbits $Y_1, \dots, Y_m$, such that if $y \in Y_i$ and $y' \in Y_j$ are such that $y < y'$, then $i < j$. (Note that we can never have $y < y'$ with $i = j$.) Let $X_i = X \cup Y_i \cup \dots \cup Y_m$. It remains to see that if we remove a whole orbit Y_i , then we get a G -homotopy equivalence $X_{i+1} \hookrightarrow X_i$. Hence, it is enough to establish the case $B(f) = Z = X \cup Y_0$, where Y_0 is a G -orbit, and for all $y \in Y_0$ we have that $\text{Lk}_Z(y) = X_{<y} * X_{>y}$ is G_y -contractible.

At this point, we pass through the posets of chains $\text{sd}(X)$ and $\text{sd}(Z)$. Since the natural maps $\text{sd}(X) \rightarrow X$ and $\text{sd}(Z) \rightarrow Z$, which send a chain to its maximal element, are G -homotopy equivalences, we see that $i : X \hookrightarrow Z$ is a G -homotopy equivalence if and only if the inclusion $j : \text{sd}(X) \hookrightarrow \text{sd}(Z)$ is. We show that the latter map is a G -homotopy equivalence. To this aim, we prove that the maps $j_1 : \text{sd}(Z) \setminus \text{sd}(Y_0) \hookrightarrow \text{sd}(Z)$ and $j_2 : \text{sd}(X) \hookrightarrow \text{sd}(Z) \setminus \text{sd}(Y_0)$ are G -homotopy equivalences. Since $j = j_1 \circ j_2$, this will conclude the proof.

First, note that an element $s \in \text{sd}(Y_0)$ is just a chain of the form $s = s_y := (y)$ for some $y \in Y_0$. Then, we see that $(\text{sd}(Z) \setminus \text{sd}(Y_0))_{>s_y} = \text{sd}(Z)_{>s_y} = \text{sd}(\text{Lk}_Z(y))$ is G_y -contractible by hypothesis. Therefore, j_1 is a G -homotopy equivalence by the Thévenaz–Webb result [26, Theorem 1].

Second, we show that j_2 is a G -homotopy equivalence. If $\tau \in \text{sd}(Z) \setminus \text{sd}(Y_0)$, then $r(\tau) := \tau \setminus Y_0$ is a non-empty chain lying in $\text{sd}(Z \setminus Y_0) = \text{sd}(X)$. Thus, the map $r : \text{sd}(Z) \setminus \text{sd}(Y_0) \rightarrow \text{sd}(Z \setminus Y_0)$ is an equivariant map that satisfies $r(\tau) \leq \tau$. To finish the argument, note that $r \circ j_2 = \text{Id}_{\text{sd}(X)}$ and $j_2 \circ r \leq \text{Id}_{\text{sd}(Z) \setminus \text{sd}(Y_0)}$. Since all these maps are G -equivariant, we conclude that j_2 is a G -homotopy equivalence with inverse r . ■

Lemma A.2. *Let X be a finite poset, and $x \in X$ such that $X_{<x} * X_{>x}$ is $(n-1)$ -connected. Then, the inclusion $X \setminus \{x\} \hookrightarrow X$ is an n -equivalence.*

Proof. (Sketch) For simplicity, we only show the homology version of the result. The homotopy version follows from the homotopy excision theorem.

Consider the covering $X = (X \setminus \{x\}) \cup \text{St}_X(x)$, where $\text{St}_X(x) = \{y \in X : y \leq x \text{ or } y \geq x\}$. Then, $(X \setminus \{x\}) \cap \text{St}_X(x) = X_{<x} * X_{>x}$. Note that $\text{St}_X(x)$ is contractible since every element is comparable with x . By the Mayer–Vietoris sequence applied to this decomposition, and since the intersection is $(n-1)$ -connected, if $m \leq n$ then we have

$$\tilde{H}_m(X_{<x} * X_{>x}) \rightarrow \tilde{H}_m(X - x) \rightarrow \tilde{H}_m(X) \rightarrow \tilde{H}_{m-1}(X_{<x} * X_{>x}) = 0.$$

$=0 \text{ when } m < n$

Hence, the inclusion $X \setminus \{x\} \hookrightarrow X$ is an isomorphism in the homology groups of degree at most $n-1$, and it is an epimorphism in degree n . ■

B. Field and graph automorphisms in the same characteristic p

In this section, we collect some more-technical aspects of the automorphisms in the set Φ_L —which we had indicated in Definition 6.3 as our preferred “field-like” outer automorphisms of L under $(\text{sLie-}p)$.

Introduction: Motivation for the terminology. Our first goal in the section will be to lead up to the general naming-conventions for “field” and “graph” automorphisms in Φ_L —we will follow the conventions used in [12, Section 2.5].

Caution: We do not assume that $p = r$, where p is the prime used for (H-QC) and r is the characteristic of the underlying field of a group of Lie type. This initial discussion will hold for any prime r .

Recall that the group automorphisms of L in Φ_L are defined, at the level of the overlying algebraic group $\bar{L}$, using powers of the generator $x \mapsto x^r$ of the Galois

group of the algebraic closure $\overline{\mathbb{F}_r}$ of the prime field $\mathbb{F}_r$. So, from that algebraic-group viewpoint, it would be natural to call the members of Φ_L by the name “field automorphisms” of L .

And indeed, this is “mostly” the convention in [12, Section 2.5]. Namely, we will see in Proposition B.2 (1) below that usually the members B of $O_G^\Phi(L)$ have the intuitively natural property of centralizing in L a subgroup of the same Lie type—but defined instead over the fixed subfield of B . However, we’ll also see that in some of the d -twisted groups L , certain members of Φ_L (a subset that we will call Φ_d) exhibit a further diagram-symmetry aspect, causing them to centralize in L a subgroup which is defined instead by a “twisted” Dynkin diagram, as we will record in Proposition B.2 (2). These two cases are correspondingly called “field” and “graph” automorphisms in [12]. This post-classical terminology, determined by the above centralizer behavior, appears in Definition 2.5.13 there. We will then indicate our particular special subcase, under (sLie- p), in Definition B.1 below.

We turn to providing some details implementing the above overview.

We’ll now make use of the following expanded notation of [12]. For q denoting the order of the characteristic- r field of definition, we can write L in the form

$$L \cong {}^d\Sigma(q),$$

where the case $d = 1$ indicates that L is the untwisted group with Dynkin diagram of type Σ ; but $d = 2$ or 3 indicates that L is the twisted group corresponding to Σ , constructed using a diagram-symmetry of order d .

In particular, the value of $d > 1$ is one of the possibilities s in Remark 2.6 (g) (i) for elements of the group of diagram-symmetries $\Delta \cong C_2$, or Sym_3 for D_4 . We emphasize these distinct notations as follows. Here, $d > 1$ indicates the order of a symmetry from Δ that was *previously fixed* and used in the construction of twisted L , and indeed identifying it by name. By contrast, as we go on to further consider the possible automorphisms $x \in \text{Aut}(L)$, the various order- s symmetries in Δ are still available as potential ingredients in the action of x on L . In particular, the d -notation does *not* mean that L itself has a graph automorphism of order d . Indeed, we saw in Remark 6.2 (0) that for twisted L (where $d > 1$), we have $\Gamma_L = 1$ —so that even though Δ has order at least d , it makes no graph-contribution to $x \in \text{Aut}(L)$.

We now examine the automorphisms in Φ_L in the above d -context.

Case $d = 1$. So, L is untwisted, and Σ can be any Dynkin diagram.

Here, Φ_L consists just of the usual field automorphisms. And since in our context we mainly work under $(p-\Phi)$ (i.e., the only possible outer automorphisms of order p are field automorphisms), this is basically what we need to know for untwisted L .

But for completeness, and especially as background to the twisted groups below, we now examine the conventions related to classical graph automorphisms for the

untwisted groups $L = {}^1\Sigma(q)$. So, we consider the subcases for Σ , namely, with $\Delta > 1$, where such automorphisms might arise from diagram symmetries.

We saw for the single-bond diagrams $\Sigma = A_n, D_n, E_6$ in Remark 6.2(2) that $\text{Out}(L)^* = \Phi_L \times \Gamma_L$, with $\Gamma_L \cong \Delta \cong C_2$, or Sym_3 for D_4 . Here, the non-trivial members of $\Gamma_L \cong \Delta$ give graph automorphisms in the classical sense of Steinberg. Their products with non-trivial members of Φ_L give graph-field automorphisms.

We also saw in Remark 6.2(1) for the multiple-bond diagrams $\Sigma = B_2, F_4, G_2$ with the further restriction $r = 2, 2, 3$ that we somewhat-similarly have $\text{Out}(L)^* = \Phi_L \Gamma_L$ cyclic, with Φ_L of index 2. Now, in the special case where Φ_L is an r' -group, we get $\Phi_L \Gamma_L = \Phi_L \times \Gamma_L$, so that the order- r subgroup Γ_L as above gives rise to the classical graph automorphisms in the Steinberg-sense. These also determine graph-field automorphisms via products as in the previous paragraph. But more generally, a generator of Γ_L has r th power given by a non-trivial member of Φ_L , and so, should be considered a graph-field automorphism. Indeed, the later terminology-convention in [12, Definition 2.5.13 (b)] is to just call *all* the members of $\Phi_L \Gamma_L \setminus \Phi_L$ by the name of graph-field automorphisms (that is, even the classical Steinberg-sense graph automorphisms in the above special case); so there are *never* any graph automorphisms in this multiple-bond case, in the new convention.

Now, consider the remaining case.

Case $d > 1$. So that $L = {}^d\Sigma(q)$ is twisted (for the relevant Σ above).

Here, we saw in Remark 6.2(0) that $\Gamma_L = 1$. Thus, there are no Steinberg-sense graph automorphisms, and $\text{Out}(L)^* = \Phi_L$, which classically would be called field automorphisms. So, we examine the revised convention in [12, Definition 2.5.13 (c)], in which the name “graph automorphism” is used for certain members of Φ_L .

In the multiple-bond cases $\Sigma = B_2, F_4, G_2$ with $r = 2, 2, 3$, we have $d = 2$; and here there is no change of terminology: all members of Φ_L are still called “field automorphisms”. Furthermore, we recall (e.g., [12, Theorem 2.2.3 (b)]) that our twisted L in these cases is defined over a field of order an *odd* power of r ; so we see that for $d = 2$ here, we have Φ_L a d' -group.

So, now consider the single-bond cases $\Sigma = A_n, D_n, E_6$, with $d = 2$ (or possibly 3 in case D_4). Again, there is no change of terminology, for elements $a \in \Phi_L$ of order coprime to d —they are still called “field automorphisms”. However, if d does divide the order of a , the revised terminology of [12] is to call a a “graph automorphism”. And correspondingly, for this $d > 1$, we set

$$\Phi_d := \{a \in \Phi_L : d \text{ divides } |a|\},$$

which is non-empty if and only if $L = {}^d\Sigma(q)$ with single-bond Σ (since Φ_L is a d' -group for the twisted multiple-bond cases).

The motivation for this choice of terminology is indicated in the remark after [12, Definition 2.5.13], and we expand now on one aspect which suggests the “graph-like” behavior of such a . Namely, our d -twisted group L arises as the fixed point subgroup $C_{\bar{L}}(\bar{f}\bar{g})$, where $\bar{L}$ is an overlying algebraic group of (untwisted) type Σ , for a suitable order- d field automorphism $\bar{f} \in \Phi_{\bar{L}}$, and a commuting order- d graph automorphism $\bar{g} \in \Gamma_{\bar{L}}$. In particular, we can regard L as the subgroup on which $\bar{f}$ acts as $\bar{g}^{-1}$. Since $\Phi_{\bar{L}}$ is cyclic, we can expect a to be influenced by this graph behavior of $\bar{f}$ (notably when $\bar{a} = \bar{f}$, which leads to the “if” part of the “if and only if” statement above). And indeed, we will see in Proposition B.2 (2) (b) below that we can get $C_L(a)$ of Lie type—for a diagram *different* from Σ for $L = {}^d\Sigma(q)$.

With the above discussion of general naming-conventions in place, in the remainder of the section, we will obtain some results which are related to our further hypothesis of L satisfying (sLie- p). Recall this means that L is a simple group of Lie type in characteristic r , where now $r = p$ for the prime p we use in studying (H-QC). Note since L is simple that L is adjoint (cf. [12, Theorems 2.2.6 and 2.2.7]).

So, for this context, we will first specialize our naming-conventions to outer automorphisms of order p . Recall that Φ_L is a cyclic and normal subgroup of $\text{Out}(L)^* = \text{Out}(L)/\text{Outdiag}(L)$ (see (6.1)). Thus, for $x \in \text{Aut}(L)$, we see that we have x $\text{Aut}(L)$ -conjugate to y with $y^* \in \Phi_L$ if and only if we already have $x^* \in \Phi_L$. In particular, for such an x of order p , its location in Φ_L is determined, since

Cyclic Φ_L has a unique subgroup $\Omega_1(O_p(\Phi_L))$ of order p (which is also normal).

We can now check, using the discussion above, that such non-trivial elements $x \in \Omega_1(O_p(\Phi_L))$ lie in Φ_d if and only if $p = d$; recall we already had $p = r$. (Here, we are again using the fact that for the twisted groups with multiple-bond diagrams, Φ_L is a d' -group for $d = 2$.)

Below, we indicate the [12]-naming for such order p automorphisms x of L . We will use the case division on whether x^* lies in $\Phi_L \setminus \Phi_d$ or in Φ_d .

Definition B.1 (“Field” and “graph” naming-conventions for order- p automorphisms in Φ_L). Assume that $L = {}^d\Sigma(q)$ satisfies (sLie- p), so that $p = r$, and we have some $x \in \text{Aut}(L)$ of order p , with x conjugate under $\text{Aut}(L)$ to an element of Φ_L . Then, in fact x^* itself lies in Φ_L , and so, generates $\Omega_1(O_p(\Phi_L))$. And the naming-convention of [12] gives us the following two cases.

(1) $(\Phi_L \setminus \Phi_d = \text{field})$. When $x^* \in \Phi_L \setminus \Phi_d$, x is called a *field automorphism*. Notice, when we now consider the specific diagram types Σ for $L = {}^d\Sigma(q)$, that this field-type is the “usual” case: since we saw $\Phi_d = \emptyset$ for untwisted L (i.e., $d = 1$); and also for many twisted cases L —namely, with $d > 1$ but not indicated in (2) below. Indeed, we saw in our preliminary discussion above that for twisted L , where Σ has

multiple bonds (namely, B_2, F_4, G_2 with $d = 2$), we have Φ_L is a $2'$ -group in each case. Hence, since we are now further assuming (sLie- p) so that $p = r = 2, 2, 3$, such an x exists if and only if $p = 3$ and L has type 2G_2 . By contrast, when Σ has single-bonds, our twisted L here is in the proper-subcase where $p \neq d$, namely, $p \neq 2$ in all types A_n, D_n, E_6 , or $p \neq 3$ in type 3D_4 .

(2) (Φ_d =graph). Otherwise, x^* is in Φ_d , and x is called a *graph automorphism*. Recall that $\Phi_d \neq \emptyset$ requires L twisted, with single-bond diagram Σ (namely, of type A_n, D_n, E_6), and d dividing the order of Φ_L ; while $p = d$ by our assumption that x has order p . Thus, the possibilities for the group L are

$$L = {}^2A_n(q) = \text{PSU}_{n+1}(q) \ (n \geq 3); \quad {}^2D_n(q) \ (n \geq 4); \quad {}^2E_6(q); \text{ and } {}^3D_4(q),$$

with $p = d = 2, 2, 2, 3$, respectively. ■

Next, we summarize the structure of the centralizers of order- p elements of type Φ_L . Recall the poset $\mathcal{O}_{\text{Aut}(L)}^\Phi(L)$ from Definition 6.3.

Proposition B.2. *Let $L \cong {}^d\Sigma(q)$ be of type (sLie- p). Assume $B \in \mathcal{O}_{\text{Aut}(L)}^\Phi(L)$, and note since Φ_L is cyclic that $B^* = \Omega_1(O_p(\Phi_L))$.*

- (1) *Assume B as in case (1) of Definition B.1, i.e., $B^* \subseteq \Phi_L \setminus \Phi_d$. Then, the elements of $\mathcal{O}_{\text{Aut}(L)}^\Phi(L)$ are all $\text{Inndiag}(L)$ -conjugate to B . Furthermore, $O^{p'}(C_L(B)) \cong {}^d\Sigma(q^{1/p})$, that is, the (adjoint) group of the same Lie type (and hence the same Lie rank), but over the B -fixed subfield. We further have $C_{\text{Inndiag}(L)}(B) = \text{Inndiag}({}^d\Sigma(q^{1/p}))$, and, in particular, $O_p(C_L(B)) = 1$.*
- (2) *Assume instead B is in case (2) of Definition B.1, so $x^* \in \Phi_d$ for a generator x of B . Then, B satisfies one of the following two mutually exclusive conditions:*
 - (1) *$F^*(C_L(B)) = O_p(C_L(B)) > 1$ is a non-trivial p -group; or else*
 - (2) *B is $\text{Inndiag}(L)$ -conjugate to $\langle \gamma \rangle$, where $\gamma \in \Phi_d$ has order p , and the centralizer $C_L(B)$ is a subgroup of the same Lie rank m below as L , but of the indicated different, untwisted Lie type over $\mathbb{F}_q$ ²¹:*

p	L	m	$C_L(B)$
2	$\text{PSU}_n(q) \cong {}^2A_{n-1}(q)$	$\lfloor \frac{n}{2} \rfloor$	$C_{\lfloor \frac{n}{2} \rfloor}(q) \cong \text{PSp}_{2\lfloor \frac{n}{2} \rfloor}(q)$
2	${}^2D_n(q)$	$n - 1$	$C_{n-1}(q) \cong \text{PSp}_{2(n-1)}(q)$
2	${}^2E_6(q)$	4	$F_4(q)$
3	${}^3D_4(q)$	2	$G_2(q)$.

In particular, $O_p(C_L(B)) = 1$.

²¹The Lie type of $C_L(B)$ can be regarded as that for a suitable quotient-diagram of the Dynkin diagram for L , under the order- d diagram symmetry.

- (3) For any $C \in \mathcal{O}_{\text{Aut}(L)}^\Phi(L)$, we have $\mathcal{O}_{LC}(L) = \mathcal{O}_{LC}^\Phi(L)$ so that we can take C in the role of “ B ” above. Further, there is $D \in \mathcal{O}_{LC}(L)$ (hence of type Φ) with $O_p(C_L(D)) = 1$. In particular, if (2)(a) holds for C , then (2)(b) holds for D .

Proof. Item (1) follows directly from [12, Proposition 4.9.1]. Here, we deduce that $O_p(C_L(B)) = 1$ from the Lie-type structure of $O^{p'}(C_L(B))$. (Indeed, that Lie-type group is usually even simple.) Item (2) follows from [12, Proposition 4.9.2]—using similar Lie-type remarks for $O_p(C_L(B)) = 1$ in case (2)(b). The first part of (3) then follows from the observation early in the statement that $B^* = \Omega_1(O_p(\Phi_L))$. And then by that first part, we can—and do—now take B in the role of the original “ C ”.

If (1) holds for B , then the second part of (3) already holds, with B itself in the role of “ D ”. So, we may assume (2) holds for B ; and, in particular, L is twisted, for Σ with single-bonds (hence all roots are “long”). If in fact case (2)(b) already holds for B , then again the second part of (3) holds with B in the role of “ D ”. So, we may assume that case (2)(a) holds for B .

Exclude for a moment the case $p = 2$ and $L = {}^2D_n(q)$, with q even. With the notation of [12, Proposition 4.9.2], let Z be a long-root subgroup of L . Then, $Z \leq L$; and as in the proof of that proposition, any member of $\mathcal{O}_{\text{Aut}(L)}^\Phi(L)$ is $\text{Inndiag}(L)$ -conjugate to $\langle \gamma \rangle$ or $\langle \gamma z \rangle$ (for $z \in Z^\#$), where $\gamma \in \Phi_L$ of order p is as in item (2)(b). That is, LB contains an element of the form γ^g or $(\gamma z)^g$, with $g \in \text{Inndiag}(L)$. If we have the latter, that is $(\gamma z)^g \in LB$, then since $z^g \in Z^g \leq L^g = L$, we also get $\gamma^g \in LB$; so we have $O_p(C_L(\gamma^g)) \cong O_p(C_L(\gamma)) = 1$, as we had noted for case (2)(b). Hence, we can take “ D ” to be $\langle \gamma^g \rangle \in \mathcal{O}_{LB}(L)$.

Finally, suppose that $p = 2$ and $L = {}^2D_n(q)$. Then, $\text{Out}(L)$ is cyclic, and so, $LB = LD$ for every $B, D \in \mathcal{O}_{\text{Aut}(L)}^\Phi(L)$. That is, $\mathcal{O}_{\text{Aut}(L)}^\Phi(L) = \mathcal{O}_{LB}^\Phi(L) = \mathcal{O}_{LB}(L)$. In particular, we can take “ D ” in $\mathcal{O}_{\text{Aut}(L)}^\Phi(L)$ as in (2)(b). This final case completes the proof of the second part of (3). (See also the remark right after the statement of [12, Proposition 4.9.2].) ■

Note that item (3) of the previous proposition provides a “non-conical complement”, as in [4, Theorem 2.3]—we had mentioned these via (someNC) in Remark 3.10, and we need them for the corresponding hypothesis (someNC+ Φ) in Theorem 6.4.

The starting point for our homology propagation in Theorem 6.4 will be the non-zero homology of $\mathcal{A}_p(LB)$ provided by Theorem 6.6. And to bound the vector-space dimension of that homology group above zero, we will want the following easy but somewhat-technical numerical observation. Roughly, we check below, based on our determination of centralizer-structures in Proposition B.2, and using just the standard group-order formulas for the individual simple groups L of Lie type, that the index

$|L : C_L(B)|_{p'}$ (which basically records the number of tori) grows very strongly with increasing powers of p ; and, in particular, exceeds the very modest bound of $(p - 1)$ that we require. This result may be well known, or clear to those familiar with the polynomials in the orders.

Lemma B.3. *Let L be of type (sLie- p). If $B \in \mathcal{O}_{\text{Aut}(L)}^\Phi(L)$ satisfies $O_p(C_L(B)) = 1$ (that is, case (1) or (2) (b) of Proposition B.2 holds), then $|L : C_L(B)|_{p'} > (p - 1)$.*

Proof. For the order-formulas discussed below, we will use [12, Table 2.2] as our basic source. And rather than checking each detailed formula, we will try to exploit common features of the formulas as products of cyclotomic polynomials in q , which are easy to read from that table.

Recall that we write $L \cong {}^d\Sigma(q)$. We split the verification of the bound into the two cases in the hypothesis: the field-automorphism case $B^* \subseteq \Phi_L \setminus \Phi_d$ of (1) of Proposition B.2; and the graph-automorphism case $x^* \in \Phi_d$, $\langle x \rangle = B$, of (2) (b) there.

The field-case $B^ \subseteq \Phi_L \setminus \Phi_d$.* We now deviate from our earlier notational convention of using r for the characteristic of L —of course p supplies that characteristic for us under (sLie- p)—to instead write r for $q^{1/p}$, the order of the fixed subfield under B . Thus, $q = r^p$, where r itself is a power of p (so, in particular, $r \geq p$).

We first treat the case of untwisted L ($d = 1$). The formula for $|L|_{p'}$ for the adjoint simple group L has the form

$$\frac{1}{f} \prod_e (q^e - 1),$$

where the exponents e and the divisor f are specified in [12, Table 2.2]. In type A_n , we have $f = \gcd(n + 1, q - 1) \leq q - 1$, while $f \leq 4 \leq q$ for all other Lie types; so for crude bounding, it will usually suffice to just say in summary that

$$f \leq q.$$

Now, we will need to compare the above product for $|L|_{p'}$ with the similar product for $|C_L(B)|_{p'}$. By case (1) of Proposition B.2, $C_L(B)$ lies between the adjoint version of ${}^d\Sigma(r)$ and $\text{Inndiag}({}^d\Sigma(r))$. Here, the possible extra order in the Inndiag -form is controlled by f ; that is, for our bounding-purposes, the Inndiag -order formula we need to use is the analogue for r of the polynomial-product for $|L|_{p'}$ above. Note that

$$\Pi := \prod_e \frac{(r^p)^e - 1}{r^e - 1} = \prod_e \frac{(r^e)^p - 1}{r^e - 1} = \prod_e ((r^e)^{p-1} + (r^e)^{p-2} + \cdots + 1); \quad (\text{B.1})$$

so it will suffice to verify the following variant form of our original bound

$$\Pi = \prod_e (r^{e(p-1)} + r^{e(p-2)} + \cdots + 1) \quad \text{should be} > f(p - 1). \quad (\text{B.2})$$

We now observe that the number of exponents e is equal to the Lie rank n —giving the number of factors in Π . Furthermore, the smallest exponent always takes the value $e = 2$. In particular, note then that each factor in Π exceeds $r^2 + 1 > 1$. So, it will suffice to find a sub-product satisfying our bound—and it turns out that the first factor (for $e = 2$) already works. Namely, since we saw $f \leq q$, it suffices for the purposes of (B.2) to set $e = 2$ there, and get a value $> q(p - 1)$.

To that end, consider first the case that $p = 2$. Then, (B.2) gives $r^2 + 1 > r^2 = q(p - 1)$ —as required. So, now we assume the remaining case, that is, $p \geq 3$. Then, $2(p - 1) \geq p + 1$, so the leading term $r^{2(p-1)}$ in the $e = 2$ factor in (B.2) is at least $r^{p+1} = qr > q(p - 1)$, since r is a power of p . This completes the proof of the bound for field-automorphisms of untwisted L .

So, we turn to field-automorphisms for the case of twisted L , where $d > 1$. We will use similar ideas. Let us first recall the cases here.

We saw in the corresponding case (1) of Definition B.1 that the only multiple-bond case arising here is type 2G_2 for an odd power of $p = 3$; while we can have the single-bond twisted types when $p \neq d$, namely, ${}^2A_n, {}^2D_n, {}^2E_6$ when $p \neq 2$, and type 3D_4 when $p \neq 3$.

Again, we obtain a product Π of polynomials, in an analogue of (B.1) and (B.2), but this time there are some differences from the untwisted case above. The number of exponents is now the Lie rank of the overlying Lie-type group $\Sigma(q)$, which exceeds the Lie rank of the twisted group $L = {}^d\Sigma(q)$; and this number is always ≥ 2 —so we have no case where Π has just one single factor (even when L has Lie rank 1). And now, the exponents e come with signs $\varepsilon = \pm 1$ (or cube roots of unity for type 3D_4), giving the polynomial $q^e - \varepsilon$. We still have minimum exponent-value $e = 2$, now always with sign $\varepsilon = -1$. One effect is to give, in type 2A_n , the adjusted value $f = \gcd(n + 1, q + 1) \leq q + 1$ —and, correspondingly, now

$$f \leq q + 1.$$

Another effect is that for $\varepsilon \neq 1$, we do not get the even division of $q^e + 1$ by $r^e + 1$ (nor analogously for cube roots); however, we still get the property that each factor in Π exceeds 1, just using crude estimates, such as

$$\frac{r^{pe} + 1}{r^e + 1} > \frac{r^{pe}}{2r^e} \geq \frac{r^{p(e-1)}}{2} \geq 2 > 1.$$

So, again, we may establish our bound on a sub-product (and again it suffices that we have two factors).

Namely, consider any of the above types other than 3D_4 . Then, in each case, there is a choice of a non-minimal exponent $e \geq 3$ with sign $\varepsilon = 1$. So, we recall the calculation in the untwisted case, showing this second factor contributes at least qr to

the product. This time using $r < q$ since L is twisted, we get

$$qr > qr + (r - q) + (-1) = (q + 1)(r - 1) \geq f(p - 1),$$

again giving the bound desired in the analogue of (B.2).

So, we turn to the remaining case of type 3D_4 . Here, $f = 1$; so we can just quote our earlier crude bound $\frac{r^{p(e-1)}}{2}$ with $e = 2$, giving at least $\frac{r^p}{2} \geq r^{p-1} \geq r > p - 1$, as desired.

This completes the proof for field automorphisms of twisted L , and hence for field automorphisms of all L —the case $B^* \in \Phi_L \setminus \Phi_d$.

The graph-case $d = p$. Notice here we do not consider the fixed subfield under B , and so, our quotient calculations in Π will involve only the power q of p (that is, no “ r ”).

The four types for L in Proposition B.2 (2)(b) are ${}^2A_{n-1}(q)$, ${}^2D_n(q)$, ${}^2E_6(q)$, ${}^3D_4(q)$, where we have $p = 2, 2, 2, 3$; and $C_L(B) = C_{\lfloor \frac{n}{2} \rfloor}(q)$, $C_{n-1}(q)$, $F_4(q)$, $G_2(q)$. Here, as before in type ${}^2A_{n-1}(q)$ we have $f \leq \gcd(n, q + 1) \leq q + 1$, and $f \leq 4 \leq q$ in other types, so that again

$$f \leq q + 1;$$

and though this estimate might be sharpened by dividing out f -type terms in $C_L(B)$, it will suffice to use the above rough bound.

Again, we get a product Π , in an analogue of (B.2)—now with still more differences. First, the number of exponents for twisted $L = {}^d\Sigma(q)$ is the Lie rank for untwisted $\Sigma(q)$, namely, $n, n, 6, 4$, which strictly exceeds the number of exponents for untwisted $C_L(B)$, i.e., its Lie rank $\lfloor \frac{n}{2} \rfloor, n - 1, 4, 2$. Now, we check in each L (again excepting type ${}^3D_4(q)$, which varies slightly) that we may pair each exponent e for $C_L(B)$ (which has positive sign $\varepsilon = +1$) with an exponent e of the same value, but negative sign $\varepsilon = -1$, starting at the minimum value $e = 2$. As a result, the factor at e in the product Π has the value $\frac{q^e + 1}{q^e - 1}$, only slightly larger than 1. Thus, to get our bound, we focus instead on the exponents e of L which are *not* paired with those of $C_L(B)$; these give “numerator-only” terms in the product Π .

And just one is enough: note we have $e \geq 3$ for such an exponent, giving a term at least

$$q^3 + 1 > q^2 - 1 = (q + 1)(q - 1) \geq f(p - 1),$$

which is the bound that we needed.

To adapt the above proof to cover the remaining type $L = {}^3D_4(q)$, just “pair” the exponent $e = 6$ for $C_L(B) = G_2(q)$ with the two exponents $e = 4$ (and their “signs” ω and $\bar{\omega}$)—the quotient of $(q^4 - \omega)(q^4 - \bar{\omega}) = q^8 + q^4 + 1$ by $q^6 - 1$ certainly exceeds 1. This leaves an unpaired numerator-term of $q^8 + q^4 + 1$, which exceeds the desired bound $p - 1$, since $f = 1$ in this case.

We have now verified the bound of Lemma B.3 in all cases. ■

C. Additional lemma(s)

The following lemma provides an alternative way of eliminating certain components L of a minimal counterexample G to (H-QC). Namely, those simple components L for which $O_p(C_L(B)) > 1$ for every p -outer B . The proof of this lemma invokes Theorem 3.6 with a non-standard p -subgroup poset $\mathcal{B}$. Indeed, we take $\mathcal{B}$ to be the image-poset $\mathcal{A}_{G,L}$, for a simple component L of a group G .

For example, if $L = \text{PSL}_2(2^3), \text{PSU}_3(2^3), \text{Sz}(2^5)$ with $p = 3, 3, 5$, respectively, then every p -outer B of L satisfies that $O_p(C_L(B)) > 1$. Therefore, Lemma C.1 provides an alternative argument, via Theorems 2.24 and 3.6, to the elimination of these components (cf. [16, Theorem 5.1].)

Lemma C.1. *Let G be a finite group, and $L \leq G$ a simple subgroup. If $O_p(C_L(B)) > 1$ for all $B \in O_G(L)$, then the inclusion $\mathcal{A}_p(L) \hookrightarrow \mathcal{A}_{G,L}$ is a homotopy equivalence.*

In particular, if in addition L is a component of G of order divisible by p and $C_G(\hat{L})$ satisfies (H-QC), then G satisfies (H-QC)²².

Proof. Since L is simple, we can regard $\mathcal{A}_p(L) \subseteq \mathcal{A}_{G,L} \subseteq \mathcal{A}_p(\text{Aut}(L))$. Let $\mathcal{B} := \mathcal{A}_{G,L}$. Then,

$$\mathcal{F}_{\mathcal{B}}(L) = \{DC_G(L)/C_G(L) : D \in O_G(L)\} = O_{\text{Aut}(L)}(L) \cap \mathcal{A}_{G,L},$$

using Remark 2.18, again since $Z(L) = 1$ by the hypothesis of simplicity of L . Fix $D \in O_G(L)$, and let B be the image of D in $\mathcal{B}$. Suppose that $E \in \mathcal{N}_{\mathcal{B}}(L)$ is such that $C_{E \cap L}(B) > 1$. We may pick some $F \in \mathcal{A}_p(N_G(L))$ with $FC_G(L)/C_G(L) = E$. In particular, $F \cap L > 1$. Then, it is straightforward to verify that

$$\begin{aligned} C_E(B) &= C_F(D)C_G(L)/C_G(L) \in \mathcal{B}, \\ C_E(B)B &= (C_F(D)D)C_G(L)/C_G(L) \in \mathcal{B}. \end{aligned}$$

Therefore, $\mathcal{B}$ satisfies the requirements of Theorem 3.6, so we have a homotopy equivalence $\mathcal{B} \simeq X_{\mathcal{B}}(L) = \mathcal{A}_p(L) \cup \mathcal{F}_{\mathcal{B}}(L)$.

Now, if $B \in \mathcal{F}_{\mathcal{B}}(L)$, then $O_p(C_L(B)) > 1$ by our hypothesis on such B . Then, we have

$$X_{\mathcal{B}}(L)_{>B} \cap \mathcal{A}_p(L) = \{E \in \mathcal{A}_p(L) : C_E(B) > 1\} \simeq \mathcal{A}_p(C_L(B)) \simeq *.$$

Therefore, $X_{\mathcal{B}}(L)$ collapses to the subposet $\mathcal{A}_p(L)$; that is, the inclusion $\mathcal{A}_p(L) \hookrightarrow X_{\mathcal{B}}(L)$ is a homotopy equivalence by the Quillen fiber Theorem 2.10. Since the

²²Recall from Definition 2.2 that $\hat{L}$ denotes the product of the G -orbit of L .

equivalence $\alpha_{\mathcal{B},L} : \mathcal{B} \rightarrow X_{\mathcal{B}}(L)$ is the identity when restricted to $\mathcal{A}_p(L)$ by Theorem 3.6(1), we conclude that $\mathcal{A}_p(L) \hookrightarrow \mathcal{B}$ is also a homotopy equivalence. This finishes the proof of the first part.

The “In particular” part follows from Theorem 2.24, since $\mathcal{A}_p(L) \rightarrow \mathcal{A}_{G,L}$ is an isomorphism in homology, and $\tilde{H}_*(\mathcal{A}_p(L)) \neq 0$ by the almost simple case of the conjecture (recall the result of [3] quoted as Theorem 2.20.) ■

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